

# Quantifying the Trendiness of Trends (or how I learned to stop worrying and love the uncertainty of changes)

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## Abstract

A statement often seen in the news concerning some public health outcome is that some trend has changed or been broken. Such statements are often based on longitudinal data from e.g., surveys, and the change in the trend is claimed to have occurred at the time of the latest data collection. These types of statistical assessments are very important as they may potentially influence public health decisions on a national level.

Instead we propose a statistical model using methods from Functional Data Analysis under the assumption that reality evolves in continuous time. Under this assumption we define what constitutes a trend and a change in a trend, and we introduce a probabilistic Trend Direction Index. This index has the intuitive interpretation of the probability that a function has changed monotonicity at any given time conditional on observed data. We also define a global index of Expected Trend Instability that quantifies the expected number of times a trend changes on an interval.

We show how the Trend Direction Index and the Expected Trend Instability can be estimated from data in a Bayesian framework and give an application to development of the proportion of smokers in Denmark during the last 20 years.

**Keywords:** Functional Data Analysis, Gaussian Processes, Trends, Bayesian Statistics

## 1 Introduction

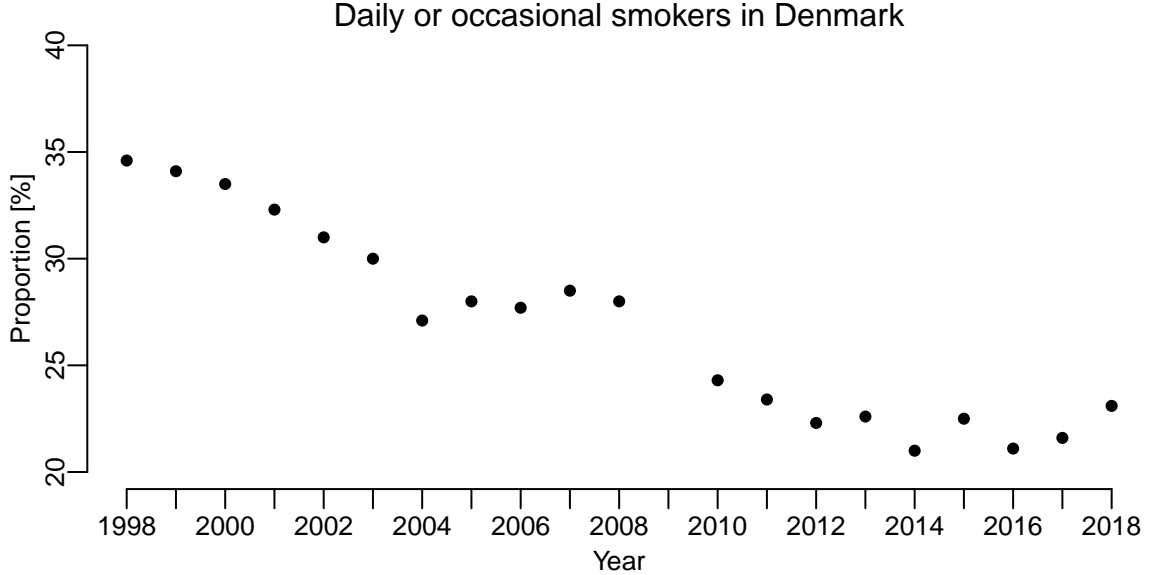
This manuscript is concerned with the fundamental problem of estimating an underlying trend based on random variables observed repeatedly over time. In addition to this problem we also wish to assess points in time where it is possible that such a trend is changing. Our motivation is a recent example from the news in Denmark where it was stated that the trend in the proportion of smokers in Denmark has changed at the end of the year 2018. This statement was based on survey data collected yearly since 1998 and reported by the Danish Health Authority (The Danish Health Authority 2019). It is not immediately obvious what constitutes a trend and more so what a change in a trend is. To elucidate these questions and provide a clear specification we start from the ground by defining the problem.

[**TODO:** Claus tilbyder at b tte sig igennem resten af introduktionen].

[**TODO:** What do we propose? We work in continuous time and introduce TDI and ETI informally at this point. TDI gives the local probability of the monotonicity of  $f$  and can answer the question „What is the probability that  $f$  is increasing?“. ETI gives the expected number of changes in the monotonicity of  $f$  on an interval and can answer the question „Is it the first time in 20 years that  $f$  has changed trend?].

[**TODO:** We are not doing a hypothesis test and compute a p-value. I.e., we do not calculate the probability that an observed change in trend is different than would have been expected if there was

no change in trend. We estimate the actual probability that the trend is changing.]



**Figure 1:** The proportion of daily or occasional smokers in Denmark during the last 20 years estimated from survey data and reported by the Danish Health Authority. [TODO: Mention in text why 2009 is missing]

[TODO: Assert that both TDI and EDI are scale free. They don't tell anything about the magnitude of a trend. They should therefore always be accompanied by plots of the posterior of  $df$ ].

[TODO: Reference some previous work]. Gottlieb and Müller (2012) define a stickiness coefficient for longitudinal data. Our ETI has some similarities to the tracking index of Foulkes and Davis (1981).

The manuscript is structured as follows: In section 2 we present our statistical model based on a latent Gaussian process formulation giving rise to analytic expressions for the Trend Direction Index and the Expected Trend Instability index conditional on observed data. We then extend this basic model to enable a data-driven adaptation to non-stationarity in the underlying dynamics. Section 2.4 is concerned with estimating the models parameters, and we give an application to the proportion of smokers in Denmark during the last 20 years in section 3, and we conclude with a discussion.

## 2 Methods

We start by defining the underlying assumptions for our method. We assume that reality evolves in continuous time  $t \in \mathcal{T} \subset \mathbb{R}$  and that there exists a random, latent function  $f = \{f(t) : t \in \mathcal{T}\}$  defined on a compact subset of the real line,  $\mathcal{T}$ , that governs the underlying evolution of some observable characteristic in a population. We can observe this latent characteristic with noise by sampling  $f$  at discrete time points according to the additive model  $Y_i = f(t_i) + \varepsilon_i$  where  $\varepsilon_i$  is a zero mean random variable independent of both  $Y_i$  and  $f(t_i)$ . Given observations of the form  $(Y_i, t_i)_{i=1}^n$  we are interested in estimating the dynamical properties of  $f$ .

The trend of  $f$  is defined as its instantaneous slope given by the function  $df(t) = \left(\frac{df(s)}{ds}\right)(t)$ , and  $f$  is increasing and has a positive trend at  $t$  if  $df(t) > 0$ , and  $f$  is decreasing with a negative trend at  $t$  if  $df(t) < 0$ . A change in trend is defined as a change in the sign of  $df$ , i.e., when  $f$  goes from increasing to decreasing or vice versa. As  $f$  is a random function there are no exact points in time where  $df$  changes sign. There is instead a gradual and continuous change in the local monotonicity of  $f$ , and an assessment of a change in trend is defined by the probability of the sign of  $df$ . This stands in contrast to traditional change-point models which assume that there are one or more exact time points where sudden a change in deterministic function occurs (Picard 1985).

To quantify the probability that a trend is changing we introduce the local probabilistic Trend Direction Index

$$\text{TDI}(t, \delta) = P(df(t + \delta) > 0 \mid \mathcal{F}_t), \quad t \in \mathcal{T} \quad (1)$$

where  $\mathcal{F}_t$  is a  $\sigma$ -algebra of available information observed up until time  $t$ . The value of  $\text{TDI}(t, \delta)$  is equal to the probability that  $f$  is increasing at time  $t + \delta$  given everything know about the data generating process up until and including time  $t$ . A similar definition can be given for a negative trend but that is equal to  $1 - \text{TDI}(t, \delta)$  and therefore redundant. The sign of  $\delta$  determines whether the Trend Direction Index estimates the past ( $\delta \leq 0$ ) or forecast into the future ( $\delta > 0$ ). Most of the examples seen in the news concerning public health outcomes is concerned with  $t$  being equal to the current calendar time and  $\delta = 0$ . This excludes the usage of both change-point and segmented regression models (Quandt 1958) as there are no observations available beyond the stipulated change-point. A useful parametrization of the Trend Direction Index is  $\text{TDI}(\max \mathcal{T}, t - \max \mathcal{T})$  with  $t \leq \max \mathcal{T}$ . In this formulation we condition on the full observation period and looks back in time, and  $t = \max \mathcal{T}$  corresponds to the Trend Direction Index at the end of the observation period.

In addition to the Trend Direction Index we also define a global measure of trend instability. Informally we say that a function  $f$  is *trend stable* on an interval  $\mathcal{I} \subseteq \mathcal{T}$  if it maintains its monotonicity so that its trend does not change on the interval. To quantify the trend instability of a random function  $f$  we propose to use the expected number of zero-crossings by  $df$  on  $\mathcal{I}$ . Informally at this point, we define the Expected Trend Instability as

$$\text{ETI}(\mathcal{I}) = \mathbb{E}[\#\{t \in \mathcal{I} : df(t) = 0\} \mid \mathcal{F}] \quad (2)$$

which is the expected value of the size of the random point set of zero crossings by  $df$  on  $\mathcal{I}$  conditional on a suitable  $\sigma$ -algebra  $\mathcal{F}$ . The usual case is when  $\mathcal{F}$  is generated by all observed data on  $\mathcal{T}$  and  $\mathcal{I} \subseteq \mathcal{T}$ . The lower  $\text{ETI}(\mathcal{I})$  is, the more trend stable  $f$  is on  $\mathcal{I}$  and vice versa.

These general definitions of trendiness will be evaluated in light of a concrete statistical model in the next section leading up to expressions of their estimates.

## 2.1 Latent Gaussian Process Model

The definitions in the previous section impose the existence of an latent function  $f$ . We shall assume that  $f$  is a Gaussian process on  $\mathcal{T}$ . From a Bayesian perspective this is equivalent to imposing

an infinite dimensional prior distribution on the latent function. Statistical models with Gaussian process priors are a flexible approach for non-parametric regression (Radford 1999). Further, it presents an analytically tractable way for performing statistical inference on its derivatives. The general idea of our model is to apply the properties of the Gaussian process prior on  $f$  to update its finite dimensional distributions by conditioning on the observed data. This results in a posterior Gaussian process from which estimates of the trendiness indices are derived.

A random function  $f$  on is a Gaussian process if and only if the vector  $(f(t_1), \dots, f(t_n))$  has a multivariate normal distribution for every finite set of evaluation points  $(t_1, \dots, t_n)$ , and we write  $f \sim \mathcal{GP}(\mu(\cdot), C(\cdot, \cdot))$  where  $\mu$  is the mean function and  $C$  is a symmetric, positive definite covariance function (Cramer and Leadbetter 1967). We have observed data of the form  $(Y_i, t_i)_{i=1}^n$  and assume that the observations are generating model by the following hierarchical model

$$\begin{aligned} f \mid \beta, \theta &\sim \mathcal{GP}(\mu_\beta(\cdot), C_\theta(\cdot, \cdot)) \\ Y_i \mid t_i, f(t_i), \Theta &\stackrel{iid}{\sim} N(f(t_i), \sigma^2), \quad \Theta = (\beta, \theta, \sigma) \end{aligned} \quad (3)$$

where  $\beta$  is a vector of parameters for the mean function of  $f$ ,  $\theta$  is a vector of parameters governing the covariance of  $f$ , and  $\sigma$  is the conditional standard deviation of the observations. We assume the following regularity conditions holds for  $f$ .

**Assumption 1.** *We assume that  $f$  is a separable Gaussian process and that*

*A1: ?*

*A2: ?*

*This ensures that the sample paths of  $f$  are in  $C^r(\mathcal{T})$ . We require that  $r \geq 1$  for the Trend Direction Index and  $r \geq 2$  for the index of Expected Trend Instability. [s@scheuerer2010regularity]*

Under the above assumptions, an important property of a Gaussian process is that it together with its first and second derivative functions are multivariate Gaussian process with explicit expressions for the mean, covariance and cross-covariance functions. Specifically, the joint distribution of the latent function and its first and second derivative is the multivariate Gaussian process

$$\begin{bmatrix} f(s) \\ df(t) \\ d^2f(u) \end{bmatrix} \mid \beta, \theta \sim \mathcal{GP} \left( \begin{bmatrix} \mu_\beta(s) \\ d\mu_\beta(t) \\ d^2\mu_\beta(u) \end{bmatrix}, \begin{bmatrix} C_\theta(s, s') & \partial_2 C_\theta(s, t) & \partial_2^2 C_\theta(s, u) \\ \partial_1 C_\theta(t, s) & \partial_1 \partial_2 C_\theta(t, t') & \partial_1 \partial_2^2 C_\theta(t, u) \\ \partial_1^2 C_\theta(u, s) & \partial_1^2 \partial_2 C_\theta(u, t) & \partial_1^2 \partial_2^2 C_\theta(u, u') \end{bmatrix} \right) \quad (4)$$

where  $d^k \mu_\beta$  is the  $k$ 'th derivative of  $\mu_\beta$  and  $\partial_j^k$  denotes the  $k$ 'th order partial derivatives with respect to the  $j$ 'th variable (Cramer and Leadbetter 1967). Proposition 1 gives the joint posterior distribution of  $(f, df, d^2f)$  condition on observed data  $(Y_i, t_i)_{i=1}^n$ .

**Proposition 1.** *Let the data generating model be defined as in Equation (3) and  $\mathbf{Y} = (Y_1, \dots, Y_n)$  the vector of observed outcomes together with its sampling times  $\mathbf{t} = (t_1, \dots, t_n)$ . Then under Assumptions ??? the posterior distribution of  $(f, df, d^2f)$  evaluated at the vector  $\mathbf{t}^*$  of  $p$  time points is*

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \end{bmatrix} \mid \mathbf{Y}, \mathbf{t}, \Theta \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where  $\boldsymbol{\mu} \in \mathbb{R}^{3p}$  is the column vector of posterior expectations and  $\boldsymbol{\Sigma} \in \mathbb{R}^{3p \times 3p}$  is the joint posterior covariance matrix. Partitioning these as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_f(\mathbf{t}^* | \Theta) \\ \mu_{df}(\mathbf{t}^* | \Theta) \\ \mu_{d^2f}(\mathbf{t}^* | \Theta) \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_f(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{f,d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) \\ \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* | \Theta)^T & \Sigma_{df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{df,d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) \\ \Sigma_{f,d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta)^T & \Sigma_{df,d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta)^T & \Sigma_{d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) \end{bmatrix}$$

the individual components are given by

$$\begin{aligned} \mu_f(\mathbf{t}^* | \Theta) &= \mu_\beta(\mathbf{t}^*) + C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t})) \\ \mu_{df}(\mathbf{t}^* | \Theta) &= d\mu_\beta(\mathbf{t}^*) + \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t})) \\ \mu_{d^2f}(\mathbf{t}^* | \Theta) &= d^2\mu_\beta(\mathbf{t}^*) + \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t})) \\ \Sigma_f(\mathbf{t}^*, \mathbf{t}^* | \Theta) &= C_\theta(\mathbf{t}^*, \mathbf{t}^*) - C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \Sigma_{df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) &= \partial_1 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) - \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \Sigma_{d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) &= \partial_1^2 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) - \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) &= \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) - C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \Sigma_{f,d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) &= \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) - C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \Sigma_{df,d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) &= \partial_1 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) - \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \left( C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \right)^{-1} \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) \end{aligned}$$

Proposition 1 grants the foundation of the rest of the methodological development. In the following two section we show how the Trend Direction Index and the Expected Trend Instability can be expressed under the data generating model in Equation (3) using the results of Proposition 1.

## 2.2 Trend Direction Index

The Trend Direction Index was defined generally in Equation (1). Conditioning on the  $\sigma$ -algebra of all observed data we may express the Trend Direction Index under the model in Equation (3) through the posterior distribution of  $df$ . The following proposition states this result.

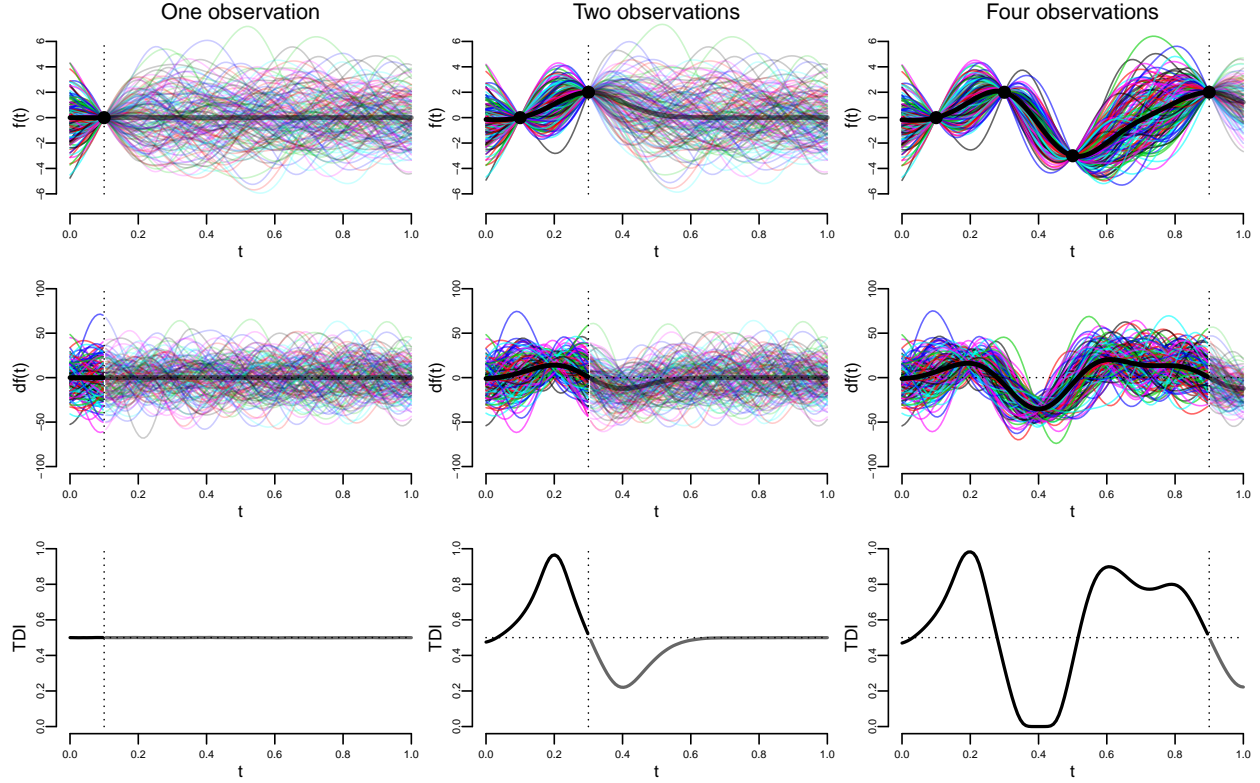
**Proposition 2.** *Let  $\mathcal{F}_t$  be the  $\sigma$ -algebra generated by  $(\mathbf{Y}, \mathbf{t})$ . The Trend Direction Index defined in Equation (1) can then be written in terms of the posterior distribution of  $df$  as*

$$\begin{aligned} \text{TDI}(t, \delta | \Theta) &= P(df(t + \delta) > 0 | \mathbf{Y}, \mathbf{t}, \Theta) \\ &= \frac{1}{2} + \frac{1}{2} \text{Erf} \left( \frac{\mu_{df}(t + \delta | \Theta)}{2^{1/2} \Sigma_{df}(t + \delta, t + \delta | \Theta)^{1/2}} \right) \end{aligned}$$

where  $\text{Erf}: x \mapsto 2\pi^{-1/2} \int_0^x \exp(-u^2) du$  is the error function and  $\mu_{df}$  and  $\Sigma_{df}$  are the posterior mean and covariance of the trend defined in Proposition 1.

From Proposition 2 it is seen that TDI it is equal to  $\frac{1}{2}$  when  $\mu_{df}(t + \delta | \Theta) = 0$ . This corresponds to the case where the expected value of the posterior of  $f$  is constant at time  $t + \delta$ . A decision

rule based on  $\text{TDI}(t, \delta \mid \Theta) \leq 50\%$  is therefore a natural choice for assessing the trendiness of  $f$ . However, different thresholds based on external loss or utility functions can be used depending on the application.



**Figure 2:** 150 realizations from the posterior distribution of  $f$  (top row),  $df$  (middle row) with expected values in bold and the Trend Direction Index (bottom row) conditional on one, two and four noise free observations. Dotted vertical lines show the points in time after which forecasting occurs.

To illustrate the Trend Direction Index we show an example in the noise free case with known model parameters in Figure 2. The first and second rows of the plot show 150 random realizations from the posterior distribution of  $f$  and  $df$  with the expected functions in bold lines, and the third row shows the Trend Direction Index. In this example we simulate using known model parameters and therefore the Trend Direction Index in Proposition 2 is a deterministic function. The three columns of the plot show how the posterior of  $f$  and  $df$  and the Trend Direction Index are updated after one, two or four observed data points. It is seen how updating the Gaussian process prior with observed data results in a continuous change in the posterior distribution of  $f$  and its derivative  $df$ . The posterior knowledge about the sign of the derivative is correspondingly reflected in the Trend Direction Index. Going further and further away from the last observed data point it is seen that the posterior of  $f$  becomes dominated by its prior distribution. In the same sense  $df$  converges to a point-wise symmetric distribution around zero and therefore the Trend Direction Index stabilizes around 50%. This is clearly seen in the second column of the figure. [TODO: More explaining]

### 2.3 Expected Trend Instability

The Expected Trend Instability was informally defined in Equation (2) as the expected number of roots of the trend on an interval. Now we make this concept more precise and frame it in the context of the latent Gaussian process model. Let  $\mathcal{I}$  be a compact subset of the real line and consider the random càdlàg function  $N_{\mathcal{I}}(t) = \# \{s \leq t : df(s) = 0, s \in \mathcal{I}\}$  counting the cumulative number of points in  $\mathcal{I}$  up to time  $t$  where the trend is equal to zero. [TODO: Something about tangents and way it's the number of crossings] The Expected Trend Instability on  $\mathcal{I}$  is equal to

$$\text{ETI}(\mathcal{I} \mid \Theta) = \mathbb{E}[N_{\mathcal{I}}(\max \mathcal{I}) \mid \Theta, \mathcal{F}]$$

giving the expected total number of zero-crossings by  $df$  on  $\mathcal{I}$ . In the definition we condition on both a vector of parameters,  $\Theta$ , and a  $\sigma$ -algebra,  $\mathcal{F}$ . Usually,  $\mathcal{F}$  will be generated by the observed data,  $(\mathbf{Y}, \mathbf{t})$ , and  $\mathcal{I}$  will be equal to  $\mathcal{T}$  or some subset of it. The following proposition gives the expression for the Expected Trend Instability under the data generating model in Equation (3).

**Proposition 3.** *Let  $\mathcal{F}$  be the  $\sigma$ -algebra generated by the observed data  $(\mathbf{Y}, \mathbf{t})$  and  $\mu_{df}$ ,  $\mu_{d^2f}$ ,  $\Sigma_{df}$ ,  $\Sigma_{d^2f}$  and  $\Sigma_{df, d^2f}$  the moments of the joint posterior distribution of  $(df, d^2f)$  as stated in Proposition 1. [TODO: Assumptions.] The Expected Trend Instability is then*

$$\text{ETI}(\mathcal{I} \mid \Theta) = \int_{\mathcal{I}} d\text{ETI}(t \mid \Theta) dt$$

where  $d\text{ETI}$  is the local Expected Trend Instability given by

$$d\text{ETI}(t, \mathcal{T} \mid \Theta) = \lambda(t \mid \Theta) \phi \left( \frac{\mu_{df}(t \mid \Theta)}{\Sigma_{df}(t, t \mid \Theta)^{1/2}} \right) \left( 2\phi(\zeta(t \mid \Theta)) + \zeta(t \mid \Theta) \text{Erf} \left( \frac{\zeta(t \mid \Theta)}{2^{1/2}} \right) \right)$$

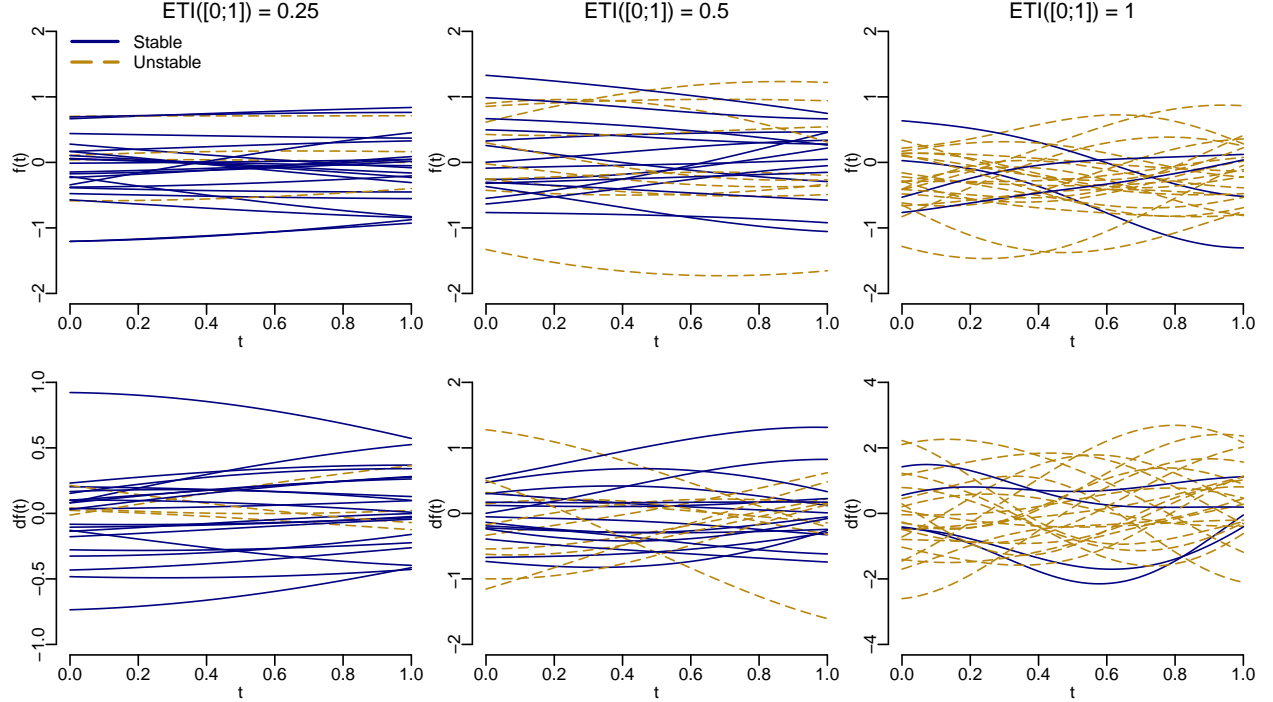
and  $\phi: x \mapsto 2^{-1/2} \pi^{-1/2} \exp(-\frac{1}{2}x^2)$  is the standard normal density function,  $\text{Erf}: x \mapsto 2\pi^{-1/2} \int_0^x \exp(-u^2) du$  is the error function, and  $\lambda$ ,  $\omega$  and  $\zeta$  are functions defined as

$$\begin{aligned} \lambda(t \mid \Theta) &= \frac{\Sigma_{d^2f}(t, t \mid \Theta)^{1/2}}{\Sigma_{df}(t, t \mid \Theta)^{1/2}} (1 - \omega(t \mid \Theta)^2)^{1/2} \\ \omega(t \mid \Theta) &= \frac{\Sigma_{df, d^2f}(t, t \mid \Theta)}{\Sigma_{df}(t, t \mid \Theta)^{1/2} \Sigma_{d^2f}(t, t \mid \Theta)^{1/2}} \\ \zeta(t \mid \Theta) &= \frac{\mu_{df}(t \mid \Theta) \Sigma_{d^2f}(t, t \mid \Theta)^{1/2} \omega(t \mid \Theta) \Sigma_{df}(t, t \mid \Theta)^{-1/2} - \mu_{d^2f}(t \mid \Theta)}{\Sigma_{d^2f}(t, t \mid \Theta)^{1/2} (1 - \omega(t \mid \Theta)^2)^{1/2}} \end{aligned}$$

The value of ETI is a finite [TODO: finite because of assumptions] real positive value.

To illustrate the Expected Trend Instability, figure 3 shows an example of 25 random Gaussian processes on the unit interval simulated under three different values of ETI. The sample paths are paired so that for each function in the first row, there is an associated derivative in the second row. The different values of ETI are set to 0.25, 0.5 and 1 meaning that these are the expected number of times that the function change monotonicity or equivalently that the trend crosses zero on that interval in each scenario. Sample paths that are *trend stable*, i.e always increasing/decreasing, are shown by solid blue lines, and sample paths that are *trend unstable*, i.e. the derivative crosses zero, are shown by dashed gold colored lines. We see that for low values of ETI most of the sample paths

are either always increasing or decreasing and their associated deviatives are either always postive or negative. For larger values of ETI, more of the derivatives start crossing zero implying that the functions start changing monotonicity. We note that even though we are only modeling a single curve, the Expected Trend Instability is calculated from a posterior distribution which is what the figure illustrates.



**Figure 3:** 25 random pairs sampled from the joint distribution of a Gassian Process ( $f$ ) and its derivative ( $df$ ) with different values of Expected Trend Instability (ETI). The first row shows samples from  $f$ , and the second row shows samples from  $df$ . The columns define different values of ETI. Sample paths that are trend stable are shown by solid blue lines, and unstable sample paths are shown by dashed gold colored lines.

## 2.4 Estimation

The latent Gaussian process model defined in Equation (3) requires estimates of the parameter vector  $\Theta$  in order to calculate the joint posterior distribution required for the trend indices. These parameters must be estimated from data  $(Y_i, t_i)_{i=1}^n$ . We consider two different estimation procedures: maximum likelihood estimation and a fully Bayesian estimator.

The maximum likelihood estimator for our model consists of finding the values of the parameters that maximize the marginal likelihood of the observed data and plugging these into the expressions. The marginal distribution of  $\mathbf{Y}$  is found by integrating out the distribution of the latent Gaussian process in the conditional observation model  $\mathbf{Y} | f(\mathbf{t}), \mathbf{t}, \Theta$ . Since the observation model consists of normal distributed random variables conditional on the latent Gaussian process, the marginal distribution is multivariate normal with expectation  $\mu_\beta(\mathbf{t})$  and  $n \times n$  covariance matrix  $C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I$ . The marginal log likelihood function is therefore



$$\log L(\Theta \mid \mathbf{Y}, \mathbf{t}) \propto -\frac{1}{2} \log |C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I| - \frac{1}{2} (\mathbf{Y} - \mu_\beta(\mathbf{t}))^T [C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I]^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t})) \quad (5)$$

and the maximum likelihood estimate  $\widehat{\Theta}^{\text{ML}} = \arg \sup_{\Theta=(\beta, \theta, \sigma)} \log L(\Theta \mid \mathbf{Y}, \mathbf{t})$  can be obtained by numerical optimization or found as the roots to the score equations  $\nabla_\theta \log L(\Theta \mid \mathbf{Y}, \mathbf{t}) = 0$ . This estimate can then be plugged in to the expressions for the posterior distributions of  $(f, df, d^2f)$  in Proposition 1 which enables simulation of the posterior distributions at any vector of time points. Point estimates of the Trend Direction Index and the Expected Trend Instability are then given by  $\text{TDI}(t, \delta \mid \widehat{\Theta}^{\text{ML}})$  and  $\text{ETI}(\mathcal{I} \mid \widehat{\Theta}^{\text{ML}})$  according to Propositions 2 and 3 respectively.

The maximum likelihood estimator is easy to implement and fast to compute, but it is difficult to propagate the uncertainties of the estimated parameter estimate through to the posterior distributions and the trend indices. This is disadvantageous since in order to conduct a qualified assessment of trendiness we are not only interested in point estimates but also the associated uncertainties. A Bayesian estimator, while slower to compute, facilitates a straightforward way to encompass all uncertainties in the final estimates. To specify a Bayesian estimator we must augment the data generating model in Equation (3) with another hierarchical level specifying a prior distribution of the model parameters. We therefore add the following level

$$(\beta, \theta, \sigma) \sim G(\Theta \mid \Psi, \mathbf{t})$$

to the specification where  $G$  is some family of distribution functions indexed by a vector of hyperparameters  $\Psi$ . The joint distribution of the model can be factorized as

$$P(\mathbf{Y}, f(\mathbf{t}), \Theta \mid \Psi, \mathbf{t}) = P(\mathbf{Y} \mid f(\mathbf{t}), \Theta, \Psi, \mathbf{t}) P(f(\mathbf{t}) \mid \Theta, \Psi, \mathbf{t}) G(\Theta \mid \Psi, \mathbf{t})$$

and each conditional probability is specified by a sub-model in the augmented hierarchy. We always condition on  $\Psi$  and  $\mathbf{t}$  as they are considered fixed data. The posterior distribution of  $\Theta$  given the observed outcomes is then

$$\begin{aligned} P(\Theta \mid \mathbf{Y}, \Psi, \mathbf{t}) &= \frac{G(\Theta \mid \Psi, \mathbf{t}) P(\mathbf{Y} \mid \Theta, \Psi, \mathbf{t})}{P(\mathbf{Y} \mid \Psi, \mathbf{t})} \\ &= \frac{G(\Theta \mid \Psi, \mathbf{t}) \int P(\mathbf{Y} \mid f(\mathbf{t}), \Theta, \Psi, \mathbf{t}) dP(f(\mathbf{t}) \mid \Theta, \Psi, \mathbf{t})}{\iint P(\mathbf{Y} \mid f(\mathbf{t}), \Theta, \Psi, \mathbf{t}) dP(f(\mathbf{t}) \mid \Theta, \Psi, \mathbf{t}) dG(\Theta \mid \Psi, \mathbf{t})} \end{aligned}$$

Let  $\tilde{\Theta} \sim P(\Theta \mid \mathbf{Y}, \Psi, \mathbf{t})$ . The posterior distribution of  $\Theta$  induces corresponding distributions over the trend indices according to  $\text{TDI}(t, \delta \mid \tilde{\Theta})$ ,  $d\text{EDI}(t, \mathcal{T}, \mid \tilde{\Theta})$  and  $\text{EDI}(\mathcal{I} \mid \tilde{\Theta})$ . For example, the Trend Direction Index in the Bayesian formulation is a surface in  $(t, \delta)$  where each value is a distribution over probability values. This implies that we in addition to quantify trendiness by a point estimate, we can also provide probability intervals for the trendiness. We suggest to summarize the trend indices by their posterior quantiles. For example, for the Trend Direction Index we summarize its posterior distribution by functions  $Q_\tau$  defined by

$$P\left(\text{TDI}\left(t, \delta \mid \tilde{\Theta}\right) \leq \tau\right) = Q_{\tau}(t, \delta)$$

with for example  $\tau \in \{0.025, 0.5, 0.975\}$ .

We have implemented both the maximum likelihood and the Bayesian estimator in Stan (Carpenter et al. 2017) and R (R Core Team 2018) in combination with the `rstan` package (Stan Development Team 2018). Stan is a probabilistic programming language enabling full Bayesian inference using Markov chain Monte Carlo sampling. The Stan implementation of the maximum likelihood estimator requires the marginal maximum likelihood estimates of the parameters supplied as data, and from these it will simulate random functions from the posterior distribution of  $(f, df, d^2f)$  on a user-supplied grid of time points as well as return point estimates of TDI and  $d\text{ETI}$ . The latter can then be integrated numerically to obtain the Expected Trend Instability on an interval. The Bayesian estimator requires values of the hyper-parameters  $\Psi$  supplied as data and from these it will generate random samples  $(\tilde{\Theta}_1, \dots, \tilde{\Theta}_K)$  from  $P(\Theta \mid \mathbf{Y}, \Psi, \mathbf{t})$  by Markov chain Monte Carlo. These samples are then used to approximate the posterior distribution of the Trend Direction Index by  $(\text{TDI}(t, \delta \mid \tilde{\Theta}_1), \dots, \text{TDI}(t, \delta \mid \tilde{\Theta}_K))$  and similarly for  $d\text{ETI}$ . Both implementations are available at Jensen (2019).

### 3 Application

A report published by The Danish Health Authority in January 2019 updated the estimated proportion of smokers in Denmark with new data from 2018 (The Danish Health Authority 2019). The data was based on an internet survey including 5017 participants. The report also included data on the proportion of smokers in Denmark during the last 20 years. These data were shown in Figure 1. The report was picked up by several news papers under headlines stating that the proportion of smokers in Denmark had significantly increased for the first time in two decades (Navne, Schmidt, and Rasmussen 2019). The report published no statistical analyses for this statement but stated in a note that because the study population is so large, then more or less all differences become statistically significant at the 5% level (this was erroneously written as a 95% significance level in the report).

This data set provides an instrumental way of exemplifying our two proposed trend quantification measures. In this application we wish to assess the statistical properties of following questions:

- Q1: Is the proportion of smokers increasing in the year 2018 conditional on data from the last 20 years?
- Q2: If the proportion of smokers is currently increasing, when did this increase probably start?
- Q3: Is it the first time during the last 20 years that the trend in the proportion of smokers has changed?

A simple approach for trying to answer questions Q1 and Q2 would be to apply a  $\chi^2$ -test in a  $2 \times 2$  table. Table 1 shows the p-values for the  $\chi^2$ -test (with Yates' continuity correction) of independence between the proportion of smokers in 2018 and each of the five previous years. Using a significance level of 5% the conclusion is ambiguous. Compared to the previous year, there was no significant change in the proportion in 2018. Three out of these five comparisons fail to show a significant

change in proportions. It is therefore clear that such point-wise testing is not sufficiently perceptive to catch the underlying continuous development.

2018	2017	2016	2015	2014	2013
p-value	0.074	<b>0.020</b>	0.495	<b>0.012</b>	0.576

**Table 1:** p-values obtained from  $\chi^2$ -tests of independence between the proportion of smokers in 2018 and the five previous years. Numbers in bold are significant differences at the 5% level.

Similarly, a simple approach for trying to answer question Q3 would be to look at the cumulative number of times that the difference in proportion between consecutive years changes sign. In the data set there were nine changes in the sign of the difference between the proportion in each year and the proportion in the previous year giving this very curde estimate of the number of times, that the trend has changed. This approach suffers from the facts that it is based on a finite difference approximation at the observed time points to the continous derivative, and that it uses the noisy measurements instead of the values of the latent function.

We now present an analysis of the data set using our proposed model. To complete the model specification in Equation (3) we must decide on the functional forms of the mean and covariance functions of the latent Gaussian process. We considered three different mean functions: a constant mean,  $\mu_\beta(t) = \beta_0$ , a linear mean,  $\mu_\beta(t) = \beta_0 + \beta_1 t$ , and a quadratic mean function,  $\mu_\beta(t) = \beta_0 + \beta_1 t + \beta_2 t^2$ . For the covariance function we consider the Squared Exponential (SE), the Rational Quadratic (RQ), the Matern 3/2 (M3/2), and the Matern 5/2 (M5/2) covariance functions. The expressions for these covariance functions are given by

$$\begin{aligned}
C_\theta^{\text{SE}}(s, t) &= \alpha^2 \exp\left(-\frac{(s-t)^2}{2\rho^2}\right), \quad \theta = (\alpha, \rho) \\
C_\theta^{\text{RQ}}(s, t) &= \alpha^2 \left(1 + \frac{(s-t)^2}{2\rho^2\nu}\right)^{-\nu}, \quad \theta = (\alpha, \rho, \nu) \\
C_\theta^{\text{M3/2}}(s, t) &= \alpha^2 \left(1 + \frac{\sqrt{3}\sqrt{(s-t)^2}}{\rho}\right) \exp\left(-\frac{\sqrt{3}\sqrt{(s-t)^2}}{\rho}\right), \quad \theta = (\alpha, \rho) \\
C_\theta^{\text{M5/2}}(s, t) &= \alpha^2 \left(1 + \frac{\sqrt{5}\sqrt{(s-t)^2}}{\rho} + \frac{5(s-t)^2}{3\rho^2}\right) \exp\left(-\frac{\sqrt{5}\sqrt{(s-t)^2}}{\rho}\right), \quad \theta = (\alpha, \rho)
\end{aligned} \tag{6}$$

and we note that the Rational Quadratic covariance function converges to the Squared Exponential covariance function for  $\nu \rightarrow \infty$ ,  $\lim_{\nu \rightarrow \infty} C_\theta^{\text{RQ}}(s, t) = C_\theta^{\text{SE}}(s, t)$ . [**TODO:** Something about the sample path regularity for these covariance functions. M3/2 cannot be used for ETI but okay for TDI].

The total combination of mean and covariance functions results in 12 candidate models, and to select the model providing the best fit to data we used leave-one-out cross-validation based on maximum likelihood estimation. For each model,  $\mathcal{M}$ , we turn by turn excluded a single observation  $(t_i, Y_i)$  from the vectors of observed outcomes and time points, and we let the leave-one-out data be denoted by  $(\mathbf{Y}_{-i}, \mathbf{t}_{-i})$ . Based on the leave-one-out data sets we estimated the parameters,  $\Theta_{-i}^{\mathcal{M}} = (\beta, \theta, \sigma^2)$ , of the latent Gaussian process for each model by maximizing the marginal log-likelihood function of the model in Equation (3). These estimates were given by

$$\hat{\Theta}_{-i}^{\mathcal{M}} = \arg \sup_{\Theta} \log L(\Theta \mid \mathbf{Y}_{-i}, \mathbf{t}_{-i})$$

where the log-likelihood is given by Equation (5). These estimates are then plugged into the expression for the posterior expectation of  $f$  in Proposition 1 to obtain the leave-one-out predictions. The mean squared prediction error was then calculated from the squared errors between the leave-one-out predictions and the observed values averaged across all data points:

$$\text{MSPE}_{\text{LOO}}^{\mathcal{M}} = \frac{1}{n} \sum_{i=1}^n \left( Y_i - \mathbb{E}[f(t_i) \mid \mathbf{Y}_{-i}, \mathbf{t}_{-i}, \hat{\Theta}_{-i}^{\mathcal{M}}] \right)^2$$

and the model providing the best fit to data was selected as  $\mathcal{M}_{\text{opt}} = \arg \min_{\mathcal{M}} \text{MSPE}_{\text{LOO}}^{\mathcal{M}}$ .

	SE	RQ	Matern 3/2	Matern 5/2
Constant	0.682	<b>0.651</b>	0.687	0.660
Linear	0.806	$\Leftarrow$	0.896	0.865
Quadratic	0.736	$\Leftarrow$	0.800	0.785

**Table 2:** Leave-one-out cross-validated mean squared error of prediction for each of the 12 candidate models.  $\Leftarrow$  indicates numerical convergence to the SE covariance function.

Table 2 shows the mean squared prediction error for each of the models. For the models with a Rational Quadratic covariance function and a linear and a quadratic mean function the parameter  $\nu$  diverged numerically implying convergence to the Squared Exponential covariance function. Comparing the leave-one-out mean squared prediction errors for the candidate models, the optimal model has a constant mean function and a Rational Quadratic covariance function. The marginal maximum likelihood estimates of the parameters of the optimal model was

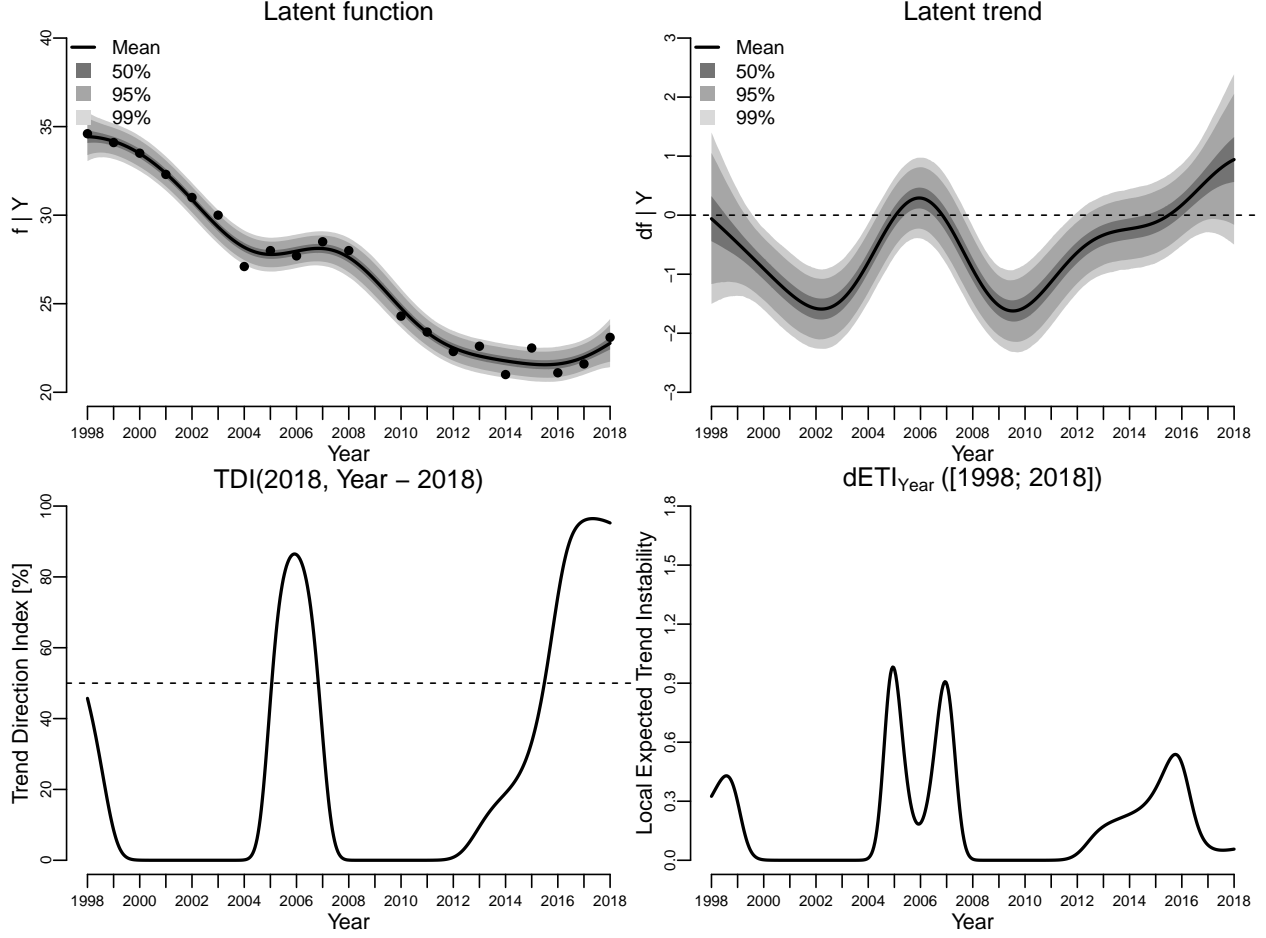
$$\widehat{\beta}_0^{\text{ML}} = 28.001, \quad \widehat{\alpha}^{\text{ML}} = 4.543, \quad \widehat{\rho}^{\text{ML}} = 4.438, \quad \widehat{\nu}^{\text{ML}} = 1.020, \quad \widehat{\sigma}^{\text{ML}} = 0.622 \quad (7)$$

Figure 4 shows the fit of the model by plugging in the values of the marginal maximum likelihood estimates into the expressions for the posterior distributions of  $f$  and  $df$  defined in Proposition 1, the Trend Direction Index in Proposition 2, and the Local Expected Trend Instability in Proposition 3.

$$\begin{aligned} \beta_0 &\sim T\left(\widehat{\beta}_0^{\text{ML}}, 3, 3\right), & \alpha &\sim \text{Half-}T\left(\widehat{\alpha}^{\text{ML}}, 3, 3\right), & \rho &\sim \text{Half-}N\left(\widehat{\rho}^{\text{ML}}, 1\right) \\ \nu &\sim \text{Half-}T\left(\widehat{\nu}^{\text{ML}}, 3, 3\right), & \sigma &\sim \text{Half-}T\left(\widehat{\sigma}^{\text{ML}}, 3, 3\right) \end{aligned}$$

where the maximum likelihood values are given in Equation (7).

We ran four chains for 25,000 iterations each with half of the iterations used for warm-up. Convergence was assessed by the potential scale reduction factor ( $\hat{R}$ ) of Gelman and Rubin (1992).



**Figure 4:** Results from fitting the latent Gaussian process model by maximum likelihood. The first row shows the posterior distributions of  $f$  (left) and  $df$  (right) with the posterior means in bold and gray areas showing point-wise probability intervals for the posterior distribution. The second row shows the estimated Trend Direction Index (left) and the local Expected Trend Instability (right).

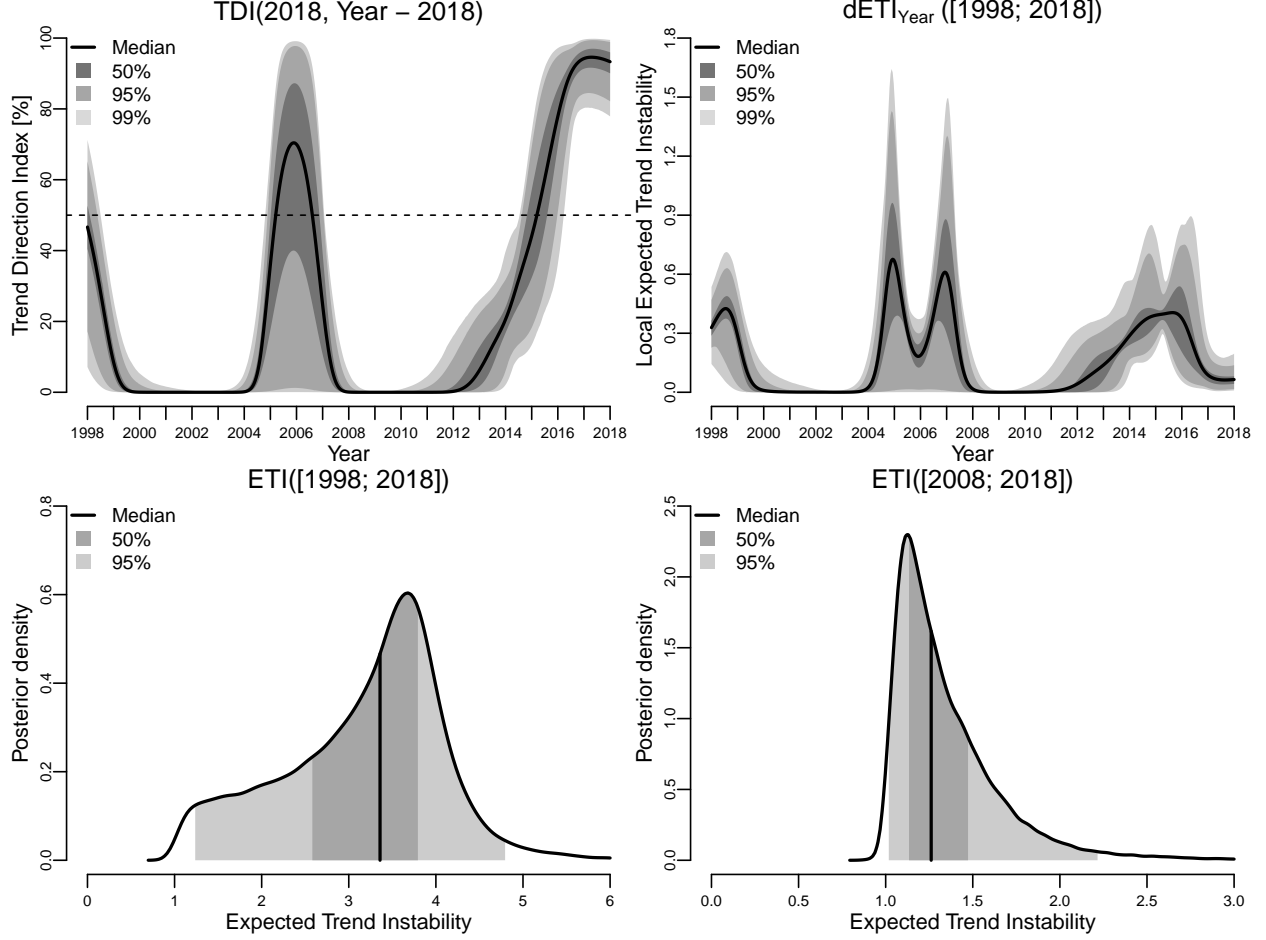
The top row of Figure 5 shows the posterior distributions of the Trend Direction Index and the Local Expected Trend Instability from the Bayesian model. The bottom row shows the posterior distributions of the Expected Trend Instability during the last twenty years and the last ten years.

Cross-point for 50% for MLE: year = 2015.48.

Cross-point for 50% for Bayes: [2014.62; 2015.96]

Note for Bayesian: The first bump drowns in undercainty. This cannot be seen from ML.

Figures 4 and 5 and the summary statistics in Table 3 indicate that the Bayesian estimates generally produce a high degree of smoothness in the model fit. This can be seen from the plot of the median local Expected Trend Instability in Figure 5 which is generally lower than its corresponding maximum likelihood point estimates in Figure 4. This is similarly reflected in the median ETI estimates which are lower than their values under maximum likelihood in Table 3. Looking at the posterior distributions of the covariance parameters  $\theta$  (not shown), we see that this is mainly a result of not restricting the parameter  $\nu$  to its maximum likelihood value. The 95% probability



**Figure 5:** hejhej

interval of the posterior distribution of  $\nu$  was  $[0.328; 10.743]$  which is highly right-skewed compared to the maximum likelihood estimate of  $\widehat{\nu^{\text{ML}}} = 1.020$ .

To understand the effect of  $\nu$  on the smoothness of the fitted models can compare the local expected number of crossings by a Gaussian process and its derivative at their mean values in the simple case with a zero mean and either the Rational Quadratic or the Squared Exponential covariance function. In this case the formula in Proposition 3 simplifies immensely so that the local expected number of mean-crossing by  $f$  is equal to  $\pi^{-1}(\frac{\partial_1 \partial_2 C_\theta(t,t)}{C_\theta(t,t)})^{1/2}$  and for  $df$  it is equal to  $\pi^{-1}(\frac{\partial_1^2 \partial_2^2 C_\theta(t,t)}{\partial_1 \partial_2 C_\theta(t,t)})^{1/2}$ . From the expressions in Equation (6) we see that the local expected number of mean-crossings by  $f$  is the same for both the Squared Exponential and the Rational Quadratic covariance functions and is equal to  $\pi^{-1} \rho^{-1}$ . However, for  $df$  we see that its local expected number of mean-crossings is equal to  $3^{1/2} \pi^{-1} \rho^{-1}$  for the Squared Exponential covariance function and  $3^{1/2} \pi^{-1} \rho^{-1} (1 + \nu^{-1})^{1/2}$  for the Rational Quadratic covariance function. As  $1 < (1 + \nu^{-1})^{1/2} < \infty$  for  $0 < \nu < \infty$  and monotonically decreasing for  $\nu \rightarrow \infty$  with a limit of one, we see that the local expected number of mean-crossings by  $df$  is always larger for the Rational Quadratic covariance functions compared to the Squared Exponential covariance function with equality in the limit. A right-skewed posterior distribution of  $\nu$  therefore generally favors fewer crossings of the trend leading to a more stable trend and a smaller value of the Expected Trend Instability.

	Maximum Likelihood	Bayesian Posterior
TDI(2018, 0)	95.24%	93.32% [82.15%; 98.86%]
TDI(2018, -1)	95.92%	94.21% [84.28%; 99.11%]
TDI(2018, -2)	74.41%	77.87% [51.02%; 94.94%]
TDI(2018, -3)	33.36%	44.11% [18.23%; 69.19%]
TDI(2018, -4)	18.96%	20.60% [6.05%; 31.82%]
TDI(2018, -5)	9.50%	6.21% [0.03%; 22.21%]
ETI([2008, 2018])	1.39	1.25 [1.02; 2.22]
ETI([1998, 2018])	3.68	3.36 [1.24; 4.79]

**Table 3:** Summary measures from the Maximum Likelihood and Bayesian analyses. The rows show the estimated Trend Direction Index for 2013 to 2018 and the Expected Trend Instability for the last 10 and 20 years all conditional on data from 1998 to 2018. For the Bayesian analysis posterior medians and 95% posterior probability intervals are given.

## 4 Discussion

[**TODO:** Her må der bættles løs]

Something about that the indices are scale-free hence TDI should be interpreted along with  $df \mid Y$  at the same time point. We could also define

$$\text{TDI}_u(t, \delta) = P(df(t + \delta) > u \mid \mathcal{F}_t)$$

where  $u$  is some trend value.

We could also have defined

$$\text{ETI}^\uparrow(\mathcal{I}) = \mathbb{E} \left[ \# \left\{ t \in \mathcal{I} : df(t) = 0, d^2f(t) > 0 \right\} \mid \mathcal{F} \right]$$

counting the expected number of up-crossings at zero by  $df$  (the number of times  $f$  has gone from being decreasing to increasing) on  $\mathcal{T}$ .

Also something about stationarity of  $C_\theta$ .

Everything including implementations and supplementary material is available at Jensen (2019).

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## 6 Appendix

### 6.1 Proof of Proposition 1

Let  $\mathbf{Y} = (Y_1, \dots, Y_n)$  and  $\mathbf{t} = (t_1, \dots, t_n)$  be the vectors of observed outcomes and associated sampling times. From the data generating model in Equation (3) we observe that the marginal distribution of the vector of observed outcomes  $\mathbf{Y} \mid \mathbf{t}, \Theta$  is



$$\begin{aligned}
P(\mathbf{Y} \mid \mathbf{t}, \Theta) &= \int P(\mathbf{Y} \mid f(\mathbf{t}), \mathbf{t}, \Theta) dP(f(\mathbf{t}) \mid \mathbf{t}, \Theta) \\
&= N(\mu_\beta(\mathbf{t}), C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I)
\end{aligned}$$

where  $\mu_\beta(\mathbf{t}) = (\mu_\beta(t_1), \dots, \mu_\beta(t_n))$ ,  $C_\theta(\mathbf{t}, \mathbf{t})$  is the  $n \times n$  covariance matrix obtained by evaluating  $C_\theta(s, t)$  at  $\{(s, t) \in \mathbf{t} \times \mathbf{t}\}$  and  $I$  is an  $n \times n$  identity matrix. This implies that the joint distribution of  $\mathbf{Y}$  and the latent functions  $(f, df, d^2f)$  evaluated at an arbitrary vector of time points  $\mathbf{t}^*$  is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \\ \mathbf{Y} \end{bmatrix} \mid \mathbf{t}, \Theta \sim N \left( \begin{bmatrix} \mu_\beta(\mathbf{t}^*) \\ d\mu_\beta(\mathbf{t}^*) \\ d^2\mu_\beta(\mathbf{t}^*) \\ \mu_\beta(\mathbf{t}) \end{bmatrix}, \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ C_\theta(\mathbf{t}, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) & C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \end{bmatrix} \right)$$

where  $\partial_j^k$  denotes the  $k$ 'th order partial derivative with respect to the  $j$ 'th variable.

By the standard formula for deriving conditional distributions in a multivariate normal model, the posterior distribution of  $(f, df, d^2f)$  evaluated at the  $p$  time points in  $\mathbf{t}^*$  is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \end{bmatrix} \mid \mathbf{Y}, \mathbf{t}, \Theta \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where  $\boldsymbol{\mu} \in \mathbb{R}^{3p}$  is the column vector of posterior expectations and  $\boldsymbol{\Sigma} \in \mathbb{R}^{3p \times 3p}$  is the joint posterior covariance matrix, and these are given by

$$\begin{aligned}
\boldsymbol{\mu} &= \begin{bmatrix} \mu_\beta(\mathbf{t}^*) \\ d\mu_\beta(\mathbf{t}^*) \\ d^2\mu_\beta(\mathbf{t}^*) \end{bmatrix} + \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\theta, \sigma}(\mathbf{t}, \mathbf{t})^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t})) \\
\boldsymbol{\Sigma} &= \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \end{bmatrix} - \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\theta, \sigma}(\mathbf{t}, \mathbf{t})^{-1} \begin{bmatrix} C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) \end{bmatrix}^T
\end{aligned}$$

where  $K_{\theta, \sigma}(\mathbf{t}, \mathbf{t}) = C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I$ . Partitioning  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_f(\mathbf{t}^* \mid \Theta) \\ \mu_{df}(\mathbf{t}^* \mid \Theta) \\ \mu_{d^2f}(\mathbf{t}^* \mid \Theta) \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_f(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{f, df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{f, d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) \\ \Sigma_{f, df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{df, d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) \\ \Sigma_{d^2f, f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{d^2f, df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) \end{bmatrix}$$

and completing the matrix algebra, we obtain the expressions of the individual components given in Proposition 1. This completes the proof.

## 6.2 Proof of Proposition 3

Rice showed in section 3.3. of Rice (1945) that the expected number of zero-crossings of a Gaussian process  $X$  on an interval  $\mathcal{I}$  is given by

$$\int_{\mathcal{I}} \int_{-\infty}^{\infty} |v| f_{X(t), dX(t)}(0, v) dv dt \quad (8)$$

where  $f_{X(t), dX(t)}$  is the joint density function of  $X$  and its derivative  $dX$  at time  $t$ . [TODO: Conditions] To derive the expression for the Expected Trend Instability in Equation (??) we must apply the Rice formula to the joint posterior distribution of  $(df, d^2f)$ . From Proposition 1 the distribution of  $(df, d^2f) \mid \mathbf{Y}, \mathbf{t}, \Theta$  is bivariate normal for each  $t$ .

Let  $\mu_{df}$ ,  $\mu_{d^2f}$ ,  $\Sigma_{df}$  and  $\Sigma_{d^2f}$  be defined as in Proposition 1 and define further

$$\omega(t \mid \Theta) = \frac{\Sigma_{df, d^2f}(t, t \mid \Theta)}{\Sigma_{df}(t, t \mid \Theta)^{1/2} \Sigma_{d^2f}(t, t \mid \Theta)^{1/2}}$$

as the posterior point-wise cross-correlation function between  $df$  and  $d^2f$ . The joint posterior density function of  $(df, d^2f)$  at any time  $t$  evaluated at  $(0, v)$  can be factorized as

$$f_{df(t), d^2f(t)}(0, v) = c_1(t) e^{c_2(t)} e^{-c_3(t)v^2 - 2c_4(t)v}$$

where  $c_1, \dots, c_4$  are functions of time given by

$$\begin{aligned} c_1(t) &= (2\pi)^{-1} \Sigma_{df}(t, t \mid \Theta)^{-1/2} \Sigma_{d^2f}(t, t \mid \Theta)^{-1/2} (1 - \omega(t \mid \Theta)^2)^{-1/2} \\ c_2(t) &= \frac{\mu_{df}(t)^2 \Sigma_{d^2f}(t, t) - 2\mu_{df}(t) \mu_{d^2f}(t) \Sigma_{df}(t, t)^{1/2} \Sigma_{d^2f}(t, t)^{1/2} \omega(t) + \mu_{d^2f}(t)^2 \Sigma_{df}(t, t)}{2\Sigma_{d^2f}(t, t)(\omega(t)^2 - 1)\Sigma_{df}(t, t)} \\ c_3(t) &= -\frac{1}{2} \Sigma_{d^2f}(t, t \mid \Theta)^{-1} (\omega(t \mid \Theta)^2 - 1)^{-1} \\ c_4(t) &= -\frac{\mu_{df}(t \mid \Theta) \Sigma_{d^2f}(t, t \mid \Theta)^{1/2} \omega(t \mid \Theta) - \mu_{d^2f}(t \mid \Theta) \Sigma_{df}(t, t \mid \Theta)^{1/2}}{2\Sigma_{d^2f}(t, t \mid \Theta)(\omega(t \mid \Theta)^2 - 1)\Sigma_{df}(t, t \mid \Theta)^{1/2}} \end{aligned}$$

Let  $d\text{ETI}_{\mathcal{I}}(t \mid \Theta)$  denote the inner integral in Equation (8). Using the factorization of the joint posterior density we may write it was

$$\begin{aligned} d\text{ETI}_{\mathcal{I}}(t \mid \Theta) &= \int_{-\infty}^{\infty} |v| f_{df(t), d^2f(t)}(0, v) dv \\ &= c_1(t) e^{c_2(t)} \int_{-\infty}^{\infty} |v| e^{-c_3(t)v^2 - 2c_4(t)v} dv \\ &= c_1(t) e^{c_2(t)} \left( \int_0^{\infty} v e^{-c_3(t)v^2 + 2c_4(t)v} dv + \int_0^{\infty} v e^{-c_3(t)v^2 - 2c_4(t)v} dv \right) \end{aligned} \quad (9)$$

Because  $c_3(t) > 0$  for all  $t$  since  $\Sigma_{df}(t, t | \Theta) > 0$  and  $|\omega(t | \Theta)| < 1$  by Assumption [TODO:??] we obtain the following solution for the type of integral in the previous display by using formula 5 in section 3.462 on page 365 of Gradshteyn and Ryzhik (2014)

$$\int_0^\infty v e^{-c_3(t)v^2 \pm 2c_4(t)v} dv = \frac{1}{2c_3(t)} \pm \frac{c_4(t)}{2c_3(t)} \frac{\pi^{1/2}}{c_3(t)^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)}} \left( 1 \pm \operatorname{Erf} \left( \frac{c_4(t)}{\sqrt{c_3(t)}} \right) \right) \quad (10)$$

where  $\operatorname{Erf}: x \mapsto 2\pi^{-1} \int_0^x e^{-u^2} du$  is the error function. Combining Equations (9) and (10) we may express  $d\text{ETI}$  as

$$d\text{ETI}(t, \mathcal{T} | \Theta) = c_1(t) e^{c_2(t)} \left( \frac{1}{c_3(t)} + \frac{c_4(t)}{c_3(t)} \frac{\pi^{1/2}}{c_3(t)^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)}} \operatorname{Erf} \left( \frac{c_4(t)}{\sqrt{c_3(t)}} \right) \right)$$

Defining  $\zeta(t | \Theta) = \sqrt{2}c_4(t)c_3(t)^{-1/2}$  and collecting some terms, the index can be rewritten as

$$d\text{ETI}(t, \mathcal{T} | \Theta) = \frac{c_1(t)}{c_3(t)} \left( e^{c_2(t)} + \frac{\pi^{1/2}}{2^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)} + c_2(t)} \zeta(t) \operatorname{Erf} \left( \frac{\zeta(t | \Theta)}{2^{1/2}} \right) \right)$$

Straightforward arithmetic calculations show that

$$\frac{c_4(t)^2}{c_3(t)} + c_2(t) = -\frac{\mu_{df}(t | \Theta)^2}{2\Sigma_{df}(t, t | \Theta)}, \quad c_2(t) = -\frac{1}{2} \left( \zeta(t | \Theta)^2 + \frac{\mu_{df}(t | \Theta)^2}{\Sigma_{df}(t, t | \Theta)} \right)$$

and by defining  $\phi: x \mapsto (2\pi)^{-1/2} e^{-x^2}$  as the density function of the standard normal distribution we may write the two exponentials in the previous display in terms of  $\phi$  and obtain

$$d\text{ETI}(t, \mathcal{T} | \Theta) = \frac{c_1(t)}{c_3(t)} \pi \phi \left( \frac{\mu_{df}(t | \Theta)}{\Sigma_{df}(t, t | \Theta)^{1/2}} \right) \left( 2\phi(\zeta(t | \Theta)) + \zeta(t | \Theta) \operatorname{Erf} \left( \frac{\zeta(t | \Theta)}{2^{1/2}} \right) \right)$$

Standard arithmetics show that

$$\frac{c_1(t)}{c_3(t)} = \frac{1}{\pi} \frac{\Sigma_{df}(t, t | \Theta)^{1/2}}{\Sigma_{df}(t, t | \Theta)^{1/2}} (1 - \omega(t | \Theta)^2)^{1/2}$$

and we finally obtain the expression

$$d\text{ETI}(t, \mathcal{T} | \Theta) = \lambda(t | \Theta) \phi \left( \frac{\mu_{df}(t | \Theta)}{\Sigma_{df}(t, t | \Theta)^{1/2}} \right) \left( 2\phi(\zeta(t | \Theta)) + \zeta(t | \Theta) \operatorname{Erf} \left( \frac{\zeta(t | \Theta)}{2^{1/2}} \right) \right)$$

where  $\lambda$  and  $\zeta$  are given by

$$\lambda(t \mid \Theta) = \frac{\Sigma_{d^2f}(t, t \mid \Theta)^{1/2}}{\Sigma_{df}(t, t \mid \Theta)^{1/2}} \left(1 - \omega(t \mid \Theta)^2\right)^{1/2}$$

$$\zeta(t \mid \Theta) = \frac{\mu_{df}(t \mid \Theta) \Sigma_{d^2f}(t, t \mid \Theta)^{1/2} \omega(t) \Sigma_{df}(t, t \mid \Theta)^{-1/2} - \mu_{d^2f}(t \mid \Theta)}{\Sigma_{d^2f}(t, t \mid \Theta)^{1/2} (1 - \omega(t \mid \Theta)^2)^{1/2}}$$

By definition

$$\text{ETI}(\mathcal{I} \mid \Theta) = \int_{\mathcal{I}} d\text{ETI}(t, \mathcal{T} \mid \Theta) dt$$

which completes the proof.