# Supplementary Material for Quantifying the Trendiness of Trends

*Andreas Kryger Jensen and Claus Thorn Ekstrøm*
*Biostatistics, Institute of Public Health, University of Copenhagen*
*aeje@sund.ku.dk, ekstrom@sund.ku.dk*

*26 December, 2019*

## A Proof of Proposition 1

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$ be the vectors of observed outcomes and associated sampling times. From the data generating model we observe that the marginal distribution of the vector of observed outcomes $\mathbf{Y} \mid \mathbf{t}, \Theta$ is

$$\begin{aligned} P(\mathbf{Y} \mid \mathbf{t}, \Theta) &= \int P(\mathbf{Y} \mid f(\mathbf{t}), \mathbf{t}, \Theta) dP(f(\mathbf{t}) \mid \mathbf{t}, \Theta) \\ &= N(\mu_\beta(\mathbf{t}), C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I) \end{aligned}$$

where $\mu_\beta(\mathbf{t}) = (\mu_\beta(t_1), \ldots \mu_\beta(t_n))$, $C_\theta(\mathbf{t}, \mathbf{t})$ is the $n \times n$ covariance matrix obtained by evaluating $C_\theta(s, t)$ at $\{(s, t) \in \mathbf{t} \times \mathbf{t}\}$ and $I$ is an $n \times n$ identity matrix. This implies that the joint distribution of $\mathbf{Y}$ and the latent functions $(f, df, d^2f)$ evaluated at an arbitrary vector of time points $\mathbf{t}^*$ is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \\ \mathbf{Y} \end{bmatrix} \mid \mathbf{t}, \Theta \sim N\left( \begin{bmatrix} \mu_\beta(\mathbf{t}^*) \\ d\mu_\beta(\mathbf{t}^*) \\ d^2\mu_\beta(\mathbf{t}^*) \\ \mu_\beta(\mathbf{t}) \end{bmatrix}, \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ C_\theta(\mathbf{t}, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) & C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \end{bmatrix} \right)$$

where $\partial_j^k$ denotes the $k$'th order partial derivative with respect to the $j$'th variable.

By the standard formula for deriving conditional distributions in a multivariate normal model, the posterior distribution of $(f, df, d^2f)$ evaluated at the $p$ time points in $\mathbf{t}^*$ is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \end{bmatrix} \mid \mathbf{Y}, \mathbf{t}, \Theta \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\mu} \in \mathbb{R}^{3p}$ is the column vector of posterior expectations and $\boldsymbol{\Sigma} \in \mathbb{R}^{3p \times 3p}$ is the joint posterior covariance matrix, and these are given by

$$
\boldsymbol{\mu} = \begin{bmatrix} \mu_\beta(\mathbf{t}^*) \\ d\mu_\beta(\mathbf{t}^*) \\ d^2\mu_\beta(\mathbf{t}^*) \end{bmatrix} + \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\theta,\sigma}(\mathbf{t}, \mathbf{t})^{-1}(\mathbf{Y} - \mu_\beta(\mathbf{t}))
$$

$$
\boldsymbol{\Sigma} = \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1\partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1\partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2\partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2\partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \end{bmatrix} - \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\theta,\sigma}(\mathbf{t}, \mathbf{t})^{-1} \begin{bmatrix} C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) \end{bmatrix}^T
$$

where $K_{\theta,\sigma}(\mathbf{t}, \mathbf{t}) = C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I$. Partitioning $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as

$$
\boldsymbol{\mu} = \begin{bmatrix} \mu_f(\mathbf{t}^* \mid \Theta) \\ \mu_{df}(\mathbf{t}^* \mid \Theta) \\ \mu_{d^2f}(\mathbf{t}^* \mid \Theta) \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_f(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{f,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) \\ \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{df,d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) \\ \Sigma_{d^2f,f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{d^2f,df}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) & \Sigma_{d^2f}(\mathbf{t}^*, \mathbf{t}^* \mid \Theta) \end{bmatrix}
$$

and completing the matrix algebra, we obtain the expressions of the individual components given in the Proposition.

# B Proof of Proposition 3

Rice showed in section 3.3. of Rice (1945) that the expected number of zero-crossings of a Gaussian process $X$ on an interval $\mathcal{I}$ is given by

$$
\int_{\mathcal{I}} \int_{-\infty}^{\infty} |v| f_{X(t),dX(t)}(0, v) \mathrm{d}v \mathrm{d}t \tag{1}
$$

where $f_{X(t),dX(t)}$ is the joint density function of $X$ and its derivative $dX$ at time $t$. To derive the expression for the Expected Trend Instability we must apply the Rice formula to the joint posterior distribution of $(df, d^2f)$. From Proposition 1 the distribution of $(df, d^2f) \mid \mathbf{Y}, \mathbf{t}, \Theta$ is bivariate normal for each $t$.

Let $\mu_{df}$, $\mu_{d^2f}$, $\Sigma_{df}$ and $\Sigma_{d^2f}$ be defined as in Proposition 1 and define further

$$
\omega(t \mid \Theta) = \frac{\Sigma_{df,d^2f}(t, t \mid \Theta)}{\Sigma_{df}(t, t \mid \Theta)^{1/2} \Sigma_{d^2f}(t, t \mid \Theta)^{1/2}}
$$

as the posterior point-wise cross-correlation function between $df$ and $d^2f$. The joint posterior density function of $(df, d^2f)$ at any time $t$ evaluated at $(0, v)$ can be factorized as

$$
f_{df(t),d^2f(t)}(0, v) = c_1(t) e^{c_2(t)} e^{-c_3(t)v^2 - 2c_4(t)v}
$$

where $c_1, \ldots, c_4$ are functions of time given by

$$
\begin{aligned}
c_1(t) &= (2\pi)^{-1}\Sigma_{df}(t,t \mid \Theta)^{-1/2}\Sigma_{d^2f}(t,t \mid \Theta)^{-1/2}(1-\omega(t \mid \Theta)^2)^{-1/2} \\
c_2(t) &= \frac{\mu_{df}(t \mid \Theta)^2}{2\Sigma_{df}(t,t \mid \Theta)(\omega(t \mid \Theta)^2-1)} + \frac{\mu_{d^2f}(t \mid \Theta)^2}{2\Sigma_{d^2f}(t,t \mid \Theta)(\omega(t \mid \Theta)^2-1)} \\
&\quad - \frac{\mu_{df}(t \mid \Theta)\mu_{d^2f}(t \mid \Theta)\omega(t \mid \Theta)}{\Sigma_{df}(t,t \mid \Theta)^{1/2}\Sigma_{d^2f}(t,t \mid \Theta)^{1/2}(\omega(t \mid \Theta)^2-1)} \\
c_3(t) &= -\frac{1}{2}\Sigma_{d^2f}(t,t \mid \Theta)^{-1}(\omega(t \mid \Theta)^2-1)^{-1} \\
c_4(t) &= -\frac{\mu_{df}(t \mid \Theta)\Sigma_{d^2f}(t,t \mid \Theta)^{1/2}\omega(t \mid \Theta) - \mu_{d^2f}(t \mid \Theta)\Sigma_{df}(t,t \mid \Theta)^{1/2}}{2\Sigma_{d^2f}(t,t \mid \Theta)(\omega(t \mid \Theta)^2-1)\Sigma_{df}(t,t \mid \Theta)^{1/2}}
\end{aligned}
$$

Let $d\mathrm{ETI}(t \mid \Theta)$ denote the inner integral in Equation (1). Using the factorization of the joint posterior density we may write it was

$$
\begin{aligned}
d\mathrm{ETI}(t \mid \Theta) &= \int_{-\infty}^{\infty} |v| f_{df(t),d^2f(t)}(0,v)\mathrm{d}v \\
&= c_1(t)e^{c_2(t)}\int_{-\infty}^{\infty} |v| e^{-c_3(t)v^2-2c_4(t)v}\mathrm{d}v \\
&= c_1(t)e^{c_2(t)}\left(\int_0^{\infty} v e^{-c_3(t)v^2+2c_4(t)v}\mathrm{d}v + \int_0^{\infty} v e^{-c_3(t)v^2-2c_4(t)v}\mathrm{d}v\right)
\end{aligned}
\tag{2}
$$

Because $c_3(t) > 0$ for all t since $\Sigma_{d^2f}(t,t \mid \Theta) > 0$ and $|\omega(t \mid \Theta)| < 1$ by Assumption A4 we obtain the following solution for the type of integral in the previous display by using formula 5 in section 3.462 on page 365 of Gradshteyn and Ryzhik (2014)

$$
\int_0^{\infty} v e^{-c_3(t)v^2 \pm 2c_4(t)v}\mathrm{d}v = \frac{1}{2c_3(t)} \pm \frac{c_4(t)}{2c_3(t)}\frac{\pi^{1/2}}{c_3(t)^{1/2}}e^{\frac{c_4(t)^2}{c_3(t)}}\left(1 \pm \mathrm{Erf}\left(\frac{c_4(t)}{\sqrt{c_3(t)}}\right)\right)
\tag{3}
$$

where $\mathrm{Erf}\colon\, x \mapsto 2\pi^{-1}\int_0^x e^{-u^2}\mathrm{d}u$ is the error function. Combining Equations (2) and (3) we may express $d\mathrm{ETI}$ as

$$
d\mathrm{ETI}(t \mid \Theta) = c_1(t)e^{c_2(t)}\left(\frac{1}{c_3(t)} + \frac{c_4(t)}{c_3(t)}\frac{\pi^{1/2}}{c_3(t)^{1/2}}e^{\frac{c_4(t)^2}{c_3(t)}}\,\mathrm{Erf}\left(\frac{c_4(t)}{\sqrt{c_3(t)}}\right)\right)
$$

Defining $\zeta(t \mid \Theta) = \sqrt{2}c_4(t)c_3(t)^{-1/2}$ and collecting some terms, the index can be rewritten as

$$
d\mathrm{ETI}(t \mid \Theta) = \frac{c_1(t)}{c_3(t)}\left(e^{c_2(t)} + \frac{\pi^{1/2}}{2^{1/2}}e^{\frac{c_4(t)^2}{c_3(t)}+c_2(t)}\zeta(t)\,\mathrm{Erf}\left(\frac{\zeta(t \mid \Theta)}{2^{1/2}}\right)\right)
$$

Straightforward arithmetic calculations show that

$$\frac{c_4(t)^2}{c_3(t)} + c_2(t) = -\frac{\mu_{df}(t \mid \Theta)^2}{2\Sigma_{df}(t,t \mid \Theta)}, \quad c_2(t) = -\frac{1}{2}\left(\zeta(t \mid \Theta)^2 + \frac{\mu_{df}(t \mid \Theta)^2}{\Sigma_{df}(t,t \mid \Theta)}\right)$$

and by defining $\phi\colon\ x \mapsto (2\pi)^{-1/2}e^{-x^2}$ as the density function of the standard normal distribution we may write $e^{\frac{c_4(t)^2}{c_3(t)}+c_2(t)} = (2\pi)^{1/2}\phi\left(\frac{\mu_{df}(t|\Theta)}{\Sigma_{df}(t,t|\Theta)^{1/2}}\right)$ and $e^{c_2(t)} = 2\pi\phi(\zeta(t))\phi\left(\frac{\mu_{df}(t|\Theta)}{\Sigma_{df}(t,t|\Theta)^{1/2}}\right)$ which leads to

$$d\mathrm{ETI}(t \mid \Theta) = \frac{c_1(t)}{c_3(t)}\pi\phi\left(\frac{\mu_{df}(t \mid \Theta)}{\Sigma_{df}(t,t \mid \Theta)^{1/2}}\right)\left(2\phi(\zeta(t \mid \Theta)) + \zeta(t \mid \Theta)\,\mathrm{Erf}\left(\frac{\zeta(t \mid \Theta)}{2^{1/2}}\right)\right)$$

Standard arithmetics show that

$$\frac{c_1(t)}{c_3(t)} = \frac{1}{\pi}\frac{\Sigma_{d^2f}(t,t \mid \Theta)^{1/2}}{\Sigma_{df}(t,t \mid \Theta)^{1/2}}\left(1-\omega(t \mid \Theta)^2\right)^{1/2}$$

and we finally obtain the expression

$$d\mathrm{ETI}(t \mid \Theta) = \lambda(t \mid \Theta)\phi\left(\frac{\mu_{df}(t \mid \Theta)}{\Sigma_{df}(t,t \mid \Theta)^{1/2}}\right)\left(2\phi(\zeta(t \mid \Theta)) + \zeta(t \mid \Theta)\,\mathrm{Erf}\left(\frac{\zeta(t \mid \Theta)}{2^{1/2}}\right)\right)$$

where $\lambda$ and $\zeta$ are given by

$$\begin{aligned}
\lambda(t \mid \Theta) &= \frac{\Sigma_{d^2f}(t,t \mid \Theta)^{1/2}}{\Sigma_{df}(t,t \mid \Theta)^{1/2}}\left(1-\omega(t \mid \Theta)^2\right)^{1/2} \\
\zeta(t \mid \Theta) &= \frac{\mu_{df}(t \mid \Theta)\Sigma_{d^2f}(t,t \mid \Theta)^{1/2}\omega(t)\Sigma_{df}(t,t \mid \Theta)^{-1/2} - \mu_{d^2f}(t \mid \Theta)}{\Sigma_{d^2f}(t,t \mid \Theta)^{1/2}\left(1-\omega(t \mid \Theta)^2\right)^{1/2}}
\end{aligned}$$

By definition

$$\mathrm{ETI}(\mathcal{I} \mid \Theta) = \int_{\mathcal{I}} d\mathrm{ETI}(t \mid \Theta)\mathrm{d}t$$

which completes the proof.

# C Zero-crossings of $f$ and $df$ in the zero-mean stationary case

Let $f \sim \mathcal{GP}(0, C_\theta(\cdot,\cdot))$ where the $C_\theta$ is either the Squared Exponential or Rational Quadratic covariance function. We look at the expected number of zero-crossings on an interval by either $f$ and $df$ as given by the Rice formula in Equation (1) with either $X(t) = f(t)$ or $X(t) = df(t)$. In this case the expressions simplifies immensely due to the zero means of both $f$, $df$, and $d^2f$ and because $\mathrm{Cov}[f(t), df(t)] = 0$ and $\mathrm{Cov}[df(t), d^2f(t)] = 0$. The latter is a result of using a stationary covariance function for the prior distribution of $f$ (Cramer and Leadbetter 1967). In this stationary case local expected number of zero-crossing of $f$ and $df$ are given by

$$\frac{\partial_1\partial_2 C_\theta(s,t)\Big|_{s=t}^{1/2}}{\pi C_\theta(t,t)^{1/2}} \quad \text{and} \quad \frac{\partial_1^2\partial_2^2 C_\theta(s,t)\Big|_{s=t}^{1/2}}{\pi \partial_1\partial_2 C_\theta(s,t)\Big|_{s=t}^{1/2}}$$

respectively. It then follows that

$$\begin{aligned}
C_\theta^{\mathrm{SE}}(t,t) = \sigma^2, \quad \partial_1\partial_2 C_\theta^{\mathrm{SE}}(s,t)\Big|_{s=t} &= \frac{\sigma^2}{\rho^2}, \quad \partial_1^2\partial_2^2 C_\theta^{\mathrm{SE}}(s,t)\Big|_{s=t} = \frac{3\sigma^2}{\rho^4} \\
C_\theta^{\mathrm{RQ}}(t,t) = \sigma^2, \quad \partial_1\partial_2 C_\theta^{\mathrm{RQ}}(s,t)\Big|_{s=t} &= \frac{\sigma^2}{\rho^2}, \quad \partial_1^2\partial_2^2 C_\theta^{\mathrm{RQ}}(s,t)\Big|_{s=t} = \frac{2\sigma^2(1+\nu)}{\nu\rho^4}
\end{aligned}$$

and the local expected number of zero-crossings of $f$ and $df$ for either the Squared Exponential and the Rational Quadratic covariance functions are

$$\begin{aligned}
\frac{\partial_1\partial_2 C_\theta^{\mathrm{SE}}(s,t)\Big|_{s=t}^{1/2}}{\pi C_\theta^{\mathrm{SE}}(t,t)^{1/2}} &= \frac{1}{\pi\rho}, \qquad \frac{\partial_1^2\partial_2^2 C_\theta^{\mathrm{SE}}(s,t)\Big|_{s=t}^{1/2}}{\pi \partial_1\partial_2 C_\theta^{\mathrm{SE}}(s,t)\Big|_{s=t}^{1/2}} = \frac{3^{1/2}}{\pi\rho} \\
\frac{\partial_1\partial_2 C_\theta^{\mathrm{RQ}}(s,t)\Big|_{s=t}^{1/2}}{\pi C_\theta^{\mathrm{RQ}}(t,t)^{1/2}} &= \frac{1}{\pi\rho}, \qquad \frac{\partial_1^2\partial_2^2 C_\theta^{\mathrm{RQ}}(s,t)\Big|_{s=t}^{1/2}}{\pi \partial_1\partial_2 C_\theta^{\mathrm{RQ}}(s,t)\Big|_{s=t}^{1/2}} = \frac{3^{1/2}}{\pi\rho}\left(1 + v^{-1}\right)^{1/2}
\end{aligned}$$

# Bibliography


Cramer, Harald, and M. R. Leadbetter. 1967. *Stationary and Related Stochastic Processes – Sample Function Properties and Their Applications.* John Wiley & Sons, Inc.

Gradshteyn, Izrail Solomonovich, and Iosif Moiseevich Ryzhik. 2014. *Table of Integrals, Series, and Products.* Academic Press.

Rice, Stephen O. 1945. "Mathematical Analysis of Random Noise, II." *Bell System Technical Journal* 24 (1): 46–156.