Supplementary Material for Quantifying the Trendiness of Trends

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A Proof of Proposition 1

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$ be the vectors of observed outcomes and associated sampling times. From the data generating model we observe that the marginal distribution of the vector of observed outcomes $\mathbf{Y} \mid \mathbf{t}, \mathbf{\Theta}$ is

$$\begin{aligned} P(\mathbf{Y} \mid \mathbf{t}, \mathbf{\Theta}) &= \int P(\mathbf{Y} \mid f(\mathbf{t}), \mathbf{t}, \mathbf{\Theta}) dP(f(\mathbf{t}) \mid \mathbf{t}, \mathbf{\Theta}) \\ &= N(\mu_{\boldsymbol{\beta}}(\mathbf{t}), C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I) \end{aligned}$$

where $\mu_{\boldsymbol{\beta}}(\mathbf{t}) = (\mu_{\boldsymbol{\beta}}(t_1), \ldots \mu_{\boldsymbol{\beta}}(t_n))$, $C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t})$ is the $n \times n$ covariance matrix obtained by evaluating $C_{\boldsymbol{\theta}}(s,t)$ at $\{(s,t) \in \mathbf{t} \times \mathbf{t}\}$ and I is an $n \times n$ identity matrix. This implies that the joint distribution of $\mathbf{Y}$ and the latent functions $(f, df, d^2 f)$ evaluated at an arbitrary vector of time points $\mathbf{t}^*$ is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2 f(\mathbf{t}^*) \\ \mathbf{Y} \end{bmatrix} \mid \mathbf{t}, \mathbf{\Theta} \sim N\left(\begin{bmatrix} \mu_{\boldsymbol{\beta}}(\mathbf{t}^*) \\ d\mu_{\boldsymbol{\beta}}(\mathbf{t}^*) \\ d^2\mu_{\boldsymbol{\beta}}(\mathbf{t}^*) \\ \mu_{\boldsymbol{\beta}}(\mathbf{t}) \end{bmatrix}, \begin{bmatrix} C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \\ C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) & \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) & \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) & C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}) + \sigma^2 I \end{bmatrix} \right)$$

where ∂_j^k denotes the k'th order partial derivative with respect to the j'th variable.

By the standard formula for deriving conditional distributions in a multivariate normal model, the posterior distribution of $(f, df, d^2 f)$ evaluated at the p time points in $\mathbf{t}^*$ is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2 f(\mathbf{t}^*) \end{bmatrix} \mid \mathbf{Y}, \mathbf{t}, \mathbf{\Theta} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma})$$

where $\boldsymbol{\mu} \in \mathbb{R}^{3p}$ is the column vector of posterior expectations and $\mathbf{\Sigma} \in \mathbb{R}^{3p \times 3p}$ is the joint posterior covariance matrix, and these are given by

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_{\boldsymbol{\beta}}(\mathbf{t}^*) \\ d\mu_{\boldsymbol{\beta}}(\mathbf{t}^*) \\ d^2\mu_{\boldsymbol{\beta}}(\mathbf{t}^*) \end{bmatrix} + \begin{bmatrix} C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\boldsymbol{\theta},\sigma}(\mathbf{t}, \mathbf{t})^{-1} (\mathbf{Y} - \mu_{\boldsymbol{\beta}}(\mathbf{t}))$$

$$\mathbf{\Sigma} = \begin{bmatrix} C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}^*) \end{bmatrix} - \begin{bmatrix} C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_{\boldsymbol{\theta}}(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\boldsymbol{\theta},\sigma}(\mathbf{t}, \mathbf{t})^{-1} \begin{bmatrix} C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \\ \partial_2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \\ \partial_2^2 C_{\boldsymbol{\theta}}(\mathbf{t}, \mathbf{t}^*) \end{bmatrix}^T$$

where $K_{\boldsymbol{\theta},\sigma}(\mathbf{t},\mathbf{t}) = C_{\boldsymbol{\theta}}(\mathbf{t},\mathbf{t}) + \sigma^2 I$. Partitioning $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_f(\mathbf{t}^* \mid \boldsymbol{\Theta}) \\ \mu_{df}(\mathbf{t}^* \mid \boldsymbol{\Theta}) \\ \mu_{d^2f}(\mathbf{t}^* \mid \boldsymbol{\Theta}) \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_f(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) & \Sigma_{f,df}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) & \Sigma_{f,d^2f}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) \\ \Sigma_{f,df}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) & \Sigma_{df}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) & \Sigma_{df,d^2f}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) \\ \Sigma_{d^2f,f}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) & \Sigma_{d^2f,df}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) & \Sigma_{d^2f}(\mathbf{t}^*,\mathbf{t}^* \mid \boldsymbol{\Theta}) \end{bmatrix}$$

and completing the matrix algebra, we obtain the expressions of the individual components given in the Proposition.

B Proof of Proposition 3

Rice showed in section 3.3. of Rice (1945) that the expected number of zero-crossings of a Gaussian process X on an interval $\mathcal{I}$ is given by

$$\int_{\mathcal{I}} \int_{-\infty}^{\infty} |v| f_{X(t),dX(t)}(0,v)\mathrm{d}v\mathrm{d}t \tag{1}$$

where $f_{X(t),dX(t)}$ is the joint density function of X and its derivative dX at time t. To derive the expression for the Expected Trend Instability we must apply the Rice formula to the joint posterior distribution of (df, d^2f). From Proposition 1 the distribution of $(df, d^2f) \mid \mathbf{Y}, \mathbf{t}, \boldsymbol{\Theta}$ is bivariate normal for each t.

Let μ_{df}, μ_{d^2f}, Σ_{df} and Σ_{d^2f} be defined as in Proposition 1 and define further

$$\omega(t \mid \boldsymbol{\Theta}) = \frac{\Sigma_{df,d^2f}(t,t \mid \boldsymbol{\Theta})}{\Sigma_{df}(t,t \mid \boldsymbol{\Theta})^{1/2}\Sigma_{d^2f}(t,t \mid \boldsymbol{\Theta})^{1/2}}$$

as the posterior point-wise cross-correlation function between df and d^2f. The joint posterior density function of (df, d^2f) at any time t evaluated at $(0,v)$ can be factorized as

$$f_{df(t),d^2f(t)}(0,v) = c_1(t)e^{c_2(t)}e^{-c_3(t)v^2-2c_4(t)v}$$

where $c_1,\ldots,c_4$ are functions of time given by

$$\begin{aligned}
c_1(t) &= (2\pi)^{-1}\Sigma_{df}(t,t \mid \boldsymbol{\Theta})^{-1/2}\Sigma_{d^2f}(t,t \mid \boldsymbol{\Theta})^{-1/2}(1-\omega(t \mid \boldsymbol{\Theta})^2)^{-1/2} \\
c_2(t) &= \frac{\mu_{df}(t \mid \boldsymbol{\Theta})^2}{2\Sigma_{df}(t,t \mid \boldsymbol{\Theta})(\omega(t \mid \Theta)^2-1)} + \frac{\mu_{d^2f}(t \mid \boldsymbol{\Theta})^2}{2\Sigma_{d^2f}(t,t \mid \boldsymbol{\Theta})(\omega(t \mid \boldsymbol{\Theta})^2-1)} \\
&\quad - \frac{\mu_{df}(t \mid \boldsymbol{\Theta})\mu_{d^2f}(t \mid \boldsymbol{\Theta})\omega(t \mid \boldsymbol{\Theta})}{\Sigma_{df}(t,t \mid \boldsymbol{\Theta})^{1/2}\Sigma_{d^2f}(t,t \mid \boldsymbol{\Theta})^{1/2}(\omega(t \mid \boldsymbol{\Theta})^2-1)} \\
c_3(t) &= -\frac{1}{2}\Sigma_{d^2f}(t,t \mid \boldsymbol{\Theta})^{-1}(\omega(t \mid \boldsymbol{\Theta})^2-1)^{-1} \\
c_4(t) &= -\frac{\mu_{df}(t \mid \boldsymbol{\Theta})\Sigma_{d^2f}(t,t \mid \boldsymbol{\Theta})^{1/2}\omega(t \mid \boldsymbol{\Theta}) - \mu_{d^2f}(t \mid \boldsymbol{\Theta})\Sigma_{df}(t,t \mid \boldsymbol{\Theta})^{1/2}}{2\Sigma_{d^2f}(t,t \mid \boldsymbol{\Theta})(\omega(t \mid \boldsymbol{\Theta})^2-1)\Sigma_{df}(t,t \mid \boldsymbol{\Theta})^{1/2}}
\end{aligned}$$

Let $d\mathrm{ETI}(t \mid \boldsymbol{\Theta})$ denote the inner integral in Equation (1). Using the factorization of the joint posterior density we may write it was

$$\begin{aligned}
d\mathrm{ETI}(t \mid \boldsymbol{\Theta}) &= \int_{-\infty}^{\infty} |v| f_{df(t),d^2f(t)}(0,v)\mathrm{d}v \\
&= c_1(t)e^{c_2(t)} \int_{-\infty}^{\infty} |v| e^{-c_3(t)v^2-2c_4(t)v}\mathrm{d}v \qquad (2) \\
&= c_1(t)e^{c_2(t)} \left(\int_0^{\infty} v e^{-c_3(t)v^2+2c_4(t)v}\mathrm{d}v + \int_0^{\infty} v e^{-c_3(t)v^2-2c_4(t)v}\mathrm{d}v \right)
\end{aligned}$$

Because $c_3(t) > 0$ for all t since $\Sigma_{d^2f}(t, t \mid \mathbf{\Theta}) > 0$ and $|\omega(t \mid \mathbf{\Theta})| < 1$ by Assumption A4 we obtain the following solution for the type of integral in the previous display by using formula 5 in section 3.462 on page 365 of Gradshteyn and Ryzhik (2014)

$$\int_0^\infty v e^{-c_3(t)v^2 \pm 2c_4(t)v} \mathrm{d}v = \frac{1}{2c_3(t)} \pm \frac{c_4(t)}{2c_3(t)} \frac{\pi^{1/2}}{c_3(t)^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)}} \left(1 \pm \operatorname{Erf}\left(\frac{c_4(t)}{\sqrt{c_3(t)}}\right)\right) \tag{3}$$

where $\operatorname{Erf}\colon x \mapsto 2\pi^{-1} \int_0^x e^{-u^2} \mathrm{d}u$ is the error function. Combining Equations (2) and (3) we may express dETI as

$$d\mathrm{ETI}(t \mid \mathbf{\Theta}) = c_1(t) e^{c_2(t)} \left(\frac{1}{c_3(t)} + \frac{c_4(t)}{c_3(t)} \frac{\pi^{1/2}}{c_3(t)^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)}} \operatorname{Erf}\left(\frac{c_4(t)}{\sqrt{c_3(t)}}\right)\right)$$

Defining $\zeta(t \mid \mathbf{\Theta}) = \sqrt{2} c_4(t) c_3(t)^{-1/2}$ and collecting some terms, the index can be rewritten as

$$d\mathrm{ETI}(t \mid \mathbf{\Theta}) = \frac{c_1(t)}{c_3(t)} \left(e^{c_2(t)} + \frac{\pi^{1/2}}{2^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)} + c_2(t)} \zeta(t) \operatorname{Erf}\left(\frac{\zeta(t \mid \mathbf{\Theta})}{2^{1/2}}\right)\right)$$

Straightforward arithmetic calculations show that

$$\frac{c_4(t)^2}{c_3(t)} + c_2(t) = -\frac{\mu_{df}(t \mid \mathbf{\Theta})^2}{2\Sigma_{df}(t, t \mid \mathbf{\Theta})}, \quad c_2(t) = -\frac{1}{2}\left(\zeta(t \mid \mathbf{\Theta})^2 + \frac{\mu_{df}(t \mid \mathbf{\Theta})^2}{\Sigma_{df}(t, t \mid \mathbf{\Theta})}\right)$$

and by defining $\phi\colon x \mapsto (2\pi)^{-1/2} e^{-x^2}$ as the density function of the standard normal distribution we may write $e^{\frac{c_4(t)^2}{c_3(t)} + c_2(t)} = (2\pi)^{1/2} \phi\left(\frac{\mu_{df}(t \mid \mathbf{\Theta})}{\Sigma_{df}(t,t \mid \mathbf{\Theta})^{1/2}}\right)$ and $e^{c_2(t)} = 2\pi\phi(\zeta(t))\phi\left(\frac{\mu_{df}(t \mid \mathbf{\Theta})}{\Sigma_{df}(t,t \mid \mathbf{\Theta})^{1/2}}\right)$ which leads to

$$d\mathrm{ETI}(t \mid \mathbf{\Theta}) = \frac{c_1(t)}{c_3(t)} \pi \phi\left(\frac{\mu_{df}(t \mid \mathbf{\Theta})}{\Sigma_{df}(t, t \mid \mathbf{\Theta})^{1/2}}\right) \left(2\phi(\zeta(t \mid \mathbf{\Theta})) + \zeta(t \mid \mathbf{\Theta}) \operatorname{Erf}\left(\frac{\zeta(t \mid \mathbf{\Theta})}{2^{1/2}}\right)\right)$$

Standard arithmetics show that

$$\frac{c_1(t)}{c_3(t)} = \frac{1}{\pi} \frac{\Sigma_{d^2f}(t, t \mid \mathbf{\Theta})^{1/2}}{\Sigma_{df}(t, t \mid \mathbf{\Theta})^{1/2}} \left(1 - \omega(t \mid \mathbf{\Theta})^2\right)^{1/2}$$

and we finally obtain the expression

$$d\mathrm{ETI}(t \mid \mathbf{\Theta}) = \lambda(t \mid \mathbf{\Theta}) \phi\left(\frac{\mu_{df}(t \mid \mathbf{\Theta})}{\Sigma_{df}(t, t \mid \mathbf{\Theta})^{1/2}}\right) \left(2\phi(\zeta(t \mid \mathbf{\Theta})) + \zeta(t \mid \mathbf{\Theta}) \operatorname{Erf}\left(\frac{\zeta(t \mid \mathbf{\Theta})}{2^{1/2}}\right)\right)$$

where λ and ζ are given by

$$\lambda(t \mid \mathbf{\Theta}) = \frac{\Sigma_{d^2f}(t, t \mid \mathbf{\Theta})^{1/2}}{\Sigma_{df}(t, t \mid \mathbf{\Theta})^{1/2}} \left(1 - \omega(t \mid \mathbf{\Theta})^2\right)^{1/2}$$

$$\zeta(t \mid \mathbf{\Theta}) = \frac{\mu_{df}(t \mid \mathbf{\Theta}) \Sigma_{d^2f}(t, t \mid \mathbf{\Theta})^{1/2} \omega(t) \Sigma_{df}(t, t \mid \mathbf{\Theta})^{-1/2} - \mu_{d^2f}(t \mid \mathbf{\Theta})}{\Sigma_{d^2f}(t, t \mid \mathbf{\Theta})^{1/2} \left(1 - \omega(t \mid \mathbf{\Theta})^2\right)^{1/2}}$$

By definition

$$\mathrm{ETI}(\mathcal{I} \mid \mathbf{\Theta}) = \int_{\mathcal{I}} d\mathrm{ETI}(t \mid \mathbf{\Theta}) \mathrm{d}t$$

which completes the proof.

C Zero-crossings of f and df in the zero-mean stationary case

Let $f \sim \mathcal{GP}\left(0, C_{\boldsymbol{\theta}}(\cdot,\cdot)\right)$ where the $C_{\boldsymbol{\theta}}$ is either the Squared Exponential or Rational Quadratic covariance function. We look at the expected number of zero-crossings on an interval by either f and df as given by the Rice formula in Equation (1) with either $X(t) = f(t)$ or $X(t) = df(t)$. In this case the expressions simplifies immensely due to the zero means of both f, df, and $d^2 f$ and because $\mathrm{Cov}[f(t), df(t)] = 0$ and $\mathrm{Cov}[df(t), d^2 f(t)] = 0$. The latter is a result of using a stationary covariance function for the prior distribution of f (Cramer and Leadbetter 1967). In this stationary case local expected number of zero-crossing of f and df are given by

$$\frac{\partial_1\partial_2 C_{\boldsymbol{\theta}}(s,t)\Big|_{s=t}^{1/2}}{\pi C_{\boldsymbol{\theta}}(t,t)^{1/2}} \quad \text{and} \quad \frac{\partial_1^2\partial_2^2 C_{\boldsymbol{\theta}}(s,t)\Big|_{s=t}^{1/2}}{\pi \partial_1\partial_2 C_{\boldsymbol{\theta}}(s,t)\Big|_{s=t}^{1/2}}$$

respectively. It then follows that

$$\begin{aligned}
C_{\boldsymbol{\theta}}^{\mathrm{SE}}(t,t) = \sigma^2, \quad \partial_1\partial_2 C_{\boldsymbol{\theta}}^{\mathrm{SE}}(s,t)\Big|_{s=t} &= \frac{\sigma^2}{\rho^2}, \quad \partial_1^2\partial_2^2 C_{\boldsymbol{\theta}}^{\mathrm{SE}}(s,t)\Big|_{s=t} = \frac{3\sigma^2}{\rho^4} \\
C_{\theta}^{\mathrm{RQ}}(t,t) = \sigma^2, \quad \partial_1\partial_2 C_{\boldsymbol{\theta}}^{\mathrm{RQ}}(s,t)\Big|_{s=t} &= \frac{\sigma^2}{\rho^2}, \quad \partial_1^2\partial_2^2 C_{\boldsymbol{\theta}}^{\mathrm{RQ}}(s,t)\Big|_{s=t} = \frac{2\sigma^2(1+\nu)}{\nu\rho^4}
\end{aligned}$$

and the local expected number of zero-crossings of f and df for either the Squared Exponential and the Rational Quadratic covariance functions are

$$\begin{aligned}
\frac{\partial_1\partial_2 C_{\boldsymbol{\theta}}^{\mathrm{SE}}(s,t)\Big|_{s=t}^{1/2}}{\pi C_{\boldsymbol{\theta}}^{\mathrm{SE}}(t,t)^{1/2}} &= \frac{1}{\pi\rho}, \qquad \frac{\partial_1^2\partial_2^2 C_{\boldsymbol{\theta}}^{\mathrm{SE}}(s,t)\Big|_{s=t}^{1/2}}{\pi\partial_1\partial_2 C_{\boldsymbol{\theta}}^{\mathrm{SE}}(s,t)\Big|_{s=t}^{1/2}} = \frac{3^{1/2}}{\pi\rho} \\
\frac{\partial_1\partial_2 C_{\boldsymbol{\theta}}^{\mathrm{RQ}}(s,t)\Big|_{s=t}^{1/2}}{\pi C_{\boldsymbol{\theta}}^{\mathrm{RQ}}(t,t)^{1/2}} &= \frac{1}{\pi\rho}, \qquad \frac{\partial_1^2\partial_2^2 C_{\boldsymbol{\theta}}^{\mathrm{RQ}}(s,t)\Big|_{s=t}^{1/2}}{\pi\partial_1\partial_2 C_{\boldsymbol{\theta}}^{\mathrm{RQ}}(s,t)\Big|_{s=t}^{1/2}} = \frac{3^{1/2}}{\pi\rho}\left(1+v^{-1}\right)^{1/2}
\end{aligned}$$

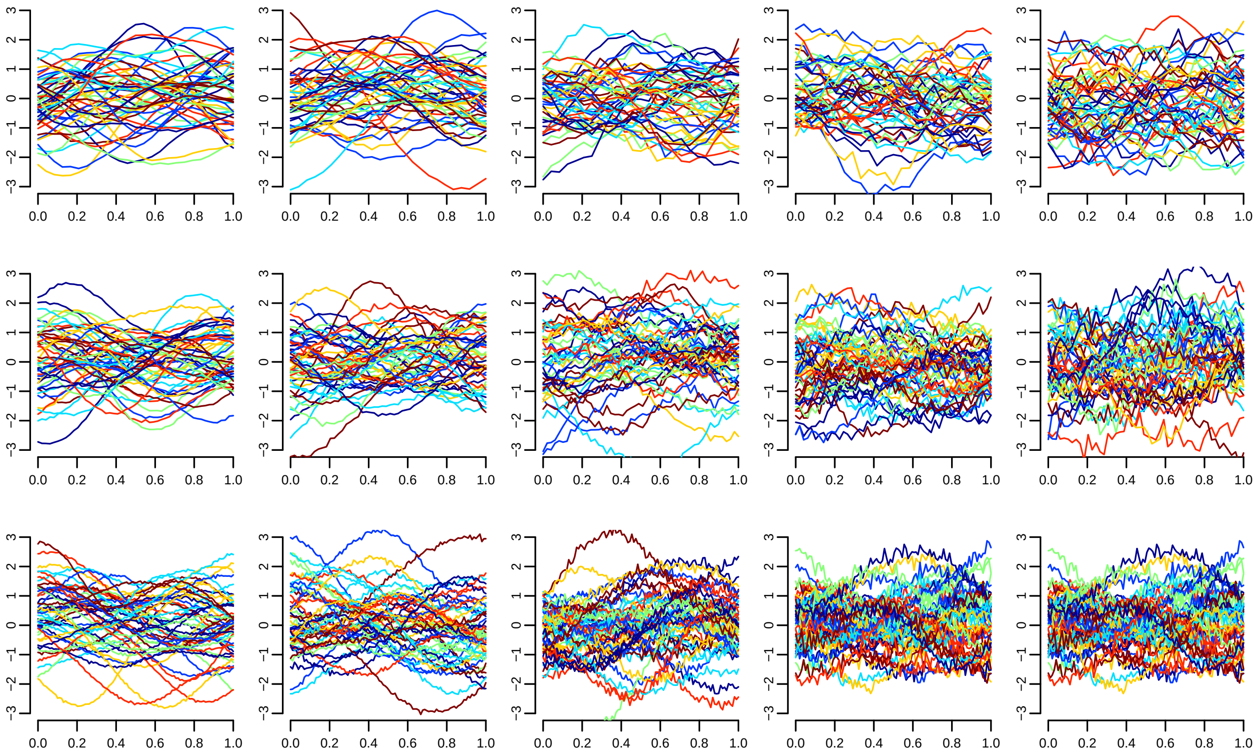

Figure 1: 50 random sample paths from each simulation scenario.

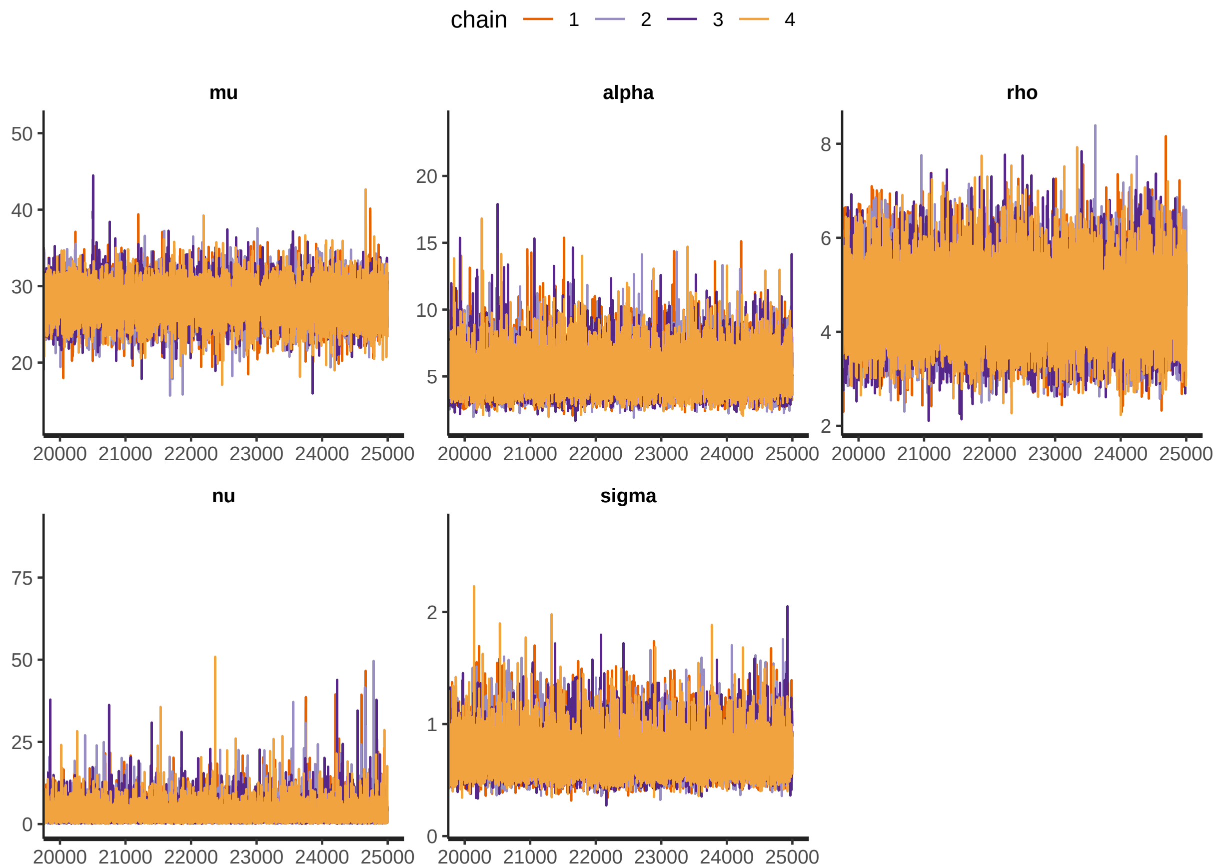

Figure 2: Trace plots of the last 5,000 MCMC iterations for the hyper-parameters in the smoking application.

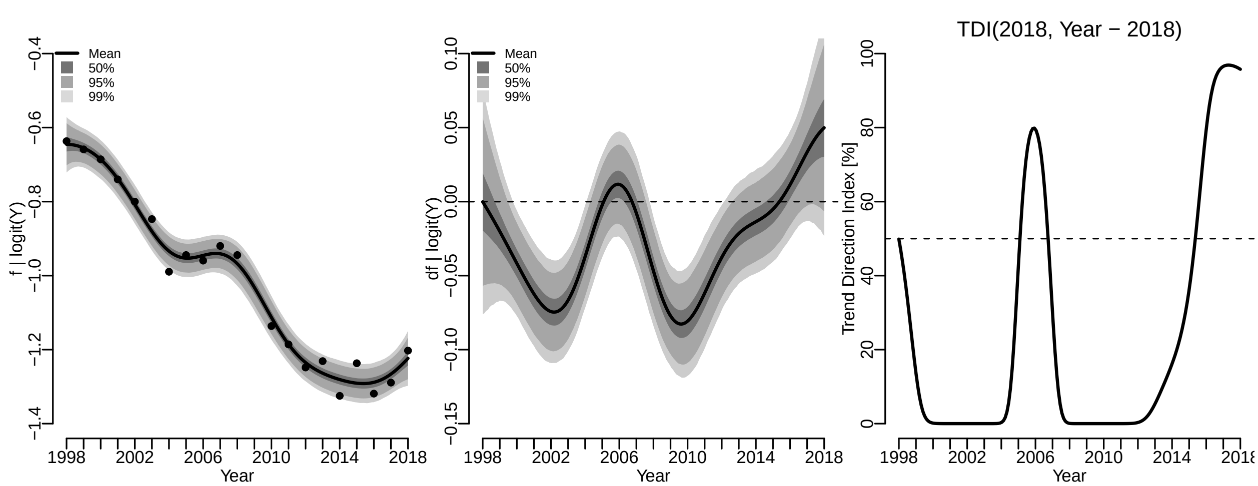

Figure 3: Trend analysis of smoking data with logit transformed outcome.

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