The Trendiness of Trends

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The problem

Changes in trends

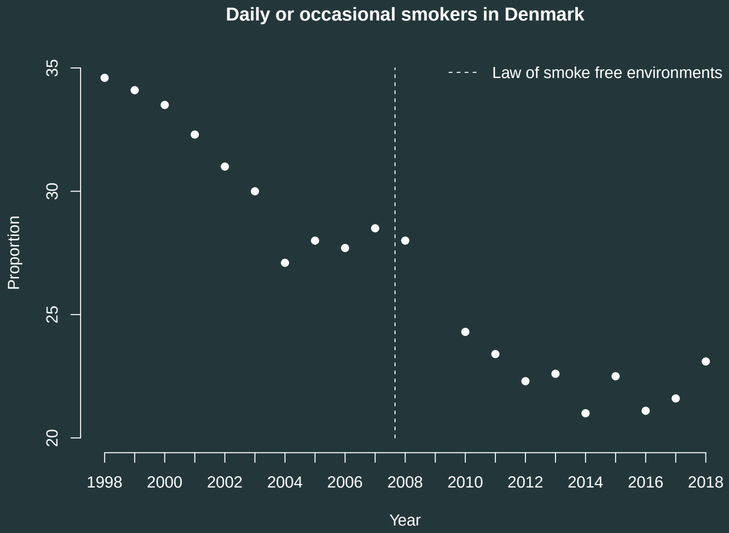
A statement often seen in the news is that at this very moment we see a significant change in the trend of something.

„The trend has broken/changed”

Recent examples from the Danish public news:

- Proportion of injuries from fireworks on New Year's Eve
- Proportion of children being baptized
- Average price of a one-family house
- Proportion of smokers

Proportion of smokers in Denmark



Data analysis objectives

We wish to:

1. Quantify the certainty by which the proportion of smokers is increasing in 2018.
2. Find out when the proportion started to increase (if it is currently increasing).
3. Assess if it is the first time in 20 years that the proportion has increased.

Six definitions

Definition 1

Definition 1
Reality evolves in continuous time $t \in \mathcal{T} \subset \mathbb{R}$.

Definition 2
There exists a latent function $f = \{f(t) : t \in \mathcal{T}\}$ governing the time evolution of some outcome. We observe random variables sampled from f at discrete time points with additive measurement noise

$$Y_i = f(t_i) + \varepsilon_i, \quad t_i \in \mathcal{T}, \quad \mathbb{E}[\varepsilon_i \mid t_i] = 0$$

We wish to understanding the dynamics of f conditional on observed data $(Y_i, t_i)_{i=1}^n$.

Definition 3
The trend of f is its instantaneous slope

$$df(t) = \left(\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right)(t)$$

- $df(t) > 0$: f is increasing at t and has a positive trend
- $df(t) < 0$: f is decreasing at t and has a negative trend

Definition 4

A change in trend of f is when df changes sign.

- f goes from increasing to decreasing or vice versa
- The trend, df, goes from positive to negative or vice versa
- f changes monotonicity

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A change in trend does not occur at a unique point in time.

Our question is:

> *„What is the probability that the trend is changing at time t given everything we know?"*

This is <u>not</u> a classical change-point model.

Definition 5
The Trend Direction Index is the conditional probability

$$\mathrm{TDI}(t,\delta) = P(df(t+\delta) > 0 \mid \mathcal{F}_t)$$

where $\mathcal{F}_t$ is the sigma algebra of all available information up until time t.

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- TDI is a local probability.
- $\delta \leq 0$ is estimation. $\delta > 0$ is forecasting.
- Alternatively: $df(t + \delta) < 0$ but that is just $1 - \mathrm{TDI}(t, \delta)$.
- Popular example: $t = \text{now}$ and $\delta = 0$ (change-point models are impossible).

Definition 6

Definition 6
The Expected Trend Instability on an interval $\mathcal{I}$ is

$$\mathrm{ETI}(\mathcal{I}) = \mathbb{E}[\#\{t \in \mathcal{I} : df(t) = 0\} \mid \mathcal{F}]$$

The value is equal to:

- The expected number of trend changes on $\mathcal{I}$
- The expected number of changes in monotonicity of f on $\mathcal{I}$
- The expected number of zero-crossings by df on $\mathcal{I}$

ETI is a global measure and the lower ETI is, the more stable the trend is on an interval.

A Latent Gaussian Process Approach for Assessing the Trendiness of Trends

Latent Gaussian Process model

Gaussian Process

A random function $\{f(t) : t \in \mathcal{T}\}$ is a Gaussian Process if and only if $(f(t_1), \ldots, f(t_n))$ is multivariate normal distributed for every $(t_1, \ldots, t_n) \subset \mathcal{T}$ with $n < \infty$.

We write $f \sim \mathcal{GP}(\mu(\cdot), C(\cdot, \cdot))$ where $\mu\colon \mathcal{T} \mapsto \mathbb{R}$ and $C\colon \mathcal{T} \times \mathcal{T} \mapsto \mathbb{R}$ are the mean and covariance functions.

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We assume that the observed data is generated by the following hierarchical model:

$$f \mid \beta, \theta \sim \mathcal{GP}(\mu_\beta(\cdot), C_\theta(\cdot, \cdot))$$

$$Y_i \mid f(t_i), \Theta, t_i \overset{iid}{\sim} N(f(t_i), \sigma^2), \quad \Theta = (\beta, \theta, \sigma^2)$$

Joint distribution

Let $\mathbf{Y}$ be the vector of observed data at time points $\mathbf{t}$, and $\mathbf{t}^* \subset \mathcal{T}$.

The model implies the following joint distribution:

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ \mathbf{Y} \end{bmatrix} \mid \mathbf{t}, \Theta \sim N\left(\begin{bmatrix} \mu_\beta(\mathbf{t}^*) \\ d\mu_\beta(\mathbf{t}^*) \\ \mu_\beta(\mathbf{t}) \end{bmatrix}, \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1\partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ C_\theta(\mathbf{t}, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) & C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \end{bmatrix} \right)$$

(sample path regularity conditions are implicitly assumed)

Joint posterior distribution

The joint posterior distribution of the latent functions is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \end{bmatrix} \mid \mathbf{Y}, \mathbf{t}, \Theta \sim N\left(\begin{bmatrix} \mu_f(\mathbf{t}^*) \\ \mu_{df}(\mathbf{t}^*) \end{bmatrix}, \begin{bmatrix} \Sigma_{f,f}(\mathbf{t}^*, \mathbf{t}^*) & \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^*) \\ \Sigma_{f,df}(\mathbf{t}^*, \mathbf{t}^*)^T & \Sigma_{df,df}(\mathbf{t}^*, \mathbf{t}^*) \end{bmatrix} \right)$$

$$\begin{aligned}
\mu_f(\mathbf{t}^*) &= \mu_\beta(\mathbf{t}^*) + C_\theta(\mathbf{t}^*, \mathbf{t}) \left[C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right]^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t})) \\
\mu_{df}(\mathbf{t}^*) &= d\mu_\beta(\mathbf{t}^*) + \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \left[C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right]^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t})) \\
\Sigma_{f,f}(\mathbf{t}^*, \mathbf{t}^*) &= C_\theta(\mathbf{t}^*, \mathbf{t}^*) - C_\theta(\mathbf{t}^*, \mathbf{t}) \left[C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right]^{-1} C_\theta(\mathbf{t}^*, \mathbf{t})^T \\
\Sigma_{df,df}(\mathbf{t}^*, \mathbf{t}^*) &= \partial_1 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) - \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \left[C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I\right]^{-1} \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t})^T \\
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\end{aligned}$$

Important: $\mathbf{t}^*$ can be any finite vector in continuous time.

Trend Direction Index

Recall the definition of the Trend Direction Index

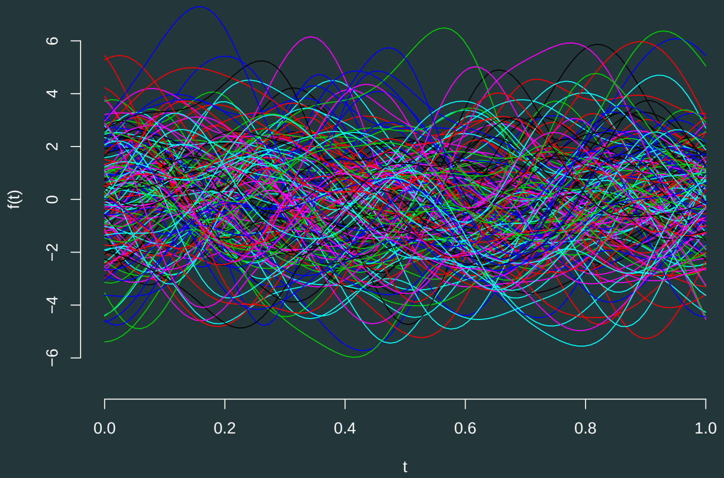
$$\mathrm{TDI}(t,\delta) = P(df(t+\delta) > 0 \mid \mathcal{F}_t)$$

Letting $\mathcal{F}_t = \{\mathbf{Y}, \mathbf{t}\}$, we may express TDI through the posterior of df as

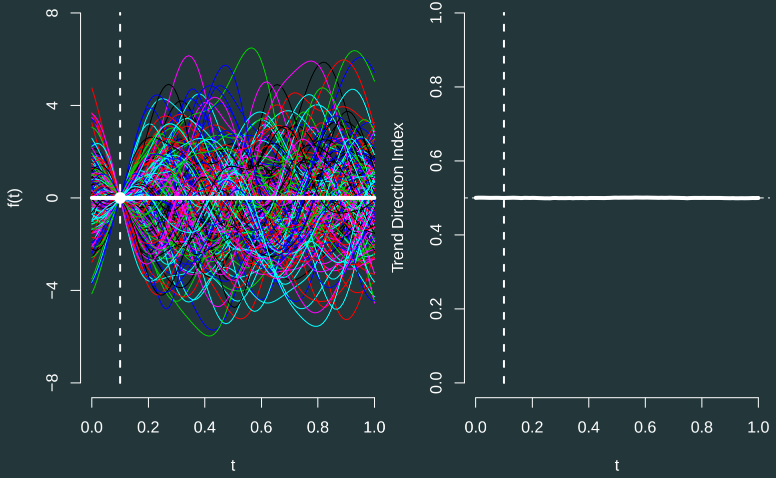
$$\begin{aligned}\mathrm{TDI}(t,\delta \mid \Theta) &= P\left(df(t+\delta) > 0 \mid \mathbf{Y}, \mathbf{t}, \Theta\right) \\ &= \int_0^\infty N\left(u, \mu_{df}(t+\delta), \Sigma_{df,df}(t+\delta, t+\delta)^{1/2}\right) \mathrm{d}u\end{aligned}$$

Example - prior distribution of f

What the world could look like with eyes closed



Example - posterior distribution of f and TDI



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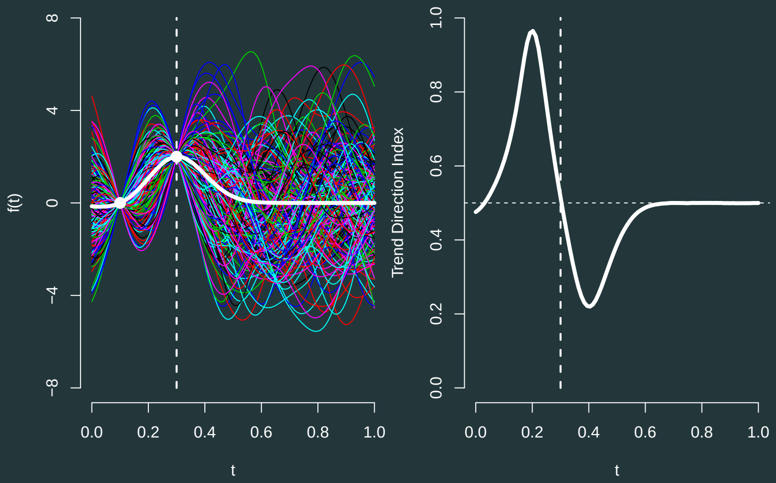

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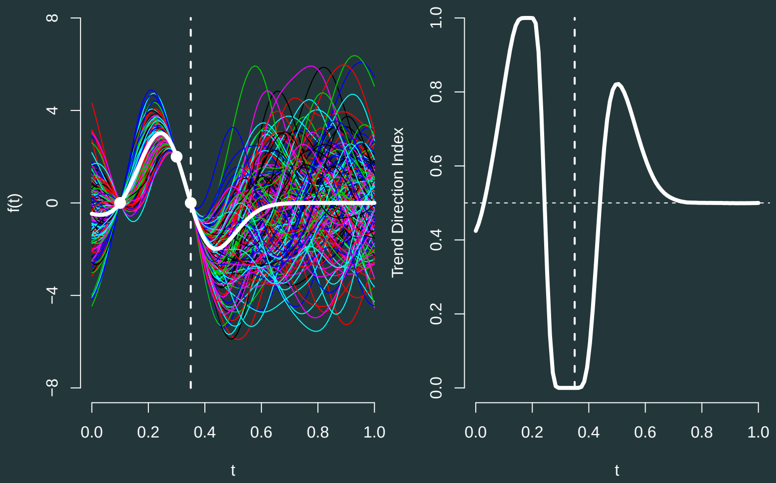

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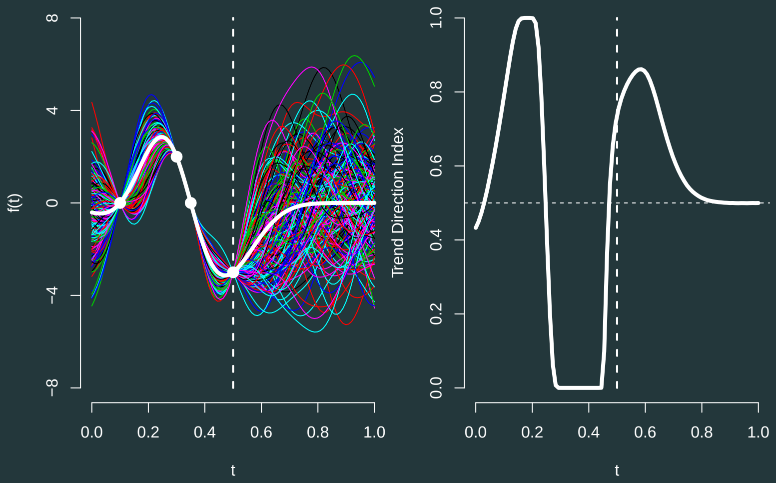

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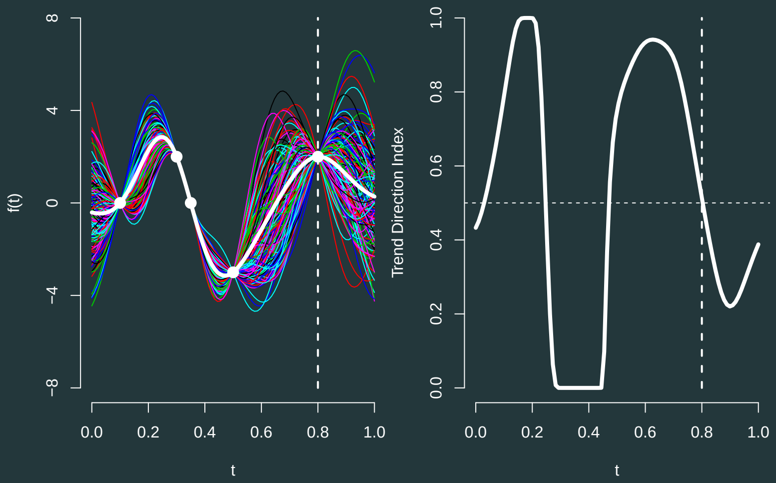

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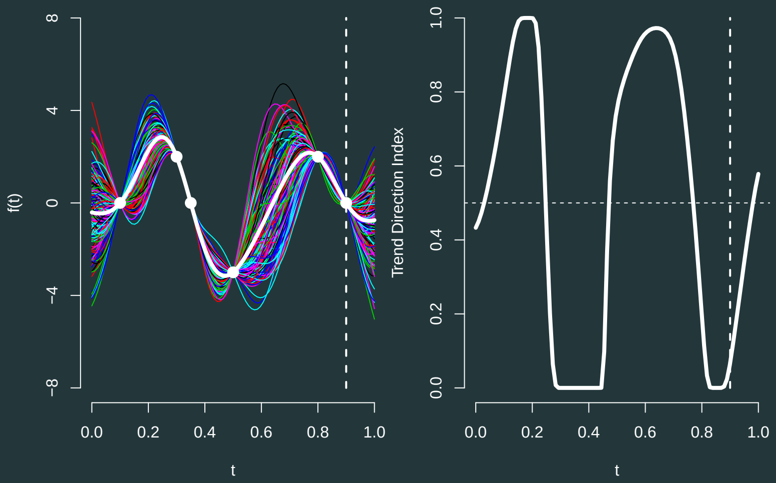

Recall the definition of the Expected Trend Instability

$$\mathrm{ETI}(\mathcal{I}) = \mathbb{E}[\#\{t \in \mathcal{I} : df(t) = 0\} \mid \mathcal{F}]$$

Rice (1944) showed that the expected number of 0-crossings of a process X under suitable regularity conditions is equal to

$$\mathrm{ETI}(\mathcal{I}) = \int_{\mathcal{I}} \int_{-\infty}^{\infty} |v| f_{X(t),dX(t)}(0, v) \mathrm{d}v \mathrm{d}t$$

where the integrand is the local crossing intensity.

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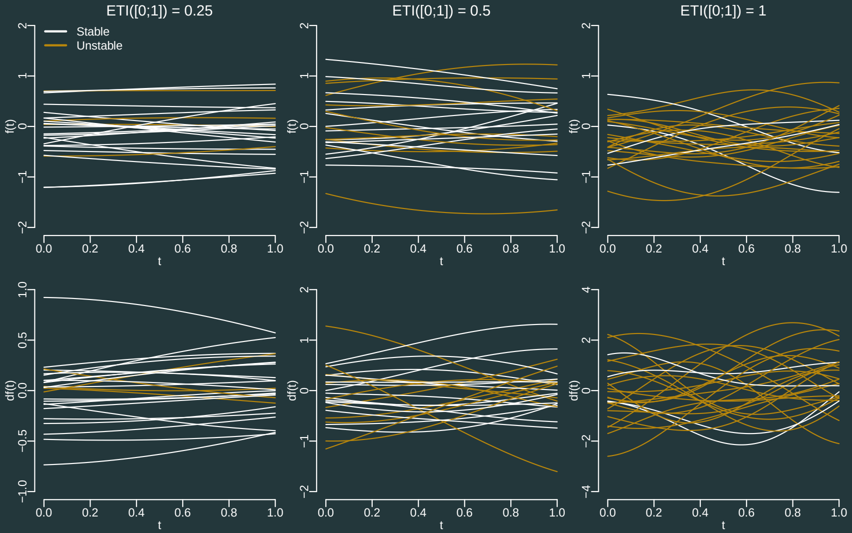
We can calculate this using the posterior of (df, d^2f):

$$\mathrm{ETI}(\mathcal{T} \mid \Theta) = \int_{\mathcal{T}} \frac{\Sigma_{d^2f}(t,t)^{1/2}}{\Sigma_{df}(t,t)^{1/2}} \sqrt{1-\omega(t)^2}\,\phi\left(\frac{\mu_{df}(t)}{\Sigma_{df}(t,t)^{1/2}}\right)\left[2\phi(\eta(t)) + \eta(t)\,\mathrm{Erf}\left(\frac{\eta(t)}{\sqrt{2}}\right)\right]\mathrm{d}t$$

$$\omega(t) = \frac{\Sigma_{df,d^2f}(t,t)}{\Sigma_{df}(t,t)^{1/2}\Sigma_{d^2f}(t,t)^{1/2}}, \quad \eta(t) = \frac{d^2f(t) - df(t)\Sigma_{d^2f}(t,t)^{1/2}\omega(t)\Sigma_{df}(t,t)^{-1/2}}{\Sigma_{d^2f}(t,t)^{1/2}(1-\omega(t)^2)^{1/2}}$$

Example - Expected Trend Instability



Estimation

Maximum Marginal Likelihood

The marginal likelihood is analytically available by integrating out the latent process

$$L(\Theta \mid \mathbf{Y}, \mathbf{t}) = N(\mathbf{Y}; \mu_\beta(\mathbf{t}), C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I)$$

leading to the estimate $\widehat{\Theta} = \arg\sup_\Theta L(\Theta \mid \mathbf{Y}, \mathbf{t})$.

This gives us the point estimates:

$$\mathrm{TDI}(t, \delta \mid \widehat{\Theta}), \quad \mathrm{ETI}(\mathcal{T} \mid \widehat{\Theta})$$

But it is difficult to obtain their distributions which we want for inference.

Bayesian estimation

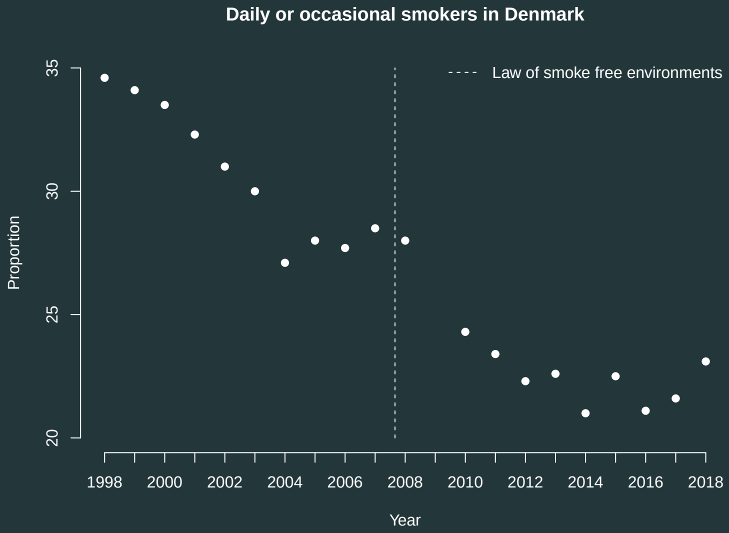
Another approach is a fully Bayesian model with a prior distribution on the parameters of the latent function, $\Theta \sim G$.

This enables posterior distributions of $\mathrm{TDI}(t, \delta \mid \widetilde{\Theta})$ and $\mathrm{ETI}(\mathcal{T} \mid \widetilde{\Theta})$ using Markov-chain Monte Carlo simulation.

We have implemented the model in Stan.

Application

Proportion of smokers in Denmark



Simple analysis - Has the proportion in 2018 changed?

A simple analysis is to test whether the proportion of smokers in 2018 has changed compared to a previous year. Just a χ^2-test.

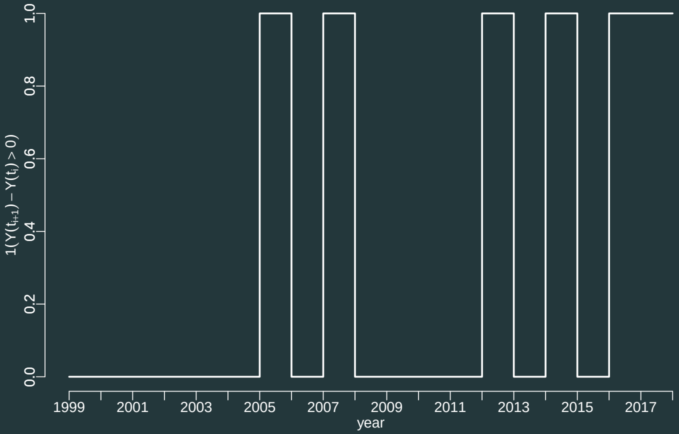
Comparison	p-value
2018 vs. 2017	0.074
2018 vs. 2016	**0.020**
2018 vs. 2015	0.495
2018 vs. 2014	**0.012**
2018 vs. 2013	0.576

Conclusion?

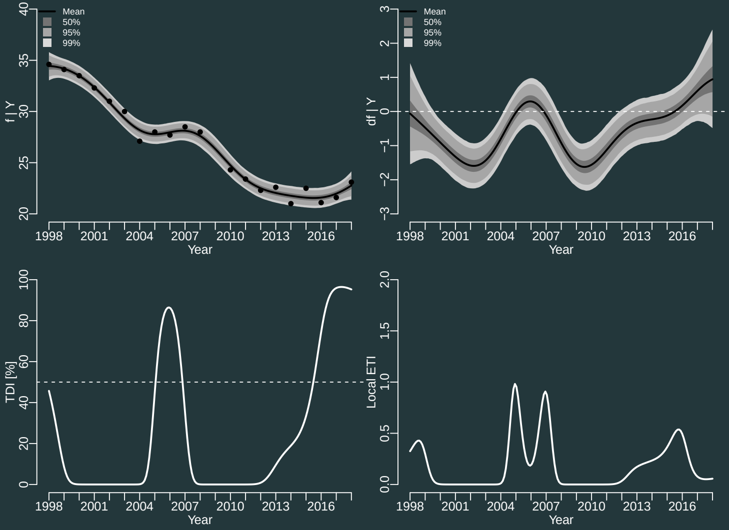
Simple analysis - It is the first time the trend has changed?

A simple approach is to look at how often $1(Y(t_{i+1}) - Y_{t_i} > 0)$ jumps.

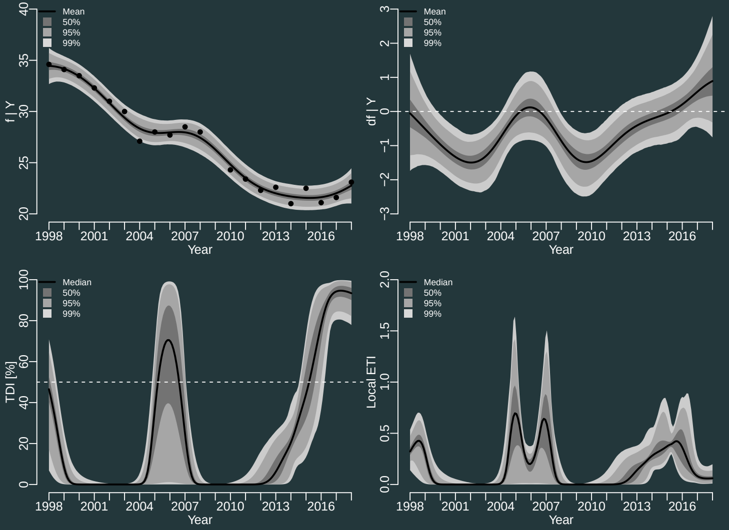
9 jumps. Many of them are probably just noise.



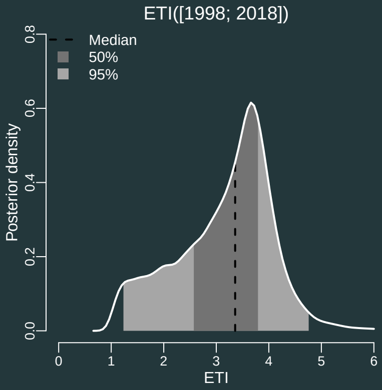
Trendiness analysis - Maximum Likelihood fit



Trendiness analysis - Bayesian fit



Trendiness analysis - Posterior ETI



Conclusions

Maximum Marginal Likelihood analysis:

- TDI(2018) = 95.24%
- ETI([1998; 2018]) = 3.68
- ETI([2008; 2018]) = 1.39

Bayesian analysis:

- Mean TDI in 2018 = 93.37% (95% CI = [82.22%; 98.87%])
- Median ETI([1998; 2018]) = 3.36 (95% CI = [1.22; 4.76])
- Median ETI([2008; 2018]) = 1.26 (95% CI = [1.02; 2.20])

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Conclusions:

- We are currently (positively) trending with a very high probability.
- We have been trending with probability > 50% since sometime between 2015 and 2016.
- The expected number of changes in trend during the last 20 years is higher than stipulated.

Thank you!

Trendiness analysis - Model selection

Before fitting the model we need to select a prior mean and covariance function for the Gaussian Process.

We consider 12 different models and compare them by Maximum Marginal Likelihood based LOO MSEP.

	SE	RQ	Matern 3/2	Matern 5/2
Constant	0.682	**0.651**	0.687	0.660
Linear	0.806	$\Leftarrow$	0.896	0.865
Quadratic	0.736	$\Leftarrow$	0.800	0.785

$$\Theta_{-i}^{\mathcal{M}} = \arg\sup_{\Theta} N\left(\mathbf{Y}_{-i}; \mu_{\beta}^{\mathcal{M}}(\mathbf{t}_{-i}), C_{\theta}^{\mathcal{M}}(\mathbf{t}_{-i}, \mathbf{t}_{-i}) + \sigma^2 I\right)$$

$$\mathrm{MSPE}_{\mathrm{LOO}}^{\mathcal{M}} = \frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \mathbb{E}[f(t_i) \mid \mathbf{Y}_{-i}, \mathbf{t}_{-i}\Theta_{-i}^{\mathcal{M}}]\right)^2$$

A provocative epilogue

We try to answer the question:

- What is the local probability of a trend being positive given everything we know?

We could also have asked:

- What is the probability that an observed difference between years is larger than what would have been expected under the assumption of no change in trend?

Which question is easier to understand and convey to the public?