

Supplementary Material for Quantifying the Trendiness of Trends

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24 May, 2020

A Proof of Proposition 1

Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ and $\mathbf{t} = (t_1, \dots, t_n)$ be the vectors of observed outcomes and associated sampling times. From the data generating model we observe that the marginal distribution of the vector of observed outcomes $\mathbf{Y} \mid \mathbf{t}, \Theta$ is

$$\begin{aligned} P(\mathbf{Y} \mid \mathbf{t}, \Theta) &= \int P(\mathbf{Y} \mid f(\mathbf{t}), \mathbf{t}, \Theta) dP(f(\mathbf{t}) \mid \mathbf{t}, \Theta) \\ &= N(\mu_\beta(\mathbf{t}), C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I) \end{aligned}$$

where $\mu_\beta(\mathbf{t}) = (\mu_\beta(t_1), \dots, \mu_\beta(t_n))$, $C_\theta(\mathbf{t}, \mathbf{t})$ is the $n \times n$ covariance matrix obtained by evaluating $C_\theta(s, t)$ at $\{(s, t) \in \mathbf{t} \times \mathbf{t}\}$ and I is an $n \times n$ identity matrix. This implies that the joint distribution of \mathbf{Y} and the latent functions (f, df, d^2f) evaluated at an arbitrary vector of time points \mathbf{t}^* is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \\ \mathbf{Y} \end{bmatrix} \mid \mathbf{t}, \Theta \sim N \left(\begin{bmatrix} \mu_\beta(\mathbf{t}^*) \\ d\mu_\beta(\mathbf{t}^*) \\ d^2\mu_\beta(\mathbf{t}^*) \\ \mu_\beta(\mathbf{t}) \end{bmatrix}, \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ C_\theta(\mathbf{t}, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) & C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I \end{bmatrix} \right)$$

where ∂_j^k denotes the k 'th order partial derivative with respect to the j 'th variable.

By the standard formula for deriving conditional distributions in a multivariate normal model, the posterior distribution of (f, df, d^2f) evaluated at the p time points in \mathbf{t}^* is

$$\begin{bmatrix} f(\mathbf{t}^*) \\ df(\mathbf{t}^*) \\ d^2f(\mathbf{t}^*) \end{bmatrix} \mid \mathbf{Y}, \mathbf{t}, \Theta \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\mu} \in \mathbb{R}^{3p}$ is the column vector of posterior expectations and $\boldsymbol{\Sigma} \in \mathbb{R}^{3p \times 3p}$ is the joint posterior covariance matrix, and these are given by

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_\beta(\mathbf{t}^*) \\ d\mu_\beta(\mathbf{t}^*) \\ d^2\mu_\beta(\mathbf{t}^*) \end{bmatrix} + \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\theta, \sigma}(\mathbf{t}, \mathbf{t})^{-1} (\mathbf{Y} - \mu_\beta(\mathbf{t}))$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) & \partial_1^2 \partial_2^2 C_\theta(\mathbf{t}^*, \mathbf{t}^*) \end{bmatrix} - \begin{bmatrix} C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1 C_\theta(\mathbf{t}^*, \mathbf{t}) \\ \partial_1^2 C_\theta(\mathbf{t}^*, \mathbf{t}) \end{bmatrix} K_{\theta, \sigma}(\mathbf{t}, \mathbf{t})^{-1} \begin{bmatrix} C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \partial_2 C_\theta(\mathbf{t}, \mathbf{t}^*) \\ \partial_2^2 C_\theta(\mathbf{t}, \mathbf{t}^*) \end{bmatrix}^T$$

where $K_{\theta, \sigma}(\mathbf{t}, \mathbf{t}) = C_\theta(\mathbf{t}, \mathbf{t}) + \sigma^2 I$. Partitioning $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_f(\mathbf{t}^* | \Theta) \\ \mu_{df}(\mathbf{t}^* | \Theta) \\ \mu_{d^2f}(\mathbf{t}^* | \Theta) \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \Sigma_f(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{f, df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{f, d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) \\ \Sigma_{f, df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{df, d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) \\ \Sigma_{d^2f, f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{d^2f, df}(\mathbf{t}^*, \mathbf{t}^* | \Theta) & \Sigma_{d^2f}(\mathbf{t}^*, \mathbf{t}^* | \Theta) \end{bmatrix}$$

and completing the matrix algebra, we obtain the expressions of the individual components given in the Proposition.

B Proof of Proposition 3

Rice showed in section 3.3. of Rice (1945) that the expected number of zero-crossings of a Gaussian process X on an interval \mathcal{I} is given by

$$\int_{\mathcal{I}} \int_{-\infty}^{\infty} |v| f_{X(t), dX(t)}(0, v) dv dt \quad (1)$$

where $f_{X(t), dX(t)}$ is the joint density function of X and its derivative dX at time t . To derive the expression for the Expected Trend Instability we must apply the Rice formula to the joint posterior distribution of (df, d^2f) . From Proposition 1 the distribution of $(df, d^2f) | \mathbf{Y}, \mathbf{t}, \Theta$ is bivariate normal for each t .

Let μ_{df} , μ_{d^2f} , Σ_{df} and Σ_{d^2f} be defined as in Proposition 1 and define further

$$\omega(t | \Theta) = \frac{\Sigma_{df, d^2f}(t, t | \Theta)}{\Sigma_{df}(t, t | \Theta)^{1/2} \Sigma_{d^2f}(t, t | \Theta)^{1/2}}$$

as the posterior point-wise cross-correlation function between df and d^2f . The joint posterior density function of (df, d^2f) at any time t evaluated at $(0, v)$ can be factorized as

$$f_{df(t), d^2f(t)}(0, v) = c_1(t) e^{c_2(t)} e^{-c_3(t)v^2 - 2c_4(t)v}$$

where c_1, \dots, c_4 are functions of time given by

$$\begin{aligned}
c_1(t) &= (2\pi)^{-1} \Sigma_{df}(t, t | \Theta)^{-1/2} \Sigma_{d^2f}(t, t | \Theta)^{-1/2} (1 - \omega(t | \Theta)^2)^{-1/2} \\
c_2(t) &= \frac{\mu_{df}(t | \Theta)^2}{2 \Sigma_{df}(t, t | \Theta) (\omega(t | \Theta)^2 - 1)} + \frac{\mu_{d^2f}(t | \Theta)^2}{2 \Sigma_{d^2f}(t, t | \Theta) (\omega(t | \Theta)^2 - 1)} \\
&\quad - \frac{\mu_{df}(t | \Theta) \mu_{d^2f}(t | \Theta) \omega(t | \Theta)}{\Sigma_{df}(t, t | \Theta)^{1/2} \Sigma_{d^2f}(t, t | \Theta)^{1/2} (\omega(t | \Theta)^2 - 1)} \\
c_3(t) &= -\frac{1}{2} \Sigma_{d^2f}(t, t | \Theta)^{-1} (\omega(t | \Theta)^2 - 1)^{-1} \\
c_4(t) &= -\frac{\mu_{df}(t | \Theta) \Sigma_{d^2f}(t, t | \Theta)^{1/2} \omega(t | \Theta) - \mu_{d^2f}(t | \Theta) \Sigma_{df}(t, t | \Theta)^{1/2}}{2 \Sigma_{d^2f}(t, t | \Theta) (\omega(t | \Theta)^2 - 1) \Sigma_{df}(t, t | \Theta)^{1/2}}
\end{aligned}$$

Let $d\text{ETI}(t | \Theta)$ denote the inner integral in Equation (1). Using the factorization of the joint posterior density we may write it was

$$\begin{aligned}
d\text{ETI}(t | \Theta) &= \int_{-\infty}^{\infty} |v| f_{df(t), d^2f(t)}(0, v) dv \\
&= c_1(t) e^{c_2(t)} \int_{-\infty}^{\infty} |v| e^{-c_3(t)v^2 - 2c_4(t)v} dv \\
&= c_1(t) e^{c_2(t)} \left(\int_0^{\infty} v e^{-c_3(t)v^2 + 2c_4(t)v} dv + \int_0^{\infty} v e^{-c_3(t)v^2 - 2c_4(t)v} dv \right)
\end{aligned} \tag{2}$$

Because $c_3(t) > 0$ for all t since $\Sigma_{d^2f}(t, t | \Theta) > 0$ and $|\omega(t | \Theta)| < 1$ by Assumption A4 we obtain the following solution for the type of integral in the previous display by using formula 5 in section 3.462 on page 365 of Gradshteyn and Ryzhik (2014)

$$\int_0^{\infty} v e^{-c_3(t)v^2 \pm 2c_4(t)v} dv = \frac{1}{2c_3(t)} \pm \frac{c_4(t)}{2c_3(t)} \frac{\pi^{1/2}}{c_3(t)^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)}} \left(1 \pm \text{Erf} \left(\frac{c_4(t)}{\sqrt{c_3(t)}} \right) \right) \tag{3}$$

where $\text{Erf}: x \mapsto 2\pi^{-1} \int_0^x e^{-u^2} du$ is the error function. Combining Equations (2) and (3) we may express $d\text{ETI}$ as

$$d\text{ETI}(t | \Theta) = c_1(t) e^{c_2(t)} \left(\frac{1}{c_3(t)} + \frac{c_4(t)}{c_3(t)} \frac{\pi^{1/2}}{c_3(t)^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)}} \text{Erf} \left(\frac{c_4(t)}{\sqrt{c_3(t)}} \right) \right)$$

Defining $\zeta(t | \Theta) = \sqrt{2} c_4(t) c_3(t)^{-1/2}$ and collecting some terms, the index can be rewritten as

$$d\text{ETI}(t | \Theta) = \frac{c_1(t)}{c_3(t)} \left(e^{c_2(t)} + \frac{\pi^{1/2}}{2^{1/2}} e^{\frac{c_4(t)^2}{c_3(t)} + c_2(t)} \zeta(t) \text{Erf} \left(\frac{\zeta(t | \Theta)}{2^{1/2}} \right) \right)$$

Straightforward arithmetic calculations show that

$$\frac{c_4(t)^2}{c_3(t)} + c_2(t) = -\frac{\mu_{df}(t | \Theta)^2}{2\Sigma_{df}(t, t | \Theta)}, \quad c_2(t) = -\frac{1}{2} \left(\zeta(t | \Theta)^2 + \frac{\mu_{df}(t | \Theta)^2}{\Sigma_{df}(t, t | \Theta)} \right)$$

and by defining $\phi: x \mapsto (2\pi)^{-1/2}e^{-x^2}$ as the density function of the standard normal distribution we may write $e^{\frac{c_4(t)^2}{c_3(t)} + c_2(t)} = (2\pi)^{1/2}\phi\left(\frac{\mu_{df}(t|\Theta)}{\Sigma_{df}(t,t|\Theta)^{1/2}}\right)$ and $e^{c_2(t)} = 2\pi\phi(\zeta(t))\phi\left(\frac{\mu_{df}(t|\Theta)}{\Sigma_{df}(t,t|\Theta)^{1/2}}\right)$ which leads to

$$d\text{ETI}(t | \Theta) = \frac{c_1(t)}{c_3(t)} \pi \phi\left(\frac{\mu_{df}(t | \Theta)}{\Sigma_{df}(t, t | \Theta)^{1/2}}\right) \left(2\phi(\zeta(t | \Theta)) + \zeta(t | \Theta) \text{Erf}\left(\frac{\zeta(t | \Theta)}{2^{1/2}}\right) \right)$$

Standard arithmetics show that

$$\frac{c_1(t)}{c_3(t)} = \frac{1}{\pi} \frac{\Sigma_{d^2f}(t, t | \Theta)^{1/2}}{\Sigma_{df}(t, t | \Theta)^{1/2}} \left(1 - \omega(t | \Theta)^2\right)^{1/2}$$

and we finally obtain the expression

$$d\text{ETI}(t | \Theta) = \lambda(t | \Theta) \phi\left(\frac{\mu_{df}(t | \Theta)}{\Sigma_{df}(t, t | \Theta)^{1/2}}\right) \left(2\phi(\zeta(t | \Theta)) + \zeta(t | \Theta) \text{Erf}\left(\frac{\zeta(t | \Theta)}{2^{1/2}}\right) \right)$$

where λ and ζ are given by

$$\begin{aligned} \lambda(t | \Theta) &= \frac{\Sigma_{d^2f}(t, t | \Theta)^{1/2}}{\Sigma_{df}(t, t | \Theta)^{1/2}} \left(1 - \omega(t | \Theta)^2\right)^{1/2} \\ \zeta(t | \Theta) &= \frac{\mu_{df}(t | \Theta) \Sigma_{d^2f}(t, t | \Theta)^{1/2} \omega(t) \Sigma_{df}(t, t | \Theta)^{-1/2} - \mu_{d^2f}(t | \Theta)}{\Sigma_{d^2f}(t, t | \Theta)^{1/2} (1 - \omega(t | \Theta)^2)^{1/2}} \end{aligned}$$

By definition

$$\text{ETI}(\mathcal{I} | \Theta) = \int_{\mathcal{I}} d\text{ETI}(t | \Theta) dt$$

which completes the proof.

C Zero-crossings of f and df in the zero-mean stationary case

Let $f \sim \mathcal{GP}(0, C_\theta(\cdot, \cdot))$ where the C_θ is either the Squared Exponential or Rational Quadratic covariance function. We look at the expected number of zero-crossings on an interval by either f and df as given by the Rice formula in Equation (1) with either $X(t) = f(t)$ or $X(t) = df(t)$. In this case the expressions simplifies immensely due to the zero means of both f , df , and d^2f and because $\text{Cov}[f(t), df(t)] = 0$ and $\text{Cov}[df(t), d^2f(t)] = 0$. The latter is a result of using a stationary covariance function for the prior distribution of f (Cramer and Leadbetter 1967). In this stationary case local expected number of zero-crossing of f and df are given by

$$\frac{\partial_1 \partial_2 C_\theta(s, t) \Big|_{s=t}^{1/2}}{\pi C_\theta(t, t)^{1/2}} \quad \text{and} \quad \frac{\partial_1^2 \partial_2^2 C_\theta(s, t) \Big|_{s=t}^{1/2}}{\pi \partial_1 \partial_2 C_\theta(s, t) \Big|_{s=t}^{1/2}}$$

respectively. It then follows that

$$\begin{aligned} C_\theta^{\text{SE}}(t, t) &= \sigma^2, \quad \partial_1 \partial_2 C_\theta^{\text{SE}}(s, t) \Big|_{s=t} = \frac{\sigma^2}{\rho^2}, \quad \partial_1^2 \partial_2^2 C_\theta^{\text{SE}}(s, t) \Big|_{s=t} = \frac{3\sigma^2}{\rho^4} \\ C_\theta^{\text{RQ}}(t, t) &= \sigma^2, \quad \partial_1 \partial_2 C_\theta^{\text{RQ}}(s, t) \Big|_{s=t} = \frac{\sigma^2}{\rho^2}, \quad \partial_1^2 \partial_2^2 C_\theta^{\text{RQ}}(s, t) \Big|_{s=t} = \frac{2\sigma^2(1 + \nu)}{\nu \rho^4} \end{aligned}$$

and the local expected number of zero-crossings of f and df for either the Squared Exponential and the Rational Quadratic covariance functions are

$$\begin{aligned} \frac{\partial_1 \partial_2 C_\theta^{\text{SE}}(s, t) \Big|_{s=t}^{1/2}}{\pi C_\theta^{\text{SE}}(t, t)^{1/2}} &= \frac{1}{\pi \rho}, & \frac{\partial_1^2 \partial_2^2 C_\theta^{\text{SE}}(s, t) \Big|_{s=t}^{1/2}}{\pi \partial_1 \partial_2 C_\theta^{\text{SE}}(s, t) \Big|_{s=t}^{1/2}} &= \frac{3^{1/2}}{\pi \rho} \\ \frac{\partial_1 \partial_2 C_\theta^{\text{RQ}}(s, t) \Big|_{s=t}^{1/2}}{\pi C_\theta^{\text{RQ}}(t, t)^{1/2}} &= \frac{1}{\pi \rho}, & \frac{\partial_1^2 \partial_2^2 C_\theta^{\text{RQ}}(s, t) \Big|_{s=t}^{1/2}}{\pi \partial_1 \partial_2 C_\theta^{\text{RQ}}(s, t) \Big|_{s=t}^{1/2}} &= \frac{3^{1/2}}{\pi \rho} (1 + \nu^{-1})^{1/2} \end{aligned}$$

D Simulation study

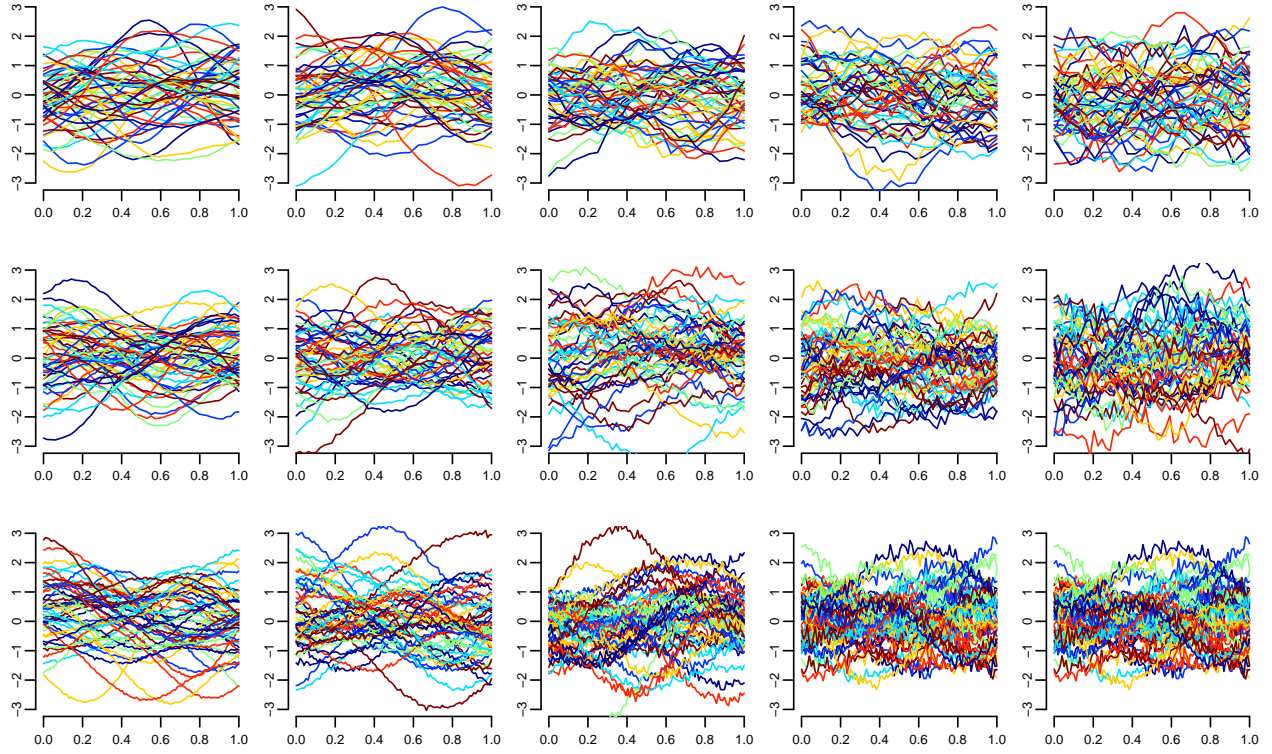
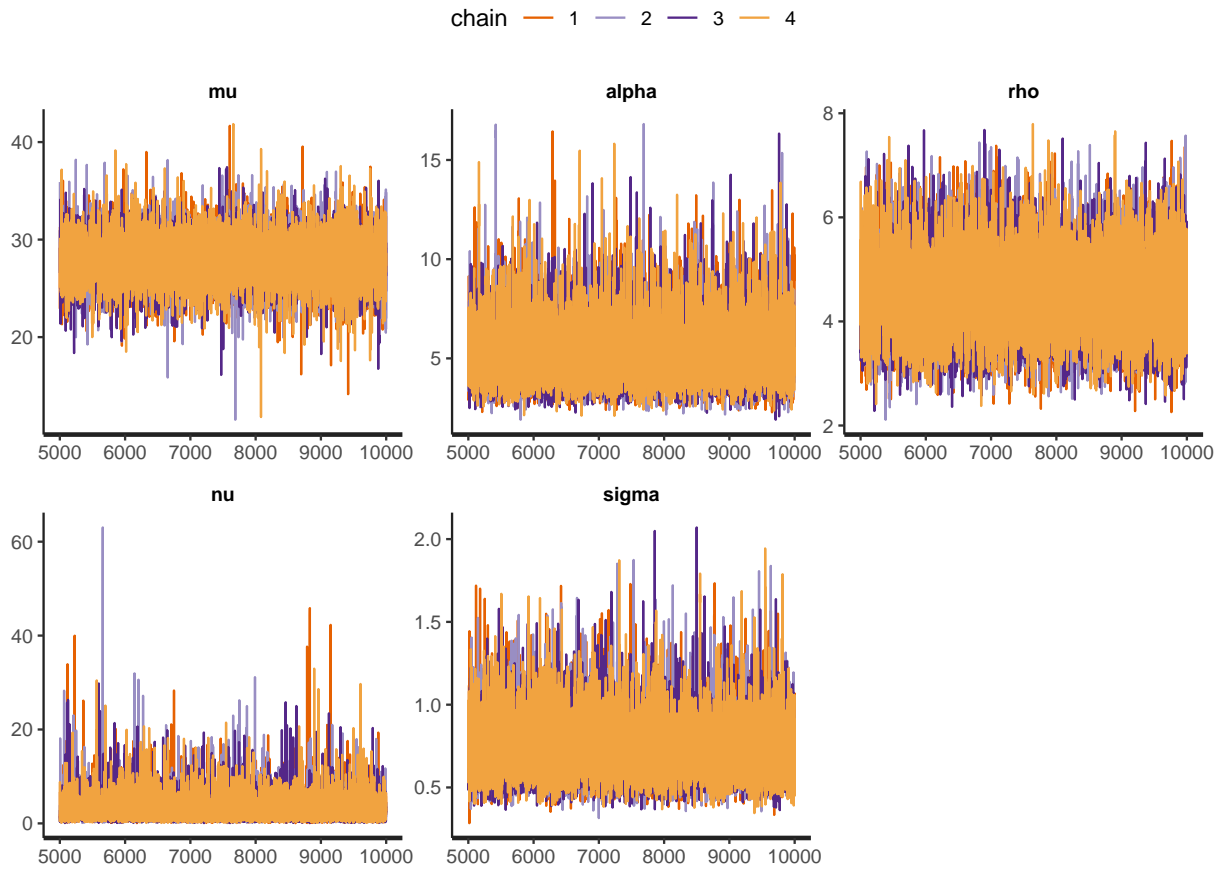
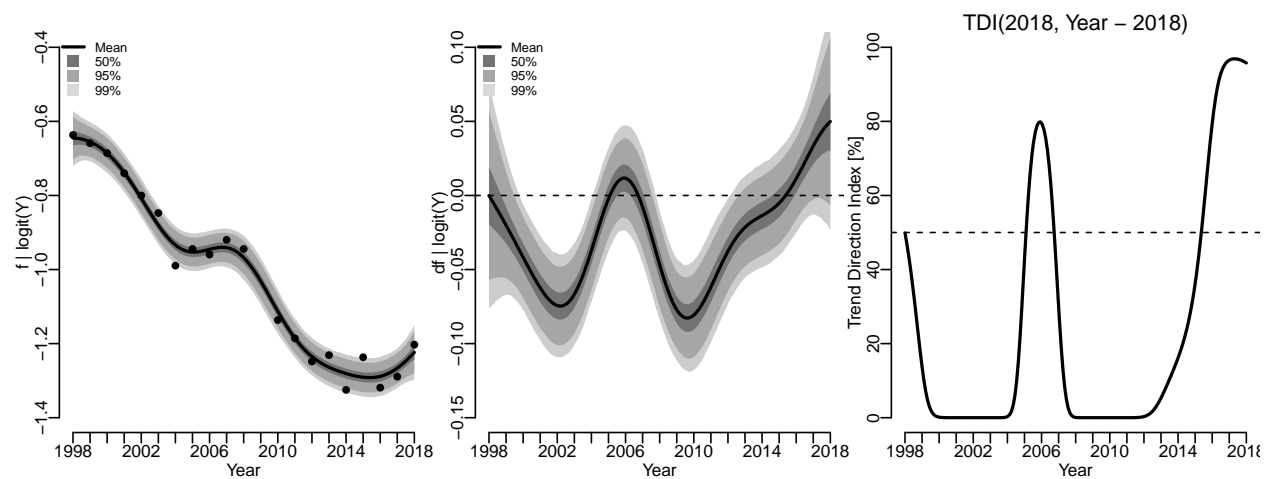


Figure 1: 50 random observations from each simulation scenario.

E Trace plot for hyper-parameters for smoking application



F Trend analysis of smoking data with logit transformed outcome



Bibliography

Cramer, Harald, and M. R. Leadbetter. 1967. *Stationary and Related Stochastic Processes – Sample Function Properties and Their Applications*. John Wiley & Sons, Inc.

Gradshteyn, Izrail Solomonovich, and Iosif Moiseevich Ryzhik. 2014. *Table of Integrals, Series, and Products*. Academic Press.

Rice, Stephen O. 1945. “Mathematical Analysis of Random Noise, II.” *Bell System Technical Journal* 24 (1): 46–156.