Libration Point Orbit Rendezvous Technique Using Linearized Relative Motion Dynamics and Nonlinear Differential Correction

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1 Introduction

The history of satellites being deployed to libration point orbits dates back to 1978, when ISEE-3 was launched to the Sun-Earth L1 point. ACE, WIND, SOHO are currently orbiting the same point, with the DSCOVR and LPF satellites set to join them in 2015. Several spacecraft have been deployed to the Sun-Earth L2 point, and the ARTEMIS mission explored L1 and L2 in the Earth-Moon system. Several additional libration point orbiting missions are currently planned, including the James Webb Space Telescope. So far, all libration point missions have consisted of a single satellite which operates independently of any other satellite. There is research on deploying a formation of satellites to fly together around libration points. However, not much research has been performed regarding rendezvous with libration point orbiters.

Libration point orbit rendezvous would be a critical component for many possible mission architectures. If a large satellite needs to be launched in components and assembled in orbit, the individual components would need to perform rendezvous and docking prior to assembly. If a valuable space asset such as a telescope requires an on-orbit repair, the satellite servicing mission will need to rendezvous with the object. A libration point orbit could be a useful place to build a space station, and in that case rendezvous capabilities would be important during the construction of the station as well as every crew and cargo mission to visit the station.

Satellite rendezvous in low-Earth orbit is well-studied, due to many years of experience operating the International Space Station and other applications. The Hill's / Clohessy-Wiltshire equations can be used to estimate the relative motion of a chaser vehicle with respect to a target vehicle in a circular orbit. These equations of relative motion can be used to compute the estimated ΔV (instantaneous change in velocity) to travel between waypoints defining an approach trajectory. However, for libration point orbits, the dynamical environment is quite different and the same equations of motion can not be used.

Luquette [1] has developed linearized equations of relative motion for formation flying in libration point orbits. Lian et al. [2] have used these linearized dynamics to compute impulses for a chaser satellite to travel between waypoints in order to approach a target orbiting a libration point. This paper discusses the results of applying this technique for test cases in the Earth-Moon L1 system, and presents an additional step in which the shooting method is used to refine the computed ΔV for use in nonlinear propagation.

2 Dynamics

2.1 Circular Restricted Three-Body Dynamics

The circular restricted three body problem (CRTBP) deals with two larger objects orbiting each other and a third object of infinitesimal mass. Examples include the Trojan asteroids orbiting in the Sun-Jupiter system, or man-made satellites in the Earth-Moon system. When dealing with the Sun-Earth system, the second body is often modeled by treating the Earth and Moon as a single object at the Earth-Moon barycenter; this is called the Sun-Earth/Moon system.

The nonlinear equations of motion for a satellite in the CRTBP are:

$$\begin{aligned}\ddot{x} &= x + 2\dot{y} + \frac{(1-\mu)(-\mu-x)}{r_1^3} + \frac{\mu(1-\mu-x)}{r_2^3} \\ \ddot{y} &= y - 2\dot{x} - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \\ \ddot{z} &= \frac{-(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}\end{aligned} \tag{1}$$

where μ is the mass ratio of the primary bodies:

$$\mu = \frac{M_2}{M_1 + M_2}$$

and r_1 and r_2 are the distances from the larger and smaller bodies to the target satellite:

$$\begin{aligned}r_1 = \|\mathbf{x} - \mathbf{X_1}\| &= \sqrt{(x - \mathbf{X_1}_x)^2 + y^2 + z^2} \\ r_2 = \|\mathbf{x} - \mathbf{X_2}\| &= \sqrt{(x - \mathbf{X_2}_x)^2 + y^2 + z^2}\end{aligned}$$

where $\mathbf{X_1}$ and $\mathbf{X_2}$ are the positions of the larger and smaller bodies along the X-axis of the rotating CRTBP frame:

$$\mathbf{X_1} = \begin{bmatrix} -\mu \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{X_2} = \begin{bmatrix} 1-\mu \\ 0 \\ 0 \end{bmatrix}$$

2.2 Linearized CRTBP Relative Motion Dynamics

Luquette [1] has derived linearized equations of relative motion of a chaser satellite with respect to a target satellite orbiting in the restricted three body problem (RTBP). He provides the equations of motion in two reference frames: the inertial frame as well as the rotating (RTBP) frame. These equations of motion are valid anywhere in the RTBP system; they are not assumed to be near any specific libration point.

The linearized equations of relative motion of a chaser satellite with respect to a target satellite, given in the RTBP reference frame, are:

$$\dot{\boldsymbol{\xi}}_R = \mathbf{A}_R(t)\boldsymbol{\xi}_R \tag{2}$$

where $\boldsymbol{\xi}_R$ is the offset state of the chaser vehicle with respect to the target vehicle in the rotating frame:

$$\boldsymbol{\xi}_R = \begin{bmatrix} \mathbf{x}_R \\ \dot{\mathbf{x}}_R \end{bmatrix}$$

and $\mathbf{A}_R(t)$ is the 6×6 linearized relative motion dynamics matrix in the rotating frame:

$$\mathbf{A}_R(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I_3} \\ \boldsymbol{\Xi}_R(t) & -2[\boldsymbol{\omega}\times] \end{bmatrix} \tag{3}$$

where

$$\boldsymbol{\Xi}_R(t) = -(c_1 + c_2)\mathbf{I_3} + 3c_1\hat{\mathbf{r}}_\mathbf{1}(t)\hat{\mathbf{r}}_\mathbf{1}(t)^T + 3c_2\hat{\mathbf{r}}_\mathbf{2}(t)\hat{\mathbf{r}}_\mathbf{2}(t)^T + [\dot{\boldsymbol{\omega}}\times] - [\boldsymbol{\omega}\times][\boldsymbol{\omega}\times]$$

and

$$c_1 = \frac{1-\mu}{r_1^3}$$
$$c_2 = \frac{\mu}{r_2^3}$$

Note that $\boldsymbol{\omega}$ is the rotation rate of the rotating RTBP frame with respect to the inertial frame:

$$\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

And $[\boldsymbol{\omega}\times]$ is the cross-product matrix of $\boldsymbol{\omega}$:

$$[\boldsymbol{\omega}\times] = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also note that if we assume that the rotation rate of the RTBP frame is constant (that is, assume we can use the CRTBP where the massive bodies are in circular orbits around each other), then the $\dot{\boldsymbol{\omega}}$ term cancels to zero.

In order to integrate a chaser satellite's trajectory using the equations of relative motion given in Equation 2, the integration state vector must contain the absolute state of the target satellite in the CRTBP frame (with respect to the origin of the CRTBP frame) and the relative state of the chaser satellite in the CRTBP frame (with the origin located at the target satellite). The target satellite's state over time can be integrated using the classical CRTBP equations of motion as given in Equation 1, which must be done concurrently with the integration of the relative motion of the chaser so that the time-dependent linearized relative motion dynamics matrix $\mathbf{A}_R(t)$ given in Equation 3 can be computed.

3 Traveling Between Waypoints with Impulsive ΔV's

The concept of waypoints can be used to divide a rendezvous approach trajectory into a series of shorter segments. The starting and ending waypoints of a segment are defined with respect to the location of a target satellite. When considering rendezvous with a satellite in a circular orbit, the Clohessy-Wiltshire or Hill's equations can be used as a linearized dynamics model for a chaser satellite with respect to a target satellite. The inverse of the linear dynamics matrix is used to compute the velocity required to travel from one waypoint to the next within a specified amount of time. This approach can also be applied in the RTBP using the linear dynamics matrix presented in Equation 3, by following the procedure below.

3.1 Using the Linearized Relative Motion Dynamics Matrix to Compute ΔV

The linearized relative motion dynamics matrix given in Equation 3 can be used to numerically accumulate a State Transition Matrix (STM), $\mathbf{\Phi}$, of the chaser satellite with respect to the target satellite over time:

$$\dot{\mathbf{\Phi}} = \mathbf{A}_R(t)\mathbf{\Phi} \tag{4}$$

The initial "state vector" for the STM passed to the integration process should be the 6×6 identity matrix, $\mathbf{I_6}$. As above, the STM must be integrated concurrently with the target satellite's state because of the time-dependence in $\mathbf{A}_R(t)$. When integrated from the initial state at time t_i to a future time t_{i+1}, this accumulated State Transition Matrix represents the relative position and velocity of the chaser at time t_{i+1} with respect to its relative position and velocity at time t_i. More explicitly:

$$\begin{bmatrix} \mathbf{r}_{i+1} \\ \mathbf{v}_{i+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{21} & \mathbf{\Phi}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{r}_i \\ \mathbf{v}_i \end{bmatrix} \tag{5}$$

As with the use of the Clohessy-Wiltshire or Hill's equations in the two-body problem, this can be used to compute the required velocity $\mathbf{v}_i^+$ for the chaser satellite to travel from waypoint $\mathbf{r}_i$ to waypoint

$\mathbf{r}_{i+1}$ in time $(t_{i+1} - t_i)$. In [2], Lian et. al. used this approach using the RTBP relative motion dynamics derived by Luquette.

$$\mathbf{v}_i^+ = \mathbf{\Phi}_{12}^{-1}(\mathbf{r}_{i+1} - \mathbf{\Phi}_{11}\mathbf{r}_i) \tag{6}$$

The instantaneous change in velocity (ΔV) for this maneuver is then simply the difference between the required velocity and the velocity that the chaser satellite had before the maneuver:

$$\Delta\mathbf{v}_i = \mathbf{v}_i^+ - \mathbf{v}_i^- \tag{7}$$

3.2 Shooting Method with Nonlinear Dynamics

The approach described above for computing ΔV is based on the linearized equations of relative motion developed by Luquette. Of course, the true dynamical environment in the CRTBP is nonlinear, as seen in Equation 1.

The linear-based estimate of ΔV can be "corrected" for the nonlinear propagation model using the shooting method of differential correction. The linear-based estimated velocity is used as an initial guess for the iterative process. Following the procedure below, each of the three components of the velocity vector is varied in order to achieve convergence on the desired three-dimensional waypoint $\mathbf{w}_{\text{desired}}$ within some specified tolerance.

Step 1: Using the initial guess for the chaser relative velocity $\mathbf{v}_i^+$ from Equation 6, propagate both the target and chaser satellite from time t_i to t_{i+1} using the nonlinear CRTBP equations of motion given in Equation 1 and compute the nominally achieved waypoint $\mathbf{w}$.

Step 2: Add to the x-component of the velocity a pre-chosen scalar value, called the perturbation. Propagate the satellites from time t_i to t_{i+1} a second time using this perturbed velocity and compute the achieved waypoint $\mathbf{w}'$.

Step 3: Compute the difference between $\mathbf{w}$ and $\mathbf{w}'$, $\frac{d\mathbf{w}}{dv_x}$.

Step 4: Reset the x-component of the velocity to its original value, and repeat steps 2 and 3 for the y-component of the velocity.

Step 5: Reset the y-component of the velocity to its original value, and repeat steps 2 and 3 for the z-component of the velocity.

Step 6: Gather the results into a partial derivatives matrix, $\mathbf{M}$:

$$\mathbf{M} = \left[\frac{d\mathbf{w}}{d\mathbf{v}}\right] = \begin{bmatrix} \frac{dw_x}{dv_x} & \frac{dw_y}{dv_x} & \frac{dw_z}{dv_x} \\ \frac{dw_x}{dv_y} & \frac{dw_y}{dv_y} & \frac{dw_z}{dv_y} \\ \frac{dw_x}{dv_z} & \frac{dw_y}{dv_z} & \frac{dw_z}{dv_z} \end{bmatrix} \tag{8}$$

Step 7: Compute an updated estimate of the velocity required to travel to the desired location $\mathbf{w}_{\text{desired}}$:

$$\mathbf{v}_i^+ = \mathbf{v}_i^+ + [\mathbf{M}]^{-1}(\mathbf{w}_{\text{desired}} - \mathbf{w}) \tag{9}$$

Repeat steps 1 through 7 until all three components of $\mathbf{w}$ converge to $\mathbf{w}_{\text{desired}}$ within some tolerance.

3.3 Definition of Waypoint Reference Frames

In this work, two local reference frames are defined with respect to the target satellite's orbit around its libration point. Each of these frames will rotate once for each full orbit of the target around the libration point.

The two libration points of most interest for this work are L_1 and L_2. The position of L_1 in the CRTBP frame is found by computing the real root l_1 of this polynomial in x:

$$(1-\mu)(p^3)(p^2 - 3p + 3) - \mu(p^2 + p + 1)(1-p)^3 = 0 \tag{10}$$

where:

$$p = 1 - \mu - x$$

The coordinates of L_1 are then $[l_1, 0, 0]$ in the CRTBP frame. Likewise, the position of L_2 in the CRTBP frame is found by computing the real root l_2 of this polynomial in x:

$$(1-\mu)(p^3)(p^2+3p+3) - \mu(p^2+p+1)(1-p)(p+1)^2 = 0 \tag{11}$$

where:

$$p = \mu - 1 + x$$

The coordinates of L_2 are then $[l_2, 0, 0]$ in the CRTBP frame.

Local RIC and VNB reference frames are then defined with the origin of both of these frames located at the target satellite's position. The RIC, or "radial-intrack-crosstrack," frame as has its first primary axis the vector pointing from the libration point radially out to the target satellite in the RTBP frame. For example, in the case of a target satellite orbiting L_1, the fundamental $\hat{\mathbf{R}}$, $\hat{\mathbf{I}}$, and $\hat{\mathbf{C}}$ axes of the RIC frame are defined as:

$$\begin{aligned} \hat{\mathbf{R}} &= \widehat{\mathbf{x} - \mathbf{L}_1} \\ \hat{\mathbf{C}} &= \widehat{\mathbf{R} \times \dot{\mathbf{x}}} \\ \hat{\mathbf{I}} &= \widehat{\mathbf{C} \times \mathbf{R}} \end{aligned} \tag{12}$$

where $\mathbf{x}$ and $\dot{\mathbf{x}}$ are the target satellite's position and velocity vectors in the CRTBP frame.

The VNB, or "velocity-normal-binormal," frame has as its first primary axis the target satellite's velocity vector in the RTBP frame. In the case of a target satellite orbiting L_1, the fundamental $\hat{\mathbf{V}}$, $\hat{\mathbf{N}}$, and $\hat{\mathbf{B}}$ axes of the VNB frame are defined as:

$$\begin{aligned} \hat{\mathbf{V}} &= \hat{\dot{\mathbf{x}}} \\ \hat{\mathbf{N}} &= \widehat{\mathbf{R} \times \mathbf{V}} \\ \hat{\mathbf{B}} &= \widehat{\mathbf{V} \times \mathbf{N}} \end{aligned} \tag{13}$$

4 Results

4.1 Performance of Linear ΔV Estimate and Shooting Method

To begin evaluating the performance of the techniques presented above, consider the following scenario. The target satellite is in a planar Lyapunov orbit around the Earth-Moon L1 point as shown in Figure 1. The target satellite's initial conditions are presented in Table 1. For this test, the rendezvous starts with the target satellite on the CRTBP +X-axis, that is, crossing the XZ-plane.

Parameter	Value
X	0.862307159058101
Z	0.0
$\dot{Y}$	-0.187079489569182
Period	2.79101343456226

Table 1: Initial Conditions of the Target Satellite in the CRTBP Frame

The initial conditions in Table 1 are provided in non-dimensional units. To convert between non-dimensional and dimensional units, the relevant properties of the Earth-Moon CRTBP system are presented in Table 2.

Three waypoints are defined for the chaser satellite to travel along on its rendezvous with the target, with a final waypoint located at the target satellite itself. The waypoint locations are presented in Table

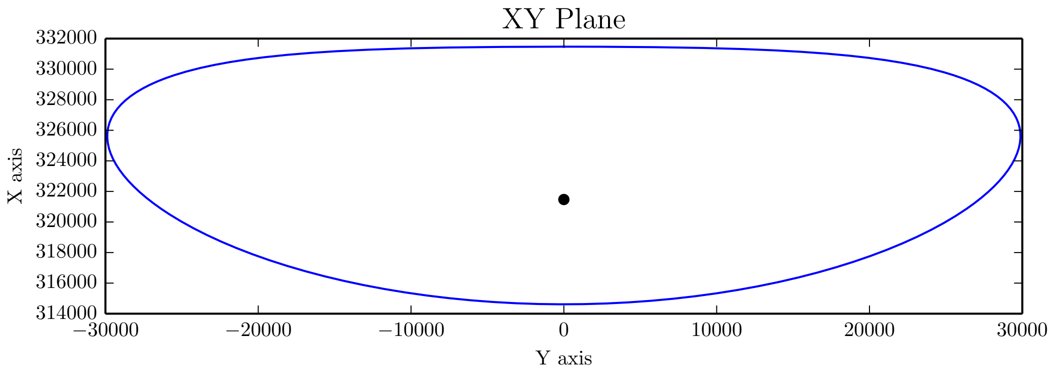

Figure 1: Target Satellite Orbit in Earth-Moon L1 CRTBP Frame (km)

Parameter	Value
Earth mass (kg)	5.97219e24
Moon mass (kg)	7.34767309e22
Mass ratio μ	0.012277471
Combined mass (kg), or 1 MU	6.045667e24
r_{12} (km), or 1 DU	384400.0
Time Constant (s), or 1 TU	375201.9
Period of Moon around Earth (s)	2πTU

Table 2: Earth-Moon CRTBP Parameters

3 in the RIC reference frame. The waypoints are converted into the CRTBP frame for the propagation. Note that these waypoints represent an approach along the **I** axis of the RIC frame.

Waypoint	Time (days)	R (km)	I (km)	C (km)
1	0.00	0.0	15.0	0.0
2	0.36	0.0	5.0	0.0
3	0.97	0.0	1.0	0.0
4	1.59	0.0	0.0	0.0

Table 3: Waypoints in RIC Frame

Figure 2 shows the result of propagating the target and chaser satellite through a rendezvous using these waypoints. The integration is performed in the CRTBP frame, and the results are converted to the RIC frame for visualization, with the target spacecraft at the origin and the chaser spacecraft offset with respect to the target shown in kilometers. The green curve shows the path of the chaser satellite when propagated using the linear relative motion dynamics as given in Equation 2, with the application of impulsive ΔV's as computed using Equation 6. The model used to compute the impulsive maneuvers matches the linear propagation model, and so the chaser satellite travels precisely to each waypoint. The red curve shows the result when the chaser satellite is propagated using the nonlinear CRTBP dynamics as shown in Equation 1 when the nominal ΔV's as computed using the linear model are still applied. It is easily seen that the nominally planned maneuvers do not bring the chaser exactly to the desired waypoints when propagating with the nonlinear model. Finally, the blue curve shows the result when the chaser satellite is propagated using the nonlinear CRTBP dynamics and the ΔV's have been corrected through the iterative shooting method procedure described in Section 3.2. The path does not precisely match the original green curve due to the different dynamical models in use; however, each waypoint is achieved successfully.

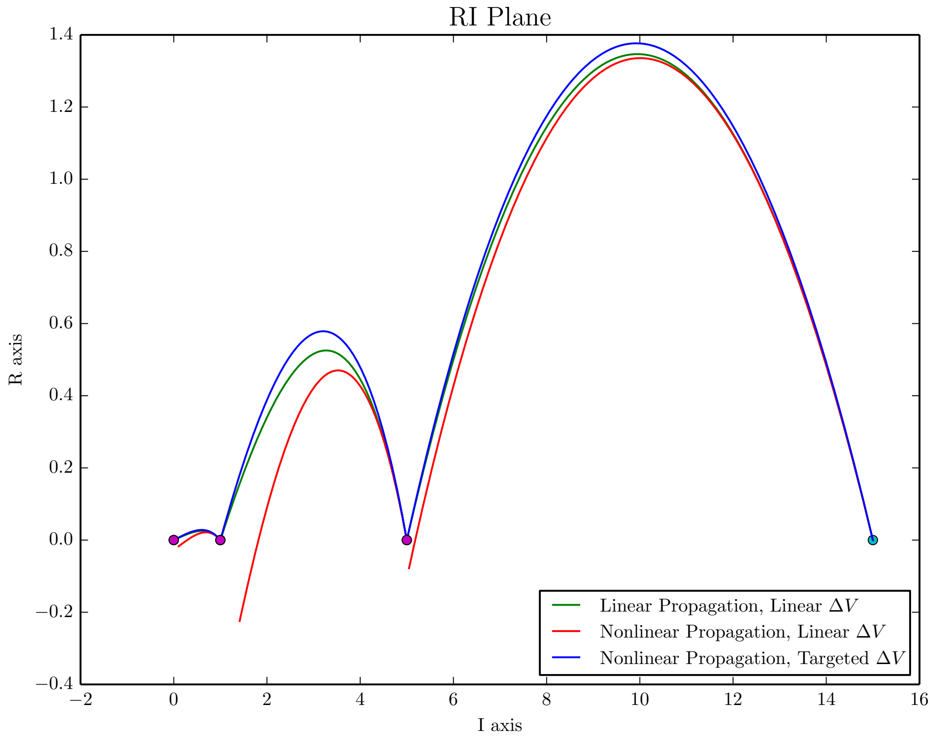

Figure 2: Relative Motion of a Chaser Satellite with respect to a Target (km)

Waypoint	Linear ΔV (m/s)	Targeted ΔV (m/s)	ΔV Angle Difference (deg)	$\|\Delta V\|$ Difference (m/s)	Linear Position Error (m)	Targeted Position Error (m)
1	0.346	0.345	0.466	-0.001	N/A	N/A
2	0.293	0.295	2.609	0.002	91.394	0.011
3	0.064	0.059	5.890	-0.005	470.653	0.063
4	0.019	0.018	0.445	-0.001	107.663	0.056

Table 4: ΔV and Position Error at each Waypoint

The results for this case are also summarized in Table 4. To compute the first ΔV, the chaser satellite is assumed to start precisely from waypoint 1 with its pre-maneuver velocity at waypoint 1 equal to the target satellite's initial velocity in the CRTBP frame. The final maneuver at waypoint 4 is computed to make the chaser's velocity equal to the target satellite's velocity for a completed rendezvous.

We can note that, while the difference in ΔV magnitude is quite small between the linear computation and the targeted computation, on the order of millimeters/second, the angular difference between the two ΔV vectors reaches almost 6°at waypoint 3. These differences result in achieved position errors on the order of hundreds of meters at each waypoint when the linear estimate is applied in the nonlinear propagation model; the errors are reduced to the order of centimeters when the shooting method is applied.

4.2 Rendezvous Approach Directions

This technique can be used to evaluate the relative ΔV cost of performing a rendezvous at different points along a target spacecraft's orbit. The following results present twelve test cases that each begin with the target satellite at a different "clock angle" in its orbit; each test case is separated by one twelfth of an orbit in time, with the starting positions indicated by the asterisks shown in Figure 3.

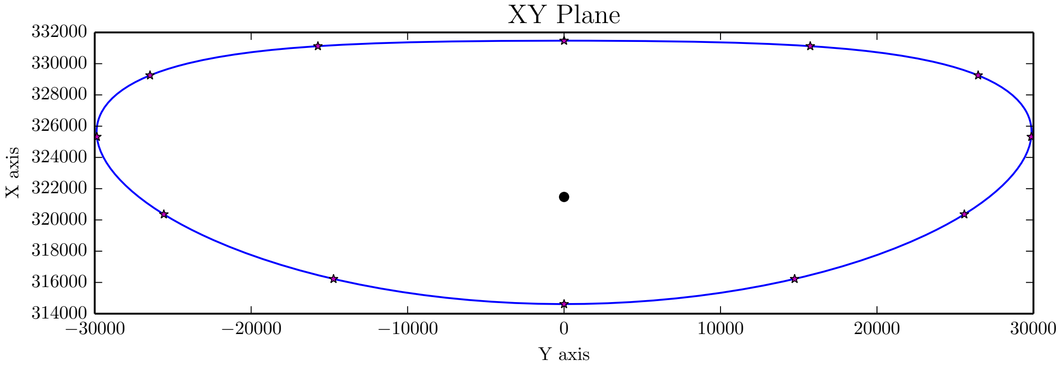

Figure 3: Twelve Target Satellite Initial Conditions (km)

Each of these twelve rendezvous sequences are computed using the waypoint locations defined in the RIC frame as shown in Table 3. However, due to the differing local curvature of the target orbit at each of the twelve starting locations, the relative motion patterns in the RIC frame for each rendezvous are different, as shown in Figure 4. When viewed using the CRTBP reference frame's rotating axes as in Figure 5, it is more intuitively seen that each of these in-track rendezvous sequences approach the target satellite from a different direction in the CRTBP frame, due to the rotation of the RIC frame with the target satellite's orbit.

References

[1] Richard J Luquette and Robert M Sanner. Linear state-space representation of the dynamics of relative motion, based on restricted three body dynamics. In *AIAA Guidance, Navigation, and Control Conference and Exhibit, AIAA*, volume 4783, 2004.

[2] Yijun Lian, Luhua Liu, Yunhe Meng, Guojian Tang, and Kejun Chen. Constant thrust glideslope guidance algorithm for rendezvous in multi-body realm. In *Recent Advances in Space Technologies (RAST), 2011 5th International Conference on*, pages 232–236. IEEE, 2011.

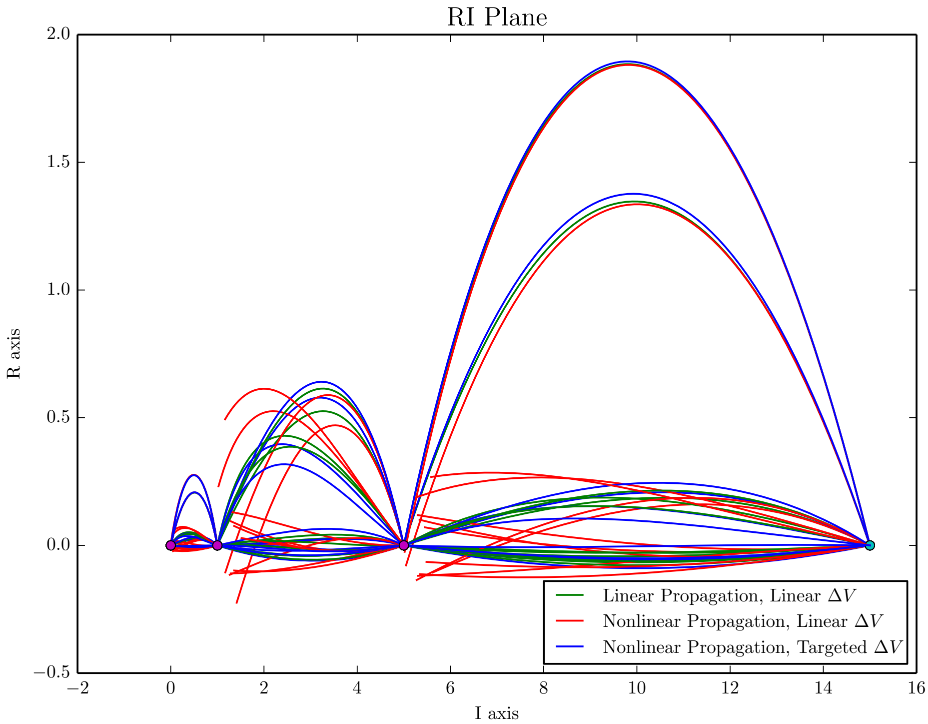

Figure 4: Relative Motion of Chaser Satellite with respect to a Target using 12 Different Initial Clock Angles; RIC Frame (km)

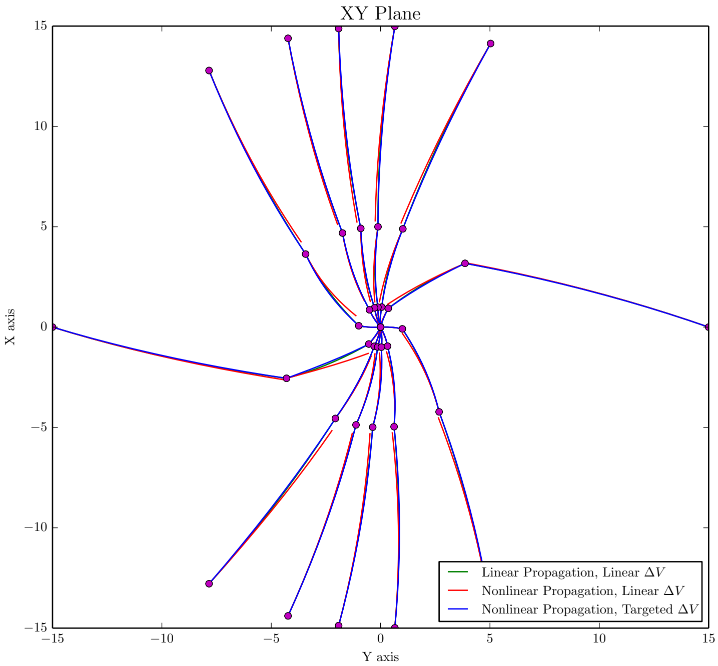

Figure 5: Relative Motion of Chaser Satellite with respect to a Target using 12 Different Initial Clock Angles; CRTBP Frame (km)