CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Sections**
Mathematics	203	All
Examination	**Date**	**Duration**
Midterm Test	24 February, 2013	1 h 30 min
Special Instructions:	Only approved calculators are allowed **Show all your work**	

1. (6 marks): Solve for x (find the *exact* values, do not approximate)

(a) $\log_3(3\,x) - \log_3(x^5) = 3$

(b) $4^{\log_2 x} - 6\,x = 0$

2. (6 marks) **(a)** Let $f(x) = \sqrt{x^2 - 1}$ and $g(x) = \dfrac{1}{x}$. Find the composite functions $f \circ g$ and $g \circ g$, and determine their domains.

(b) Given the function $f(x) = \dfrac{1}{1 + 2\,e^x}$, find the inverse function f^{-1} and the domain and the range of f^{-1}.

3. (8 marks) Find the limit or explain why the limit does not exist:

(a) $\displaystyle\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1}$

(b) $\displaystyle\lim_{x \to -1} \frac{10\,x + 10}{|\,x + 1|}$

4. (5 marks) Find all horizontal and vertical asymptotes, as well as $x-$ and $y-$intercepts of the graph $y = \dfrac{3x + 1}{\sqrt{4x^2 - 16}}$.

(continued on the other side)

5. (16 marks): Find the derivatives of the following functions
(you don't need to simplify the final answer, but you must show how you calculate it):

(a) $f(x) = \dfrac{4 + 2\sqrt{x} + 3x^2}{x}$

(b) $f(x) = \dfrac{\cos x \, \sin x}{1 - \tan x}$

(c) $f(x) = (\pi + e^x)\, 2^x$

(d) $f(x) = \cos^3(\sin\sqrt{x^2 + 9})$

6. (9 marks) Given the function $f(x) = x^2 - 2x$:

(a) Calculate $f'(x)$ using its definition as a limit (of difference quotient).

Check that your calculation is correct using standard rules for differentiation of power functions.

(b) Compute the average rate of change of $f(x)$ on the interval $[0, 3]$, call it m.

(c) Find whether there is a point a on that interval such that the instantaneous rate of change of f at that point is equal to the average rate of change, $f'(a) = m$. Calculate a if it exists.

Bonus Question (3 marks).

A cylinder is inscribed in a right circular cone with height $h = 8\,\text{m}$ and radius at the cone's base $r = 4\,\text{m}$. Express the volume of the cylinder V as a function of its radius x.

1a.

$$\log_3(3x) - \log_3 x^5 = 3 \Rightarrow \log_3 \frac{3}{x^4} = 3 \Rightarrow \frac{3}{x^4} = 3^3 = 27$$

$$x^4 = 1/9 \Rightarrow x = \frac{1}{\sqrt{3}} \text{ as } -\frac{1}{\sqrt{3}} \text{ is extraneous.}$$

1b.

$$4^{\log_2 x} - 6x = 0$$

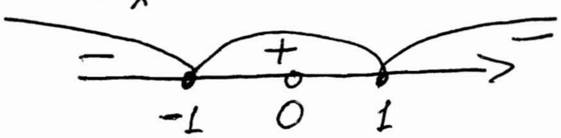
$$2^{2\log_2 x} - 6x = 0 \Rightarrow \left(2^{\log_2 x}\right)^2 - 6x = 0$$

$$\Rightarrow x^2 - 6x = 0 \Rightarrow \cancel{x_1 = 0} \quad \boxed{x_2 = 6}$$

2a.

$$f(x) = \sqrt{x^2 - 1}, \quad g(x) = \frac{1}{x}$$

$$f \circ g = \sqrt{\frac{1}{x^2} - 1} \quad \text{domain: } \frac{1}{x^2} - 1 \geq 0 \Rightarrow \frac{1 - x^2}{x^2} \geq 0$$



$$\Rightarrow x \in [-1, 1] \setminus \{0\}$$

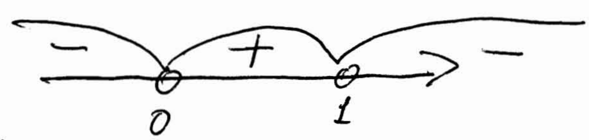
$$g \circ g = \frac{1}{1/x} = x \quad \text{domain: } x \neq 0.$$

2b.

$$f(x) = \frac{1}{1 + 2e^x} \Rightarrow x = \frac{1}{1 + 2e^y} \Rightarrow 1 + 2e^y = \frac{1}{x}$$

$$\Rightarrow e^y = \frac{1}{2x} - \frac{1}{2} = \frac{1 - x}{2x} \Rightarrow y = \ln\left(\frac{1 - x}{2x}\right)$$

$$f^{-1}(x) = \ln\left(\frac{1 - x}{2x}\right) \quad \text{Domain: } \frac{1 - x}{2x} > 0$$



$$\Rightarrow x \in (0, 1)$$

$$\text{Range}(f^{-1}) = \text{Domain}(f) = \mathbb{R}.$$

3a.

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}=\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}=$$

$$=\lim_{x\to 1}\frac{x+3-4}{(x-1)[\sqrt{x+3}+2]}=\lim_{x\to 1}\frac{1}{\sqrt{x+3}+2}=\frac{1}{4}.$$

3b.

$\lim_{x\to -1}\frac{10x+10}{|x+1|}$ does not exist as

$$\lim_{x\to -1^+}\frac{10x+10}{|x+1|}=\lim_{x\to -1^+}\frac{10(x+1)}{x+1}=\lim_{x\to -1^+}10=10 \quad \text{and}$$

$$\lim_{x\to -1^-}\frac{10x+10}{|x+1|}=\lim_{x\to -1^-}\frac{10(x+1)}{-(x+1)}=\lim_{x\to -1^-}(-10)=-10.$$

4.

$y=\frac{3x+1}{\sqrt{4x^2-16}}$ There are no intercepts as $x=0$ is outside the domain. Also $y=0 \Rightarrow x=-1/3$ which is not in the domain either.

Consider now $4x^2-16=0 \Rightarrow x^2-4=0 \Rightarrow x=\pm 2$. The numerator is not zero at $x=\pm 2 \Rightarrow x=2$ & $x=-2$ are vertical asymptotes.

$$\lim_{x\to +\infty}\frac{3x+1}{\sqrt{4x^2-16}}=\lim_{x\to +\infty}\frac{3+1/x}{\sqrt{4-16/x^2}}=\frac{3}{2}$$

$$\lim_{x\to -\infty}\frac{3x+1}{\sqrt{4x^2-16}}=\lim_{x\to -\infty}\frac{3+1/x}{-\sqrt{4-16/x^2}}=-3/2$$

$\Rightarrow y=3/2$ & $y=-3/2$ are horizontal asymptotes

5a.

$$f(x) = \frac{4 + 2\sqrt{x} + 3x^2}{x} = \frac{4}{x} + \frac{2}{\sqrt{x}} + 3x$$

$$f'(x) = -\frac{4}{x^2} - \frac{1}{x^{3/2}} + 3.$$

5b.

$$f(x) = \frac{\cos x \sin x}{1 - \tan x} \Rightarrow f'(x) = \frac{(\cos^2 x - \sin^2 x)(1 - \tan x) + \cos x \sin x \cdot \frac{1}{\cos^2 x}}{(1 - \tan x)^2}$$

5c.

$$f(x) = (\pi + e^x)2^x \Rightarrow f'(x) = e^x 2^x + (\pi + e^x)2^x \ln 2.$$

5d.

$$f(x) = \cos^3(\sin\sqrt{x^2+9}) \Rightarrow f'(x) = 3\cos^2(\sin\sqrt{x^2+9})(-\sin(\sin\sqrt{x^2+9}))\cos(\sqrt{x^2+9}) \cdot \frac{x}{\sqrt{x^2+9}}.$$

6.

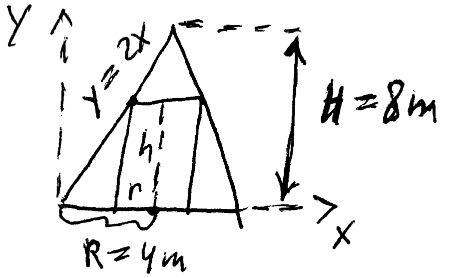
a) $$\lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - x^2 + 2x}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} =$$

$$= \lim_{h \to 0} \frac{2xh - 2h + h^2}{h} = \lim_{h \to 0}(2x - 2 + h) = 2x - 2.$$

b) $$m = \frac{f(3) - f(0)}{3 - 0} = \frac{3^2 - 2\cdot 3 - 0}{3} = \frac{3}{3} = 1$$

c) $2a - 2 = 1 \Rightarrow a = 3/2.$

BONUS Question



$\Rightarrow h = 2(4 - r)$ [r is denoted x in the problem statement]

$$V = \pi r^2 h$$

$$V = 2\pi r^2(4 - r).$$

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Course	**Number**	**Sections**
Mathematics	203	All
Examination	**Date**	**Duration**
Alternate Midterm	2 March, 2013	1 h 30 min
Special Instructions:	Only approved calculators are allowed **Show all your work**	

1. (6 marks): Solve for x (find the *exact* values, do not approximate)

(a) $\log_2(x) + \log_2(x-2) - 3 = 0$

(b) $5^x - \frac{1}{5}25^{2x} = 0$

2. (6 marks) **(a)** Let $f(x) = \ln(x^2+1)$ and $g(x) = \sqrt{x+1}$. Find the composite functions $f \circ g$ and $g \circ g$, and determine their domains.

(b) Given the function $f(x) = \dfrac{3^x}{1+3^x}$, find the inverse function f^{-1} and the domain and the range of f^{-1}.

3. (8 marks) Find the limit or explain why the limit does not exist:

(a) $\displaystyle\lim_{x\to-2} \frac{\sqrt{x^2+5}-3}{x^2+2x}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt[3]{4+x^3-8x^6}}{3x^2-6x+2}$

4. (5 marks) Let $f(x) = \dfrac{x\,|x-5|}{x^2-6x+5}$.

Find the equations of all horizontal and vertical asymptotes.

(continued on the other side)

5. (16 marks). Find the derivatives of the following functions
(you don't need to simplify the final answer, but you must show how you calculate it):

(a) $f(x) = \dfrac{1 + 2x\sqrt{x} - 3x^2}{x^2}$

(b) $f(x) = \dfrac{x^2\, 2^x}{\sin(x) + \tan(x)}$

(c) $f(x) = e^2 + e^x\,(x + x^2)$

(d) $f(x) = \arctan(1 + \arctan(\sqrt{x^2+1}))$

6. (9 marks) Given the function $f(x) = x^2 + 2x$:

(a) Calculate $f'(x)$ using its definition as a limit (of difference quotient).

Check that your calculation is correct using standard rules for differentiation of power functions.

(b) Compute the average rate of change of $f(x)$ on the interval $[-3, 2]$, call it m.

(c) Find whether there is a point a on that interval such that the instantaneous rate of change of f at that point is equal to the average rate of change, $f'(a) = m$. Calculate a if it exists.

Bonus Question (3 marks).

A cylinder is inscribed in a right circular cone with height $h = 10\,\text{m}$ and radius at the cone's base $r = 5\,\text{m}$. Express the volume of the cylinder V as a function of its height x.

1a.

$$\log_2 x + \log_2 (x-2) - 3 = 0 \Rightarrow \log_2 (x^2 - 2x) = 3$$

$$x^2 - 2x = 2^3 = 8 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow x_{1,2} = 1 \pm \sqrt{1+8} = 1 \pm 3 \Rightarrow \boxed{x=4}$$

1b.

$$5^x - \frac{1}{5} \cdot 25^{2x} = 0 \Rightarrow 5^x = 5^{4x-1} \Rightarrow 4x - 1 = x \Rightarrow x = 1/3.$$

2a.

$$f = \ln(x^2+1), \quad g(x) = \sqrt{x+1}.$$

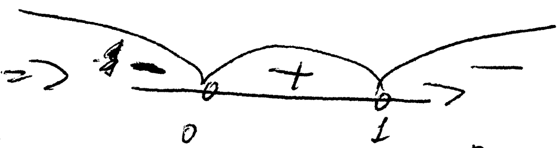
$$f \circ g = \ln(x+2) \Rightarrow x > -2 \ \& \ x \geq -1 \Rightarrow x \geq -1 \text{ (domain)}.$$

$$g \circ g = \sqrt{\sqrt{x+1}+1} \Rightarrow x > -1 \text{ (domain)}$$

2b.

$$f(x) = \frac{3^x}{1+3^x} \Rightarrow x = \frac{3^y}{1+3^y} \Rightarrow x(1+3^y) = 3^y \Rightarrow 3^y(1-x) = x$$

$$\Rightarrow y = \log_3\left(\frac{x}{1-x}\right) \Rightarrow \frac{x}{1-x} > 0 \Rightarrow$$



$\Rightarrow$ Domain: $x \in (0,1)$. $\quad f^{-1}(x) = \log_3\left(\frac{x}{1-x}\right)$. $\quad$ Range$(f^{-1}) =$ $=$ Domain$(f) = \mathbb{R}$

3a.

$$\lim_{x \to -2} \frac{\sqrt{x^2+5}-3}{x^2+2x} = \lim_{x \to -2} \frac{\sqrt{x^2+5}-3}{x(x+2)} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} =$$

$$= \lim_{x \to -2} \frac{x^2+5-9}{x(x+2)(\sqrt{x^2+5}+3)} = \lim_{x \to -2} \frac{(x+2)(x-2)}{x(x+2)[\sqrt{x^2+5}+3]} = \lim_{x \to -2} \frac{x-2}{x(\sqrt{x^2+5}+3)} = \frac{-4}{-2 \cdot 6} =$$

$$= 1/3$$

3b.

$$\lim_{x \to -\infty} \frac{\sqrt[3]{4+x^3-8x^6}}{3x^2-6x+2} = \lim_{x \to -\infty} \frac{\sqrt[3]{4/x^6 + 1/x^3 - 8}}{3 - 6/x + \frac{2}{x^2}} = \frac{\sqrt[3]{-8}}{3} = -2/3.$$

(4)

$$f(x) = \frac{x|x-5|}{x^2-6x+5} = \frac{x|x-5|}{(x-5)(x-1)}$$

$\Rightarrow$ Vertical asymptote $x=1$.

$$\lim_{x\to+\infty} \frac{x(x-5)}{(x-5)(x-1)} = \lim_{x\to+\infty} \frac{x}{x-1} = 1$$

$$\lim_{x\to-\infty} \frac{-x(x-5)}{(x-5)(x-1)} = -\lim_{x\to-\infty} \frac{x}{x-1} = -1$$

$\Rightarrow$ Horizontal asymptotes are $y=1$ and $y=-1$.

(5a.)

$$f(x) = \frac{1+2x\sqrt{x}-3x^2}{x^2} = \frac{1}{x^2} + \frac{2}{\sqrt{x}} - 3 \Rightarrow f'(x) = -\frac{2}{x^3} - \frac{1}{x^{3/2}}.$$

(5b.)

$$f(x) = \frac{x^2 2^x}{\sin x + \tan x} \Rightarrow f'(x) = \frac{(2x2^x + x^2 2^x \ln 2)(\sin x + \tan x) - x^2 2^x(\cos x + \sec^2 x)}{(\sin x + \tan x)^2}$$

(5c.)

$$f(x) = e^2 + e^x(x+x^2) \Rightarrow f'(x) = e^x(x+x^2) + e^x(1+2x)$$

(5d.)

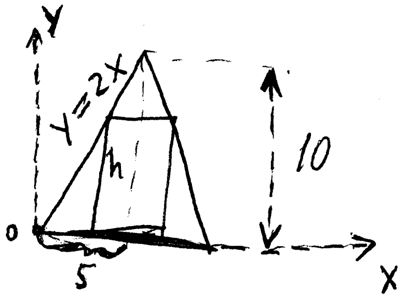
$$f(x) = \arctan(1+\arctan(\sqrt{x^2+1})) \Rightarrow f'(x) = \frac{1}{1+(1+\arctan\sqrt{x^2+1})^2} \cdot \frac{1}{1+x^2+1} \cdot \frac{x}{\sqrt{x^2+1}}$$

(6.)

a) $$\lim_{h\to 0} \frac{(x+h)^2+2(x+h)-x^2-2x}{h} = \lim_{h\to 0} \frac{2xh+2h+h^2}{h} = \lim_{h\to 0}(2x+2+h) = 2x+2.$$

b) $$m = \frac{f(2)-f(-3)}{2-(-3)} = \frac{8-3}{2+3} = \frac{5}{5} = 1$$

c) $2a+2=1 \Rightarrow a = -1/2$.



Bonus Question

$$h = 2(5-r) \Rightarrow 5-r = \frac{h}{2} \Rightarrow r = 5-\frac{h}{2}$$

$$V = \pi r^2 h = \pi h\left(5-\frac{h}{2}\right)^2.$$

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Sections**
Mathematics	203	All
Examination	**Date**	**Duration**
Midterm	21 October, 2012	1 h 30 min
Special Instructions:	Only approved calculators are allowed **Show all your work**	

1. (6 marks): Solve for x (find the *exact* values, do not approximate)

(a) $\log_7(x+3) + \log_7(x-3) = 1$

(b) $e^{5-3x} = 10$

2. (6 marks) (a) Let $f(x) = \sqrt{4-2x}$ and $g(x) = 2 - x^2$. Find the composite functions $f \circ g$ and $g \circ f$, and determine their domains.

(b) For the given one-to-one function $f(x) = \dfrac{4-x}{3+x}$ find the inverse function f^{-1} and the domain of f^{-1}.

3. (6 marks) Evaluate the limits:

(a) $\displaystyle\lim_{x\to 1} \frac{x^2+x-2}{x-1}$

(b) $\displaystyle\lim_{x\to -7} \frac{2x+14}{|x+7|}$

4. (4 marks) Find all horizontal and vertical asymptotes of the graph of the function $f(x) = \dfrac{2x+3}{\sqrt{x^2-2x-3}}$.

(continued on the other side)

5. (13 marks). Find the derivatives of the following functions (you don't need to simplify the final answer, but you must show how you calculate it):

(a) $f(x) = 5x^3 - 7\sqrt{x} + \dfrac{1}{x}$

(b) $f(x) = \dfrac{1+\sin x}{\tan x}$

(c) $f(x) = (1-x)\,e^{2x}$

(d) $f(x) = \cos(\sin(2x + \cos^2 x))$

6. (5 marks) **(a)** Let $g\,(x) = \sqrt{9-x}$. Find $g'(5)$ using the definition of the derivative as the limit of the difference quotient.

(b) Use the answer to part (a) to find the equation of the tangent line to $g\,(x)$ at the point $(5, g\,(5))$.

Bonus Question (2 marks).

Recall that $f\,(x)$ is an even function if it does not change when x is changed to $-x$, and an odd function if it changes sign, but not the absolute value, when x is changed to $-x$. Use the Chain Rule to prove that the derivative of an even function is an odd function and the derivative of an odd function is an even function.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	203	01, A, AA, B, C, G
Examination	**Date**	**Duration**
Midterm	October 22, 2011	1 h 30 min
Special Instructions:	Only approved calculators are allowed **Show all your work**	

MARKS

[9] **1.** Solve for x (Note: guessing the answer and checking by substitution is not an acceptable solution.)

(a) $\log_3(x) + \log_3(2+x) = 1$

(b) $2^{2x-1} = 8$

[9] **2.** (a) Let $f(x) = \sqrt{x+2}$ and $g(x) = 2 - \frac{1}{x^2}$. Find the following functions and their domains:

(i) $f \circ g$

(ii) $g \circ f$

(b) Given the function $g(x) = \frac{2}{1+e^x}$, find the inverse function g^{-1}

and the domain of g^{-1}. (Hint: assume $e^x = u$ to see how to find g^{-1})

[8] **3.** Evaluate the limits:

(a) $\lim\limits_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$

(b) $\lim\limits_{x \to \infty} \frac{3e^x + 2\,e^{-x}}{e^x - 5\,e^{-x}}$

(continued on the other side)

[8] **4.** Given a function $f(x) = \frac{|x-3|}{x^2-9}$, find

(a) one-sided limits $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3^+} f(x)$.

(b) equations of all horizontal and vertical asymptotes.

[9] **5.** Find the derivatives of the following functions
(you don't need to simplify the final answer, but you must show how you got it):

(a) $f(x) = \dfrac{\cos(x)}{x^2+1}$

(b) $f(x) = x^e + x^2e^x$

(c) $f(x) = \sin(\tan\sqrt{x^3-1})$

[7] **6.** Let $g(x) = \sqrt{x+3} + x$.

(a) Use the definition of derivative

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

to find $g'(1)$.

(b) Find an equation of the tangent line $T(x)$ to the graph of $g(x)$ at the point $(1,3)$.

[3] **Bonus Question**

Suppose f is continuous on the interval [1,5], and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$ explain why $f(3) > 6$.