MATHEMATICS 203/4 Sample Midterm Test Solutions

Total: 40 marks

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1. *(5 marks)* **(a)** Solve for x and explain how you got the solution (getting the answer with a calculator is not an acceptable solution).

(i) $\ln(x) = -1$ **(ii)** $2^{x-4} = 8$

(b) Let $f(x) = \sqrt{3-x}$, $g(x) = 4 - x^2$. Find the following functions **and their domains**:

(i) $f \circ g$ **(ii)** $g \circ f$

Solution (a) (i)

$$\ln(x) = -1 \Rightarrow e^{\ln(x)} = e^{-1}$$

but $e^{\ln(x)} = x$ because $\ln(x)$ and e^x are inverse functions, so

$$x = e^{-1}$$

(ii)

$$\begin{aligned} 2^{x-4} &= 8 = 2^3 \text{ and so} \\ x - 4 &= 3 \\ x &= 7 \end{aligned}$$

(b) The domain of f is given by $3 - x \geq 0$, i.e. $x \leq 3$ The domain of g is consists of all real numbers, i.e. $(-\infty, \infty)$ or $\mathbb{R}$. Now for **(i)** we have

$$\begin{aligned} f \circ g(x) = f(g(x)) &= \sqrt{3 - g(x)} \\ &= \sqrt{3 - (4 - x^2)} = \sqrt{x^2 - 1} \end{aligned}$$

Since the quantity under the root sign can't be negative, this means $x^2 - 1 \geq 0$ or $|x| \geq 1$, which means $x \geq 1$ or $x \leq -1$. Therefore the domain of $f \circ g$ is just $(-\infty, -1] \cup [1, \infty)$ in interval notation. For **(ii)** we have

$$g \circ f(x) = 4 - (f(x))^2 = 4 - (3 - x) = 1 + x$$

For the domain it is **not** correct to say that this is defined for all real numbers, so the domain is $\mathbb{R}$. We also need x in the domain of f, i.e. $x \leq 3$. So that is the domain. In interval notation it is $(-\infty, 3]$.

2. *(4 marks)* **(a)** Let $f(x) = x + \cos(\pi x)$ Find the average rate of change of $f(x)$ (i) over the interval $[0, 1/2]$; (ii) over the interval $[-1, 1]$;.

(b) Evaluate the following limits:

$$\textbf{(i)} \lim_{t \to 3} \frac{3-t}{5-\sqrt{t^2+16}} \qquad \textbf{(ii)} \lim_{x \to \infty} \frac{2x^{-2}+3x^{-4}}{5x^{-1}-x^{-3}}$$

Solution **(a)** The average rate of change of $f(x)$ over the interval $[a, b]$ is $\dfrac{f(b) - f(a)}{b-a}$ so (i)

$$\frac{1/2 + \cos(\pi/2) - (0 + \cos(0))}{1/2 - 0} = \frac{1/2 - (1)}{1/2} = -1$$

(ii)

$$\frac{1 + \cos(\pi) - (-1 + \cos(-\pi))}{1 - (-1)} = \frac{0 - (-2)}{2} = 1$$

(b) **(i)** $\displaystyle\lim_{t \to 3} \frac{3-t}{5-\sqrt{t^2+16}} = \lim_{t \to 3} \frac{3-t}{5-\sqrt{t^2+16}} \cdot \frac{5+\sqrt{t^2+16}}{5+\sqrt{t^2+16}}$

$\displaystyle= \lim_{t \to 3} \frac{3-t}{(3-t)(3+t)} \cdot \left(5+\sqrt{t^2+16}\right) = \lim_{t \to 3} \frac{\left(5+\sqrt{t^2+16}\right)}{(3+t)} = \frac{5}{3}$

(ii) $\displaystyle\lim_{x \to \infty} \frac{2x^{-2}+3x^{-4}}{5x^{-1}-x^{-3}} = \lim_{x \to \infty} \frac{2x^{-1}+3x^{-3}}{5-x^{-2}} = \frac{0}{5} = 0$

3. *(6 marks)* Find the following limits for the function $f(x) = \dfrac{1}{(x-1)(x+2)}$. **Explain your answers** in a sentence or two.

(a) $\displaystyle\lim_{x \to -2^-} f(x)$ **(b)** $\displaystyle\lim_{x \to 0} f(x)$

(c) $\displaystyle\lim_{x \to 1^+} f(x)$ **(d)** $\displaystyle\lim_{x \to \infty} f(x)$

(e) The equations of all asymptotes

Solution **(a)** $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \dfrac{1}{(x-1)} \cdot \lim_{x \to -2^-} \dfrac{1}{(x+2)} = -\dfrac{1}{3} \lim_{x \to -2^-} \dfrac{1}{(x+2)} = +\infty$ [$x \to -2^-$ means we approach -2 from the left] because the graph of $\dfrac{1}{(x+2)}$ near and to the left of -2 goes as far as you please below the

$x-$ axis ($x = -2$ is a vertical asymptote) but when we multiply by the negative number $-\dfrac{1}{3}$ the graph is reflected in the $x-$ axis and goes as far as you please **above** the $x-$ axis. **(b)** $\lim_{x\to 0} f(x) = f(0) = -1/2$ because by the limit laws, since the denominator is not zero, f is continuous at 0; **(c)** $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \dfrac{1}{(x-1)} \cdot \lim_{x\to 1^+} \dfrac{1}{(x+2)} = \dfrac{1}{3} \lim_{x\to 1^+} \dfrac{1}{(x-1)} = +\infty$ [$x \to 1^+$ means we approach 1 from the right] because this shows that the graph of f near and to the right of 1 goes as far as you please above the $x-$ axis; **(d)** from the equation, $\lim_{x\to\infty} f(x) = 0$ because after dividing numerator and denominator by x^2 we get

$$f(x) = \frac{1/x^2}{1 + 1/x - 2/x^2}$$

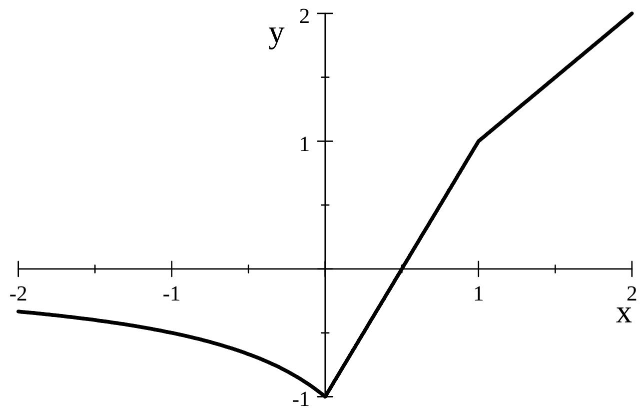
and the numerator $\to 0$ while the denominator $\to 1$, so the limit is 0; **(e)** the vertical asymptotes have already been identified: $x = -2$ and $x = 1$. $y = 0$ (the $x-$ axis) is a horizontal asymptote because of (d).

4. *(5 marks)* Find the numbers a and b so that the function $f(x)$ is continuous.

$$f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x < 0 \\ ax + b & \text{if } 0 \le x < 1 \\ x & \text{if } 1 \le x \end{cases}$$

(it may help to sketch the graph).

Solution Put $x = 0 : f(0) = 0 \cdot a + b = b$ and also $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \dfrac{1}{x-1} = -1$ because $\dfrac{1}{x-1}$ is a continuous function at 0. Since these must be equal, $b = -1$. Now consider $x = 1$. $f(1) = 1$ and also $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (ax - 1) = a - 1$. Since these must be equal, $a - 1 = 1$ and so $a = 2$.



5. *(12 marks)* For each of the following find $\dfrac{dy}{dx}$(show all steps and do not simplify your answer):

(a) $y = \dfrac{x^2 + 2x^{3/2} - 5\sqrt{x}}{\sqrt{x}}$

(b) $y = e^{x^2} (x+7)^8$

(c) $y = \cos(x^2 - \tan x)$

(d) $y = \sqrt{1 + \sqrt{x + x^2}}$

Solution **(a)** First write $y = \dfrac{x^2 + 2x^{3/2} - 5\sqrt{x}}{\sqrt{x}} = x^{3/2} + 2x - 5$. Now it is very easy to differentiate: $\dfrac{dy}{dx} = \frac{3}{2}x^{1/2} + 2$ (power rule)

(b) $\dfrac{dy}{dx} = 2xe^{x^2} (x+7)^8 + e^{x^2} \cdot 8 (x+7)^7$ (product, chain rules). Note that $e^{x^2} \neq (e^x)^2$, which is e^{2x}. For example, $e^{3^2} = 8103.1$ but $(e^3)^2 = 403.43$

(c) Put $y = \cos(u)$ and $u = x^2 - \tan x$ Then$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{dv}{dx} = -\sin u \cdot (2x - \sec^2 x) = -\sin(x^2 - \tan x)(2x - \sec^2 x)$ (chain rule)

(d) Write $y = u^{1/2}$ and $u = 1+v^{1/2}$ where $v = x+x^2$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dv}\dfrac{dv}{dx} = \frac{1}{2}u^{-1/2}\frac{1}{2}v^{-1/2}(1+2x) = \dfrac{1}{2\sqrt{1+\sqrt{x+x^2}}}\dfrac{1}{2\sqrt{x+x^2}}(1+2x)$ (chain rule)

6. *(8 marks)* Let $g(x) = 1 + \dfrac{1}{\sqrt{x}}$.

(a) Use the definition of the derivative :

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

to find $g'(4)$

(b) Check that your answer is correct by using the appropriate differentiation rule(s) to find $g'(x)$, and then substituting $x = 4$.

(c) Find the equation of the tangent line $T(x)$ to the graph of $g(x)$ at $(4, 3/2)$

Solution (a)

$$\begin{aligned}
\lim_{h\to 0} \frac{g(4+h) - g(4)}{h} &= \lim_{h\to 0} \frac{1 + \dfrac{1}{\sqrt{4+h}} - \left(1 + \dfrac{1}{\sqrt{4}}\right)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{4+h}} - \dfrac{1}{\sqrt{4}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4} - \sqrt{4+h}}{h\sqrt{4}\sqrt{4+h}} \cdot \frac{\sqrt{4} + \sqrt{4+h}}{\sqrt{4} + \sqrt{4+h}} \\
&= \lim_{h\to 0} \frac{4 - (4+h)}{h \cdot 2\sqrt{4+h}} \cdot \frac{1}{2 + \sqrt{4+h}} \\
&= \lim_{h\to 0} \frac{-h}{h \cdot 2\sqrt{4+h}} \cdot \frac{1}{2 + \sqrt{4+h}} \\
&\lim_{h\to 0} \frac{-1}{2\sqrt{4+h}} \cdot \frac{1}{2 + \sqrt{4+h}} \\
&= \frac{-1}{2\lim_{h\to 0}\left(\sqrt{4+h}\right)\left(\lim_{h\to 0}\sqrt{4+h} + 2\right)} \\
&= \frac{-1}{2\,(2)^3} = -\frac{1}{2}4^{-3/2}
\end{aligned}$$

or, one can do it by using x instead of 4 and substituting at the end:

$$
\begin{aligned}
\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} &= \lim_{h\to 0}\frac{1+\dfrac{1}{\sqrt{x+h}}-\left(1+\dfrac{1}{\sqrt{x}}\right)}{h}\\
&= \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\\
&= \lim_{h\to 0}\frac{x-(x+h)}{h\cdot\sqrt{x}\sqrt{x+h}}\cdot\frac{1}{\sqrt{x}+\sqrt{x+h}}\\
&= \lim_{h\to 0}\frac{-h}{h\cdot\sqrt{x}\sqrt{x+h}}\cdot\frac{1}{\sqrt{x}+\sqrt{x+h}}\\
&\lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}}\cdot\frac{1}{\sqrt{x}+\sqrt{x+h}}\\
&= \frac{-1}{\sqrt{x}\lim_{h\to 0}\left(\sqrt{x+h}\right)\left(\lim_{h\to 0}\sqrt{x+h}+\sqrt{x}\right)}\\
&= \frac{-1}{2\left(\sqrt{x}\right)^3} = -\frac{1}{2}4^{-3/2} \text{ when } x=4
\end{aligned}
$$

(b) As a check we see that $g(x) = 1 + x^{-1/2}$ and so $g'(x) = 0 - \frac{1}{2}x^{-3/2}$. Therefore $g'(4) = -\frac{1}{2}4^{-3/2} = -1/16$

(c) An equation of the tangent line is

$$
\begin{aligned}
y - 3/2 &= -\frac{1}{16}(x-4), \text{ or}\\
y &= -\frac{1}{16}x + \frac{7}{4}
\end{aligned}
$$