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Subject **CSI Tip of the Week: QR Damped Eigenvalue Extraction Method**
Keywords Modal Analysis: Damped: QR Damped

1. Introduction:

The QR Damped eigenvalue extraction method introduced at 5.6 provides significant speed improvements over the older damped method for modal analyses involving damping. This memo will cover some details regarding the QR Damped solution method. This memo assumes that the reader is familiar with the basics of modal analysis and damping.

2. Background Discussion:

For any free, undamped system, the equations of motion can be written as:

$$[M]\{\ddot{x}\}+[K]\{x\}=0$$

where it is assumed that the motion is harmonic:

$$\{x_i\}=\{\phi_i\}e^{j\omega_i t}$$

The equations of motion can then be rewritten as:

$$\left(-\omega_i^2[M]+[K]\right)\{\phi_i\}e^{j\omega_i t}=0$$

The eigenvalue problem posed above is solved in order to obtain the natural frequencies and mode shapes of the system.[1]

If damping is added, the equations of motion are:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=0$$

and the behavior is expressed by complex eigenvectors and eigenvalues:

$$\{x_i\}=\{\phi_i \pm \vartheta_i j\}e^{(\sigma_i \pm \omega_i j)t}$$

In the above case, $(\sigma_i \pm \omega_i j)t$ express both a time-decay term as well as the damped, free vibration term. $e^{\sigma_i t}$ represents the time-decay if σ_i is negative, which is representative of stable systems. On the other hand, if σ_i is positive, this indicates unbounded exponential increase in amplitude and represents an unstable system. The $e^{\omega_i jt}$ term is the damped, free vibration term similar to an undamped system as noted above.

On the other hand, the complex eigenvectors represent the mode shapes. In certain cases, the various nodes (DOF) in the system may be out of phase with one another. As a result, the real and imaginary values can be used to determine the actual response of the system.

To understand better when the nodes (DOF) may be out of phase with one another, it is instructive to look at the damping matrix. If proportional (Raleigh) damping is used, the damping matrix can be expressed as a linear combination of the mass and stiffness matrices:

$$[C]=\alpha[M]+\beta[K]$$

The equations of motion result in only complex eigenvalues, not complex eigenvectors.

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=0$$

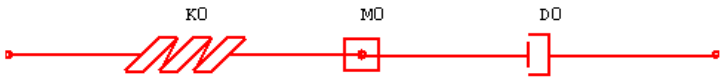
$$\left(\left(-\omega_i^2+\alpha\omega_i j\right)[M]+\left(1+\beta\omega_i j\right)[K]\right)\{\phi_i\}=0$$

However, for non-proportional damping, the damping matrix $[C]$ cannot be simplified in this manner. Thus, the eigenvectors and eigenvalues solved for are complex. This means that the nodes (DOF) will be out of phase with one another. A simple way to view this is that with a vibration of a beam, the nodes and anti-nodes will not be stationary (discussed later).

[1] The angular frequencies are, of course, not the same as the eigenvalues, but, for simplification, only the former will be used in equations throughout most of this memo, although eigenvalues will still be mentioned.

3. Simplification to a 1-D System:

The above discussion provided a framework with which to work. A 1-D spring-mass-damper system will be examined first to provide better insight into the aforementioned equations.



The figure above shows a spring [K0], mass [M0], and damper [D0] in series. Assuming that the ends are fixed and no damping is present, the natural frequency can be determined:

$$m\ddot{x} + kx = 0$$

$$x = \phi e^{j\omega t}$$

$$\left(-\omega^2 m + k\right)\phi = 0$$

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f$$

If damping is added, the equations can then be written as:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x = \phi e^{j\omega t}$$

$$\left((\omega j)^2 m + \omega j c + k\right)\phi = 0$$

$$\left(k + c(\omega j) + m(\omega j)^2\right)\phi = 0$$

This is a familiar quadratic equation, the solution of which is as follows:

$$\omega j = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Consider the following cases:

a) $c^2 \geq 4km$

In this case, the solution ωj is comprised of two real values, both of which are always negative for positive values of damping. This means that the displacement is $x = e^{yt}$. If y is negative, the displacement decays as a function of time and no vibration takes place (i.e., stable). This corresponds to the *overdamped* situation. If y is positive (i.e., if negative damping exists), the displacement increases without bound, and the system is unstable.

b) $c^2 = 4km$

If the above is true, there is one unique solution to the problem. This *critically damped* system has a real, negative root. This corresponds to a stable system as noted in case (a) where the displacement will decay as a function of time and no vibration takes place.

c) $c^2 \leq 4km$

For this situation, the solution is complex (the eigenvalues are complex). The imaginary portion will represent the oscillatory part of the motion, i.e. the damped free vibration frequency is $\sqrt{4km - c^2}/2m$. Recall that oscillatory behavior is represented by the imaginary portion of the exponent, $e^{j\omega t}$.
On the other hand, the real, negative part represents the exponential decay of the system. Recall that $e^{-\sigma t}$ represents exponential decay.

d) $c^2 = 0$

This is the *undamped* case as explained above where $j\omega = \sqrt{-k/m}$.

Attached is an input file, "springdamp.inp" which demonstrates this. Open the file in any text editor and modify the parameter R_DAMP, which represents the damping constant. For example, if R_DAMP is ¼ of the critical damping value (as defined above), ANSYS will print out the solution

```
***** FREQUENCIES FROM LANCZOS DAMPED EIGENSOLVER *****

MODE          COMPLEX FREQUENCY (HERTZ)

  1      -2.5000000           9.6824584    j
  2      -2.5000000          -9.6824584    j
```

For a 1 DOF, underdamped system, two solutions are expected (complex conjugate). For the DAMP eigenvalue extraction method, four result sets will be available per complex eigenvalue: the real and imaginary of the first eigenvalue, the real and the imaginary of its conjugate. The input file calculates the real and imaginary portion of the solution using the equations mentioned above, so a user can compare this with ANSYS's calculated values. The results match exactly.

```
***** DAMPED FREQUENCIES FROM REDUCED DAMPED EIGENSOLVER *****

MODE          COMPLEX FREQUENCY (HERTZ)           MODAL DAMPING RATIO

  1      -2.5000000           9.6824584    j       0.25000000
         -2.5000000          -9.6824584    j       0.25000000
```

In the case of the QRDAMP eigenvalue extraction method, only one result set per frequency will be available. This is because QRDAMP method uses a different technique than the DAMP method, as will be explained in the next section. The complex conjugate roots as well as the modal damping ratio are reported for QRDAMP. The results agree very well with the DAMP method and with hand calculations.

Lastly, it is useful to note that the undamped frequency for the above case is 10 Hz (as noted in the input file). Damping, as expected, shifts the frequency by lowering it. Recall that the angular frequency for a 1 DOF underdamped system is given by $\sqrt{4km - c^2}\,/2m$. For any positive value of c, the damped free vibration frequency will always be lower than the undamped value.

4. Differences Between DAMP and QRDAMP eigenvalue extraction methods:

Although a detailed technical discussion is outside of the scope of this memo, it may be instructive to go over the basics of the ways in which the eigenvalue problem is solved since the two methods (DAMP and QRDAMP) are quite different.

The DAMP method essentially solves the eigenvalue problem with traditional methods.[2] This means that the original problem

$$[M]\ddot{x} + [C]\dot{x} + [K]x = 0$$

is transformed into a more familiar eigenvalue problem:

$$[B]\{y_i\} = \mu_i\{y_i\}$$

and the eigenvalues and eigenvectors are then solved for. Unfortunately, this process is very CPU- and memory-intensive. Sturm sequence checking is also not available to check for missed modes. On the other hand, this method handles critically and overdamped systems as well as unsymmetric matrices (as often arises in acoustics with fluid-structure interaction).

The output usually consists of a complex conjugate pair, each with a real and imaginary set (i.e., 4 results set per solution). The real and imaginary eigenvectors and corresponding complex frequencies are available.

[2] For a more detailed treatment of traditional methods of (tridiagonal) transformation & (QR) iterations, the reader is referred to the ANSYS Theory Manual, Ch. 15.10 "Eigenvalue and Eigenvector Extraction" as well as K.J. Bathe's "Finite Element Procedures", Ch. 11 "Solution Methods for Eigenproblems"

The QRDAMP method, introduced at ANSYS 5.6, provides a much more efficient way of dealing with symmetric, lightly damped systems. Recall the equation for an undamped system

$$[M]\ddot{x} + [K]x = 0$$

Assuming $x = \phi e^{j\omega t}$, we get the eigenvalues and eigenvectors:

$$[K]\phi_i = \lambda_i [M]\phi_i$$

Using this information, one can transform the equation of motion into modal coordinates:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = 0$$

$$[M][\Phi]\{\ddot{y}\} + [C][\Phi]\{\dot{y}\} + [K][\Phi]\{y\} = 0$$

$$[\Phi]^T[M][\Phi]\{\ddot{y}\} + [\Phi]^T[C][\Phi]\{\dot{y}\} + [\Phi]^T[K][\Phi]\{y\} = 0$$

$$[I]\{\ddot{y}\} + [\Phi]^T[C][\Phi]\{\dot{y}\} + [\Lambda]\{y\} = 0$$

$$\begin{bmatrix} 0 & I \\ I & \Phi^T C \Phi \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} -I & 0 \\ 0 & \Lambda \end{bmatrix} \begin{Bmatrix} \dot{y} \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The last equation is a familiar-looking form of the eigenvalue problem (similar to $[M]\ddot{x} + [K]x = 0$).

The basic idea behind this method is not to solve for the eigenvalue problem explicitly. Instead, using the Block Lanczos method, *n* number of undamped eigenvalues and eigenvectors are solved for. Then, the eigenvalue problem with damping is converted in the modal subspace. This results in a *2n* system, but these are much fewer number of equations than one would otherwise have (*n* < number of DOF). Hence, the QRDAMP method is much faster than the DAMP method and supports both proportional and non-proportional damping. At 5.7, the QRDAMP solution can be used in subsequent modal superposition analyses (harmonic or transient) as well. However, QRDAMP only supports symmetric matrices and is recommended for lightly-damped systems (because it is using undamped modal coordinates), so acoustic fluid-structure interaction problems cannot be solved with this method.

With the QRDAMP method, complex frequencies are reported, but only one results set is available. This includes the real eigenvector solution. In other words, even for non-proportional damping, the phase difference between nodes is not available. The main reason for this is because the undamped (real) eigenvectors are used in the solution. Consequently, despite the fact that non-proportional damping is supported, the phase difference is assumed to be negligible.

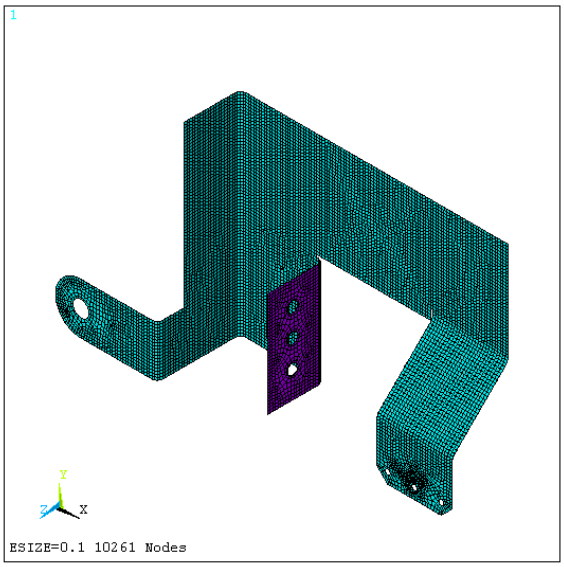
5. Comparison of DAMP and QRDAMP Solutions:

Shown on the right is a sample model containing approximately 60k DOF. Raleigh damping was used in the model. 25 modes were requested using the DAMP and QRDAMP methods.[3]

On an HP C3000 running HP-UX 10.20, the DAMP method took 14700 CPU seconds whereas the QRDAMP method took only 126 CPU seconds. The QRDAMP method is over 100 times faster than the DAMP method for this particular model.

All of the calculated modes match well, with the largest error being 0.3%.

The author has found that this model is illustrative of the accuracy and efficiency of the QRDAMP solver for lightly-damped systems. The author has seen very similar trends between the two methods for other models, with the QRDAMP method solving in the fraction of the time of the DAMP method.



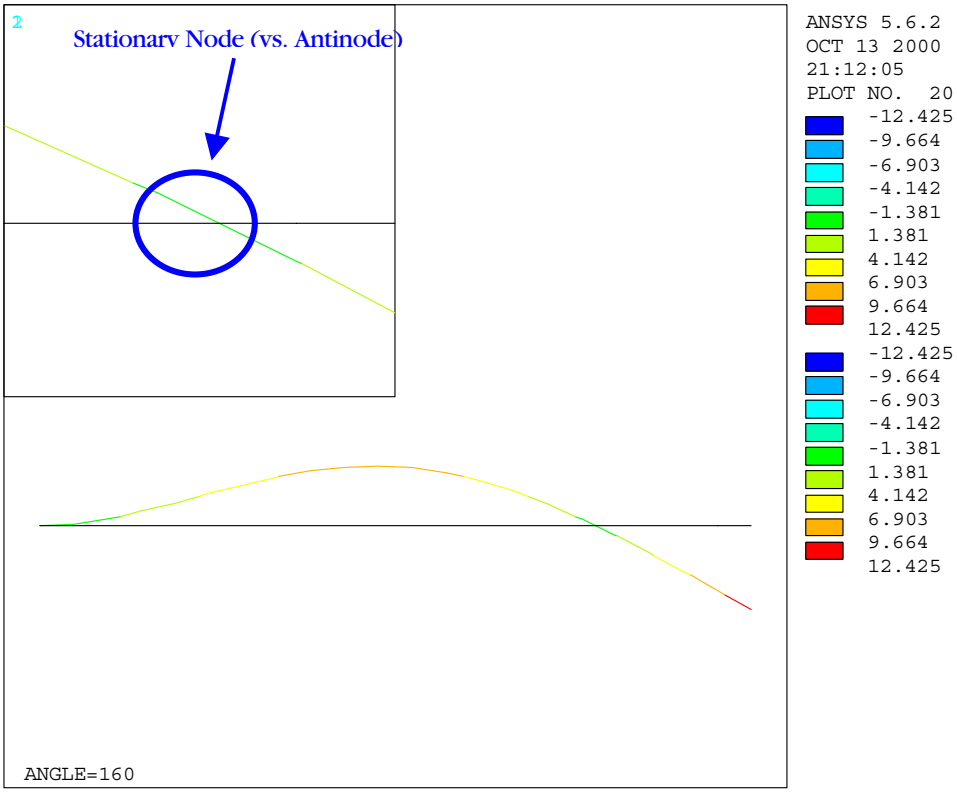
[3] Although 25 modes were requested with the DAMP eigenvalue extraction method, recall that the complex conjugate is also returned. Hence, only 13 modes were retrieved with the DAMP method.

6. Comparison of Proportional and Non-Propotional Damping:

The significance of the real and imaginary eigenvectors was mentioned earlier. As with a harmonic analysis, the nodes (DOF) of the model may not be in-phase with one another, hence the eigenvectors are represented in the complex plane. While, for proportional damping, the nodes will still be in-phase, for non-proportional damping nodes are often out-of-phase with one another.

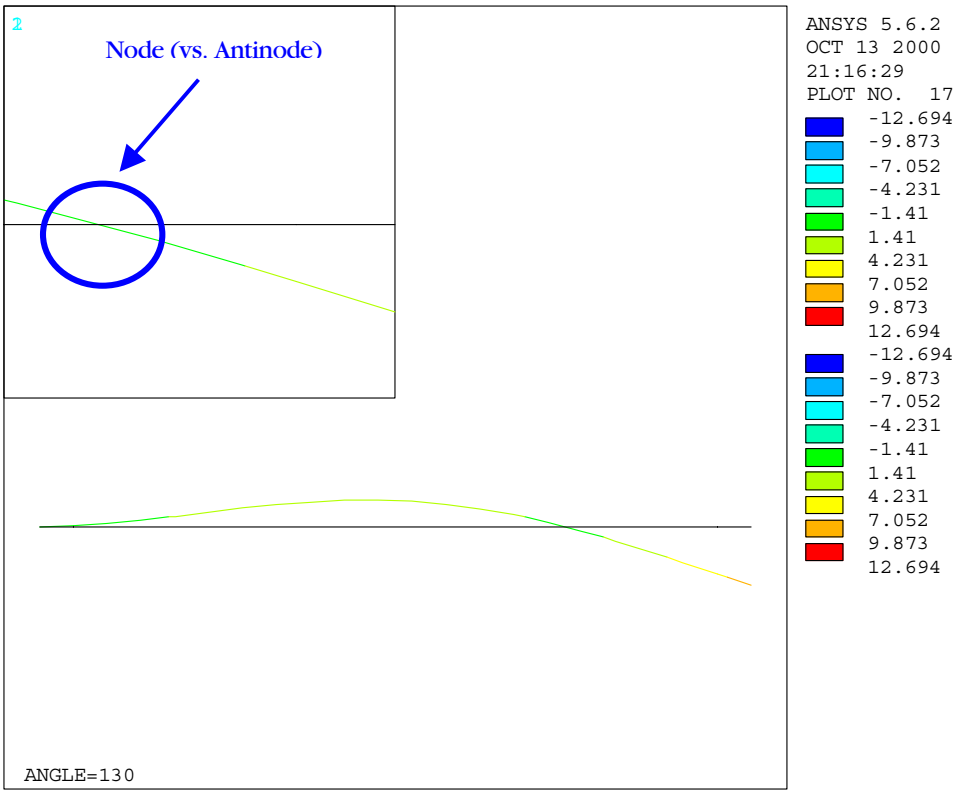
Attached are input files "beam.inp" and "beam2.inp" showing cases of both proportional and non-proportional damping. Proportional damping is modeled via beta constants whereas non-proportional damping is modeled with discrete damping elements (COMBIN14). Animations of these two cases are also provided with "beam.avi" and "beam2.avi".

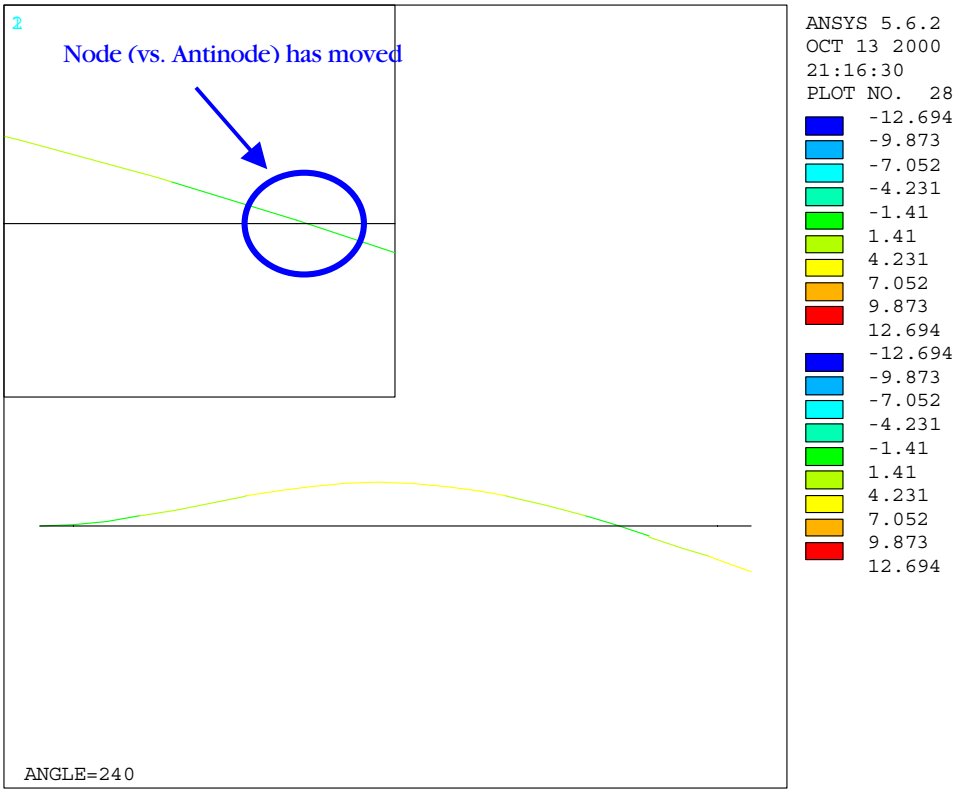
"beam2.inp" demonstrates proportional damping. A snapshot of the mode shape is shown below. The upper-left corner shows a magnified view of the 'node' (node/antinode). The 'node' is stationary, which is better illustrated by the included animation file.



On the other hand, the next two snapshots show the case of non-proportional damping (available with DAMP method only). Notice that the 'node' (point where displacement is zero) is not stationary but moves. This is representative of non-proportional damping where the nodes are out-of-phase with one another. Another way to view this is that for *proportional* damping, the mode shape is the same as the undamped case, but there is a shift in frequency as well as a time-decay component. For *non-proportional* damping, the mode shape itself is a function of time, so the mode shape is changing and not necessarily the same as the undamped case.

To view the mode shapes for complex eigenvectors, the author recommends using the HRCPLX macro. This is similar to a harmonic analysis where the nodal responses are also out-of-phase with one another. Please note that at ANSYS 5.5, the HRCPLX macro has a wrong sign, so contact CSI to obtain the corrected HRCPLX macro (Class3 Error 1999-26). Also, there is a another error related to using *AFUN,DEG prior to issuing the HRCPLX macro which produces incorrect results (Class3 Error 2000-36), so please be aware of this when using the HRCPLX macro to interpret both damped eigenvector and harmonic results.


2
Node (vs. Antinode)
ANGLE=130
ANSYS 5.6.2
OCT 13 2000
21:16:29
PLOT NO. 17
-12.694
-9.873
-7.052
-4.231
-1.41
1.41
4.231
7.052
9.873
12.694
-12.694
-9.873
-7.052
-4.231
-1.41
1.41
4.231
7.052
9.873
12.694


2
Node (vs. Antinode) has moved
ANGLE=240
ANSYS 5.6.2
OCT 13 2000
21:16:30
PLOT NO. 28
-12.694
-9.873
-7.052
-4.231
-1.41
1.41
4.231
7.052
9.873
12.694
-12.694
-9.873
-7.052
-4.231
-1.41
1.41
4.231
7.052
9.873
12.694

7. Conclusions:

The DAMP and QRDAMP eigenvalue extraction methods are useful for individuals wanting to extract the damped frequencies of the structure. The QRDAMP method, when applicable, is a much more efficient method of obtaining the frequencies, although the user should keep in mind that for non-proportional damping, the phase difference between nodes will not be available. At 5.7, the QRDAMP method has been extended to support mode superposition harmonic and transient analyses as well.

To keep the present memo short, the basics of eigenvalue extraction, transformation into modal coordinates, and obtaining damping constants have not been discussed. The reader is referred to the ANSYS Structural Analysis Guide as well as the ANSYS Theory manual for more details concerning these topics. Any dynamics text can also provide useful insights into damped modal analyses.

8. References:

- ANSYS Theory Manual, Rev 5.6
- ANSYS Structural Analysis Guide, Rev 5.6
- Hurty, W.C. and Rubinstein, M.F., "Dynamics of Structures", Prentice Hall, 1964
- Thomson, W.T. and Dahleh, M.D., "Theory of Vibration with Applications", 5th ed., Prentice Hall, 1993
- Bathe, K.J. "Finite Element Procedures", Prentice-Hall, 1996

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