Hybrid Finite Element - Trefftz Method for Open Boundary Analysis

Miklos Gyimesi, Igor Tsukerman and Doug Lavers
University of Toronto, Department of Electrical Engineering, Toronto, ON, M5S-1A4, Canada

Tim Pawlak and Dale Ostergaard
Swanson Analysis Systems, Inc. P.O.Box 65, Houston, PA 15342, USA

***Abstract* — This paper discusses a new hybrid finite element - Trefftz formulation to solve open boundary potential problems with an economic number of degrees of freedom.**

I. Introduction

Electromagnetic field computation often involves open boundary analysis. Discretization of the exterior region by the finite element method (FEM) is cumbersome and computationally inefficient. Of the several methods in the literature (see Emson [1]), the hybrid FEM - boundary element method (BEM) seems to be the most general. However, the BEM leads to a large non-symmetric dense matrix prohibiting practical applications. For example, a roughly meshed cube with a $30 \times 30 \times 30$ grid would need 5,400 BEM degrees of freedoms (DOFs), (the number of surface nodes).

As an alternative to the FEM-BEM, we propose the hybrid FEM - Trefftz method (TRM). Similar to BEM, TRM is theoretically sound in describing the field in the exterior region. The FEM-TRM offers the following advantages:

(1) Symmetric FEM style stiffness matrix

(2) No singular integral

(3) Number of TRM DOFs is independent of the FEM mesh

(1) results in less storage, easier equation solution and convenient coding since the exterior region can be treated as a 'super' element by conventional FEM techniques. (2) makes computation faster and more robust. Since reasonable precision can be obtained with 20-500 TRM DOFs (3) means that the matrix size no longer prohibits practical analysis.

After the overview of literature, the paper describes the most important theoretical features of the proposed hybrid FEM-TRM formulation which is based on the synthesis of several proposed methods. Then it demonstrates the application of the formulation on a simple example which clearly shows its excellent convergence behavior and reasonable precision both within the FEM region and very far from it. The verification has been carried out by a commercially available general purpose FEM package.

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II. Overview of Literature

Boundary-type approximations have been known since the classical work of Trefftz [2]. BEM constitutes a special case with singular (Green type) trial functions. TRM uses non-singular Trefftz complete function sets called T-sets (see Herrera [3]). TRM has been applied with success to:

- 2D interior Laplace [4] and Helmholtz equation [5] by Cheung, Jin and Zienkiewitz
- 2D Poisson equation by Zielinski and Zienkiewitzs [6] (*generalized* FEM: FEM style stiffness matrix)
- 3D exterior harmonic problems by Mayergoyz, Chari and Konrad [7] (*quasi* FEM).
- air gap of machines by Lee, DeBortoli and Salon [8]
- plate bending at corners, holes by Jirousek, Guex [9] as well as Zienkiewitz, Kelly and Bettess [10,11,12].
- 2D geoelectric structure positioning by Gyimesi [19]

There are numerous examples of analytic T-sets:

- Cylindrical harmonics exterior to a circle FEM region by Lee and Csendes [13] (*transfinite elements*) as well as Chari and Bedrosian [14].
- Legendre polynomials exterior to a sphere for axisymmetric 3D problems by Chari [15].
- Helical functions by Igarashi and Honma [16].
- Toroidal functions by Gyimesi and Lavers [20].

Excellent accuracy can be achieved with analytic T-sets over simple domains. For complex geometries, however, numerical difficulties occur (see Gyimesi [19]). A more robust T-set can be constructed similar to BEM: The exterior field is expressed by the potential of representing sources.

In a hybrid FEM-BEM, there is a representing source at each node on the boundary. Since the Green functions are singular, improper integrals are evaluated. This shortcoming can be overcome by positioning the sources within the FEM region (see McDonald and Wexler [17] as well as Mayergoyz, Chari and Konrad [7]). Fortunately the number of TRM representing sources can be chosen independently of the number of surface nodes of the FEM mesh (see Murakami et al [18].) Since reasonable precision can be obtained with 50-500 TRM DOFs, the size of the dense TRM matrix in not an obstacle in practical computations.

The continuity of the normal derivative of the potential on the surface separating the FEM and TRM domains can be satisfied as a natural boundary condition. The continuity of the potential is not automatic. The most efficient way to satisfy this condition is point matching according to *transfinite* elements [13].

III. Hybrid FEM-Trefftz Formulation

We propose the synthesis of the methods discussed above. The hybrid formulation follows the *generalized* FEM [4,5,6] to obtain symmetric FEM matrices. The non-singular T-set is derived from the potential of point sources according to *quasi* FEM [7]. The representing sources are located in the FEM region; their number is independent of the FEM mesh according to Murakami et al [18]. The continuity of the potential is satisfied according to *transfinite* elements [13].

The governing equations are:

$$\nabla(\sigma_1 \nabla U_1) = g_1 \quad \nabla(\sigma_2 \nabla U_2) = g_2$$

where σ is the conductivity, U is the potential, g is the source, finally subscripts 1 and 2 denote the FEM and TRM domains, respectively. Source terms in the TRM domain can take into account sources far from the FEM region by the general potential formulation (Gyimesi and Lavers [21]). The TRM domain may be inhomogeneous allowing analysis, for example, in stratified medium. The potential is sought as:

$$U_1 = \sum c_i N_i \quad U_2 = \sum c_i f_i$$

where N_i are the nodal shape functions, c_i are the nodal potentials or intensity of representing sources and f_i are members of a T-set satisfying:

$$\nabla(\sigma_2 \nabla f_i) = 0$$

This selection of the shape functions constitutes the 1st difference between FEM and TRM.

The energy of the whole system can be expressed as:

$$W = W_1 + W_2$$

$$W_1 = \frac{1}{2}\mathbf{c}_1^+ \mathbf{K}_1 \mathbf{c}_1 - \mathbf{c}_1^+ \mathbf{g}_1 \quad W_2 = \frac{1}{2}\mathbf{c}_2^+ \mathbf{K}_2 \mathbf{c}_2 - \mathbf{c}_2^+ \mathbf{g}_2$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are the stiffness matrices, $\mathbf{g}_1$ and $\mathbf{g}_2$ are the load vectors and $\mathbf{c}_1$ and $\mathbf{c}_2$ are the DOFs of the FEM and TRM domains, respectively. These can be computed as:

$$K_{1ij} = \int \sigma_1 \nabla N_i \nabla N_j \, dV \quad K_{2ij} = \int \sigma_2 \nabla f_i \nabla f_j \, dV$$

$$g_{1i} = \int N_i g_1 \, dV \quad g_{2i} = \int f_i g_2 \, dV$$

The 2nd difference is that the continuity of the potential is achieved by weighting the residual with functions, λ_i.

$$\int \lambda_i (U_1 - U_2) \, dS = 0$$

The best choice of weighting functions, λ_i, is still to be analyzed. We applied the normal derivatives of the Trefftz shape functions because this way it is consistent with the Trefftz formulation providing constraints related to energy. The integral should be evaluated on the exterior surface (see Fig. 1.) separating the FEM and TRM regions. This leads to the following constraints:

$$\mathbf{Q}_1 \mathbf{c}_1 + \mathbf{Q}_2 \mathbf{c}_2 = 0$$

$$Q_{1ij} = \int \lambda_i N_j \, dS \quad Q_{2ij} = -\int \lambda_i f_j \, dS$$

The constrained variation of the energy provides the satisfaction of the governing equations and the continuity of the normal derivative as a natural boundary condition.

The 3rd difference is that the volume integrals of the TRM stiffness matrix can be reduced to surface integrals using identities of vector analysis and the Gauss theorem:

$$K_{2ij} = \int \sigma_2 \nabla f_i \nabla f_j \, dV = \int \left[\nabla(f_i \, \sigma_2 \nabla f_j) - f_i \nabla\sigma_2 \nabla f_j \right] dV$$

$$K_{2ij} = \int f_i \sigma_2 \frac{\partial}{\partial n} f_j \, dS = \int f_j \sigma_2 \frac{\partial}{\partial n} f_i \, dS$$

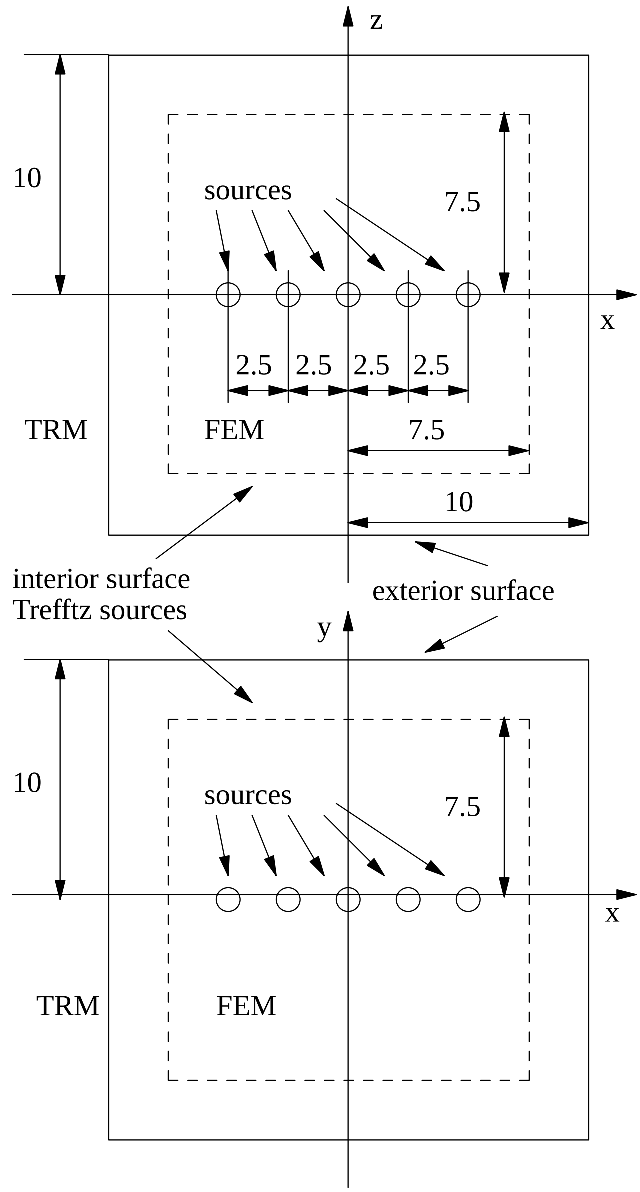
This puts TRM into the group of boundary methods albeit the shape functions are defined in the space.

The method can be easily implemented in a general purpose FEM package with open architecture. The exterior Trefftz region can be treated as a superelement where the master DOFs are the intensity of the representing sources. Thanks to the Trefftz formulation, the stiffness matrix of the superelement is symmetric. The exterior superelement can be linked to a conventional FEM mesh by the discussed constraint equations.

IV. Demonstration Example

The geometry of the demonstration example is shown in Fig. 1. The FEM domain is a cube within the exterior surface. However, there is no artificial boundary condition introduced on the exterior surface since the potential in the exterior domain is taken into account by the TRM.

Point sources are located in the FEM domain at 5 points. Their intensity changes according to 5 different cases. (see Fig. 1.) The field in the Trefftz region is described by the potential of representing point sources scattered on the interior surface. (The analysis of the optimal selection of Trefftz sources will be presented in another paper.) In this example the interior surface is also a cube. Its side is 3/4-th of the exterior cube.



case	source intensities				
1st	0	0	+1	0	0
2nd	0	0	0	+1	0
3rd	0	0	0	0	+1
4th	0	-1	0	+1	0
5th	-1	0	0	0	+1

Fig. 1. Studied arrangements

In the 1st case the source is in the middle of the cube. In the 2nd and 3rd cases, the source is moved on the x axis by 1/4-th and a half of the half side length of the exterior cube, respectively. In these cases the sum of sources is not zero. Consequently, the potential distribution far from the FEM domain decreases as the inverse of the distance. The representation of the 1st case is rather easy; the 3rd is difficult and the 2nd falls somewhere in between.

In the 4th and 5th cases, a source and a sink are located symmetrically with respect to the middle of the cube on the x axis. The distance of the sources from the middle of the cube is 1/4-th and a half of the half side length of the exterior cube, respectively. The sum of sources in the 4th and 5th cases is zero. The terms decreasing as the inverse of the distance due to the sources and sinks cancel each other. Consequently, the potential distribution far from the FEM region is similar to that of a dipole source; it decreases as the square of the distance from the FEM domain. Therefore, its representation is more difficult.

Note, that even if there were separate infinite elements for the 1st and 2nd order decays, it would still be a question which of them should be applied since the order of decay depends on if the sources and sinks cancel each other or not. If the dipole moments cancelled each other then the potential would decrease even with a higher order. The Trefftz representation finds the appropriate order automatically.

Since the locations of the actual sources and representing sources are different, the Trefftz representation introduces errors even if the evaluation of the surface integrals were exact. The major goal of this example is to analyze the Trefftz representation error against the number of Trefftz DOFs.

The discretization of the exterior surface has an influence on the precision of the surface integrals. In this example the sides of the exterior surface are subdivided into 8 parts providing satisfactory precision. This results in 384 square surface elements and 512 cube volume elements.

The sides of the interior surface are subdivided into m parts. By changing the number of subdivisions, Trefftz representations with different source densities can be generated. The number of TRM DOFs are about $6m^2$. Three different discretizations of the interior surface by setting the subdivision parameter to m=1, m=2 and m=4 have been analyzed.

Table 1. Convergence of the 1st case

m	DOFs	Total src	Mx	My	Mz	error [%]
	exact	1	0	0	0	
1	8	0.999	0	0	0	38.0
2	24	0.999	0	0	0	6.6
4	98	1.001	0	0	0	1.5

Table 2. Convergence of the 2nd case

m	DOFs	Total src	Mx	My	Mz	error [%]
	exact	1	2.5	0	0	
1	8	0.999	1.47	0	0	44.2
2	24	0.999	2.42	0	0	8.4
4	98	1.001	2.49	0	0	2.7

Table 3. Convergence of the 3rd case

m	DOFs	Total src	Mx	My	Mz	error [%]
	exact	1	5	0	0	
1	8	0.999	2.75	0	0	61.0
2	24	0.999	4.80	0	0	8.5
4	98	1.001	4.99	0	0	3.5

Table 4. Convergence of the 4th case

m	DOFs	Total src	Mx	My	Mz	error [%]
	exact	0	5	0	0	
1	8	0	2.94	0	0	84.2
2	24	0	4.83	0	0	14.9
4	98	0	4.99	0	0	5.8

Table 5. Convergence of the 5th case

m	DOFs	Total src	Mx	My	Mz	error [%]
	exact	0	10	0	0	
1	8	0	5.49	0	0	88.2
2	24	0	9.61	0	0	14.9
4	98	0	9.98	0	0	6.4

The most important results of the computation are summarized in Table 1-5 for the five cases. The 1st column shows the subdivision parameter, *m*; the 2nd the number of TRM DOFs; the 3rd the exact and computed sum of sources and sinks; the 4rd, 5th and 6th the exact and computed moments of the sources; the 7th the relative error of the computed potential on the exterior surface. The comparison basis for the relative error calculation is the exact value of the potential on the exterior surface. This is quite severe since the potential takes much larger values within the FEM domain. The total of sources and moments describe the precision at far region. The error on the exterior surface characterizes the worst case error in the FEM region.

The excellent convergence of the solution can be observed as m increases. Note that practically reasonable precision can be obtained with about 20-100 Trefftz DOFs. Compare this number to the order of 400 required by the hybrid FEM-BEM method. Note also the excellent precision in the description of the far field. The results confirm the feasibility of hybrid FEM-TRM method.

SUMMARY

The paper describes a hybrid FEM-TRM method which is the synthesis of the *generalized* FEM [4,5,6], *quasi* FEM

[7] and *transfinite* FEM [13] methods. The formulation provides symmetric stiffness matrix which can be evaluated without singular integrals. As opposed to hybrid FEM-BEM formulations, the number of DOFs to describe the potential in the far region are small and can be selected independently of the FEM mesh. With a few Trefftz DOFs excellent precision can be obtained both in the FEM and the far regions. The method can be easily implemented in a general purpose FEM package with open architecture.

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