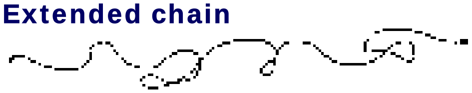
Hyperelasticity

Background on Elastomers

Unextended chain

Extended chain

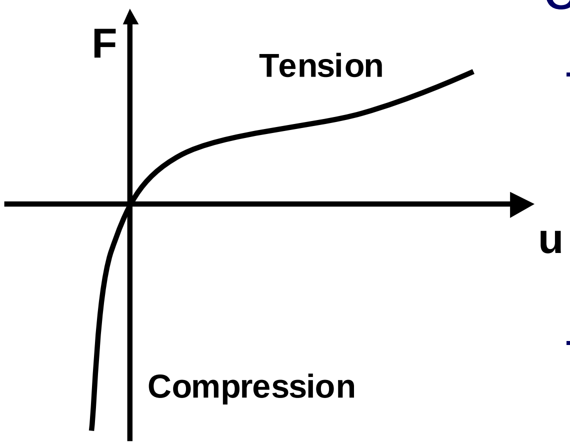


Schematic of single molecular chain. In network, these chains are randomly oriented and often have crosslinks.

Example of Rubber boot, o-rings/seals

- Elastomers are a class of polymers with the following properties
  - Elastomers involve natural and synthetic rubbers, which are amorphous and are comprised of long molecular chains
  - The molecular chains are highly twisted, coiled, and randomly oriented in an undeformed state
  - These chains become partially straightened and untwisted under a tensile load
  - Upon removal of the load, the chains revert back to its original configuration
  - Strengthening of the rubber is achieved by forming crosslinks between molecular chains through a vulcanization process

Background on Elastomers (cont'd)



- On a macroscopic level, rubber behavior exhibits certain characteristics
  - They can undergo large elastic (recoverable) deformations, anywhere on the order of 100-700%. As noted previously, this is due to the untwisting of cross-linked molecular chains.
  - There is little volume change under applied stress since the deformation is related to straightening of chains. Hence, elastomers are nearly incompressible.
  - Their stress-strain relationship can be highly nonlinear
  - Usually, in tension, the material softens then stiffens again. On the other hand, in compression, the response becomes quite stiff.

Background on Hyperelasticity

- There are some key assumptions related to the hyperelastic constitutive models in ANSYS
  - Material response is isotropic, isothermal, and elastic
    - Thermal expansion is isotropic
    - Deformations are fully recoverable (conservative)
  - Material is fully or nearly incompressible
    - Requires element formulations discussed earlier such as B-Bar or Mixed U-P to handle incompressibility condition
- The constitutive hyperelastic models are defined through a strain energy density function
  - Unlike plasticity, hyperelasticity is not defined as a rate formulation $\dot{\sigma} = D : \dot{\varepsilon}$
  - *Instead, total-stress vs. total-strain relationship is defined through a strain energy potential (W)*

Definition of Stretch Ratio

Before proceeding to a detailed discussion on different forms of the strain energy potential, some terms will be defined:

- The *stretch ratio* (or simply 'stretch') is defined as

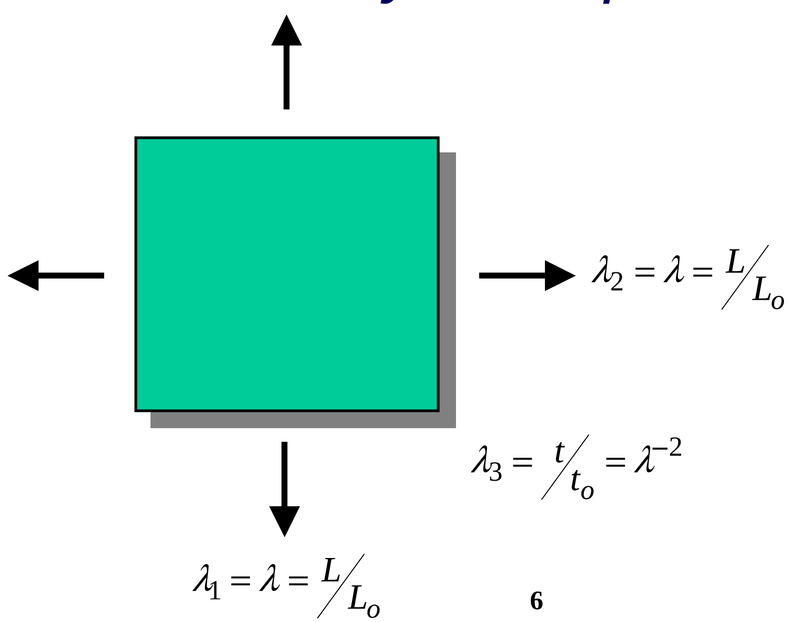
$$\lambda = \frac{L}{L_o} = \frac{L + \Delta u}{L_o} = 1 + \varepsilon_E$$

The above is an example of stretch ratio as defined for uniaxial tension of a rubber specimen, where ε_E is engineering strain.

There are three principal stretch ratios λ_1, λ_2, and λ_3 which will provide a measure of the deformation. These will also be used in defining the strain energy potential.

Definition of Stretch Ratio (cont'd)

- To illustrate the definition of the principal stretch ratios by an example, consider a thin square rubber sheet in biaxial tension. The principal stretch ratios λ_1 and λ_2 characterize in-plane deformation. On the other hand, λ_3 defines the thickness variation (t/t_o). Additionally, if the material is assumed to be *fully incompressible*, then λ_3 will equal λ^{-2}.



Definition of Strain Invariants

- The three *strain invariants* are commonly used to define the strain energy density function.

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

  If a material is fully incompressible, $I_3 = 1$.

- Because we assume that the material is isotropic, some forms of the strain energy potential are expressed as a function of these scalar invariants. In other words, strain invariants are measures of strain which are independent of the coordinate system used to measure the strains.

Definition of Volume Ratio

- The *volume ratio* J is defined as

$$J = \lambda_1 \lambda_2 \lambda_3 = \frac{V}{V_o}$$

As shown above, J can be thought of as the ratio of deformed to undeformed volume of the material.

- In the case of thermal expansion, the thermal volumetric deformation is

$$J_{th} = \left(1 + \varepsilon_{th}\right)^3$$

The elastic volumetric deformation is related to the total and thermal volumetric deformation by the following:

$$J_{el} = J = \frac{J_{total}}{J_{th}}$$

Definition of Strain Energy Potential

- The strain energy potential (or strain energy function) is usually denoted as W
  - Strain energy potential can either be a direct function of the principal stretch ratios or a function of the strain invariants

$$W = W(I_1, I_2, I_3)$$

or

$$W = W(\lambda_1, \lambda_2, \lambda_3)$$

    The particular forms of the strain energy potential will be discussed shortly. These forms determine whether stretch ratios or strain invariants are used.

  - Based on W, second Piola-Kirchoff stresses (and Green-Lagrange strains) are determined:

$$S_{ij} = \frac{dW}{dE_{ij}}$$

Definition of Strain Energy Potential

- Because of material incompressibility, we split the deviatoric (subscript d or with 'bar') and volumetric (subscript b) terms of the strain energy function. As a result, the volumetric term is a function of volume ratio J only.

$$W = W_d\left(\overline{I_1}, \overline{I_2}\right) + W_b(J)$$

$$W = W_d\left(\overline{\lambda_1}, \overline{\lambda_2}, \overline{\lambda_3}\right) + W_b(J)$$

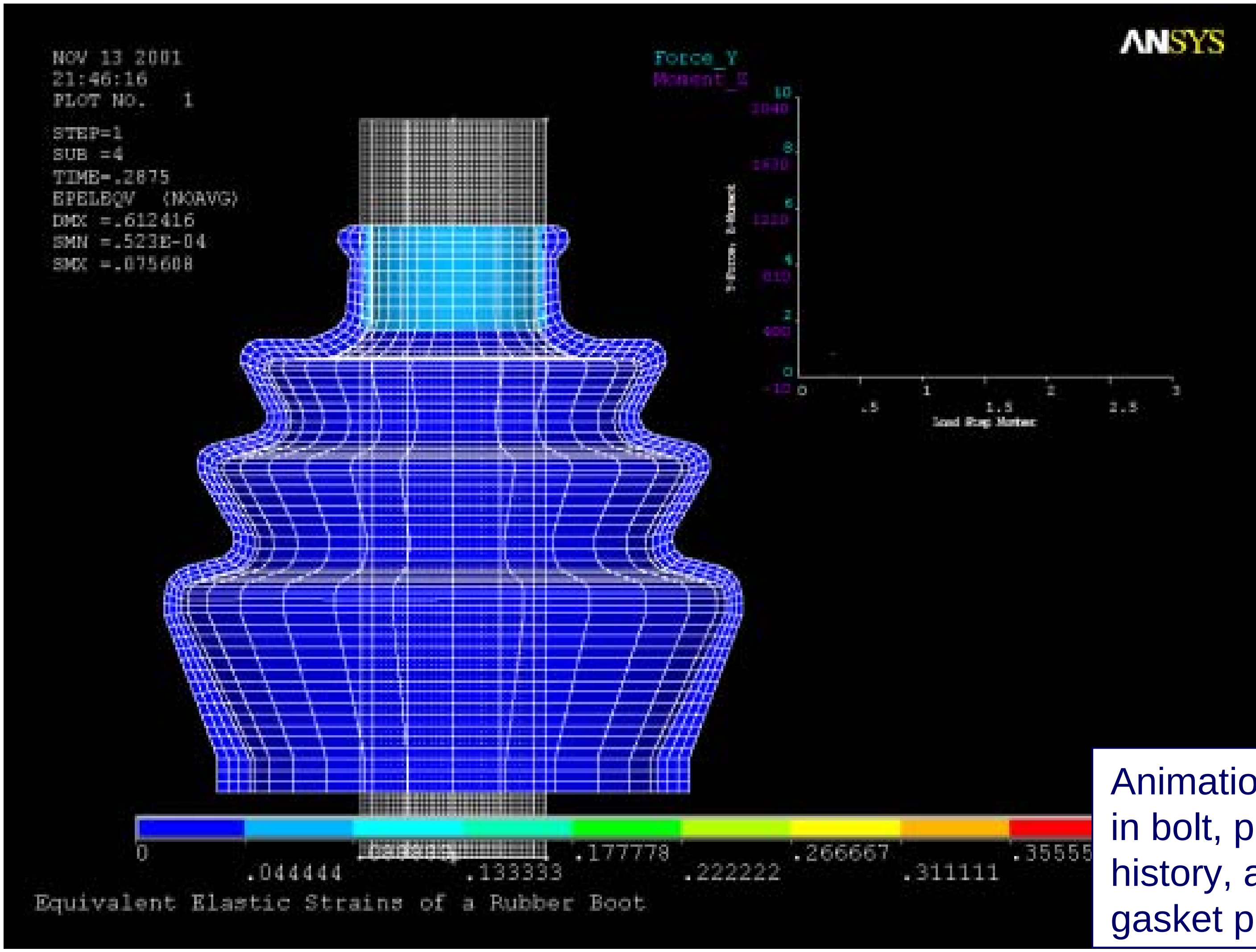
where the deviatoric principal stretches and deviatoric invariants are defined as (for p=1,2,3):

$$\overline{\lambda_p} = J^{-1/3}\lambda_p$$

$$\overline{I_p} = J^{-2/3} I_p$$

Note that $I_3=J^2$, so I_3 is not used in the definition of W.

Example of Rubber Boot



Animation of reaction forces in bolt, pressure/closure history, and contour of gasket pressure.

Particular Forms of W

Strain Energy Functions

Polynomial

Mooney-Rivlin

Neo-Hookean

Yeoh

Arruda-Boyce

Gent

Ogden

Hyperfoam

Blatz-Ko

Items shown in green are available from 5.7 onwards, although *still under development.* When contact is used with these materials prior to 6.1, be sure to use absolute specification of FKN.

- In this section, the different hyperelastic models for the 18x series of elements will be presented, as listed on the left. Each is a particular form of W, based either on the strain invariants or on the principal stretch ratios directly.
- The strain energy potential W will require certain types of parameters input as material constants.
  - The number of material constants will depend on the strain energy function W chosen.
  - The choice of W will depend on the user, although some very general guidelines will be presented to aid the user in selection of W.
  - From the selection of W and material constants which are input, stress and strain behavior are calculated by ANSYS.

Polynomial Form

- The *polynomial form* is based on the first and second strain invariants. It is a phenomenological model of the form

$$W = \sum_{i+j=1}^{N} c_{ij} \left(\overline{I}_1 - 3\right)^i \left(\overline{I}_2 - 3\right)^j + \sum_{k=1}^{N} \frac{1}{d_k} \left(J_{el} - 1\right)^{2k}$$

  where the initial bulk modulus and initial shear modulus are

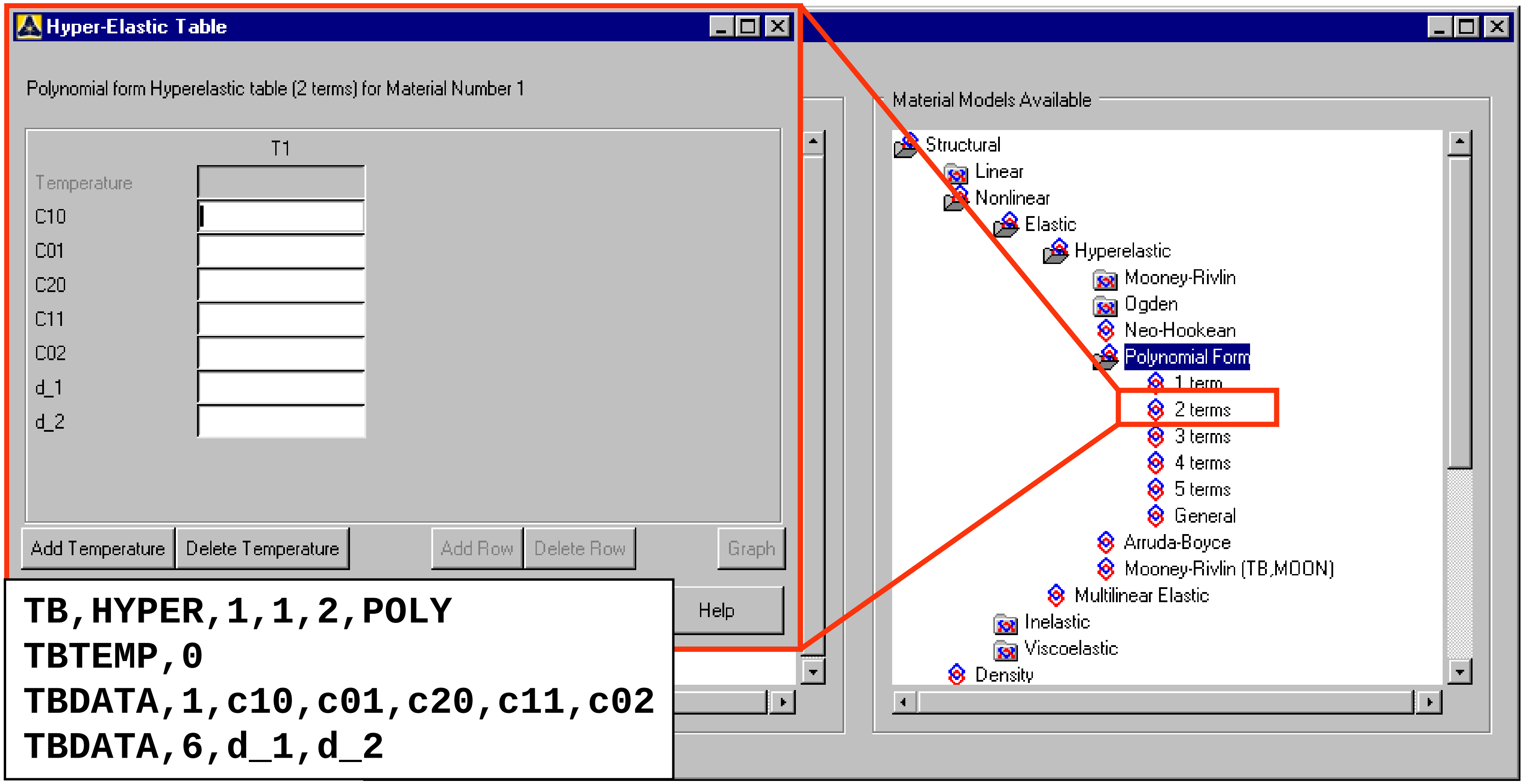
$$\mu_o = 2\left(c_{10} + c_{01}\right)$$

$$\kappa_o = \frac{2}{d_1}$$

- This option is defined via TB,HYPER,,,N,POLY. c_{ij} and d_i are input via TBDATA. Usually, values of N greater than 3 are rarely used. It may be applicable for strains up to 300%.

Polynomial Form (cont'd)

Sample definition of 2-term Polynomial form shown below.
Constants c_{10}, c_{01}, c_{20}, c_{11}, c_{02}, d_1, d_2 to be defined.




```
TB,HYPER,1,1,2,POLY
TBTEMP,0
TBDATA,1,c10,c01,c20,c11,c02
TBDATA,6,d_1,d_2
```

Mooney-Rivlin

- There are two-, three-, five-, and nine-term *Mooney Rivlin models* available in ANSYS. These can also be thought of as particular cases of the polynomial form.
  - The two-term Mooney-Rivlin model is equivalent to polynomial form with N=1:

$$W = c_{10}\left(\overline{I}_1 - 3\right) + c_{01}\left(\overline{I}_2 - 3\right) + \frac{1}{d}(J_{el} - 1)^2$$

  - The three-term Mooney-Rivlin model is similar to the polynomial form when N=2 and $c_{20}=c_{02}=0$:

$$W = c_{10}\left(\overline{I}_1 - 3\right) + c_{01}\left(\overline{I}_2 - 3\right) + c_{11}\left(\overline{I}_1 - 3\right)\left(\overline{I}_2 - 3\right) + \frac{1}{d}(J_{el} - 1)^2$$

  - The five-term Mooney-Rivlin model is same as the polynomial form when N=2:

$$W = \sum_{i+j=1}^{2} c_{ij}\left(\overline{I}_1 - 3\right)^i\left(\overline{I}_2 - 3\right)^j + \frac{1}{d}(J_{el} - 1)^2$$

  - The nine-term Mooney-Rivlin model is same as the polynomial form when N=3:

$$W = \sum_{i+j=1}^{3} c_{ij}\left(\overline{I}_1 - 3\right)^i\left(\overline{I}_2 - 3\right)^j + \frac{1}{d}(J_{el} - 1)^2$$

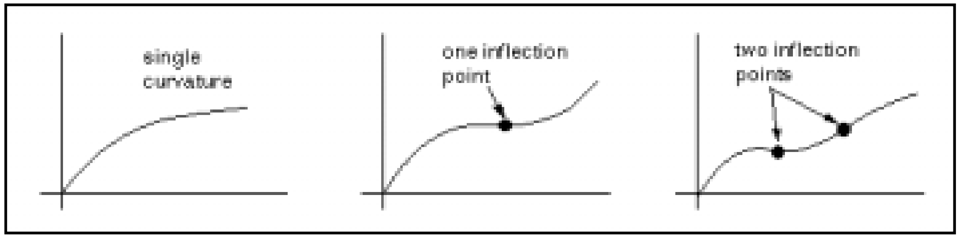
- For all of the preceding Mooney-Rivlin forms, the initial shear and initial bulk moduli are similar to that of the polynomial form, defined as:

$$\mu_o = 2(c_{10} + c_{01})$$

$$\kappa_o = \frac{2}{d}$$

Mooney-Rivlin (cont'd)

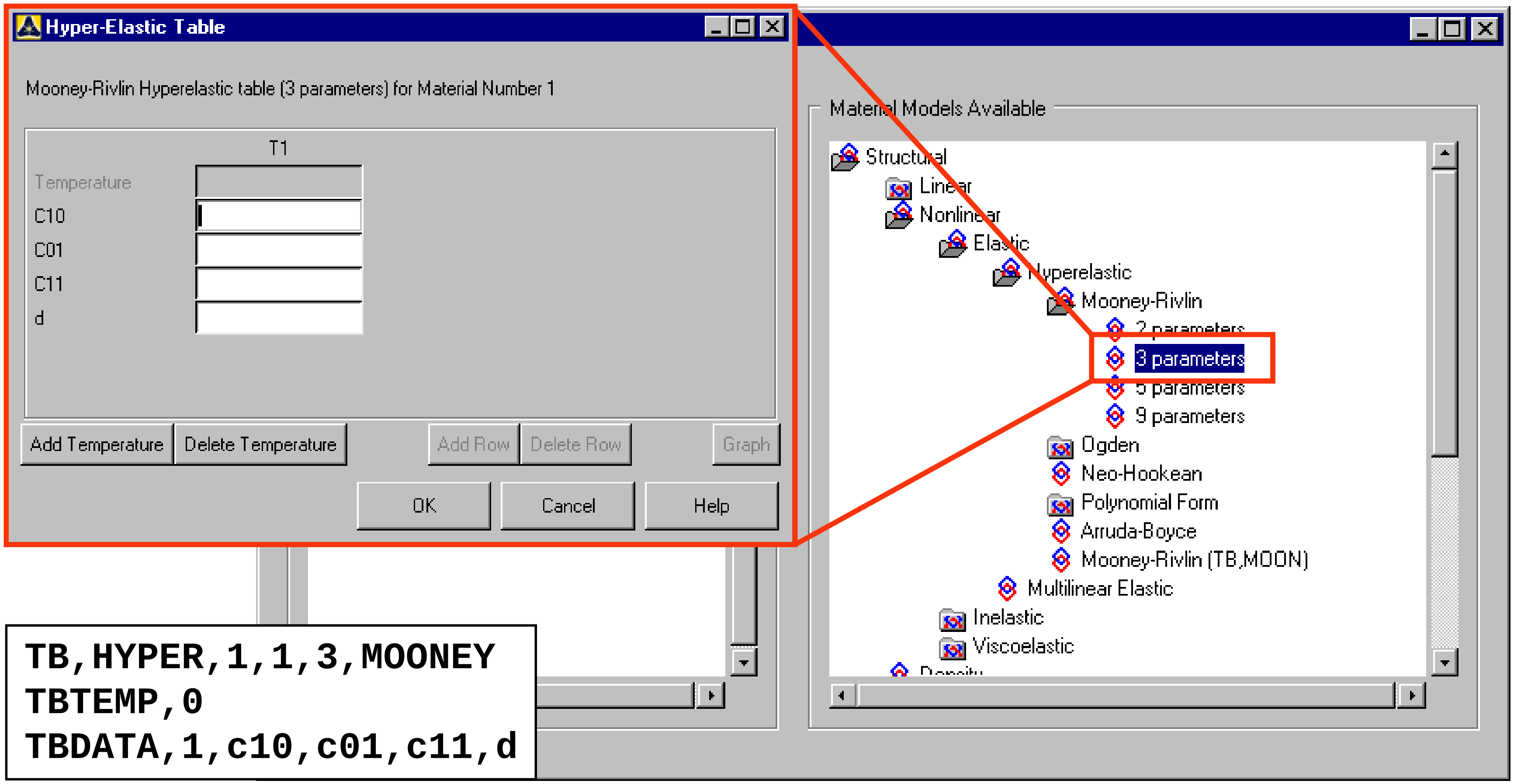
- For the *18x series of elements*, this option is defined via TB,HYPER,,,N,MOONEY. Constants c_{ij} and d are input via TBDATA.
  - As a very general guideline, the two-term MR form may be valid up to 90-100% tensile strains, although it will not account for stiffening effects of the material, usually present at large strains. Compression behavior may also not be characterized well with only two-term MR.
  - As noted in the figure below, more terms may capture any inflection points in the engineering stress-strain curve. As with the polynomial form, the user must ensure that enough data is supplied with inclusion of higher-order terms. Five- or Nine-term MR may be used up to 100-200% strains (general guideline).



Two-Term MR **Five-Term MR** **Nine-Term MR**

Mooney-Rivlin (cont'd)

Sample definition of 3-term Mooney-Rivlin form shown below. Constants c_{10}, c_{01}, c_{11}, d to be defined.




```
TB,HYPER,1,1,3,MOONEY
TBTEMP,0
TBDATA,1,c10,c01,c11,d
```

Yeoh

This is an unreleased, undocumented feature.

- The *Yeoh model* (a.k.a. *reduced polynomial form*) is similar to the polynomial form but is based on first strain invariant only.

$$W = \sum_{i=1}^{N} c_{i0}\left(\overline{I}_1 - 3\right)^i + \sum_{i=1}^{N} \frac{1}{d_i}\left(J_{el} - 1\right)^{2i}$$

  The Yeoh model is commonly considered with N=3, although ANSYS allows for any value of N.

- The initial shear and bulk moduli are defined similar to other invariant-based models:

$$\mu_o = 2c_{10}$$

$$\kappa_o = \frac{2}{d_1}$$

- This option is defined via TB,HYPER,,,N,YEOH. c_{i0} and d_i are input via TBDATA. The 3-term Yeoh model generally provides a good fit over large strain ranges.

Yeoh (cont'd)

This is an unreleased, undocumented feature.

- Although the Yeoh model is available from 5.7 onwards, it is *undocumented*. Hence, input for Yeoh parameters must be done through command-line only.

  TB,HYPER,mat,ntemp,N,YEOH

  TBTEMP,temp1

  TBDATA,1,c_{i0}(i=1…N),d_i(i=1…N)

  where

  N = number of terms

  ci0 (i=1…N) are coefficients

  di (i=1…N) are compressibility terms.

  For example, a three-term Yeoh model (MAT 1) would be input as:

  TB,HYPER,1,1,3,YEOH

  TBTEMP,0

  TBDATA,1,c10,c20,c30,d1,d2,d3

Neo-Hookean

- The *Neo-Hookean form* can be thought of as a subset of the polynomial form for N=1, c_{01}=0, and c_{10}=μ/2:

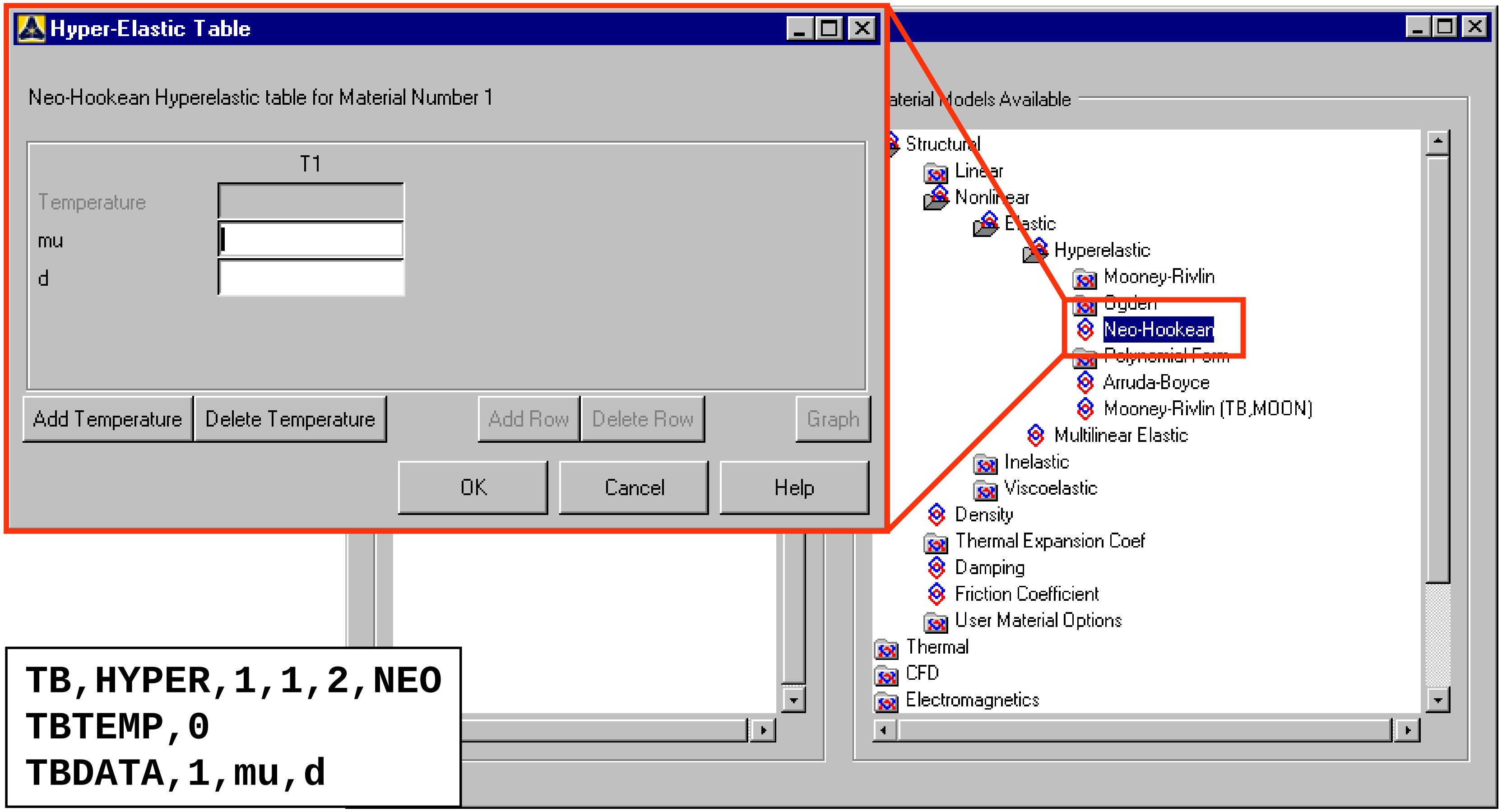
$$W = \frac{\mu}{2}\left(\overline{I}_1 - 3\right) + \frac{1}{d}\left(J_{el} - 1\right)^2$$

where the initial bulk modulus is defined as $\kappa_o = \frac{2}{d}$

- This option is defined via TB,HYPER,,,,NEO. The constants μ and d are input via TBDATA.
  - This is the simplest hyperelastic model which can serve as a good starting point, using a constant shear modulus. However, it is limited to strains up to 30-40% in uniaxial tension and up to 80-90% in pure shear (these are general guidelines).

Neo-Hookean (cont'd)

Sample definition of Neo-Hookean form shown below.
Constants μ and d to be defined.




```
TB,HYPER,1,1,2,NEO
TBTEMP,0
TBDATA,1,mu,d
```

Arruda-Boyce

- The *Arruda-Boyce form*, also known as the eight-chain model, is a statistical mechanics-based model. This means that the form was developed as a statistical treatment of non-Gaussian chains emanating from the center of the element to its corners (eight-chain network).

$$W = \mu \sum_{i=1}^{5} \frac{C_i}{\lambda_L^{2i-2}} \left(\overline{I_1^i} - 3^i \right) + \frac{1}{d} \left(\frac{J_{el}^2 - 1}{2} - \ln J_{el} \right)$$

where the constants C_i are defined as

$$C_1 = \frac{1}{2}, C_2 = \frac{1}{20}, C_3 = \frac{11}{1050}, C_4 = \frac{19}{7050}, C_5 = \frac{519}{673750}$$

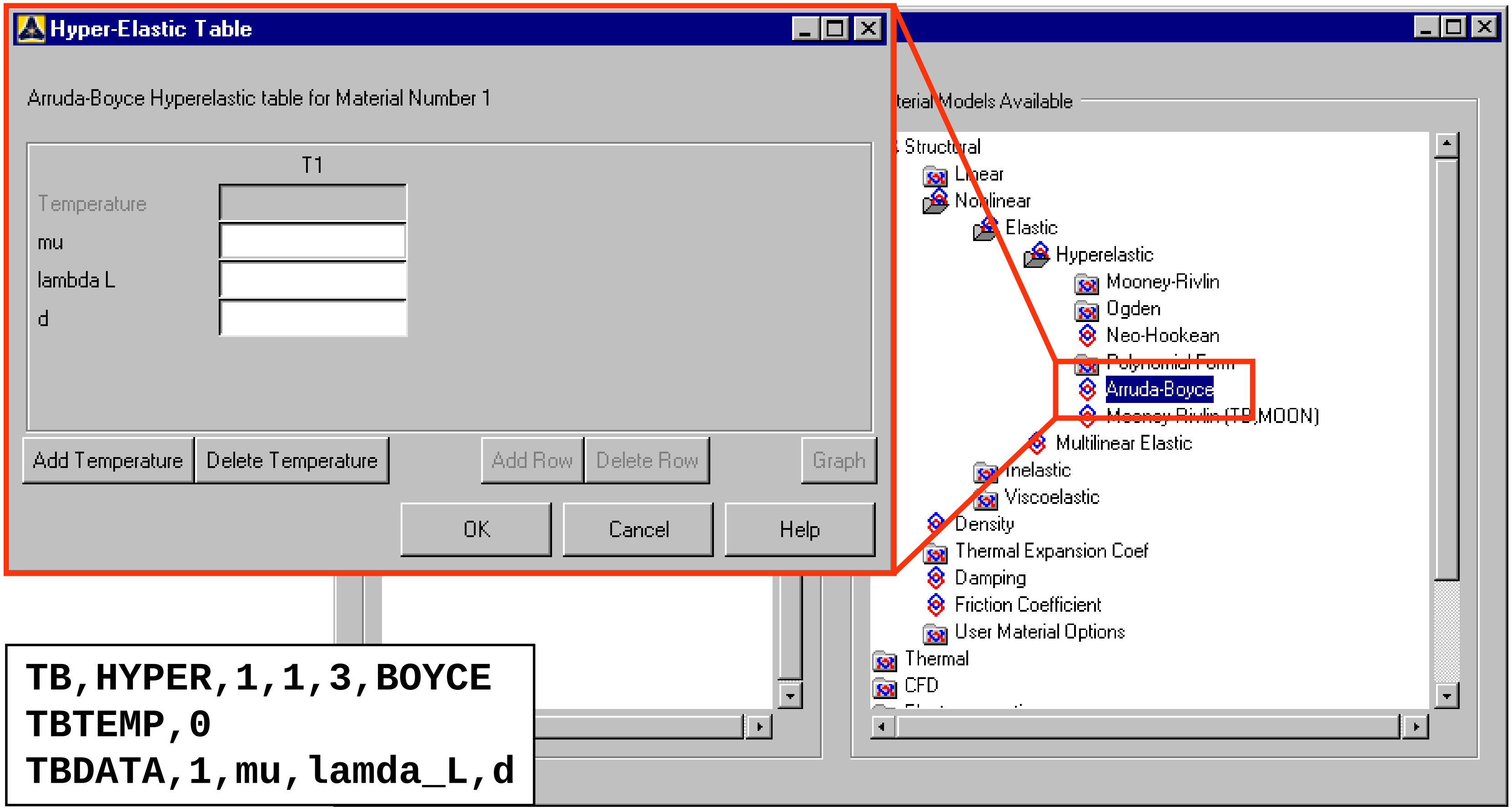
- From the above equation, it is clear that the Arruda-Boyce form can be thought of as a Yeoh model with N=5, although the coefficients are *predefined functions of the limiting network stretch* λ_L.

Arruda-Boyce (cont'd)

- This option is defined via TB,HYPER,,,,BOYCE. Constants μ, λ_L, and d are input via TBDATA.
  - The initial shear modulus is μ.
    - In the Arruda-Boyce paper, this rubbery modulus is defined as $nk\Theta$, which is a function of chain density (n), Boltzmann's constant (k), and temperature (Θ).
  - The limiting network stretch λ_L is the stretch at which stress starts to increase without limit. Note that as λ_L becomes infinite, the Arruda-Boyce form becomes the Neo-Hookean form.
    - The Arruda-Boyce equation on the previous slide is actually the first five terms of the series expansion of the strain energy. The original equation contains an inverse Langevin function which is expanded. This series expansion, however, may cause the limiting network stretch to be less pronounced.
  - Generally limited to 300% strain at most.

Arruda-Boyce (cont'd)

Sample definition of Arruda-Boyce form shown below. Constants μ, λ_L, and d to be defined.




```
TB,HYPER,1,1,3,BOYCE
TBTEMP,0
TBDATA,1,mu,lamda_L,d
```

Gent

This is an unreleased, undocumented feature.

- The *Gent model* is also micromechanical model, similar to Arruda-Boyce, which utilizes the concept of limiting network stretch:

$$W = -\frac{EI_m}{6}\ln\left(1 - \frac{\bar{I}_1 - 3}{I_m}\right) + \frac{1}{d}\left(\frac{J_{el}^2\text{-}1}{2} - \ln J_{el}\right)$$

  where the constants E, I_m, and d are input. E is the initial elastic modulus, which, for incompressible materials, is $3\mu_0$. I_m is the limiting value of (I_1-3), analogous to λ_L for Arruda-Boyce.
- If the natural logarithm is expanded, the resulting form will be similar to the Yeoh model. The coefficients, however, are predefined functions of I_m.
  - It is quite clear that there are many similarities between the Gent and Arruda-Boyce models.

Gent (cont'd)

This is an unreleased, undocumented feature.

- Although the Gent model is available from 5.7 onwards, it is *undocumented*. Hence, input for Gent parameters must be done through command-line only.

```
TB,HYPER,mat,ntemp,3,GENT
TBTEMP,temp1
TBDATA,1,E,Im,d
```

where

E = small strain tensile modulus

Im = limiting value of (I_1-3)

d = material compressibility term (=2/initial_bulk_modulus)

The following is a sample input (incompressible, material 1):

```
TB,HYPER,1,1,3,GENT
TBTEMP,0
TBDATA,1,0.81,101,0.0
```

Ogden

- The *Ogden form*, another phenomenological model, is directly based on the principal stretch ratios rather than the strain invariants:

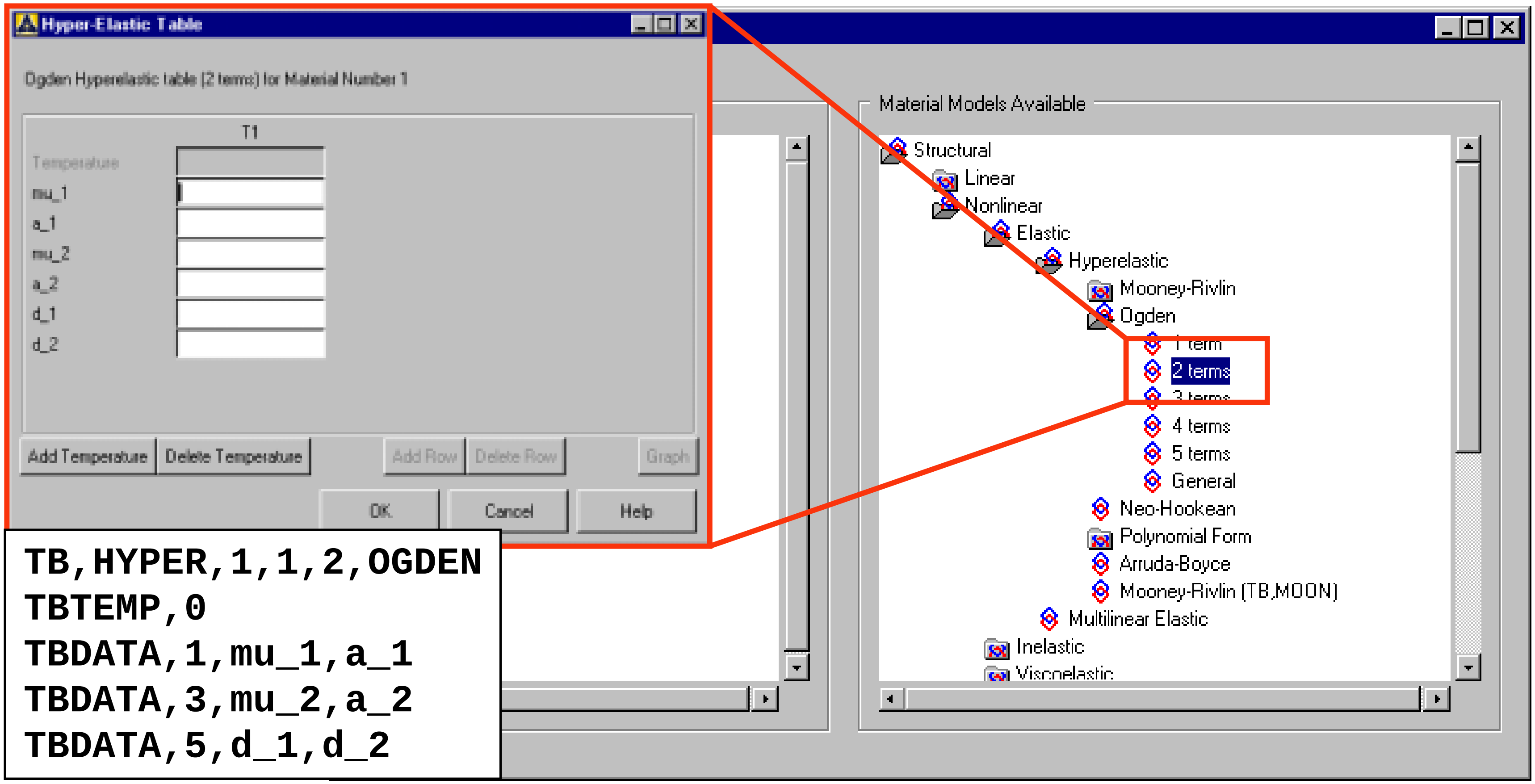
$$W = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}\left(\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i} - 3\right) + \sum_{i=1}^{N} \frac{1}{d_i}\left(J_{el} - 1\right)^{2i}$$

  where the initial bulk and shear moduli are $\mu_o = \frac{\sum_{i=1}^{N} \mu_i \alpha_i}{2}$ $\kappa_o = \frac{2}{d_1}$

- This option is defined via TB,HYPER,,,N,OGDEN. μ_i, α_i, and d_i need to be supplied via TBDATA.
  - Degenerates to the Neo-Hookean form: N=1 $\mu_1=\mu$ $\alpha_1=2$
  - Equivalent to the 2-term M-R form: N=2 $\mu_1=2c_{10}$ $\alpha_1=2$ $\mu_2=-2c_{01}$ $\alpha_2=-2$
  - Since Ogden is based on principal stretch ratios directly, it may be more accurate and may provide better data fitting. However, it is more computationally expensive.
  - In general, Ogden form may be applicable for strains up to 700%.

Ogden (cont'd)

Sample definition of 2-term Ogden model shown below.
Constants μ_1, α_1, μ_2, α_2, d_1, d_2 to be defined.




```
TB,HYPER,1,1,2,OGDEN
TBTEMP,0
TBDATA,1,mu_1,a_1
TBDATA,3,mu_2,a_2
TBDATA,5,d_1,d_2
```

Hyperfoam Model

This is an unreleased, undocumented feature.

- The *Hyperfoam model* (a.k.a. Ogden foam model) is very similar to the Ogden model for incompressible materials.

$$W = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i} \left(J_{el}^{\alpha_i/3} \left(\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i} \right) - 3 \right) + \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i \beta_i} \left(J_{el}^{-\alpha_i \beta} - 1 \right)$$

where the initial bulk and shear moduli are

$$\mu_o = \frac{\sum_{i=1}^{N} \mu_i \alpha_i}{2} \qquad \kappa_o = \sum_{i=1}^{N} \mu_i \alpha_i \left(\frac{1}{3} + \beta_i \right)$$

- However, unlike the regular Ogden model, in the hyperfoam model, the volumetric and deviatoric terms are tightly coupled. Hence, this model is meant to model highly compressible rubber behavior.

Hyperfoam Model (cont'd)

This is an unreleased, undocumented feature.

- Although the Hyperfoam model is available from 5.7 onwards, it is *undocumented*. Hence, input for Hyperfoam parameters must be done through command-line only.

  TB,HYPER,mat,ntemp,npts,FOAM

  TBTEMP,temp1

  TBDATA,1,μ_i,α_i(i=1…N),β_i(i=1…N)

  where

  npts = number of terms for hyperfoam model

  μ_i, α_i, β_i = material input parameters, as defined on previous slide

  The following is a sample input:

```
TB,HYPER,1,1,3,FOAM
TBTEMP,0
TBDATA,1,mu_1,alpha_1
TBDATA,3,mu_2,alpha_2
TBDATA,5,mu_3,alpha_3
TBDATA,7,beta_1,beta_2,beta_3
```

Blatz-Ko

This is an unreleased, undocumented feature.

- The *Blatz-Ko model* is specifically for compressible polyurethane foam rubber with the following form:

$$W = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2\sqrt{I_3} - 5\right)$$

where μ is the shear modulus.

  - This is the same model as available with HYPER84/86 with KEYOPT(2)=1, although the input is different (not MP)
  - The Blatz-Ko model can be thought of as a subset of hyperfoam model, with N=1, α_1=-2, β_1=0.5, μ_1=-μ.

$$W = \frac{-\mu}{-2}\left(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right) + \frac{-\mu}{2\cdot 0.5}\left(J_{el}^{2\cdot 0.5} - 1\right)$$

$$W = \frac{\mu}{2}\left(\frac{I_2}{I_3} - 3\right) + \frac{\mu}{2}\left(2\sqrt{I_3} - 2\right)$$

  - The effective Poisson's ratio can be determined from β, which leads to the assumption of ν=0.25 for Blatz-Ko model.

Blatz-Ko (cont'd)

- Although the Blatz-Ko model is available from 5.7 onwards, it is *undocumented*. Hence, input for Blatz-Ko parameters must be done through command-line only.

```
TB,HYPER,mat,ntemp,1,BLATZ
TBTEMP,temp1
TBDATA,1,μ
```

where

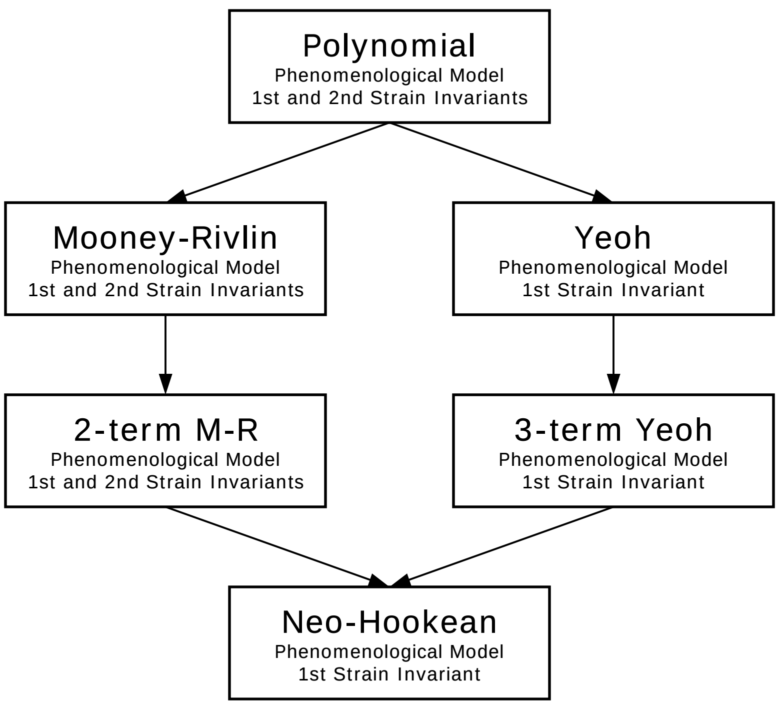
μ = shear modulus

The following is a sample input:

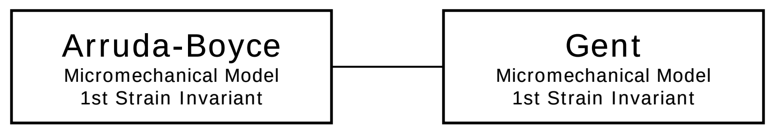
```
TB,HYPER,1,1,1,BLATZ
TBTEMP,0
TBDATA,1,mu
```

Categorization of Models

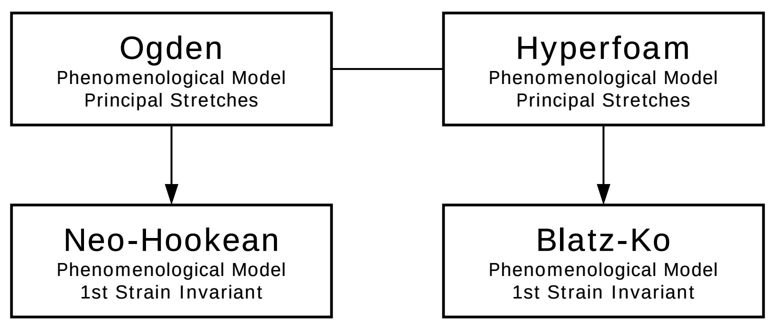
It may be useful to attempt to establish a relationship between various hyperelastic models discussed.



Nearly-/fully-incompressible phenomenological hyperelastic models based on strain invariants.



Nearly-/fully-incompressible micromechanical models based on 1st strain invariant



Incompressible (left) and compressible (right) models based on principal stretches

Notes: Mooney-Rivlin is similar to General Polynomial form. Yeoh model is also known as “Reduced Polynomial” form. Hyperfoam is also referred to as “Rubber foam” or “Ogden foam” model.

Why So Many Models?

- A common concern which arises is the selection of an appropriate hyperelastic model
  - For nearly-/fully-incompressible elastomers, user must select between phenomenological vs. micromechanical, strain invariant-based vs. principal stretch-based models.
    - Neo-Hookean model is most simple and a good way to start.
    - Two-term Mooney-Rivlin is one of the most widely used models, although not suitable to capture the stiffening effect. General M-R models (i.e., Polynomial) are an extension of this for larger strains
    - Yeoh proposed omitting second invariant term, as it is harder to measure and provides less accurate fit for limited test data. 3-term Yeoh model provides good fit for large stretch values, though maybe not so for low strain.
    - Arruda-Boyce and Gent models can be thought of as extensions of Yeoh, where the constants have physical meaning. Good for small + large stretch.
    - Ogden is based on principal stretches, usually provide much better curve fitting. Usually a little more computationally intenstive because of this.
  - For compressible polyurethane foam-type rubbers, Blatz-Ko is suitable. Other highly compressible elastomeric foams require Hyperfoam model.

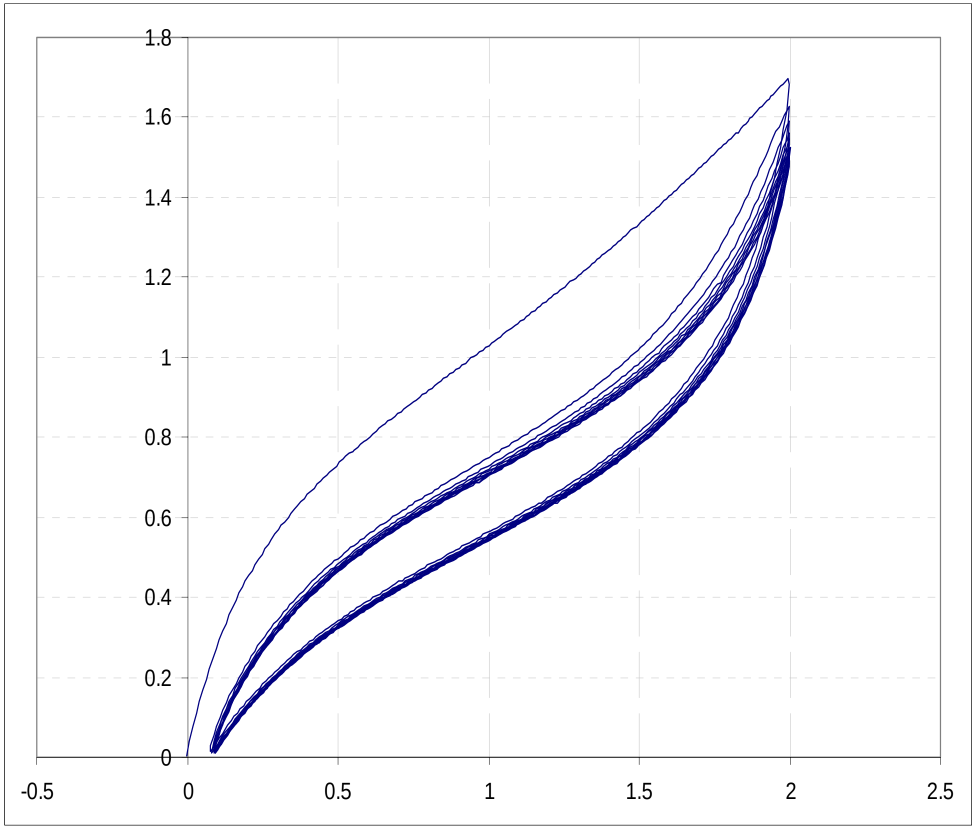
Material Characterization

- Collected data may need to be adjusted to account for effects such as hysteresis and stress-softening behavior.

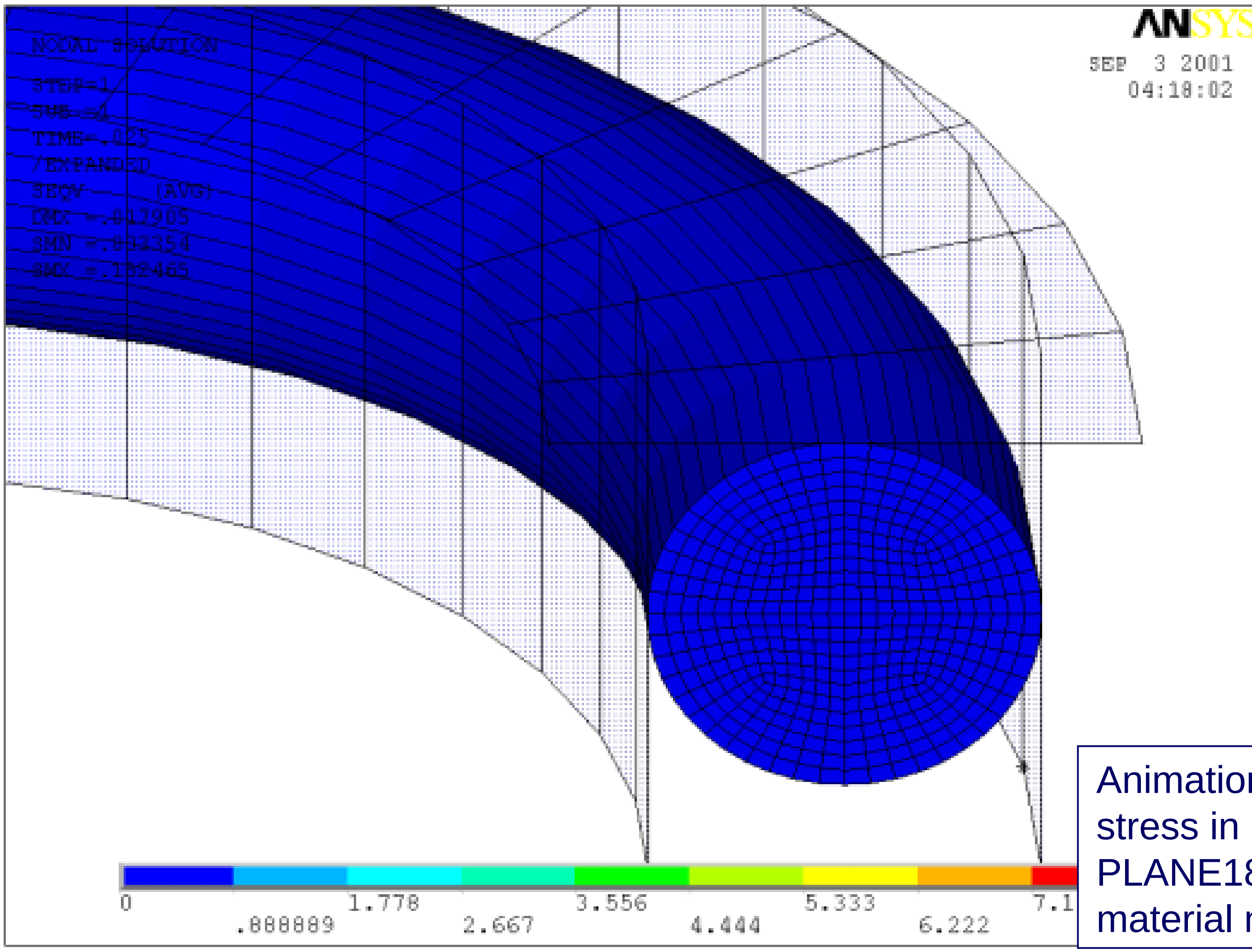
A typical engineering stress-strain curve for a rubber sample under cyclic loading is shown on the right.

Note that hysteresis is present. Stress-softening effects (such as Mullins effect) are also present.

A stabilized curve needs to be chosen:
1) Loading or unloading path
2) Initial or Nth repetition
This curve would then be shifted to the origin (zero stress for zero strain) and used for curve-fitting procedures.



Example of O-ring Compression



Animation of contour of stress in axisymmetric PLANE182 using Ogden material model.