A NON-LINEAR DOMAIN DECOMPOSITION METHOD

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ABSTRACT

A new domain decomposition technique for parallelization of physically non-linear problems in solid mechanics is developed. It is extended for any stress-strain diagrams. The approach is based on the approximating generalized models of subdomains that leads to a considerably fewer number of iterations requiring updating of substructure parameters. The analytical proof of the convergence of the theoretical iterative procedure and the demonstration of convergence of its numerical implementation with the help of ANSYS are presented.

INTRODUCTION

NOMENCLATURE

L	potential energy
N	number of subdomains
Q_s, F_k	generalized forces
Q_a	effective generalized stress
c_{rsa}	constant matrix
d_{rsa}	constant matrix
u_s, w_k	generalized displacements
u_a	effective generalized
x_i	spacial co-ordinates
Ψ	work
Ω,Ω_a	domains and subdomains
ε_{ij}	strain tensor
λ	parameter
σ_{ij}	stress tensor
σ_e	equivalent stress
σ_y	yield stress

The domain decomposition technique has been extended to parallelize analysis of non-linear problems in solid mechanics i.e., physically and geometrically non-linear problem analyses, shape optimization, coupling phenomena, contact problem analysis etc (Toping and Khan 1996, Ast et al. 1998, Chiang and Fulton 1990, Storaasli and Bergan 1987, Utku et al. 1982, Lund 1997). The way, which has been used to introduce parallel computation, is successive solution of the non-linear problems based on iterative or step-by-step procedures, linearizing and solving the problems in parallel at each iteration or step. The non-linear software is often a modification to a linear program and practically retains the substructuring technique used in its linear version. Such an approach proves its value in those cases when, for example, a respective sequential technique is based on the step-by-step computing. However, if the parallelization is connected with the usage of a step-by-step solving procedure instead of a more effective iterative one or with updating of substructure parameters at each of numerous iterations, the parallel efficiency is low.

Physically non-linear static problems in solid mechanics, which parallelization is carried out in the above non-effective way, include the problems of elasticity, steady state creep and some plasticity problems. Sequentially these problems is invariably solved by linearizing the non-linear equations and iteratively solving the resulting linear equations until a solution to the non-linear problem is found. The Newton-Raphson solution is the most popular example of such a technique. The approach keeping advantages of the above procedures at parallelization was introduced for structures of power law materials (Klebanov and Davydov 1998). It is based on non-linear generalized models, which represent the behavior of a complete deformable solid, or part of it (Kachanov 1967, Boyle and Spence 1983, Samarin and Klebanov 1994). The high efficiency of the software based on the approach introduced was shown through the study of several examples. This paper considers the expansion of the approach for any stress-strain diagrams.

BASIC EQUATIONS

Consider a structure or a solid body at static loading. It occupies volume Ω, which is fictitiously divided into N subdomains Ω_a, $a = 1,2,\ldots,N$.

The displacements of all the points in the given body as well as their gradients are assumed to be sufficiently small that changes in geometry are neglected in the equilibrium equations and strain-displacement relations are assumed in a linear form.

The constitutive relationships are

$$\sigma_{ij} = \frac{\partial \Psi(x_k, \varepsilon_{lm})}{\partial \varepsilon_{ij}} \quad ; i,j,k,l,m = 1,2,3 \ (1)$$

where σ_{ij} is the stress tensor, Ψ the work, a convex homogeneous function of strains ε_{ij}. For non-homogeneous materials it depends on the spatial co-ordinates x_i. The materials obey the condition of local stability. At $\sigma_e < \sigma_Y$, where σ_Y is the yield stress and σ_e the effective stress, the material possesses linear properties.

Let us introduce the functional

$$\Psi_\alpha = \int_{\Omega_\alpha} \Psi d\Omega \quad ; a = 1,\ldots,N$$

which is the work in the whole subdomain.

Assume that displacements exerted on the subdomain can be represented as linear functions of parameters named generalized displacements u_s ($s = 1,2,\ldots$), which forms denumerable set.

According to Lagrange's theorem we have

$$Q_s = \frac{\partial \Psi_\alpha}{\partial u_s}; \quad \Psi_\alpha = \Psi_\alpha(u_r); \quad s,r = 1,2,\ldots$$

where Q_s are the generalized forces corresponding to generalized displacements.

The summation is not taken over the subscript *a* here and below.

Hypersurfaces Ψ_a = *const* considered in the space of generalized displacements are the surfaces of equal work in the subdomain

$$\Psi_\alpha = \int_{\Omega_\alpha} \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}\, d\Omega$$

For a linear dependence between stress and strain these hypersurfaces are hyperellipsoids. For non-linear stress-strain diagram we shall approximate the actual hypersurfaces in neighborhood of the point under consideration by a tangential hyperellipsoid passed through this point.

The approximating hyperellipsoid corresponds to a fictitious subdomain, which is of the same shape as the subdomain under consideration but of proper non-homogeneous liner properties. The effective generalized strain for the fictitious subdomain can be introduced as

$$u_\alpha = \left(c_{rs\alpha}\, u_r\, u_s\right)^{\frac{1}{2}}$$

Coefficients $c_{rs\alpha}$ are functionals of the strain field corresponding to the point of approximation. For the linear fictitious subdomain $Q_s = c_{rs\alpha} u_r$ and it is possible to assume the effective generalized stress as $Q_a = u_a$.

A generalized substructure model for non-linear subdomain associated with the above approximation can be represented as

$$Q_s = d_{rs\alpha}\, u_r \frac{1}{u'_\alpha} \frac{\partial \hat{\Psi}_\alpha}{\partial u'_\alpha} \quad (2)$$

where

$$d_{rs\alpha} = \lambda c_{rs\alpha}; \quad u'_\alpha = \left(d_{rs\alpha}\, u_r\, u_s\right)^{\frac{1}{2}} \quad (3)$$

and

$$\hat{\Psi}_\alpha\left(u'_\alpha\right) = \Psi_\alpha\left(u_s\right)$$

ALGORITHM

The iterative solution of the above non-linear boundary value problem includes a subsequent approximation of

substructure diagrams Q_a - u_a at each of the iterations. Calculated points of the diagrams are connected by straight-line segments.

- Step 0: Solve a boundary value problem for the body under consideration at given boundary conditions for the initial linear properties using a known domain decomposition method.

- Step 1: Calculate the first point of the non-linear part of Q_a - u_a diagrams - the yield point. The corresponding effective generalized stress is obtained by re-scaling all the generalized forces with factor $\underset{\Omega_\alpha}{Max}(\sigma_Y/\sigma_e)$

  One non-linear iteration proceeds as following.

- Step 2: Solve directly the non-linear local problem for each subdomain at interface values $u_s^{(p-1)}$ calculated at the previous iteration. Here p is the iteration index.

- Step 3: Find $c_{rs\alpha}^{(p-1)}$ and update approximating generalized constitutive equation (2) providing successive approximation of the actual diagrams Q_a - u_a from the above; reduce external forces, initial stresses and strains to interfaces. Parameter λ is obtained from the condition of normalization

$$\lambda c_{rs\alpha}^{(p-1)} u_r^{(p-1)} u_s^{(p-1)} = d_{rs\alpha}^{(p-2)} u_r^{(p-1)} u_s^{(p-1)} \quad (4)$$

- Step 4: Solve the non-linear problem for the whole domain regarded as a composition of the subdomains at given boundary conditions.

- Step 5: Check for convergence; if needed, return to Step 2.

At Step 1, Step 2 and Step 3 the analyses of subdomains can be performed completely in parallel. The last iteration provides a solution corresponding to given accuracy. Output within subdomains is available as the result of Step 2 implementation at this iteration.

CONVERGENCE PROOF

We consider the body subjected to the prescribed generalized force vector F_k (k = 1,2,...) over a part of its surface. There are generalized displacement rates, imposed over the remainder. The actual solution is given by the vector u_s^*. Let Ψ_α^* denote the actual work for the subdomain a. The solution for a current non-linear iteration obtained at Step 4 are denoted by $u_s^{(p)}$. The matrix $c_{rs\alpha}^{(p-1)}$ is obtained at Step 3 of the current iteration p in accordance with interface values $u_s^{(p-1)}$ calculated at the previous iteration.

In accordance with the principle of minimum potential energy

$$\sum_{\alpha=1}^{N}\hat{\Psi}_{\alpha}^{(p-1)}\left[\left(d_{rs\alpha}^{(p-1)}u_r^{(p-1)}u_s^{(p-1)}\right)^{\frac{1}{2}}\right]-F_k\,w_k^{(p-1)}$$
$$\geq\sum_{\alpha=1}^{N}\hat{\Psi}_{\alpha}^{(p-1)}\left[\left(d_{rs\alpha}^{(p-1)}u_r^{(p)}u_s^{(p)}\right)^{\frac{1}{2}}\right]-F_k\,w_k^{(p)} \qquad p = 1,2,... \quad (5)$$

where generalized displacements $w_k^{(p)}$ of the whole body correspond to the generalized displacement $u_s^{(p)}$ of the subdomains, which are calculated at Step 4.

Taking (3) and (4) into account, rewrite (5) in the form

$$\sum_{\alpha=1}^{N}\hat{\Psi}_{\alpha}^{(p-1)}\left[\left(d_{rs\alpha}^{(p-2)}u_r^{(p-1)}u_s^{(p-1)}\right)^{\frac{1}{2}}\right]-F_k\,w_k^{(p-1)}$$
$$\geq\sum_{\alpha=1}^{N}\hat{\Psi}_{\alpha}^{(p-1)}\left[\left(d_{rs\alpha}^{(p-1)}u_r^{(p)}u_s^{(p)}\right)^{\frac{1}{2}}\right]-F_k\,w_k^{(p)}$$

Since the procedure of Q_a - u_a diagram approximation guaranties

$$\sum_{\alpha=1}^{N}\hat{\Psi}_{\alpha}^{(p-2)}\left(u_{\alpha}\right)\geq\sum_{\alpha=1}^{N}\hat{\Psi}_{\alpha}^{(p-1)}\left(u_{\alpha}\right)$$

(6)

we obtain from the previous inequality that

$L_{p-2} \geq L_{p-1}$ (7)

where

$$L_{p-1}=\sum_{\alpha=1}^{N}\hat{\Psi}_{\alpha}^{(p-1)}\left[\left(d_{rs\alpha}^{(p-1)}u_r^{(p)}u_s^{(p)}\right)^{\frac{1}{2}}\right]$$

It is easy to be convinced that $\lambda > 1$. Then from the condition of approximation

$$\hat{\Psi}_{\alpha}^{(p-1)}\left[\left(d_{rs\alpha}^{(p-1)}u_r^{(p-1)}u_s^{(p-1)}\right)^{\frac{1}{2}}\right]=\Psi_{\alpha}^{*}\left(u_r^{(p-1)}\right)$$

and (7) we have $L_{p-2}\geq L_{*}$ where $L_{*}=\sum_{\alpha=1}^{N}\Psi_{\alpha}^{*}-F_k\,w_k^{*}$. It means that regular sequence Lp (p = 1,2,…) is convergent to L_{*} uniformly. The obtained result provides convergence of $u_s^{(p)}$ to u^*_s since the latest corresponds to the absolute minimum of the potential energy (Klebanov and Davydov 1998).

TEST CASES

The above parallel procedure has been incorporated in a newly developed non-linear static FE program. Non-linear problems at Step 2 and Step 4 are solved by the Newton-Raphson method. Test cases were studied on the PVM installed on a net of Pentium 200 MHz PCs. Each computer is equipped with a processor of 32 Megabytes of RAM. The net is 10-Megabit Ethernet with twisted cable connection.

As in the case of power-law diagram multiple subdomains are created from the initial definition with the help of ANSYS's pre-processor subroutines. Figure 1 shows the geometry the FE mesh and the boundary conditions for a holed structure. It was divided into 8 and 16 subdomains (Figure 2). The stress-strain diagram employed is represented in Figure 3.

Figure 1. FE DISCRETIZATION AND BOUNDARY CONDITIONS OF THE HOLED STRUCTURE.

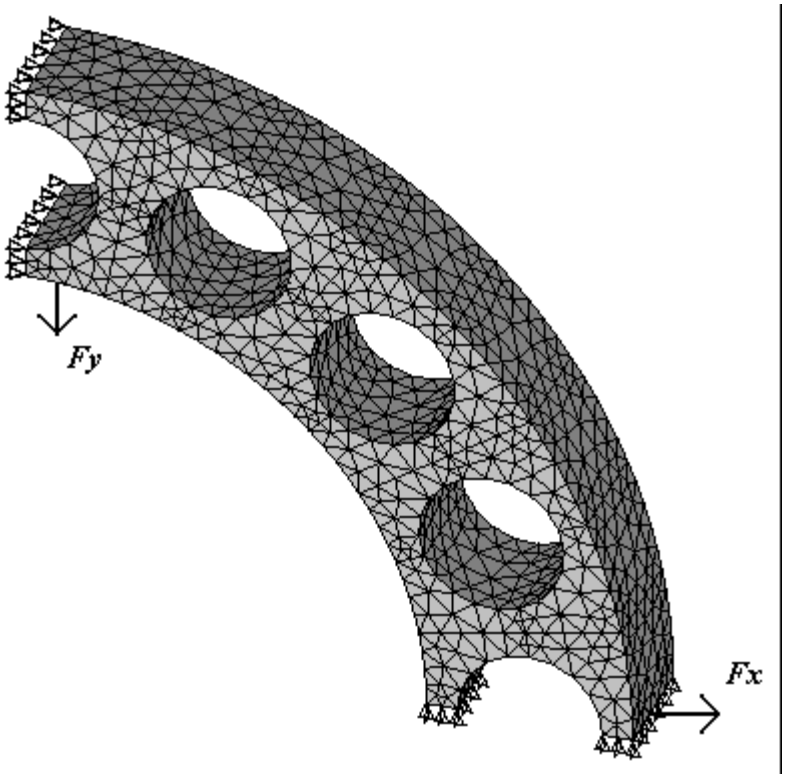


Figure 2. THE HOLED STRUCTURE DIVIDED INTO SUBSTRUCTURES.

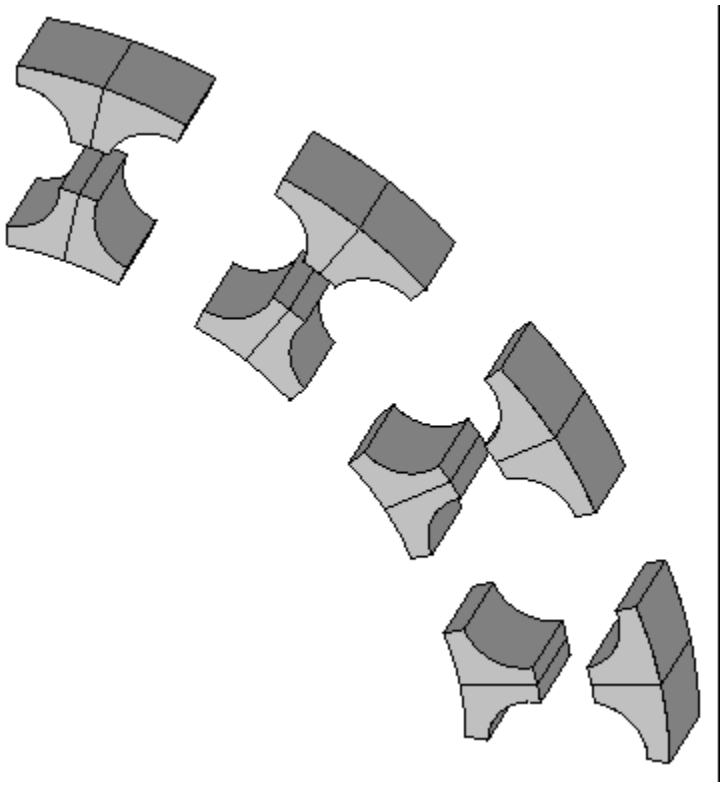


Figure 3. POLYGONAL STRESS-STRAIN DIAGRAM.

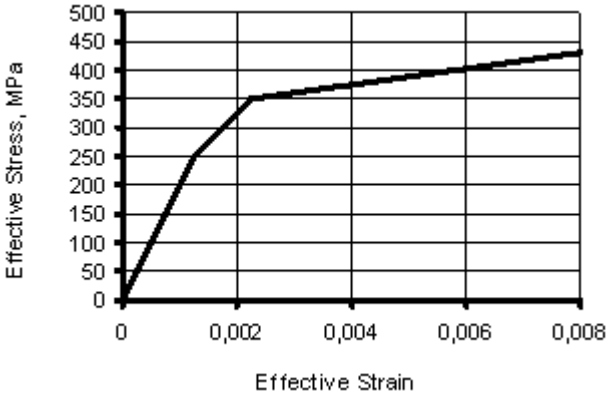


Figure 4. CONVERGENCE OF THE RELATIVE STRESS-STRAIN STATE CHARACTERISTICS WITH ITERATIONS.

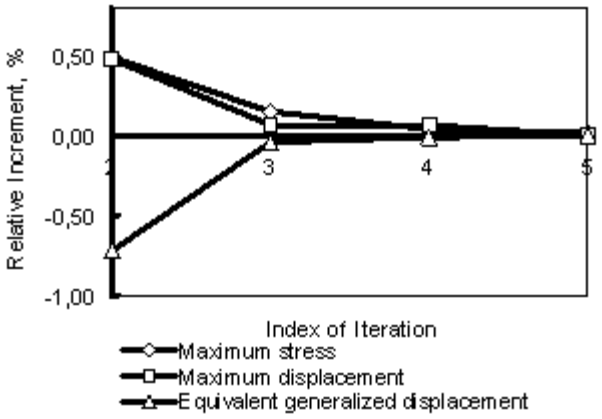
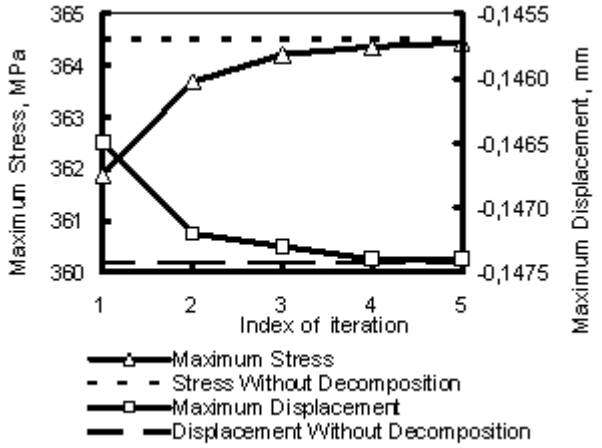


Figure 5. CONVERGENCE OF THE MAXIMUM STRESS AND THE MAXIMUM DISPLACEMENT FOR THE STRUCTURE WITH ITERATIONS.



The convergence of the iterative procedure introduced is illustrated at Figures 4 and 5 for eight-subdomain decomposition. Step 0 and Step 1 are considered as the first iteration. The five iterations is necessary to reach reasonable accuracy. The value of accuracy is practically equals that obtained without domain decomposition with the same original FE mesh in this example and the others considered.

The parallel speed up and efficiency considered in comparison with the sequent algorithm without domain decomposition and the domain decomposition algorithm introduced but computed with the only processor. The results are shown in Figures 6 - 9.

It is clearly from the diagrams that both the characteristics are higher at 16 subdomains. It is due to reduction of the time of static condensation at Step 3, which is the major time consuming operation. That number of subdomains is close to the optimal one.

Figure 6. PARALLEL SPEEDUP AT 8 SUBSTRUCTURES.

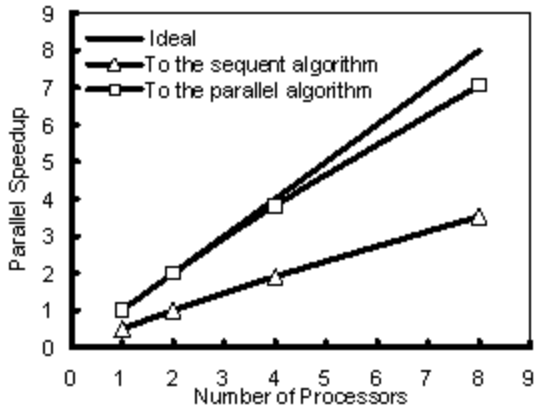


Figure 7. PARALLEL EFFICIENCY AT 8 SUBSTRUCTURES.

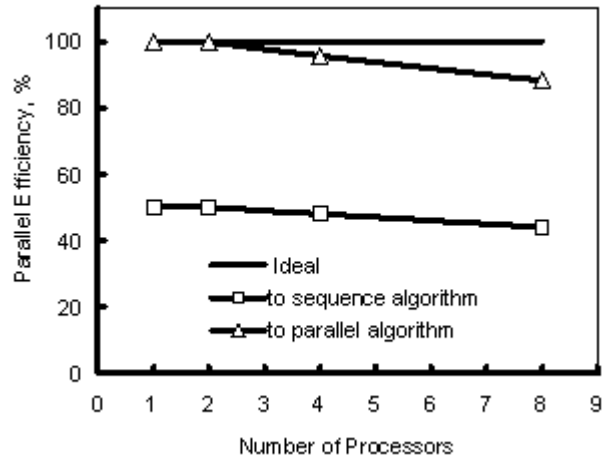


Figure 8. PARALLEL SPEEDUP AT 16 SUBSTRUCTURES.

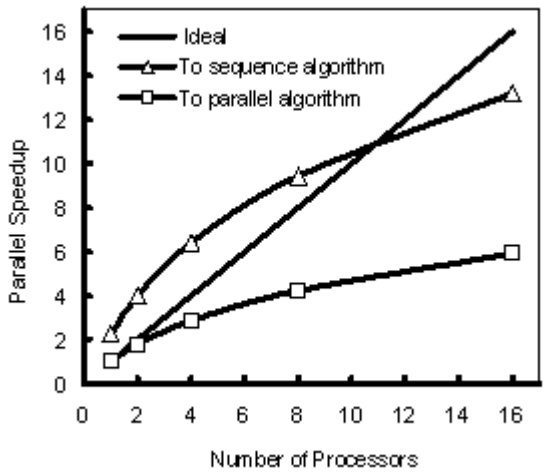
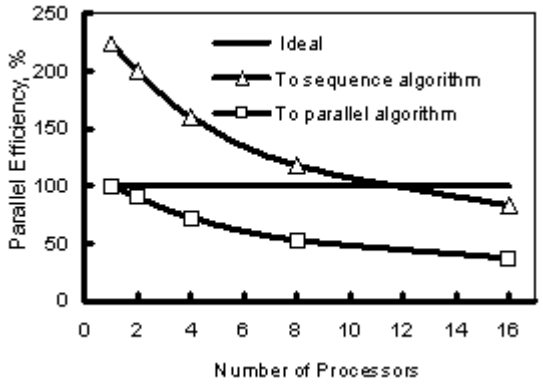


Figure 9. PARALLEL EFFICIENCY AT 16 SUBSTRUCTURES.



The optimal number corresponds to the equal times for analyses of subdomains at Step 2 and Step 3, and for the whole structure at Step 4 if each processor holds active systems corresponding to an individual subdomain. The domain decomposition into 16 subdomain calculated sequentially reduces down the computational time, if even the only processor is used, in comparison with no dividing into parts - the computational time of 88.2 min is reduced down to 39.2 min.

The same structure was computed with the power-law diagram represented in Figure 10. The two methods were used i.e., the method for power-law diagrams (Klebanov and Davydov 1998) and the more general method introduced in this paper. For the latest an initial segment of the power-law diagram has to be approximated as a linear one. The convergence of both the methods is illustrated in Figure 11 and Figure 12 for the eight-subdomain decomposition. The algorithm, specialized for the power-law materials, is more effective for such a diagram. It involves 2 or 3 less iterations in comparison with the general algorithm.

Figure 10. POWER-LAW STRESS-STRAIN DIAGRAM.

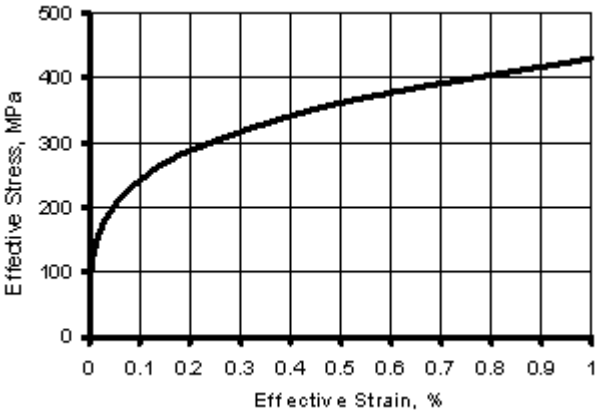


Figure 11. CONVERGENCE OF THE GENERAL ALGORITHM.

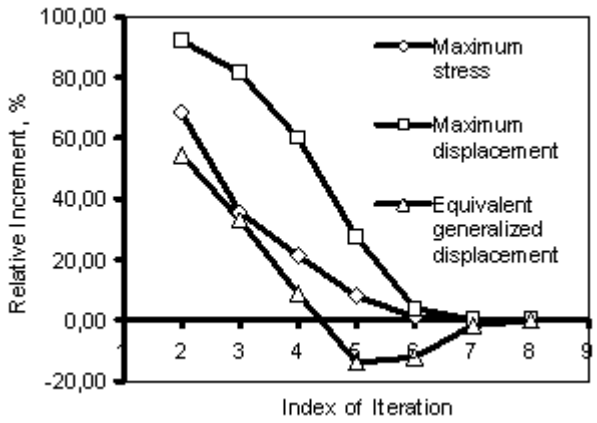
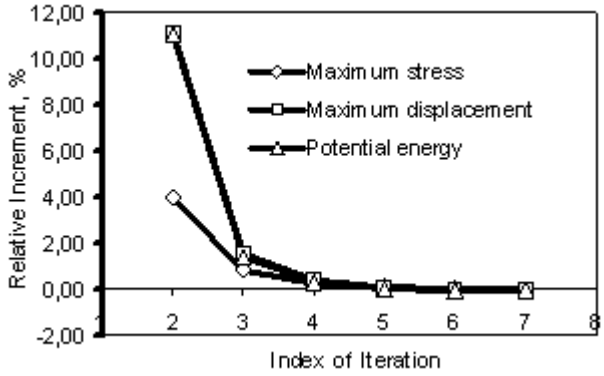


Figure 12. CONVERGENCE OF THE SPECIALIZED ALGORITHM.



CONCLUSION

The domain decomposition technique for physically non-linear problems in solid mechanics is developed on the base of the generalized structural model concept. The approach was previously introduced for power-law materials. This paper has considered its extension for any stress-strain diagrams.

The convergence proof of the theoretical iterative algorithm uses the principle of minimum potential energy and the properties of the approximating substructure models. The convergence and effectiveness of the numerical implementation with the help of ANSYS as a pre- and post-processor have been demonstrated through several examples. In comparison with the other methods, this approach involves a considerably fewer number of iterations requiring updating of substructure parameters through the static condensation procedure, which is the major time-consuming operation.

The above approach extended for geometrically non-linear problems are currently being investigated and intended for the next publication.

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