# 15CS301 THEORY OF COMPUTATION

## UNIT-1 FINITE AUTOMATA

- Introduction : Basic Mathematical Notation and techniques
- Finite state systems, Basic definitions, Finite automaton : DFA
- Finite Automaton : NDFA, Finite Automaton with $\epsilon$- moves
- Regular Languages - Regular Expressions
- Equivalence of NFA and DFA
- Equivalence of NDFA's with and without $\epsilon$-moves
- Equivalence of finite automaton and regular expressions
- Minimization of DFA .
- pumping lemma for regular sets, problems based on pumping lemma.

### Introduction:

⇒ Theory of computation is the branch that deals with how efficiently problems can be solved on a model of computation using using an algorithm.

⇒ The field is divided into three major branches

1. Automata theory
2. Computability theory
3. Computational Complexity theory.

### Introduction to formal proof

The formal proof can be using deductive proof and inductive proof.

1) Deductive proof ⇒ Sequence of statements given with logical reasoning in order to prove the first or initial statement.

Initial statement - Hypothesis.

2) Inductive proof ⇒ Recursive kind of proof which consists of sequence of parameterized statements that use the statement itself with lower value of its parameter.

Example: Statement B is true because statement A is true

Here Statement A ⇒ Hypothesis

Statement B ⇒ conclusion

Various forms of proof:

1) proofs about sets ⇒ set is a collection of elements or items.

Thus proof is of the kind "if and only if" that means an elemen

X is in A if and only if it is in B.

Example: Let us prove P∪Q = Q∪P

L.H.S ⇒

| Statement | Justification |
|---|---|
| X is in P∪Q | Given |
| X is in P or X is in Q | 1) And by definition of union |
| X is in Q or X is in P | 2) And by definition of union |
| X is in Q∪P | 3) Rule 2) And by definition of union |

RHS ⇒

| Statement | Justification |
|---|---|
| X is in Q∪P | Given |
| X is in Q or X is in P | 1) And by definition of union |
| X is in P or X is in Q | 2) And by definition of union |
| X is in P∪Q | 3) Rule 2) And by definition of union |

Hence P∪Q = Q∪P. Thus A=B is true as element X is in B if and only if X is in A.

2) Proof by Contradiction ⇒ The statement of the form if A and B we start with statement A is not true and thus by assuming false A we try to get the conclusion of statement B.

Example: prove P∪Q = Q∪P

Proof:

Assume P∪Q = Q∪P is not true

(i.e) P∪Q ≠ Q∪P

Consider X is in Q or X is in P ⇒ X is in PUQ then

X is in QUP according to definition of Union

Hence, The assumption which made initially is false.

Thus, PUQ = QUP is proved.

3) Proof by counter example

"All possible conditions" in which the statement remains true

Ex: There is no such pair of integers such that

$a \bmod b = b \bmod a$

Proof: Consider a=2 and b=3, then

$2 \bmod 3 \neq 3 \bmod 2$, thus the given pair is true for any pair of integers but if a=b then naturally $\boxed{a \bmod b = b \bmod a,\ a = b}$

## Inductive proof:

⇒ Inductive proofs are special proof based on some observations.

⇒ It is used to prove recursively defined objects.

The proof by mathematical induction can be carried out using following steps.

1. Basis ⇒ In this step we assume the lowest possible value. This is an initial step in the proof of mathematical induction.

   (ie) prove the result is true for n=0 or n=1

2. Induction hypothesis ⇒ In this step we assign value of n to some other value k.

   (ie) Check whether the result is true for n=k or not.

3. Inductive step ⇒ In this step, if n=k is true, then we check whether the result is true for n=k+1 or not.

   If we get the same result at n=k+1 then we can state that given proof is true by principle of mathematical Induction.

Problem1 : prove by induction on 'n' that $\sum_{i=0} i = \frac{n(n+1)}{2}$

Solution:

1) Basis of induction

Assume n=1, Then $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

$$L.H.S = \sum_{i=0}^{1} 1 = 1$$

$$R.H.S = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

2) Induction hypothesis

Assume n=k, Then $1+2+3+\ldots.+k = \frac{k(k+1)}{2}$ is true.

3) Inductive step

Assume n=k+1,

N.K.T $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

$$L.H.S = \sum_{i=1}^{k+1} i$$

$$= \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{k^2+k+2k+2}{2}$$

$$LHS = \frac{k^2+3k+2}{2} \quad \text{——} (1)$$

$$RHS = \frac{n(n+1)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{k^2+2k+k+2}{2}$$

$$RHS = \frac{k^2+3k+2}{2} \quad \text{——} (2)$$

From (1) and (2) LHS = RHS

$\therefore$ $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Hence, the given hypothesis is proved.

**Problem 2:** Prove that $1^2+2^2+3^2+\dots+n^2=\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$ using mathematical induction.

**Solution:**

1) **Basis of induction** $\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$

Let $n=1$, $LHS=1^2=1$

$$RHS=\frac{n(n+1)(2n+1)}{6}=\frac{1(1+1)(2(1)+1)}{6}=\frac{1\times2\times3}{6}=1$$

$LHS=RHS$

2) **Induction hypothesis**

Let $n=k$, $1^2+2^2+3^2+\dots+k^2=\sum_{i=1}^{k} i^2=\frac{k(k+1)(2k+1)}{6}$ is true.

3) **Inductive step**

Let $n=k+1$, $\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$

$$LHS=\sum_{i=1}^{k+1} i^2$$

$$=\sum_{i=1}^{k} i^2+(k+1)^2$$

$$=\frac{k(k+1)(2k+1)}{6}+k^2+2k+1$$

$$=\frac{(k^2+k)(2k+1)+6k^2+12k+6}{6}$$

$$=\frac{2k^3+k^2+2k^2+k+6k^2+12k+6}{6}$$

$$LHS=\frac{2k^3+9k^2+13k+6}{6} \quad \text{——①}$$

$$RHS=\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$=\frac{(k+1)(k+2)(2k+2+1)}{6}$$

$$=\frac{(k^2+2k+k+2)(2k+3)}{6}$$

$$=\frac{2k^3+3k^2+4k^2+6k+2k^2+3k+4k+6}{6}$$

$$RHS=\frac{2k^3+9k^2+13k+6}{6} \quad \text{——②}$$

From ① and ② $LHS=RHS$

$$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6}$$

Hence, the given statement is proved using mathematical induction.

**Problem 3:** Prove that for every integer $n \geq 0$ the number $4^{2n+1} + 3^{n+2}$ is multiple of 13.

**Solution:**

1) Basis of induction, Let $n=0$

$$4^{2n+1} + 3^{n+2} = 4^{2(0)+1} + 3^{0+2}$$
$$= 4 + 3^2$$
$$= 4+9$$
$$= 13$$

$4^{2n+1} + 3^{n+2}$ = Multiple of 13 $\therefore$

2) Induction hypothesis

Let $n=k$, $4^{2k+1} + 3^{k+2}$, for all $k \geq 0$ it is multiple of 13.

3) Inductive step.

Let $n=k+1$, $4^{2n+1} \cdot 3^{n+2} = 4^{2(k+1)+1} \cdot 3^{k+1+2}$

$$= 4^{2k+3} + 3^{k+3}$$
$$= 4^{2k+1} \cdot 4^2 + 3^{k+2} \cdot 3^1$$
$$= 16\left(4^{2k+1}\right) + 3\left(3^{k+2}\right)$$
$$= 13\left(4^{2k+1}\right) + 3\left(4^{2k+1}\right) + 3\left(3^{k+2}\right)$$
$$= 13\left(4^{2k+1}\right) + 3\left(4^{2k+1} + 3^{k+2}\right)$$
$$= 13\left(4^{2k+1}\right) + 3\,(13k)$$
$$= 13\left(4^{2k+1} + 3k\right)$$

$4^{2n+1} + 3^{n+2}$ = Multiple of 13

$\therefore$ For every integer $n \geq 0$, the number $4^{2n+1} + 3^{n+2}$ is multiple of 13.

**Problem 4:** Prove that for every integer $n \geq 0$ the number $n^4 - 4n^2$ is Divisible by 3.

**Problem 5:** Prove that every tree has 'e' edges and 'e+1' nodes.

## Basic concept of Automata theory

- Automata theory has a basic fundamental unit called set.
- Set is used to represent the mathematical model.

**Set** => It is defined as collection of objects enclosed within { }.

Ex: A set of element which are less than 5,

$A = \{0, 1, 2, 3, 4\}$

Set of vowels $A = \{a, e, i, o, u\}$

**Recursion** => To define the elements of the set $A = \{x \mid x \text{ is a square of } n\}$ where $n \leq 10$

(ie) $A = \{0, 1, 4, 9, 16, \ldots 100\}$

**Subset** => Subset A is called subset of B if every element of set A is present in set B, denoted by $A \subseteq B$

Ex: $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$ then $A \subseteq B$.

**Empty set** => Set having no element, $A = \{\}$ written as $\phi$ (phi).

**Null String** => Null element is denoted by $\varepsilon$

**Power set** => Set of all the subsets of its elements

Ex: $A = \{1, 2, 3\}$ power set $Q = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{3,2\}, \{2,3\}, \{1,2,3\}\}$

**Operation on set** =>

(i) **Union** => combination of both sets.

$A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$

(ii) **Intersection** => Common elements from both the sets

$A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$ then $A \cap B = \{2\}$

(iii) **Difference** => Elements there in one set but not there in another

$A = \{1, 2, 3\}$, $B = \{1, 2, 4\}$ then $A - B = \{3\}$.

(iv) **Complement** => $\overline{A} = U - A$. $U = \{10, 20, 30, 40, 50\}$, $A = \{10, 20\}$

$\overline{A} = U - A = \{30, 40, 50\}$

**Cartesian product** => Cartesian product of two sets A and B is a set of all possible ordered pairs.

Ex: $A = \{a, b\}$ and $B = \{0, 1, 2\}$

$A \times B = \{\{a, b\} \mid a \in A, b \in B\}$ $A \times B = \{(a,0), (a,1), (a,2), (b,0), (b,1), (b,2)\}$

Relations => The relation R is a collection for the set S which represents the pair of elements.

Properties: If $A = \{a, b\}$ then,

Reflexive relation R can be $= \{(a,a), (b,b)\}$

Irreflexive relation R can be $= \{(a,b)\}$

Transitive relation R can be $= \{(a,b), (b,a), (a,a)\}$

Symmetric relation R can be $= \{(a,b), (b,a)\}$

Asymmetric relation R can be $= \{(a,b)\}$

Equivalence relation => A relation is said to be equivalence relation if it is, reflexive, symmetric and transitive.

Kleen closure => set of strings of any length including null string $\epsilon$.

Ex: let $\Sigma = \{a, b\}$ obtain $\Sigma^*$

$$\Sigma^* = \{\epsilon, a, b, aa, bb, ab, ba, aaa, bbb, aba, \ldots\ldots\}$$

Positive closure => consists of all the strings of any length except a null string.

Ex: let $\Sigma = \{a, b\}$ $\Sigma^+ = \{a, b, aa, bb, ab, ba, aaa, bbb, aba, \ldots\}$

Alphabets, Strings and Languages

Alphabet: An alphabet is a finite collection of symbols.

Ex: $S = \{a, b, c, \ldots z\}$, $W = \{0, 1\}$ Here 0, 1 are alphabets, denoted by W.

Strings: Finite collection of symbols from alphabet.

Ex: $\Sigma = \{a, b\} = \{ab, bb, aa, aaa, bbb, \ldots\}$

Language: It is a collection of appropriate strings. The language is defined using an input set.

Ex: $\Sigma = \{\epsilon, 0, 00, 000, \ldots\}$

Here, the language is defined as 'any number of zeros'.

Operations => Union $L_1 \cup L_2$, Intersection $L_1 \cap L_2$

Concatenation $L_1 L_2$, Difference $L_1 - L_2$.

Example: Describe the following language over the input set $A=\{a,b\}$

(i) $L_1=\{a, ab, ab^2\}$ ⇒ All the strings with letter a followed by any number of b's.

(ii) $L_2=\{a^n b^n \mid n\geq 1\}$ ⇒ strings having equal number of a's followed by equal number of b's.

(iii) $L_3=\{a^m b^n \mid n>0\}$ ⇒ All the strings with any numbers of a's followed by atleast one b.

## Application of Automata theory

1) It is the base for the formal languages and useful for programming languag
2) It plays an important role in compiler Design
3) To prove the correctness of the program automata theory is used.
4) Switching theory and design and analysis of digital circuits automata Theory is applied.
5) Automata theory deals with the design finite state machine.

## Finite State Systems

⇒ Finite state system represents a mathematical model of a system with certain input. The model finally gives certain output.

⇒ The input when is given to the machine it is processed by various states, These states are called as intermediate states.

Example: Control mechanism of elevator. This mechanism only remembers the current floor number pressed, it does not remember all the previously pressed numbers.

## Finite Automata - Definition

A finite automata is a collection of 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where,

$Q$ - finite set of states

$\Sigma$ - input alphabet

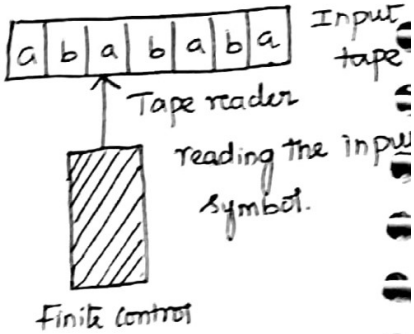
$q_0$ - initial state and $q_0$ is in $Q$ (ie) $q_0 \in Q$

$F$ - Final states

$\delta$ - Transition function - used to determine next state.

## Finite Automata Model.

The finite automata can be represented using




(i) Input tape => Linear tape having number of cells. Each i/p symbol is placed in each cell.

(ii) Finite control => It decides the next state on receiving particular i/p from input tape.

The tape reader reads the cells one by one from left to right and at a time only one i/p symbol is read.

## Acceptance of strings and Languages

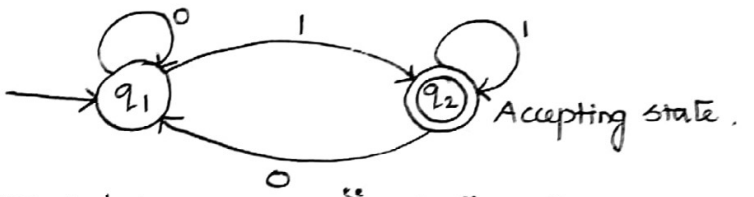
The strings and languages can be accepted by a finite automata, when it reaches to a final state.

Notations => (i) Transition diagram (ii) Transition table.

Transition diagram => Directed graph associated with the vertices of the graph correspond to the states of the finite automata.

Start State => $\rightarrow q_1$ Final state => $(q_2)$

Example:




If a string ends in a 0, it is "rejected" and "accepted" only if the string ends in 1 ∴ The language $L(M) = \{ w \mid w \text{ ends in } 1 \}$.

Transition table => Tabular representation of finite automata.

| Input / States | 0 | 1 |
|---|---|---|
| $\rightarrow q_1$ | $q_1$ | $q_2$ |
| $*q_2$ | $q_1$ | $q_2$ |

## Language Acceptance by FA:

A string 'x' is accepted by FA, $M = (Q, \Sigma, \delta, q_0, F)$ only if $\delta(q_0, x) = P$ for some P in F.

$$L(M) = \{ x \mid \delta(q_0, x) \text{ is in } F \}$$

## DFA ( Deterministic finite Automata)

The finite Automata is called Deterministic finite Automata if there is only one path for a specific input from current state to next state.

A Deterministic finite automata is a Quintuple

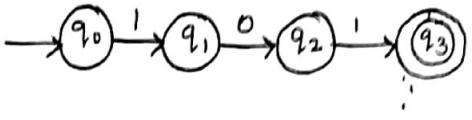
$$M = (Q, \Sigma, \delta, q_0, F)$$

$Q$ - Finite set of states
$\Sigma$ - Finite set of I/p symbols
$\delta$ - Transition function
$q_0$ - Start state, $q_0 \in Q$
$F$ - Final state, $F \subseteq Q$

Problems:

1) Design an FA which accepts the only input 101 over the input set $\Sigma = \{0,1\}$

Solution:



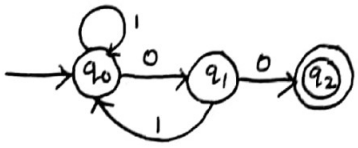

Here $M = (\{q_0, q_1, q_2, q_3\}, \{0,1\}, \delta, q_0, \{q_3\})$

$\delta$: $\delta(q_0,1) = q_1$ $\quad \delta(q_0,1) = q_3$
$\delta(q_1,0) = q_2$

2) Design DFA for a) set of all strings ending in 00
b) Set of all strings with three consecutive Xero's.

Solution:

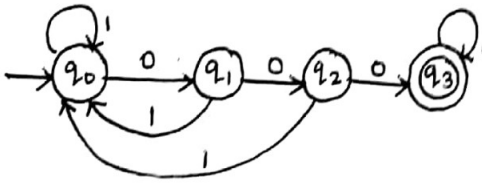
a)




Here $M = (\{q_0, q_1, q_2\}, \{0,1\}, \delta, q_0, \{q_2\})$

$\delta$: $\delta(q_0,0) = q_1$ | $\delta(q_1,0) = q_2$
$\delta(q_0,1) = q_0$ | $\delta(q_1,1) = q_0$

b)

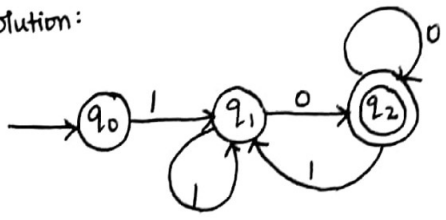



Here $M = (\{q_0, q_1, q_2, q_3\}, \{0,1\}, \delta, q_0, \{q_3\})$

$\delta$: $\delta(q_0,0) = q_1$ | $\delta(q_1,0) = q_2$ | $\delta(q_2,0) = q_3$ | $\delta(q_3,0) = q_3$
$\delta(q_0,1) = q_0$ | $\delta(q_1,1) = q_0$ | $\delta(q_2,1) = q_0$ | $\delta(q_3,1) = q_3$

3) Design FA which accepts only those strings which start with 1 and ends with 0.

Solution:

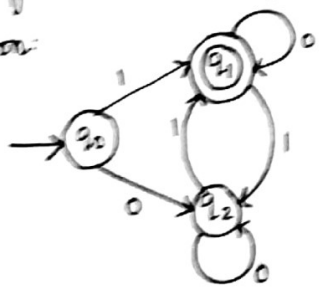



Here $M = (\{q_0, q_1, q_2\}, \{0,1\}, q_0, \delta, \{q_2\})$

$\delta$: $\delta(q_0,1) = q_1$ | $\delta(q_2,0) = q_2$
$\delta(q_1,0) = q_2$ | $\delta(q_2,1) = q_1$
$\delta(q_1,1) = q_1$ |

4) Design FA which accepts odd number of 1's and any number of 0's.

Solution:




Here $M = (\{q_0, q_1, q_2\}, \{0,1\}, q_0, \delta, \{q_1\})$

$\delta$: $\delta(q_0,0) = q_1$ | $\delta(q_1,0) = q_1$ | $\delta(q_2,0) = q_2$
$\delta(q_0,1) = q_2$ | $\delta(q_1,1) = q_2$ | $\delta(q_2,1) = q_1$

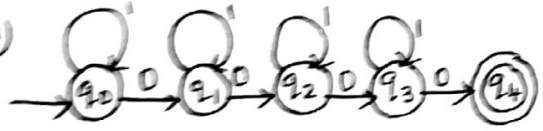
Testing: 10101 $\Rightarrow$ 1 0 1 0 1

$$\begin{aligned}
\delta(q_0, 10101) &= \delta(\delta(q_0,1), 0101) \\
&= \delta(q_1, 0101) \\
&= \delta(\delta(q_1,0), 101) \\
&= \delta(q_1, 101) \\
&= \delta(\delta(q_1,1), 01) \\
&= \delta(q_2, 01) \\
&= \delta(\delta(q_2,0), 1) \\
&= \delta(q_2, 1) \\
&= q_1 \text{ Accepting state.}
\end{aligned}$$

5) Design DFA with a) All string that contain exactly 4 zeros.
b) All string that doesn't contain the substring 110.

Solution:

a)

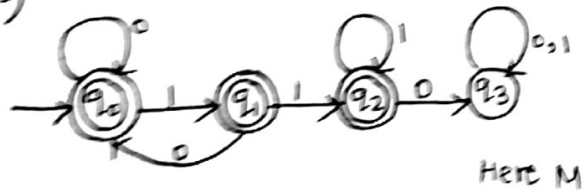



Here $M = (\{q_0, q_1, q_2, q_3, q_4\}, \{0,1\}, q_0, \delta, \{q_4\})$

$\delta$: $\delta(q_0,0) = q_1$ | $\delta(q_1,0) = q_2$ | $\delta(q_2,0) = q_3$
$\delta(q_0,1) = q_0$ | $\delta(q_1,1) = q_1$ | $\delta(q_2,1) = q_2$

$\delta(q_3,0) = q_4$, $\delta(q_3,1) = q_3$

b)

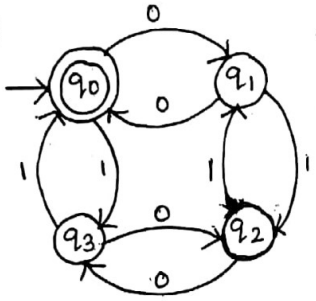



Here $M = (\{q_0, q_1, q_2, q_3\}, \{0,1\}, q_0, \delta, \{q_0, q_1, q_2\})$

$\delta$: $\delta(q_0,0) = q_0$ | $\delta(q_1,0) = q_0$ | $\delta(q_2,0) = q_3$ | $\delta(q_3,0) = q_3$
$\delta(q_0,1) = q_1$ | $\delta(q_1,1) = q_2$ | $\delta(q_2,1) = q_2$ | $\delta(q_3,1) = q_3$

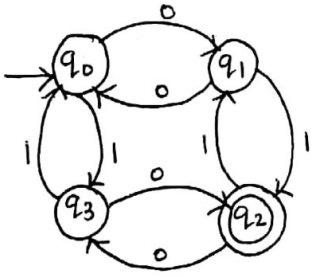
b) Design FA which accepts i) even number of 0's and even number of 1's.
ii) odd number of 0's and odd number of 1's.
iii) even number of 0's and odd number of 1's.
iv) odd number of 0's and even number of 1's.

(i)



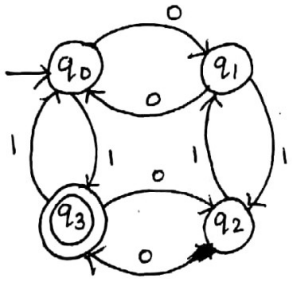

$\delta(q_0, 0011) = \delta(\delta(q_0,0), 011)$
$= \delta(q_1, 011)$
$= \delta(\delta(q_1,0), 11)$
$= \delta(q_0, 11)$
$= \delta(\delta(q_0,1), 1)$
$= \delta(q_3, 1) = q_0$ Accepted.

(ii)



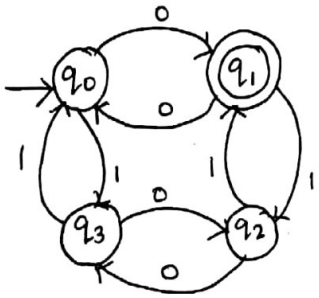

$\delta(q_0, 101010) = \delta(\delta(q_0,1), 01010)$
$= \delta(q_3, 01010)$
$= \delta(\delta(q_3,0), 1010)$
$= \delta(q_2, 1010)$
$= \delta(\delta(q_2,1), 010)$
$= \delta(q_1, 010)$
$= \delta(\delta(q_1,0), 10)$
$= \delta(q_0, 10) = \delta(\delta(q_0,1), 0) = \delta(q_3, 0)$
$= q_2$ Accepted.

(iii)




$\delta(q_0, 010) = \delta(\delta(q_0,0), 10)$
$= \delta(q_1, 10)$
$= \delta(\delta(q_1,1), 0)$
$= \delta(q_2, 0) = q_3$ Accepted.

(iv)




$\delta(q_0, 101) = \delta(\delta(q_0,1), 01)$
$= \delta(\delta(q_3,0), 1)$
$= \delta(q_2, 1)$
$= q_1$ Accepted.

7) Design FA for the following

a) Number of 1's in W is 3 mod 4

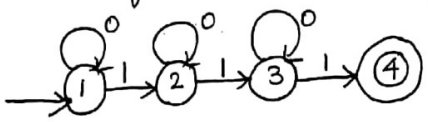
b) $L = \{0^n \mid n \bmod 3 = 2\}$

Solution:

a) Consider W = 0100 1 0001

No of 1's = 3 (ie) 3 mod 4 = 3

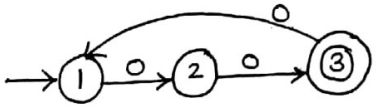
The no. of 1's should be exactly 3.




b) $L = \{0^n \mid n \bmod 3 = 2\}$

Here 1 mod 3 = 1
2 mod 3 = 2 → Accepted
3 mod 3 = 0
5 mod 3 = 2 → Accepted

$L = \{00, 0000, 00000000, \ldots\}$



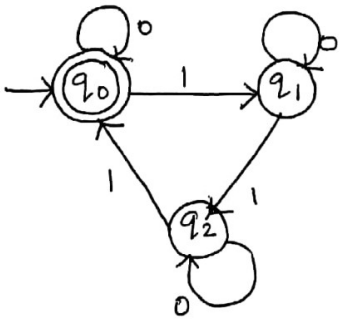

8) Construct DFA for the following

a) Number of 1's is a multiple of 3.

b) Number of 1's is not a multiple of 3.

Solution

a)




$\delta(q_0, 1110) = \delta(\delta(q_0, 1), 110)$

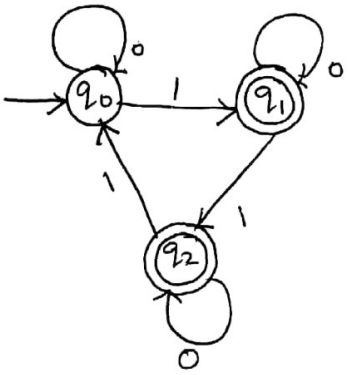
$= \delta(\delta(q_1, 1), 10)$

$= \delta(\delta(q_2, 1), 0)$

$= \delta(q_0, 0)$

$= q_0$ Accepted $\Rightarrow$ No of 1's is multiple of 3

b)




$\delta(q_0, 11110) = \delta(\delta(q_0, 1), 1110)$

$= \delta(\delta(q_1, 1), 110)$

$= \delta(\delta(q_2, 1), 10)$

$= \delta(\delta(q_0, 1), 0)$

$= \delta(q_1, 0)$

$= q_1$ Accepted.

Here no. of 1's is not a multiple of 3.

## NFA (Non-Deterministic Finite Automata)

The finite Automata is called NFA when there exists many paths for a specific input from current state to next state.

A non-deterministic finite Automata is defined by,

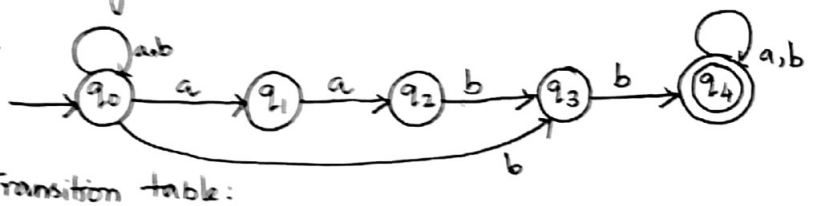
$$M = (Q, \Sigma, \delta, q_0, F)$$

Where

$Q$ - Finite set of internal states
$\Sigma$ - Finite set of I/p symbols
$q_0$ - Start state, $q_0 \in Q$
$\delta$ - Transition function
$F$ - Finite set of final states $F \subseteq Q$.

**Example:** Draw state transition diagram for FA over $\{a,b\}$ containing substring aabb.

Solution:




Transition table:

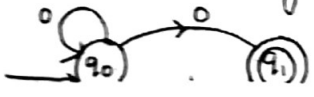
| State \ I/p | 0 | 1 |
|---|---|---|
| → $q_0$ | $\{q_0, q_1\}$ | $\{q_0, q_3\}$ |
| $q_1$ | – | $q_2$ |
| $q_2$ | $q_2$ | $q_2$ |
| $q_3$ | $q_4$ | – |
| ← $q_4$ | $q_4$ | $q_4$ |

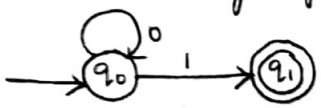
Start State → $q_0$

Final State ← $q_4$

## DFA Vs NFA

| | DFA | NFA |
|---|---|---|
| 1. | Every i/p string leads to the unique state of finite automata. | For some i/p there can be more than one next states. |
| 2. | Requires more memory for storing the state information. | Requires more computation time. |
| 3. | It is not possible to move to next state without reading any symbol. | Possible to move to next state without reading any symbol. |



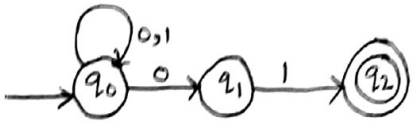
## Extended Transition function: $\hat{\delta}$

This is to represent transition functions with a string of input symbols 'w' and returns a set of states.

Suppose $w = xa$,

$$\hat{\delta}(q, xa) = \hat{\delta}(\delta(q, x), a)$$

Example:



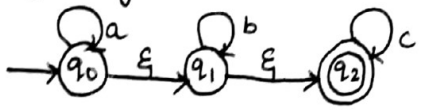

Process the input 001

$$\begin{aligned}
\hat{\delta}(q_0, 001) &= \delta(\hat{\delta}(q_0, 00), 1) \\
&= \delta(\delta(\hat{\delta}(q_0, 0), 0), 1) \\
&= \delta(\delta(\{q_0, q_1\}, 0), 1) \\
&= \delta((\delta(q_0, 0) \cup \delta(q_1, 0)), 1) \\
&= \delta(\{q_0, q_1\} \cup \phi), 1) \\
&= \delta(\{q_0, q_1\}, 1) \\
&= \delta(q_0, 1) \cup \delta(q_1, 1) \\
&= \{q_0\} \cup \{q_2\} \\
&= \{q_0, q_2\}
\end{aligned}$$

## Finite Automaton with ε-moves

In the non-deterministic finite state machine, the transition that does not require input symbols for the state transition and is capable of transiting to zero or more states with ε is called epsilon transition.

Example: Find ε-closure for the following NFA.




ε-closure($q_0$) = $\{q_0, q_1, q_2\}$

ε-closure($q_1$) = $\{q_1, q_2\}$

ε-closure($q_2$) = $\{q_2\}$

### ε-closure

The ε-closure(p) is a set of all states which are reachable from state p on ε-moves such that

(i) ε-closure(p) = p where $p \in Q$

(ii) If there exists ε-closure(p) = $\{q\}$ and $\delta(q, \varepsilon) = r$ then ε-closure(p) = $\{q, r\}$.

## Equivalence of NFA and DFA

**Theorem:** Let L be a set accepted by NFA, then there exists a DFA that accepts L

**Proof:**

This can be proved by subset construction method

Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$

$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

The states of D are all the subset of the set states of N.

$$Q_D = 2^{Q_N}$$

An element of $Q_D$ will be denoted by $[q_1, q_2, \ldots q_i]$

where $q_1, q_2, \ldots q_i \in Q$

Define $\delta_D([q_1, q_2, \ldots q_i], a) = [P_1, P_2, \ldots P_j]$

if and only if

$\delta_N([q_1, q_2, \ldots q_i], a) = \{P_1, P_2, \ldots P_j\}$

$\delta_D$ is computed by applying $\delta_N$ to each states of $\delta_D$ represented by $[q_1, q_2, \ldots q_i]$ on applying to each of $q_1, q_2, \ldots q_i$ and taking the union.

we get, $[P_1, P_2, \ldots P_j]$ in $Q_D$

To show $\delta_D(\{q_0\}, x) = [q_1, q_2, \ldots q_j]$ if and only if

$\delta_N(q_0, x) = \{q_1, q_2, \ldots q_i\}$

This is proved by the method of induction.

$F_D$ is the set of subset 'S' of $Q_N$ such that $S_n \cap F_n \neq \phi$.

**Problem:** Construct DFA equivalent to the NFA $M = (\{a, b, c, d\}, \{0, 1\}, \delta, a, \{b, d\})$ where $\delta$ is defined in the following table.

| $\delta$ | 0 | 1 |
|---|---|---|
| a | {b, d} | b |
| b | c | {b, c} |
| c | d | a |
| d | $\phi$ | a |

Solution:

(i) Find the states of $Q_N = \{a, b, c, d\}$

(ii) Construct subsets $Q_D = 2^{Q_N} = 2^4 = 16$

$$Q_D = \{ [a], [b], [c], [d], [a,b], [a,c], [a,d], [b,c], [b,d], [c,d], [a,b,c], [a,c,d], [b,c,d], [a,b,c,d], \phi \}$$

(iii) Find $\delta_D$ for all states

$\delta_D([a],0) = [b,d]$
$\delta_D([a],1) = [b]$
$\delta_D([b],0) = [c]$
$\delta_D([b],1) = [b,c]$
$\delta_D([c],0) = [d]$
$\delta_D([c],1) = [a]$
$\delta_D([d],0) = \phi$
$\delta_D([d],1) = [a]$

$\delta_D([a,b],0) = [b,c,d]$
$\delta_D([a,b],1) = [b,c]$

$\delta_D([a,c],0) = [b,d]$
$\delta_D([a,c],1) = [a,b]$
$\delta_D([a,d],0) = [b,d]$
$\delta_D([a,d],1) = [a,b]$
$\delta_D([b,c],0) = [c,d]$
$\delta_D([b,c],1) = [a,b,c]$
$\delta_D([b,d],0) = [c]$
$\delta_D([b,d],1) = [a,b,c]$
$\delta_D([c,d],0) = [d]$
$\delta_D([c,d],1) = [a]$

$\delta_D([a,b,c],0) = [b,c,d]$
$\delta_D([a,b,c],1) = [a,b,c]$
$\delta_D([a,c,d],0) = [b,d]$
$\delta_D([a,c,d],1) = [a,b]$
$\delta_D([b,c,d],0) = [c,d]$
$\delta_D([b,c,d],1) = [a,b,c]$
$\delta_D([a,b,c,d],0) = [b,c,d]$
$\delta_D([a,b,c,d],1) = [a,b,c]$

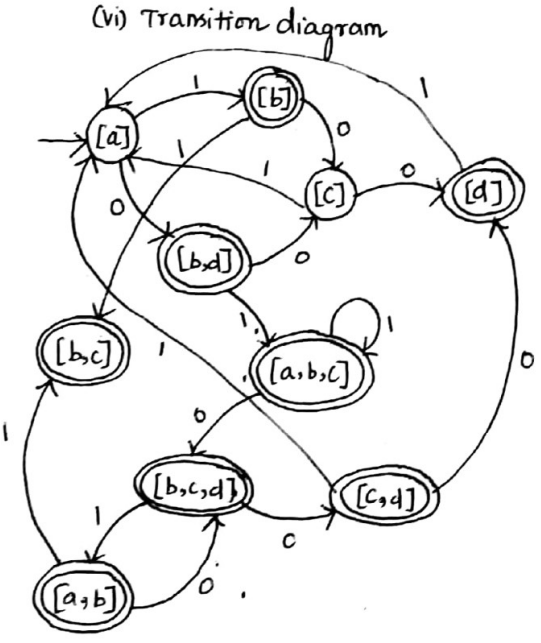
(iv) $F_N = \{b,d\}$ (Final state) $F_D \Rightarrow$ all set containing $\{b,d\}$

$$F_D = \{ [b], [d], [a,d], [b,d], [c,d], [a,b], [b,c], [a,b,c], [b,c,d], [a,b,d], [a,b,c,d] \}$$

(v) Transition table:

| | 0 | 1 |
|---|---|---|
| [a] | [b,d] | [b] |
| * [b] | [c] | [b,c] |
| [c] | [d] | [a] |
| * [d] | $\phi$ | [a] |
| * [a,b] | [b,c,d] | [b,c] |
| [a,c] | [b,d] | [a,b] |
| * [a,d] | [b,d] | [a,b] |
| * [b,c] | [c,d] | [a,b,c] |
| * [b,d] | [c] | [a,b,c] |
| * [c,d] | [d] | [a] |
| * [a,b,c] | [b,c,d] | [a,b,c] |
| * [a,c,d] | [b,d] | [a,b] |
| * [b,c,d] | [c,d] | [a,b,c] |
| * [a,b,c,d] | [b,c,d] | [a,b,c] |

(vi) Transition diagram



## Equivalence of NDFA's with and without $\epsilon$-moves

**Theorem:** If L is accepted by NFA with $\epsilon$-transitions, then there exist L which is accepted by NFA without $\epsilon$-transitions.

**Proof:**

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA with $\epsilon$-transition

Construct M' which is NFA without $\epsilon$-transition.

$M' = (Q, \Sigma, \delta', q_0, F')$ where,

$$F' = \begin{cases} F \cup \{q_0\}, & \text{if } \epsilon\text{-closure}(q_0) \text{ contains a state of } F \\ F, & \text{otherwise.} \end{cases}$$

By induction: $\begin{cases} \delta' \text{ and } \hat{\delta} \text{ are same} \\ \delta \text{ and } \hat{\delta} \text{ are different} \end{cases}$

Let $x$ be any string $\delta'(q_0, x) = \hat{\delta}(q_0, x)$

This statement is not true if $x = \epsilon$ because,

$\delta'(q_0, \epsilon) = \{q_0\}$ and $\hat{\delta}(q_0, \epsilon) = \epsilon$-closure$(q_0)$.

**Basis:** $|x| = 1$, $x$ is a symbol whose value is $a$ $\delta'(q_0, a) = \hat{\delta}(q_0, a)$ [By definition of $\hat{\delta}$]

**Inductive:** Let $X = wa$ where $a$ is in $\Sigma$

$$\delta'(q_0, wa) = \delta'(\hat{\delta}(q_0, w), a)$$
$$= \delta'(P, a)$$

Now we must know that

$$\delta'(P, a) = \hat{\delta}(q_0, wa)$$

But

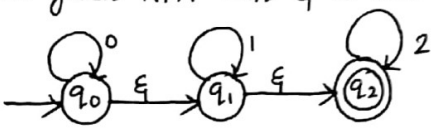
$$\delta'(P, a) = \bigcup_{q \text{ in } P} \delta'(q, a) = \bigcup_{q \text{ in } P} \hat{\delta}(q, a)$$

$$\therefore = \hat{\delta}(\hat{\delta}(q_0, w), a)$$
$$= \hat{\delta}(q_0, wa)$$
$$= \hat{\delta}(q_0, x)$$

Hence $\delta'(q_0, x) = \hat{\delta}(q_0, x)$

Hence proved.

**Problem:** Convert the given NFA with ε to NFA without ε.




**Solution:**

(i) Find ε-closure of all the states

$\varepsilon\text{-closure}(q_0) = \{q_0, q_1, q_2\}$

$\varepsilon\text{-closure}(q_1) = \{q_1, q_2\}$

$\varepsilon\text{-closure}(q_2) = \{q_2\}$

(ii) Find $\delta'$ transitions for each state on each input symbol.

$\delta'(q_0,0) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_0,\varepsilon),0))$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_0),0)$

$= \varepsilon\text{-closure}(\delta(q_0,q_1,q_2),0)$

$= \varepsilon\text{-closure}(q_0 \cup \phi \cup \phi)$

$= \varepsilon\text{-closure}(q_0)$

$\delta'(q_0,0) = \{q_0, q_1, q_2\}$

$\delta'(q_0,1) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_0,\varepsilon),1)$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_0),1)$

$= \varepsilon\text{-closure}(\delta(q_0,q_1,q_2),1)$

$= \varepsilon\text{-closure}(\phi \cup q_1 \cup \phi)$

$= \varepsilon\text{-closure}(q_1)$

$\delta'(q_0,1) = \{q_1, q_2\}$

$\delta'(q_1,0) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_1,\varepsilon),0))$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_1),0)$

$= \varepsilon\text{-closure}(\delta(q_1,q_2),0)$

$= \varepsilon\text{-closure}(\phi \cup \phi)$

$= \varepsilon\text{-closure}(\phi)$

$\delta'(q_1,0) = \phi$

$\delta'(q_1,1) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_1,\varepsilon),1)$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_1),1)$

$= \varepsilon\text{-closure}(\delta(q_1,q_2),1)$

$= \varepsilon\text{-closure}(q_1 \cup \phi)$

$= \varepsilon\text{-closure}(q_1)$

$\delta'(q_1,1) = \{q_1, q_2\}$

$\delta'(q_2,0) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_2,\varepsilon),0)$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_2),0)$

$= \varepsilon\text{-closure}(\delta(q_2,0))$

$= \varepsilon\text{-closure}(\phi)$

$\delta'(q_2,0) = \phi$

$\delta'(q_2,1) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_2,\varepsilon),1)$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_2),1)$

$= \varepsilon\text{-closure}(\delta(q_2,1))$

$= \varepsilon\text{-closure}(\phi)$

$\delta'(q_2,1) = \phi$

$\delta'(q_0,2) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_0,\varepsilon),2)$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_0),2))$

$= \varepsilon\text{-closure}(\delta(q_0,q_1,q_2),2)$

$= \varepsilon\text{-closure}(\phi \cup \phi \cup q_2)$

$= \varepsilon\text{-closure}(q_2)$

$\delta'(q_0,2) = \{q_2\}$

$\delta'(q_1,2) = \varepsilon\text{-closure}(\delta(\hat{\delta}(q_1,\varepsilon),2)$

$= \varepsilon\text{-closure}(\delta(\varepsilon\text{-closure}(q_1),2)$

$= \varepsilon\text{-closure}(\delta(q_1,q_2),2)$

$= \varepsilon\text{-closure}(\phi \cup q_2)$

$\delta'(q_1,2) = \{q_2\}$

$\delta'(q_2,2) = \epsilon\text{-closure}(\delta(\hat{\delta}(q_2,\epsilon),2))$

$= \epsilon\text{-closure}(\delta(\epsilon\text{-closure}(q_2),2)$

$= \epsilon\text{-closure}(\delta(q_2,2))$

$= \epsilon\text{-closure}(q_2)$

$\delta'(q_2,2) = \{q_2\}$

(iii) Summarize all the computed $\delta'$ transitions.

$\delta'(q_0,0) = \{q_0,q_1,q_2\}$ $\quad \delta'(q_1,2) = \{q_2\}$

$\delta'(q_0,1) = \{q_1,q_2\}$ $\quad \delta'(q_2,0) = \phi$

$\delta'(q_0,2) = \{q_2\}$ $\quad \delta'(q_2,1) = \phi$

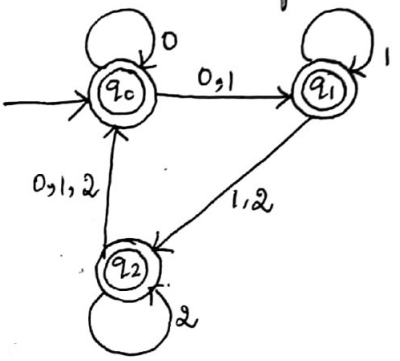
$\delta'(q_1,0) = \phi$ $\quad \delta(q_2,2) = \{q_2\}$.

$\delta'(q_1,1) = \{q_1,q_2\}$

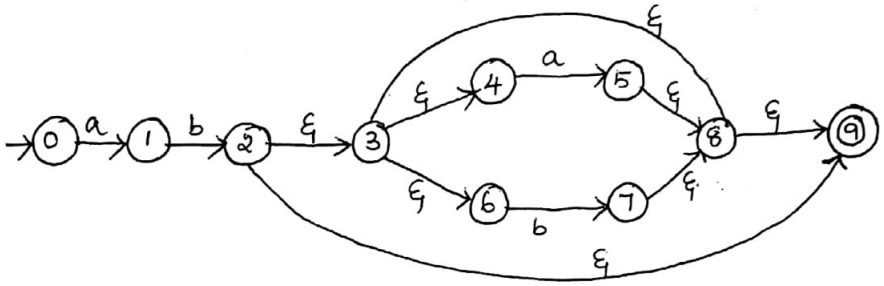
(iv) Transition table

| Input / State | 0 | 1 | 2 |
|---|---|---|---|
| $q_0$ | $\{q_0,q_1,q_2\}$ | $\{q_1,q_2\}$ | $\{q_2\}$ |
| $q_1$ | $\phi$ | $\{q_1,q_2\}$ | $\{q_2\}$ |
| $q_2$ | $\phi$ | $\phi$ | $\{q_2\}$ |

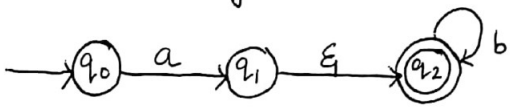
(v) Transition diagram.



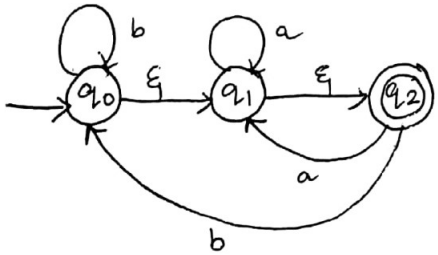

Problem2: Convert the following NFA with $\epsilon$ to NFA without $\epsilon$.




Problem3: Convert the following NFA with $\epsilon$ to NFA without $\epsilon$.




Problem4: Construct NFA with $\epsilon$ to ordinary NFA.



# Regular Language - Regular Expression

⇒ Regular languages are those languages which are accepted by finite automaton.

Ex: Set of strings of 0's and 1's $L = \{(0+1)^* \mid (0,1) \in \Sigma\}$

⇒ The languages accepted by finite automata are easily described by simple expression called regular expressions.

Properties: (closure)

1. $\phi$ is a regular expression which denotes the empty set.
2. $\varepsilon$ is a RE and denotes the set $\{\varepsilon\}$ and it is a null string.
3. For each 'a' in $\Sigma$ 'a' is a RE and denotes the set $\{a\}$
4. If r and s are regular expression denoting the language $L_1$ and $L_2$ respectively then

   $r+s$ is equivalent to $L_1 \cup L_2$ (union)

   $rs$ is equivalent to $L_1 L_2$ (concatenation)

   $r^*$ is equivalent to $L_1^*$ (closure)

Problem 1: Describe the following by regular expression.

a) $L_1$ = The set of all strings of 0's and 1's ending in 00

b) $L_2$ = The set of all strings of 0's and 1's beginning with 0 and ending with 1.

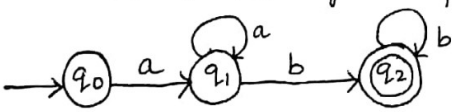
Solution

a) Regular Expression $L_1 = (0+1)^*00$

b) Regular expression $L_2 = 0(0+1)^*1$

Problem 2: Construct a DFA for the regular expression $aa^*bb^*$

Solution:




Problem 3: Find out the language generated by the regular expression $(0+1)^*$

Solution: $L = \{\varepsilon, 0, 1, 00, 11, 01, 10, 000, 111, \ldots\ldots\}$

$L = \{$Any number of 0's and 1's including null string$\}$

Problem 4: Give regular expression to describe an identifier and positive integer.

Solution: identifier = (a-z A-Z) (a-z A-Z 0-9)*

Positive integer = $(0-9)^+$

## Equivalence of finite Automaton and regular expressions

Every language defined by the regular expressions can be also be defined by the finite state automata.

**Theorem:** If $L = L(A)$ for some DFA, Then there is a regular expression R such that $L = L(R)$.

**Proof:** Let A be a DFA which defines the language L.

Let the states be $\{1, 2, \ldots n\}$

⇒ Construct RE of the form $R_{ij}^{(k)}$ – set of string w that takes the FA from state 'i' to state 'j' without going through any states labeled greater than k.

⇒ Let us prove this theorem by induction for $R_{ij}^{(k)}$.

**Basis:** If k=0, there will be no intermediate node between i and j

(i) 

An arc from node 'i' to 'j'

(ii) 

An arc from the node 'i' itself.

The transition may be (i) $R_{ij}^{(0)} = \phi$ (ii) $R_{ij}^{(0)} = a$ (iii) $R_{ij}^{(0)} = a_1 + a_2 + \cdots + a_l$

if $i = j$ then $R_{ij}^{(0)} = \varepsilon + a$

if $i \neq j$ then $R_{ij}^{(0)} = a_1 + a_2 + \cdots + a_l$

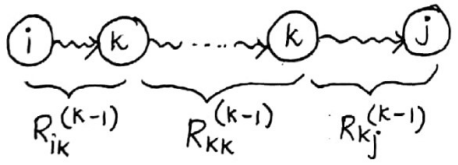
If there is no arc form i to j then, $R_{ij}^{(0)} = \phi$.

**Induction:** Let there is a path from state i to state j that goes through 'i' which is not greater than k.

Here it is assumed that we have already got $R_{ij}^{(k-1)}$

(i) Suppose if path does not goes through the state k, then $R_{ij}^{(k)} = R_{ij}^{(k-1)}$

(ii) If the path goes through the states k atleast one.




The set of lables for all path is represented by the RE

$$R_{ik}^{(k-1)} \left(R_{kk}^{(k-1)}\right)^* R_{kj}^{(k-1)}$$

By combining the above two path going through k and not going through k, we have

$$R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \left(R_{kk}^{(k-1)}\right)^* R_{kj}^{(k-1)}$$

∴ Regular expression $R_{ij}^{(k)}$ can be constructed for all values of i, j and k.

## ARDEN'S THEOREM: (FA → RE)

Let P and Q be two regular expressions over $\Sigma$. If P does not contain $\epsilon$, Then the equation in $R = Q + RP$ has a solution $R = QP^*$

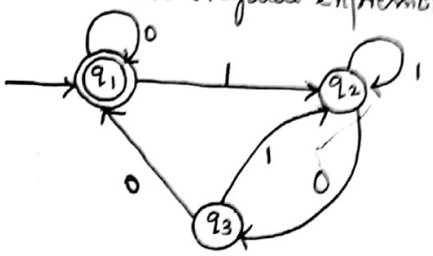
Using this theorem, it is easy to find the regular expression

The condition to apply this theorem are,

(i) Finite automata does not have $\epsilon$-moves.

(ii) It has only 1 start state.

**Example:** Construct the regular expression for the given finite automata




**Solution:** The transitions are defined by the equations,

$$q_1 = q_10 + q_30 + \epsilon \quad \text{——(1)}$$

$$q_2 = q_11 + q_21 + q_31 \quad \text{——(2)}$$

$$q_3 = q_20 \quad \text{——(3)}$$

Substitute (3) in (2)

$$q_2 = q_11 + q_21 + q_201$$

$$q_2 = q_11 + q_2(1+01)$$

$$q_2 = q_11 + (1+01)^* \quad \text{——(4) [By theorem]}$$

Substitute (4) in (1)

$$q_1 = q_10 + q_20.0 + \epsilon$$

$$= q_10 + q_11(1+01)^*0.0 + \epsilon$$

$$= q_10 + q_1(1+(1+01)^*00) + \epsilon$$

$$= q_1(0+1(1+01)^*00) + \epsilon$$

$$= \epsilon(0+1(1+01)^*00)^* \quad \text{[By theorem]}$$

$$q_1 = (0+1(1+01)^*00)^*$$

Since $q_1$ is the final state, the required regular expression is

$$= (0+1(1+01)^*00)^*$$

## THAMSON'S CONSTRUCTION (RE → FA)

Every language defined by a regular expression is also defined by a finite automaton.

Proof: To prove $L = L(E)$ for some $\xi$-NFA.

Basis: (Zero operator)

(i) Exactly one accepting state → $q_0$ $r = \xi$

(ii) No arcs into the initial state → $q_0$ $q_1$ $r = \phi$

(iii) No arcs out of the accepting state → $q_0$ —a→ $q_1$ $r = a$

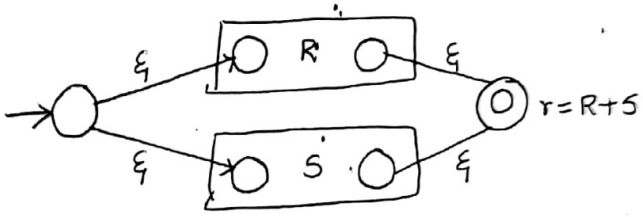
Induction: This theorem can be true for $n$ number of operators.

In any type of regular expression there are only three cases possible.

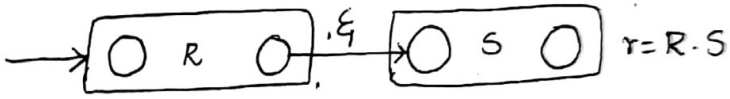
1. Union 2. Concatenation 3. Closure

Case(i) Union. $r = R + S$

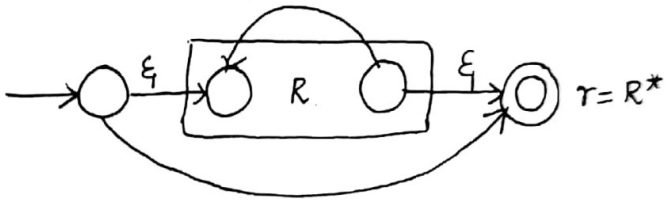
Let $r = R + S$, where $r_1$ and $r_2$ be the regular expressions.




Case(ii) $r = R \cdot S$ Concanetation

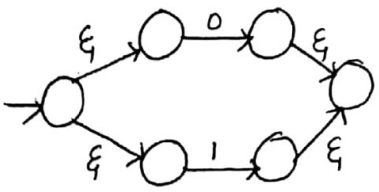


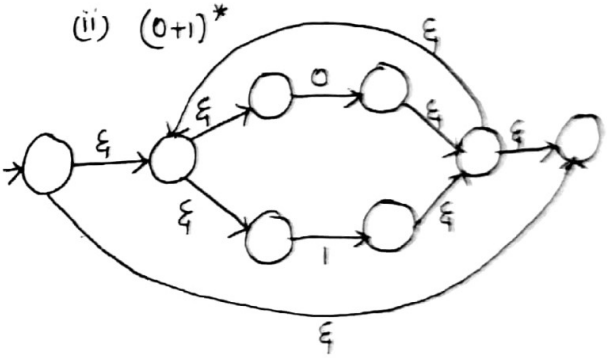

Case(iii) Closure



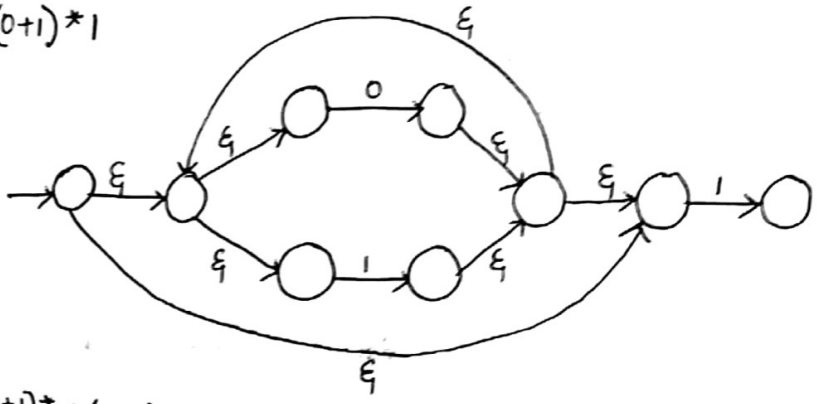

Problem 1: Convert the regular expression $(0+1)^*1(0+1)$ to an $\xi$-NFA.

Solution: (i) Find the RE for 0+1 (ii) $(0+1)^*$

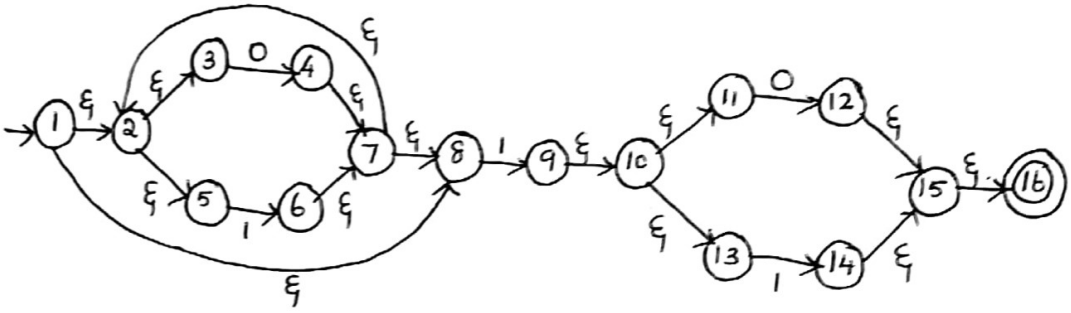


(iii) $(0+1)^*1$




(iv) $(0+1)^*1(0+1)$




Problem 2: Construct a NFA equivalent $(0+1)^*(00+11)$

Problem 3: Construct an NFA accepting L given by

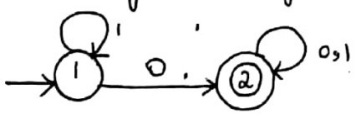
$L = \{ X \mid (a,b)^*\}$, $|X| \geq 3$ and the third symbol of $x$ from right is 'b'

$[RE = (a+b)^* b (a+b)(a+b)]$

Problem 4: Construct an NFA to accept the language indicated by the following regular expression $((01+001)^*0^*)^*$

Problems: FA → RE

1) Convert the following to a regular expression.




Solution: Find $R_{ij}^{(k)}$ for all the values of i, j, k.

Let k=0, $R_{ij}^{(0)} = \begin{cases} \xi + a_1 + a_2 + \dots + a_k & \text{if } i=j \\ a_1 + a_2 + \dots + a_r & \text{if } i \neq j \end{cases}$

$R_{11}^{(0)} = \xi + 1 \quad R_{12}^{(0)} = 0 \quad R_{21}^{(0)} = 0 \quad R_{22}^{(0)} = \xi + 0 + 1$

If k=1, Then $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \left(R_{kk}^{(k-1)}\right)^* R_{kj}^{(k-1)}$ ——— (1)

$R_{ij}^{(1)} = R_{ij}^{(0)} + R_{i1}^{(0)} \left(R_{11}^{(0)}\right)^* R_{1j}^{(0)}$ ——— (2)

Find $R_{11}^{(1)}, R_{12}^{(1)}, R_{21}^{(1)}, R_{22}^{(1)}$

(2) $\Rightarrow R_{11}^{(1)} = R_{11}^{(0)} + R_{11}^{(0)} \left(R_{11}^{(0)}\right)^* R_{11}^{(0)}$

$= (\xi+1) + (\xi+1)(\xi+1)^* (\xi+1)$

$= (\xi+1) + (\xi+1)\, 1^* (\xi+1)$

$= (\xi+1) + 1^*$

$\boxed{R_{11}^{(1)} = 1^*}$

(2) $\Rightarrow R_{21}^{(1)} = R_{21}^{(0)} + R_{21}^{(0)} \left(R_{11}^{(0)}\right)^* R_{11}^{(0)}$

$= \phi + \phi (\xi+1)^* (\xi+1)$

$= \phi + (\xi+1)^* (\xi+1)\, \phi$

$\boxed{R_{21}^{(1)} = \phi}$

(2) $\Rightarrow R_{12}^{(1)} = R_{12}^{(0)} + R_{11}^{(0)} \left(R_{11}^{(0)}\right)^* R_{12}^{(0)}$

$= 0 + (\xi+1)(\xi+1)^* 0$

$= 0 + (\xi+1)\, 1^* 0$

$= 0 + 1^* 0$

$\boxed{R_{12}^{(1)} = 1^* 0}$

(2) $\Rightarrow R_{22}^{(1)} = R_{22}^{(0)} + R_{21}^{(0)} \left(R_{11}^{(0)}\right)^* R_{12}^{(0)}$

$= (\xi+0+1) + \phi\, (\xi+1)^* 0$

$= (\xi+0+1) + \phi$

$\boxed{R_{22}^{(1)} = \xi + 0 + 1}$

Let k=2

$R_{ij}^{(2)} = R_{ij}^{(1)} + R_{i2}^{(1)} \left(R_{22}^{(1)}\right)^* R_{2j}^{(1)}$ ——— (3)

Find $R_{11}^{(2)}, R_{12}^{(2)}, R_{21}^{(2)}, R_{22}^{(2)}$

③ $\Rightarrow R_{11}^{(2)} = R_{11}^{(1)} + R_{12}^{(1)} \left(R_{22}^{(1)}\right)^* R_{21}^{(1)}$

$= 1^* + 1^* 0 \, (\xi+0+1)^* \phi$

$= 1^* + \phi$

$\boxed{R_{11}^{(2)} = 1^*}$

③ $\Rightarrow R_{21}^{(2)} = R_{21}^{(1)} + R_{22}^{(1)} \left(R_{22}^{(1)}\right)^* R_{21}^{(1)}$

$= \phi + (\xi+0+1) \, (\xi+0+1)^* \phi$

$= \phi + \phi$

$\boxed{R_{21}^{(2)} = \phi}$

③ $\Rightarrow R_{12}^{(2)} = R_{12}^{(1)} + R_{12}^{(1)} \left(R_{22}^{(1)}\right)^* R_{22}^{(1)}$

$= 1^* 0 + 1^* 0 \, (\xi+0+1)^* (\xi+0+1)$

$= 1^* 0 + 1^* 0 \, (0+1)^* (\xi+0+1)$

$= 1^* 0 + 1^* 0 \, (0+1)^*$

$\boxed{R_{12}^{(2)} = 1^* 0 \, (0+1)^*}$

③ $\Rightarrow R_{22}^{(2)} = R_{22}^{(1)} + R_{22}^{(1)} \left(R_{22}^{(1)}\right)^* R_{22}^{(1)}$

$= (\xi+0+1) + (\xi+0+1) \, (\xi+0+1) \, (\xi+0+1)$

$= (\xi+0+1) + (\xi+0+1) \, (0+1)^*$

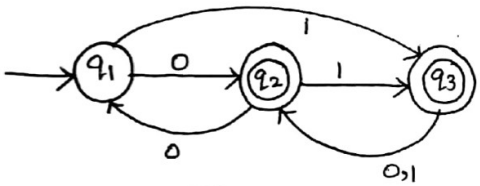
$= (\xi+0+1) + (0+1)^*$

$\boxed{R_{22}^{(2)} = (0+1)^*}$

Regular expression for the accepting state

$\boxed{R_{12}^{(2)} = 1^* 0 \, (0+1)^*}$

**Problem 2:** Convert the following to a regular expression.




**Solution:** To find $R_{12}^{(3)} + R_{13}^{(3)}$

$$R_{ij}^{(0)} = \begin{cases} \xi + a_1 + a_2 + \cdots + a_k & \text{if } i = j \\ a_1 + a_2 + \cdots + a_r & \text{if } i \neq j \end{cases}$$

**k=0**

$R_{11}^{(0)} = \xi$ $\quad R_{21}^{(0)} = 0$ $\quad R_{31}^{(0)} = \phi$

$R_{12}^{(0)} = 0$ $\quad R_{22}^{(0)} = \xi$ $\quad R_{32}^{(0)} = 0+1$

$R_{13}^{(0)} = 1$ $\quad R_{23}^{(0)} = 1$ $\quad R_{33}^{(0)} = \xi$

**k=1**

$$R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} \left(R_{kk}^{(k-1)}\right)^* R_{kj}^{(k-1)}$$

$$R_{ij}^{(1)} = R_{ij}^{(0)} + R_{i1}^{(0)} \left(R_{11}^{(0)}\right)^* R_{1j}^{(0)}$$

$R_{11}^{(1)} = R_{11}^{(0)} + R_{11}^{(0)}\left(R_{11}^{(0)}\right)^* R_{11}^{(0)}$

$= \varepsilon + \varepsilon(\varepsilon)^* \varepsilon$

$= \varepsilon + \varepsilon$

$\boxed{R_{11}^{(1)} = \varepsilon}$

$R_{12}^{(1)} = R_{12}^{(0)} + R_{11}^{(0)}\left(R_{11}^{(0)}\right)^* R_{12}^{(0)}$

$= 0 + \varepsilon(\varepsilon)^* 0$

$= 0 + 0$

$\boxed{R_{12}^{(1)} = 0}$

$R_{13}^{(1)} = R_{13}^{(0)} + R_{11}^{(0)}\left(R_{11}^{(0)}\right)^* R_{13}^{(0)}$

$= 1 + \varepsilon(\varepsilon)^* 1$

$\boxed{R_{13}^{(1)} = 1}$

$R_{21}^{(1)} = R_{21}^{(0)} + R_{21}^{(0)}\left(R_{11}^{(0)}\right)^* R_{11}^{(0)}$

$= 0 + 0(\varepsilon)^* \varepsilon$

$= 0 + 0 \cdot \varepsilon$

$= 0 + 0$

$\boxed{R_{21}^{(1)} = 0}$

$R_{22}^{(1)} = R_{22}^{(0)} + R_{21}^{(0)}\left(R_{11}^{(0)}\right)^* R_{12}^{(0)}$

$= \varepsilon + 0(\varepsilon)^* 0$

$\boxed{R_{22}^{(1)} = \varepsilon + 00}$

$R_{23}^{(1)} = R_{23}^{(0)} + R_{21}^{(0)}\left(R_{11}^{(0)}\right)^* R_{13}^{(0)}$

$= 1 + 0(\varepsilon)^* 0$

$\boxed{R_{23}^{(1)} = 1 + 01}$

$R_{31}^{(1)} = R_{31}^{(0)} + R_{31}^{(0)}\left(R_{11}^{(0)}\right)^* R_{11}^{(0)}$

$= \phi + \phi(\varepsilon)^* \varepsilon$

$\boxed{R_{31}^{(1)} = \phi}$

$R_{32}^{(1)} = R_{32}^{(0)} + R_{31}^{(0)}\left(R_{11}^{(0)}\right)^* R_{12}^{(0)}$

$= 0 + 1 + \phi(\varepsilon)^* 0$

$= 0 + 1 + \phi$

$\boxed{R_{32}^{(1)} = 0 + 1}$

$R_{33}^{(1)} = R_{33}^{(0)} + R_{31}^{(0)}\left(R_{11}^{(0)}\right)^* R_{13}^{(0)}$

$= \varepsilon + \phi(\varepsilon)^* 0$

$\boxed{R_{33}^{(1)} = \varepsilon}$

**k = 2** $\quad R_{ij}^{(2)} = R_{ij}^{(1)} + R_{i2}^{(1)}\left(R_{22}^{(1)}\right)^* R_{2j}^{(1)}$

$R_{11}^{(2)} = R_{11}^{(1)} + R_{12}^{(1)}\left(R_{22}^{(1)}\right)^* R_{21}^{(1)}$

$= \varepsilon + 0\ (\varepsilon + 00)^* 0$

$= \varepsilon + 0\ (00)^* 0$

$= \varepsilon + (00)^*$

$\boxed{R_{11}^{(2)} = (00)^*}$

$R_{12}^{(2)} = R_{12}^{(1)} + R_{12}^{(1)}\left(R_{22}^{(1)}\right)^* R_{22}^{(1)}$

$= 0 + 0(\varepsilon + 00)^* (\varepsilon + 00)$

$= 0 + 0\ (00)^* (\varepsilon + 00)$

$= 0 + 0\ (00)^*$

$\boxed{R_{12}^{(2)} = 0\ (00)^*}$

$R_{13}^{(2)} = R_{13}^{(1)} + R_{12}^{(1)}\left(R_{22}^{(1)}\right)^* R_{23}^{(1)}$

$= 1 + 0\ (\varepsilon + 00)^* (1 + 01)$

$= 1 + 0\ (00)^* (1 + 01)$

$= 1 + (00)^* (\varepsilon + 0)\,1 \quad [(00)^*(\varepsilon + 0) = 0^*]$

$= 1 + 0^* 1$

$\boxed{R_{13}^{(2)} = 0^* 1}$

$R_{21}^{(2)} = R_{21}^{(1)} + R_{22}^{(1)} \left(R_{22}^{(1)}\right)^* R_{21}^{(1)}$

$= 0 + (\varepsilon + 00)\ (\varepsilon + 00)^* 0$

$= 0 + (\varepsilon + 00)\ (00)^* 0$

$= 0 + (00)^* 0$

$\boxed{R_{21}^{(2)} = (00)^* 0}$

$R_{22}^{(2)} = R_{22}^{(1)} + R_{22}^{(1)} \left(R_{22}^{(1)}\right)^* R_{22}^{(1)}$

$= (\varepsilon + 00) + (\varepsilon + 00)\ (\varepsilon + 00)^* (\varepsilon + 00)$

$= (\varepsilon + 00) + (\varepsilon + 00)\ (00)^* (\varepsilon + 00)$

$= (\varepsilon + 00) + (00)^*$

$\boxed{R_{22}^{(2)} = (00)^*}$

$R_{23}^{(2)} = R_{23}^{(1)} + R_{22}^{(1)} \left(R_{22}^{(1)}\right)^* R_{23}^{(1)}$

$= (1 + 01) + (\varepsilon + 00)\ (\varepsilon + 00)^* (1 + 01)$

$= (1 + 01) + (\varepsilon + 00)\ (00)^* (1 + 01)$

$= (1 + 01) + (00)^* (1 + 01)$

$= (1 + 01) + (00)^* (\varepsilon + 0)1$

$= (1 + 01) + 0^* 1$

$= 1 + 0^* 1$

$\boxed{R_{23}^{(2)} = 0^* 1}$

$R_{31}^{(2)} = R_{31}^{(1)} + R_{32}^{(1)} \left(R_{22}^{(1)}\right)^* R_{21}^{(1)}$

$= \phi + (0 + 1)\ (\varepsilon + 00)^* 0$

$\boxed{R_{31}^{(2)} = (0 + 1)(00)^* 0}$

$R_{32}^{(2)} = R_{32}^{(1)} + R_{32}^{(1)} \left(R_{22}^{(1)}\right)^* R_{22}^{(1)}$

$= (0 + 1) + (0 + 1)\ (\varepsilon + 00)^* (\varepsilon + 00)$

$= (0 + 1) + (0 + 1)\ (00)^* (\varepsilon + 00)$

$= (0 + 1) + (0 + 1)\ (00)^*$

$\boxed{R_{32}^{(2)} = (0 + 1)\ (00)^*}$

$R_{33}^{(2)} = R_{33}^{(1)} + R_{32}^{(1)} \left(R_{22}^{(1)}\right)^* R_{23}^{(1)}$

$= \varepsilon + (0 + 1)\ (\varepsilon + 00)^* (1 + 01)$

$= \varepsilon + (0 + 1)\ (00)^* (1 + 01)$

$= \varepsilon + (0 + 1)\ (00)^* (\varepsilon + 0)1$

$\boxed{R_{33}^{(2)} = \varepsilon + (0 + 1)\ 0^* 1}$

Calculate $\gamma_{12}^{(3)} + \gamma_{13}^{(3)}$, to find regular expression

$\gamma_{12}^{(3)} = \gamma_{12}^{(2)} + \gamma_{13}^{(2)} \left(\gamma_{33}^{(2)}\right)^* \gamma_{32}^{(2)}$

$= 0\ (00)^* + 0^* 1\ (\varepsilon + (0 + 1) 0^* 1)^* (0 + 1)\ (00)^*$

$= 0\ (00)^* + 0^* 1\ ((0 + 1)\ 0^* 1)^*\ (0 + 1)\ (00)^*$

$\gamma_{13}^{(3)} = \gamma_{13}^{(2)} + \gamma_{13}^{(2)} \left(\gamma_{33}^{(2)}\right)^* \gamma_{33}^{(2)}$

$= 0^* 1 + 0^* 1\ (\varepsilon + (0 + 1)\ 0^* 1)^*\ (\varepsilon + (0 + 1) 0^* 1)$

$= 0^* 1\ ((0 + 1) 0^* 1)^*$

$\therefore \gamma_{12}^{(3)} + \gamma_{13}^{(3)} = 0^* 1\ ((0 + 1) 0^* 1 + (\varepsilon + (0 + 1)\ 00)^*) + 0\ (00)^*$

## Minimization of DFA.

The minimization of FSM means reducing the number of states from given FA. While minimizing FSM, first find out which two states are equivalent, we can represent those two states by one representative state.

**Pbm:** Give the minimized DFA for the following expression $(a/b)^* abb$

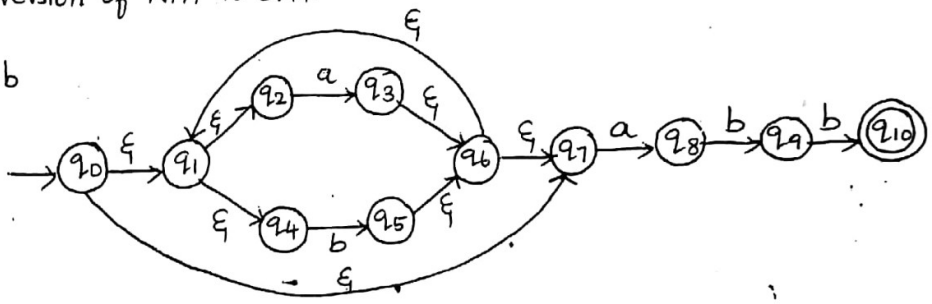
**Solution:** **Steps:**

(1) Design of NFA with ε for given regular expression

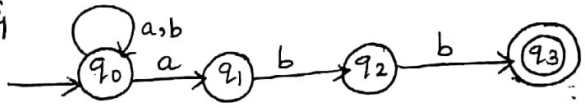
(2) Conversion of NFA with ε to without ε

(3) Conversion of NFA to DFA.

$(a/b)^* abb$




NFA without ε




Transition table:

| | a | b |
|---|---|---|
| $q_0$ | $\{q_0, q_1\}$ | $q_0$ |
| $q_1$ | $\phi$ | $q_2$ |
| $q_2$ | $\phi$ | $q_3$ |
| $q_3$ | $\phi$ | $\phi$ |

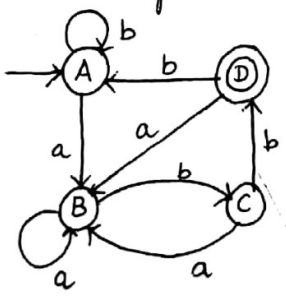
Equivalent DFA

| | | a | b |
|---|---|---|---|
| → | $q_0$ | $\{q_0, q_1\}$ | $q_0$ |
| | $q_1$ | $\phi$ | $q_2$ |
| | $q_2$ | $\phi$ | $q_3$ |
| * | $q_3$ | $\phi$ | $q$ |
| | $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
| | $\{q_0, q_2\}$ | $\{q_0, q_1\}$ | $\{q_0, q_3\}$ |
| * | $\{q_0, q_3\}$ | $\{q_0, q_1\}$ | $q_0$ |

Assume $q_0 = A$, $\{q_0, q_1\} = B$

$\{q_0, q_2\} = C$, $\{q_0, q_3\} = D$

Transition diagram,




Transition table,

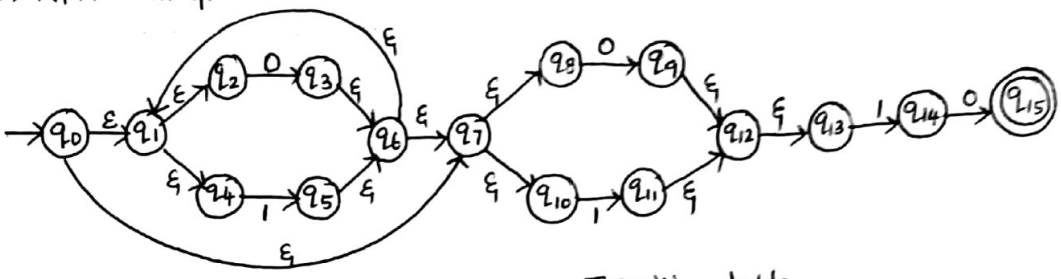
| I/P / States | a | b |
|---|---|---|
| A | B | A |
| B | B | C |
| C | A | D |
| D | B | A |

Problem 2: Construct the minimized DFA for the regular expression

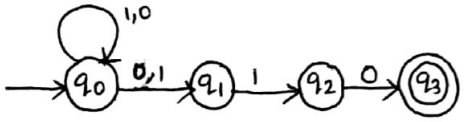
$(0+1)^*(0+1)10$

Solution:

(i) NFA with ε.




(ii) NFA without ε




Transition table

| S \ I/P | 0 | 1 |
|---|---|---|
| $q_0$ | $\{q_0,q_1\}$ | $\{q_0,q_1\}$ |
| $q_1$ | $\phi$ | $q_2$ |
| $q_2$ | $q_3$ | $\phi$ |
| $q_3$ | $\phi$ | $\phi$ |

Equivalent DFA

| State \ I/P | 0 | 1 |
|---|---|---|
| → $q_0$ | $[q_0,q_1]$ | $[q_0,q_1]$ |
| $q_1$ | $\phi$ | $[q_2]$ |
| $q_2$ | $[q_3]$ | $\phi$ |
| $q_3$ | $\phi$ | $\phi$ |
| $[q_0,q_1]$ | $[q_0,q_1]$ | $[q_0,q_1,q_2]$ |
| $[q_0,q_1,q_2]$ | $[q_0,q_1,q_3]$ | $[q_0,q_1,q_2]$ |
| * $[q_0,q_1,q_3]$ | $[q_0,q_1]$ | $[q_0,q_1,q_2]$ |

Assume

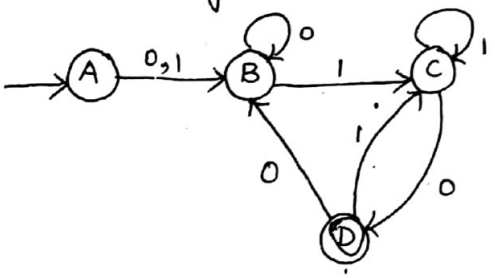
$q_0 = A$

$[q_0,q_1] = B$

$[q_0,q_1,q_2] = C$

$[q_0,q_1,q_3] = D$

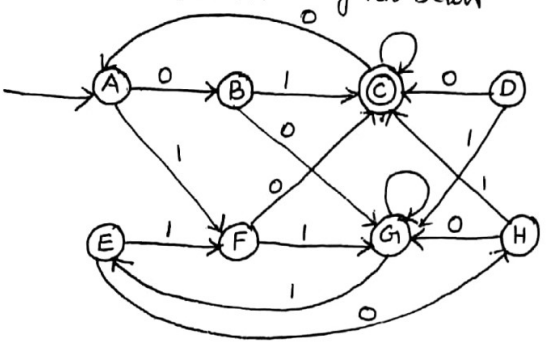
Transition Diagram (Min-DFA)




Transition table:

| | 0 | 1 |
|---|---|---|
| → A | B | B |
| B | B | C |
| C | D | C |
| D | B | C |

Problem: 3 Minimize the DFA as given below




Solution:

Transition table

| | 0 | 1 |
|---|---|---|
| → A | B | F |
| B | G | C |
| (C) | A | C |
| D | C | G |
| E | H | F |
| F | C | G |
| G | G | E |
| H | G | C |

Find the equivalent state

Consider pair (B, H)

$\delta(B,0) = G$ | $\delta(B,1) = C$

$\delta(H,0) = G$ | $\delta(H,1) = C$

∴ (B, H) are equivalent.

Since (B, H) are equivalent, consider the pair (A, E)

$\delta(A,0) = B$ (or) $H$ | $\delta(A,1) = F$

$\delta(E,0) = H$ (or) $B$ | $\delta(E,1) = F$

∴ (A, E) are equivalent

Consider the pair (D, F)

$\delta(D,0) = C$ | $\delta(D,1) = G$ ∴ (D, F) are equivalent.

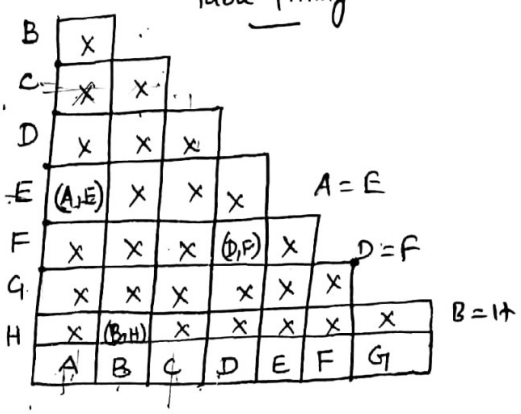
$\delta(F,0) = C$ | $\delta(F,1) = G$

∴ The equivalent pairs are (A, E), (B, H) and (D, F)

∴ The minimized DFA will be

| | 0 | 1 |
|---|---|---|
| → A | B | D |
| B | G | C |
| (C) | A | C |
| D | C | G |
| G | G | A |

Table filling



**Problem 1:** Define distinguishable and indistinguishable States. Using table filling method, minimize the following DFA. Draw the transition diagram for resulting DFA.

| | 0 | 1 |
|---|---|---|
| → A | B | E |
| B | C | F |
| * C | D | H |
| D | E | H |
| E | F | I |
| * F | G | B |
| G | H | B |
| H | I | C |
| * I | A | E |

**Distinguishable States**

↳ If for some input string w, $\hat{\delta}(p,w)$ gives an accepting state and $\hat{\delta}(q,w)$ gives a non-accepting state then states, p and q are called distinguishable or non-equivalent state.

**Indistinguishable states**

↳ If for some input string w, $\hat{\delta}(p,w)$ and $\hat{\delta}(q,w)$ both produces either accepting or non accepting states.

Solution:

Here Final states are C, F, I

Other states are A, B, D, E, G, H

Since, C, F, I are final state

Consider the pair (E, H)

$\delta(E,0) = (F)$ | $\delta(E,1) = (I)$

$\delta(H,0) = (I)$ | $\delta(H,1) = (C)$

∴ (E, H) is equivalent state

Consider the pair (B, E)

$\delta(B,0) = (C)$ | $\delta(B,1) = (F)$

$\delta(E,0) = (F)$ | $\delta(E,1) = (I)$

∴ (B, E) is equivalent state.

Consider the pair (B, H)

$\delta(B,0) = (C)$ | $\delta(B,1) = (F)$

$\delta(H,0) = (I)$ | $\delta(H,1) = (C)$

∴ (B, H) is equivalent state.

Since (B, H) and (B, E) is equivalent,

Consider the pair (A, G)

$\delta(A,0) = B$ or $H$ | $\delta(A,1) = E$ or $B$

$\delta(G,0) = H$ or $B$ | $\delta(G,1) = B$ or $E$

∴ (A, G) is equivalent state.

Since (B,E) and (E,H) are equivalent

Consider the pair (A,D)

$\delta(A,0) = B$ or $E$ | $\delta(A,1) = E$ or $H$

$\delta(D,0) = E$ or $B$ | $\delta(D,1) = H$ or $E$

∴ (A,D) is equivalent state

Since (E,H) and (B,H) are equivalent

Consider the pair (D,G)

$\delta(D,0) = E$ or $H$ | $\delta(D,1) = H$ or $B$

$\delta(G,0) = H$ or $E$ | $\delta(G,1) = B$ or $H$

∴ (D,G) are equivalent.

Since (D,G) and (B,H) are equivalent

Consider the pair (C,F)

$\delta(C,0) = D$ or $G$ | $\delta(C,1) = H$ or $B$

$\delta(F,0) = G$ or $D$ | $\delta(F,1) = B$ or $H$

∴ (C,F) are equivalent.

Since (A,G) and (B,E) are equivalent

Consider the pair (F,I)

$\delta(F,0) = G$ or $A$ | $\delta(F,1) = B$ or $E$

$\delta(I,0) = A$ or $G$ | $\delta(I,1) = E$ or $B$

∴ (F,I) is equivalent state.

Since (A,D) and (E,H) are equivalent

Consider the pair (C,I)

$\delta(C,0) = D$ or $A$ | $\delta(C,1) = H$ or $E$

$\delta(I,0) = A$ or $D$ | $\delta(I,1) = E$ or $H$

∴ (C,I) is equivalent.

Table filling:

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| B | X | | | | | | | |
| C | X | X | | | | | | |
| D | (A,D) | X | X | | | | | |
| E | X | (B,E) | X | X | | | | |
| F | X | X | (C,F) | X | X | | | |
| G | X | X | X | (D,G) | X | X | | |
| H | X | (B,H) | X | X | (E,H) | X | X | |
| I | X | X | (C,I) | X | X | (F,I) | X | X |

Here

State A = G = D

State B = H = E

State C = I = F

Minimized DFA is

| State \ I/P | 0 | 1 |
|---|---|---|
| A | B | B |
| B | C | C |
| ⓒ | A | B |

## Pumping lemma for Regular sets

⇒ Used for checking whether given string is accepted by regular expression or not.

⇒ Whether given language is regular or not.

**Theorem:** Let L be a regular set. Then there is a constant n such that if z is any word in L and $|z| \geq n$ we can write $z = uvw$ such that $|uv| \leq n$, $|v| \geq 1$ for all $i \geq 0$, $uv^iw$ is in L. The n should not be greater than the number of states.

**Proof:** If the language L is regular it is accepted by a DFA.

$M = (Q, \Sigma, \delta, q_0, F)$ with some particular number of states say n

Consider the input can be $a_1, a_2, \ldots a_m$, $m \geq n$. The mapping function $\delta$ could be written as $\delta(q_0, q_1, \ldots q_i) = q_i$.

If $q_m$ is in F (i.e) $q_1, q_2, \ldots q_m$ is in $L(M)$ then $a_1, a_2 \ldots a_j\ a_{k+1}\ a_{k+2} \ldots a_m$ is also in $L(M)$.

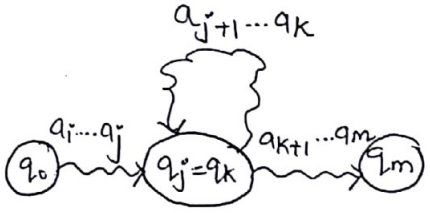



Fig: Pumping lemma

Since there is path from $q_0$ to $q_m$ that goes through $q_j$ but not around the loop labelled $a_{j+1} \ldots a_k$. Thus.

$$\begin{aligned}\delta(q_0, a_1, a_j\ a_{k+1} \ldots a_m) &= \delta(\delta(q_0, q_1, \ldots q_j), a_{k+1}, \ldots a_m) \\ &= \delta(q_j, q_{k+1} \ldots q_m) \\ &= \delta(q_k, q_{k+1} \ldots q_m) \\ &= q_m\end{aligned}$$

Thus $a_1 \ldots a_j (a_{j+1} \ldots a_k)^i a_{k+1} \ldots a_m$ is in $L(M)$ for any $i \geq 0$

**Problem 1:** Show that $L = \{0^{n^2} \mid n \geq 1\}$ is not regular.

**Solution**

(i) Assume $L$ is regular.

(ii) No. of states $= n^2 > n$

(iii) Let $z = 0^{n^2}$, consider $n^2 = r$, $z = 0^r$

(iv) According to the pumping lemma,

$z$ can be written as,

$z = uvw$ such that $|uv| \leq n$, $|v| \geq 1$

$\therefore z = 0^r$ can be expressed as,

Assume, $uv = 0^m$, where $m < r$

$v = 0^q$, where $q < m$

and $w = 0^{r-m}$

(v) check the condition.

(a) $|uv| \leq n$, $|0^m| \leq n$, $m \leq n$

(b) $|v| \geq 1$, $|0^q| \geq 1$, $q \geq 1$

(c) for all $i \geq 0$, the string $uv^iw$ is in $L$

$$uv^iw = uv^{i-1}(v)w$$
$$= (uv)(v)^{i-1}w$$
$$= 0^m (0^q)^{i-1} 0^{r-m}$$
$$= 0^{m+q(i-1)+r-m}$$
$$\boxed{uv^iw = 0^{q(i-1)+r}}$$

For $i = 0$,

$$uv^0w = uw = 0^{r-q} \neq 0^r \notin L$$

$i = 1$, $uvw = 0^r \in L$

$i = 2$ $uv^2z = 0^{q+r} \notin L$

Since for $i = 0, 2$, we have the string that does not belongs to the language $L$.

$\therefore L = \{0^{n^2} \mid n \geq 1\}$ is not regular.

Problem 2: Show that $L = \{a^n b^n \mid n \geq 1\}$ is not regular.

Solution:

i) Assume L is regular.

ii) No. of. states $L = \{a^n b^n \mid n \geq 1\}$ (ie) $n+n = 2n > n$

iii) Let $z = a^n b^n$

According to the pumping lemma, z can be written as,

$z = uvw$ such that $|uv| \leq n$, $|v| \geq 1$

iv) Consider $z = a^n b^n = a^m a^{n-m} b^n$

$z = a^m a^{n-m} b^n$ can be expressed as,

$$\boxed{uv = a^m,\ v = a^j \quad w = a^{n-m} b^n}$$

v) Check the conditions,

(a) $|uv| \leq n \Rightarrow |a^m| \leq n,\ m \leq n$

(b) $|v| \geq 1 \Rightarrow |a^j| \geq 1,\ j \geq 1$

(c) for all $i \geq 0$ the string $uv^i w$ is in L.

$$uv^i w = u\, v^{i-1} (v)\, w$$
$$= (uv)\, (v)^{i-1} w$$
$$= a^m\, (a^j)^{i-1}\, a^{n-m} b^n$$

$$\boxed{uv^i w = a^{n+j(i-1)} b^n}$$

for $i = 0$, $uw = a^{n-j} b^i \notin L$

$i = 1$, $uvw = a^n b^n \in L$

$i = 2$, $uv^2 w = a^{n+j} b^n \notin L$

Since for $i = 0, 2$, we have the string that does not belongs to the language L.

$\therefore L = \{a^n b^n \mid n \geq 1\}$ is not regular.

Problem 3: Show that $L = \{a^m b^n a^{m+n} \mid m \geq 1, n \geq 1\}$ is not regular.

Solution:

i) Assume L is regular.

ii) Let $z = a^m b^n a^{m+n}$

iii) According to pumping lemma

$z = uvw$, $|uv| \leq n$, $|v| \geq 1$

$z = a^m b^n a^{m+n}$ is expressed as,

$$\boxed{uv = a^m,\ v = a^q,\ w = a^{P-m} b^n a^O} \quad [O = m+n]$$

iv) Check the condition.

(a) $|uv| \leq n$, $|a^m| \leq n$, $m \leq n$

(b) $|v| \geq 1$, $|a^q| \geq 1$, $q \geq 1$

(c) for all $i \geq 0$ the string $uv^iw$ is in L

$$uv^iw = uv^{i-1}vw$$
$$= (uv)(v)^{i-1}w$$
$$= a^m (a^q)^{i-1} a^{P-m} b^n a^O$$
$$\boxed{uv^iw = a^{q(i-1)+P} b^n a^O}$$

For $i=0$, $uw = a^{P-q} b^n a^O \notin L$

$i=1$, $uvw = a^P b^n a^O \in L$

$i=2$, $uv^2w = a^{P+q} b^n a^O$

Take some random value for P, q, n and O

$P=4$, $n=3$, $O=10$, $q=1$

$a^{4+1} b^3 a^{10} = a^5 b^3 a^{10} \notin L$ [Here $m=5$, $n=3$, $m+n=8 \neq 10$]

∴ Since for $i=0$ and $i=2$, string does not belongs to L.

∴ $L = \{a^m b^n a^{m+n} \mid m \geq 1, n \geq 1\}$ is not regular.

Problem 4: Is the following language regular? Justify your answer.

$$L = \{0^{2n} \mid n \geq 1\}$$

Solution:

This is a language length of string is always even.

(ie) $n = 1$, $L = 00$

$n = 2$, $L = 0000$ and so on.

Let $L = uvw$

$L = 0^{2n}$

$|z| = 2n = uv^i w$

If we add $2n$ to this string length

$|x| = 4n = uv.vw$ = even length of string.

Thus even after pumping $2n$ to the string we get the even length. So the language $L = \{0^{2n} \mid n \geq 1\}$ is regular language.

Problem 5: prove $L = \{a^p \mid p$ is a prime$\}$ is not regular.

Solution: Let, assume L is regular and p is a prime number.

$$L = a^p$$

$$|x| = uvw, \; i = 1$$

Consider $L = uv^i w$ where $i = 2$

$= uv.vw$

Adding 1 to p, we get

$$p < |uvvw|$$

$$p < p+1$$

But $p+1$ is not prime number.

Assumption is contradictory. Thus $L = \{a^p \mid p$ is a prime$\}$ behaves as it is not a regular language.