# Bayesian Calculations: Simulation of Coin Flipping

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## 1 Background

Bayes Theorem allows us to state our *prior* beliefs, to calculate the *likelihood* of our data given those beliefs, and then to update those beliefs with data, thus arriving at a set of *posterior* beliefs. However, Bayesian calculations can be difficult to understand. This document attempts to provide a simple walkthrough of some Bayesian calculations.

```
library(pander) # nice tables

library(tibble) # data frames

library(ggplot2) # beautiful graphs
```

## 2 Bayes Rule

Mathematically Bayes Theorem is as follows:

$$P(H \mid D) = \frac{P(D \mid H)P(H)}{P(D)}$$

In words, Bayes Theorem may be written as follows:

$$posterior = \frac{likelihood * prior}{data}$$

Our posterior beliefs are proportional to our prior beliefs, multiplied by the likelihood of those beliefs, given the data.

# 3 This Example

In this example, we provide an example of using Bayes Theorem to examine our conclusions about the proportion of heads when a coin is flipped 10 times.

Conventionally, we call this proportion that we are trying to estimate $\theta$.

For the sake of simplicity, this example uses a relatively simple set of prior beliefs about 3 possible values for the proportion $\theta$.

R code in this example is adapted and simplified from Kruschke (2011), p. 70

# 4 Prior

We set a simple set of prior beliefs, concerning 3 values of $\theta$, the proportion of heads.

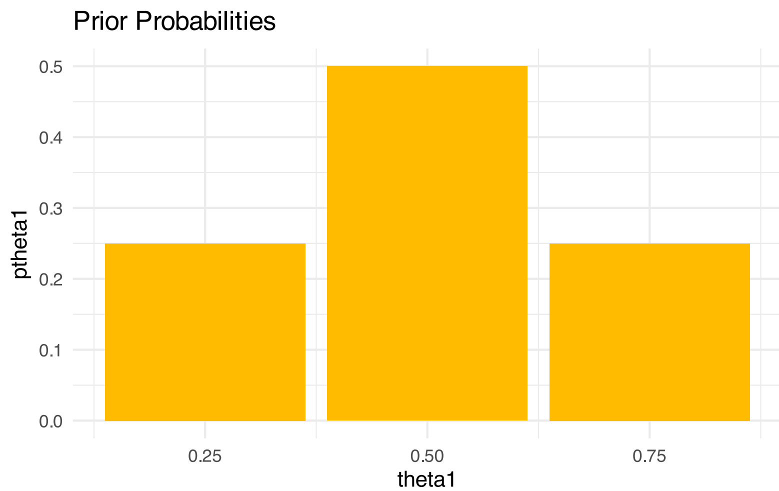
```
theta1 <- c(0.25, 0.50, 0.75) # candidate parameter values

ptheta1 <- c(.25, .50, .25) # prior probabilities

ptheta1 <- ptheta1/sum(ptheta1) # normalize
```

Our values of $\theta$ are *0.25*, *0.5* and *0.75*, with probabilities $P(\theta)$ of *0.25*, *0.5* and *0.25*.

```
ggplot(data = NULL,
       aes(x = theta1,
           y = ptheta1)) +
  geom_bar(stat = "identity",
           fill = "#FFBB00") +
  labs(title = "Prior Probabilities") +
  theme_minimal()
```




```
myBayesianEstimates <- tibble(theta1, ptheta1) # data frame

pander(myBayesianEstimates) # nice table
```

| theta1 | ptheta1 |
|---|---|
| 0.25 | 0.25 |
| 0.5 | 0.5 |
| 0.75 | 0.25 |

# 5 The Data

10 coin flips. 1 Heads. 9 Tails.

```
data1 <- c(1, 0, 0, 0, 0, 0, 0, 0, 0, 0) # the data

data1_factor <- factor(data1,
                       levels = c(0, 1),
                       labels = c("T", "H"))
```

```
n_heads <- sum(data1 == 1) # number of heads

n_tails <- sum(data1 == 0) # number of tails
```

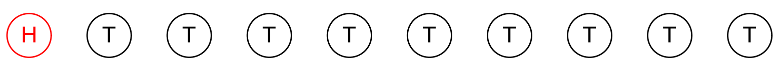
```
x <- seq(1, 10) # x goes from 1 to 10

y <- rep(1, 10) # y is a sequence of 10 1's
```

```
coindata <- data.frame(x, y, data1_factor) # data for visualization

ggplot(coindata,
       aes(x = x,
           y = y,
           label = data1_factor,
           color = data1_factor)) +
  geom_point(size = 10, shape = 1, pch=19) +
  geom_text() +
  labs(x = "",
       y = "") +
  scale_color_manual(values = c("black", "red")) +
  theme_void() +
  theme(legend.position = "none")
```

```
Warning: Duplicated aesthetics after name standardisation: shape
```




# 6 Likelihood

The likelihood is the probability that a given value of $\theta$ would produce this number of heads.

The probability of multiple independent events $A$, $B$, $C$, etc. is $P(A, B, C, ...) = P(A) * P(B) * P(C) * ....$

Therefore, in this case, the likelihood is proportional to $[P(heads)]^{\text{number of heads}}$ and multiply this by $[P(tails)]^{\text{number of tails}}$.

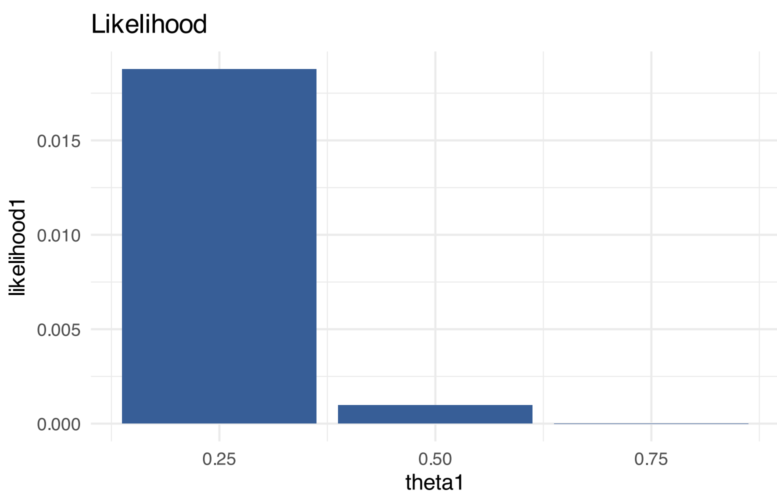
Thus:

$$\mathcal{L}(\theta) \propto \theta^{\text{number of heads}} * (1-\theta)^{\text{number of tails}}$$

```
likelihood1 <- theta1^n_heads * (1 - theta1)^n_tails # likelihood
```

```
ggplot(data = NULL,
       aes(x = theta1,
           y = likelihood1)) +
```

```
  geom_bar(stat = "identity",
           fill = "#375E97") +
  labs(title = "Likelihood") +
  theme_minimal()
```




At this point our estimates include not only a value of $\theta$ and $P(\theta)$, but also the likelihood, $\mathcal{L}(\theta)$.

```
myBayesianEstimates <- tibble(theta1, ptheta1, likelihood1)

pander(myBayesianEstimates) # nice table
```

| theta1 | ptheta1 | likelihood1 |
|---|---|---|
| 0.25 | 0.25 | 0.01877 |
| 0.5 | 0.5 | 0.0009766 |
| 0.75 | 0.25 | 2.861e-06 |

# 7 Posterior

We then calculate the denominator of Bayes theorem:

$$\Sigma[\mathcal{L}(\theta) * P(\theta)]$$

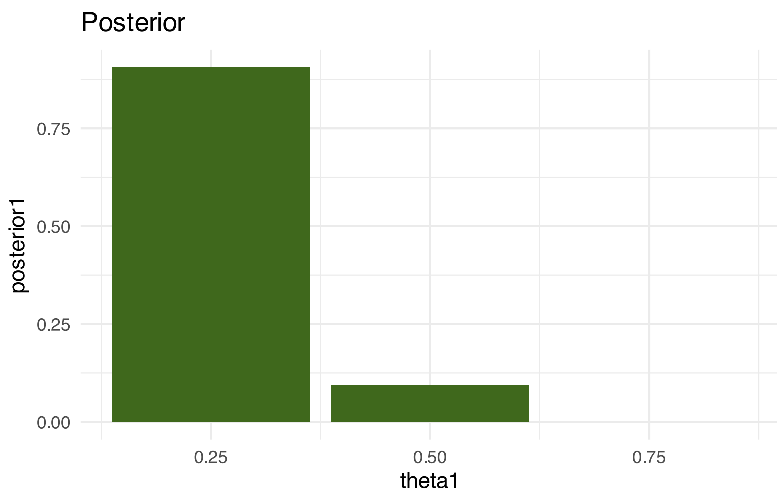
```
pdata1 <- sum(likelihood1 * ptheta1) # normalize
```

We then use Bayes Rule to calculate the posterior:

$$P(H \mid D) = \frac{P(D \mid H)P(H)}{P(D)}$$

```r
posterior1 <- likelihood1 * ptheta1 / pdata1 # Bayes Rule
```

```r
ggplot(data = NULL,
       aes(x = theta1,
           y = posterior1)) +
  geom_bar(stat = "identity",
           fill = "#3F681C") +
  labs(title = "Posterior") +
  theme_minimal()
```




Our estimates now include $\theta$, $P(\theta)$, $\mathcal{L}(\theta)$ and $P(\theta \mid D)$.

```r
myBayesianEstimates <- tibble(theta1, ptheta1, likelihood1, posterior1)

pander(myBayesianEstimates) # nice table
```

| theta1 | ptheta1 | likelihood1 | posterior1 |
|---|---|---|---|
| 0.25 | 0.25 | 0.01877 | 0.9056 |
| 0.5 | 0.5 | 0.0009766 | 0.09423 |
| 0.75 | 0.25 | 2.861e-06 | 0.000138 |

# 8 Credits

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Questions, comments and corrections are most welcome.