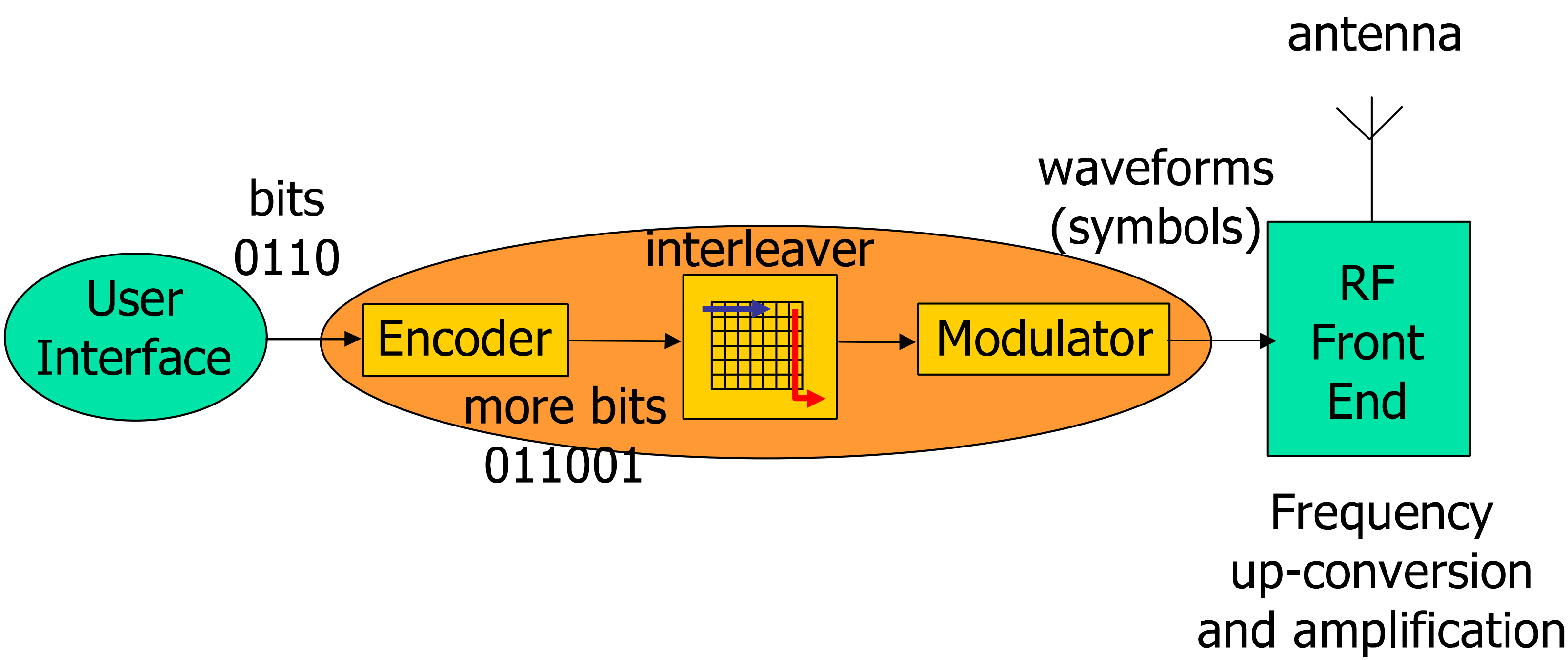
MODULATION

Introduction, Motivation and Geometry of Signals

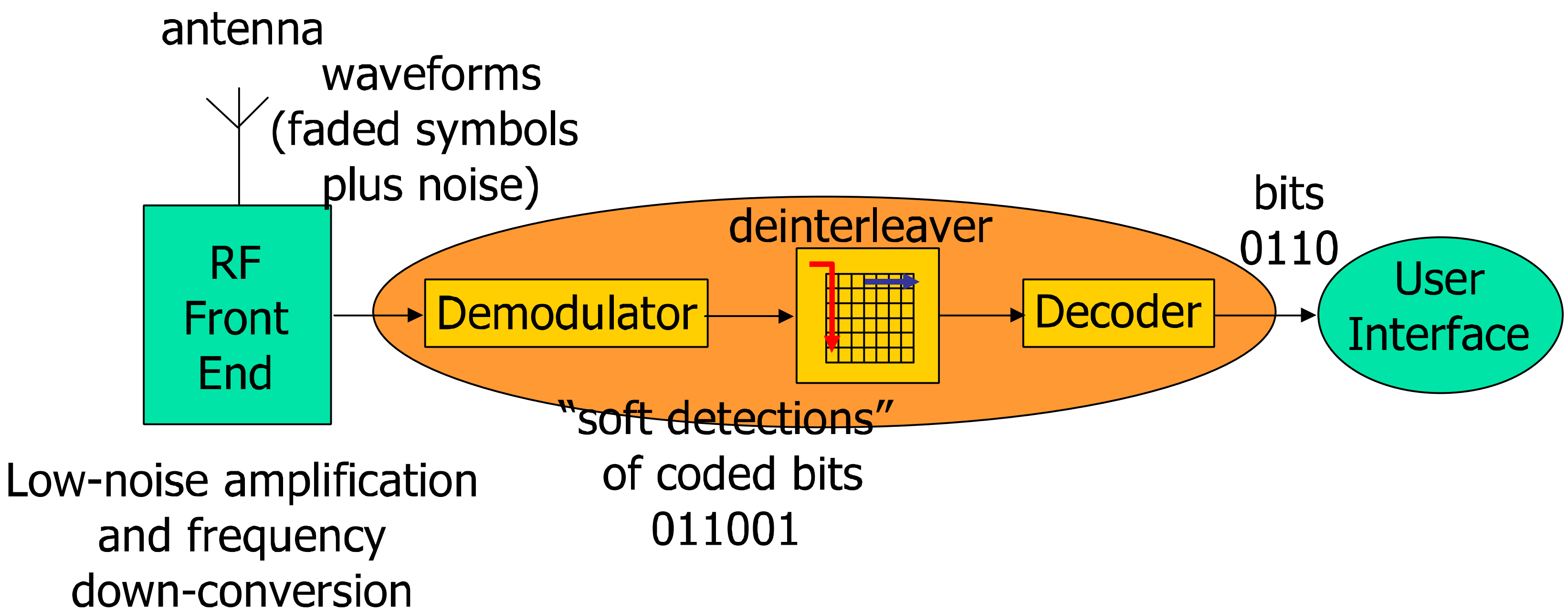
Digital Transmission

- A simplified block diagram:



Digital Reception

- A simplified block diagram:



Symbols

- In each symbol period, T_s, a digital modulator maps N coded bits word to a transmitted waveform from a set of $M=2^N$ possible waveforms
- Each waveform corresponds to an information symbol, x_n
- For Binary symbols, $N=1$
- The job of the receiver is to determine which symbols were sent and to reconstruct the bit stream that created them

Definitions

- Bit Rate (bits per sec or bps)

$$R = N / T_S$$

- Bandwidth Efficiency (bps/Hz)

$$\eta_B = R / B$$

where B is the bandwidth occupied by the signal

Shannon Theorem

- In a non-fading channel, the maximum bandwidth efficiency, or Shannon Capacity is

$$\eta_{B_{MAX}} = \log_2\left(1 + SNR\right)$$

SNR = signal-to-noise ratio

Pulses

- A symbol period, T_s, suggests a localization in time
- Localization in frequency is also necessary to enable frequency division multiplexing
- Regulatory agencies provide spectral masks to limit the distribution of power in the frequency domain

Example: Linear Modulation

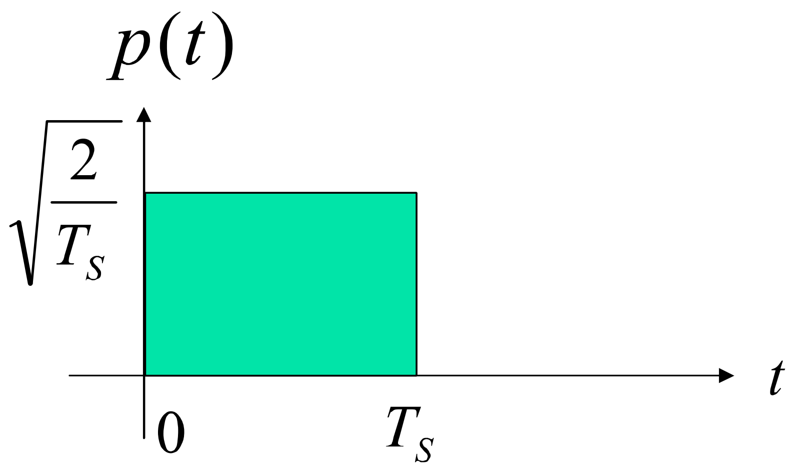
- $s(t)$ is the output of the modulator
- $g(t)$ is the complex envelope
- $p(t)$ is the basic pulse
- x_n is the nth symbol
- A is the amplification in the transmitter

$$s(t) = \mathrm{Re}\left\{ g(t) e^{j 2\pi f_c t} \right\}$$

$$g(t) = A \sum_n x_n p(t - nT_S)$$

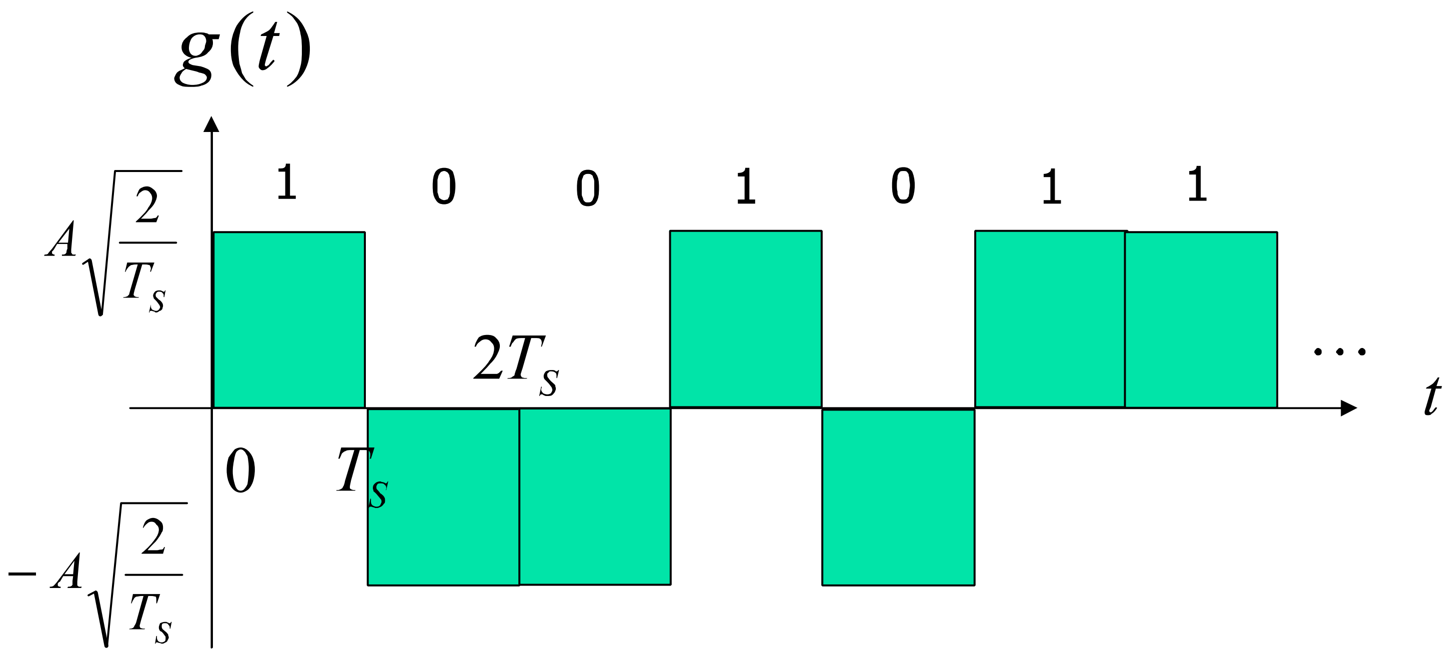
Rectangular Pulses

- Suppose $p(t)$ is a rectangular pulse
- This pulse is not used in practice, but is OK for illustration



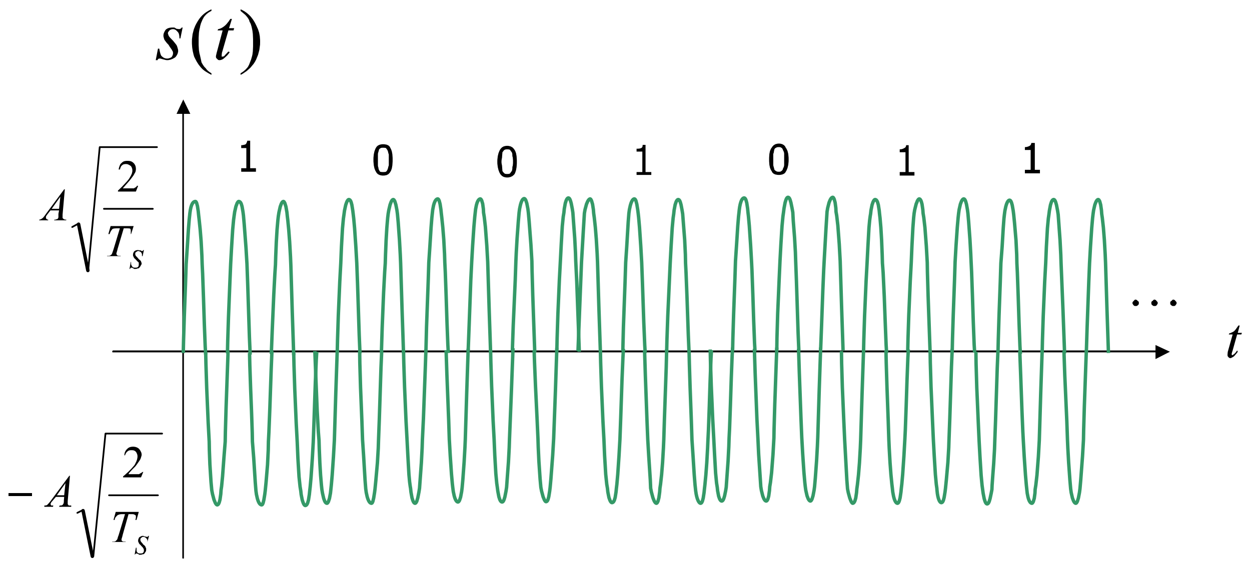
Binary Phase Shift Keying (BPSK)

- For BPSK, each symbol carries one bit of information $x_n \in \{-1,1\}$



BPSK Modulated Carrier

- The information is in the phase of the carrier



Power Spectral Density (PSD) of Linear Modulation

- Assume that the symbol sequence $\{x_n\}$ is iid and zero mean. Then, the PSD of $s(t)$ is

$$S_s(f) = \frac{1}{2}\left[S_g(f - f_c) + S_g(-f - f_c)\right]$$

where

$$S_g(f) = \frac{A^2 E\left\{|x_n|^2\right\}}{2T_S}|P(f)|^2$$

and $P(f)$ is the Fourier Transform of $p(t)$

Bandwidth Properties

- The RF bandwidth of the modulated carrier is two times the baseband bandwidth of

$$S_g(f) = \frac{A^2 E\left\{\left|x_n\right|^2\right\}}{2T_S}\left|P(f)\right|^2$$

which is clearly seen to depend on the bandwidth of the pulse

Fourier Transform of Rectangular Pulses

$$P(f) = \int_{-\infty}^{+\infty} p(t)\exp(-j2\pi ft)dt = \int_{0}^{T_S} \sqrt{\frac{2}{T_S}} \exp(-j2\pi ft)dt$$

$$= \sqrt{\frac{2}{T_S}} \frac{\exp(-j2\pi ft)}{-j2\pi f}\Bigg|_{0}^{T_S} = \sqrt{\frac{2}{T_S}} \frac{\exp(-j2\pi fT_S)-1}{-j2\pi f}$$

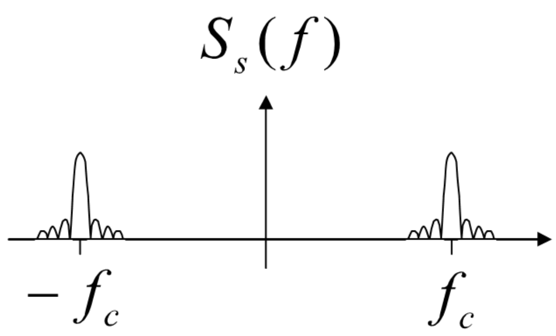
$$= \frac{\exp(-j\pi fT_S)}{\pi f}\sqrt{\frac{2}{T_S}}\sin(\pi fT_S) = \exp(-j\pi fT_S)\sqrt{2T_S}\,\mathrm{sinc}(fT_S)$$

PSD of BPSK for Rectangular Pulses

- $\left|P(f)\right|^2 = 2T_S \text{sinc}^2\left(fT_S\right)$
- $|x_n|^2 = 1$

$$S_g(f) = \frac{A^2 E\left\{\left|x_n\right|^2\right\}}{2T_S}\left|P(f)\right|^2$$

$$= A^2 \text{sinc}^2\left(fT_S\right)$$



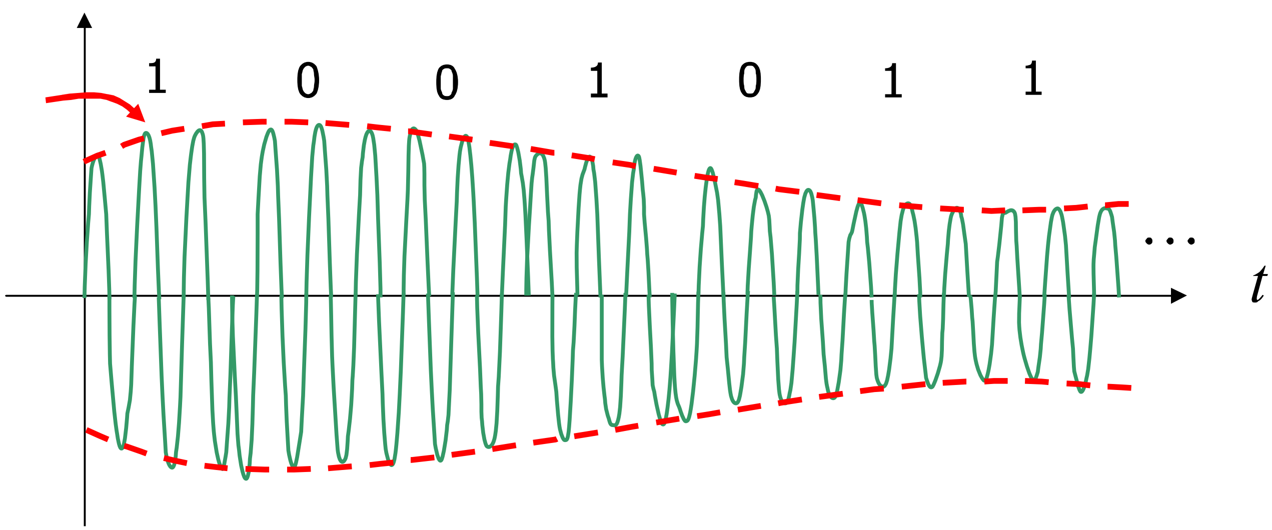
$$S_s(f) = \frac{A^2}{2}\left[\text{sinc}^2\left(\left[f - f_c\right]T_S\right) + \text{sinc}^2\left(\left[-f - f_c\right]T_S\right)\right]$$

Received Signal

- The received signal has been attenuated by path loss and multipath fading and has added noise

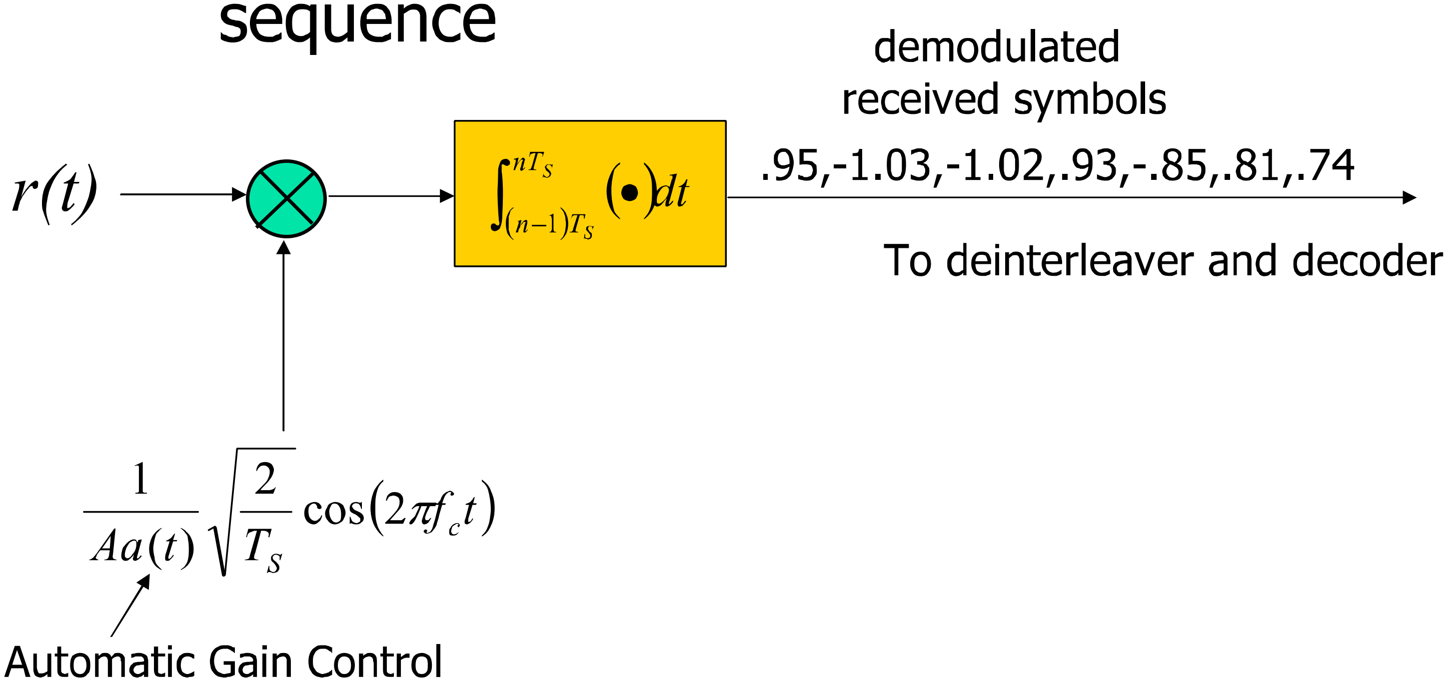
$$r(t) = a(t)A\sum_n x_n p(t - nT_S)\cos(2\pi f_c t + \theta_c) + n(t)$$

$$\underbrace{a(t)}_{\text{Channel Gain}} A\sqrt{\frac{2}{T_S}}$$



BPSK Demodulator

- The output of the correlator is a sequence of noisy versions of the transmitted symbol sequence



Integrate and Dump Output

- Consider one symbol period and the rectangular pulse:

channel gains and TX power lumped into $\mathcal{E}_b$

$$R = \int_0^{T_S} r(t)\sqrt{\frac{2}{T_S}}\cos(2\pi f_c t)dt = \int_0^{T_S}\left(x_n\sqrt{\frac{2\mathcal{E}_b}{T_S}}\cos(2\pi f_c t) + n(t)\right)\sqrt{\frac{2}{T_S}}\cos(2\pi f_c t)dt$$

$$= \frac{x_n}{T_S}\sqrt{\mathcal{E}_b}\int_0^{T_S} 1\,dt + \sqrt{\frac{2}{T_S}}\int_0^{T_S} n(t)\cos(2\pi f_c t)dt = x_n\sqrt{\mathcal{E}_b} + v_n$$

- v_n is a zero-mean Gaussian RV with variance

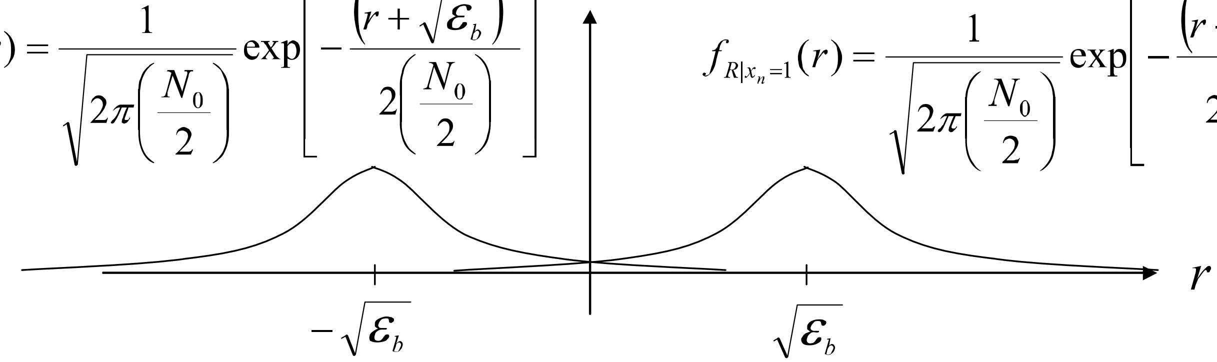
$$\sigma_v^2 = \frac{N_0}{2}$$

Conditional PDFs

- Recall $R = x_n\sqrt{\mathcal{E}_b} + v_n$

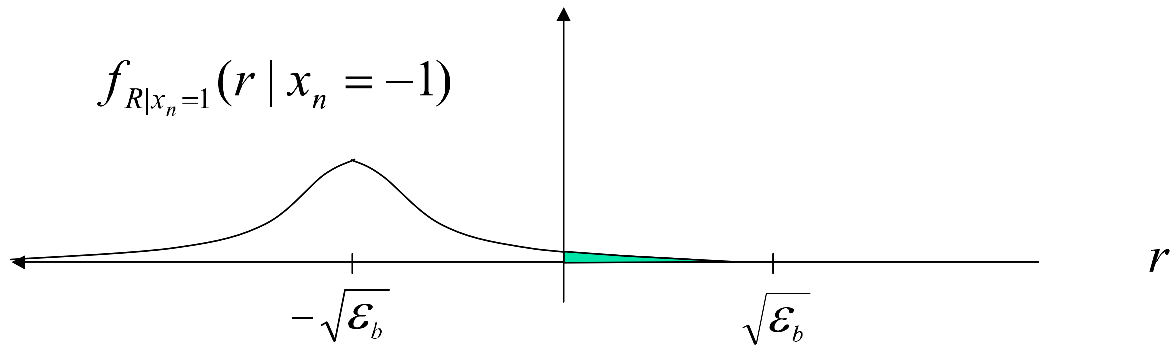
$$f_{R|x_n=-1}(r) = \frac{1}{\sqrt{2\pi\left(\frac{N_0}{2}\right)}}\exp\left[-\frac{\left(r+\sqrt{\mathcal{E}_b}\right)^2}{2\left(\frac{N_0}{2}\right)}\right]$$

$$f_{R|x_n=1}(r) = \frac{1}{\sqrt{2\pi\left(\frac{N_0}{2}\right)}}\exp\left[-\frac{\left(r-\sqrt{\mathcal{E}_b}\right)^2}{2\left(\frac{N_0}{2}\right)}\right]$$



Conditional Probability of Bit Error

- If x_n is -1, then an error happens if $R>0$



$$P(error \mid x_n = -1) = P(R > 0 \mid x_n = -1)$$

$$= \int_0^{+\infty} f_{R|x_n=-1}(r \mid x_n = -1)\,dr$$

Conditional Probability of Error: Expression

$$\int_{0}^{+\infty} f_{R|x_n=-1}(r \mid x_n = -1)\,dr = Q\left(\frac{0-\left(-\sqrt{\mathcal{E}_b}\right)}{\sqrt{\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

- Same for other kind of error, by symmetry

Unconditional Probability of Error

- Assume the two possible values of x_n are equally likely

$$P(error) = \frac{1}{2}P(error \mid x_n = -1) + \frac{1}{2}P(error \mid x_n = 1)$$

$$= Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

How to Improve System Performance

- Increase symbol energy $\mathcal{E}_b$
- Decrease average noise power $N_0 / 2$

$$P(error) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$