3D-Transformations
CS-477 Computer Vision

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1 Transformation of Lines

2 3D Transformations

3 Decomposition of 2D Transformations

1 Transformation of Lines

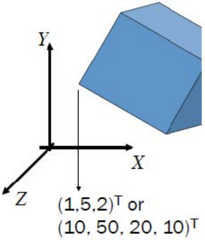
2 3D Transformations

3 Decomposition of 2D Transformations

- Consider the projective transform which transforms vectors points as
$$x' = HX$$
- What affect will this transform have on lines?
- **Theorem:** Under a projective transformation , a line transforms as
$$l' = H^{-T} l$$
- **Proof:** Consider point x_i, that lie on line l. Then all such points satisfy $l^T x_i = 0$. Therefore they must also satisfy $l^T H^{-1} H x_i = 0$. Now note that Hx_i are transformed points x_i', and hence, must lie on the line l'. Therefore, $l'^T = l^T H^{-1}$ and hence, $l' = H^{-T} l$.
- Thus, if points transform by H, lines transform by H^{-T}

Points and Planes in 3D

- World points at 3-dimensional entities
$$\tilde{\mathbb{X}} = (\tilde{X}, \tilde{Y}, \tilde{Z})^T \in \mathbb{R}^3$$
- In homogeneous coordinates,
$$\mathbb{X} = (w\tilde{X}, w\tilde{Y}, w\tilde{Z}, w)^T \in \mathbb{R}^3$$
- or in general $\mathbb{X} = (X_1, X_2, X_3, X_4)^T$
- where $\tilde{X} = \frac{X_1}{X_4} \quad \tilde{Y} = \frac{X_2}{X_4} \quad \tilde{Z} = \frac{X_3}{X_4}$
- $X_4 = 0$ denotes ideal points
- The vector space $\mathbb{P}$ is defined as
$$\mathbb{P}^3 = \mathbb{R}^4 - (0, 0, 0, 0)^T$$



Planes in 3D

- Planes in 3D are natural extension of lines in 2D
- A plane in 3D-space may be written as

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

- Thus the representation of a plane is a homogeneous 4-vector

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4)^T$$

- Homogenizing by replacements $X = \frac{X_1}{X_4} \quad Y = \frac{X_2}{X_4} \quad Z = \frac{X_3}{X_4}$
- Equation of Plane:

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$
$$\pi^T X = 0$$

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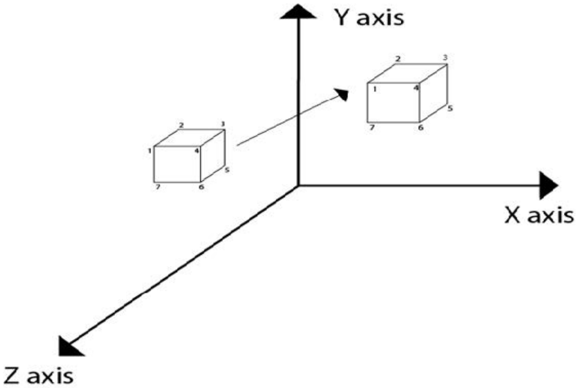
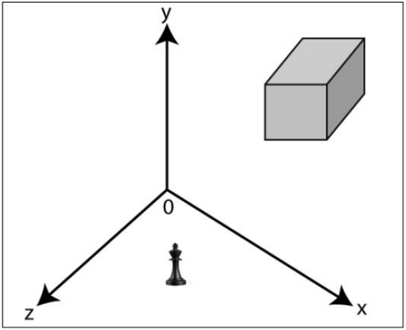
3D Translation

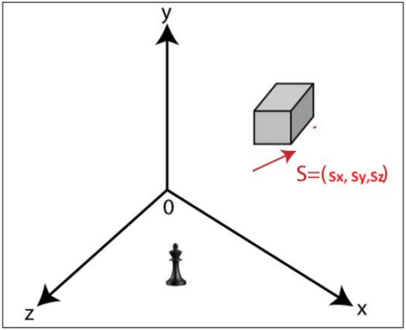
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



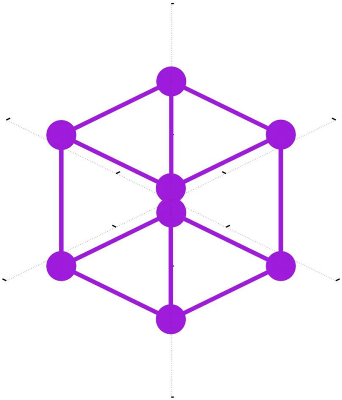
3D Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

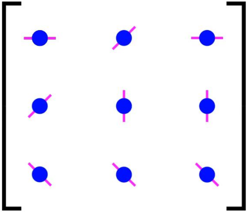




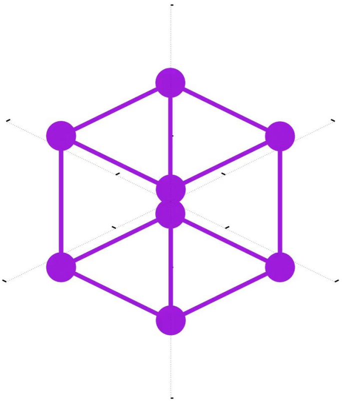
3D Scaling along x-axis



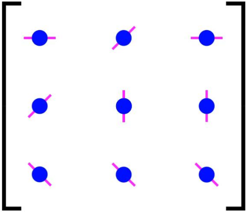
$$\begin{bmatrix} 1.00 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v}_0$$



3D Scaling along y-axis

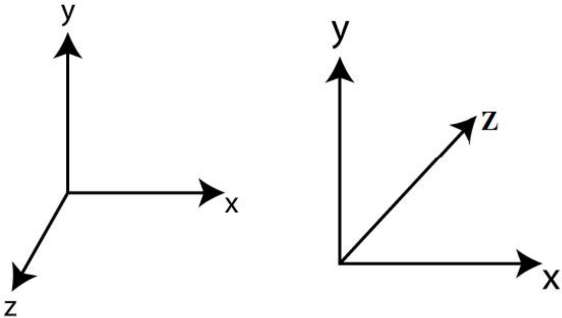


$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.00 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v}_0$$



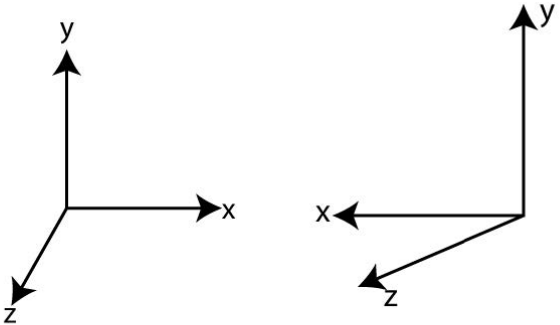
Reflection relative to XY plane

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



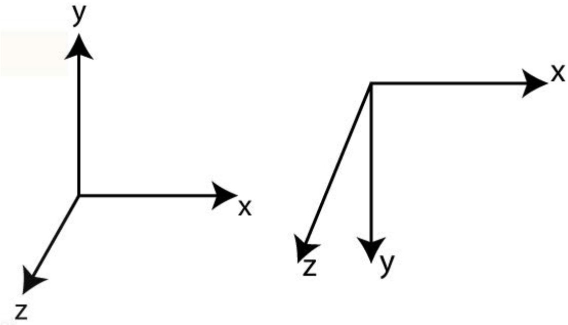
Reflection relative to YZ plane

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



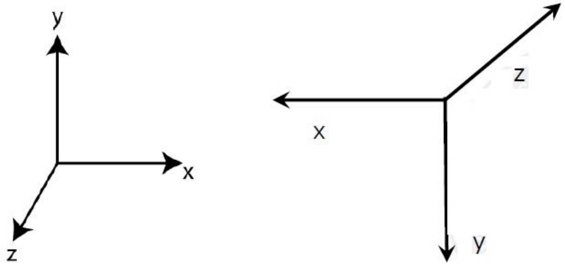
Reflection relative to XZ plane

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Reflection

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



3D Shearing: shearing along x-axis

Proportional to y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

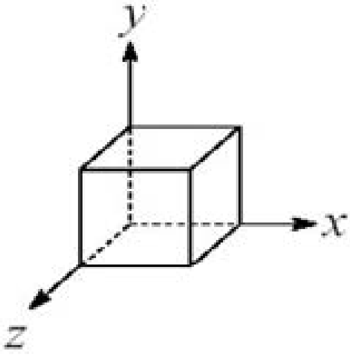
Proportional to z-axis

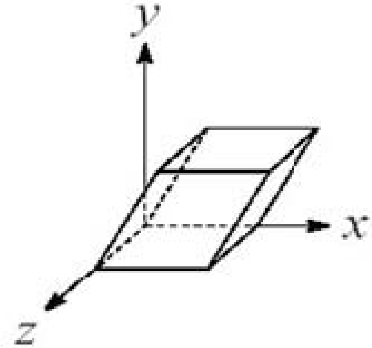
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Proportional to both y and z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & s & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Shearing: shearing along x-axis





3D Shearing: shearing along y-axis

Proportional to x-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Proportional to z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Proportional to both x and z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & 1 & s & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Shearing: shearing along z-axis

Proportional to x-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

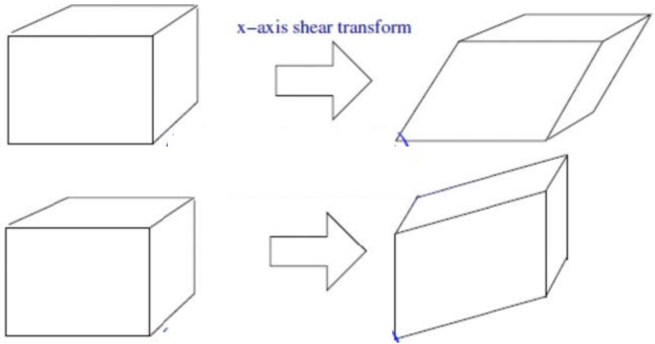
Proportional to y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Proportional to both x and y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s & s & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Shearing Transformations



Rotation in 3D

- 3D rotation about z-axis
  - z-coordinate will not change
  - $z' = z$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- 3D rotation about x-axis
  - Relate (y', z') to (y, z)

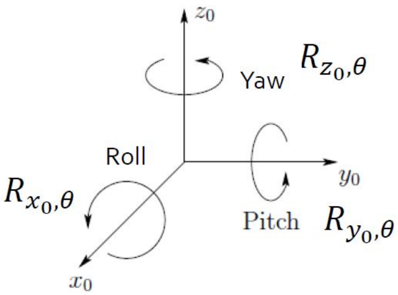
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Orthonormal Rotation Matrices

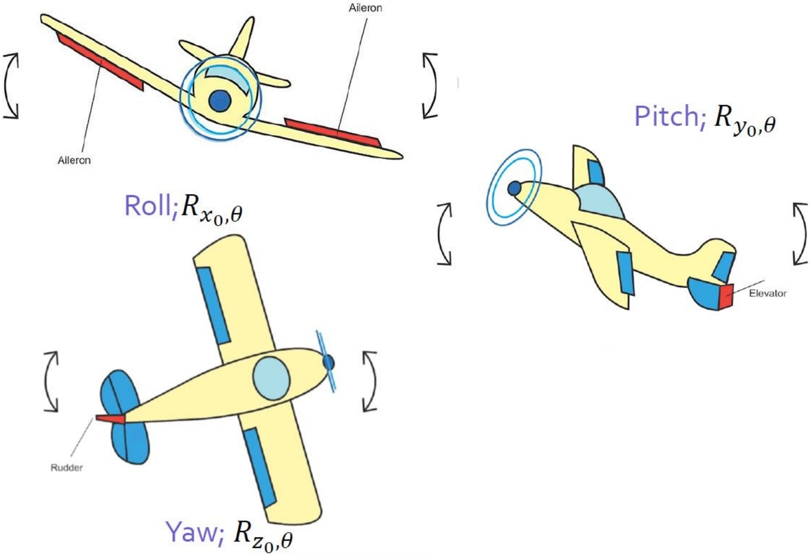
$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation in 3D



Do it!

Mathematically derive the following rotation matrix

Rotation about Z

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$X' = X\cos\theta - Y\sin\theta$$
$$X' = X\sin\theta - Y\cos\theta$$
$$Z' = Z$$

Rotation about X

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y' = Y\cos\theta - Z\sin\theta$$
$$X' = Y\sin\theta - Z\cos\theta$$
$$X' = X$$

Rotation about Y

$$\begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z' = Z\cos\theta - X\sin\theta$$
$$X' = Z\sin\theta - X\cos\theta$$
$$Y' = Y$$

Properties of Rotation Matrix

- Rotation matrices are orthonormal with a determinant of 1
- Inverse of a rotation matrix is its transpose, i.e.,
$$R^{-1} = R^T$$
- $RR^T = R^T R = I$
- $R_{Z,0} = I$
- $R_{Z,\theta} R_{Z,\phi} = R_{Z,\theta+\phi}$
- $R_{Z,\theta}^{-1} = R_{Z,-\theta}$

Concatenation of Rotations

- Example: Rotation about X by γ, followed by rotation about Y by β, followed by rotation about Z by θ

$$R = R_{Z,\theta} R_{Y,\beta} R_{X,\gamma} = R_{\theta}^{Z} R_{\beta}^{Y} R_{\gamma}^{X}$$

$$\mathrm{R} = \begin{bmatrix} \cos\beta\cos\theta & \sin\beta\cos\theta\sin\gamma - \cos\gamma\sin\theta & \sin\gamma\sin\theta + \cos\gamma\sin\beta\cos\theta & 0 \\ \cos\beta\sin\theta & \cos\gamma\cos\theta + \sin\beta\sin\gamma\sin\theta & \cos\gamma\sin\beta\sin\theta - \cos\theta\sin\gamma & 0 \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Concatenation of rotation matrices is also a rotation matrix, i.e., matrix remains orthonormal with determinant of 1.

Do it!

- Concatenation of rotation matrices is also a rotation matrix, i.e. matrix remains orthonormal with determinant of 1.
- Proof???

Interpreting Rotation

Given a general rotation matrix, how can we interpret the transformation?

$$R = \begin{bmatrix} -0.56325 & 0.75604 & 0.33338 \\ -0.68219 & -0.19784 & -0.7039 \\ -0.46622 & -0.62391 & 0.6272 \end{bmatrix}$$

Rotation about Arbitrary Axis: Inverse Problem

- An arbitrary rotation matrix with r_{ij}

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \rightarrow \text{Rotation matrix}$$

- The equivalent angle θ

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{Tr(R)-1}{2}\right) \\ &= \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right) \end{aligned}$$

- Equivalent axis k are given by the expressions

$$k = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

- Limitation: θ is always in between [0,180]. A rotation of $-\theta$ about $-k$ is the same as a rotation of θ about k

$$R_{k,\theta} = R_{-k,-\theta}$$

- Fails if $\theta = 0$ or $\theta = 180$

Axis/angle Representation

- Example: Suppose R is generated by a rotation of 90 degree about z_0 followed by a rotation of 30 degree about y_0 followed by a rotation of 60 degree about x_0. Then

$$R = R_{x,60} R_{y,30} R_{z,90}$$
$$= \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{4} & \frac{3}{4} \\ \frac{\sqrt{3}}{2} & \frac{1}{4} & \frac{\sqrt{3}}{4} \end{bmatrix}$$

- Compute theta and arbitrary axis?

$$\theta = \cos^{-1}(-\tfrac{1}{2}) = 120^o \qquad k = (\tfrac{1}{\sqrt{3}}, \tfrac{1}{2\sqrt{3}} - \tfrac{1}{2}, \tfrac{1}{2\sqrt{3}} + \tfrac{1}{2})^T$$

Do it!

Question: Given an arbitrary 3D rotation matrix, how can we find out the axis k and the angle θ that represents this rotation?

$$\begin{bmatrix} -0.8256 & 0.40388 & -0.39404 \\ -0.20084 & -0.86294 & -0.46367 \\ -0.5273 & -0.30367 & 0.79356 \end{bmatrix}$$

Pure rotation transformations

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pure translation transformations

$$Trans_x(d) = \begin{bmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_y(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Trans_z(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary: Properties of Transformation

$$SS^{-1} = I$$

$$TT^{-1} = I$$

$$RR^{-1} = I$$

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Factorizing Transformations

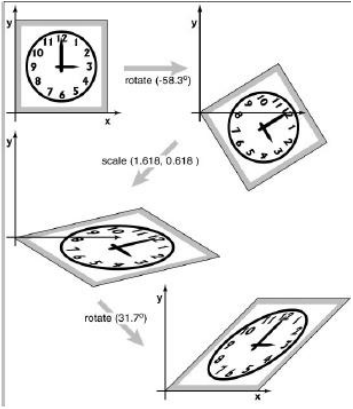
- Opposite of Concatenation of Transformations
- Given a transformation matrix, decompose it into a sequence of simpler transformations
- Example:

$$\begin{bmatrix} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Question: How to factorize the multiplicative part?

Decomposition of 2D Transformations

- An interesting result
- Any 2D transformation can be written as RSR^T
- This is called the Factorization of the matrix



Decomposition of 2D Transformations

- Vector is defined by:
  - Magnitude
  - Direction
- Transformation can either change
  - magnitude of a vector
  - direction of a vector
  - or both

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Decomposition of 2D Transformations

Rotation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

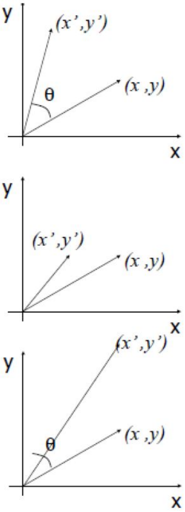
Non-uniform scaling:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation + Scaling | Non-uniform scaling:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_1\cos\theta & -s_1\sin\theta \\ s_2\sin\theta & s_2\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Decomposition of 2D Transformations

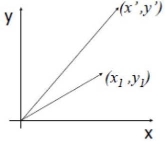
- Given a transformation matrix M
  - Decompose it into various matrix products i.e., $M = M_1 M_2 M_3 = M_3 M_4 M_5$ etc.
- If transformation matrix is symmetric
  - $M = M^T$
  - Eigen Value decomposition
- If transformation matrix is not symmetric
  - Singular Value decomposition

Eigenvalues and Eigenvectors

- A matrix act on a vector by changing both its magnitude and direction

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_1 \cos\theta & -s_1 \sin\theta \\ s_2 \sin\theta & s_2 \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

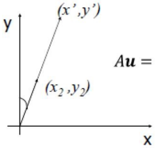
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$



- However,same matrix may act on certain vectors by changing only their magnitudes and leaving their directions unchanged (or possibly reversing it).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_1 \cos\theta & -s_1 \sin\theta \\ s_2 \sin\theta & s_2 \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$



$$A\boldsymbol{u} = \lambda\boldsymbol{u}$$

- These vectors are the eigenvectors of the matrix
- A matrix act on an eigenvector u by multiply its magnitude by a factor. This factor is the eigenvalue λ associated with that eigenvector

Eigenvalues and Eigenvectors

- Eigenvector u of matrix A satisfies the following equation:

$$Au = \lambda u$$

or

$$(A - \lambda I)u = 0$$

- where λ is a scalar called eigenvalue associated to the eigenvector

Example: Eigenvalues and Eigenvectors

- Eigenvalues

$$A = \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix}$$

$$det(A - \lambda I) = 0)$$

$$det\left(\begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} 4-\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix}\right) = 0$$

$$(4-\lambda)(-1-\lambda) - 6 = 0$$
$$\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda^2 + 2\lambda - 5\lambda - 10 = 0$$

$$(\lambda + 2)(\lambda - 5) = 0$$

$$\lambda_1 = -2, \qquad \lambda_2 = 5$$

Eigenvectors

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = 0$$

$$\begin{bmatrix} 4+2 & 3 \\ 2 & -1+2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = 0$$

$$6u_{11} + 3u_{21} = 0$$
$$2u_{11} + u_{21} = 0$$

$$\begin{bmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = 0$$

$$\begin{bmatrix} 4-5 & 3 \\ 2 & -1-5 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = 0$$

$$-u_{12} + 3u_{22} = 0$$
$$2u_{12} - 6u_{22} = 0$$

$$u_{11}^2 + u_{21}^2 = 1 \qquad u_{12}^2 + u_{22}^2 = 1$$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$$

Do it!

- For example, the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

- has the eigenvectors;

$$u_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ with eigenvalue } \lambda_1 = 4$$

and

$$u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ with eigenvalue } \lambda_2 = -1$$

Singular Value Decomposition

- Symmetric matrices ($A = A^T$) can be decomposed as $A = U \sum U^T$
- Non-symmetric real matrix A can be decomposed as $A = U \sum V^T$
  - U and V are orthonormal ($UU^T = I$) and $\sum$ is diagonal
  - U and V are matrices of Eigen vectors of AA^T and $A^T A$, respectively
  - Diagonal entries of $\sum$ consist of the square root of the Eigen values of AA^T or $A^T A$

https://www.youtube.com/watch?v=4tvw-1HI45s

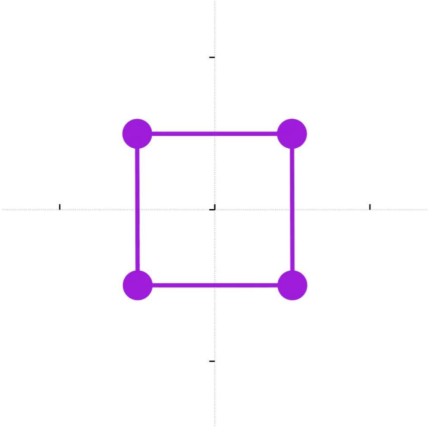
Singular Value Decomposition

- Let A be a m-by-n matrix whose entries are real numbers. Then A may be decomposed as
$$A = U \sum V^T$$
where
  - U is an m-by-m orthonormal matrix
  - $\sum$ is an m-by-n matrix with non-negative numbers on the main diagonal and zeros elsewhere
  - V is an n-by-n orthonormal matrix
- Example:

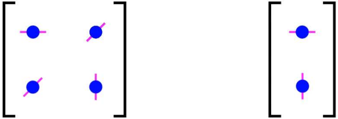
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

Decomposition of 2D Transformations

Rotation is a combination of shearing (horizontal), shearing (vertical) and scaling

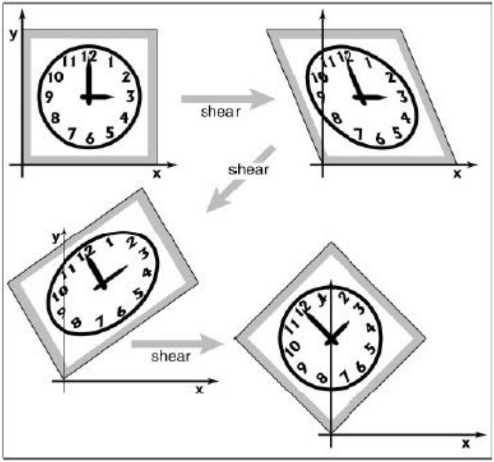


$$\begin{bmatrix} 1 & 0.00 \\ 0 & 1 \end{bmatrix} \mathbf{v}_0 + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Decomposition of 2D Transformations

Decomposing a rotation about origin into 3 shears or 2 shears and scaling



Decomposition of 2D Transformations

Decomposing a rotation about origin into 3 shears

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & \frac{\cos\phi - 1}{\sin\phi} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sin\phi & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\cos\phi - 1}{\sin\phi} \\ 0 & 1 \end{bmatrix}$$

Summary

- Every matrix can be decomposed via SVD
- Only symmetric matrices can be decomposed via Eigen value decomposition
- Such matrices are a simple scale in an arbitrary direction.
- The SVD of a symmetric matrix will lead to same result as Eigen value decomposition.