Geometry of Signals

Motivation

- Signals can be viewed as points in a space
- Notions of distance and angle have practical meaning
- Probability of bit error can be determined *graphically*

Euclidean Space

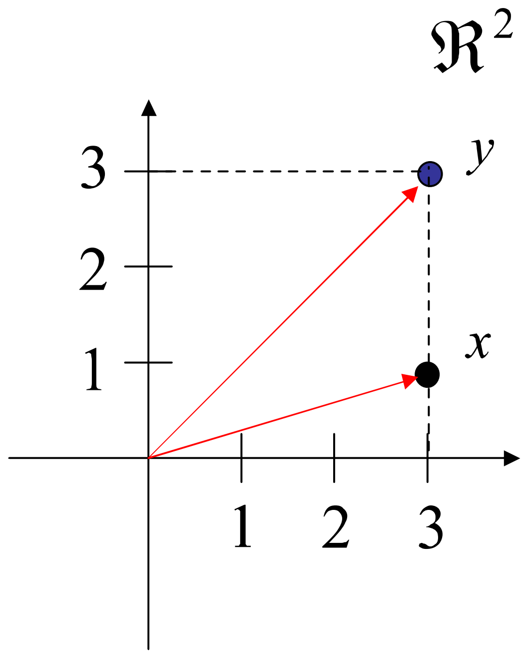
- The space: $\Re^2$
- The basis: $(1,0)$ and $(0,1)$
- Two points:

$$x = (3,1) \text{ and } y = (3,3)$$

- Inner product:

$$\langle x, y \rangle = 3 \cdot 3 + 1 \cdot 3 = 12$$

$$= |x||y| \cos\theta$$



The Space of Signals

- The space: All square-integrable (finite energy) functions, L_2
- Example basis functions:

$$e_1(t) = \sqrt{\frac{2}{T_S}} \cos\left(2\pi f_c t\right) \quad 0 \le t \le T_S$$

$$e_2(t) = \sqrt{\frac{2}{T_S}} \cos\left(2\pi f_c t + \frac{\pi}{2}\right) \quad 0 \le t \le T_S$$

Inner Product and Magnitude

$$\langle x, y \rangle = \int_{-\infty}^{+\infty} x(t)\, y(t)\, dt$$

$$|x|^2 = \langle x, x \rangle = \int_{-\infty}^{+\infty} x^2(t)\, dt$$

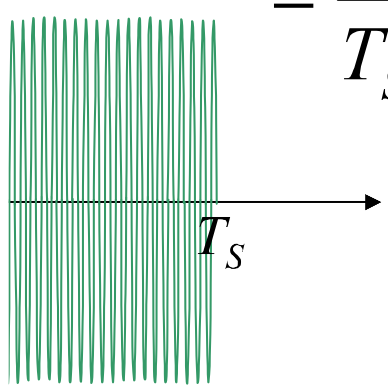
$$|x| = \sqrt{\int_{-\infty}^{+\infty} x^2(t)\, dt}$$

Basis Functions should have a zero Inner Product

$$\langle e_1, e_2 \rangle = \frac{2}{T_S}\int_0^{T_S} \cos(2\pi f_c t)\cos\left(2\pi f_c t + \frac{\pi}{2}\right)dt$$

$$= \frac{2}{T_S}\int_0^{T_S} \frac{1}{2}\left[\cos\left(-\frac{\pi}{2}\right) + \cos\left(2\pi 2 f_c t + \frac{\pi}{2}\right)\right]dt$$

$$= \frac{1}{T_S}\int_0^{T_S} \cos\left(2\pi 2 f_c t + \frac{\pi}{2}\right)dt \approx 0$$



Basis Function should have a unit Norm

$$\langle e_1, e_1 \rangle = \frac{2}{T_S} \int_0^{T_S} \cos^2(2\pi f_c t)\, dt$$

$$= \frac{2}{T_S} \int_0^{T_S} \frac{1}{2} \left[1 + \cos(2\pi 2 f_c t)\right] dt$$

$$\approx 1$$

Two Points in Space

$$x(t) = \sqrt{\frac{2\mathcal{E}_b}{T_S}} \cos\left(2\pi f_c t + \frac{\pi}{4}\right) \quad 0 \le t \le T_S$$

$$y(t) = \sqrt{\frac{2\mathcal{E}_b}{T_S}} \cos\left(2\pi f_c t - \frac{\pi}{6}\right) \quad 0 \le t \le T_S$$

Signal Energy

- The energy of *x(t)* is

$$\int_0^{T_S} x^2(t)dt = |x|^2 = \frac{2\mathcal{E}_b}{T_S}\int_0^{T_S}\cos^2\left(2\pi f_c t + \frac{\pi}{4}\right)dt = \mathcal{E}_b$$

- The energy of *y(t)* is the same
- The energy of a signal is its norm squared

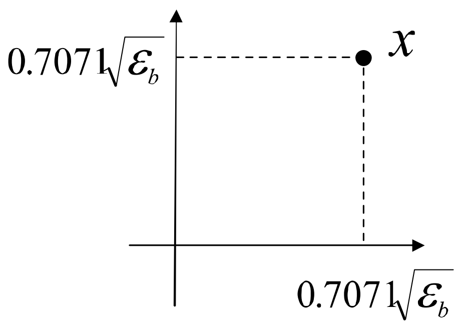
Find Coordinates of x

$$\langle x, e_1 \rangle = \frac{2\sqrt{\mathcal{E}_b}}{T_S}\int_0^{T_S}\cos\left(2\pi f_c t + \frac{\pi}{4}\right)\cos(2\pi f_c t)\,dt$$

$$= \frac{2\sqrt{\mathcal{E}_b}}{T_S}\int_0^{T_S}\frac{1}{2}\left[\cos\left(\frac{\pi}{4}\right) + \cos\left(2\pi 2 f_c t + \frac{\pi}{4}\right)\right]dt$$

$$= \sqrt{\frac{\mathcal{E}_b}{2}} = 0.7071\sqrt{\mathcal{E}_b}$$

$$\langle x, e_2 \rangle = \sqrt{\frac{\mathcal{E}_b}{2}} = 0.7071\sqrt{\mathcal{E}_b}$$



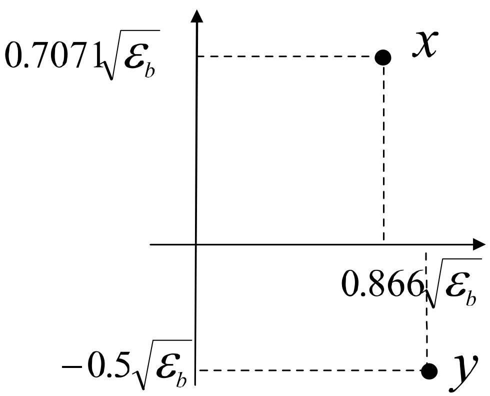
Find Coordinates of y

$$\langle y, e_1 \rangle = \frac{2\sqrt{\mathcal{E}_b}}{T_S} \int_0^{T_S} \cos\left(2\pi f_c t - \frac{\pi}{6}\right) \cos(2\pi f_c t) dt$$

$$= \frac{2\sqrt{\mathcal{E}_b}}{T_S} \int_0^{T_S} \frac{1}{2} \left[\cos\left(-\frac{\pi}{6}\right) + \cos\left(2\pi 2 f_c t - \frac{\pi}{6}\right) \right] dt$$

$$= \sqrt{\mathcal{E}_b} \cos\left(-\frac{\pi}{6}\right) = 0.866\sqrt{\mathcal{E}_b}$$

$$\langle y, e_2 \rangle = \sqrt{\mathcal{E}_b} \cos\left(\frac{2\pi}{3}\right) = -0.5\sqrt{\mathcal{E}_b}$$



Distance between Points

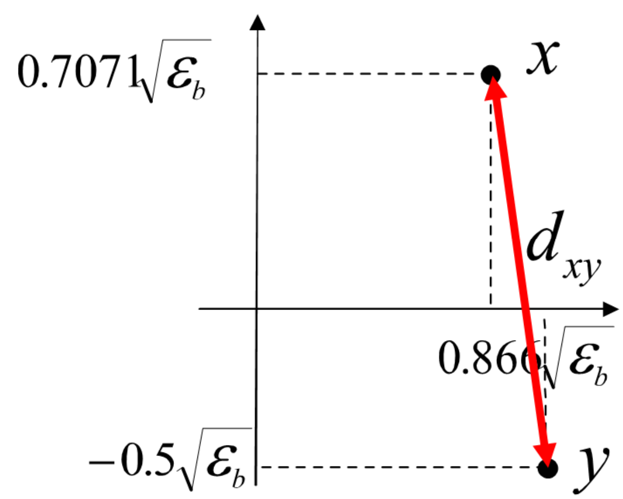
- Can find using the integral:

$$\left|x-y\right| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\int_{-\infty}^{+\infty}\left[x(t)-y(t)\right]^2}$$

- or using the signal space diagram:

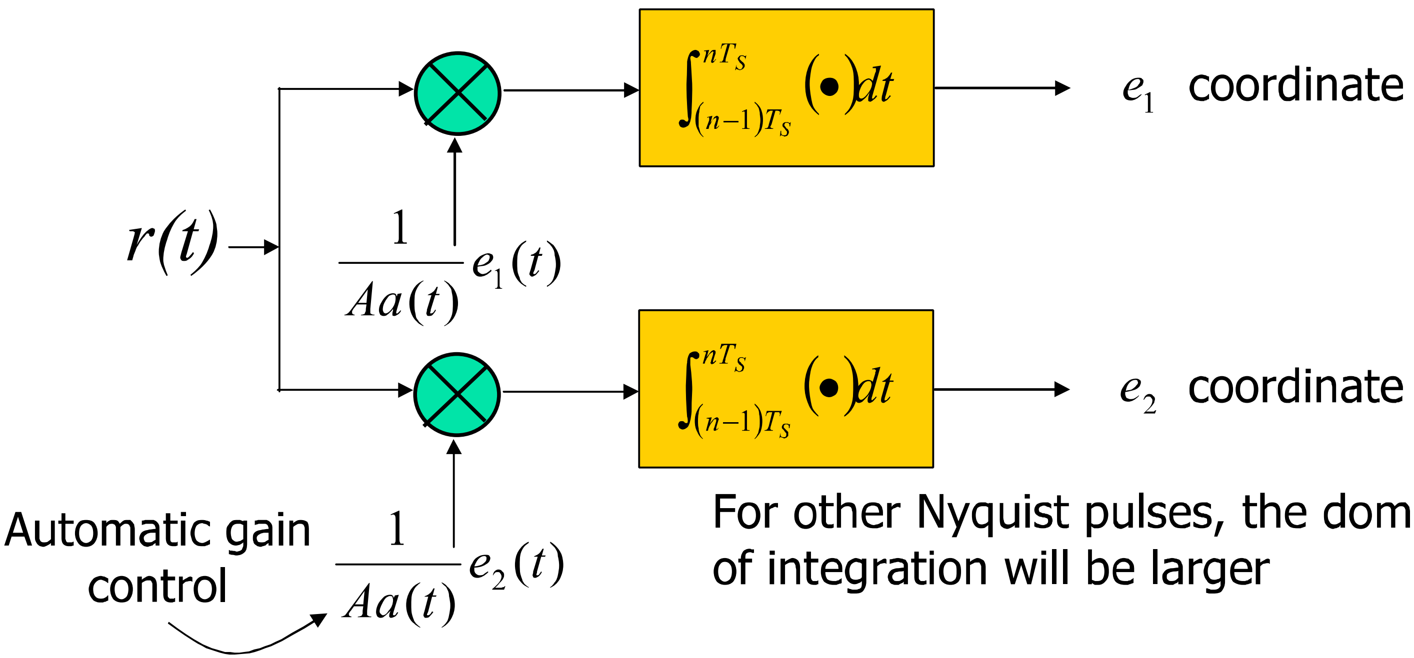
$$\begin{aligned} x-y &= (0.7071-0.866)\sqrt{\varepsilon_b}\,e_1 \\ &\quad + (0.7071-[-0.5])\sqrt{\varepsilon_b}\,e_2 \\ &= -1.589\sqrt{\varepsilon_b}\,e_1 + 1.2071\sqrt{\varepsilon_b}\,e_2 \end{aligned}$$

$$\begin{aligned} \left|x-y\right| &= \sqrt{(-1.589)^2\varepsilon_b + (1.2071)^2\varepsilon_b} \\ &= 1.751\sqrt{\varepsilon_b} = d_{xy} \end{aligned}$$



The Demodulator

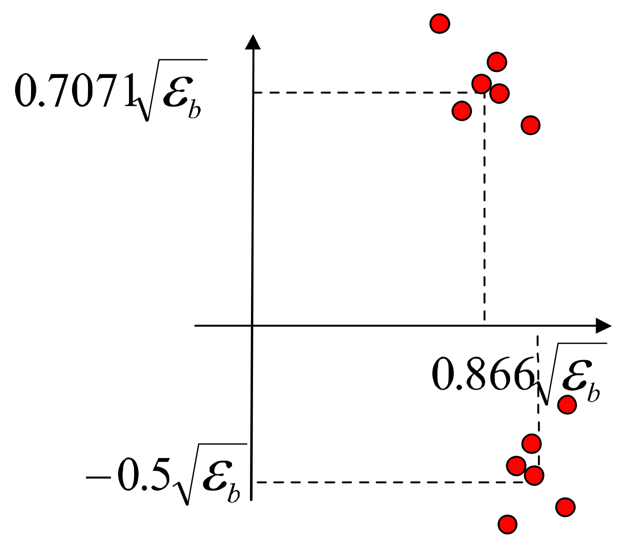
- The demodulator computes the coordinates of each received symbol



For other Nyquist pulses, the domain of integration will be larger

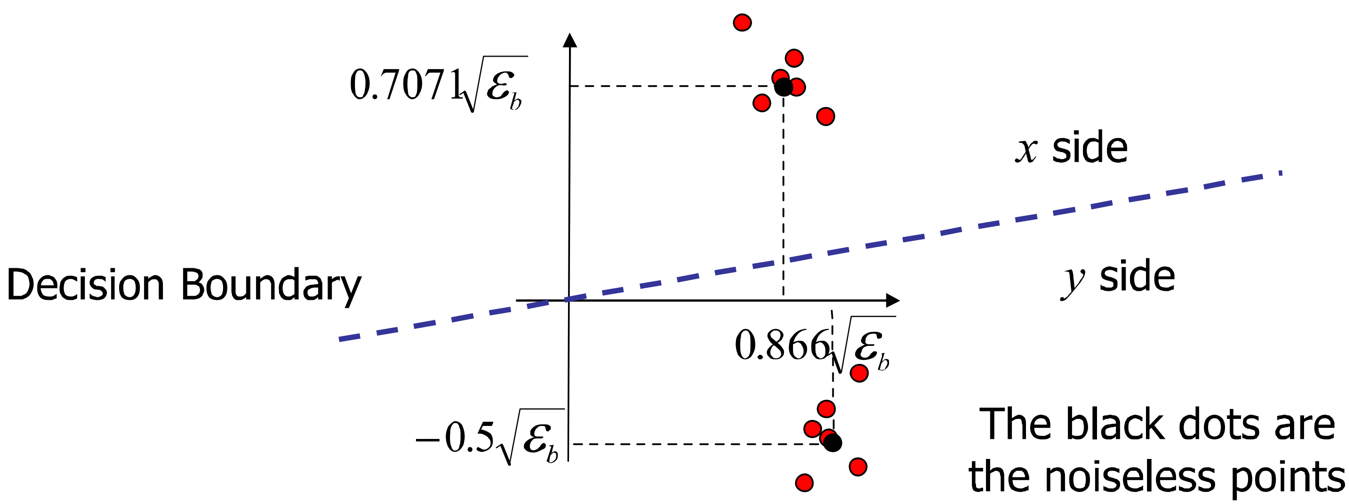
Receiver Signal Space

- Because of noise, the received points are scattered



Optimal Detection

- The optimal binary receiver effectively draws a threshold line between the two noiseless points



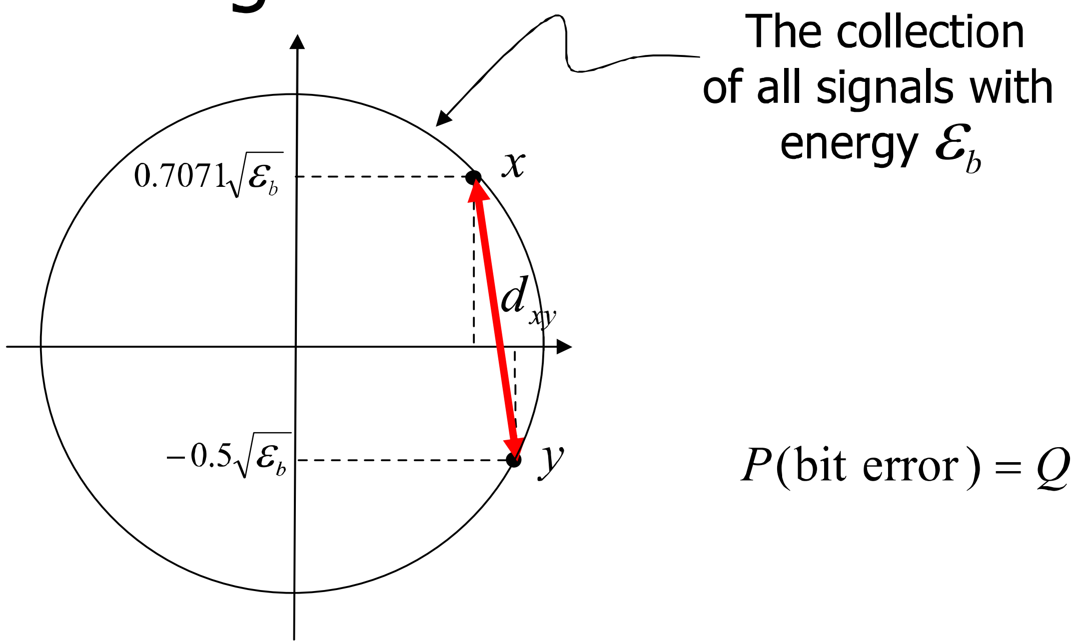
Binary Probability of Bit Error

- The probability of the received signal point being on the wrong side of the line
- It depends on the distance between the two noiseless points and the noise spectral height

$$P(\text{bit error}) = Q\left(\frac{d_{xy}}{\sqrt{2N_0}}\right)$$

Energy Efficient Signals

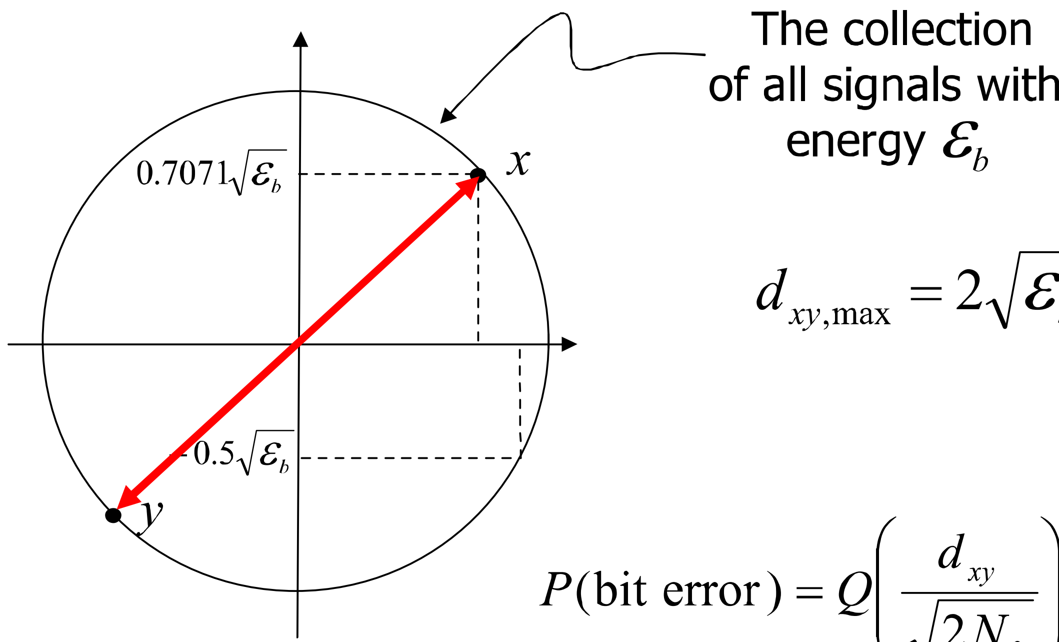
- If we constrain each signal to have energy $\mathcal{E}_b$, what is the best choice for the two signals?



$$P(\text{bit error}) = Q\left(\frac{d_{xy}}{\sqrt{2N_0}}\right)$$

Maximize the Distance

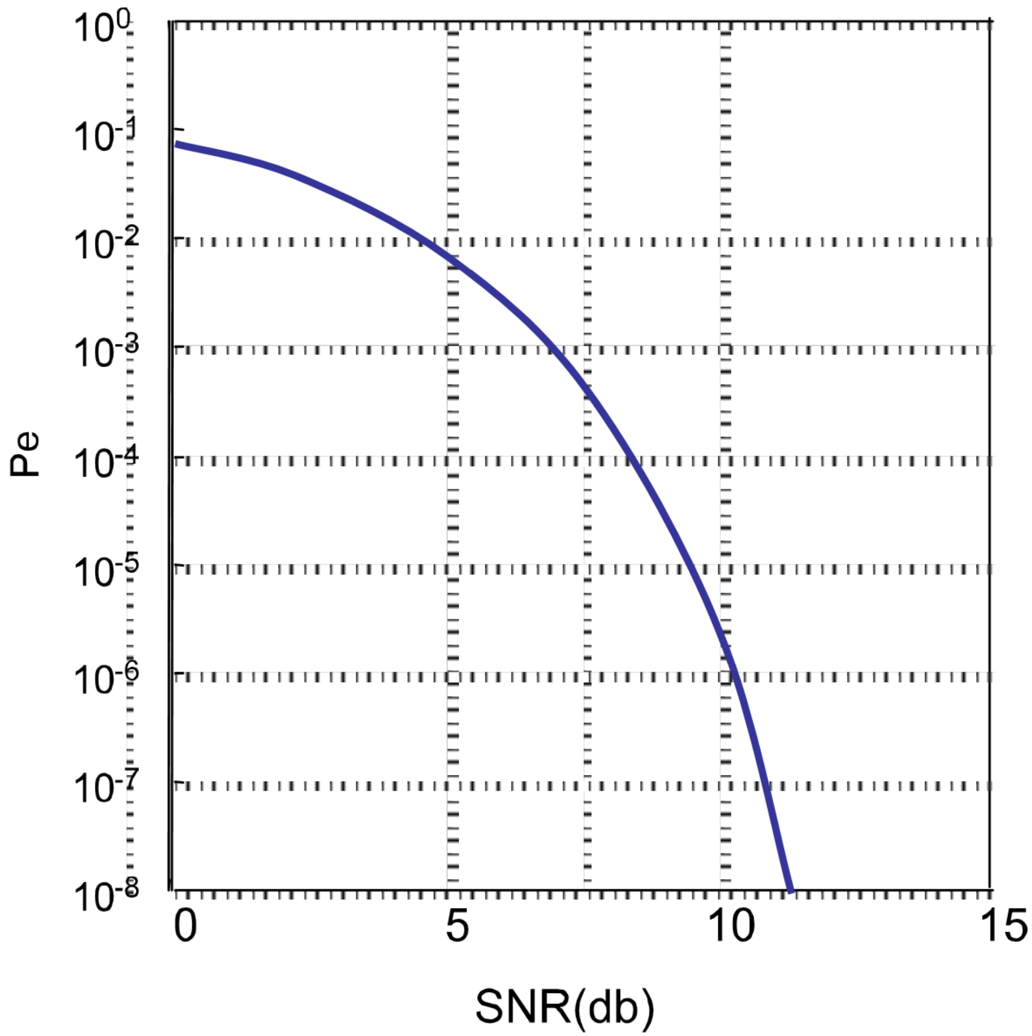
- $y(t) = -x(t)$ (BPSK)



$$d_{xy,\max} = 2\sqrt{\mathcal{E}_b}$$

$$P(\text{bit error}) = Q\left(\frac{d_{xy}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

BPSK BER Plot



$$SNR = \frac{\mathcal{E}_b}{N_0}$$

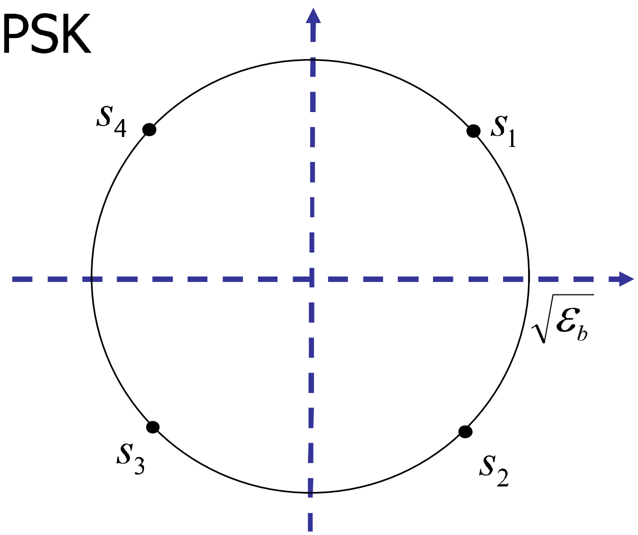
M-ary Modulation

- Can have M signals or symbols in the set

  $$M = 2^k$$

  where k is the number of bits per symbol
  - Example: QPSK

    $k=2,\ M=4$



Union Bound

- When M>2, a union bound may be used for the probability of symbol error

$$P(\text{ symbol error }) \leq \frac{1}{M}\sum_{i=1}^{M} P\left(\text{symbol error} \mid s_i \text{ sent}\right)$$

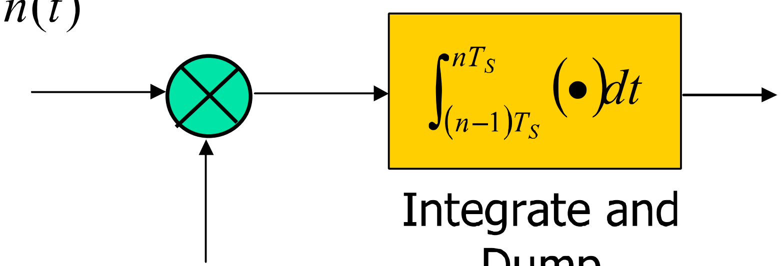
$$= \frac{1}{M}\sum_{i=1}^{M}\sum_{\substack{j=1 \\ j\neq i}}^{M} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

BPSK Receiver

- The local oscillator must match the phase of the incoming carrier

$$r(t) = Aa(t)\sqrt{\frac{2}{T_S}}\cos\left[2\pi f_c t + \theta_c + \phi(t)\right] + n(t)$$

Either 0 or π during each symbol period, depending on the symbol



Integrate and Dump

$$\frac{1}{Aa(t)}\cos\left(2\pi f_c t + \theta_c\right)$$

Local Oscillator

QPSK

- QPSK is like BPSK, except the phase can take four values instead of just two

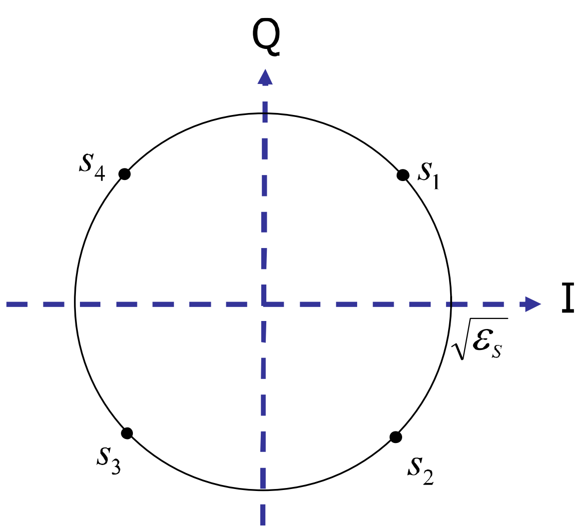
$$s(t) = \mathrm{Re}\left\{ g(t) e^{j2\pi f_c t} \right\}$$

$$g(t) = A \sum_n x_n p(t - nT_S)$$

$$x_n \in \left\{ e^{j\pi/4}, e^{j3\pi/4}, e^{-j\pi/4}, e^{-j3\pi/4} \right\}$$

QPSK Signal Space Diagram

- The bases functions are cosine and sine
  - The cosine component is called the “In Phase” component
  - The sine component is called the “Quadrature” component



Square Pulse Case

- A QPSK waveform:

$$s_{QPSK}(t)=\sqrt{\mathcal{E}_S}\cos\phi_n\sqrt{\frac{2}{T_S}}\cos(2\pi f_c t)-\sqrt{\mathcal{E}_S}\sin\phi_n\sqrt{\frac{2}{T_S}}\sin(2\pi f_c t)$$

  where $\phi_n \in \{\pi/4, 3\pi/4, -\pi/4, -3\pi/4\}$

  and $nT_S < t \le (n+1)T_S$

- $\mathcal{E}_S$ is the symbol energy
- $\mathcal{E}_b = \frac{1}{2}\mathcal{E}_S$ is the bit energy

QPSK=2BPSK

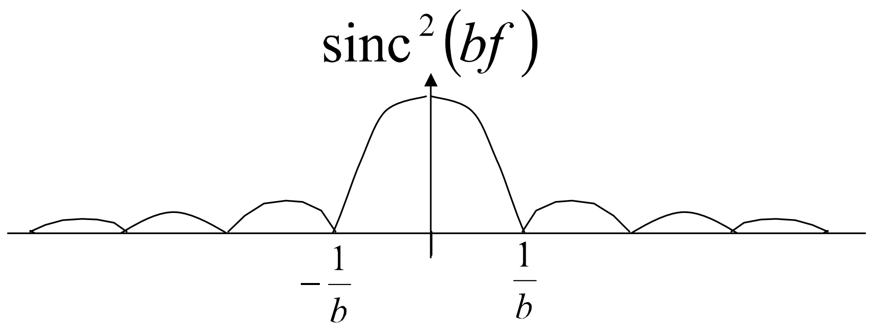
$$s_{QPSK}(t) = \underbrace{\sqrt{\mathcal{E}_S}\cos\phi_n\sqrt{\frac{2}{T_S}}\cos(2\pi f_c t)}_{\text{One BPSK Signal}} - \underbrace{\sqrt{\mathcal{E}_S}\sin\phi_n\sqrt{\frac{2}{T_S}}\sin(2\pi f_c t)}_{\text{Another BPSK Signal}}$$

- The two BPSK signals are separated in the receiver using two LOs, one a cosine, the other a sine

QPSK Performance

- BER same as BPSK for the same $\mathcal{E}_b$
- For the same data rate, QPSK has a bandwidth half of that of BPSK

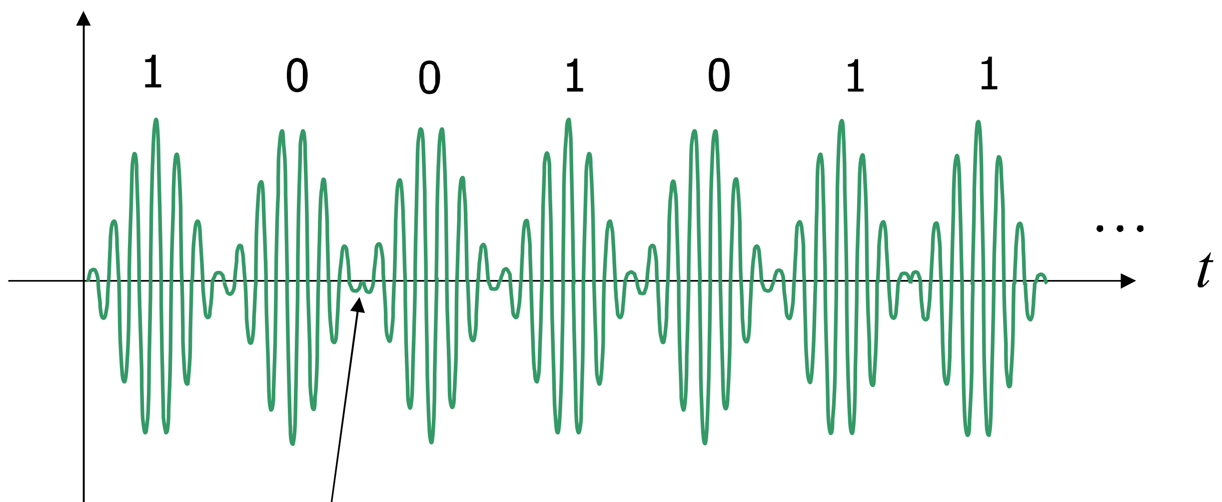
$$S_s(f) = \frac{\mathcal{E}_S}{2}\left[\operatorname{sinc}^2([f - f_c]T_S) + \operatorname{sinc}^2([-f - f_c]T_S)\right]$$
$$= \mathcal{E}_b\left[\operatorname{sinc}^2([f - f_c]2T_B) + \operatorname{sinc}^2([-f - f_c]2T_B)\right]$$



Pulse Shaping Effects

- Because of realistic pulse-shaping, the envelope of BPSK or QPSK is not constant
- Undesirable because linear amplifiers, which are not as power efficient and more expensive, are required

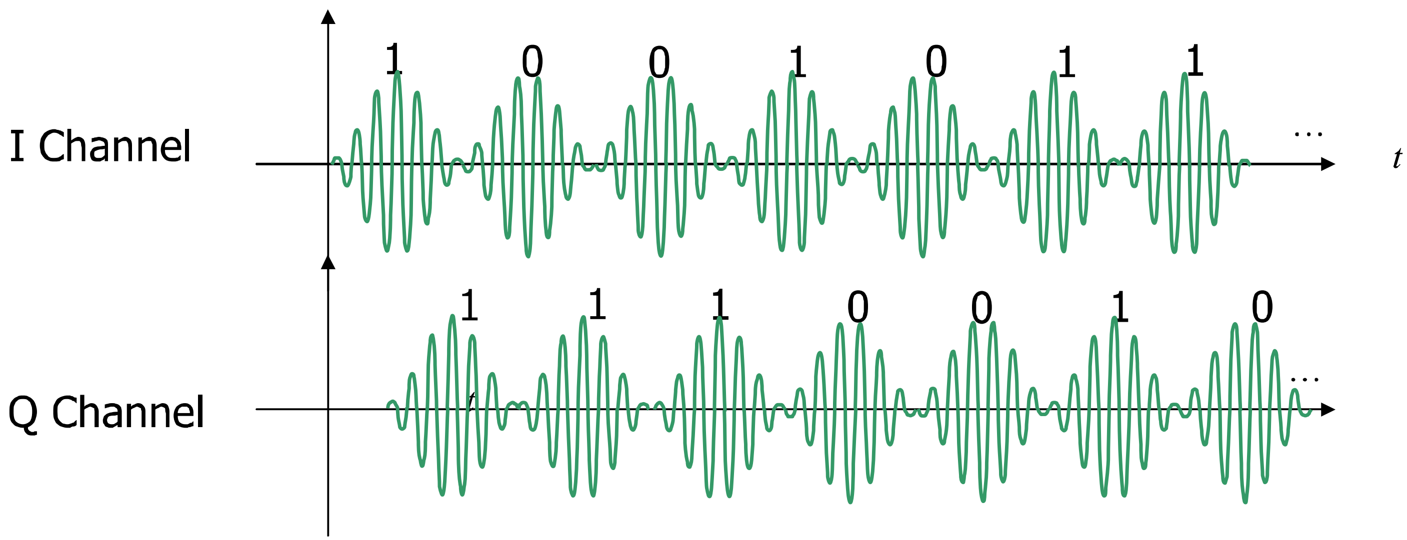
BPSK Example



Bandlimiting causes amplitude at 180° phase changes to go to zero

Offset QPSK

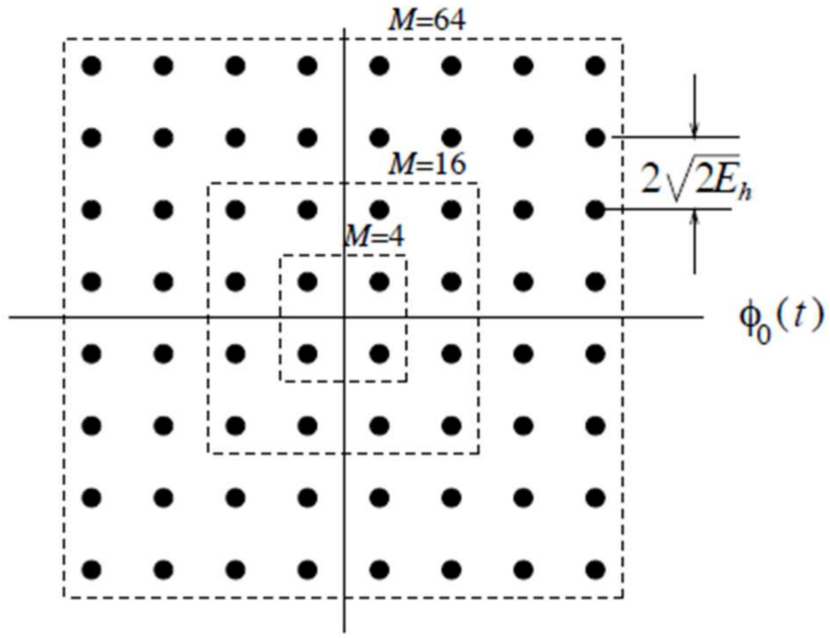
- Since the QPSK waveform is just a superposition of two independent BPSK waveforms, why not shift one relative to the other by half a symbol period to make the envelope more constant?



Offset QPSK Performance

- By switching the phase twice as often as QPSK, the max phase change becomes +/- 90 degrees instead of 180 degrees, so amplitude not forced to zero by bandlimiting
- Same spectrum as QPSK
- Same BER as QPSK

M-QAM



FSK

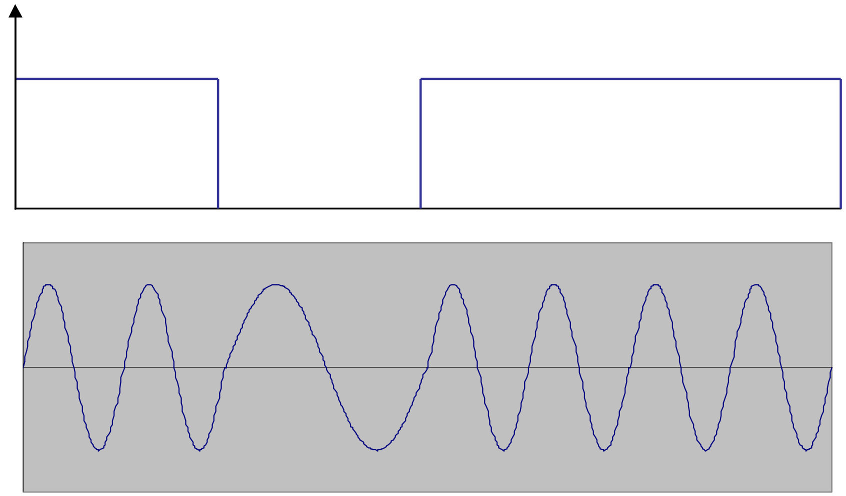
- Constant envelope
  - More efficient, less costly power amplifiers
- Gaussian minimum shift keying (GMSK), a special type of FSK, is used in the European digital cellular communications system (GSM)

Continuous Phase BFSK

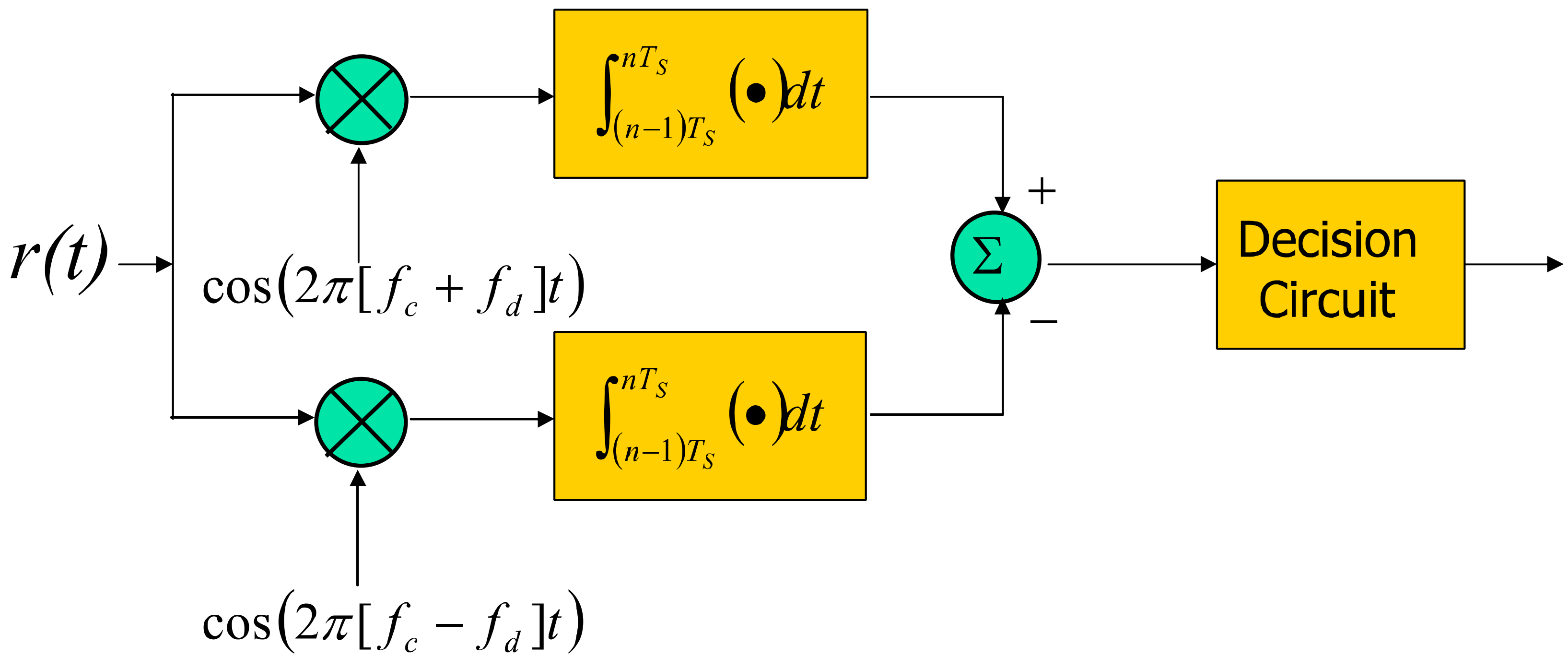
- Phase continuity is important to reduce bandwidth

NRZ Binary data

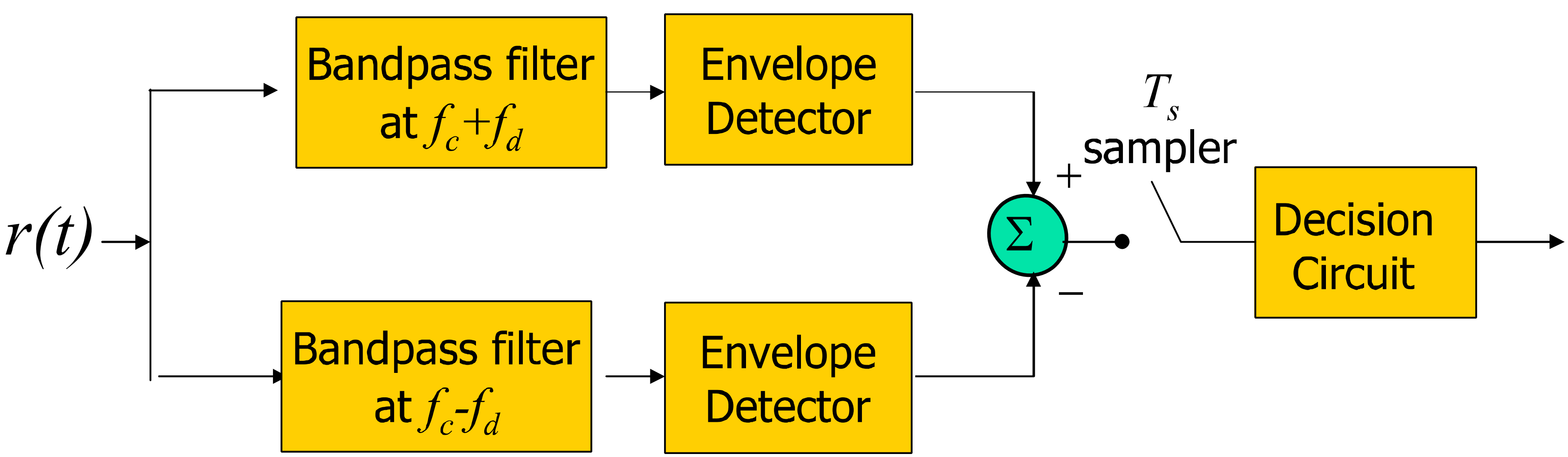
BFSK waveform



BFSK Coherent Detection



BFSK Non-Coherent Detection



Summary

- Digital modulation is a way of transforming information bits to symbols
- These symbols can be produced in a variety of ways by changing the amplitude, phase, or frequency of the carrier
- M-ary modulation increases the data rate of the system at the expense of high modulated power