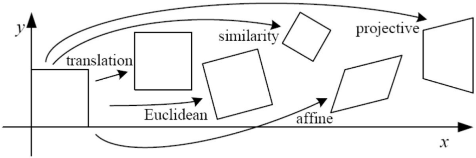
Fundamental Matrix
CS-477 Computer Vision

Dr. Mohsin Kamal
Associate Professor
dr.mohsinkamal@seecs.edu.pk

School of Electrical Engineering and Computer Science (SEECS)
National University of Sciences and Technology (NUST), Pakistan

Transformations Between Two Images

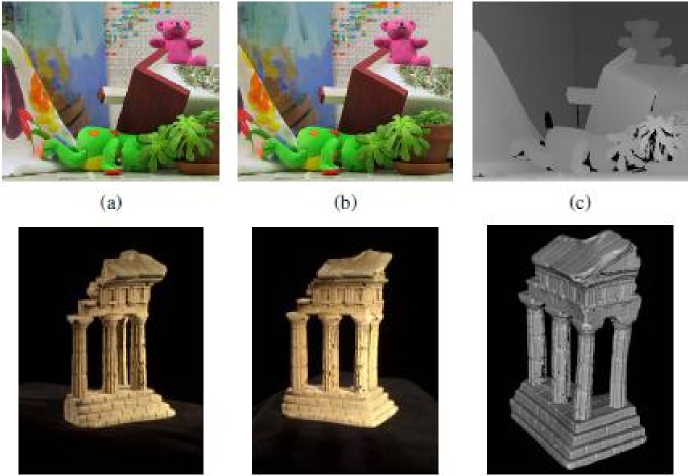
- Translation
- Rotation
- Rigid
- Similarity (scaled rotation)
- Affine
- Projective



Applications

- Stereo
- Structure from motion
- View invariant action recognition

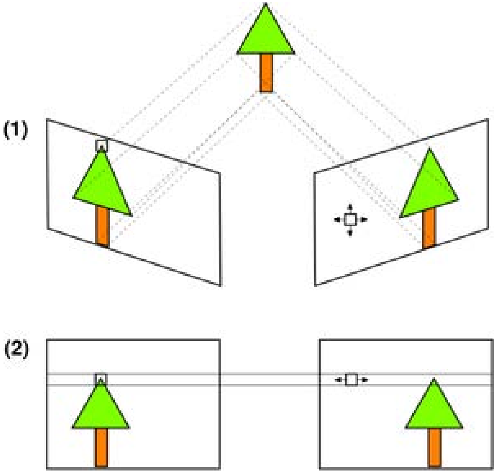
Stereo Pairs and Depth Maps



(a) (b) (c)

(d) (e) (f)

Image Rectification For Stereo



- Linear Independence
- Rank of a Matrix
- Matrix Norm
- Singular Value Decomposition
- Vector Cross product to Matrix Multiplication
- RANSAC

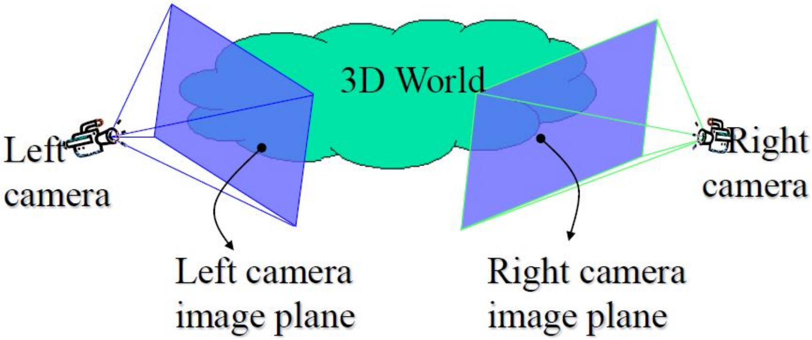
Vector Cross product to Matrix Multiplication

$$A \times B = \begin{bmatrix} 1 & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} = \begin{bmatrix} -A_z B_y + A_y B_z & A_z B_x - A_x B_z & -B_x A_y + B_y A_x \end{bmatrix}$$

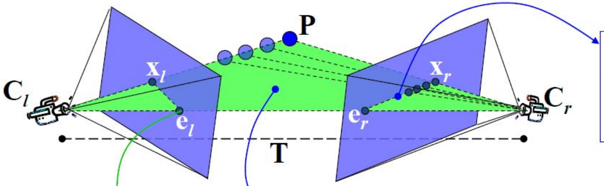
$$A \times B = S.B = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} -A_z B_y + A_y B_z & A_z B_x - A_x B_z & -B_x A_y + B_y A_x \end{bmatrix} \quad (1)$$

Epipolar Geometry

Defined for two static cameras



Epipolar Geometry

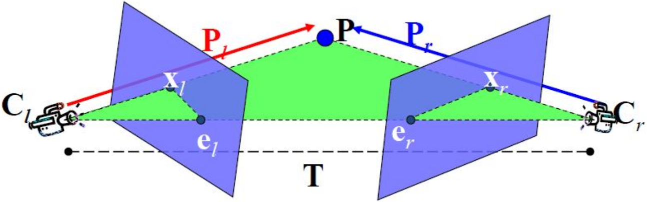


Epipolar line: set of world points that project to the same point in left image, when projected to right image forms a line.

Epipolar plane: plane defined by the camera centers and world point.

Epipole: intersection of image plane with line connecting camera centers. Image of a left camera center in the right, and vice versa.

Essential Matrix



Coplanarity constraint between vectors $(P_l - T)$, T, P_l.

$$(P_l - T)^T T \times P_l = 0 \tag{2}$$

$$P_r = R(P_l - T)$$

Above equation becomes

$$R^T P_r = (P_l - T)$$
$$P_r^T R = (P_l - T)^T$$

Put in the equation 2, which becomes

$$P_r^T RT \times P_l = 0 \tag{3}$$

Essential Matrix

As per equation 1, equation 3 can be written as:

$$P_r^T RSP_l = 0 \tag{4}$$

Where,

$$S = \begin{bmatrix} 0 & -T_z & T_y \\ T_z & 0 & -T_x \\ -T_y & T_x & 0 \end{bmatrix}$$

- S is the translation anti-symmetrical matrix because 0s are on the diagonal and T_x, T_y and T_z are negatives of each other.
- There are two matrices in equation 4, i.e., R and S.
- Hence, they can be represented by a multiplication of R and S matrices as E which is called **Essential Matrix** i.e., $E = RS$.

Equation 4 becomes

$$P_r^T EP_l = 0 \tag{5}$$

Using this equation, we are capturing the transformation between two cameras in 3D by a simple equation. The coordinates of point P in the right camera coordinate system, the coordinates of the same point in left camera coordinate system, multiplied with E should result in a 0.

Fundamental Matrix

We know that from the camera model that

$$x_l = M_l P_l$$
$$x_r = M_r P_r$$

where M represents the camera matrix.

Using the above two equations, we can find P_r^T and P_l for equation 5. Hence

$$M_l^{-1} x_l = P_l$$
$$M_r^{-1} x_r = P_r$$
$$x_r^T M_r^{-T} = P_r^T$$

Equation 5 becomes

$$x_r^T M_r^{-T} E M_l^{-1} x_l = 0$$
$$x_r^T \left(M_r^{-T} E M_l^{-1} \right) x_l = 0$$

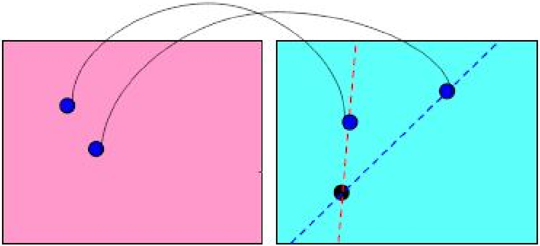
$$x_r^T F x_l = 0 \tag{6}$$

F is the **fundamental matrix.**

Fundamental Matrix

$$x'^T F x = x'^T \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & m \end{bmatrix} x = 0$$

- F is a 3×3 matrix with 9 components
- Rank is 2 (due to S)
- Given a point in left camera x, epipolar line in the right camera is: $u_r = Fx$



Fundamental Matrix

Fundamental matrix captures the relationship between the corresponding points in two views.

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x_i' \\ y_i' \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11}x' + f_{12}y_i' + f_{13} \\ f_{21}x' + f_{22}y_i' + f_{23} \\ f_{31}x' + f_{32}y_i' + f_{33} \end{bmatrix} = 0$$

Expand the matrix to get linear equation:

$$x_i(f_{11}x' + f_{12}y_i' + f_{13}) + y_i(f_{21}x' + f_{22}y_i' + f_{23}) + f_{31}x' + f_{32}y_i' + f_{33} = 0$$

$$x_i x' f_{11} + x_i y_i' f_{12} + x_i f_{13} + y_i x' f_{21} + x' y_i' f_{22} + y_i' f_{23} + x' f_{31} + y_i' f_{32} + f_{33} = 0$$

Fundamental Matrix

The last equation shows one equation for one point correspondence. Hence we can elaborate it as:

$$Mf = \begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 & 1 \\ x'_2x_2 & x'_2y_2 & x'_2 & y'_2x_2 & y'_2y_2 & y'_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_nx_n & x'_ny_n & x'_n & y'_nx_n & y'_ny_n & y'_n & x_n & y_n & 1 \end{bmatrix} f = 0$$

where M is 9 $\times$ n matrix and

$$f = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{21} & f_{22} & f_{23} & f_{31} & f_{32} & f_{33} \end{bmatrix}$$