Optical Flow
CS-477 Computer Vision

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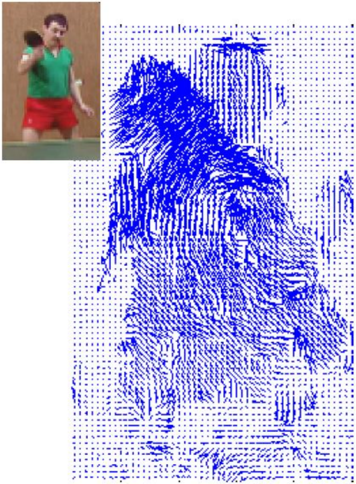
1 Optical Flow

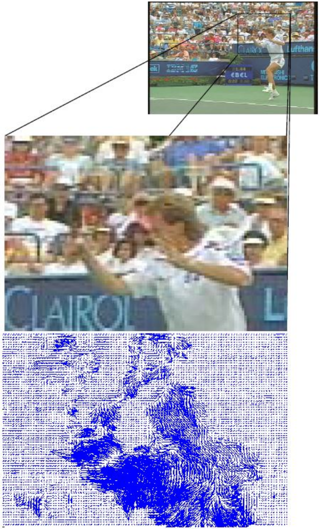
2 Pyramids

3 Lucas Kanade with Pyramids

- Optical flow is a technique used to describe image motion.
- It is usually applied to a series of images that have a small time step between them, for example, video frames.
- Optical flow calculates a velocity for points within the images, and provides an estimation of where points could be in the next image sequence.
- Applications:
  - object tracking,
  - video stabilization, and
  - motion analysis.

Motion of the brightness patterns in an image is referred to as the "Optical Flow".

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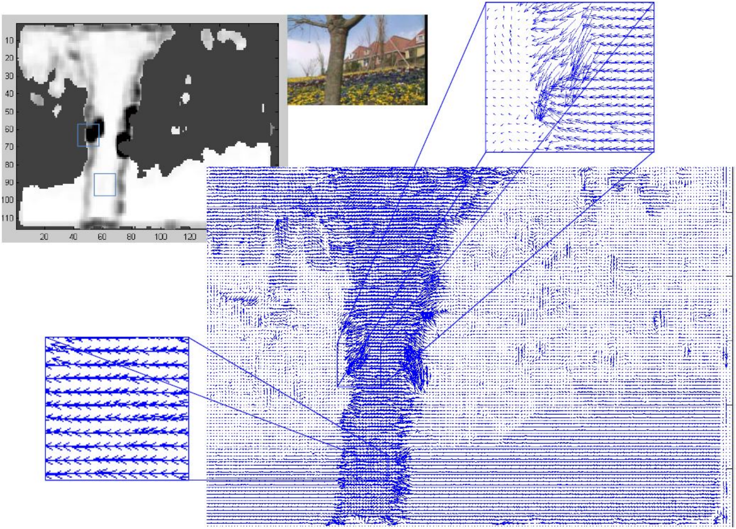
Optical flow equation



Optical flow equation

A1: Brightness constancy assumption

$$f(x, y, t) = f(x + dx, y + dy, t + dt)$$

A2: Displacements and time step are small

Taylor series approximation gives

$$f(x + dx, y + dy, t + dt) = f(x, y, t) + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial t}dt$$

let $\frac{\partial f}{\partial x}$ be denoted as f_x and so on. Above equation can then be written as:

$$f(x + dx, y + dy, t + dt) = f(x, y, t) + f_x dx + f_y dy + f_t dt$$

Subtracting **A1** from **A2**, we get

$$f_x dx + f_y dy + f_t dt = 0$$

Divide both sides by dt, we get

$$f_x u + f_y v + f_t = 0$$

Interpretation of optical flow equation

As we know

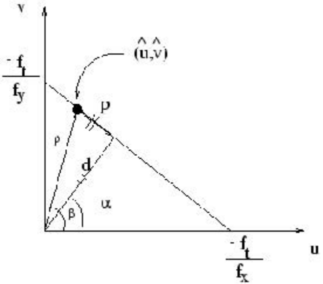
$$f_x u + f_y v + f_t = 0$$

This optical flow equation can further be modified as,

$$v = -\frac{f_x}{f_y} u - \frac{f_t}{f_y} \quad (1)$$

which is representing the equation of a straight line

$d =$ normal flow
$p =$ parallel flow
where $d = \frac{f_t}{\sqrt{f_x^2 + f_y^2}}$



Derivative Masks

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \text{first image} \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \text{first image} \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \text{first image}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \text{second image} \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \text{second image} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{second image}$$

$$f_x \qquad\qquad f_y \qquad\qquad f_t$$

- Apply first mask to 1st image
- Apply second mask to 2nd image
- Add the response to get f_x, f_y and f_t

Lucas & Kanade

The Lucas Kanade method uses the assumption that the optical flow in a very small neighborhood in a scene is the same for all the points within that neighborhood.

Lucas & Kanade

Method 1:

- Optical flow equation

$$f_x u + f_y v = -f_t$$

- Consider a 3 by 3 window

$$f_{x1} u + f_{y1} v = -f_{t1}$$
$$\vdots$$
$$f_{x9} u + f_{y9} v = -f_{t9}$$

- This can be written as:

$$\begin{bmatrix} f_{x1} & f_{y1} \\ \vdots & \vdots \\ \\ f_{x9} & f_{y9} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -f_{t1} \\ \vdots \\ \\ -f_{t9} \end{bmatrix}$$

$$Au = f_t$$

Lucas & Kanade

$$Au = f_t$$

To make it a squared matrix, multiply A^T on both sides

$$A^T A u = A^T f_t$$
$$u = (A^T A)^{-1} A^T f_t$$

Method 2:

Apply least square fit as:

$$\min \sum_i (f_{xi} u + f_{yi} v + f_{ti})^2$$

$$\sum (f_{xi} u + f_{yi} v + f_{ti}) f_{xi} = 0$$
$$\sum (f_{xi} u + f_{yi} v + f_{ti}) f_{yi} = 0$$

$$\sum f_{xi}^2 u + \sum f_{xi} f_{yi} v = -\sum f_{xi} f_{ti}$$
$$\sum f_{xi} f_{yi} u + \sum f_{yi}^2 v = -\sum f_{yi} f_{ti}$$

$$\begin{bmatrix} \sum f_{xi}^2 & \sum f_{xi} f_{yi} \\ \sum f_{xi} f_{yi} & \sum f_{yi}^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\sum f_{xi} f_{ti} \\ -\sum f_{yi} f_{ti} \end{bmatrix}$$

Lucas & Kanade

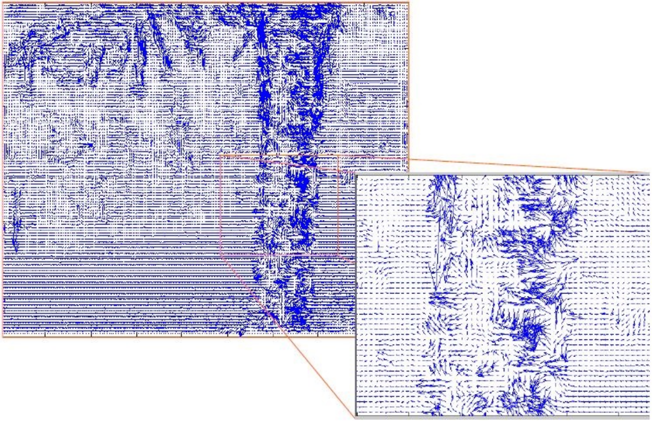
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum f_{xi}^2 & \sum f_{xi} f_{yi} \\ \sum f_{xi} f_{yi} & \sum f_{yi}^2 \end{bmatrix}^{-1} \begin{bmatrix} -\sum f_{xi} f_{ti} \\ -\sum f_{yi} f_{ti} \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sum f_{xi}^2 \sum f_{yi}^2 - (\sum f_{xi} f_{yi})^2} \begin{bmatrix} \sum f_{yi}^2 & -\sum f_{xi} f_{yi} \\ -\sum f_{xi} f_{yi} & \sum f_{xi}^2 \end{bmatrix} \begin{bmatrix} -\sum f_{xi} f_{ti} \\ -\sum f_{yi} f_{ti} \end{bmatrix}$$

$$u = \frac{-\sum f_{yi}^2 \sum f_{xi} f_{ti} + \sum f_{xi} f_{yi} \sum f_{yi} f_{ti}}{\sum f_{xi}^2 \sum f_{yi}^2 - (\sum f_{xi} f_{yi})^2}$$

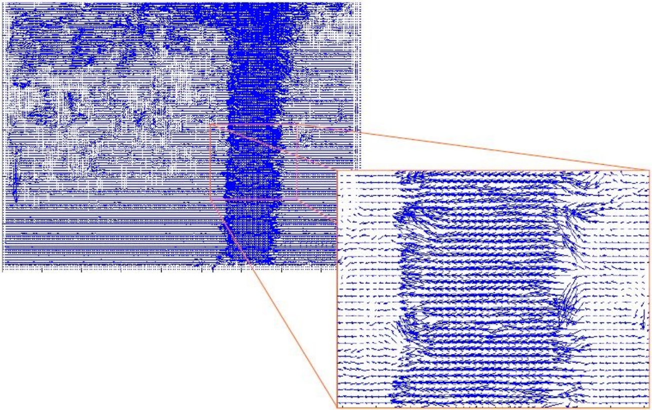
$$v = \frac{\sum f_{xi} f_{ti} \sum f_{xi} f_{yi} - \sum f_{xi}^2 \sum f_{yi} f_{ti}}{\sum f_{xi}^2 \sum f_{yi}^2 - (\sum f_{xi} f_{yi})^2}$$

Without pyramids



Fails in areas of large motion

With pyramids



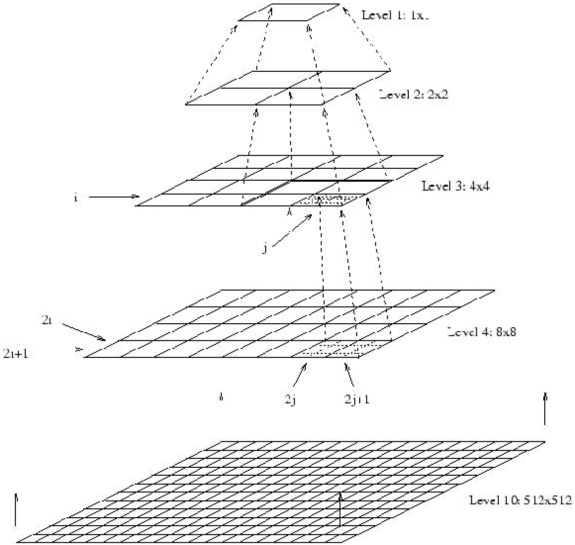
Comments

- Lucas-Kanade optical method works only for small motion.
- If object moves faster, the brightness changes rapidly,
  - 2×2 or 3×3 masks fail to estimate spatiotemporal derivatives.
- Pyramids can be used to compute large optical flow vectors.

Pyramids

- Very useful for representing images.
- Pyramid is built by using multiple copies of image.
- Each level in the pyramid is 1/4 of the size of previous level.
- The lowest level is of the highest resolution.
- The highest level is of the lowest resolution.

Pyramids


Level 1: 1x1
Level 2: 2x2
i
Level 3: 4x4
j
2i
2i+1
Level 4: 8x8
2j
2j+1
Level 10: 512x512

Reduce

$$g_l(i,j) = \sum_{m=-2}^{2} \sum_{n=-2}^{2} w(m,n) g_{l-1}(2i+m, 2j+n) \tag{2}$$

- where l represents the level.
- $g_l = Reduce[g_{l-1}]$
- w represents the mask
- g is the image

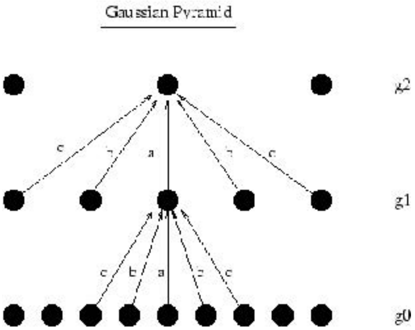
Reduce (1D)

$$g_l(i) = \sum_{m=-2}^{2} w(m) g_{l-1}(2i+m) \tag{3}$$

$$g_l(2) = w(-2)g_{l-1}(4-2) + w(-1)g_{l-1}(4-1) + w(0)g_{l-1}(4) + w(1)g_{l-1}(4+1) + w(2)g_{l-1}(4+2)$$

$$g_l(2) = w(-2)g_{l-1}(2) + w(-1)g_{l-1}(3) + w(0)g_{l-1}(4) + w(1)g_{l-1}(5) + w(2)g_{l-1}(6)$$

Reduce



$g0 = \text{IMAGE}$

$g1 = \text{REDUCE}[g_{l-1}]$

Expand

$$g_{l,n}(i,j) = \sum_{p=-2}^{2} \sum_{q=-2}^{2} w(p,q) g_{l,n-1}\left(\frac{i-p}{2}, \frac{j-q}{2}\right) \tag{4}$$

- where l represents the level.
- $g_l = Expand[g_{l,n-1}]$
- w represents the mask
- g is the image

Expand (1D)

$$g_{l,n}(i) = \sum_{p=-2}^{2} w(p) g_{l,n-1}\left(\frac{i-p}{2}\right) \quad (5)$$

$$g_{l,n}(4) = w(-2)g_{l,n-1}\left(\tfrac{4+2}{2}\right) + w(-1)g_{l,n-1}\left(\tfrac{4+1}{2}\right) + w(0)g_{l,n-1}\left(\tfrac{4}{2}\right) + w(1)g_{l,n-1}\left(\tfrac{4-1}{2}\right) + w(2)g_{l,n-1}\left(\tfrac{4-2}{2}\right)$$

$$g_{l,n}(4) = w(-2)g_{l,n-1}(3) + w(0)g_{l,n-1}(2) + w(2)g_{l,n-1}(1)$$

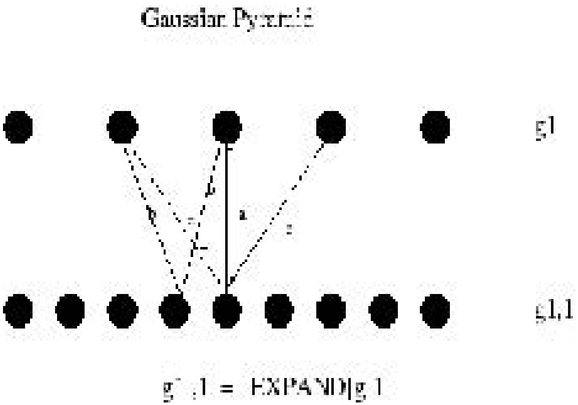
Expand (1D) - Another example

$$g_{l,n}(i) = \sum_{p=-2}^{2} w(p) g_{l,n-1}\left(\frac{i-p}{2}\right) \tag{6}$$

$$g_{l,n}(3) = w(-2)g_{l,n-1}(\tfrac{3+2}{2}) + w(-1)g_{l,n-1}(\tfrac{3+1}{2}) + w(0)g_{l,n-1}(\tfrac{3}{2}) + w(1)g_{l,n-1}(\tfrac{3-1}{2}) + w(2)g_{l,n-1}(\tfrac{3-2}{2})$$

$$g_{l,n}(3) = w(-1)g_{l,n-1}(2) + w(1)g_{l,n-1}(1)$$

Expand



Convolution mask

We have: $[w(-2), w(-1), w(0), w(1), w(2)]$

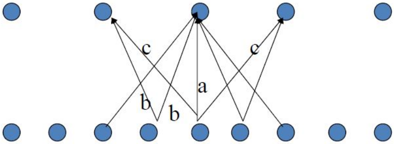
Properties of convolution mask

1. Separable, i.e., $w(m, n) = w(m)w(n)$
2. Symmetric, i.e., $w(i) = w(-i)$

Convolution mask

- The sum of mask should be equal to 1.
$$a + 2b + 2c = 1$$
- All nodes at a given level must contribute the same total weight to the nodes at the next higher level.
$$a + 2c = 2b$$

Convolution mask



From the above, the following can be observed

$$a + 2c = 2b$$
$$a + 2b + 2c = 1$$

Solving the above two equations, we get

$$b = \frac{1}{4}$$
$$c = \frac{1}{2}(2b - a)$$
$$c = \frac{1}{4} - \frac{a}{2}$$

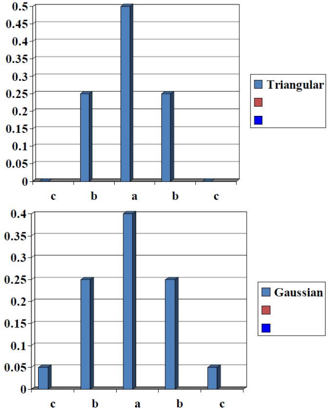
Convolution mask

So,

$$w(0) = a$$
$$w(-1) = w(1) = \frac{1}{4}$$
$$w(-2) = w(2) = \frac{1}{4} - \frac{a}{2}$$

We can get approximation of gaussian Gaussian and Triangular by using $a = 0.4$ and $a = 0.5$, respectively.

Convolution mask



Algorithm

- Apply 1-D mask to alternate pixels along each row of image.
- Apply 1-D mask to each pixel along alternate columns of resultant image from previous step.

Gaussian Pyramids

Laplacian Pyramids

- Similar to edge detected images.
- Most pixels are zero.
- Can be used for image compression

$$L_1 = g_1 - Expand[g_2]$$
$$L_2 = g_2 - Expand[g_3]$$
$$L_3 = g_3 - Expand[g_4]$$

Laplacian Pyramids

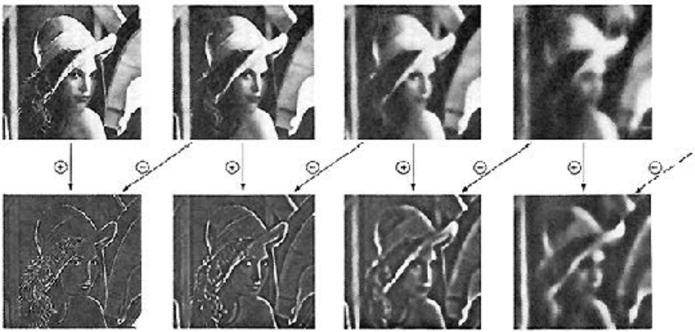


Fig. 5. First four levels of the Gaussian and Laplacian pyramid. Gaussian images, upper row, were obtained by expanding pyramid arrays (Fig. 4) through Gaussian interpolation. Each level of the Laplacian pyramid is the difference between the corresponding and next higher levels of the Gaussian pyramid.

Steps of implementing Laplacian Pyramid

- Compute Gaussian Pyramid

$$g_1, g_2, g_3, g_4$$

- Compute Laplacian pyramid

$$\begin{aligned} L_1 &= g_1 - Expand[g_2] \\ L_2 &= g_2 - Expand[g_3] \\ L_3 &= g_3 - Expand[g_4] \\ L_4 &= g_4 \end{aligned}$$

- Code Laplacian pyramid

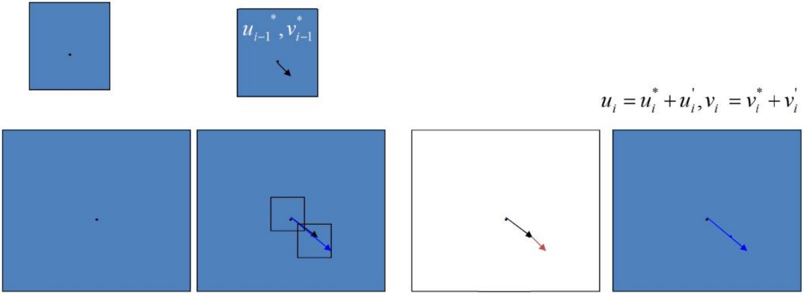
Decoding using Laplacian Pyramid

- Decode Laplacian pyramid
- Compute Gaussian pyramid from Laplacian pyramid

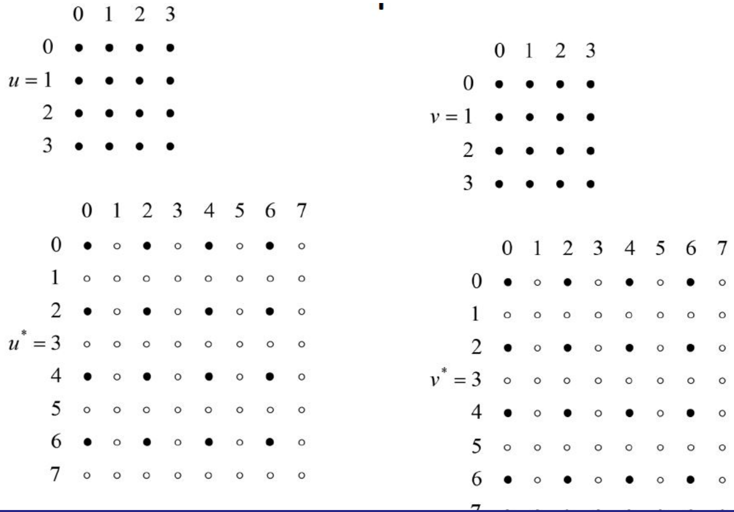
$$\begin{aligned} g_4 &= L_4 \\ g_3 &= Expand[g_4] + L_3 \\ g_2 &= Expand[g_3] + L_2 \\ g_1 &= Expand[g_2] + L_1 \end{aligned}$$

- g_1 is the reconstructed image

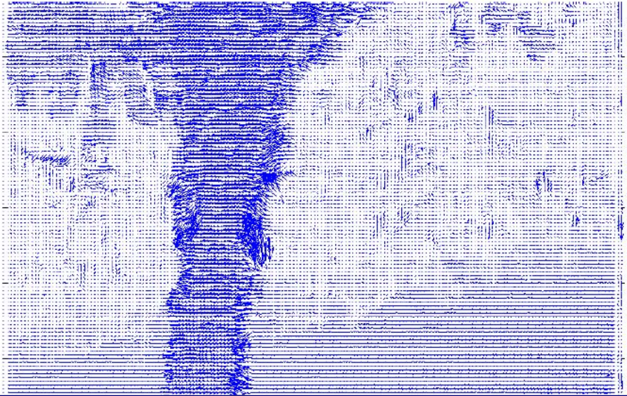
- Compute the pyramids of frames i.e., frame 1, frame 2, ...
- Compute "simple" LK optical flow at highest level (because we assume that the motion will be small)
- At level i
  - Take flow u_{i-1}, v_{i-1} from level $i-1$
  - bilinear interpolate it to create u_i^*, v_i^* matrices of twice resolution for level i
  - multiply u_i^*, v_i^* by 2
  - compute f_t from a block displaced by $u_i^*(x, y), v_i^*(x, y)$
  - Apply LK to get $u_i^{'}(x, y), v_i^{'}(x, y)$ (the correction in flow)
  - Add corrections $u_i^{'} v_i^{'}$, i.e., $u_i = u_i^* + u_i^{'}, v_i = v_i^* + v_i^{'}$.


u_{i-1}^*, v_{i-1}^*
$u_i = u_i^* + u_i', v_i = v_i^* + v_i'$

Interpolation



LK



LK