Bernoulli-Euler beam theory

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1 Introduction

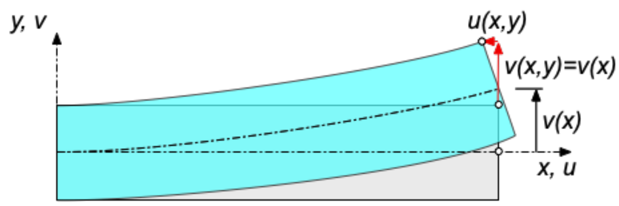

Figure 1: Deformation of the Bernoulli-Euler beam. Definition of coordinate axes and components of displacement.

This tool employs the Bernoulli-Euler beam theory. This theory, also known as *shear rigid beam theory*, is based on the kinematic assumption that

Any plane cross section perpendicular to the undeformed beam's axis remain plane and perpendicular to the axis throughout the deformation.

This allows us to reduce the three-dimensional problem to a single unknown function, $v(x)$, known as *deflection* of the beam.

2 Kinematics

Navier's assumption leads to

$$u(x, y) = -y\, v'(x) \qquad v(x, y) = v(x) \tag{1}$$

This displacement field induces an axial strain of

$$\varepsilon(x, y) = \frac{\partial u(x, y)}{\partial x} = -y\, v''(x)\ . \tag{2}$$

Equation (2) states that a fiber parallel to the beam axis stretches in the bottom portion of the beam ($y < 0$) and contracts if the fiber is located above the beam axis ($y > 0$). The beam axis itself does not stretch.

3 Constitutive relations

For a slender beam, we can ignore stress components acting perpendicular to the beam's axis. Thus, the constitutive relations can be simplified as the 1D-version of Hooke's law:

$$\sigma(x, y) = E\, \varepsilon(x, y) \tag{3}$$

where E is the modulus of elasticity.

The imposed state of deformation induces normal stress proportional to the strain field (2) as

$$\sigma(x, y) = -Ey\, v''(x)\ . \tag{4}$$

This relation states that (i) the stress varies linearly with the distance from the beam's axis, vanishing at the axis, and (ii) the stress is proportional to the curvature of the beam.

4 Stress resultants

The beam sees two stress resultants: the internal moment,

$$M(x) = -\int_A y\sigma(x, y)\, dA \tag{5}$$

and the transverse shear force,

$$V(x) = -\int_A \tau_{xy}(x, y)\, dA \tag{6}$$

Substituting (4) into (5) yields

$$M(x) = \int_A Ey^2\, v''(x)\, dA = E\, v''(x) \int_A y^2\, dA = EI\, v''(x) \tag{7}$$

where

$$I = \int_A y^2 \, dA \tag{8}$$

is the *area moment of inertia* or, short, *moment of inertia*.

Note that the modulus of elasticity, E, characterizes the material, the moment of inertia, I, characterized the shape of the cross section, and the second derivative of the deflection, $v''(x)$, characterizes the deformation (curvature) of the beam.

5 Equilibrium

Equilibrium is formulated in terms of shear forces, $V(x)$, and internal moments, $M(x)$. Equilibrium of forces on an beam element of infinitesimal length, formulated in the y-direction, yields

$$V'(x) = -w(x) \tag{9}$$

where $w(x)$ is the distributed lateral load per length. $w(x)$ is defined positive if pointing against the (upward) positive y-axis.

Moment equilibrium around the out-of-plane axis on the same element yields

$$M'(x) = V(x) \ . \tag{10}$$

A system for which equations (9) and (10) are sufficient to determine the internal moment and shear functions is called *statically determinate*. Otherwise, the system is called *statically indeterminate*. Solving these equations for the latter requires consideration of the kinematic relation (7) and respective boundary conditions.

Equations (9) and (10) may be combined into one equation as

$$M''(x) = V'(x) = -w(x) \tag{11}$$

Equation (11) replaces both equilibrium equations (9) and (10).

6 Governing equation

The governing equation is obtained by assuming the displacement function, $v(x)$, as the primary unknown and expressing $M(x)$ in (11) using (7) to obtain

$$\big(EI(x)\, v''(x)\big)'' + w(x) = 0 \tag{12}$$

This equation is known as the governing equation of the Bernoulli-Euler beam.

If the beam possesses a constant cross section and is made of one material, then $EI(x) = EI =$ *const*. and (12) simplifies to

$$EI\, v''''(x) + w(x) = 0 \tag{13}$$

Equation (13) is what is implemented in this program.

7 Finding moment, shear force, and slope from the displacement function

Solving (13) and applying suitable boundary conditions yields the displacement function, $v(x)$, for the beam. The slope, $\theta(x)$, is obtained through differentiation as

$$\theta(x) = v'(x) \ . \tag{14}$$

It is positive if the cross section rotates counter-clockwise during deformation.

The moment follows from (7) as

$$M(x) = EI(x)\, v''(x) = EI(x)\, \theta'(x) \ . \tag{15}$$

The transverse shear force follows from (11) as

$$V(x) = M'(x) = \big(EI(x)\, v''(x)\big)' \tag{16}$$

or, for constant EI, simplifies to

$$V(x) = EI\, v'''(x) \ . \tag{17}$$

8 Examples

8.1 Single span beam with constant distributed force

Both bending stiffness, EI, and distributed load, $w(x) = w_0$, are constant over the length of the beam. Thus, (13) simplifies to

$$v''''(x) = -\frac{w_0}{EI} \tag{18}$$

$$v'''(x) = -\frac{w_0\, x}{EI} + c_1 \tag{19}$$

$$v''(x) = -\frac{w_0\, x^2}{2EI} + c_1\, x + c_2 \tag{20}$$

$$\theta(x) = v'(x) = -\frac{w_0\, x^3}{6EI} + \frac{1}{2}\, c_1\, x^2 + c_2\, x + c_3 \tag{21}$$

$$v(x) = -\frac{w_0\, x^4}{24EI} + \frac{1}{6}\, c_1\, x^3 + \frac{1}{2}\, c_2\, x^2 + c_3\, x + c_4 \tag{22}$$

$$M(x) = EI\, v''(x) = -\frac{w_0\, x^2}{2} + EI\, c_1\, x + EI\, c_2 \tag{23}$$

$$V(x) = M'(x) = EI\, v'''(x) = -w_0\, x + EI\, c_1 \tag{24}$$

Pinned on both ends yields the boundary conditions

$$\left\{ \begin{array}{c} v(0) \\ M(0) \\ v(\ell) \\ M(\ell) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} \tag{25}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & EI & 0 & 0 \\ \ell^3/6 & \ell^2/2 & \ell & 1 \\ EI\,\ell & EI & 0 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{w_0 \ell^4}{24\,EI} \\ \dfrac{w_0 \ell^2}{2} \end{Bmatrix} \tag{26}$$

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} \dfrac{w_0 \ell}{2\,EI} \\ 0 \\ -\dfrac{w_0 \ell^3}{24\,EI} \\ 0 \end{Bmatrix} \tag{27}$$

$$v(x) = \frac{w_0\,\ell^4}{24EI}\,\frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)\left(\frac{x^2}{\ell^2} - \frac{x}{\ell} - 1\right) \tag{28}$$

$$M(x) = EI\,v''(x) = \frac{w_0\,\ell^2}{2}\,\frac{x}{\ell}\left(1 - \frac{x}{\ell}\right) \tag{29}$$

$$V(x) = M'(x) = EI\,v'''(x) = \frac{w_0\,\ell}{2}\left(1 - 2\frac{x}{\ell}\right) \tag{30}$$

Shear vanishes at $x = \ell$ and, thus,

$$\max M = M(\ell/2) = \frac{w_0\,\ell^2}{8} \tag{31}$$

By symmetry, rotation vanishes at $x = \ell$ and, thus,

$$\max |v| = -\min v = -v(\ell/2) = \frac{5w_0\,\ell^4}{384EI} \tag{32}$$

8.2 Single span beam with a single concentrated force

$$\begin{aligned}
v_1''''(x) &= 0 && (33)\\
v_1'''(x) &= c_1 && (34)\\
v_1''(x) &= c_1\,x + c_2 && (35)\\
\theta_1(x) = v_1'(x) &= \frac{1}{2}\,c_1\,x^2 + c_2\,x + c_3 && (36)\\
v_1(x) &= \frac{1}{6}\,c_1\,x^3 + \frac{1}{2}\,c_2\,x^2 + c_3\,x + c_4 && (37)\\
M_1(x) = EI\,v_1''(x) &= EI\,c_1\,x + EI\,c_2 && (38)\\
V_1(x) = M_1'(x) = EI\,v_1'''(x) &= EI\,c_1 && (39)
\end{aligned}$$

$$\begin{aligned}
v_2''''(x) &= 0 && (40)\\
v_2'''(x) &= d_1 && (41)\\
v_2''(x) &= d_1\,x + d_2 && (42)\\
\theta_2(x) = v_2'(x) &= \frac{1}{2}\,d_1\,x^2 + d_2\,x + d_3 && (43)\\
v_2(x) &= \frac{1}{6}\,d_1\,x^3 + \frac{1}{2}\,d_2\,x^2 + d_3\,x + d_4 && (44)\\
M_2(x) = EI\,v_2''(x) &= EI\,d_1\,x + EI\,d_2 && (45)\\
V_2(x) = M_2'(x) = EI\,v_2'''(x) &= EI\,d_1 && (46)
\end{aligned}$$

Boundary conditions

$$\left\{\begin{array}{c} v(0) = v_1(0) \\ M(0) = M_1(0) \\ v(\ell) = v_2(\ell) \\ M(\ell) = M_2(\ell) \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right\} \tag{47}$$

Continuity conditions:

$$v_1(a) = v_2(a) \qquad \text{and} \qquad \theta_1(a) = \theta_2(a) \tag{48}$$

Equilibrium of forces for the interval $[a-\epsilon, a+\epsilon]$:

$$\lim_{\epsilon\to 0}\,[V(a-\epsilon) - P - V(a+\epsilon)] = 0 \qquad \Rightarrow \quad V_1(a) - V_2(a) = P \tag{49}$$

Moment equilibrium for the interval $[a-\epsilon, a+\epsilon]$:

$$\lim_{\epsilon\to 0}\,[M(a-\epsilon) + \epsilon V(a-\epsilon) - M(a+\epsilon) + \epsilon V(a+\epsilon)] = 0 \quad \Rightarrow \quad M_1(a) = M_2(a) \tag{50}$$

$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & EI & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ell^3/6 & \ell^2/2 & \ell & 1 \\
0 & 0 & 0 & 0 & EI\,\ell & EI & 0 & 0 \\
a^3/6 & a^2/2 & a & 1 & -a^3/6 & -a^2/2 & -a & -1 \\
a^2/2 & a & 1 & 0 & -a^2/2 & -a & -1 & 0 \\
EI\,a & EI & 0 & 0 & -EI\,a & -EI & 0 & 0 \\
EI & 0 & 0 & 0 & -EI & 0 & 0 & 0
\end{bmatrix}
\left\{\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P \end{array}\right\} \tag{51}$$

Using $\alpha = a/\ell$, the integration constants are obtained as

$$\{c_1\,,\ c_2\,,\ c_3\,,\ c_4\} = \left\{\frac{(1-\alpha)P}{EI},\ 0,\ -\frac{\alpha\left(\alpha^2 - 3\alpha + 2\right)P\ell^2}{6\,EI},\ 0\right\} \tag{52}$$

and

$$\{d_1\,,\ d_2\,,\ d_3\,,\ d_4\} = \left\{\frac{-\alpha P}{2EI},\ \frac{\alpha P\ell}{EI},\ -\frac{\alpha\left(2 + \alpha^2\right)P\ell^2}{6\,EI},\ \frac{\alpha^3 P\ell^3}{6\,EI}\right\} \tag{53}$$

$$v(x) = \begin{cases} \frac{(1-\alpha)Px(x^2-\alpha(2-\alpha)\ell^2)}{6\,EI} & x < a \\ -\frac{\alpha^2(1-\alpha)^2P\ell^3}{3\,EI} & x = a \\ \frac{\alpha P(\ell-x)(x^2-2\ell x+\alpha^2\ell^2)}{6\,EI} & x > a \end{cases} \tag{54}$$

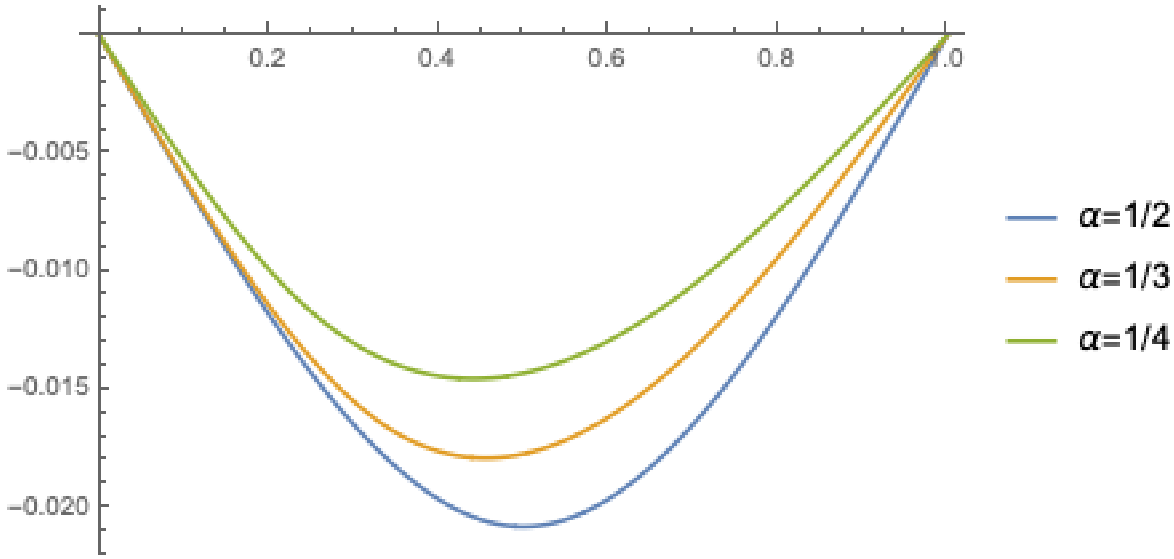

Figure 2: Deformed shape for different locations, $\alpha = a/\ell$, for $EI = 1$, $\ell = 1$, $P = 1$

$$\theta(x) = \begin{cases} \frac{(1-\alpha)P(3x^2-\alpha(2-\alpha)\ell^2)}{6\,EI} & x < a \\ \frac{\alpha(1-\alpha)(2\alpha-1)P\ell^2}{3\,EI} & x = a \\ \frac{-\alpha P(3x^2-6\ell x+(2+\alpha^2)\ell^2)}{6\,EI} & x > a \end{cases} \tag{55}$$

$$M(x) = \begin{cases} (1-\alpha)Px & x < a \\ \alpha(1-\alpha)P\ell & x = a \\ \alpha P(\ell - x) & x > a \end{cases} \tag{56}$$

$$V(x) = \begin{cases} (1-\alpha)P & x < a \\ \text{undefined} & x = a \\ -\alpha P & x > a \end{cases} \tag{57}$$

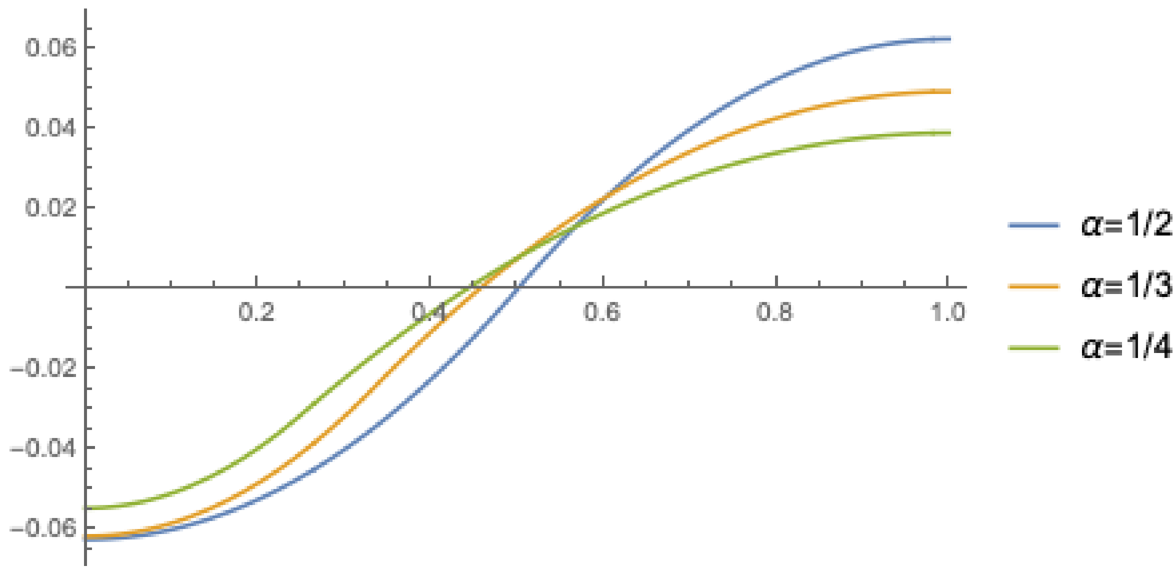

Figure 3: Rotation for different locations, $\alpha = a/\ell$, for $EI = 1$, $\ell = 1$, $P = 1$

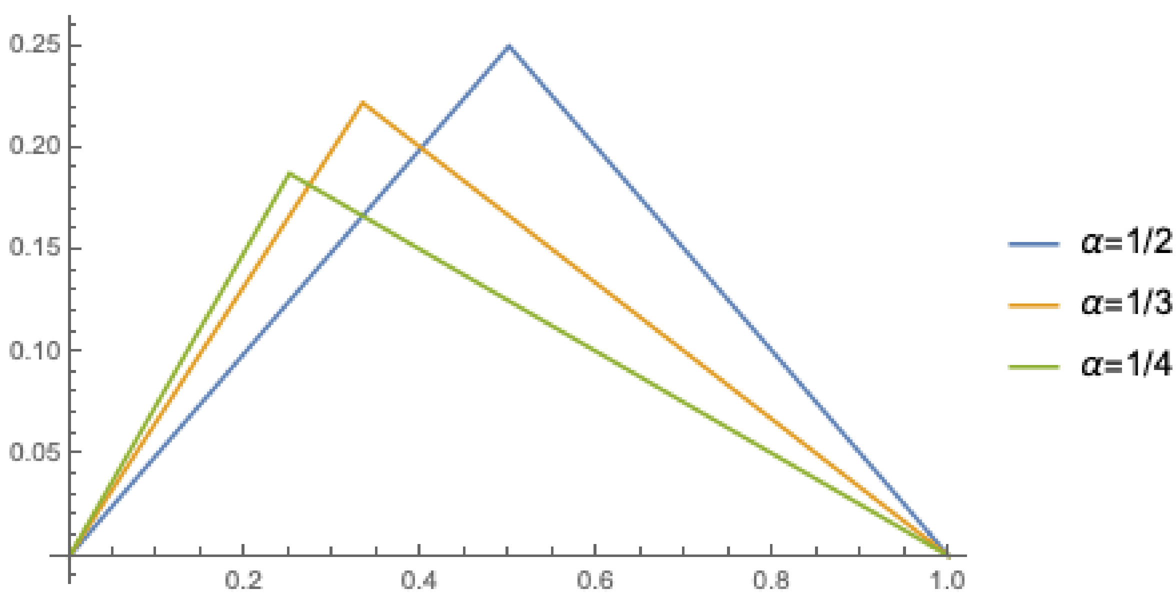

Figure 4: Moment diagrams for different locations, $\alpha = a/\ell$, for $EI = 1$, $\ell = 1$, $P = 1$

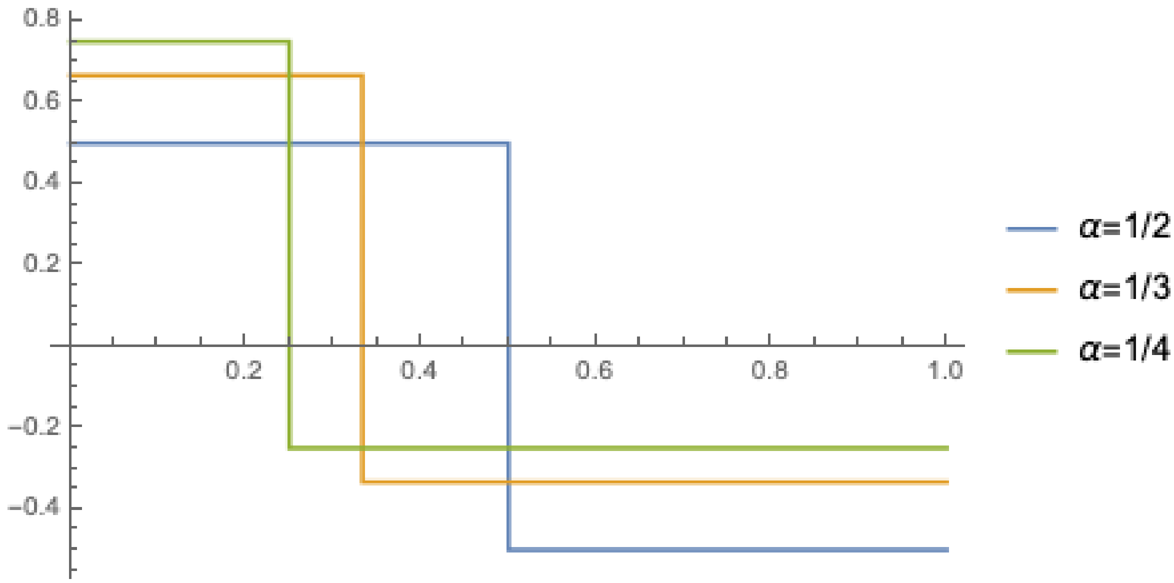

Figure 5: Shear diagrams for different locations, $\alpha = a/\ell$, for $EI = 1$, $\ell = 1$, $P = 1$

8.3 Fundamentals of the stiffness method

$$
\begin{aligned}
v''''(x) &= -\frac{w_0}{EI} && (58)\\
v'''(x) &= -\frac{w_0 x}{EI} + c_1 && (59)\\
v''(x) &= -\frac{w_0 x^2}{2EI} + c_1\,x + c_2 && (60)\\
\theta(x) = v'(x) &= -\frac{w_0 x^3}{6EI} + \frac{1}{2}\,c_1\,x^2 + c_2\,x + c_3 && (61)\\
v(x) &= -\frac{w_0 x^4}{24EI} + \frac{1}{6}\,c_1\,x^3 + \frac{1}{2}\,c_2\,x^2 + c_3\,x + c_4 && (62)
\end{aligned}
$$

$$
\left\{ \begin{array}{c} v(0) \\ \theta(0) = v'(0) \\ v(\ell) \\ \theta(\ell) = v'(\ell) \end{array} \right\} = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \ell^3/6 & \ell^2/2 & \ell & 1 \\ \ell^2/2 & \ell & 1 & 0 \end{array} \right] \left\{ \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ 0 \\ -\frac{w_0 \ell^3}{6EI} \\ -\frac{w_0 \ell^4}{24EI} \end{array} \right\} = \left\{ \begin{array}{c} v_i \\ \theta_i \\ v_j \\ \theta_j \end{array} \right\} \quad (63)
$$

$$(64)$$

$$
\begin{aligned}
v(x) &= v_h(x) + v_p(x) && (65)\\
\theta(x) &= \theta_h(x) + \theta_p(x) && (66)\\
M(x) &= M_h(x) + M_p(x) && (67)\\
V(x) &= V_h(x) + V_p(x) && (68)
\end{aligned}
$$

$$v_h(x) = \left(\frac{2x^3}{\ell^3} - \frac{3x^2}{\ell^2} + 1\right) v_i + \left(\frac{x^3}{\ell^2} - \frac{2x^2}{\ell} + x\right)\theta_i + \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3}\right) v_j + \left(\frac{x^3}{\ell^2} - \frac{x^2}{\ell}\right)\theta_j \tag{69}$$

$$\theta_h(x) = \left(\frac{6x^2}{\ell^3} - \frac{6x}{\ell^2}\right) v_i + \left(\frac{3x^2}{\ell^2} - \frac{4x}{\ell} + 1\right)\theta_i + \left(\frac{6x}{\ell^2} - \frac{6x^2}{\ell^3}\right) v_j + \left(\frac{3x^2}{\ell^2} - \frac{2x}{\ell}\right)\theta_j \tag{70}$$

$$M_h(x) = \frac{6EI}{\ell^2}\left(2\frac{x}{\ell} - 1\right)(v_i - v_j) + \frac{2EI}{\ell}\left(3\frac{x}{\ell} - 2\right)\theta_i + \frac{2EI}{\ell}\left(3\frac{x}{\ell} - 1\right)\theta_j \tag{71}$$

$$V_h(x) = \frac{12EI}{\ell^3}(v_i - v_j) + \frac{6EI}{\ell^2}(\theta_i + \theta_j) \tag{72}$$

$$v_p(x) = -\frac{w_0\ell^4}{24EI}\left(\frac{x}{\ell}\right)^2\left(1 - \frac{x}{\ell}\right)^2 \tag{73}$$

$$\theta_p(x) = -\frac{w_0\ell^3}{12EI}\left(\frac{x}{\ell}\right)\left(1 - \frac{x}{\ell}\right)\left(1 - 2\frac{x}{L}\right) \tag{74}$$

$$M_p(x) = +\frac{w_0\ell^2}{12}\left(1 - 6\frac{x}{\ell} + 6\frac{x^2}{\ell^2}\right) \tag{75}$$

$$V_p(x) = +\frac{w_0\ell}{2}\left(1 - 2\frac{x}{\ell}\right) \tag{76}$$

$$\underbrace{\begin{Bmatrix} \bar{V}_i = V(0) \\ \bar{M}_i = -M(0) \\ \bar{V}_j = -V(\ell) \\ \bar{M}_j = M(\ell) \end{Bmatrix}}_{=:\boldsymbol{f}^e} = \underbrace{\begin{bmatrix} \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & -\frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} \\ \frac{6EI}{\ell^2} & \frac{4EI}{\ell} & -\frac{6EI}{\ell^2} & \frac{2EI}{\ell} \\ -\frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} & \frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} \\ \frac{6EI}{\ell^2} & \frac{2EI}{\ell} & -\frac{6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix}}_{=:\boldsymbol{K}^e} \underbrace{\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}}_{=:\boldsymbol{q}^e} + \underbrace{\begin{Bmatrix} \frac{w_0\ell}{2} \\ -\frac{w_0\ell^2}{12} \\ \frac{w_0\ell}{2} \\ \frac{w_0\ell^2}{12} \end{Bmatrix}}_{=:\boldsymbol{P}^e} \tag{77}$$

(78)

(79)

(80)

(81)

(82)

(83)

8.4 Single span beam with a concentrated force and distributed load using the stiffness method

$$\{\boldsymbol{q}\} = \{q_1,\ q_2,\ q_3,\ q_4,\ q_5,\ q_6\}^t = \{v_1,\ \theta_1,\ v_2,\ \theta_2,\ v_3,\ \theta_3\}^t \tag{84}$$

$$\{\boldsymbol{q}^{(1)}\} = \{v_1,\ \theta_1,\ v_2,\ \theta_2\}^t = \{q_1,\ q_2,\ q_3,\ q_4\}^t \tag{85}$$

$$\{\boldsymbol{q}^{(2)}\} = \{v_2,\ \theta_2,\ v_3,\ \theta_3\}^t = \{q_3,\ q_4,\ q_5,\ q_6\}^t \tag{86}$$

$$[\boldsymbol{K}^{(1)}] = \begin{bmatrix} \frac{12EI}{a^3} & \frac{6EI}{a^2} & -\frac{12EI}{a^3} & \frac{6EI}{a^2} \\ \frac{6EI}{a^2} & \frac{4EI}{a} & -\frac{6EI}{a^2} & \frac{2EI}{a} \\ -\frac{12EI}{a^3} & -\frac{6EI}{a^2} & \frac{12EI}{a^3} & -\frac{6EI}{a^2} \\ \frac{6EI}{a^2} & \frac{2EI}{a} & -\frac{6EI}{a^2} & \frac{4EI}{a} \end{bmatrix} \tag{87}$$

$b = \ell - a$:

$$[\boldsymbol{K}^{(2)}] = \begin{bmatrix} \frac{12EI}{b^3} & \frac{6EI}{b^2} & -\frac{12EI}{b^3} & \frac{6EI}{b^2} \\ \frac{6EI}{b^2} & \frac{4EI}{b} & -\frac{6EI}{b^2} & \frac{2EI}{b} \\ -\frac{12EI}{b^3} & -\frac{6EI}{b^2} & \frac{12EI}{b^3} & -\frac{6EI}{b^2} \\ \frac{6EI}{b^2} & \frac{2EI}{b} & -\frac{6EI}{b^2} & \frac{4EI}{b} \end{bmatrix} \tag{88}$$

$$[\boldsymbol{K}] = \begin{bmatrix} \frac{12EI}{a^3} & \frac{6EI}{a^2} & -\frac{12EI}{a^3} & \frac{6EI}{a^2} & 0 & 0 \\ \frac{6EI}{a^2} & \frac{4EI}{a} & -\frac{6EI}{a^2} & \frac{2EI}{a} & 0 & 0 \\ -\frac{12EI}{a^3} & -\frac{6EI}{a^2} & \left(\frac{12EI}{a^3} + \frac{12EI}{b^3}\right) & \left(-\frac{6EI}{a^2} + \frac{6EI}{b^2}\right) & -\frac{12EI}{b^3} & \frac{6EI}{b^2} \\ \frac{6EI}{a^2} & \frac{2EI}{a} & \left(-\frac{6EI}{a^2} + \frac{6EI}{b^2}\right) & \left(\frac{4EI}{a} + \frac{4EI}{b}\right) & -\frac{6EI}{b^2} & \frac{2EI}{b} \\ 0 & 0 & -\frac{12EI}{b^3} & -\frac{6EI}{b^2} & \frac{12EI}{b^3} & -\frac{6EI}{b^2} \\ 0 & 0 & \frac{6EI}{b^2} & \frac{2EI}{b} & -\frac{6EI}{b^2} & \frac{4EI}{b} \end{bmatrix} \tag{89}$$

$$\{\boldsymbol{P}\} = \{0,\ 0,\ -P,\ 0,\ 0,\ 0\}^t \tag{90}$$

Boundary conditions $v(0) = v_1 = q_1 = 0$ and $v(\ell) = v_3 = q_5 = 0$ tell us that the 1^{st} and 5^{th} columns and rows are to be eliminated, resulting in

$$[\boldsymbol{K}_{ff}] = \begin{bmatrix} \frac{4EI}{a} & -\frac{6EI}{a^2} & \frac{2EI}{a} & 0 \\ -\frac{6EI}{a^2} & \left(\frac{12EI}{a^3} + \frac{12EI}{b^3}\right) & \left(-\frac{6EI}{a^2} + \frac{6EI}{b^2}\right) & \frac{6EI}{b^2} \\ \frac{2EI}{a} & \left(-\frac{6EI}{a^2} + \frac{6EI}{b^2}\right) & \left(\frac{4EI}{a} + \frac{4EI}{b}\right) & \frac{2EI}{b} \\ 0 & \frac{6EI}{b^2} & \frac{2EI}{b} & \frac{4EI}{b} \end{bmatrix} \tag{91}$$

$$\{\boldsymbol{P}_f\} = \{0,\ -P,\ 0,\ 0\}^t \tag{92}$$

$$\{\boldsymbol{q}_f\} = [\boldsymbol{K}_{ff}]^{-1}\{\boldsymbol{P}_f\} = \left\{-\frac{(\ell+b)abP}{6EI\ell},\ -\frac{(a^2b^2P)}{3EI\ell},\ \frac{(a-b)abP}{3EI\ell},\ \frac{(a+\ell)abP}{6EI\ell}\right\}^t \tag{93}$$

$$\{\boldsymbol{q}^{(1)}\} = \left\{0,\ -\frac{(\ell+b)abP}{6EI\ell},\ -\frac{(a^2b^2P)}{3EI\ell},\ \frac{(a-b)abP}{3EI\ell}\right\}^t \tag{94}$$

$$\{\boldsymbol{q}^{(2)}\} = \left\{-\frac{(a^2b^2P)}{3EI\ell},\ \frac{(a-b)abP}{3EI\ell},\ 0,\ \frac{(a+\ell)abP}{6EI\ell}\right\}^t \tag{95}$$

$$\{\boldsymbol{f}^{(1)}\} = [\boldsymbol{K}^{(1)}]\{\boldsymbol{q}^{(1)}\} = \{\underbrace{\frac{bP}{\ell}}_{V_1^{(1)}},\ \underbrace{0}_{-M_1^{(1)}},\ \underbrace{-\frac{bP}{\ell}}_{-V_2^{(1)}},\ \underbrace{\frac{abP}{\ell}}_{M_2^{(1)}}\}^t \tag{96}$$

Measuring $\bar{x}$ from the left end of segment 1, i.e., $\bar{x} = x$, we get

$$M(\bar{x}) = \frac{\bar{x}bP}{\ell} \qquad \text{where } 0 \leq \bar{x} \leq a \tag{97}$$

$$V(\bar{x}) = \frac{bP}{\ell} \qquad \text{where } 0 \leq \bar{x} \leq a \tag{98}$$

$$\{\boldsymbol{f}^{(2)}\} = [\boldsymbol{K}^{(2)}]\{\boldsymbol{q}^{(2)}\} = \{\underbrace{-\frac{aP}{\ell}}_{V_2^{(2)}}, \underbrace{-\frac{abP}{\ell}}_{-M_2^{(2)}}, \underbrace{\frac{aP}{\ell}}_{-V_3^{(2)}}, \underbrace{0}_{M_3^{(2)}}\}^t \tag{99}$$

Measuring $\bar{x}$ from the left end of segment 2, i.e., $\bar{x} = (x - a)$, we get

$$M(\bar{x}) = \frac{a(b - \bar{x})P}{\ell} \qquad \text{where } 0 \leq \bar{x} \leq b \tag{100}$$

$$V(\bar{x}) = -\frac{aP}{\ell} \qquad \text{where } 0 \leq \bar{x} \leq b \tag{101}$$