# Cartesian Coordinates, Points, and Transformations

CIS - 600.445

Russell Taylor

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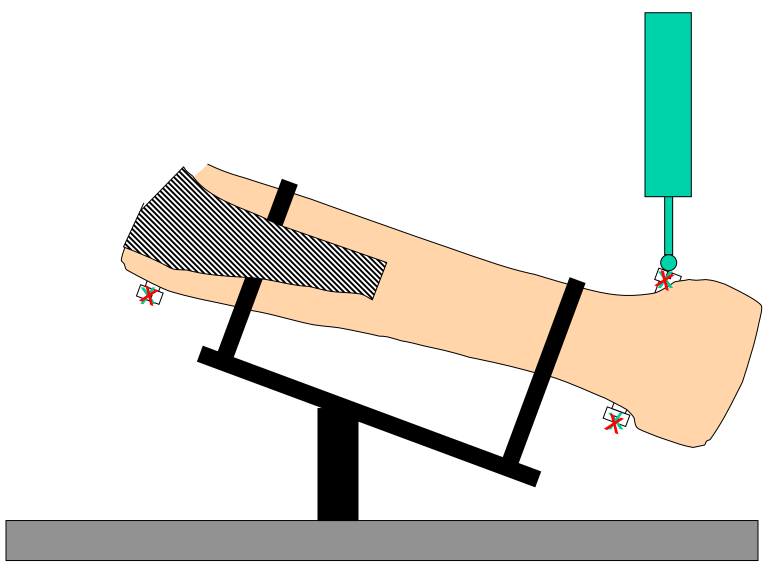

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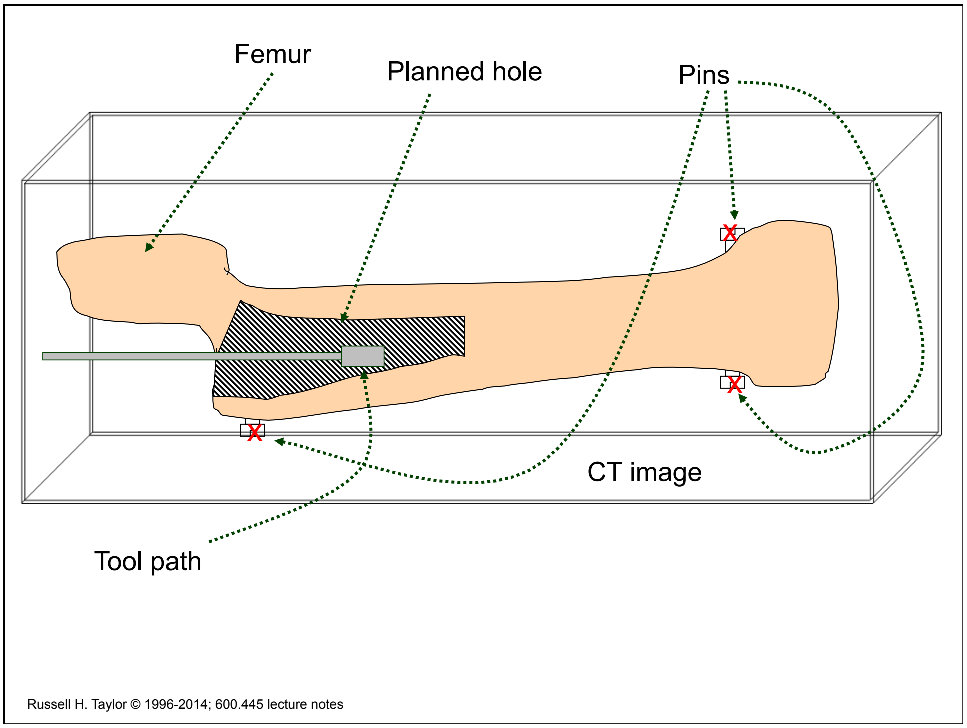
Femur
Planned hole
Pins
CT image
Tool path
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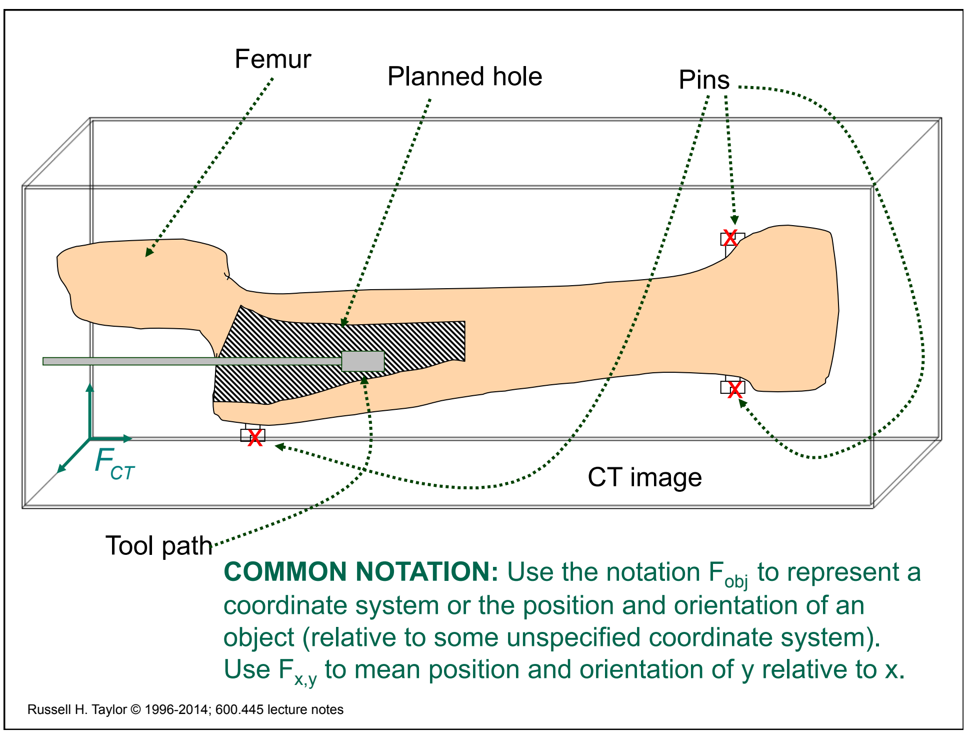
Femur
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$F_{CT}$
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**COMMON NOTATION:** Use the notation $F_{obj}$ to represent a coordinate system or the position and orientation of an object (relative to some unspecified coordinate system). Use $F_{x,y}$ to mean position and orientation of y relative to x.
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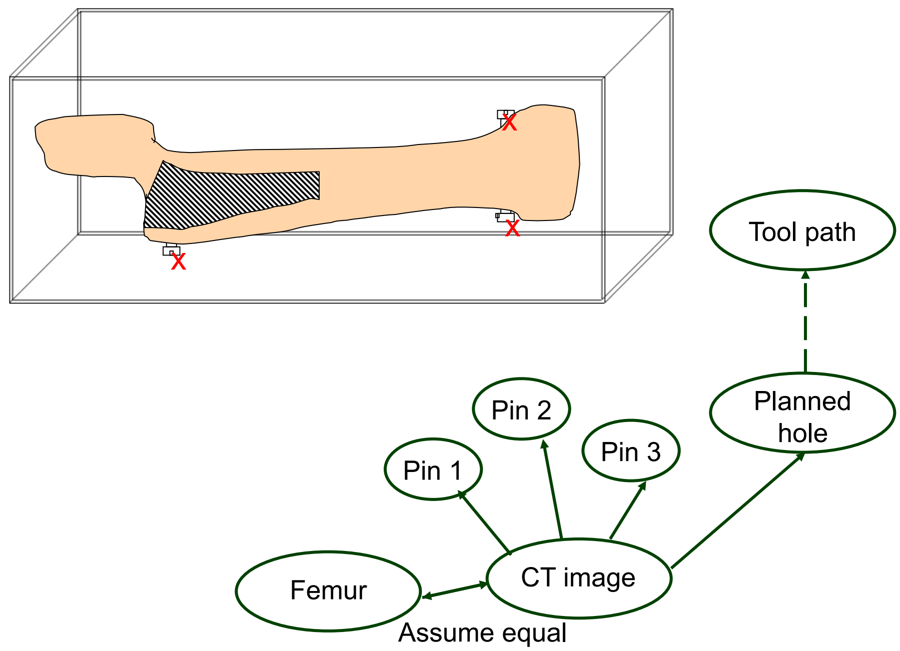


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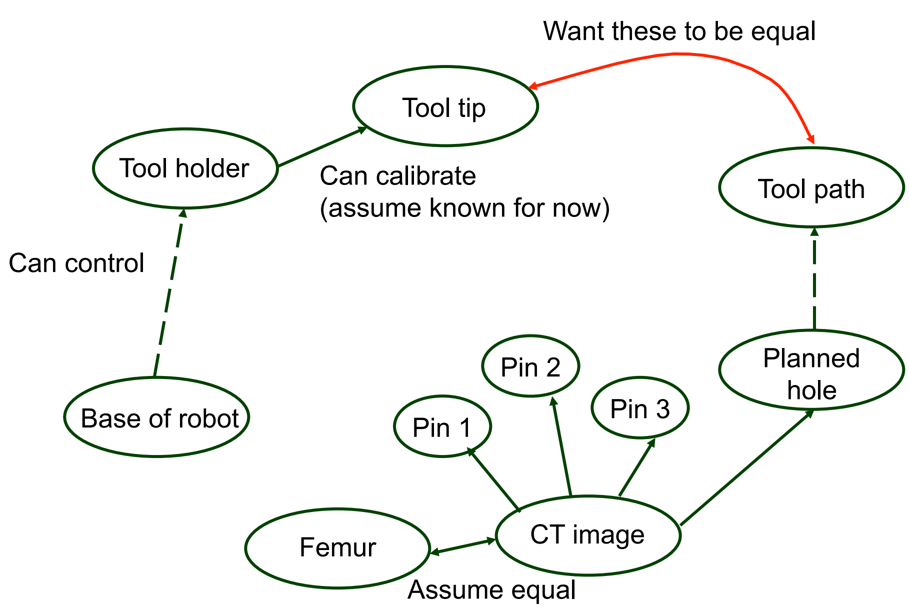


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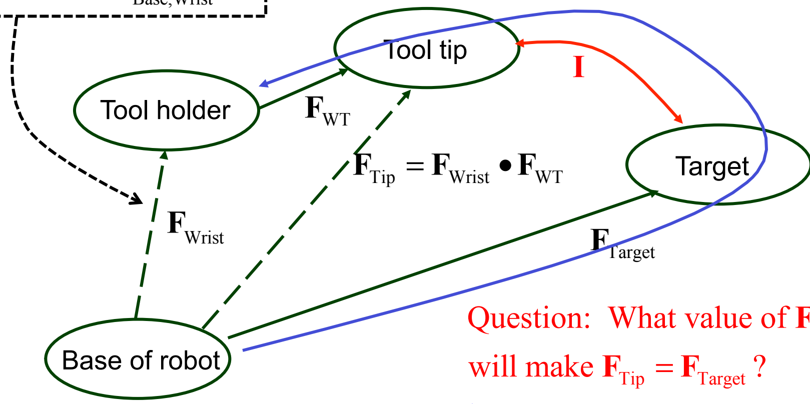
More correctly, this would be $\mathbf{F}_{\text{Base,Wrist}}$

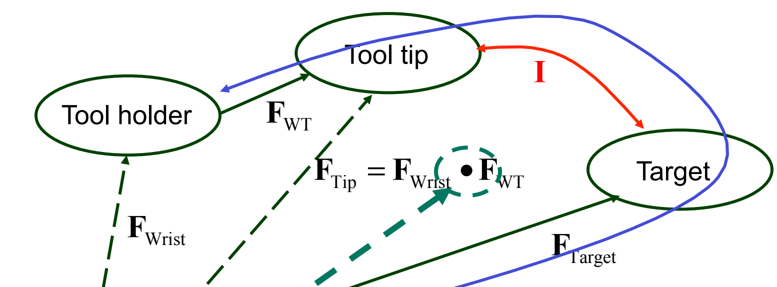



Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make $\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Target}}$ ?

Answer:

$$\mathbf{F}_{\text{Wrist}} = \mathbf{F}_{\text{Target}} \bullet \mathbf{I} \bullet \mathbf{F}_{\text{WT}}^{-1}$$
$$= \mathbf{F}_{\text{Target}} \bullet \mathbf{F}_{\text{WT}}^{-1}$$






**Notational Note**

We use the notation $\mathbf{A} \bullet \mathbf{B}$ to represent composition or transformation. Where the context is clear, we may also use $\mathbf{AB}$ for the same thing.

Question: What value of $\mathbf{F}_{\text{Wrist}}$ will make $\mathbf{F}_{\text{Tip}} = \mathbf{F}_{\text{Target}}$ ?

Answer:

$$\mathbf{F}_{\text{Wrist}} = \mathbf{F}_{\text{Target}} \bullet \mathbf{I} \bullet \mathbf{F}_{\text{WT}}^{-1}$$
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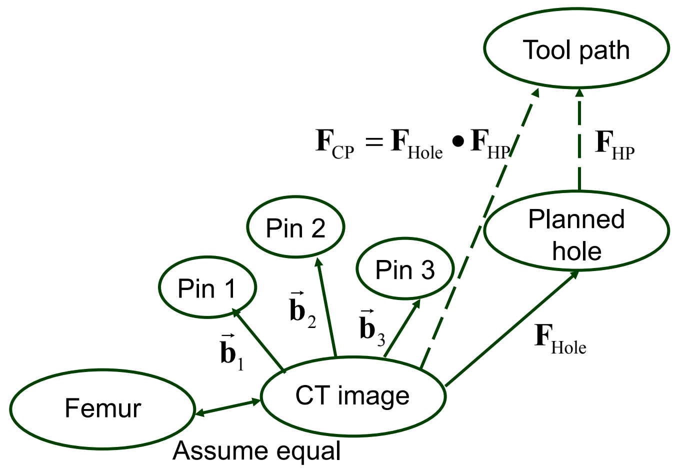


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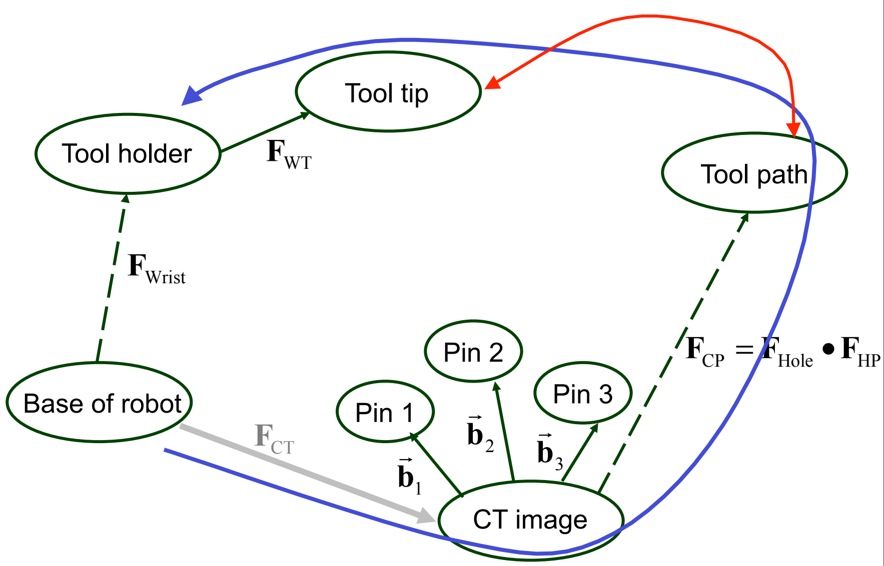
Question: What value of $\mathbf{F}_{Wrist}$ will make these equal?

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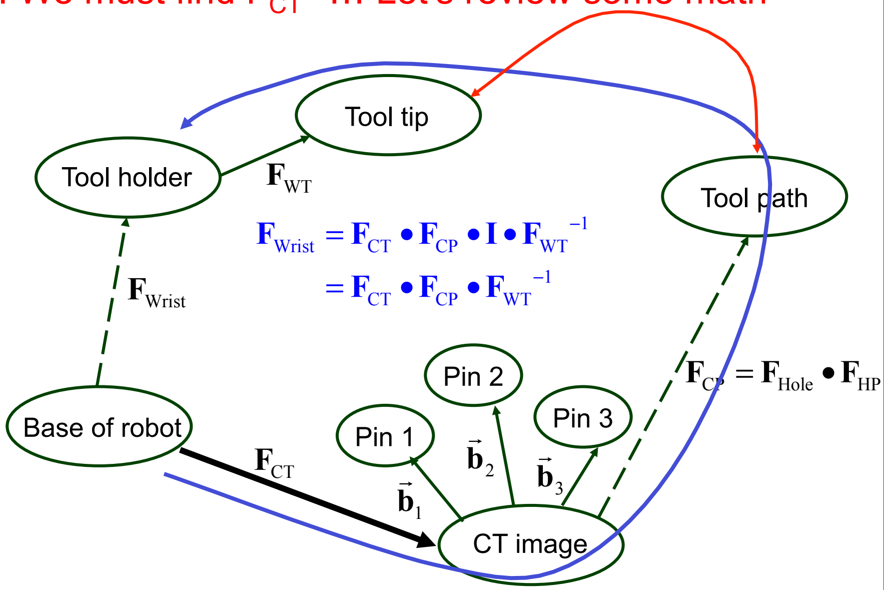
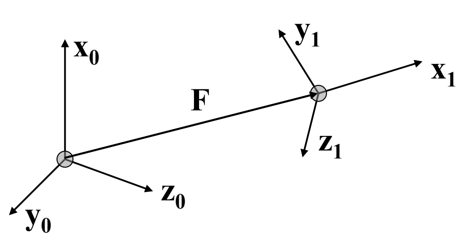
## But: We must find $F_{CT}$ … Let's review some math






## Coordinate Frame Transformation

$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$



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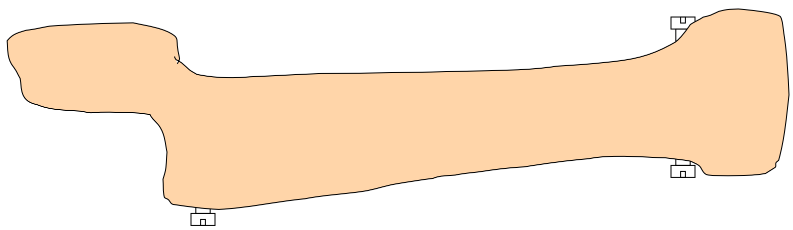

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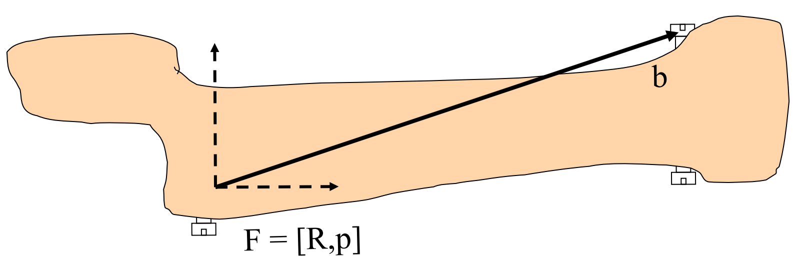


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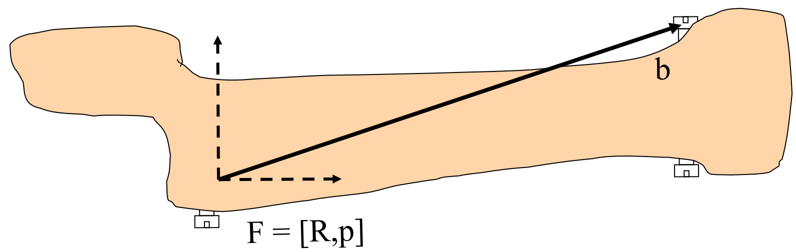


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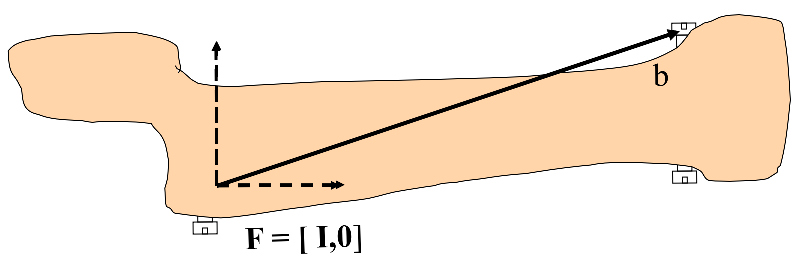


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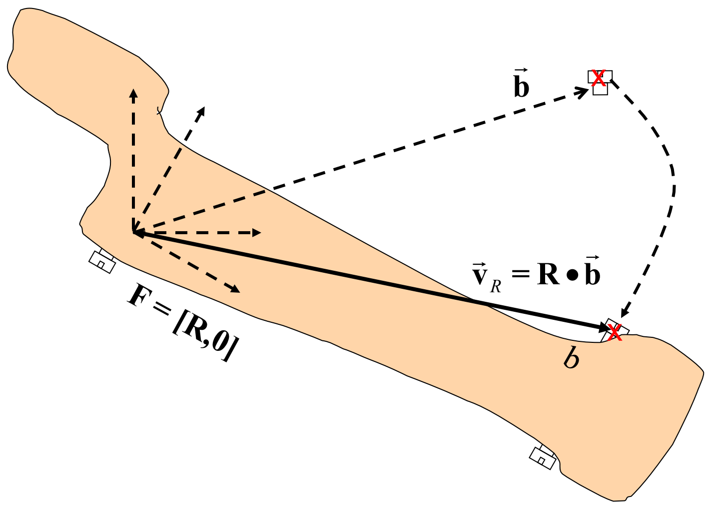


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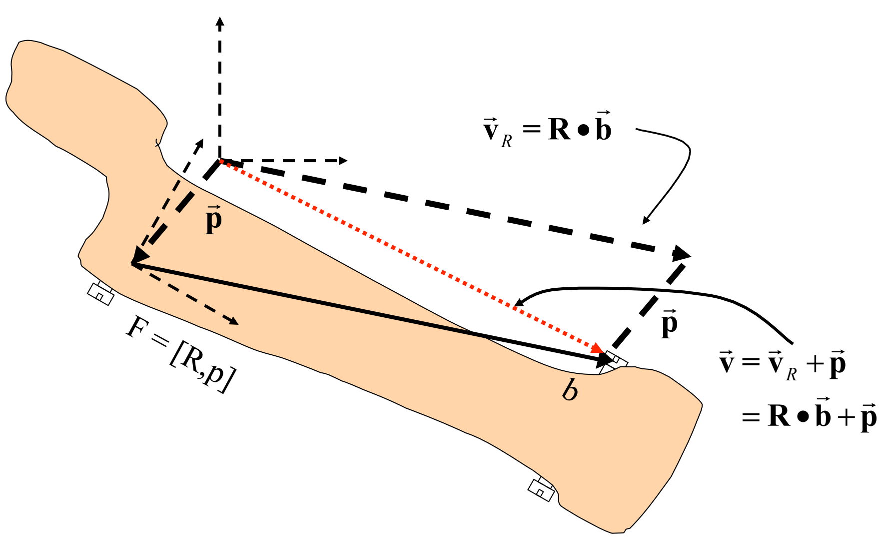


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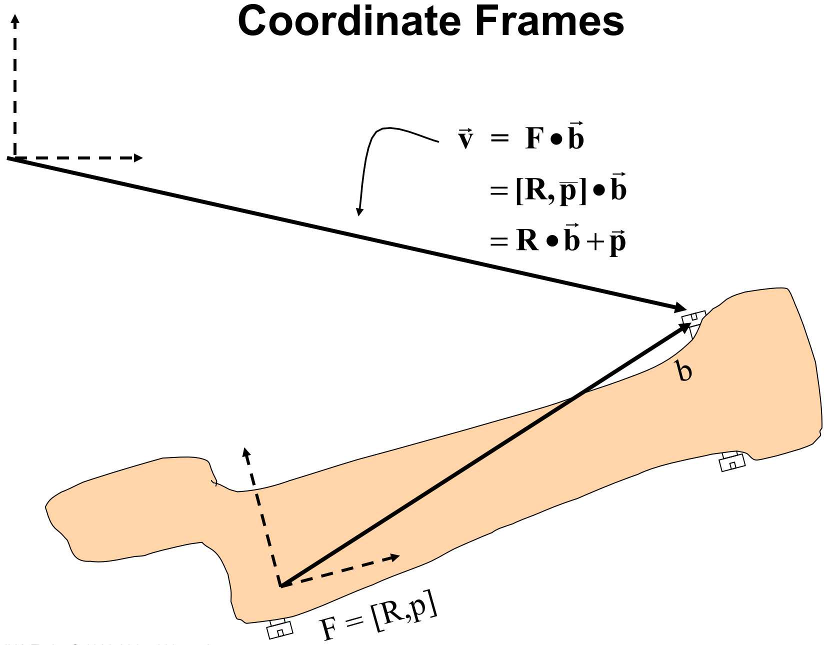
## Coordinate Frames

$$\vec{v} = F \bullet \vec{b} = [R, \vec{p}] \bullet \vec{b} = R \bullet \vec{b} + \vec{p}$$

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## Forward and Inverse Frame Transformations

**Forward**

$$\mathbf{F} = [\mathbf{R}, \mathbf{p}]$$

$$\begin{aligned} \mathbf{v} &= \mathbf{F} \bullet \mathbf{b} \\ &= [\mathbf{R}, \mathbf{p}] \bullet \mathbf{b} \\ &= \mathbf{R} \bullet \mathbf{b} + \mathbf{p} \end{aligned}$$

**Inverse**

$$\mathbf{F}^{-1}\mathbf{v} = \mathbf{b}$$

$$\begin{aligned} \mathbf{b} &= \mathbf{R}^{-1} \bullet (\mathbf{v} - \mathbf{p}) \\ &= \mathbf{R}^{-1} \bullet \mathbf{v} - \mathbf{R}^{-1} \bullet \mathbf{p} \end{aligned}$$

$$\mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1} \bullet \mathbf{p}]$$

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# Composition

Assume $\mathbf{F}_1 = [\mathbf{R}_1, \vec{\mathbf{p}}_1], \quad \mathbf{F}_2 = [\mathbf{R}_2, \vec{\mathbf{p}}_2]$

Then

$$
\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 \bullet \vec{\mathbf{b}} &= \mathbf{F}_1 \bullet (\mathbf{F}_2 \bullet \vec{\mathbf{b}}) \\
&= \mathbf{F}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) \\
&= \mathbf{R}_1 \bullet (\mathbf{R}_2 \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}}_2) + \vec{\mathbf{p}}_1 \\
&= \mathbf{R}_1 \bullet \mathbf{R}_2 \bullet \vec{\mathbf{b}} + \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1 \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1] \bullet \vec{\mathbf{b}}
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbf{F}_1 \bullet \mathbf{F}_2 &= [\mathbf{R}_1, \vec{\mathbf{p}}_1] \bullet [\mathbf{R}_2, \vec{\mathbf{p}}_2] \\
&= [\mathbf{R}_1 \bullet \mathbf{R}_2, \mathbf{R}_1 \vec{\mathbf{p}}_2 + \vec{\mathbf{p}}_1]
\end{aligned}
$$

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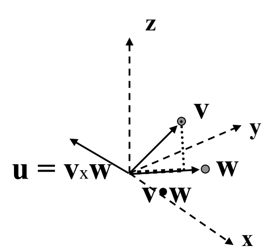
# Vectors

$$
\mathbf{v}_{col} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \end{bmatrix}
$$

$$
\mathbf{v}_{row} = \begin{bmatrix} \mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \end{bmatrix}
$$




$$
\text{length}: \|\mathbf{v}\| = \sqrt{\mathbf{v}_x{}^2 + \mathbf{v}_y{}^2 + \mathbf{v}_z{}^2}
$$

$$
\text{dot product}: a = \mathbf{v} \cdot \mathbf{w} = \left(\mathbf{v}_x \mathbf{w}_x + \mathbf{v}_y \mathbf{w}_y + \mathbf{v}_z \mathbf{w}_z\right) = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta
$$

$$
\text{cross product}: \mathbf{u} = \mathbf{v} \times \mathbf{w} = \begin{bmatrix} \mathbf{v}_y \mathbf{w}_z - \mathbf{v}_z \mathbf{w}_y \\ \mathbf{v}_z \mathbf{w}_x - \mathbf{v}_x \mathbf{w}_z \\ \mathbf{v}_x \mathbf{w}_y - \mathbf{v}_y \mathbf{w}_x \end{bmatrix}, \|\mathbf{u}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta
$$

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Slide acknowledgment: Sarah Graham and Andy Bzostek

## Matrix representation of cross product operator

Define

$$\hat{\vec{\mathbf{a}}} \stackrel{\Delta}{=} skew(\vec{\mathbf{a}}) \stackrel{\Delta}{=} \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Then

$$\vec{\mathbf{a}} \times \vec{\mathbf{v}} = skew(\vec{\mathbf{a}}) \bullet \vec{\mathbf{v}}$$

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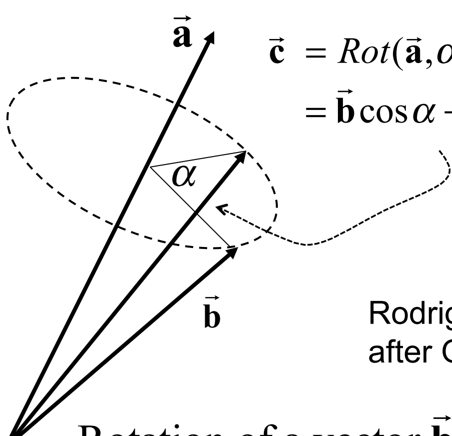
## Axis-angle Representations of Rotations




$$\begin{aligned} \vec{\mathbf{c}} &= Rot(\vec{\mathbf{a}},\alpha) \bullet \vec{\mathbf{b}} \\ &= \vec{\mathbf{b}}\cos\alpha + \vec{\mathbf{a}} \times \vec{\mathbf{b}} \sin\alpha + \vec{\mathbf{a}}(\vec{\mathbf{a}} \bullet \vec{\mathbf{b}})\left(1-\cos\alpha\right) \end{aligned}$$

Rodrigues' rotation formula (named after Olinde Rodrigues)

Rotation of a vector $\vec{\mathbf{b}}$ by angle $\alpha$ about axis $\vec{\mathbf{a}}$

(Assumes that $\vec{\mathbf{a}}$ is a unit vector, $\left\|\vec{\mathbf{a}}\right\| = 1$)

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## Exponential representation

Consider a rotation about axis $\vec{\mathbf{n}}$ by angle $\theta$. Then

$$e^{skew(\vec{\mathbf{n}})\theta} = \mathbf{I} + \theta skew(\vec{\mathbf{n}}) + \frac{\theta^2}{2!} skew(\vec{\mathbf{n}})^2 + \ldots$$

By doing some manipulation, you can show

$$\begin{aligned} Rot(\vec{\mathbf{n}},\theta) &= e^{skew(\vec{\mathbf{n}})\theta} \\ &= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + skew(\vec{\mathbf{n}})^2(1-\cos\theta) \\ &= \mathbf{I} + skew(\vec{\mathbf{n}})\sin\theta + (\vec{\mathbf{n}} \bullet \vec{\mathbf{n}}^T - \mathbf{I})(1-\cos\theta) \\ &= \mathbf{I}\cos\theta + skew(\vec{\mathbf{n}})\sin\theta + \vec{\mathbf{n}} \bullet \vec{\mathbf{n}}^T(1-\cos\theta) \end{aligned}$$

Note that for small $\theta$, this reduces to

$$Rot(\vec{\mathbf{n}},\theta) \approx \mathbf{I} + skew(\theta\vec{\mathbf{n}})$$

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## Rotations: Some Notation

$Rot(\vec{\mathbf{a}},\alpha) =$ Rotation by angle $\alpha$ about axis $\vec{\mathbf{a}}$

$\mathbf{R}_{\vec{\mathbf{a}}}(\alpha) =$ Rotation by angle $\alpha$ about axis $\vec{\mathbf{a}}$

$\mathbf{R}(\vec{\mathbf{a}}) = Rot(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|)$

$$\mathbf{R}_{\mathbf{xyz}}(\alpha,\beta,\gamma) = \mathbf{R}(\vec{\mathbf{x}},\alpha) \bullet \mathbf{R}(\vec{\mathbf{y}},\beta) \bullet \mathbf{R}(\vec{\mathbf{z}},\gamma)$$

$$\mathbf{R}_{\mathbf{zyz}}(\alpha,\beta,\gamma) = \mathbf{R}(\vec{\mathbf{z}},\alpha) \bullet \mathbf{R}(\vec{\mathbf{y}},\beta) \bullet \mathbf{R}(\vec{\mathbf{z}},\gamma)$$

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# Rotations: A few useful facts

$$Rot(s\vec{\mathbf{a}},\alpha)\bullet\vec{\mathbf{a}}=\vec{\mathbf{a}} \quad \text{and} \quad \left\|Rot(\vec{\mathbf{a}},\alpha)\bullet\vec{\mathbf{b}}\right\|=\left\|\vec{\mathbf{b}}\right\|$$

$$Rot(\vec{\mathbf{a}},\alpha)=Rot(\hat{\mathbf{a}},\alpha) \ \text{ where } \hat{\mathbf{a}}=\frac{\vec{\mathbf{a}}}{\left\|\vec{\mathbf{a}}\right\|}$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet Rot(\hat{\mathbf{a}},\beta)=Rot(\hat{\mathbf{a}},\alpha+\beta)$$

$$Rot(\hat{\mathbf{a}},\alpha)^{-1}=Rot(\hat{\mathbf{a}},-\alpha)$$

$$Rot(\vec{\mathbf{a}},0)\bullet\vec{\mathbf{b}}=\vec{\mathbf{b}} \quad \text{i.e., } Rot(\vec{\mathbf{a}},0)=\mathbf{I}_{Rot}= \text{ the identity rotation}$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet\vec{\mathbf{b}}=\left(\hat{\mathbf{a}}\bullet\vec{\mathbf{b}}\right)\hat{\mathbf{a}}+Rot(\hat{\mathbf{a}},\alpha)\bullet\left(\vec{\mathbf{b}}-\left(\hat{\mathbf{a}}\bullet\vec{\mathbf{b}}\right)\hat{\mathbf{a}}\right)$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet Rot(\hat{\mathbf{b}},\beta)=Rot(\hat{\mathbf{b}},\beta)\bullet Rot(Rot(\hat{\mathbf{b}},-\beta)\bullet\hat{\mathbf{a}},\alpha)$$

$$Rot(\hat{\mathbf{a}},\alpha)\bullet\mathbf{R}_\beta=\mathbf{R}_\beta\bullet Rot(\mathbf{R}_\beta^{-1}\bullet\hat{\mathbf{a}},\alpha)$$

$$\mathbf{R}_\alpha\bullet Rot(\hat{\mathbf{b}},\beta)=Rot(\mathbf{R}_\alpha\bullet\hat{\mathbf{b}},\beta)\bullet\mathbf{R}_\alpha$$



# Rotations: more facts

If $\vec{\mathbf{v}}=[v_x,v_y,v_z]^T$ then a rotation $\mathbf{R}\bullet\vec{\mathbf{v}}$ may be described in terms of the effects of $\mathbf{R}$ on orthogonal unit vectors, $\vec{\mathbf{e}}_x=[1,0,0]^T$, $\vec{\mathbf{e}}_y=[0,1,0]^T$, $\vec{\mathbf{e}}_z=[0,0,1]^T$

$$\mathbf{R}\bullet\vec{\mathbf{v}}=v_x\vec{\mathbf{r}}_x+v_y\vec{\mathbf{r}}_y+v_z\vec{\mathbf{r}}_z$$

where

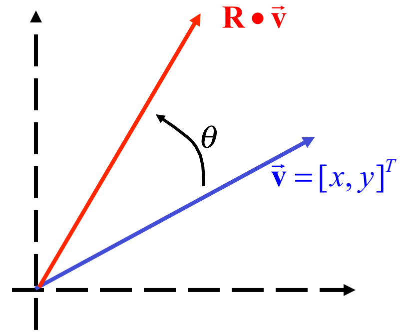
$$\vec{\mathbf{r}}_x=\mathbf{R}\bullet\vec{\mathbf{e}}_x$$
$$\vec{\mathbf{r}}_y=\mathbf{R}\bullet\vec{\mathbf{e}}_y$$
$$\vec{\mathbf{r}}_z=\mathbf{R}\bullet\vec{\mathbf{e}}_z$$

Note that rotation doesn't affect inner products

$$\left(\mathbf{R}\bullet\vec{\mathbf{b}}\right)\bullet\left(\mathbf{R}\bullet\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\bullet\vec{\mathbf{c}}$$

## Rotations in the plane

$$\mathbf{R}\bullet\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\cos\theta-y\sin\theta\\x\sin\theta+y\cos\theta\end{bmatrix}$$
$$=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\bullet\begin{bmatrix}x\\y\end{bmatrix}$$

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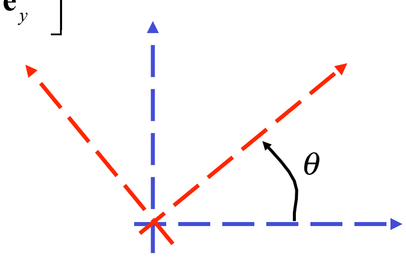
## Rotations in the plane

$$\mathbf{R}\bullet\begin{bmatrix}\vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\bullet\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}\mathbf{R}\bullet\vec{\mathbf{e}}_x & \mathbf{R}\bullet\vec{\mathbf{e}}_y\end{bmatrix}$$
$$=\begin{bmatrix}\vec{\mathbf{r}}_x & \vec{\mathbf{r}}_y\end{bmatrix}$$

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## 3D Rotation Matrices

$$\mathbf{R} \bullet \begin{bmatrix} \vec{\mathbf{e}}_x & \vec{\mathbf{e}}_y & \vec{\mathbf{e}}_z \end{bmatrix} = \begin{bmatrix} \mathbf{R} \bullet \vec{\mathbf{e}}_x & \mathbf{R} \bullet \vec{\mathbf{e}}_y & \mathbf{R} \bullet \vec{\mathbf{e}}_z \end{bmatrix}$$
$$= \begin{bmatrix} \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z \end{bmatrix}$$

$$\mathbf{R}^T \bullet \mathbf{R} = \begin{bmatrix} \hat{\mathbf{r}}_{\mathbf{x}}^{\ T} \\ \hat{\mathbf{r}}_y^{\ T} \\ \hat{\mathbf{r}}_z \end{bmatrix} \bullet \begin{bmatrix} \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z \end{bmatrix}$$
$$= \begin{bmatrix} \vec{\mathbf{r}}_{\mathbf{x}}^{\ T} \bullet \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_{\mathbf{x}}^{\ T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_{\mathbf{x}}^{\ T} \bullet \vec{\mathbf{r}}_z \\ \vec{\mathbf{r}}_y^{\ T} \bullet \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_y^{\ T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_y^{\ T} \bullet \vec{\mathbf{r}}_z \\ \vec{\mathbf{r}}_z^{\ T} \bullet \vec{\mathbf{r}}_{\mathbf{x}} & \vec{\mathbf{r}}_z^{\ T} \bullet \vec{\mathbf{r}}_y & \vec{\mathbf{r}}_z^{\ T} \bullet \vec{\mathbf{r}}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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## Properties of Rotation Matrices

Inverse of a Rotation Matrix equals its transpose:

$\mathbf{R}^{-1} = \mathbf{R}^{\mathbf{T}}$

$\mathbf{R}^{\mathbf{T}}\,\mathbf{R} = \mathbf{R}\,\mathbf{R}^{\mathbf{T}} = \mathbf{I}$

The Determinant of a Rotation matrix is equal to +1:

**det(R)= +1**

Any Rotation can be described by consecutive rotations about the three primary axes, x, y, and z:

$\mathbf{R} = \mathbf{R}_{\mathbf{z},\theta}\, \mathbf{R}_{\mathbf{y},\varphi}\, \mathbf{R}_{\mathbf{x},\psi}$

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# Canonical 3D Rotation Matrices

***Note:*** **Right-Handed Coordinate System**

$$\mathbf{R}_{\vec{\mathbf{x}}}(\theta) = Rot(\vec{\mathbf{x}},\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{\mathbf{y}}}(\theta) = Rot(\vec{\mathbf{y}},\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\vec{\mathbf{z}}}(\theta) = Rot(\vec{\mathbf{z}},\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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# Homogeneous Coordinates

- Widely used in graphics, geometric calculations
- Represent 3D vector as 4D quantity
- For our current purposes, we will keep the "scale" *s = 1*

$$\vec{\mathbf{v}} \equiv \begin{bmatrix} xs \\ ys \\ zs \\ s \end{bmatrix} \cong \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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## Representing Frame Transformations as Matrices

$$\mathbf{v}+\mathbf{p} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \mathbf{p}_x \\ 0 & 1 & 0 & \mathbf{p}_y \\ 0 & 0 & 1 & \mathbf{p}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ \mathbf{v}_z \\ 1 \end{bmatrix} = \mathbf{P} \bullet \mathbf{v}$$

$$\mathbf{R} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$$

$$\mathbf{P} \bullet \mathbf{R} \rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \bullet \begin{bmatrix} \mathbf{R} & 0 \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} = [\mathbf{R},\mathbf{p}] = \mathbf{F}$$

$$\mathbf{F} \bullet \mathbf{v} \rightarrow \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} = \begin{bmatrix} (\mathbf{R} \bullet \mathbf{v}) + \mathbf{p} \\ 1 \end{bmatrix}$$

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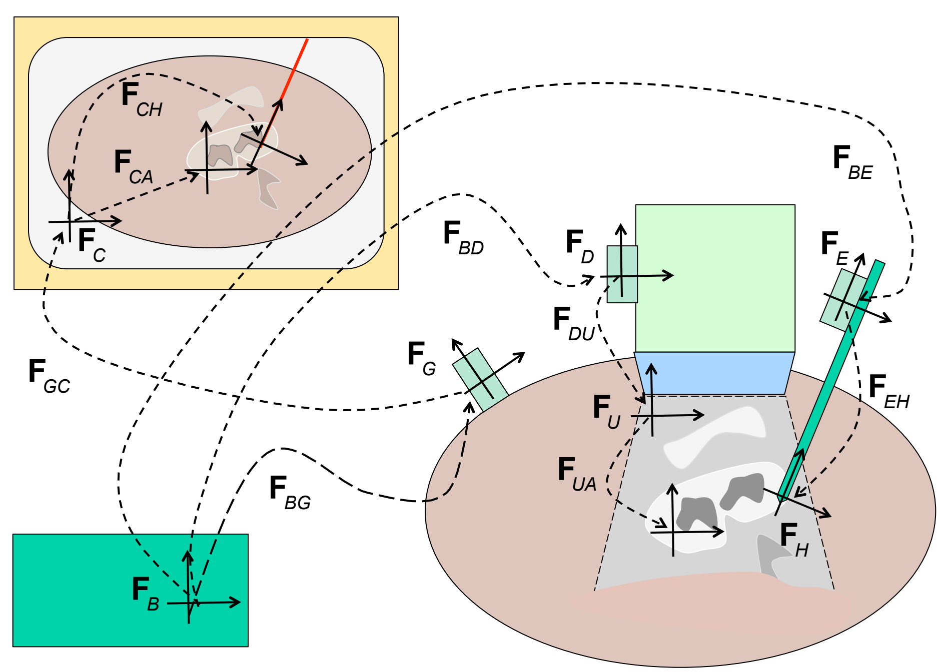


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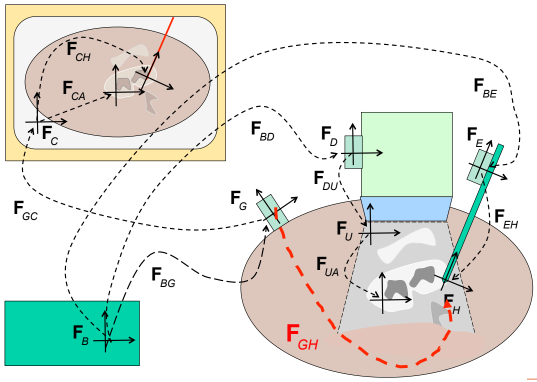


Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_{G}$

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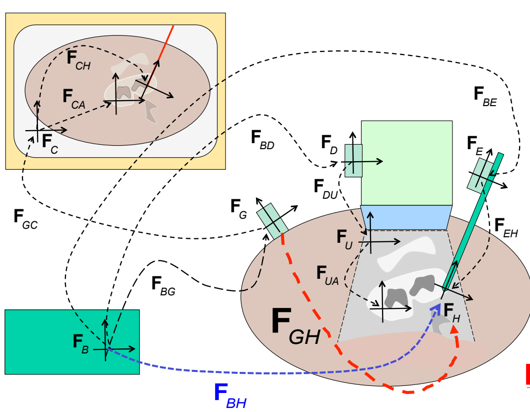


$$
\begin{aligned}
\mathbf{F}_{GH} &= \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH} \\
&= \mathbf{F}_{BG}^{-1}\mathbf{F}_{BH}
\end{aligned}
$$

$$\mathbf{R}_{GH} = \mathbf{R}_{BG}^{-1}\mathbf{R}_{BE}\mathbf{R}_{EH}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}\vec{\mathbf{p}}_{BH} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\vec{\mathbf{p}}_{EH}$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{F}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE})$$

$$\vec{\mathbf{p}}_{GH} = \mathbf{R}_{BG}^{-1}(\mathbf{R}_{BE}\vec{\mathbf{p}}_{EH} + \vec{\mathbf{p}}_{BE} - \vec{\mathbf{p}}_{BG})$$

Give a formula for computing the pose $\mathbf{F}_{GH}$
of the surgical tool coordinate system
relative to the patient rigid body coordinate system $\mathbf{F}_{G}$ ?

What are the components $\mathbf{F}_{GH} = [\mathbf{R}_{GH}, \vec{\mathbf{p}}_{GH}]$?

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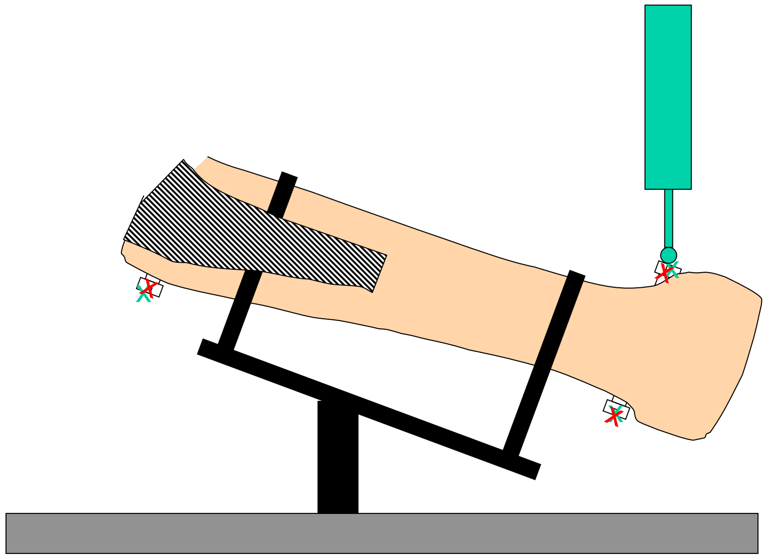

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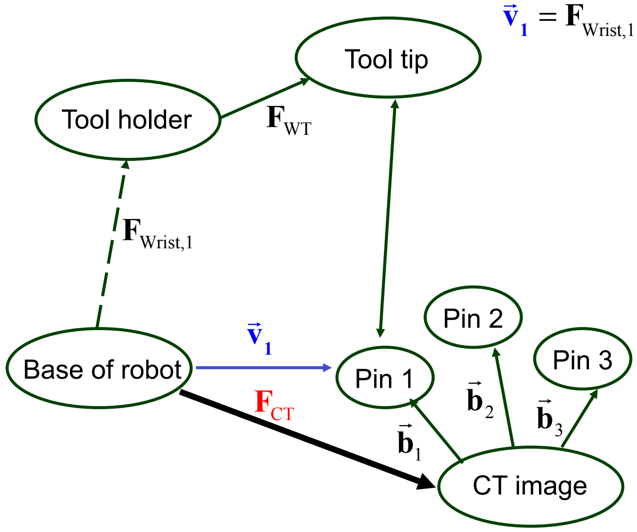
$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\text{Wrist},1} \bullet \vec{\mathbf{p}}_{\text{WT}} = \mathbf{F}_{\text{CT}} \bullet \vec{\mathbf{b}}_1$$

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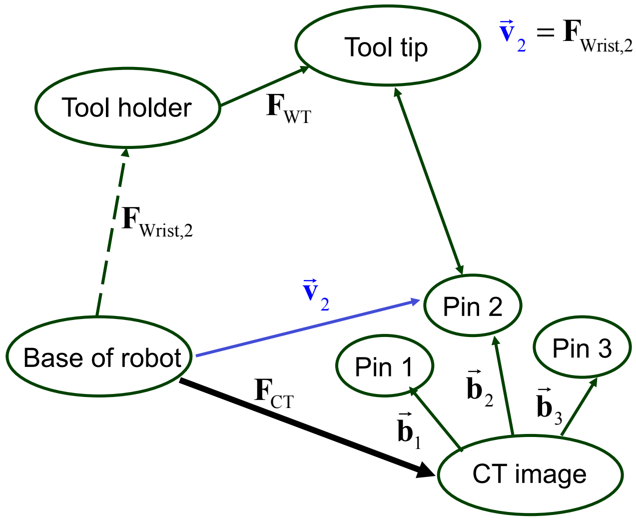
$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\mathrm{Wrist},1} \bullet \vec{\mathbf{p}}_{WT} = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_1$$
$$\vec{\mathbf{v}}_2 = \mathbf{F}_{\mathrm{Wrist},2} \bullet \vec{\mathbf{p}}_{WT} = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_2$$

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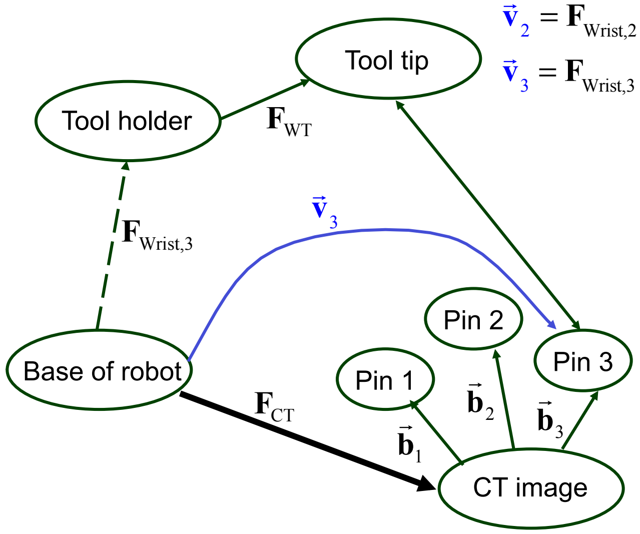
$$\vec{\mathbf{v}}_1 = \mathbf{F}_{\mathrm{Wrist},1} \bullet \vec{\mathbf{p}}_{WT} = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_1$$
$$\vec{\mathbf{v}}_2 = \mathbf{F}_{\mathrm{Wrist},2} \bullet \vec{\mathbf{p}}_{WT} = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_2$$
$$\vec{\mathbf{v}}_3 = \mathbf{F}_{\mathrm{Wrist},3} \bullet \vec{\mathbf{p}}_{WT} = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_3$$

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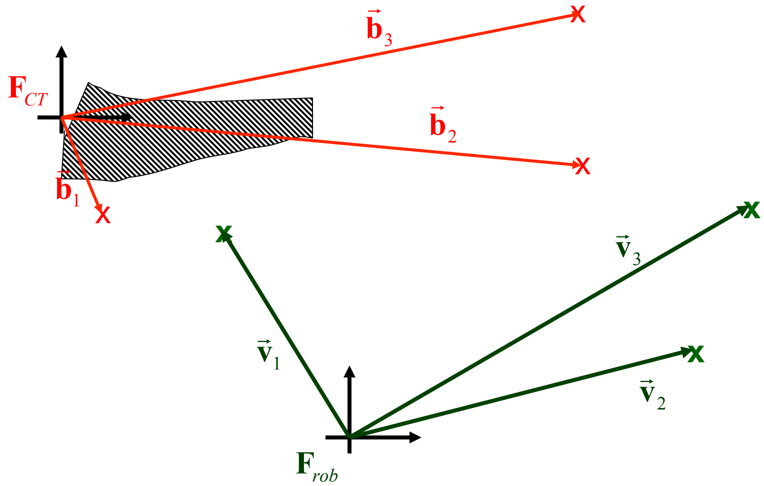
## Frame transformation from 3 point pairs

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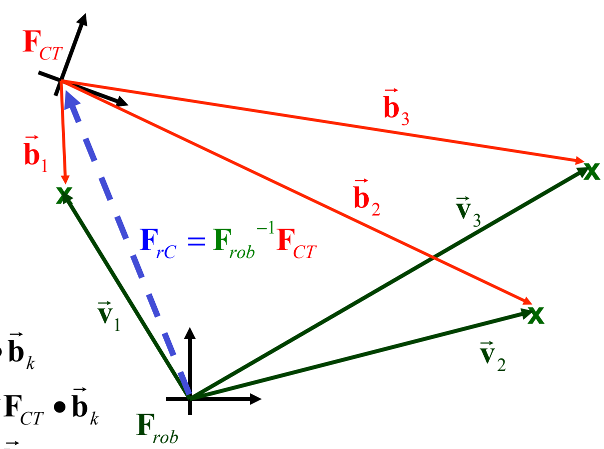
## Frame transformation from 3 point pairs




$$\mathbf{F}_{rob} \bullet \vec{\mathbf{v}}_k = \mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_k$$
$$\vec{\mathbf{v}}_k = \mathbf{F}_{rob}^{-1}\mathbf{F}_{CT} \bullet \vec{\mathbf{b}}_k$$
$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC} \bullet \vec{\mathbf{b}}_k$$

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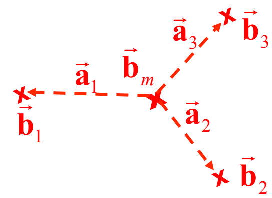
## Frame transformation from 3 point pairs

$$\vec{\mathbf{v}}_k = \mathbf{F}_{rC}\vec{\mathbf{b}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC}$$

Define

$$\vec{\mathbf{v}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{v}}_k \quad \vec{\mathbf{b}}_m = \frac{1}{3}\sum_1^3 \vec{\mathbf{b}}_k$$

$$\vec{\mathbf{u}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m \quad \vec{\mathbf{a}}_k = \vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m$$




$$\mathbf{F}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC}$$

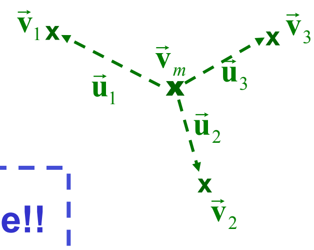
$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k + \vec{\mathbf{p}}_{rC} = \mathbf{R}_{rC}\left(\vec{\mathbf{b}}_k - \vec{\mathbf{b}}_m\right) + \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \mathbf{R}_{rC}\vec{\mathbf{b}}_k + \vec{\mathbf{p}}_{rC} - \mathbf{R}_{rC}\vec{\mathbf{b}}_m - \vec{\mathbf{p}}_{rC}$$

$$\mathbf{R}_{rC}\vec{\mathbf{a}}_k = \vec{\mathbf{v}}_k - \vec{\mathbf{v}}_m = \vec{\mathbf{u}}_k$$

$$\vec{\mathbf{p}}_{rC} = \vec{\mathbf{v}}_m - \mathbf{R}_{rC}\vec{\mathbf{b}}_m$$

**Solve These!!**

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## Rotation from multiple vector pairs

Given a system $\mathbf{R}\vec{\mathbf{a}}_k = \vec{\mathbf{u}}_k$ for $k = 1, \cdots, n$ the problem is to estimate $\mathbf{R}$. This will require at least three such point pairs. Later in the course we will cover some good ways to solve this system. Here is a not-so-good way that will produce roughly correct answers:

Step 1: Form matrices $\mathbf{U} = [\vec{\mathbf{u}}_1 \ \cdots \ \vec{\mathbf{u}}_n]$ and $\mathbf{A} = [\vec{\mathbf{a}}_1 \ \cdots \ \vec{\mathbf{a}}_n]$

Step 2: Solve the system $\mathbf{RA} = \mathbf{U}$ for $\mathbf{R}$. E.g., by $\mathbf{R} = \mathbf{U}\mathbf{A}^{-1}$

Step 3: Renormalize $\mathbf{R}$ to guarantee $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

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## Renormalizing Rotation Matrix

Given "rotation" matrix $\mathbf{R} = \left[ \vec{\mathbf{r}}_x \mid \vec{\mathbf{r}}_y \mid \vec{\mathbf{r}}_z \right]$, modify it so $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.

Step 1: $\vec{\mathbf{a}} = \vec{\mathbf{r}}_y \times \vec{\mathbf{r}}_z$

Step 2: $\vec{\mathbf{b}} = \vec{\mathbf{r}}_z \times \vec{\mathbf{a}}$

Step 3: $\mathbf{R}_{normalized} = \left[ \frac{\vec{\mathbf{a}}}{\left\|\vec{\mathbf{a}}\right\|} \;\middle|\; \frac{\vec{\mathbf{b}}}{\left\|\vec{\mathbf{b}}\right\|} \;\middle|\; \frac{\vec{\mathbf{r}}_z}{\left\|\vec{\mathbf{r}}_z\right\|} \right]$

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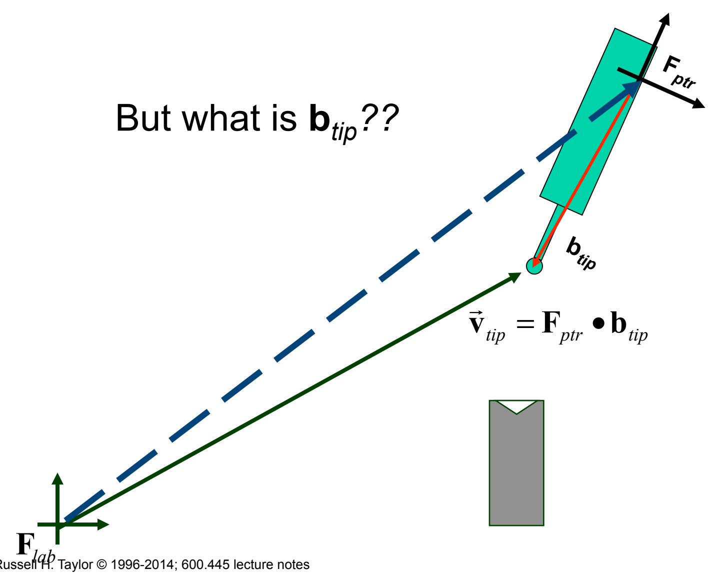
## Calibrating a pointer

But what is $\mathbf{b}_{tip}$??

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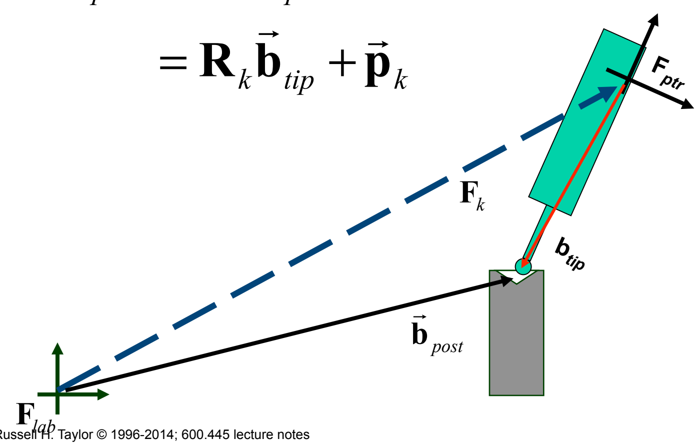
## Calibrating a pointer

$$\vec{\mathbf{b}}_{post} = \mathbf{F}_k \vec{\mathbf{b}}_{tip}$$
$$= \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

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## Calibrating a pointer

For each measurement $k$, we have

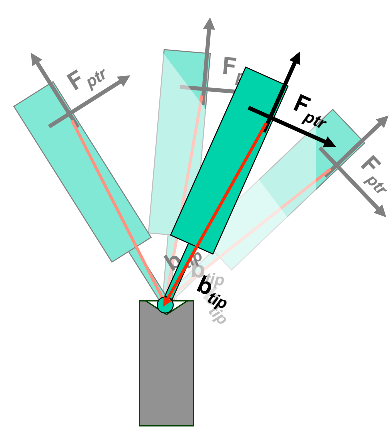
$$\vec{\mathbf{b}}_{post} = \mathbf{R}_k \vec{\mathbf{b}}_{tip} + \vec{\mathbf{p}}_k$$

I. e.,

$$\mathbf{R}_k \vec{\mathbf{b}}_{tip} - \vec{\mathbf{b}}_{post} = -\vec{\mathbf{p}}_k$$

Set up a least squares problem

$$\begin{bmatrix} \vdots & \vdots \\ \mathbf{R}_k & -\mathbf{I} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}}_{tip} \\ \vec{\mathbf{b}}_{post} \end{bmatrix} \cong \begin{bmatrix} \vdots \\ -\vec{\mathbf{p}}_k \\ \vdots \end{bmatrix}$$

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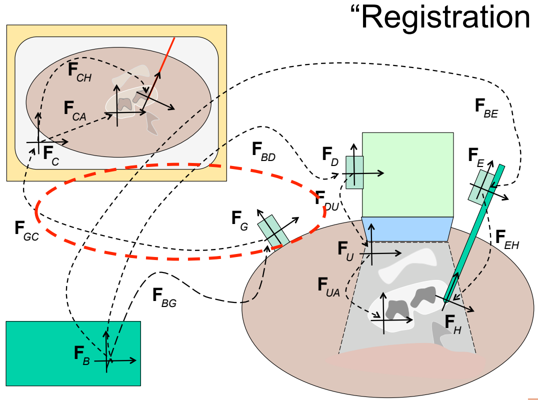
## “Registration Transformations”




Given a coordinate system $\mathbf{F}_C$ and another coordinate system $\mathbf{F}_G$ (e.g., a CT scan and a tracked "rigid body" attached to the patient, and points $\vec{\mathbf{c}}_i$ in the coordinate system $\mathbf{F}_C$ and points $\vec{\mathbf{g}}_i$ in the coordinate system $\mathbf{F}_G$, then the "registration transformation" $\mathbf{F}_{GC}$ between $\mathbf{F}_G$ and $\mathbf{F}_C$ one in which for $\mathbf{F}_{GC}\vec{\mathbf{c}}_i = \vec{\mathbf{g}}_i$ if and only if $\vec{\mathbf{c}}_i$ and $\vec{\mathbf{g}}_i$ refer to the same or corresponding points.

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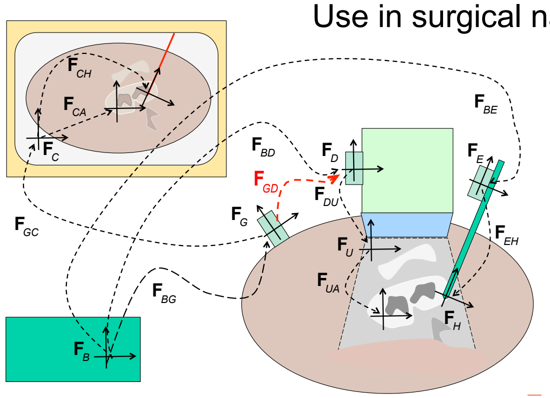
## Use in surgical navigation




$$\mathbf{F}_{BA} = \mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$
$$\mathbf{F}_{BA} = \mathbf{F}_{BG}\mathbf{F}_{GC}\mathbf{F}_{CA}$$

$$\mathbf{F}_{BG}\mathbf{F}_{GC}\mathbf{F}_{CA} = \mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$
$$\mathbf{F}_{CA} = \left(\mathbf{F}_{BG}\mathbf{F}_{GC}\right)^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}$$

$$\mathbf{F}_{GC} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}\mathbf{F}_{DU}\mathbf{F}_{UA}\mathbf{F}_{CA}^{-1}$$
$$\mathbf{F}_{GC} = \mathbf{F}_{GD}\mathbf{F}_{DU}\mathbf{F}_{UA}\mathbf{F}_{CA}^{-1}$$

where $\mathbf{F}_{GD} = \mathbf{F}_{BG}^{-1}\mathbf{F}_{BD}$

If an anatomic structure is identified at pose $\mathbf{F}_{UA}$ in ultrasound image coordinates give the formula for computing the corresponding pose $\mathbf{F}_{CA}$ in CT coordinates

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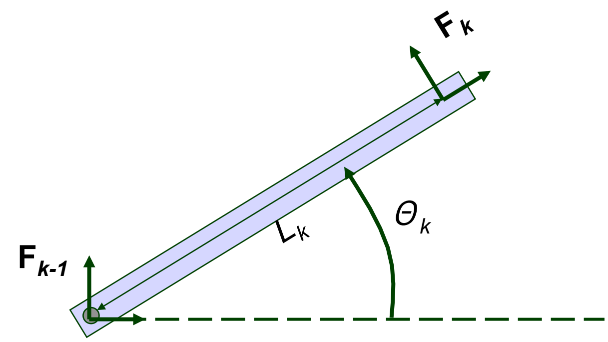
# Kinematic Links




$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$[\mathbf{R}_k, \vec{\mathbf{p}}_k] = [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}]$$

$$= [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [Rot(\vec{\mathbf{r}}_k, \theta_k), L_k Rot(\vec{\mathbf{r}}_k, \theta_k) \bullet \vec{\mathbf{x}}]$$

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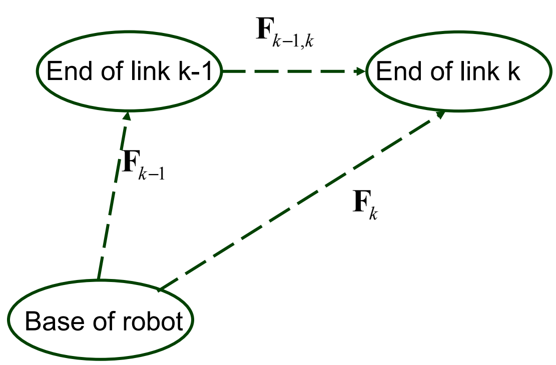
# Kinematic Links




$$\mathbf{F}_k = \mathbf{F}_{k-1} \bullet \mathbf{F}_{k-1,k}$$

$$[\mathbf{R}_k, \vec{\mathbf{p}}_k] = [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [\mathbf{R}_{k-1,k}, \mathbf{p}_{k-1,k}]$$

$$= [\mathbf{R}_{k-1}, \mathbf{p}_{k-1}] \bullet [Rot(\vec{\mathbf{r}}_k, \theta_k), L_k Rot(\vec{\mathbf{r}}_k, \theta_k) \bullet \vec{\mathbf{x}}]$$

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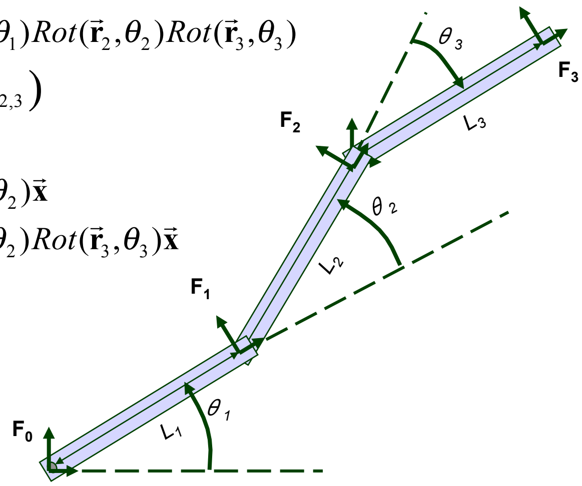
# Kinematic Chains

$$\mathbf{F}_0 = [\mathbf{I}, \vec{\mathbf{0}}]$$

$$\mathbf{R}_3 = \mathbf{R}_{0,1}\mathbf{R}_{1,2}\mathbf{R}_{2,3} = Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)$$

$$\begin{aligned}\vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right)\\ &= L_1 Rot(\vec{\mathbf{r}}_1,\theta_1)\vec{\mathbf{x}}\\ &\quad + L_2 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)\vec{\mathbf{x}}\\ &\quad + L_3 Rot(\vec{\mathbf{r}}_1,\theta_1)Rot(\vec{\mathbf{r}}_2,\theta_2)Rot(\vec{\mathbf{r}}_3,\theta_3)\vec{\mathbf{x}}\end{aligned}$$

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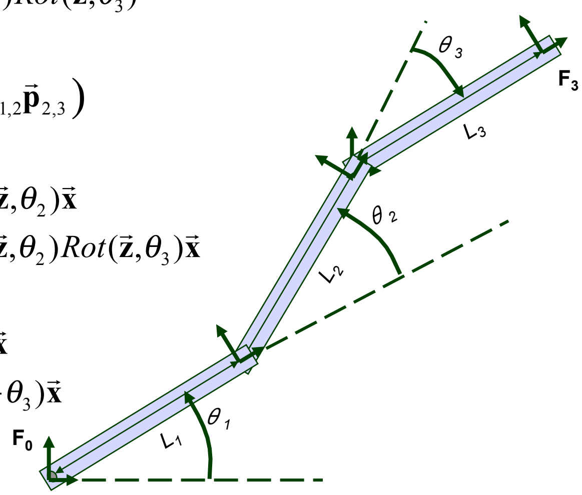
# Kinematic Chains

If $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}}$,

$$\begin{aligned}\mathbf{R}_3 &= Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3)\\ &= Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)\end{aligned}$$

$$\begin{aligned}\vec{\mathbf{p}}_3 &= \vec{\mathbf{p}}_{0,1} + \mathbf{R}_{0,1}\left(\vec{\mathbf{p}}_{1,2} + \mathbf{R}_{1,2}\vec{\mathbf{p}}_{2,3}\right)\\ &= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}}\\ &\quad + L_2 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)\vec{\mathbf{x}}\\ &\quad + L_3 Rot(\vec{\mathbf{z}},\theta_1)Rot(\vec{\mathbf{z}},\theta_2)Rot(\vec{\mathbf{z}},\theta_3)\vec{\mathbf{x}}\\ &= L_1 Rot(\vec{\mathbf{z}},\theta_1)\vec{\mathbf{x}}\\ &\quad + L_2 Rot(\vec{\mathbf{z}},\theta_1+\theta_2)\vec{\mathbf{x}}\\ &\quad + L_3 Rot(\vec{\mathbf{z}},\theta_1+\theta_2+\theta_3)\vec{\mathbf{x}}\end{aligned}$$

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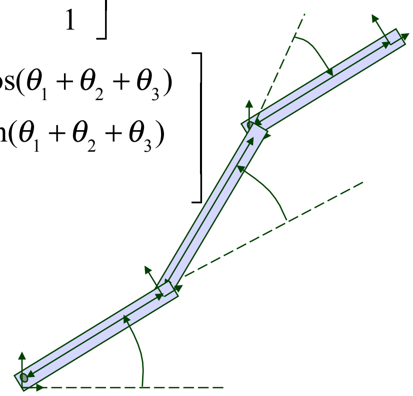
## Kinematic Chains

If $\vec{\mathbf{r}}_1 = \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_3 = \vec{\mathbf{z}},$

$$\mathbf{R}_3 = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 \end{bmatrix}$$





## "Small" Transformations

- A great deal of CIS is concerned with computing and using geometric information based on imprecise knowledge
- Similarly, one is often concerned with the effects of relatively small rotations and displacements
- Essentially, we will be using fairly straightforward linearizations to model these situations, but a specialized notation is often useful

## “Small” Frame Transformations

Represent a "small" pose shift consisting of a small rotation $\Delta\mathbf{R}$ followed by a small displacement $\Delta\vec{\mathbf{p}}$ as

$$\Delta\mathbf{F} = [\Delta\mathbf{R}, \Delta\vec{\mathbf{p}}]$$

Then

$$\Delta\mathbf{F} \bullet \vec{\mathbf{v}} = \Delta\mathbf{R} \bullet \vec{\mathbf{v}} + \Delta\vec{\mathbf{p}}$$

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## Small Rotations

$$\Delta\mathbf{R} = \text{ a small rotation}$$

$$\mathbf{R}_{\vec{\mathbf{a}}}(\Delta\alpha) = \text{ a rotation by a small angle } \Delta\alpha \text{ about axis } \vec{\mathbf{a}}$$

$$Rot(\vec{\mathbf{a}}, \|\vec{\mathbf{a}}\|) \bullet \vec{\mathbf{b}} \approx \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}} \quad \text{for } \|\vec{\mathbf{a}}\| \text{ sufficiently small}$$

$$\Delta\mathbf{R}(\vec{\mathbf{a}}) = \text{ a rotation that is small enough so that any error introduced by this approximation is negligible}$$

$$\Delta\mathbf{R}(\lambda\vec{\mathbf{a}}) \bullet \Delta\mathbf{R}(\mu\,\vec{\mathbf{b}}) \cong \Delta\mathbf{R}(\lambda\vec{\mathbf{a}} + \mu\,\vec{\mathbf{b}}) \quad \text{(Linearity for small rotations)}$$

Exercise: Work out the linearity proposition by substitution

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## Approximations to “Small” Frames

$$
\begin{aligned}
\Delta\mathbf{F}(\vec{\mathbf{a}},\Delta\vec{\mathbf{p}}) &\approx [\Delta\mathbf{R}(\vec{\mathbf{a}}),\Delta\vec{\mathbf{p}}] \\
\Delta\mathbf{F}(\vec{\mathbf{a}},\Delta\vec{\mathbf{p}})\bullet\vec{\mathbf{v}} &= \Delta\mathbf{R}(\vec{\mathbf{a}})\bullet\vec{\mathbf{v}}+\Delta\vec{\mathbf{p}} \\
&\approx \vec{\mathbf{v}}+\vec{\mathbf{a}}\times\vec{\mathbf{v}}+\Delta\vec{\mathbf{p}}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\mathbf{a}}\times\vec{\mathbf{v}} &= skew(\vec{\mathbf{a}})\bullet\vec{\mathbf{v}} \\
&= \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\bullet\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
\end{aligned}
$$

$$skew(\vec{\mathbf{a}})\bullet\vec{\mathbf{a}}=\vec{0}$$

$$
\begin{aligned}
\Delta\mathbf{R}(\vec{\mathbf{a}}) &\approx \mathbf{I}+skew(\vec{\mathbf{a}}) \\
\Delta\mathbf{R}(\vec{\mathbf{a}})^{-1} &\approx \mathbf{I}-skew(\vec{\mathbf{a}})=\mathbf{I}+skew(-\vec{\mathbf{a}})
\end{aligned}
$$

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## Approximations to “Small” Frames

Notational NOTE:

We often use $\vec{\alpha}$ to represent a vector of small angles
and $\vec{\varepsilon}$ to represent a vector of small displacements

In using these approximations, we typically ignore second order terms. I.e.,

$\vec{\alpha}_A\vec{\alpha}_B \approx \vec{0}$, $\vec{\alpha}_A\vec{\varepsilon}_B \approx \vec{0}$, $\vec{\varepsilon}_A\vec{\varepsilon}_B \approx \vec{0}$, etc.

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# Errors & sensitivity

Often, we do not have an accurate value for a transformation, so we need to model the error. We model this as a composition of a "nominal" frame and a small displacement

$$\mathbf{F}_{\text{actual}} = \mathbf{F}_{\text{nominal}} \bullet \Delta\mathbf{F}$$

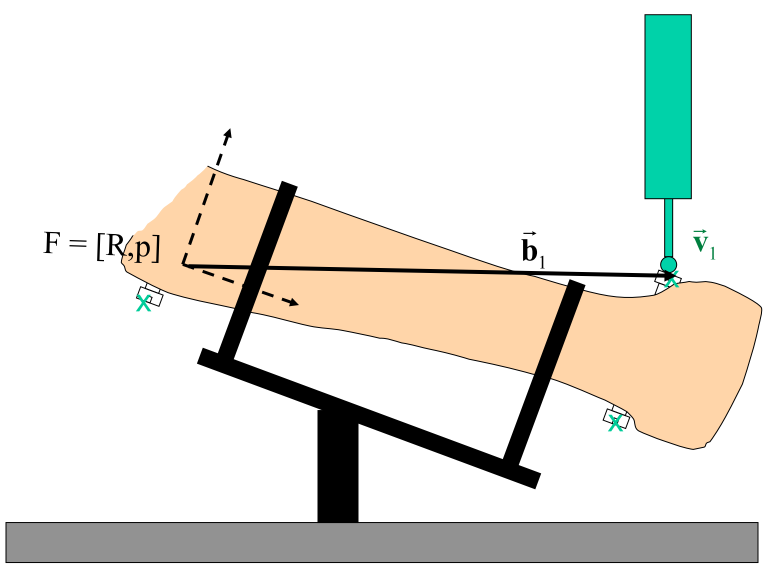
Often, we will use the notation $\mathbf{F}^*$ for $\mathbf{F}_{\text{actual}}$ and will just use $\mathbf{F}$ for $\mathbf{F}_{\text{nominal}}$. Thus we may write something like

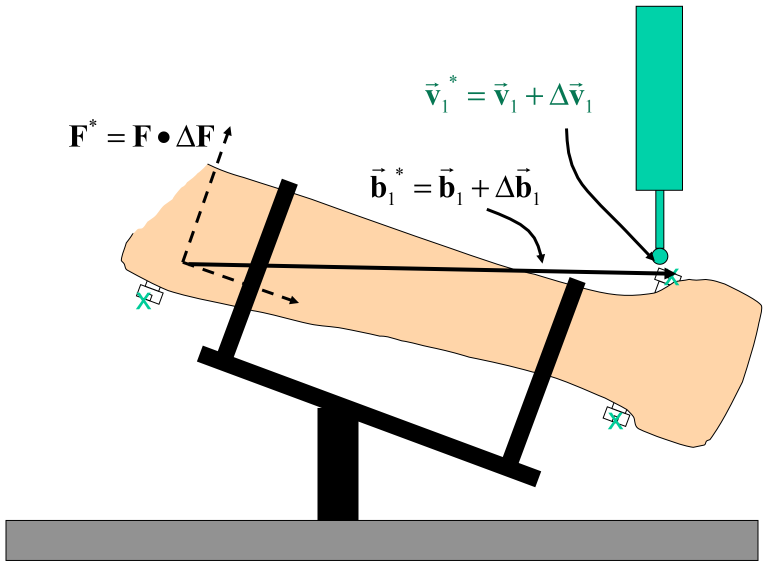
$$\mathbf{F}^* = \mathbf{F} \bullet \Delta\mathbf{F}$$

or (less often) $\mathbf{F}^* = \Delta\mathbf{F} \bullet \mathbf{F}$. We also use $\vec{\mathbf{v}}^* = \vec{\mathbf{v}} + \Delta\vec{\mathbf{v}}$, *etc*. Thus, if we use the former form (error on the right), and have nominal relationship $\vec{\mathbf{v}} = \mathbf{F} \bullet \vec{\mathbf{b}}$, we get

$$\begin{aligned}\vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\ &= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}})\end{aligned}$$





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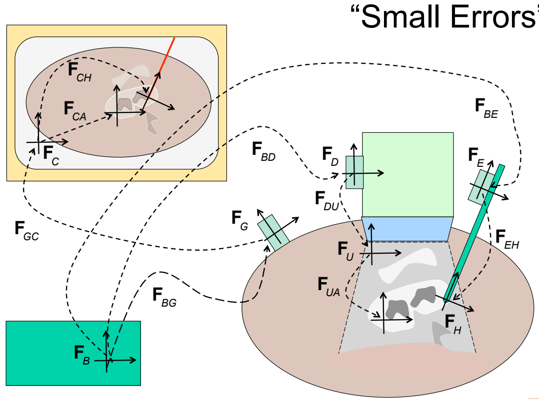
## "Small Errors"




Suppose that there is a small systematic error in the tracking system so that

$$\mathbf{F}^*_{Bx} = \Delta\mathbf{F}_B\mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}$, $\mathbf{F}_{BD}$, $\mathbf{F}_{BE}$. How does this affect the calculation of $\mathbf{F}_{GH}$ ?

$$\begin{aligned}
\mathbf{F}^*_{GH} &= (\mathbf{F}^*_{BG})^{-1}\mathbf{F}^*_{BE}\,\mathbf{F}_{EH} \\
\mathbf{F}_{GH}\Delta\mathbf{F}_{GH} &= \left(\Delta\mathbf{F}_B\mathbf{F}_{BG}\right)^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH} \\
\Delta\mathbf{F}_{GH} &= \mathbf{F}_{BG}^{-1}\Delta\mathbf{F}_B^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\mathbf{F}_{EH}\mathbf{F}_{GH}^{-1} \\
&= \mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}\mathbf{F}_{GH}^{-1} \\
&= \mathbf{F}_{GH}\mathbf{F}_{GH}^{-1} = \mathbf{I}
\end{aligned}$$

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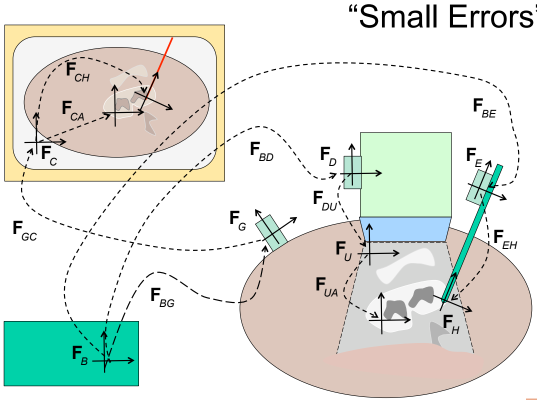
## “Small Errors”




Suppose that there are additional errors in the tracking of each tracker body so that

$$\mathbf{F}^*_{Bx} = \Delta\mathbf{F}_B\mathbf{F}_{Bx}\Delta\mathbf{F}_{Bx}$$

for $\mathbf{F}_{BG}, \mathbf{F}_{BD}, \mathbf{F}_{BE}$. How does this affect the calculation of $\mathbf{F}_{GH}$ ?

$$\mathbf{F}^*_{GH} = \mathbf{F}_{GH}\Delta\mathbf{F}_{GH} = (\mathbf{F}^*_{BG})^{-1}\mathbf{F}^*_{BE}\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \mathbf{F}_{GH}^{-1}\left(\Delta\mathbf{F}_B\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}\right)^{-1}\left(\Delta\mathbf{F}_B\mathbf{F}_{BE}\Delta\mathbf{F}_{BE}\right)\mathbf{F}_{EH}$$

$$\Delta\mathbf{F}_{GH} = \left(\mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\mathbf{F}_{EH}\right)^{-1}\Delta\mathbf{F}_{BG}^{-1}\mathbf{F}_{BG}^{-1}\Delta\mathbf{F}_B^{-1}\Delta\mathbf{F}_B\mathbf{F}_{BE}\Delta\mathbf{F}_{BG}\mathbf{F}_{EH}$$

$$= \mathbf{F}_{EH}^{-1}\mathbf{F}_{BE}^{-1}\mathbf{F}_{BG}\Delta\mathbf{F}_{BG}^{-1}\mathbf{F}_{BG}^{-1}\mathbf{F}_{BE}\Delta\mathbf{F}_{BG}\mathbf{F}_{EH}$$

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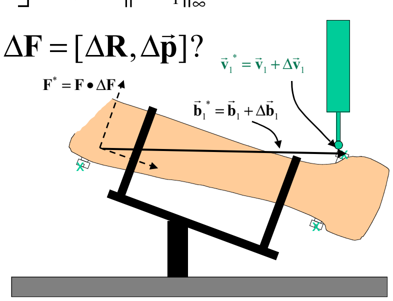
## Errors & Sensitivity

Suppose that we know nominal values for $\mathbf{F}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{v}}$ and that

$$\left[-\varepsilon,-\varepsilon,-\varepsilon\right]^T \le \Delta\vec{\mathbf{v}}_1 \le \left[\varepsilon,\varepsilon,\varepsilon\right]^T \quad (i.e., \left\|\Delta\vec{\mathbf{v}}_1\right\|_\infty \le \varepsilon)$$

What does this tell us about $\Delta\mathbf{F} = [\Delta\mathbf{R},\Delta\vec{\mathbf{p}}]$?

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## Errors & Sensitivity

$$
\begin{aligned}
\vec{\mathbf{v}}^* &= \mathbf{F}^* \bullet \vec{\mathbf{b}}^* \\
&= \mathbf{F} \bullet \Delta\mathbf{F} \bullet (\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}) \\
&= \mathbf{R} \bullet \left(\Delta\mathbf{R}(\vec{\alpha}) \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}}\right) + \Delta\vec{\mathbf{p}}\right) + \vec{\mathbf{p}} \\
&\approx \mathbf{R} \bullet \left(\vec{\mathbf{b}} + \Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \vec{\alpha} \times \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) + \vec{\mathbf{p}} \\
&= \mathbf{R} \bullet \vec{\mathbf{b}} + \vec{\mathbf{p}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \vec{\alpha} \times \Delta\vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) \\
&\approx \vec{\mathbf{v}} + \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right)
\end{aligned}
$$

if $\left\|\vec{\alpha} \times \Delta\vec{\mathbf{b}}\right\| \le \left\|\vec{\alpha}\right\| \left\|\Delta\vec{\mathbf{b}}\right\|$ is negligible (it usually is)

so

$$
\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}^* - \vec{\mathbf{v}} \approx \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}} + \vec{\alpha} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}\right) = \mathbf{R} \bullet \Delta\vec{\mathbf{b}} + \mathbf{R} \bullet \vec{\alpha} \times \vec{\mathbf{b}} + \mathbf{R} \bullet \Delta\vec{\mathbf{p}}
$$

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## Digression: "rotation triple product"

Expressions like $\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}}$ are linear in $\vec{\mathbf{a}}$, but are not always convenient to work with. Often we would prefer something like $\mathbf{M}(\mathbf{R}, \vec{\mathbf{b}}) \bullet \vec{\mathbf{a}}$.

$$
\begin{aligned}
\mathbf{R} \bullet \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= -\mathbf{R} \bullet \vec{\mathbf{b}} \times \vec{\mathbf{a}} \\
&= \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \bullet \vec{\mathbf{a}} \\
&= \left[\mathbf{R} \bullet skew(\vec{\mathbf{b}})^T\right] \bullet \vec{\mathbf{a}}
\end{aligned}
$$

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## Digression: "rotation triple product"

Here are a few more useful facts:

$$\mathbf{R} \bullet \left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) = \left(\mathbf{R} \bullet \vec{\mathbf{a}}\right) \times \left(\mathbf{R} \bullet \vec{\mathbf{b}}\right)$$

$$\vec{\mathbf{a}} \times \left(\mathbf{R} \bullet \vec{\mathbf{b}}\right) = \mathbf{R} \bullet \left(\left(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}}\right) \times \vec{\mathbf{b}}\right)$$

Consequently

$$skew(\vec{\mathbf{a}}) \bullet \mathbf{R} = \mathbf{R} \bullet skew(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}})$$

$$\mathbf{R}^{-1} skew(\vec{\mathbf{a}}) \bullet \mathbf{R} = skew(\mathbf{R}^{-1} \bullet \vec{\mathbf{a}})$$



## Errors & Sensitivity

Previous expression was

$$\Delta\vec{\mathbf{v}}_1 \approx \mathbf{R} \bullet \left(\Delta\vec{\mathbf{b}}_1 + \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \Delta\vec{\mathbf{p}}_1\right)$$

Substituting triple product and rearranging gives

$$\Delta\vec{\mathbf{v}}_1 \approx \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \end{array}\right] \bullet \begin{bmatrix} \Delta\vec{\mathbf{b}}_1 \\ \Delta\vec{\mathbf{p}} \\ \vec{\mathbf{a}} \end{bmatrix}$$

So

$$\begin{bmatrix} -\varepsilon \\ -\varepsilon \\ -\varepsilon \end{bmatrix} \le \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \end{array}\right] \begin{bmatrix} \Delta\vec{\mathbf{b}}_1 \\ \Delta\vec{\mathbf{p}} \\ \vec{\mathbf{a}} \end{bmatrix} \le \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}$$

## Errors & Sensitivity

Now, suppose we know that $\left|\Delta\vec{\mathbf{b}}_1\right| \le \beta$, this will give us a system of linear constraints

$$\begin{bmatrix} -\varepsilon \\ -\varepsilon \\ -\varepsilon \\ \hline -\beta \\ -\beta \\ -\beta \end{bmatrix} \le \left[\begin{array}{c|c|c} \mathbf{R} & \mathbf{R} & \mathbf{R} \bullet skew(-\vec{\mathbf{b}}) \\ \hline \mathbf{I} & \mathbf{0} & \mathbf{0} \end{array}\right] \begin{bmatrix} \Delta\vec{\mathbf{b}}_1 \\ \Delta\vec{\mathbf{p}}_1 \\ \vec{\mathbf{a}} \end{bmatrix} \le \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \\ \hline \beta \\ \beta \\ \beta \end{bmatrix}$$

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## Error from frame composition

Consider $\mathbf{R}_1^*\mathbf{R}_2^* = \mathbf{R}_3^*$ where $\mathbf{R}_1^* = \mathbf{R}_1\Delta\mathbf{R}_1, \mathbf{R}_2^* = \mathbf{R}_2\Delta\mathbf{R}_2,\ \mathbf{R}_3^* = \mathbf{R}_3\Delta\mathbf{R}_3$
and $\Delta\mathbf{R}_1 \approx \mathbf{I} + sk(\vec{\alpha}_1),\ \Delta\mathbf{R}_2 \approx \mathbf{I} + sk(\vec{\alpha}_2)$, estimate $\Delta\mathbf{R}_3 \approx \mathbf{I} + sk(\vec{\alpha}_3)$

$$\mathbf{R}_1\Delta\mathbf{R}_1\mathbf{R}_2\Delta\mathbf{R}_2 = \mathbf{R}_1\mathbf{R}_2\Delta\mathbf{R}_3$$

$$\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{R}_1\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_3))$$

$$(\mathbf{R}_1\mathbf{R}_2)^{-1}\mathbf{R}_1(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1}\cancel{\mathbf{R}_1^{-1}\mathbf{R}_1}(\mathbf{I} + sk(\vec{\alpha}_1))\mathbf{R}_2(\mathbf{I} + sk(\vec{\alpha}_2)) \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{I} + \mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) + \cancel{\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 sk(\vec{\alpha}_2)} \approx \mathbf{I} + sk(\vec{\alpha}_3)$$

$$\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 + sk(\vec{\alpha}_2) \approx sk(\vec{\alpha}_3)$$

Since $\mathbf{R}^{-1}\bullet(\vec{\mathbf{a}} \times \mathbf{R}\vec{\mathbf{b}}) = (\mathbf{R}^{-1}\vec{\mathbf{a}}) \times \vec{\mathbf{b}}$ for all $\mathbf{R}, \vec{\mathbf{a}}, \vec{\mathbf{b}}$ we get $\mathbf{R}_2^{-1}sk(\vec{\alpha}_1)\mathbf{R}_2 = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1)$

$$sk(\vec{\alpha}_3) \approx sk(\mathbf{R}_2^{-1}\vec{\alpha}_1) + sk(\vec{\alpha}_2) = sk(\mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2)$$

$$\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$$

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## Error from frame composition

Consider $\mathbf{F}_1^*\mathbf{F}_2^* = \mathbf{F}_3^*$ where $\mathbf{F}_1^* = \mathbf{F}_1\Delta\mathbf{F}_1, \mathbf{F}_2^* = \mathbf{F}_2\Delta\mathbf{F}_2,\ \mathbf{F}_3^* = \mathbf{F}_3\Delta\mathbf{F}_3$
and $\Delta\mathbf{F}_1 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_1\right), \vec{\varepsilon}_1\right],\ \Delta\mathbf{F}_2 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_2\right), \vec{\varepsilon}_3\right],$
estimate $\Delta\mathbf{F}_3 \approx \left[\mathbf{I} + sk\left(\vec{\alpha}_3\right), \vec{\varepsilon}_3\right]$

From before, we have $\vec{\alpha}_3 \approx \mathbf{R}_2^{-1}\vec{\alpha}_1 + \vec{\alpha}_2$. So now we just need $\vec{\varepsilon}_3$.

$$
\begin{aligned}
\vec{\mathbf{p}}_3^* &= \mathbf{R}_1^*\vec{\mathbf{p}}_2^* + \vec{\mathbf{p}}_1^* \\
\vec{\mathbf{p}}_3 + \vec{\varepsilon}_3 &\approx \mathbf{R}_1\left(\mathbf{I} + sk(\vec{\alpha}_1)\right)\left(\vec{\mathbf{p}}_2 + \vec{\varepsilon}_2\right) + \left(\vec{\mathbf{p}}_1 + \vec{\varepsilon}_1\right) \\
&= \mathbf{R}_1\vec{\mathbf{p}}_2 + \mathbf{R}_1\vec{\varepsilon}_2 + \mathbf{R}_1 \bullet \left(\vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\alpha}_1 \times \vec{\varepsilon}_2\right) + \vec{\mathbf{p}}_1 + \vec{\varepsilon}_1 \\
&= \vec{\mathbf{p}}_3 + \mathbf{R}_1\vec{\varepsilon}_2 + \mathbf{R}_1 \bullet \left(\vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\alpha}_1 \times \vec{\varepsilon}_2\right) + \vec{\varepsilon}_1 \\
\vec{\varepsilon}_3 &\approx \mathbf{R}_1\vec{\varepsilon}_2 + \mathbf{R}_1 \bullet \vec{\alpha}_1 \times \vec{\mathbf{p}}_2 + \vec{\varepsilon}_1 \\
&= \mathbf{R}_1\vec{\varepsilon}_2 - \mathbf{R}_1 \bullet \vec{\mathbf{p}}_2 \times \vec{\alpha}_1 + \vec{\varepsilon}_1 \\
&= \vec{\varepsilon}_1 - \mathbf{R}_1 sk(\vec{\mathbf{p}}_2)\vec{\alpha}_1 + \mathbf{R}_1\vec{\varepsilon}_2
\end{aligned}
$$

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## Inverse of frame transformation with errors

$$
\begin{aligned}
\mathbf{F}_i &= \mathbf{F}^{-1} = [\mathbf{R}^{-1}, -\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\mathbf{F}_i^* &= \left(\mathbf{F}\Delta\mathbf{F}\right)^{-1} \\
\mathbf{F}_i\Delta\mathbf{F}_i &= \left[\mathbf{R}\Delta\mathbf{R}, \mathbf{R}\Delta\vec{\mathbf{p}} + \vec{\mathbf{p}}\right]^{-1} \\
&= [(\mathbf{R}\Delta\mathbf{R})^{-1}, -(\mathbf{R}\Delta\mathbf{R})^{-1}\left(\mathbf{R}\Delta\mathbf{p} + \vec{\mathbf{p}}\right)] \\
&= [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\left(\mathbf{R}\Delta\mathbf{p} + \vec{\mathbf{p}}\right)] \\
&= [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
\Delta\mathbf{F}_i &= \left(\mathbf{F}^{-1}\right)^{-1}[\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
&= [\mathbf{R}, \vec{\mathbf{p}}] \bullet [\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}}] \\
&= [\mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}, -\mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} + \vec{\mathbf{p}}]
\end{aligned}
$$

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# Inverse of frame transformation with errors

Suppose we know that $\Delta\mathbf{R}$ is "small", i.e.,$\Delta\mathbf{R} \approx \mathbf{I} + sk(\vec{\alpha})$, and for notational convenience we write $\Delta\vec{\mathbf{p}} = \vec{\varepsilon}$, we get

$$
\begin{aligned}
\Delta\mathbf{R}_i &= \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1} \approx \mathbf{R}\left(\mathbf{I} + sk(\vec{\alpha})\right)^{-1}\mathbf{R}^{-1} \\
&\approx \mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\mathbf{R}^{-1} \\
&= \mathbf{R}\mathbf{R}^{-1} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\
&= \mathbf{I} - \mathbf{R}sk(\vec{\alpha})\mathbf{R}^{-1} \\
&= \mathbf{I} - sk\left(\left(\mathbf{R}^{-1}\right)^{-1}\vec{\alpha}\right) = \mathbf{I} - sk(\mathbf{R}\vec{\alpha})
\end{aligned}
$$

$$
\begin{aligned}
\Delta\vec{\mathbf{p}}_i &= -\mathbf{R}\Delta\mathbf{R}^{-1}\Delta\vec{\mathbf{p}} - \mathbf{R}\Delta\mathbf{R}^{-1}\mathbf{R}^{-1}\vec{\mathbf{p}} + \vec{\mathbf{p}} \\
&\approx -\mathbf{R}\left(\mathbf{I} - sk(\vec{\alpha})\right)\vec{\varepsilon} - \left(\mathbf{I} - sk(\mathbf{R}\vec{\alpha})\right)\vec{\mathbf{p}} + \vec{\mathbf{p}} \\
&= -\mathbf{R}\vec{\varepsilon} + \mathbf{R}\left(\vec{\alpha} \times \vec{\varepsilon}\right) - \vec{\mathbf{p}} + (\mathbf{R}\vec{\alpha}) \times \vec{\mathbf{p}} + \vec{\mathbf{p}} \\
&\approx -\mathbf{R}\vec{\varepsilon} + (\mathbf{R}\vec{\alpha}) \times \vec{\mathbf{p}} = -\mathbf{R}\vec{\varepsilon} - \vec{\mathbf{p}} \times (\mathbf{R}\vec{\alpha}) = -\mathbf{R}\vec{\varepsilon} - \mathbf{R}\left(\vec{\mathbf{p}} \times (\mathbf{R}^{-1}\mathbf{R}\vec{\alpha})\right) \\
&= -\mathbf{R}\bullet\left(\vec{\varepsilon} + \vec{\mathbf{p}} \times \vec{\alpha}\right) = -\mathbf{R}\vec{\varepsilon} + \mathbf{R}sk(\vec{\mathbf{p}})\vec{\alpha}
\end{aligned}
$$

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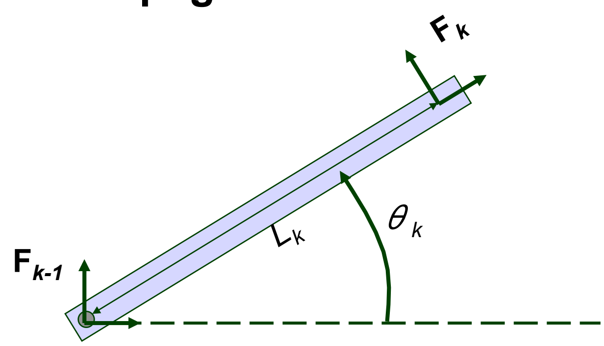
# Error Propagation in Chains




$$
\begin{aligned}
\mathbf{F}^*_k &= \mathbf{F}^*_{k-1} \bullet \mathbf{F}^*_{k-1,k} \\
\mathbf{F}_k\Delta\mathbf{F}_k &= \mathbf{F}_{k-1}\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta\mathbf{F}_{k-1,k} \\
\Delta\mathbf{F}_k &= \left(\mathbf{F}_k^{-1}\mathbf{F}_{k-1}\right)\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\Delta\mathbf{F}_{k-1,k} \\
&= \left(\mathbf{F}_{k-1,k}{}^{-1}\Delta\mathbf{F}_{k-1}\mathbf{F}_{k-1,k}\right)\Delta\mathbf{F}_{k-1,k}
\end{aligned}
$$

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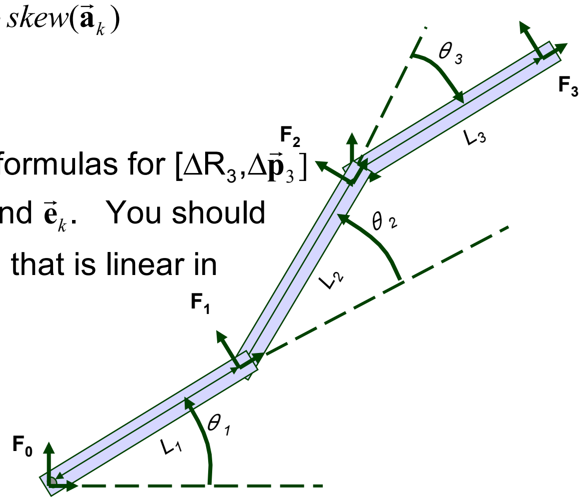
# Exercise

Suppose that you have

$$\Delta \mathbf{R}_{k-1,k} = \Delta \mathbf{R}(\vec{\mathbf{a}}_k) \cong \mathbf{I} + skew(\vec{\mathbf{a}}_k)$$
$$\Delta \vec{\mathbf{p}}_{k-1,k} = \vec{\mathbf{e}}_k$$

Work out approximate formulas for $[\Delta \mathrm{R}_3, \Delta \vec{\mathbf{p}}_3]$ in terms of $L_k, \vec{\mathbf{r}}_k, \theta_k, \vec{\mathbf{a}}_k$ and $\vec{\mathbf{e}}_k$. You should come up with a formula that is linear in $L_k$, $\vec{\mathbf{a}}_k$, and $\vec{\mathbf{e}}_k$.

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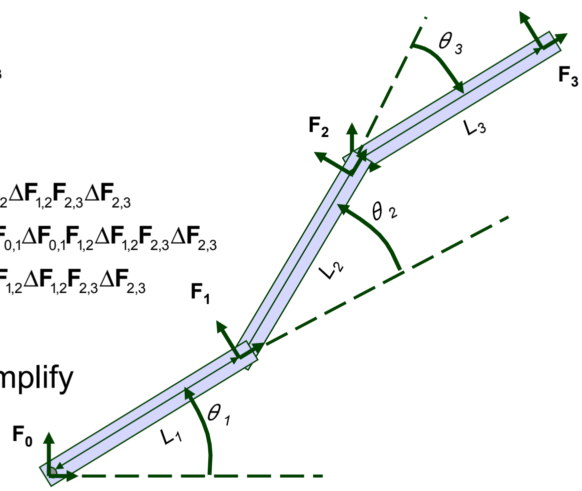
# Exercise

Suppose we want to know error in $\mathbf{F}_{0,3} = \mathbf{F}_0^{-1}\mathbf{F}_3$

$$\begin{aligned}
\mathbf{F}_{0,3} &= \mathbf{F}_0^{-1}\mathbf{F}_0\mathbf{F}_{0,1}\mathbf{F}_{1,2}\mathbf{F}_{2,3} \\
\mathbf{F}_{0,3}^* &= \mathbf{F}_{0,1}^*\mathbf{F}_{1,2}^*\mathbf{F}_{2,3}^* \\
\mathbf{F}_{0,3}\Delta\mathbf{F}_{0,3} &= \mathbf{F}_{0,1}^*\mathbf{F}_{1,2}^*\mathbf{F}_{2,3}^* \\
\Delta\mathbf{F}_3 &= \mathbf{F}_{0,3}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3} \\
&= \mathbf{F}_{2,3}^{-1}\mathbf{F}_{1,2}^{-1}\mathbf{F}_{0,1}^{-1}\mathbf{F}_{0,1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3} \\
&= \mathbf{F}_{2,3}^{-1}\mathbf{F}_{1,2}^{-1}\Delta\mathbf{F}_{0,1}\mathbf{F}_{1,2}\Delta\mathbf{F}_{1,2}\mathbf{F}_{2,3}\Delta\mathbf{F}_{2,3}
\end{aligned}$$

Now substitute and simplify

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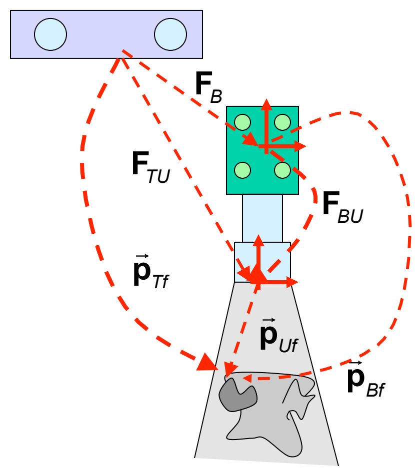
# Another Example

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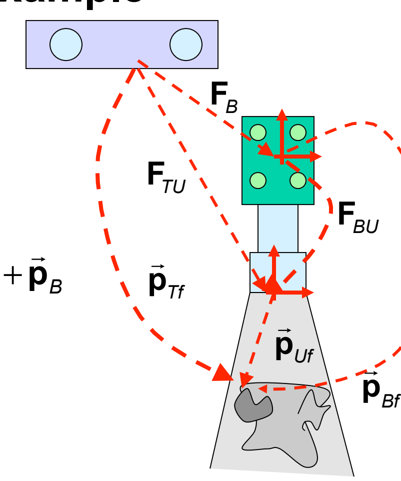
# Another Example




$$
\begin{aligned}
\vec{\mathbf{p}}_{Tf} &= \mathbf{F}_{TU} \bullet \vec{\mathbf{p}}_{Uf} \\
\mathbf{F}_{TU} &= \mathbf{F}_B \bullet \mathbf{F}_{BU} \\
&= [\mathbf{R}_B \bullet \mathbf{R}_{BU}, \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B] \\
\vec{\mathbf{p}}_{Tf} &= \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B
\end{aligned}
$$

**Also**

$$
\begin{aligned}
\vec{\mathbf{p}}_{Tf} &= \mathbf{F}_B \bullet \vec{\mathbf{p}}_{Bf} \\
\vec{\mathbf{p}}_{Bf} &= \mathbf{F}_{BU} \bullet \vec{\mathbf{p}}_{Uf} \\
&= \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \vec{\mathbf{p}}_{BU} \\
\vec{\mathbf{p}}_{Tf} &= \mathbf{R}_B \bullet \mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} + \mathbf{R}_B \bullet \vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_B
\end{aligned}
$$

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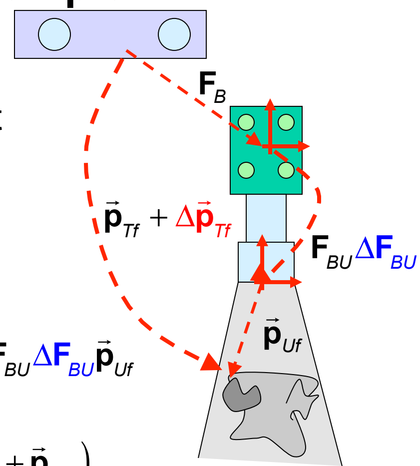
# Another Example

Suppose that the track body to US calibration is not perfect




$$\mathbf{F}_{BU}^{*} = \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}$$
$$= [\mathbf{R}_{BU}\Delta\mathbf{R}_{BU}, \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_{BU}]$$

$$\vec{\mathbf{p}}_{Bf}^{*} = \mathbf{F}_{BU}^{*} \bullet \vec{\mathbf{p}}_{Uf} \quad \text{I.e.,} \quad \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{BF} = \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}\vec{\mathbf{p}}_{Uf}$$

$$\Delta\vec{\mathbf{p}}_{Bf} = \mathbf{F}_{BU}\Delta\mathbf{F}_{BU}\vec{\mathbf{p}}_{Uf} - \vec{\mathbf{p}}_{Bf}$$
$$= \mathbf{F}_{BU}(\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \Delta\vec{\mathbf{p}}_{BU}) - \left(\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \vec{\mathbf{p}}_{BU}\right)$$
$$= \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} - \vec{\mathbf{p}}_{BU}$$
$$= \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}$$



# Another Example

Continuing …

$$\Delta\vec{\mathbf{p}}_{Bf} = \mathbf{R}_{BU}\Delta\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}$$

$$\approx \mathbf{R}_{BU}\left(\mathbf{I} + \mathbf{skew}(\vec{\alpha}_{BU})\right)\vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}$$
$$= \cancel{\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}} + \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} - \cancel{\mathbf{R}_{BU}\vec{\mathbf{p}}_{Uf}}$$
$$= \mathbf{R}_{BU} \bullet \vec{\alpha}_{BU} \times \vec{\mathbf{p}}_{Uf} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$
$$= -\mathbf{R}_{BU} \bullet \vec{\mathbf{p}}_{Uf} \times \vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$
$$= \mathbf{R}_{BU}\,skew(-\vec{\mathbf{p}}_{Uf})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

## Another Example

$$\vec{\mathbf{p}}_{Tf} + \Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right)$$

$$\Delta\vec{\mathbf{p}}_{Tf} = \mathbf{F}_B \Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) - \mathbf{F}_B \vec{\mathbf{p}}_{Bf}$$

$$\begin{aligned}
\Delta\mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) &= \Delta\mathbf{R}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B \\
&\approx \left(\mathbf{I} + skew(\vec{\alpha}_B)\right)\left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \Delta\vec{\mathbf{p}}_B \\
&= \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf}\right) + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \Delta\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B \\
&\approx \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B
\end{aligned}$$

$$\begin{aligned}
\Delta\vec{\mathbf{p}}_{Tf} &\approx \mathbf{F}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) - \mathbf{F}_B \vec{\mathbf{p}}_{Bf} \\
&= \mathbf{R}_B \left(\vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) + \vec{\mathbf{p}}_B - \left(\mathbf{R}_B \vec{\mathbf{p}}_{Bf} + \vec{\mathbf{p}}_B\right) \\
&= \mathbf{R}_B \left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)
\end{aligned}$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

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## Another Example

$$\Delta\vec{\mathbf{p}}_{Tf} \approx \mathbf{R}_B \left(\Delta\vec{\mathbf{p}}_{Bf} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right)$$

$$\Delta\vec{\mathbf{p}}_{Bf} \approx \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU}$$

$$\begin{aligned}
\Delta\vec{\mathbf{p}}_{Tf} &\approx \mathbf{R}_B \left(\mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} + \vec{\alpha}_B \times \vec{\mathbf{p}}_{Bf} + \Delta\vec{\mathbf{p}}_B\right) \\
&= \left(\begin{array}{c} \mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU})\vec{\alpha}_{BU} + \mathbf{R}_{BU}\Delta\vec{\mathbf{p}}_{BU} \\ + \mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf})\vec{\alpha}_B + \Delta\vec{\mathbf{p}}_B \end{array}\right) \\
&= \left[\mathbf{R}_{BU} skew(-\vec{\mathbf{p}}_{BU}) \;\middle|\; \mathbf{R}_{BU} \;\middle|\; \mathbf{R}_B skew(-\vec{\mathbf{p}}_{Bf}) \;\middle|\; \mathbf{I}\right] \begin{bmatrix} \vec{\alpha}_{BU} \\ \Delta\vec{\mathbf{p}}_{BU} \\ \vec{\alpha}_B \\ \Delta\vec{\mathbf{p}}_B \end{bmatrix}
\end{aligned}$$

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## Parametric Sensitivity

Suppose you have an explicit formula like

$$\vec{\mathbf{p}}_3 = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \\ 0 \end{bmatrix}$$

and know that the only variation is in parameters like $L_k$ and $\theta_k$. Then you can estimate the variation in $\vec{\mathbf{p}}_3$ as a function of variation in $L_k$ and $\theta_k$ by remembering your calculus.

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix} \begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$



## Parametric Sensitivity

Grinding this out gives:

$$\Delta\vec{\mathbf{p}}_3 \cong \begin{bmatrix} \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} & \frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} \end{bmatrix} \begin{bmatrix} \Delta\vec{L} \\ \Delta\vec{\theta} \end{bmatrix}$$

where

$$\vec{L} = [L_1, L_2, L_3]^T$$

$$\vec{\theta} = [\theta_1, \theta_2, \theta_3]^T$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{L}} = \begin{bmatrix} \cos(\theta_1) & \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2+\theta_3) \\ \sin(\theta_1) & \sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\vec{\mathbf{p}}_3}{\partial\vec{\theta}} = \begin{bmatrix} -L_1\sin(\theta_1) - L_2\sin(\theta_1+\theta_2) - L_3\sin(\theta_1+\theta_2+\theta_3) & -L_2\sin(\theta_1+\theta_2) - L_3\sin(\theta_1+\theta_2+\theta_3) & -L_3\sin(\theta_1+\theta_2+\theta_3) \\ L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) & L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) & L_3\cos(\theta_1+\theta_2+\theta_3) \\ 0 & 0 & 0 \end{bmatrix}$$

# More generally …

Suppose that we have a vector function

$$\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}}) \quad = [g_1(\vec{\mathbf{q}}),\ldots,g_m(\vec{\mathbf{q}})]^T$$

of parameters $\vec{\mathbf{q}} = [q_1,\ldots,q_n]$. Then we can estimate the value of

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} = \vec{\mathbf{g}}(\vec{\mathbf{q}} + \Delta\vec{\mathbf{q}})$$

by

$$\vec{\mathbf{v}} + \Delta\vec{\mathbf{v}} \approx \vec{\mathbf{g}}(\vec{\mathbf{q}}) + \mathbf{J}_{\mathbf{G}}(\vec{\mathbf{q}}) \bullet \Delta\vec{\mathbf{q}}$$

where

$$\mathbf{J}_{\mathbf{G}}(\vec{\mathbf{q}}) = \begin{bmatrix} \frac{\partial g_1}{\partial q_1} & & \frac{\partial g_1}{\partial q_j} & & \frac{\partial g_1}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_i}{\partial q_1} & \cdots & \frac{\partial g_i}{\partial q_j} & \cdots & \frac{\partial g_i}{\partial q_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial q_1} & & \frac{\partial g_m}{\partial q_j} & & \frac{\partial g_m}{\partial q_n} \end{bmatrix}$$

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