Instructions and Score Sheet (hand in with answers)

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Signature (required) I/We have followed the rules in completing this assignment Andrew Hundt	Signature (required) I/We have followed the rules in completing this assignment

Question	Points	Points	Totals
1A	5		
1B	5		
1C	5		
1D	10		
1E	10		
1F	10		
1G	10		
1H	10		
1I	10		
2A	5		
2B	5		
2C	5		
2D	10		
Total	100		

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or **in teams of two** and are **not to discuss the problems with anyone** other than the TAs or the instructor.
3. It is otherwise open book, notes, and web. But you should cite any references you consult.
4. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
5. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
6. Sign and hand in the score sheet as the first sheet of your assignment.
7. Remember to include a sealable 8 ½ by 11 inch self-addressed envelope if you want your assignment

Problem Scenario: Ilizarov Distraction Osteogenesis

Distraction osteogenesis[1] is a method for correcting skeletal deformities or lengthening bones. In this process, the bone is broken into segments that are held in close proximity to each other by some sort of fixation device. The bone segments are then very slowly moved apart, allowing new bone to grow in the gap. When the bone has reached its desired length, the distraction process ends and the bone is allowed to heal. Although the main use has been for lengthening long bones, it also has been applied elsewhere, notably for craniofacial and maxillofacial deformities.

Although there are a number of methods used for the distraction phase, the most commonly used is an external fixator technique developed by Gavril Ilizarov in 1951. In this apparatus (illustrated in Fig. 1), rings are attached above and below the fracture point and secured to the bone by percutaneous pins.

Fig. 1:Ilizarov fixator on leg (http://en.wikipedia.org/wiki/Ilizarov_apparatus)

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[1] See, e.g., http://en.wikipedia.org/wiki/Distraction_osteogenesis; http://en.wikipedia.org/wiki/Ilizarov_apparatus

Screw rods between the pins are used to change the relative position and orientation of the rings and (so) of the bone fragments.

Consider the simplified Ilizarov apparatus shown in Fig. 2. For this exercise, we will assume that the apparatus has been put in place and a CT scan of the bone and apparatus has been made. We have defined the following coordinate systems and designed values:

$\mathbf{F}_1$, $\mathbf{F}_2$	Coordinate systems associated with the lower and upper rings
$\mathbf{F}_D$, $\mathbf{F}_U$	Coordinate systems associated with the lower (distal) and upper bone fragments
$\mathbf{F}_{C1}$, $\mathbf{F}_{C2}$, $\mathbf{F}_{CD}$, $\mathbf{F}_{CU}$	Position and orientation of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_D$, and $\mathbf{F}_U$ in CT coordinates
$\vec{\mathbf{q}}_{1,i}$, $\vec{\mathbf{q}}_{2,i}$ $i = 1,\cdots,6$	Position of attachment points of rod i on rings 1 and 2, relative to the coordinate systems $\mathbf{F}_1$ and $\mathbf{F}_2$
L_i	Length of rod i = $\left\|\vec{\mathbf{q}}_{1,i} - \vec{\mathbf{q}}_{2,i}\right\|$
$\mathbf{F}_{12}$	Position and orientation of $\mathbf{F}_2$ relative to $\mathbf{F}_1$

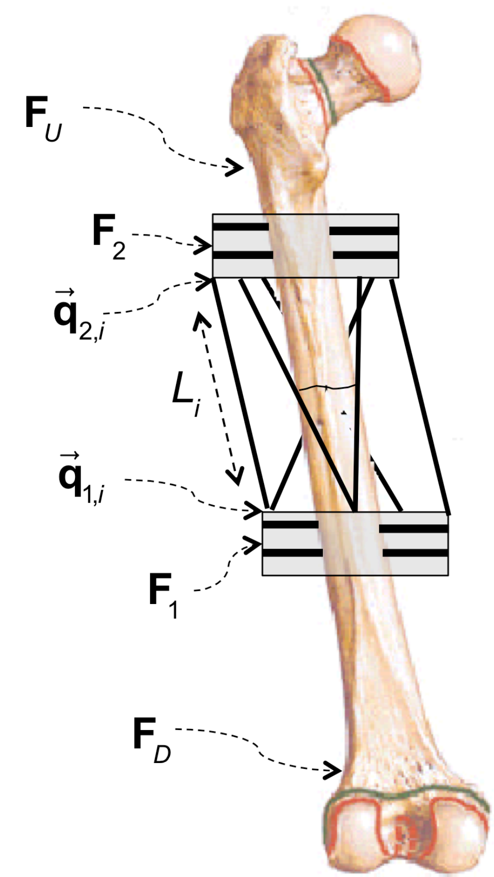


Fig. 2

Question 1

A. Suppose that the distraction process has proceeded for some time, so that the current position and orientation of $\mathbf{F}_2$ relative to $\mathbf{F}_1$ is now $\mathbf{F}_{12} = [\mathbf{R}_{12}, \vec{\mathbf{p}}_{12}]$. What is the current transformation $\mathbf{F}_{DU} = [\mathbf{R}_{DU}, \vec{\mathbf{p}}_{DU}]$?

B. Suppose that we now want to adjust the frame so that the upper bone fragment is at a position $\mathbf{F}_{DU}^{new} = \mathbf{F}_{DU}^{new} \Delta \mathbf{F}_{DU}$. What is the corresponding transformation $\mathbf{F}_{12}^{new} = \mathbf{F}_{12} \Delta \mathbf{F}_{12}^{right}$ that will accomplish this, where $\Delta \mathbf{F}_{12}^{right} = [\Delta \mathbf{R}_{12}^{right}, \Delta \vec{\mathbf{p}}_{12}^{right}]$? Give your answer in terms of the $\Delta \mathbf{R}$'s and $\Delta \vec{\mathbf{p}}$'s.

C. Suppose that we now want to adjust the frame so that the upper bone fragment is at a position $\mathbf{F}_{DU}^{new} = \mathbf{F}_{DU}^{new} \Delta \mathbf{F}_{DU}$. What is the corresponding transformation $\mathbf{F}_{12}^{new} = \Delta \mathbf{F}_{12}^{left} \mathbf{F}_{12}$ that will accomplish this, where $\Delta \mathbf{F}_{12}^{left} = [\Delta \mathbf{R}_{12}^{left}, \Delta \vec{\mathbf{p}}_{12}^{left}]$? Give your answer in terms of the $\Delta \mathbf{R}$'s and $\Delta \vec{\mathbf{p}}$'s.

D. Let $\vec{\mathbf{r}}_i = \mathbf{F}_{12} \vec{\mathbf{q}}_{2,i}$ be the position of the attachment point of rod i on the upper ring. Given a value for $\Delta \mathbf{F}_{12}^{right} = \Delta \mathbf{F}_{12} = [\Delta \mathbf{R}_{12}, \Delta \vec{\mathbf{p}}_{12}]$ (i.e., drop the superscript), what is the corresponding value for the $\Delta\vec{\mathbf{r}}_i$. Give your answer in terms of the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s.

E. Now, suppose that we have the approximations $\Delta\mathbf{R}_{xy} \approx \mathbf{I} + sk(\vec{\alpha}_{xy})$ and $\Delta\vec{\mathbf{p}}_{xy} = \vec{\varepsilon}_{xy}$ for all indices xy, develop a linearized approximation for $\Delta\vec{\mathbf{r}}_i$ in terms of $\vec{\alpha}_{12}$ and $\vec{\varepsilon}_{12}$. You should show your answer in matrix form:

$$\Delta\vec{\mathbf{r}}_i = [\mathbf{M}_{\vec{\alpha}} \mid \mathbf{M}_{\vec{\varepsilon}}]\begin{bmatrix} \vec{\alpha}_{12} \\ \vec{\varepsilon}_{12} \end{bmatrix}$$

where $\mathbf{M} = [\mathbf{M}_{\vec{\alpha}} \mid \mathbf{M}_{\vec{\varepsilon}}]$ are expressions involving known quantities.

F. Give an linearized expression for the change ΔL_i in the length of the i'th rod in terms of $\vec{\alpha}_{12}$ and $\vec{\varepsilon}_{12}$.

G. Now suppose that the initial image processing of the CT image had some errors, so that the true initial ring poses relative to their corresponding bone fragments are $\mathbf{F}^*_{C1} = \Delta\mathbf{F}_{C1}\mathbf{F}_{C1}$ and $\mathbf{F}^*_{C2} = \Delta\mathbf{F}_{C2}\mathbf{F}_{C2}$. Given a motion $\Delta\mathbf{F}^{right}_{12} = \Delta\mathbf{F}_{12} = [\Delta\mathbf{R}_{12}, \Delta\vec{\mathbf{p}}_{12}]$ what is the new pose $\mathbf{F}^{new}_{DU} = \mathbf{F}_{DU}\Delta\mathbf{F}_{DU}$? Give your answer in terms of the $\Delta\mathbf{R}$'s and $\Delta\vec{\mathbf{p}}$'s.

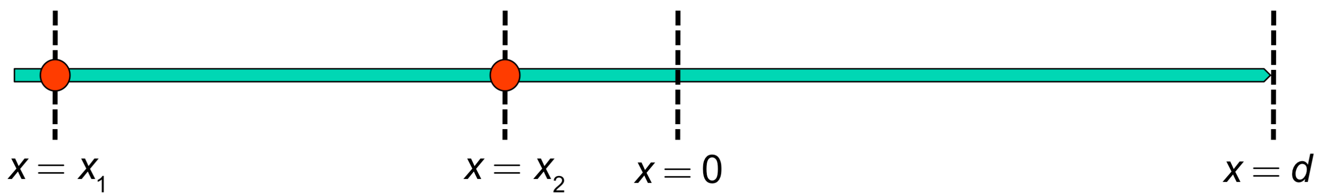
H. Now modify your answer to Question 1F to produce linearized estimates of $\vec{\alpha}_{DU}$ and $\vec{\varepsilon}_{DU}$, using the approximation assumptions $\Delta\mathbf{R}_{xy} \approx \mathbf{I} + sk(\vec{\alpha}_{xy})$ and $\Delta\vec{\mathbf{p}}_{xy} = \vec{\varepsilon}_{xy}$ for all indices xy.

I. Suppose that the rings have several fiducial markers embedded in them. Those in ring 1 are at positions $\vec{\mathbf{a}}_{1,k}$ relative to $\mathbf{F}_1$ and at $\vec{\mathbf{a}}_{2,k}$ relative to $\mathbf{F}_2$. These are located at the following positions relative to their respective ring coordinate systems:

$\vec{\mathbf{a}}_{1,1}$ $\vec{\mathbf{a}}_{2,1}$	$[100,0,0]^T$
$\vec{\mathbf{a}}_{1,2}$ $\vec{\mathbf{a}}_{2,2}$	$[0,100,0]^T$
$\vec{\mathbf{a}}_{1,3}$ $\vec{\mathbf{a}}_{2,3}$	$[-100,0,0]^T$
$\vec{\mathbf{a}}_{1,4}$ $\vec{\mathbf{a}}_{2,4}$	$[0,-100,0]^T$

Suppose that the CT image analysis software uses these fiducials to locate the rings. The fiducials can be found with the following limit on errors: $\left|\Delta\vec{\mathbf{a}}\right| \leq [\delta_{xy}, \delta_{xy}, \delta_z]$ in CT coordinates. Produce linearized estimates of bounds on $\Delta\mathbf{F}_{C1}$ (i.e., on the components of $\vec{\alpha}_{C1}, \vec{\varepsilon}_{C1}$).

Question 2



$$err_x = \delta \qquad \text{where } |\delta| \le \delta_{\text{max}}$$

$$err_{yz} = \delta + f(x)$$

$$|f(x)| \le \begin{cases} \rho & \text{if } x \le \nu \le 0 \\ \rho - \rho(x-\nu)/\mu & \text{if } \nu \le x \le \nu + \mu \le 0 \\ 0 & \text{otherwise} \end{cases}$$

Fig. 3

Consider the somewhat simplified pointer design shown in Fig. 3. The pointer is nominally designed to have two tracked markers in line with its tip. For now, we can assume that the tracking system is very accurate, but that the pointer manufacturing process is associated with some errors, which are shown in the figure. These errors include a possible misplacement of each marker by a random amount in each direction (with maximum misplacement δ_{max} in each direction), together with a possible z-shaped bend in the shaft of the tool that occurs after the markers are placed. The direction of this bend is unknown, but it is known that the total additional lateral deflection is given by $f(x)$ where

$$\left|f(x)\right| \le \begin{cases} \rho & \text{if } x \le \nu \le 0 \\ \rho - \rho(x-\nu)/\mu & \text{if } \nu \le x \le \nu + \mu \le 0 \\ 0 & \text{otherwise} \end{cases}$$

The length of the bend μ is known and it is known that the maximum magnitude is $\rho \le \rho_{\text{max}}$, but the position is not known.

A. Suppose that the tracker reports that the positions of the two markers are $\vec{\mathbf{m}}_1$ and $\vec{\mathbf{m}}_2$. Give a formula for computing the position $\vec{\mathbf{p}}_{tip}$ of the pointer tip assuming that there are no errors.

B. Assume that $\rho_{\text{max}} = 0$, give an estimate for the maximum error in the any component of the value of $\vec{\mathbf{p}}_{tip}$ in terms of δ_{max}. (I.e., assume that $\left|\Delta\vec{\mathbf{m}}_1\right| \le \delta_{\text{max}}$ and $\left|\Delta\vec{\mathbf{m}}_2\right| \le \delta_{\text{max}}$).

C. Assume that $\rho_{\text{max}} > 0$ but that the position is not known. Assume that $x_2 = 0$, what is the maximum error in any component of the value of $\vec{\mathbf{p}}_{tip}$? **Hint:** There will be different cases depending on the value of x_1.

D. Suppose that we want to place marker 1 as close as possible to marker 2, which is placed at $x_2 = 0$, while still maintaining sufficient accuracy for the pointer. I.e., we want to guarantee that no component of $\Delta\vec{\mathbf{p}}_{tip}$ is larger than some value $\varepsilon_{\max}$. What is the value of x_1 that will accomplish this?