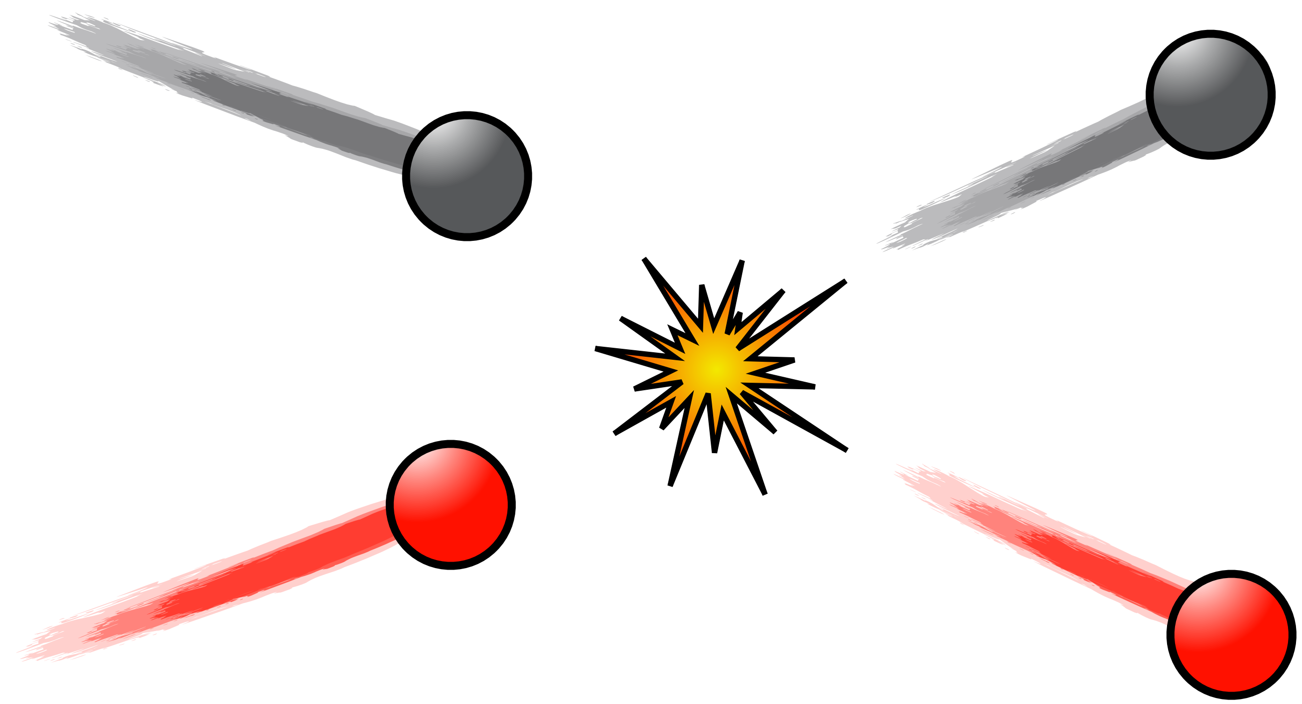
Scattering Theory: Complex Numbers Review

Andrew W. Jackura

2021 REYES Nuclear Theory Mentorship



Old Dominion University

Jefferson Lab
Thomas Jefferson National Accelerator Facility

Why Complex Numbers?

We live in a ‘real’ world, why describe it with ‘complex’ numbers?

- In most subfields of physics, complex numbers are useful shortcut

$$\begin{aligned}\Psi(x,t) &= A\cos(\omega t - kx) \\ &= \mathrm{Re}\left[Ae^{i(\omega t - kx)}\right]\end{aligned}$$

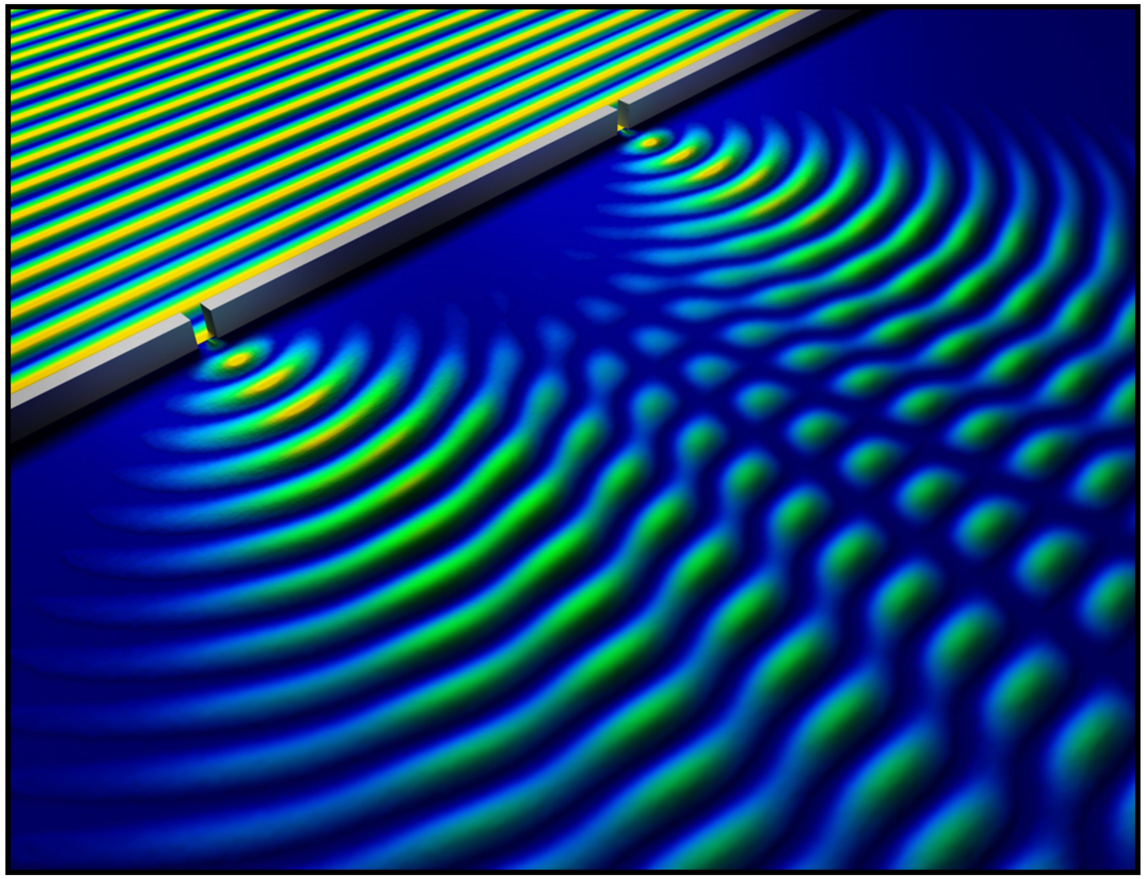
Why Complex Numbers?

We live in a ‘real’ world, why describe it with ‘complex’ numbers?

- In most subfields of physics, complex numbers are useful shortcut
- In **Quantum Physics**, complex natures are more fundamental

$$\Psi(x,t) = Ae^{i(\omega t - kx)}$$

$$\mathrm{Prob} = \int_x \left|\Psi(x,t)\right|^2$$



Why Complex Numbers?

We live in a ‘real’ world, why describe it with ‘complex’ numbers?

- In most subfields of physics, complex numbers are useful shortcut
- In **Quantum Physics**, complex natures are more fundamental

More technical…

Quantum physics needs complex numbers

Marc-Olivier Renou[1], David Trillo[2], Mirjam Weilenmann[2], Le Phuc Thinh[2], Armin Tavakoli[2], Nicolas Gisin[3,4], Antonio Acín[1,5] and Miguel Navascués[2]
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arXiv:2101.10873v1 [quant-ph] 26 Jan 2021

Fun read…

Why are complex numbers needed in quantum mechanics? Some answers for the introductory level

Ricardo Karam*

Department of Science Education, University of Copenhagen, Denmark

Abstract

Complex numbers are broadly used in physics, normally as a calculation tool that makes things easier due to Euler’s formula. In the end, it is only the real component that has physical meaning or the two parts (real and imaginary) are treated separately as real quantities. However, the situation seems to be different in quantum mechanics, since the imaginary unit i appears explicitly in its fundamental equations. From a learning perspective, this can create some challenges to newcomers. In this article, four conceptually different justifications for the use/need of complex numbers in quantum mechanics are presented and some pedagogical implications are discussed.

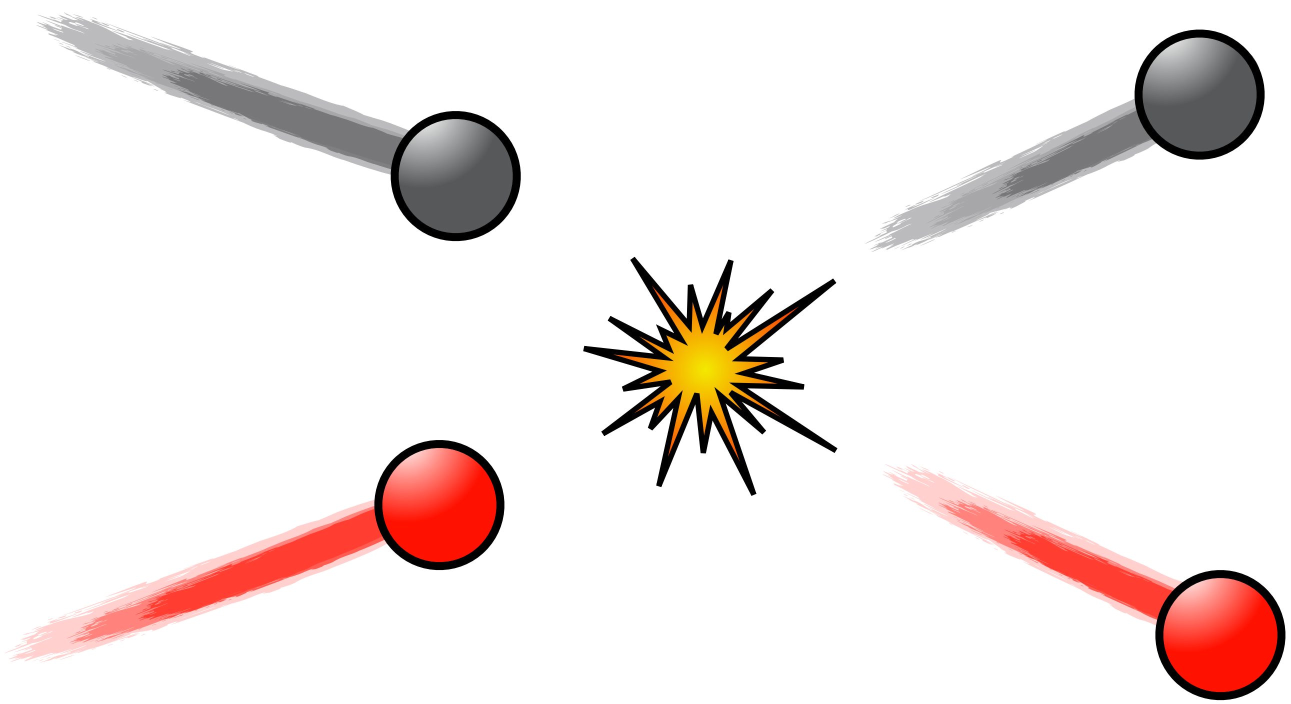
Scattering Amplitudes

An example of such an amplitude is the **Scattering Amplitude** $\mathcal{M}$

- Probability of an interaction to occur $\mathrm{Prob} \propto |\mathcal{M}|^2$

Complex conjugation

$$|\mathcal{M}|^2 = \mathcal{M}^* \mathcal{M}$$

$\mathcal{M} \ =$ 

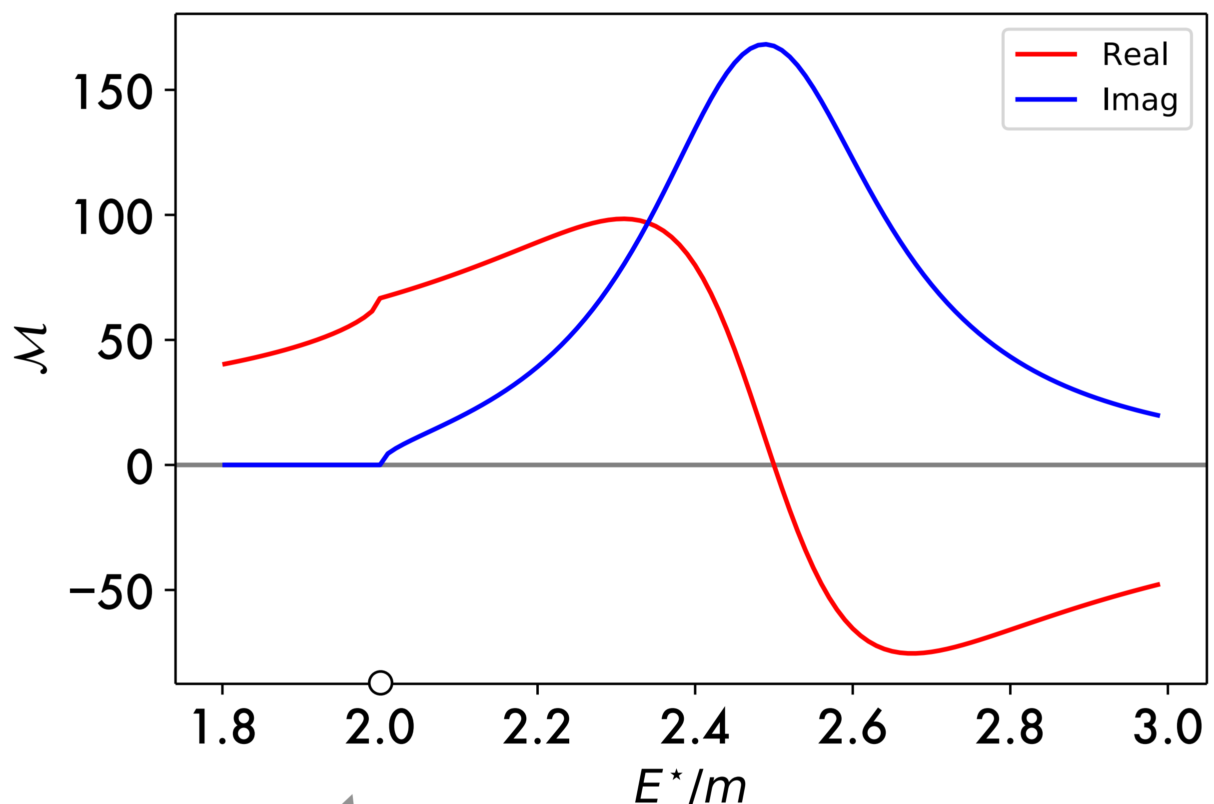
$$\mathcal{M} = \mathcal{M}(E^\star)$$

Center-of-Momentum Energy

Scattering Amplitudes

An example of a probability amplitude is the **scattering amplitude** $\mathcal{M}$

- Probability of an interaction to occur $\text{Prob} \propto |\mathcal{M}|^2$

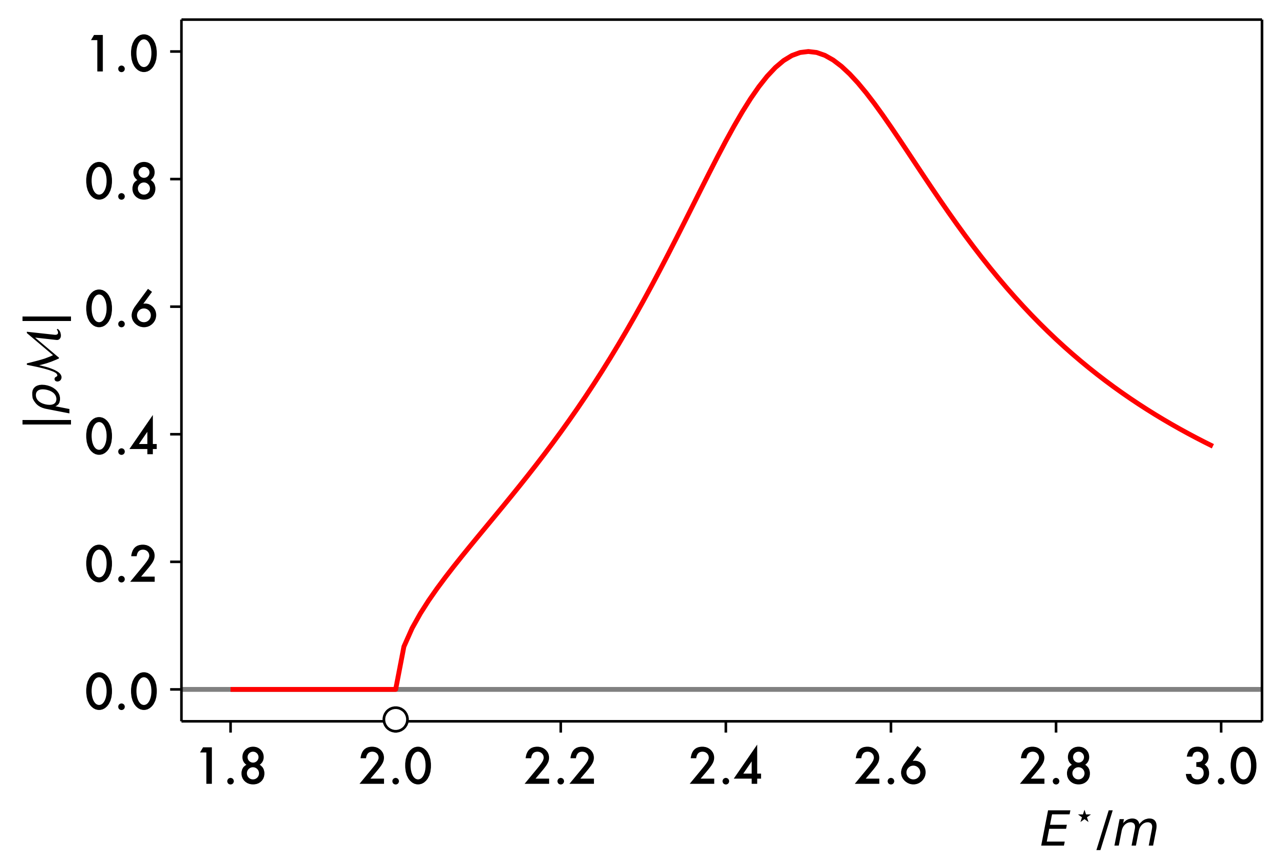


Real AND Imaginary components

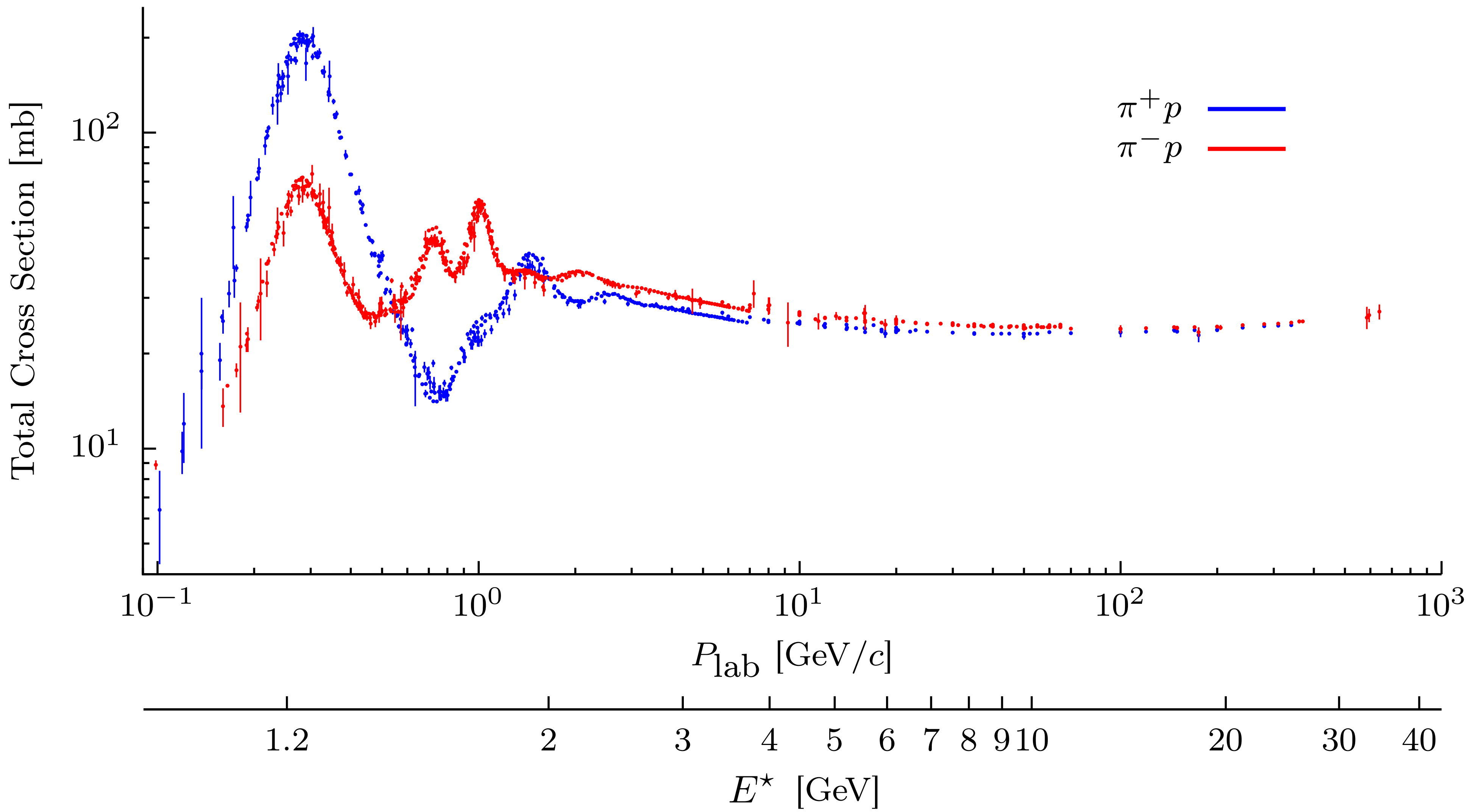
Threshold Energy

$$E^\star_{\text{th}} = 2m$$

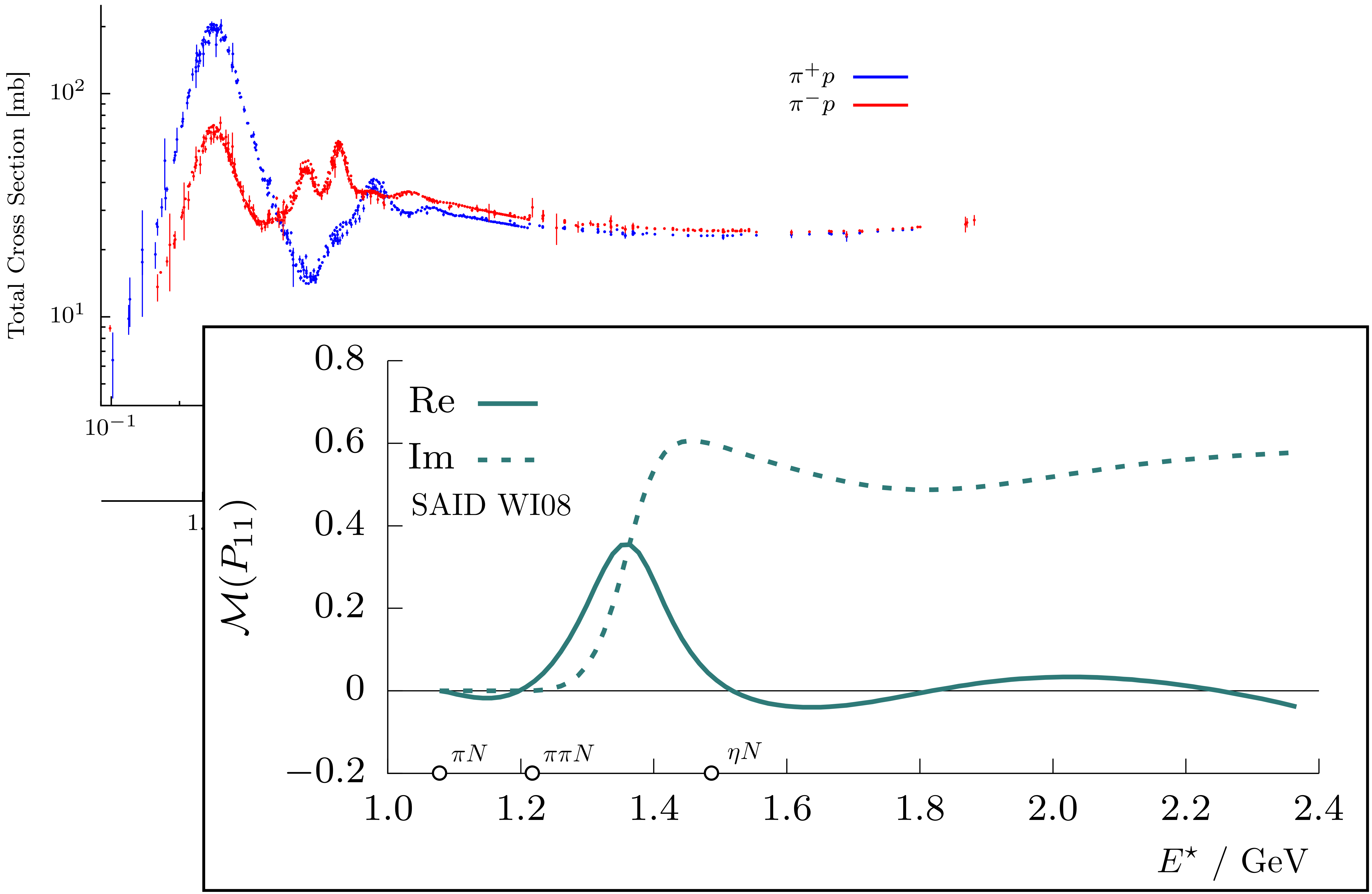
Kinematic factor "phase space" ρ



Scattering Amplitudes — Pion-Nucleon Scattering



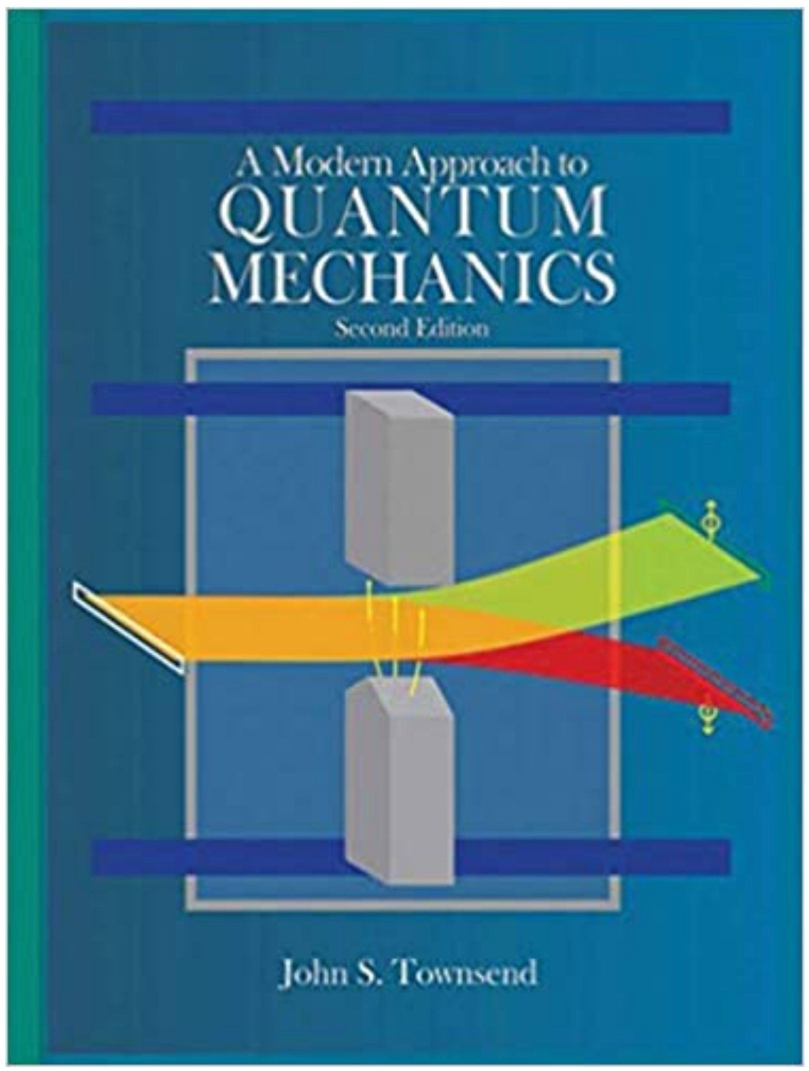
Scattering Amplitudes — Pion-Nucleon Scattering

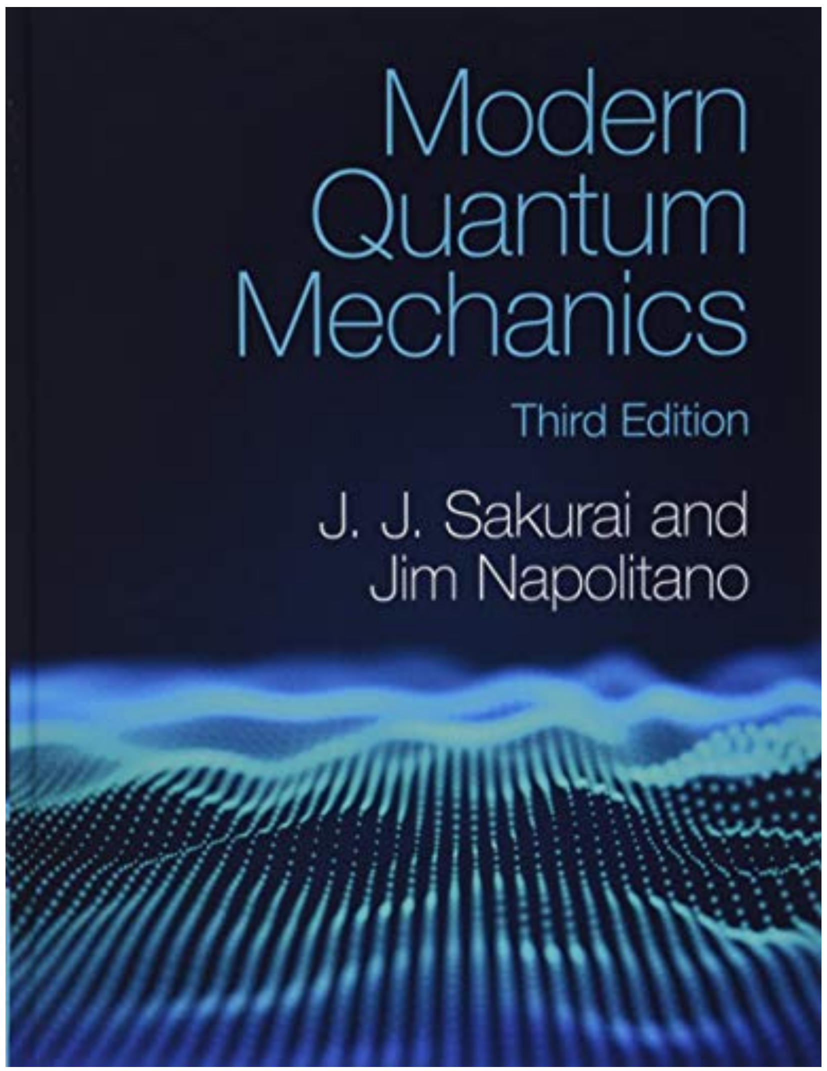


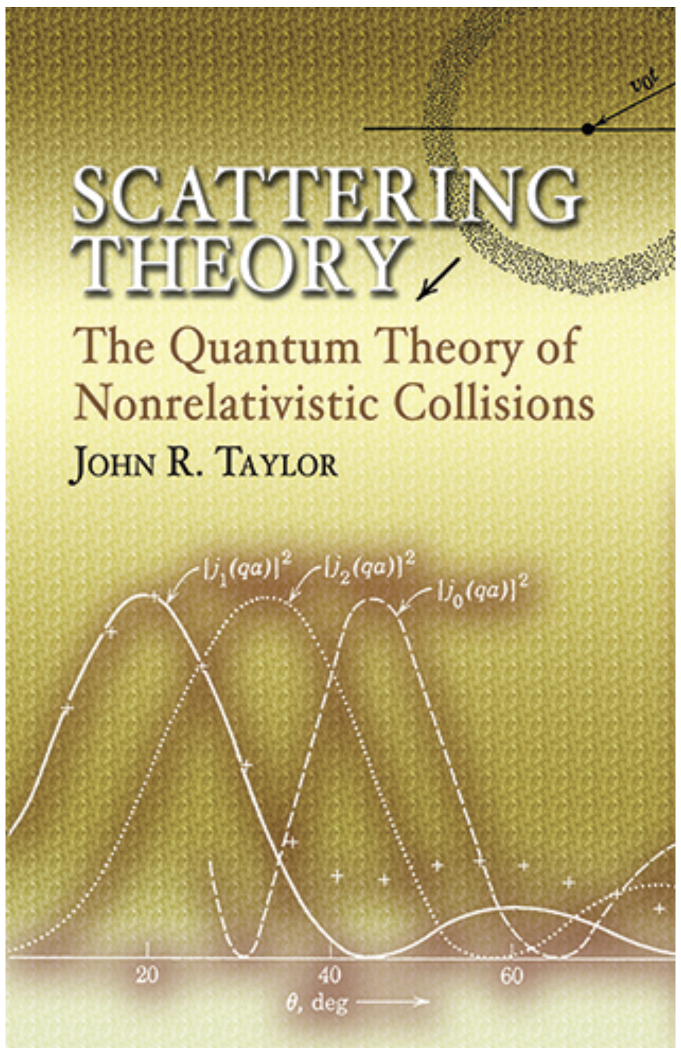
Scattering Amplitudes

Some references

- http://cgl.soic.indiana.edu/jpac/schools.html







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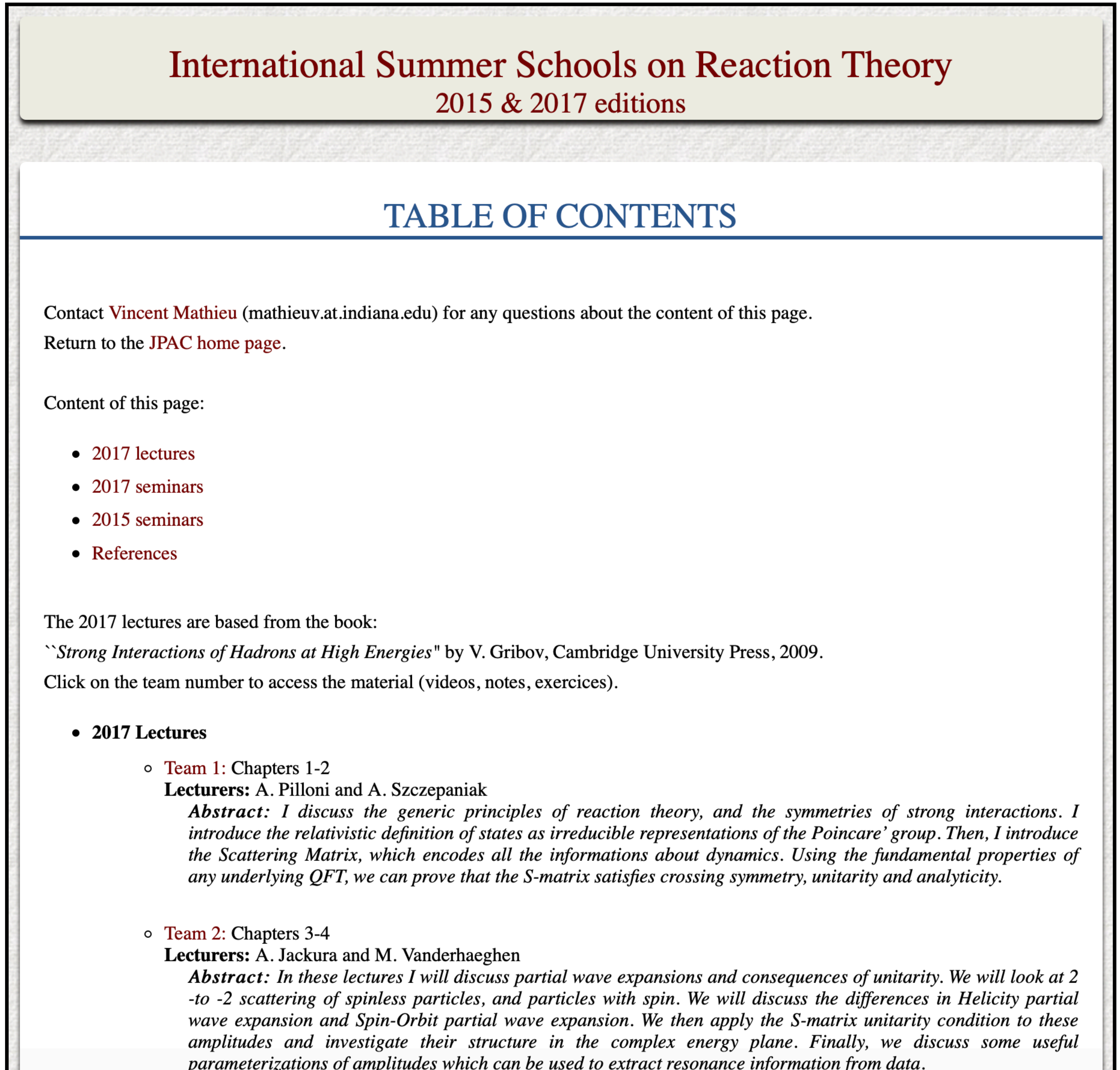

International Summer Schools on Reaction Theory
2015 & 2017 editions

TABLE OF CONTENTS

Contact Vincent Mathieu (mathieuv.at.indiana.edu) for any questions about the content of this page.
Return to the JPAC home page.

Content of this page:

- 2017 lectures
- 2017 seminars
- 2015 seminars
- References

The 2017 lectures are based from the book:
``*Strong Interactions of Hadrons at High Energies*" by V. Gribov, Cambridge University Press, 2009.
Click on the team number to access the material (videos, notes, exercices).

- **2017 Lectures**
  - Team 1: Chapters 1-2
    Lecturers: A. Pilloni and A. Szczepaniak
    Abstract: *I discuss the generic principles of reaction theory, and the symmetries of strong interactions. I introduce the relativistic definition of states as irreducible representations of the Poincare' group. Then, I introduce the Scattering Matrix, which encodes all the informations about dynamics. Using the fundamental properties of any underlying QFT, we can prove that the S-matrix satisfies crossing symmetry, unitarity and analyticity.*
  - Team 2: Chapters 3-4
    Lecturers: A. Jackura and M. Vanderhaeghen
    Abstract: *In these lectures I will discuss partial wave expansions and consequences of unitarity. We will look at 2 -to -2 scattering of spinless particles, and particles with spin. We will discuss the differences in Helicity partial wave expansion and Spin-Orbit partial wave expansion. We then apply the S-matrix unitarity condition to these amplitudes and investigate their structure in the complex energy plane. Finally, we discuss some useful parameterizations of amplitudes which can be used to extract resonance information from data.*

Scattering Amplitudes

Some references

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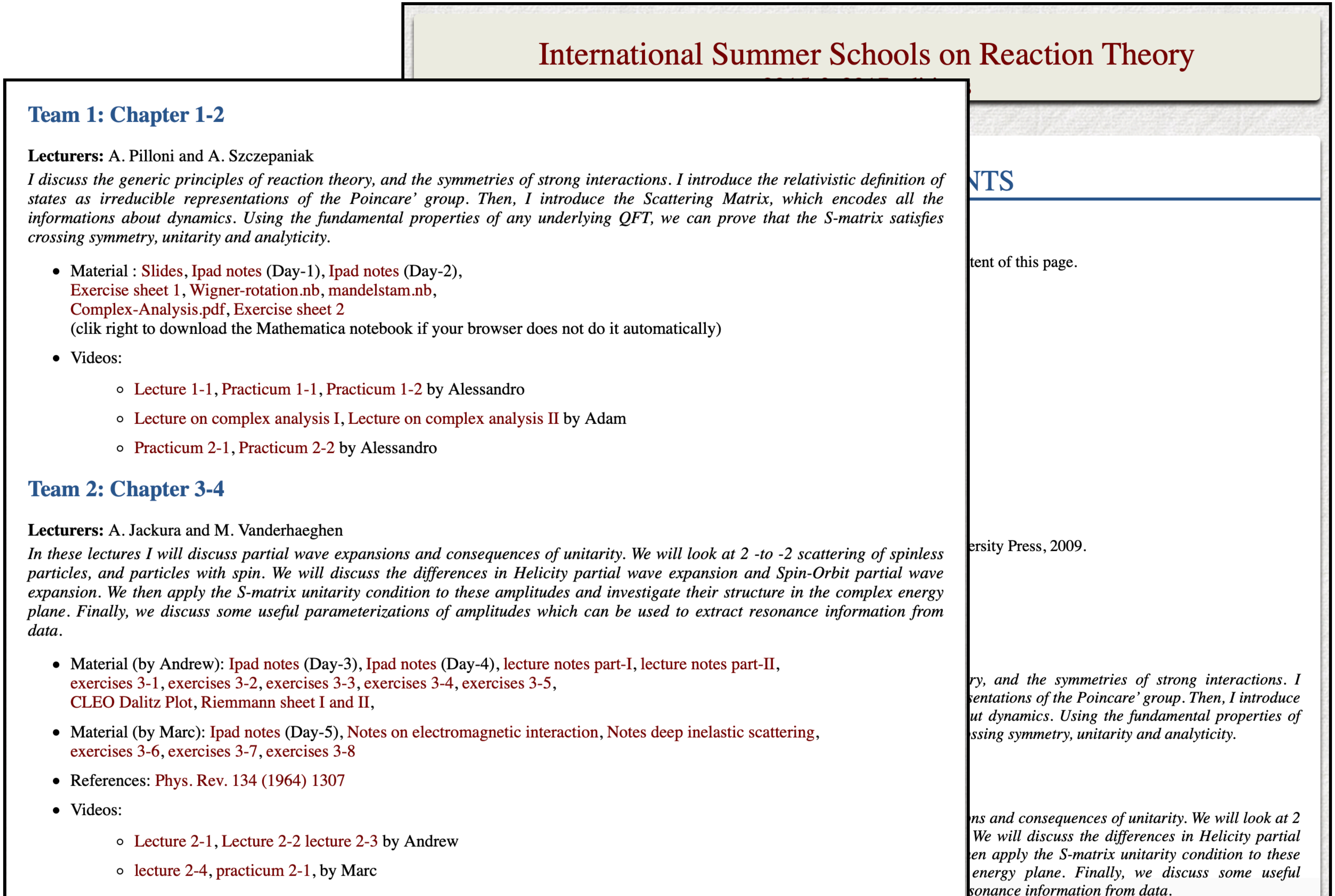

International Summer Schools on Reaction Theory

Team 1: Chapter 1-2

Lecturers: A. Pilloni and A. Szczepaniak

I discuss the generic principles of reaction theory, and the symmetries of strong interactions. I introduce the relativistic definition of states as irreducible representations of the Poincare' group. Then, I introduce the Scattering Matrix, which encodes all the informations about dynamics. Using the fundamental properties of any underlying QFT, we can prove that the S-matrix satisfies crossing symmetry, unitarity and analyticity.

- Material : Slides, Ipad notes (Day-1), Ipad notes (Day-2), Exercise sheet 1, Wigner-rotation.nb, mandelstam.nb, Complex-Analysis.pdf, Exercise sheet 2
  (clik right to download the Mathematica notebook if your browser does not do it automatically)
- Videos:
  - Lecture 1-1, Practicum 1-1, Practicum 1-2 by Alessandro
  - Lecture on complex analysis I, Lecture on complex analysis II by Adam
  - Practicum 2-1, Practicum 2-2 by Alessandro

Team 2: Chapter 3-4

Lecturers: A. Jackura and M. Vanderhaeghen

In these lectures I will discuss partial wave expansions and consequences of unitarity. We will look at 2 -to -2 scattering of spinless particles, and particles with spin. We will discuss the differences in Helicity partial wave expansion and Spin-Orbit partial wave expansion. We then apply the S-matrix unitarity condition to these amplitudes and investigate their structure in the complex energy plane. Finally, we discuss some useful parameterizations of amplitudes which can be used to extract resonance information from data.

- Material (by Andrew): Ipad notes (Day-3), Ipad notes (Day-4), lecture notes part-I, lecture notes part-II, exercises 3-1, exercises 3-2, exercises 3-3, exercises 3-4, exercises 3-5, CLEO Dalitz Plot, Riemmann sheet I and II,
- Material (by Marc): Ipad notes (Day-5), Notes on electromagnetic interaction, Notes deep inelastic scattering, exercises 3-6, exercises 3-7, exercises 3-8
- References: Phys. Rev. 134 (1964) 1307
- Videos:
  - Lecture 2-1, Lecture 2-2 lecture 2-3 by Andrew
  - lecture 2-4, practicum 2-1, by Marc

Basics of Complex Numbers

Let $z \in \mathbb{C}$

$$z = x + iy \qquad x, y \in \mathbb{R} \qquad i \equiv \sqrt{-1}$$

Complex conjugation

$$z^* \equiv x - iy \qquad (z^*)^* = z$$

Real and Imaginary parts

$$\operatorname{Re} z = x = \frac{z + z^*}{2}$$

$$\operatorname{Im} z = y = \frac{z - z^*}{2i}$$

$$\operatorname{Re} z^* = \operatorname{Re} z$$

$$\operatorname{Im} z^* = -\operatorname{Im} z$$

Basics of Complex Numbers

Magnitude

$$
\begin{aligned}
|z| &\equiv \sqrt{z \cdot z^*} \\
&= \sqrt{x^2 + y^2}
\end{aligned}
$$

$$|z^*| = |z|$$

Algebraic properties

$$
\begin{aligned}
z_1 \pm z_2 &= (x_1 + iy_1) \pm (x_2 + iy_2) \\
&= (x_1 \pm x_2) + i(y_1 \pm y_2)
\end{aligned}
$$

$$
\begin{aligned}
z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\
&= (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{z} &= \frac{z^*}{z \cdot z^*} = \frac{z^*}{|z|^2} \\
&= \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}
\end{aligned}
$$

Basics of Complex Numbers

Polar representation

$$z = r\, e^{i\varphi} \qquad 0 \leq r < \infty \qquad \varphi \in (-\pi, \pi]$$

Magnitude

$$r = |z|$$

Phase

$$\varphi = \arg(z) = \begin{cases} 2\arctan\left(\frac{y}{\sqrt{x^2+y^2}+x}\right) & \text{if } x > 0 \text{ or } y \neq 0\,, \\ \pi & \text{if } x < 0 \text{ and } y = 0\,, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

$$\arg z^* = -\arg z \pmod{2\pi}$$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$e^{i\pi} = -1$$

$$\varphi = \operatorname{atan2}(\operatorname{Im} z, \operatorname{Re} z)$$

Numerical Complex Numbers

```python
# Python code to demonstrate the working of
# complex(), real() and imag()

# importing "cmath" for complex number operations
import cmath

# Initializing real numbers
x = 5
y = 3

# converting x and y into complex number
z = complex(x,y);

# printing real and imaginary part of complex number
print ("The real part of complex number is : ",end="")
print (z.real)

print ("The imaginary part of complex number is : ",end="")
print (z.imag)
```

Python

C++

```cpp
// Program illustrating the use of real() and
// imag() function
#include <iostream>

// for std::complex, std::real, std::imag
#include <complex>

// driver function
int main()
{
  // defines the complex number: (5 + 3i)
  std::complex<double> mycomplex(5.0, 3.0);

  // prints the real part using the real function
  std::cout << "Real part: " << std::real(mycomplex) << std::endl;
  std::cout << "Imaginary part: " << std::imag(mycomplex) << std::endl;
  return 0;
}
```

Exercise 1: Complex Numbers Review

Let $z \in \mathbb{C}$

$$z = \frac{1 - i}{1 + i} \, ,$$

1. What are the real and imaginary components?

2. What is the magnitude $|z|$?

3. What is the phase of z?

4. Verify these numerically with the language of your choice (e.g. python, C++, Fortran, …)