Scattering Theory: Scattering Amplitudes

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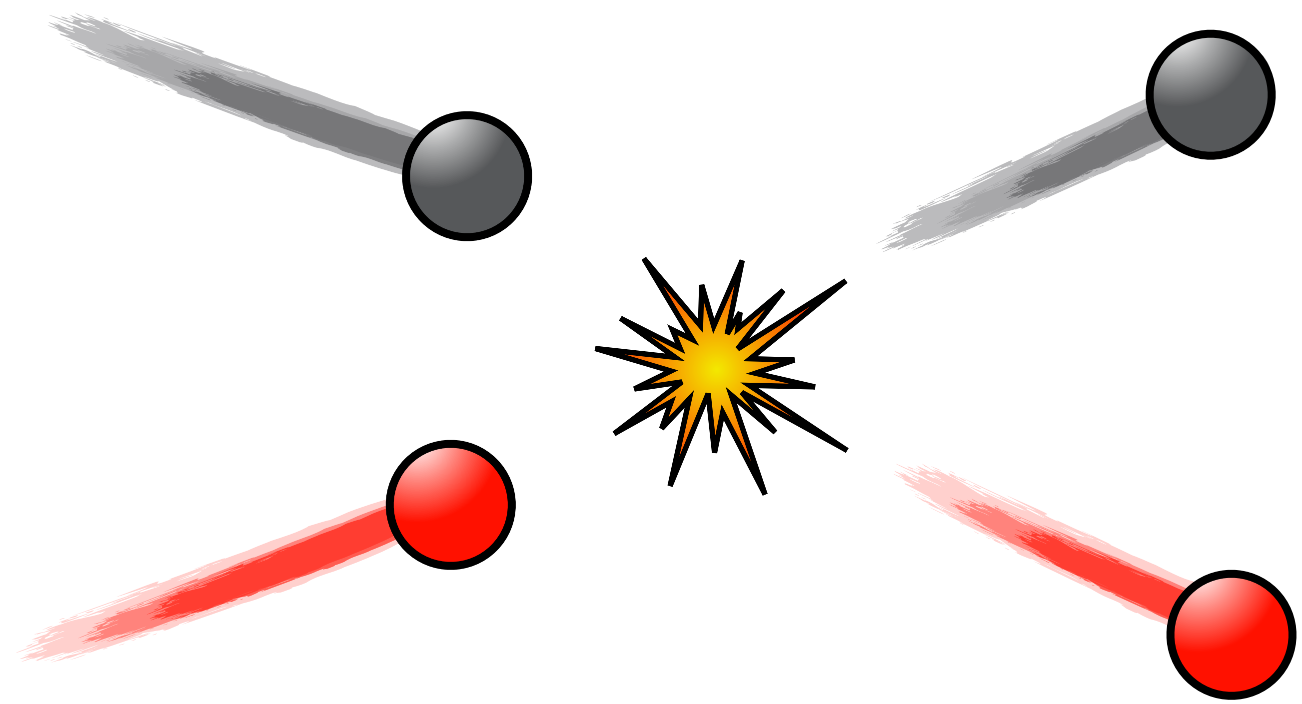
2021 REYES Nuclear Theory Mentorship



Old Dominion University

Jefferson Lab
Thomas Jefferson National Accelerator Facility

Scattering Amplitudes

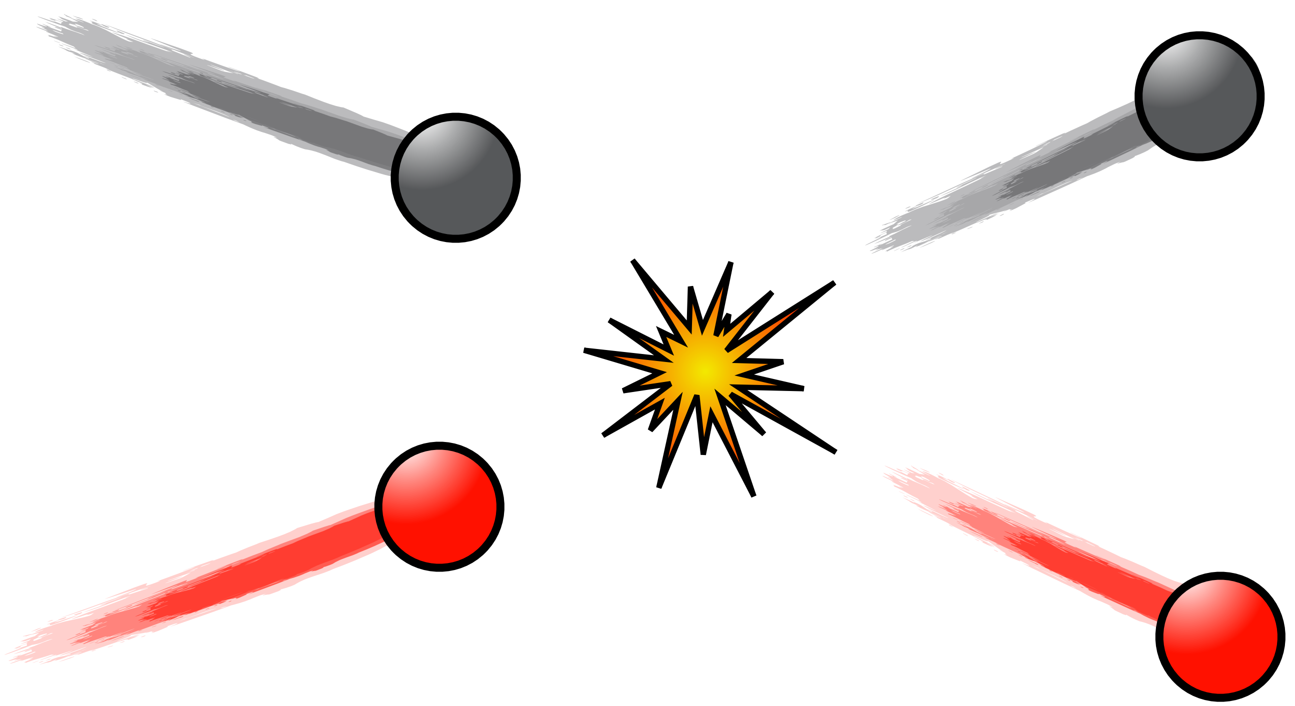
The **Scattering Amplitude** $\mathcal{M}$ — Characterizes probability of an interaction to occur

S matrix

No interaction

$$i\mathcal{M} \;\propto\; \langle \text{final}|\, S - 1 \,|\text{initial}\rangle$$

$\sim$



$$\mathcal{M} = \mathcal{M}(E^{\star})$$

Center-of-Momentum Energy

Scattering Amplitudes

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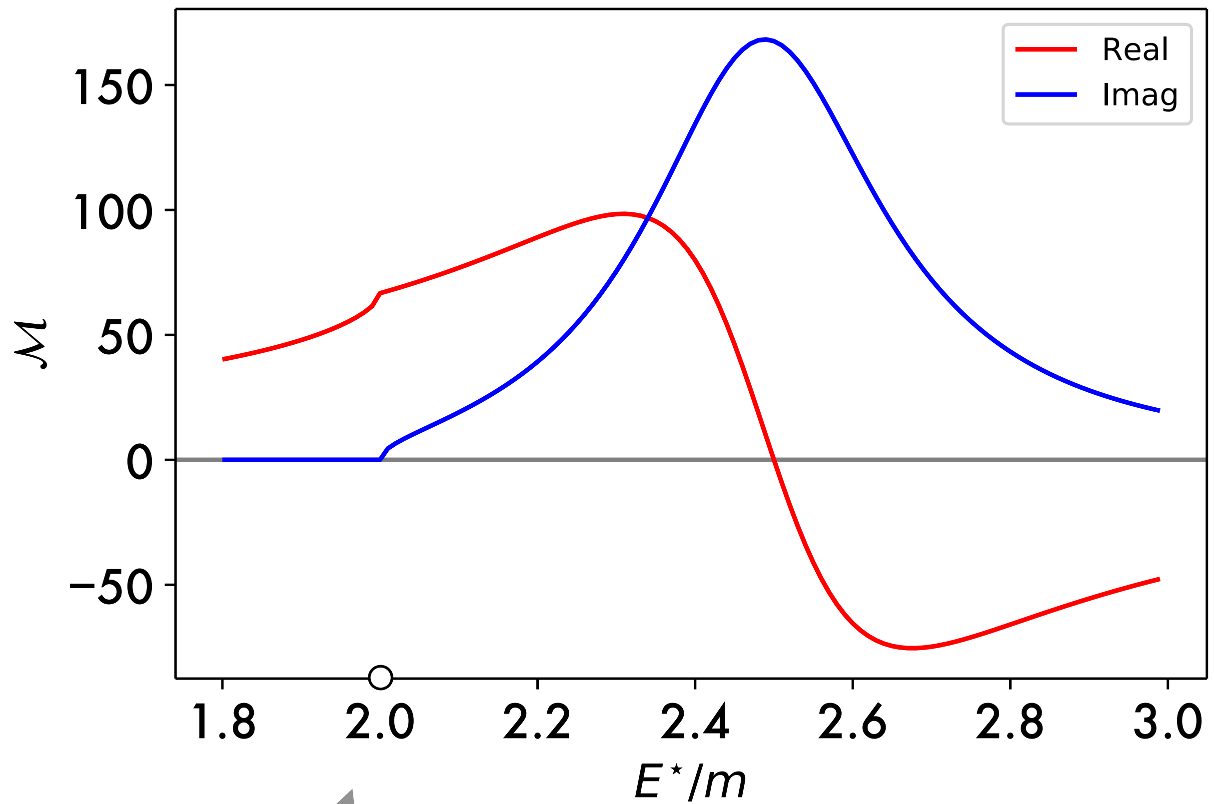
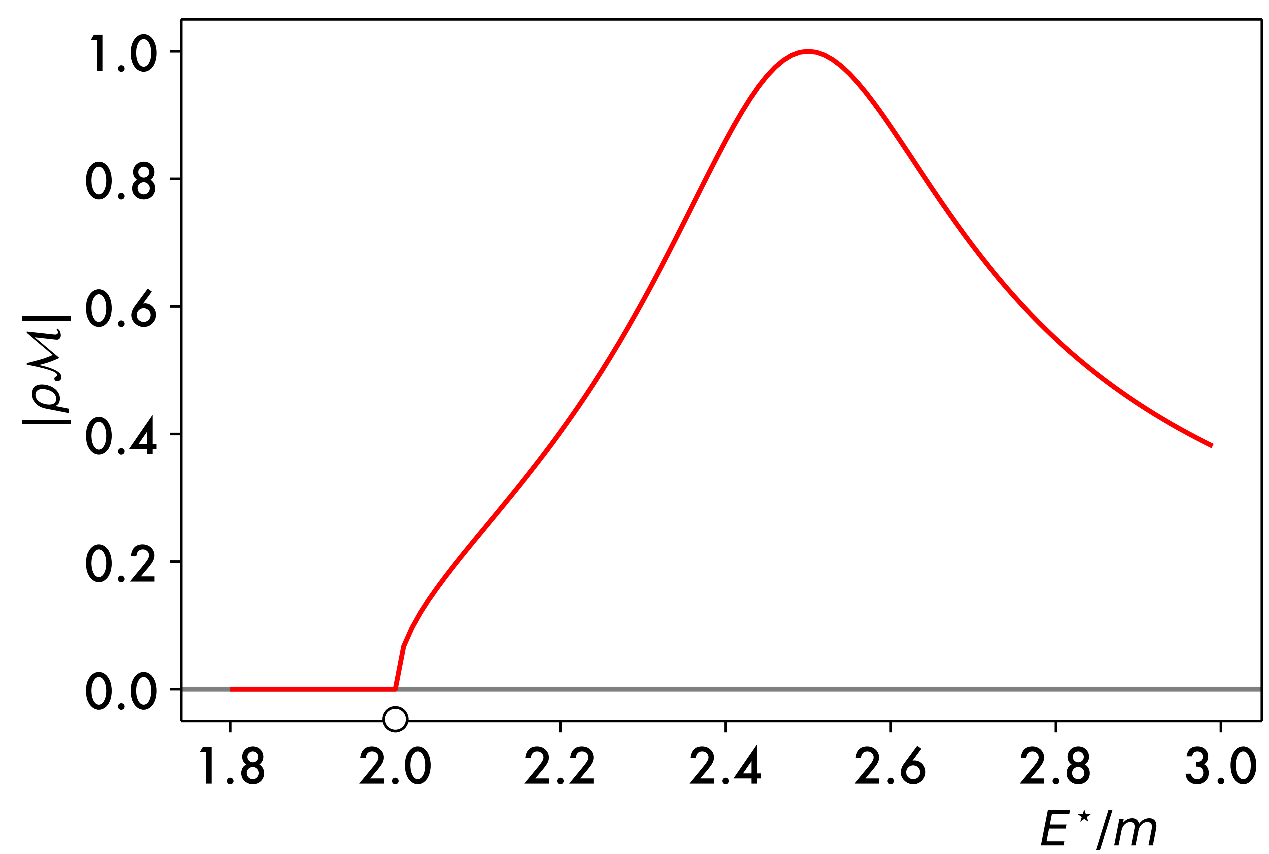


Real AND Imaginary components

Threshold Energy

$$E^\star_{\mathrm{th}} = 2m$$

$$\mathrm{Prob} \propto |\mathcal{M}|^2$$



Kinematic factor "phase space" ρ

Scattering Theory

Model independent features of scattering amplitudes

Symmetry

Spacetime — Lorentz invariance

Internal — Flavor, baryon number, …

Unitarity

Probability conservation $\Longrightarrow$ The S matrix is a unitary operator

Analyticity

Causality $\Longrightarrow$ Amplitudes are boundary values of analytic functions in complex energy plane

Crossing

CPT symmetry $\Longrightarrow$ Relates particle—anti-particles in scattering processes

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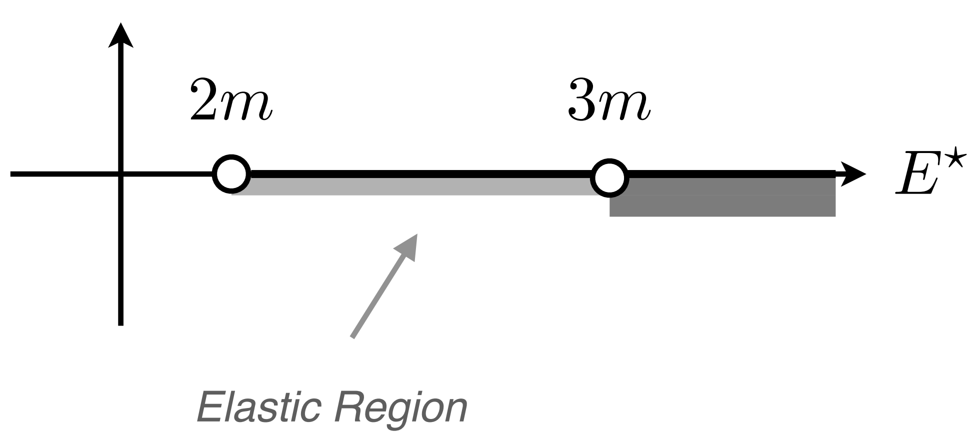
Scattering Theory - Unitarity

Probability conservation $\Longrightarrow$ The S matrix is a unitary operator

$$\sum_f \mathrm{Prob}(i \to f) = 1 \qquad \Longrightarrow \qquad S^\dagger S = \mathbb{1}$$

After some work, can show that (in a limited energy region)

$$\mathrm{Im}\,\mathcal{M} = \rho|\mathcal{M}|^2 \qquad \text{for} \;\; E^\star \geq 2m$$



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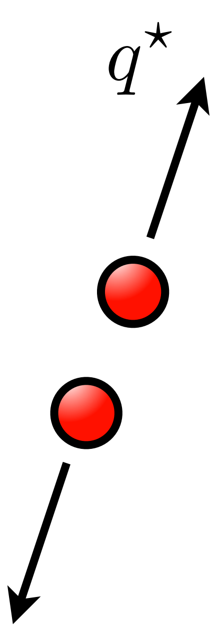
Phase space

- *Kinematic function*
- *Characterizes on-shell scattering of two-particles*



$$\rho = \frac{\xi q^\star}{8\pi E^\star}$$

Relative momentum between particles (in CM frame)

$$q^\star = \frac{1}{2}\sqrt{E^{\star\,2} - 4m^2}$$

$$\xi = \begin{cases} \frac{1}{2} & \text{identical} \\ 1 & \text{otherwise} \end{cases}$$

Scattering Theory - Phase Shift

Unitarity enforces some useful properties of the scattering amplitude

1. Phase shift representation

$$\mathcal{M} = |\mathcal{M}|\, e^{i\delta}$$

At a fixed energy,
amplitude determined by magnitude and phase (2 real numbers)

Impose unitarity $\mathrm{Im}\,\mathcal{M} = \rho|\mathcal{M}|^2$ …

…can show

$$|\mathcal{M}| = \frac{1}{\rho}\sin\delta$$

…such that

$$\mathcal{M} = \frac{1}{\rho}e^{i\delta}\,\sin\delta$$

At a fixed energy, 1 real parameter!
δ — the ***phase shift***

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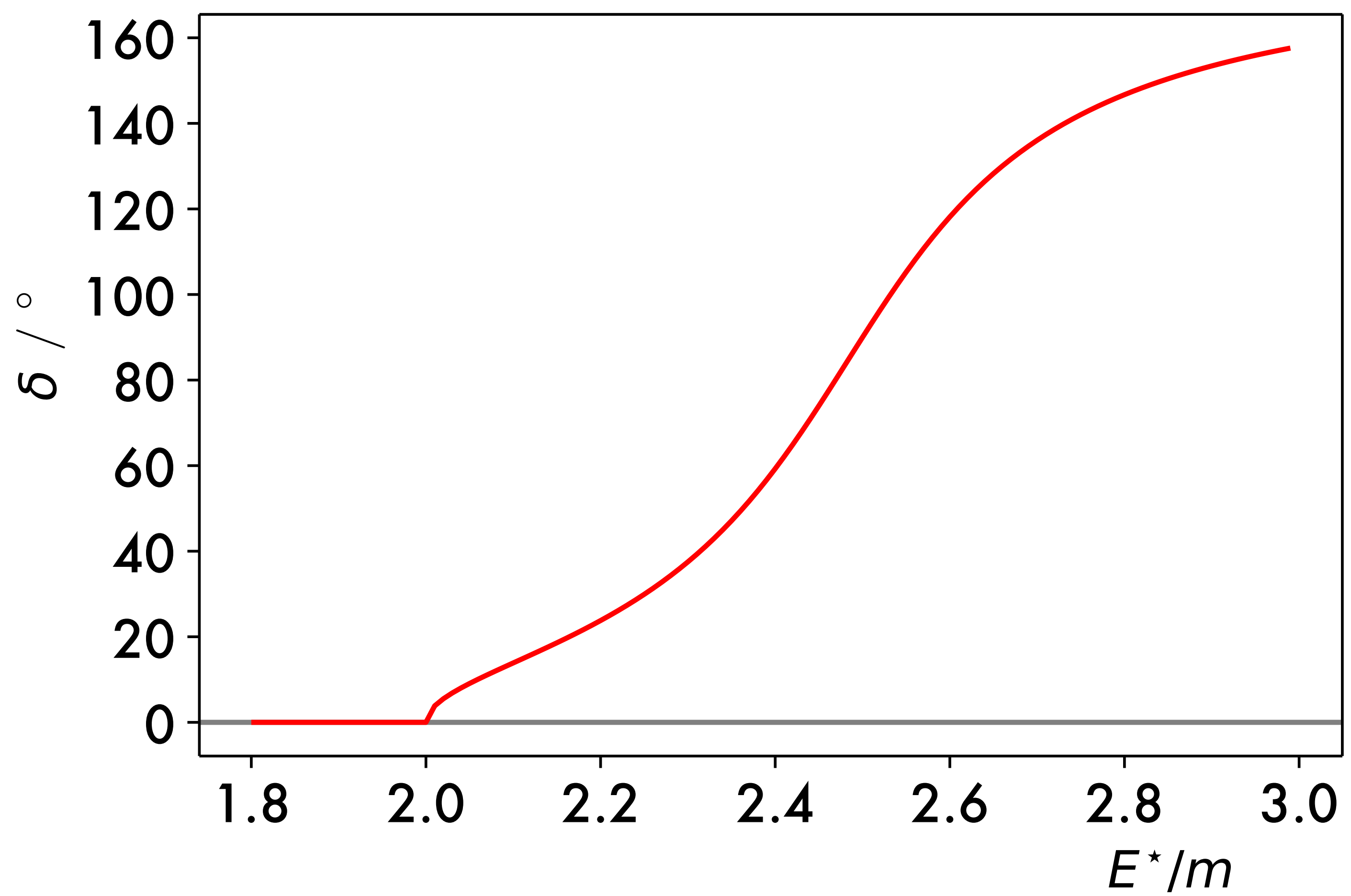

Scattering Theory - *K* matrix

Unitarity enforces some useful properties of the scattering amplitude

2. *K* matrix representation

$$\mathrm{Im}\,\mathcal{M} = \rho|\mathcal{M}|^2$$

$$\implies \qquad \mathrm{Im}\,\mathcal{M}^{-1} = -\rho$$

$$\implies \qquad \mathcal{M}^{-1} = \mathcal{K}^{-1} - i\rho$$

Real function
Characterizes 'short-range' forces between two particles
Not known a priori — Need to specify interaction

$$\mathcal{M} = \mathcal{K}\frac{1}{1 - i\rho\mathcal{K}}$$

Can relate to phase shift

$$\mathcal{K}^{-1} = \rho\cot\delta$$

Scattering Theory - Exercises

1. Plot ρ (phase space) for identical particles in the range $1.8 \leq E^\star/m \leq 3.2$
2. Derive the *phase shift representation* for the scattering amplitude
3. Show that $\mathrm{Im}\,\mathcal{M}^{-1} = -\rho$
4. Show that $\mathcal{K}^{-1} = \rho \cot \delta$