

# Hadron Spectroscopy

Andrew W. Jackura

Old Dominion University & Jefferson Lab

RPI Computational Summer School

June 20, 2022



**OLD DOMINION**  
UNIVERSITY

**Jefferson Lab**  
Thomas Jefferson National Accelerator Facility

# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

## Lattice QCD & Hadron Spectroscopy

Lattice QCD

Lüscher & the Finite-Volume

# Some Reviews

arXiv:1706.06223 (2017)

## Scattering processes and resonances from lattice QCD

Raúl A. Briceño,<sup>1,\*</sup> Jozef J. Dudek,<sup>1,2,†</sup> and Ross D. Young<sup>3,‡</sup>

<sup>1</sup>Thomas Jefferson National Accelerator Facility, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA

<sup>2</sup>Department of Physics, College of William and Mary, Williamsburg, Virginia 23187, USA

<sup>3</sup>Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia

(Dated: June 21, 2017)

The vast majority of hadrons observed in nature are not stable under the strong interaction, rather they are *resonances* whose existence is deduced from enhancements in the energy dependence of scattering amplitudes. The study of hadron resonances offers a window into the workings of quantum chromodynamics (QCD) in the low-energy non-perturbative region, and in addition, many probes of the limits of the electroweak sector of the Standard Model consider processes which feature hadron resonances. From a theoretical standpoint, this is a challenging field: the same dynamics that binds quarks and gluons into hadron resonances also controls their decay into lighter hadrons, so a complete approach to QCD is required. Presently, lattice QCD is the only available tool that provides the required non-perturbative evaluation of hadron observables. In this article, we review progress in the study of few-hadron reactions in which resonances and bound-states appear using lattice QCD techniques. We describe the leading approach which takes advantage of the periodic finite spatial volume used in lattice QCD calculations to extract scattering amplitudes from the discrete spectrum of QCD eigenstates in a box. We explain how from explicit lattice QCD calculations, one can rigorously garner information about a variety of resonance properties, including their masses, widths, decay couplings, and form factors. The challenges which currently limit the field are discussed along with the steps being taken to resolve them.

### CONTENTS

I. Introduction	2	1. The importance of “multi-hadron” operators	16
II. Resonances, composite particles, and scattering amplitudes	3	VI. Examples of resonance determination	18
A. Pole singularities	3	A. Elastic resonances in $\pi\pi$ scattering	18
B. Coupled-channel scattering	4	B. Resonances in coupled-channel meson-meson scattering	22
C. Diagrammatic representation	5	VII. Other approaches to resonance determination	23
III. Lattice QCD	5	A. Resonances in the Lüscher formalism in the narrow-width approximation	23
IV. Scattering in a finite-volume	6	B. Resonances and ‘naive’ level counting	25
A. Scattering in non-relativistic quantum mechanics in one space dimension	7	C. Finite volume EFT Hamiltonian approach	26
B. Scattering in a periodic cubic volume	7	D. Unitarized chiral perturbation theory and chiral extrapolations	27
C. Relating scattering amplitudes to finite-volume spectra	10	E. Other approaches	28
1. Dominance of the lowest partial-wave	10	VIII. Coupling resonances to external currents	29
2. Coupled-channel scattering and parameterization of scattering amplitudes	11	A. Determining matrix elements in lattice QCD	30
3. Examples of finite-volume spectra for simple scattering amplitudes	11	B. Lellouch-Lüscher formalism and its generalizations	31
C. Applications	32	C. Applications	32
V. Determining the finite-volume spectrum	12	IX. Contemporary Extensions	33
A. Variational analysis of correlation matrices	12	A. Particles with nonzero intrinsic spin	33
B. Operator construction	15	B. Three-particle systems	34
X. Outlook	35	C. Elastic form factors of resonances	35

# Some Reviews

arXiv:1706.06223 (2017)

## Scattering processes and resonances from lattice QCD

Raúl A. Briceño,<sup>1,\*</sup> Jozef J. Dudek,<sup>1,2,†</sup> and Ross D. Young<sup>3,‡</sup>

<sup>1</sup>Thomas Jefferson National Accelerator Facility, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA

<sup>2</sup>Department of Physics, College of William and Mary, Williamsburg, Virginia 23187, USA

<sup>3</sup>Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia

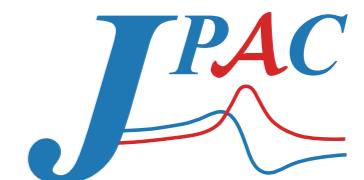
(Dated: June 21, 2017)

The vast majority of hadrons observed in nature are *bound states*, rather than *resonances* whose existence is inferred from the energy dependence of scattering amplitudes. Lattice QCD offers a window into the workings of quantum chromodynamics in the non-perturbative region, and in addition, many processes in the Standard Model consider processes where the theoretical standpoint, this is a challenging field: the conversion of gluons into hadron resonances also controls the complete approach to QCD is required. Presently, that provides the required non-perturbative evaluation of the spectrum. In this article, we review progress in the study of few-hadron scattering, where bound-states appear using lattice QCD techniques which takes advantage of the periodic finite spatial volumes to extract scattering amplitudes from the discrete box. We explain how from explicit lattice QCD calculations one can obtain information about a variety of resonance properties, decay couplings, and form factors. The challenges discussed along with the steps being taken to resolve them.

## CONTENTS

- I. Introduction
- II. Resonances, composite particles, and scattering amplitudes
  - A. Pole singularities
  - B. Coupled-channel scattering
  - C. Diagrammatic representation
- III. Lattice QCD
- IV. Scattering in a finite-volume
  - A. Scattering in non-relativistic quantum mechanics in one space dimension
  - B. Scattering in a periodic cubic volume
  - C. Relating scattering amplitudes to finite-volume spectra
    - 1. Dominance of the lowest partial-wave
    - 2. Coupled-channel scattering and parameterizations of scattering amplitudes
    - 3. Examples of finite-volume spectra for simple scattering amplitudes
- V. Determining the finite-volume spectrum
  - A. Variational analysis of correlation matrices
  - B. Operator construction

arXiv:2112.13436 (2021)



## Novel approaches in Hadron Spectroscopy

Miguel Albaladejo<sup>a,b</sup>, Łukasz Bibrzycki<sup>c</sup>, Sebastian M. Dawid<sup>d,e</sup>, César Fernández-Ramírez<sup>f,g</sup>, Sergi González-Solís<sup>d,e,h</sup>, Astrid N. Hiller Blin<sup>a</sup>, Andrew W. Jackura<sup>a,i</sup>, Vincent Mathieu<sup>j,k</sup>, Mikhail Mikhasenko<sup>l,m</sup>, Victor I. Mokeev<sup>a</sup>, Emilie Passemard<sup>a,d,e</sup>, Alessandro Pilloni<sup>n,o,\*</sup>, Arkaitz Rodas<sup>a,p</sup>, Jorge A. Silva-Castro<sup>f</sup>, Wyatt A. Smith<sup>d</sup>, Adam P. Szczepaniak<sup>a,d,e</sup>, Daniel Winney<sup>d,e,q,r</sup>,

(Joint Physics Analysis Center)

<sup>a</sup>Theory Center and Physics Division, Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA

<sup>b</sup>Instituto de Física Corpuscular (IFIC), Centro Mixto CSIC-Universidad de Valencia, E-46071 Valencia, Spain

<sup>c</sup>Pedagogical University of Krakow, 30-084 Kraków, Poland

<sup>d</sup>Department of Physics, Indiana University, Bloomington, IN 47405, USA

<sup>e</sup>Center for Exploration of Energy and Matter, Indiana University, Bloomington, IN 47403, USA

<sup>f</sup>Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Ciudad de México 04510, Mexico

<sup>g</sup>Departamento de Física Interdisciplinar, Universidad Nacional de Educación a Distancia (UNED), Madrid E-28040, Spain

<sup>h</sup>Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

<sup>i</sup>Department of Physics, Old Dominion University, Norfolk, VA 23529, USA

<sup>j</sup>Departament de Física Quàntica i Astrofísica and Institut de Ciències del Cosmos, Universitat de Barcelona, E-08028, Spain

<sup>k</sup>Departamento de Física Teórica, Universidad Complutense de Madrid and IPARCOS, E-28040 Madrid, Spain

<sup>l</sup>ORIGINS Excellence Cluster, 80939 Munich, Germany

<sup>m</sup>Ludwig-Maximilian University of Munich, Germany

<sup>n</sup>Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra, Università degli Studi di Messina, I-98122 Messina, Italy

<sup>o</sup>INFN Sezione di Catania, I-95123 Catania, Italy

<sup>q</sup>Guangdong Provincial Key Laboratory of Nuclear Science, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China

<sup>r</sup>Guangdong-Hong Kong Joint Laboratory of Quantum Matter, Southern Nuclear Science Computing Center, South China Normal University, Guangzhou 510006, China

# Python Notebooks

I have created a few Jupyter notebooks to accompany the lectures

- All notebooks use **python**, and the modules that come with the [Anaconda package manager](#)
- Includes introductory and advanced exercises
- <https://github.com/ajackura/RPISummerSchoolHadspec>

The screenshot shows a Jupyter Notebook interface with a dark theme. The left sidebar (EXPLORER) lists three notebooks: 'hadspec\_ex0\_complex.ipynb' (selected), 'hadspec\_ex1\_scattering.ipynb', and 'hadspec\_ex2\_luescher.ipynb'. Below these are 'LICENSE' and 'README.md'. The main area displays the content of 'hadspec\_ex0\_complex.ipynb'. The title 'A Brief On Complex Analysis' is followed by text about scattering processes and basic features. A 'References' section lists books and blog posts. The code cell contains imports for cmath, math, numpy, scipy, random, and matplotlib.pyplot, along with a function 'plot\_amplitude' for plotting scattering amplitudes. The plot shows the real (red line) and imaginary (blue line) parts of the amplitude as a function of  $E^*/m$  from 1.8 to 3.0. The real part has a peak at approximately 2.4, while the imaginary part has a sharp dip at approximately 2.7.

```
hadspec_ex0_complex.ipynb -- RPISummerSchoolHadspec
Notebooks > hadspec_ex1_scattering.ipynb > M+Scattering Amplitudes > M+Complex Square Root
Author - Andrew W. Jackura
Email - ajackura@odu.edu / ajackura@jlab.org

A Brief On Complex Analysis

Studying scattering processes in physics inevitably leads us to require unders integrals. In this notebook, we review some basic features which we need in o

References

I find the following references useful, you may too:


- Fundamentals of Complex Analysis with Applications to Engineering, Sc


In the context of scattering, this old (but very useful) book has a good chapter


- Dispersion Relation Dynamics - Burkhardt


This blog post has a nice discussion on moving the branch cut of the square root


- https://flothesof.github.io/branch-cuts-with-square-roots.html


And I have taken excerpts from the following website which has some numeric


- http://people.exeter.ac.uk/sh481/cauchy-theorem.html



...
Importing useful libraries
...
import cmath as cm
import math as m
import numpy as np
import scipy.special as sp
import numpy.random as rn
import matplotlib.pyplot as plt
from scipy import integrate
from mpl_toolkits import mplot3d
from matplotlib.colors import hsv_to_rgb
from colorsys import hls_to_rgb
# math library (complex)
# math library
# basic functions, lin
# special functions
# random numbers
# plotting library
# library for integrat
# for 3d plotting
# convert the color fr
# convert the color fr

# Routine to plot amplitude
def plot_amplitude( Kmatrix ):
    eps = 1e-16
    Ecm_o_m = np.arange(1.8, 3, 0.01)
    amp = Amplitude( Ecm_o_m**2+1j*eps, Kmatrix )
    plt.axhline(y=0.0, color='gray', linestyle='--')
    plt.plot(Ecm_o_m, amp.real, color='red', label="Real")
    plt.plot(Ecm_o_m, amp.imag, color='blue', label="Imag")
    plt.xlabel(r'$E^*/m$', size=15)
    plt.ylabel(r'$\mathcal{M}$', size=15)
    plt.xticks(fontname="Futura", fontsize=15)
    plt.yticks(fontname="Futura", fontsize=15)
    plt.legend(loc="upper right")
    plt.figure(figsize=(2,1), dpi= 100, facecolor='w', edgecolor='k')

# Sample Effective Range parameters
a = 2.0 # /m
r = 0.0 # /m

# Sample Breit-Wigner parameters
m0 = 2.5 # /m
g0 = 3.0

# Plot Breit-Wigner
plot_amplitude( lambda s:Kmatrix_BreitWigner(s,m0,g0) )
```

A line plot showing the real (red line) and imaginary (blue line) parts of the scattering amplitude  $\mathcal{M}$  as a function of  $E^*/m$ . The x-axis ranges from 1.8 to 3.0, and the y-axis ranges from -50 to 150. The real part has a peak at approximately 2.4, while the imaginary part has a sharp dip at approximately 2.7.

# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

## Lattice QCD & Hadron Spectroscopy

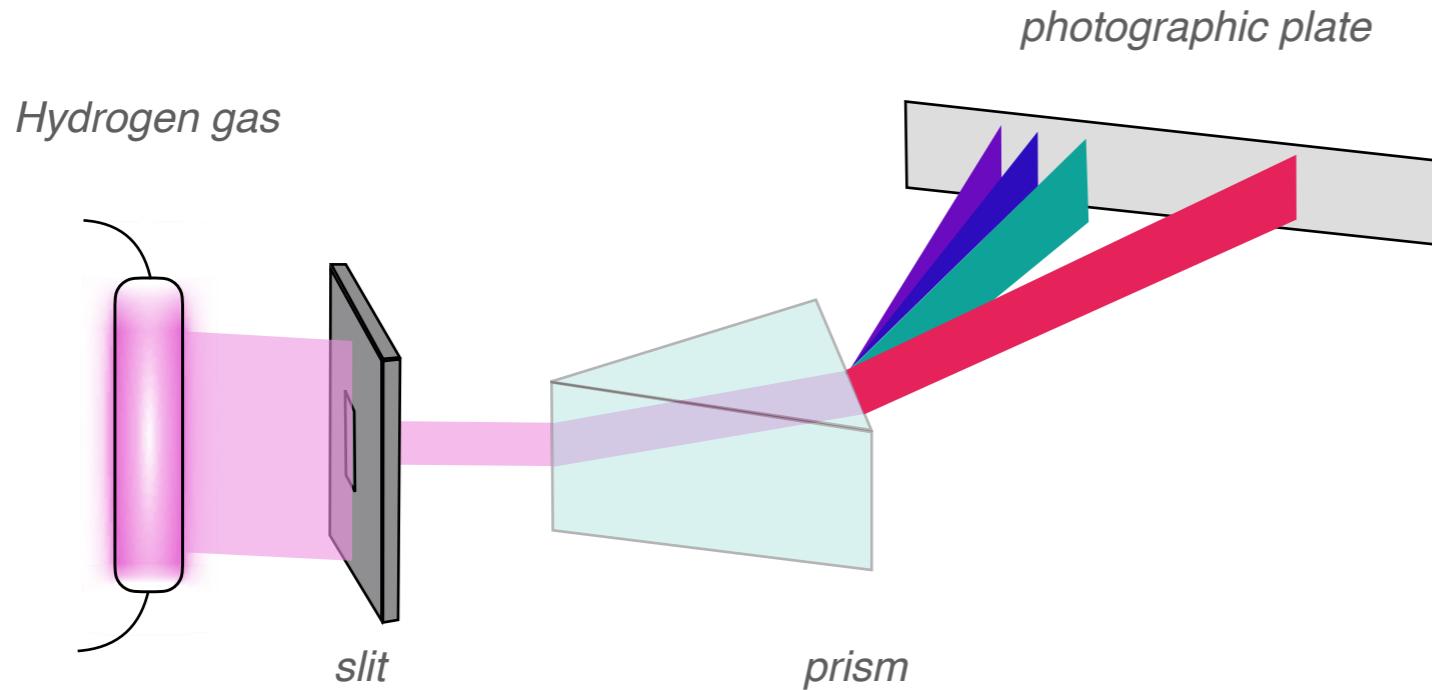
Lattice QCD

Lüscher & the Finite-Volume

# Spectroscopy

Studying the **spectrum** is key to understanding physical phenomena

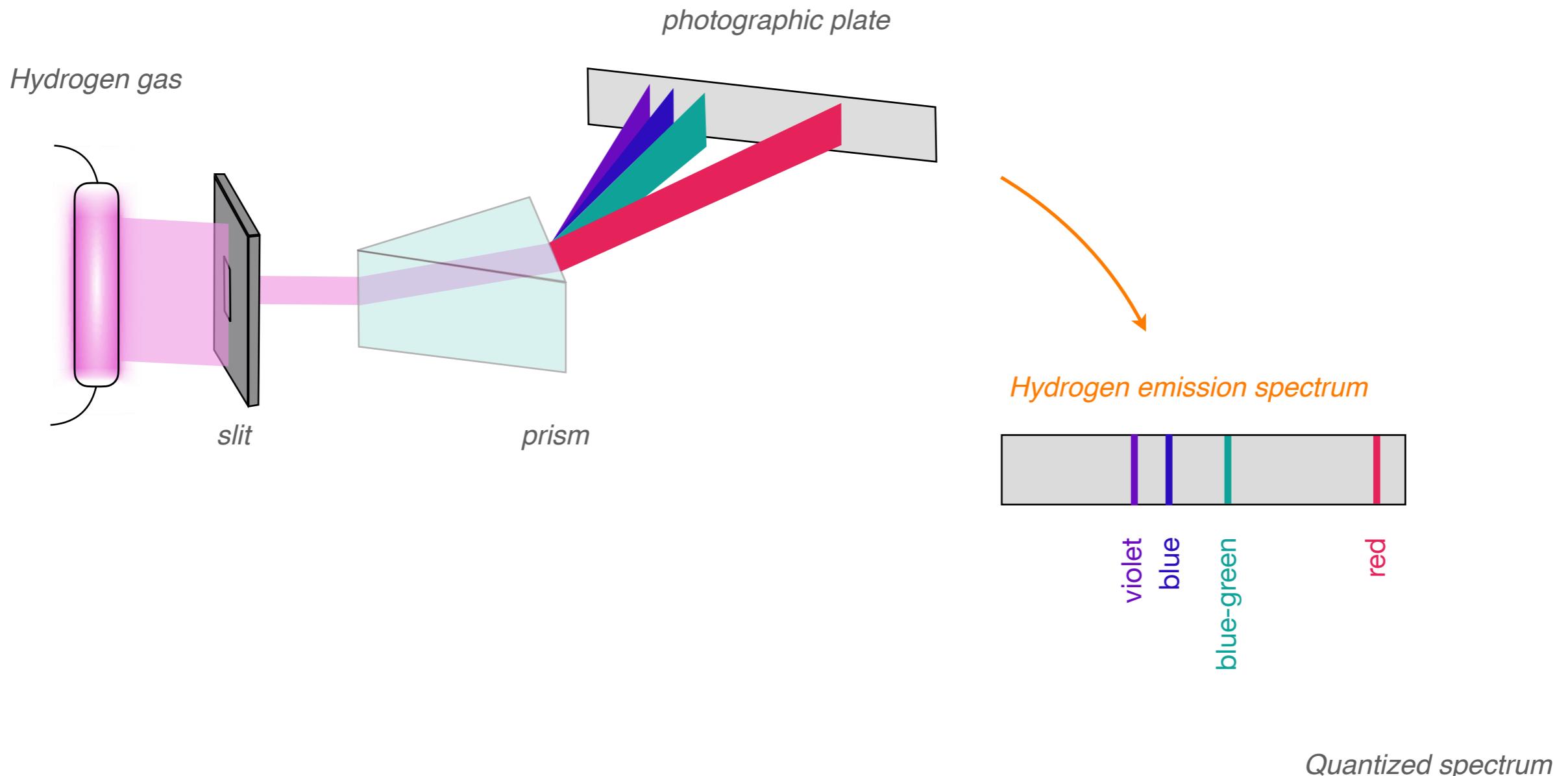
- e.g., the Hydrogen atom led to the discovery of Quantum Mechanics



# Spectroscopy

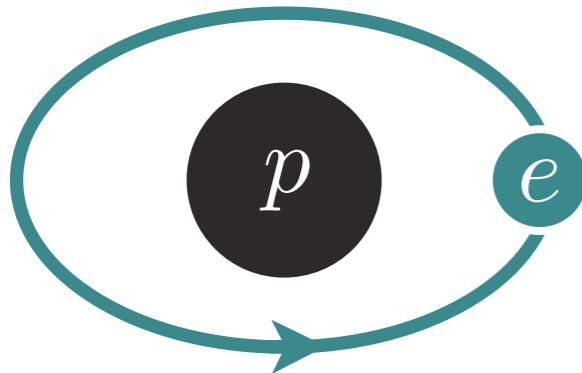
Studying the **spectrum** is key to understanding physical phenomena

- e.g., the Hydrogen atom led to the discovery of Quantum Mechanics



Studying the **spectrum** is key to understanding physical phenomena

- e.g., the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for modern particle physics



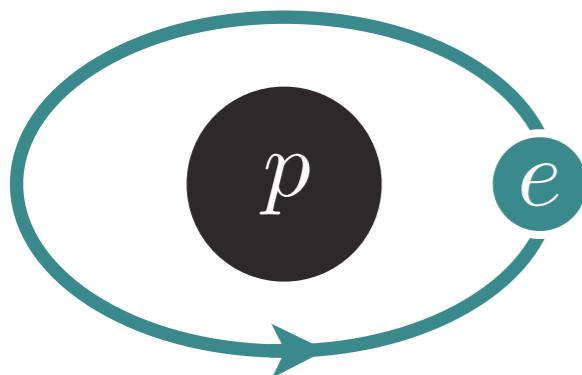
$$\hat{H} |\psi\rangle = E_n |\psi\rangle \quad \textit{Schrödinger equation}$$

# Spectroscopy

$$\hbar = c = 1$$

Studying the **spectrum** is key to understanding physical phenomena

- e.g., the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for modern particle physics

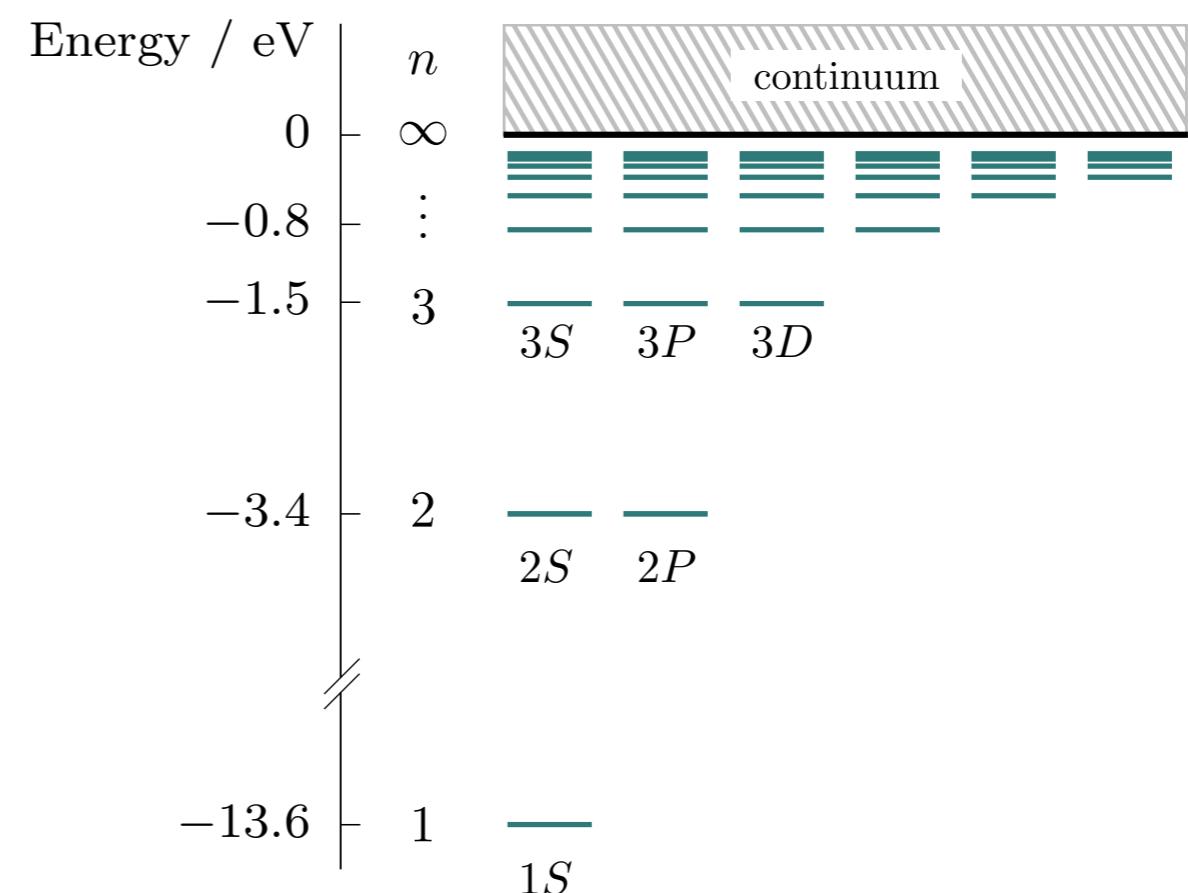


$$\hat{H} |\psi\rangle = E_n |\psi\rangle \quad \text{Schrödinger equation}$$

*spectrum*

$$E_n = -\frac{m\alpha^2}{2n}$$

$$\alpha = \frac{e^2}{4\pi}$$

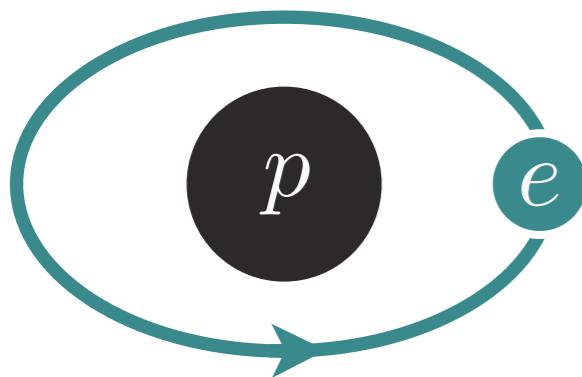


# Spectroscopy

$$\hbar = c = 1$$

Studying the **spectrum** is key to understanding physical phenomena

- e.g., the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for modern particle physics

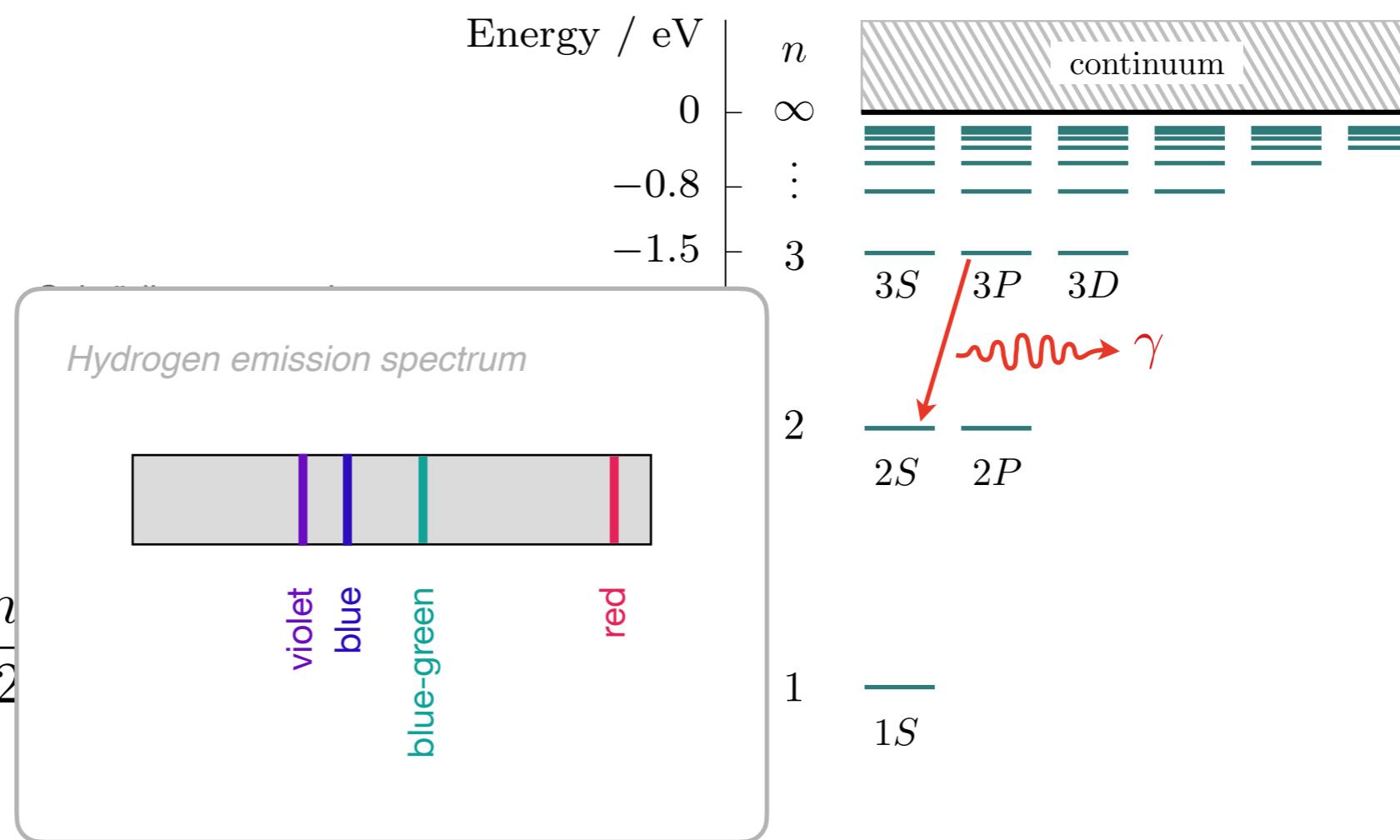


$$\hat{H} |\psi\rangle = E_n |\psi\rangle$$

spectrum

$$E_n = -\frac{m}{2}$$

$$\alpha = \frac{e^2}{4\pi}$$

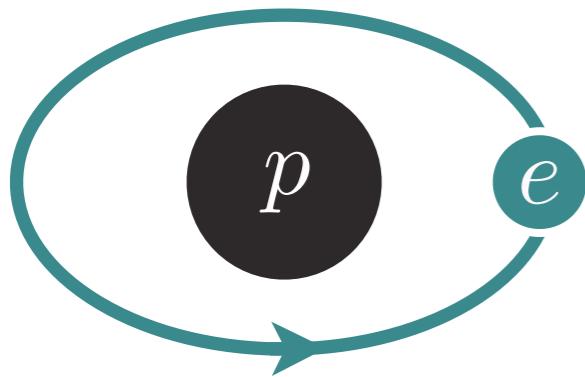


# Spectroscopy

$$\hbar = c = 1$$

Studying the **spectrum** is key to understanding physical phenomena

- e.g., the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for modern particle physics



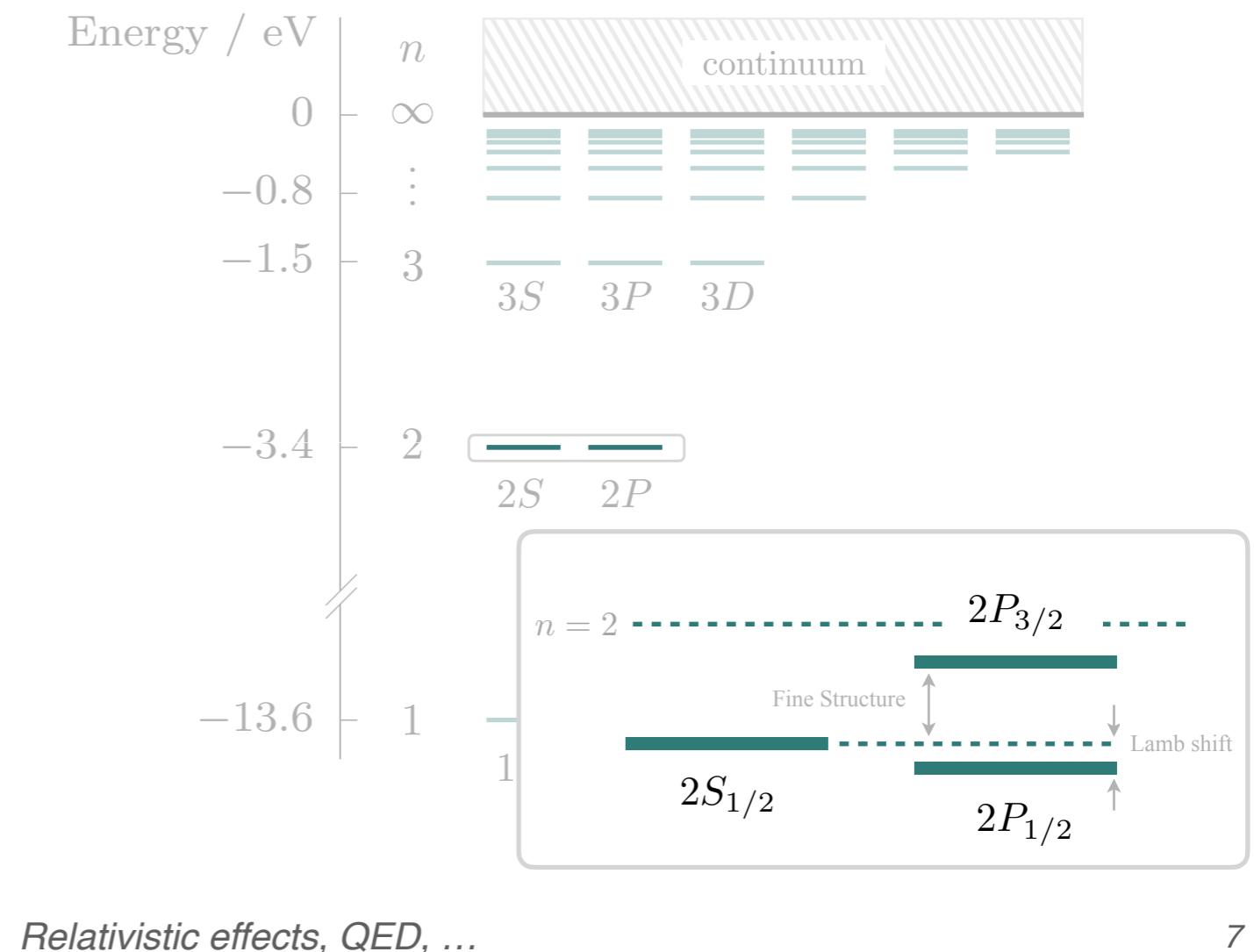
$$\hat{H} |\psi\rangle = E_n |\psi\rangle$$

*Schrödinger equation*

*spectrum*

$$E_n = -\frac{m\alpha^2}{2n} + \mathcal{O}(\alpha^4)$$

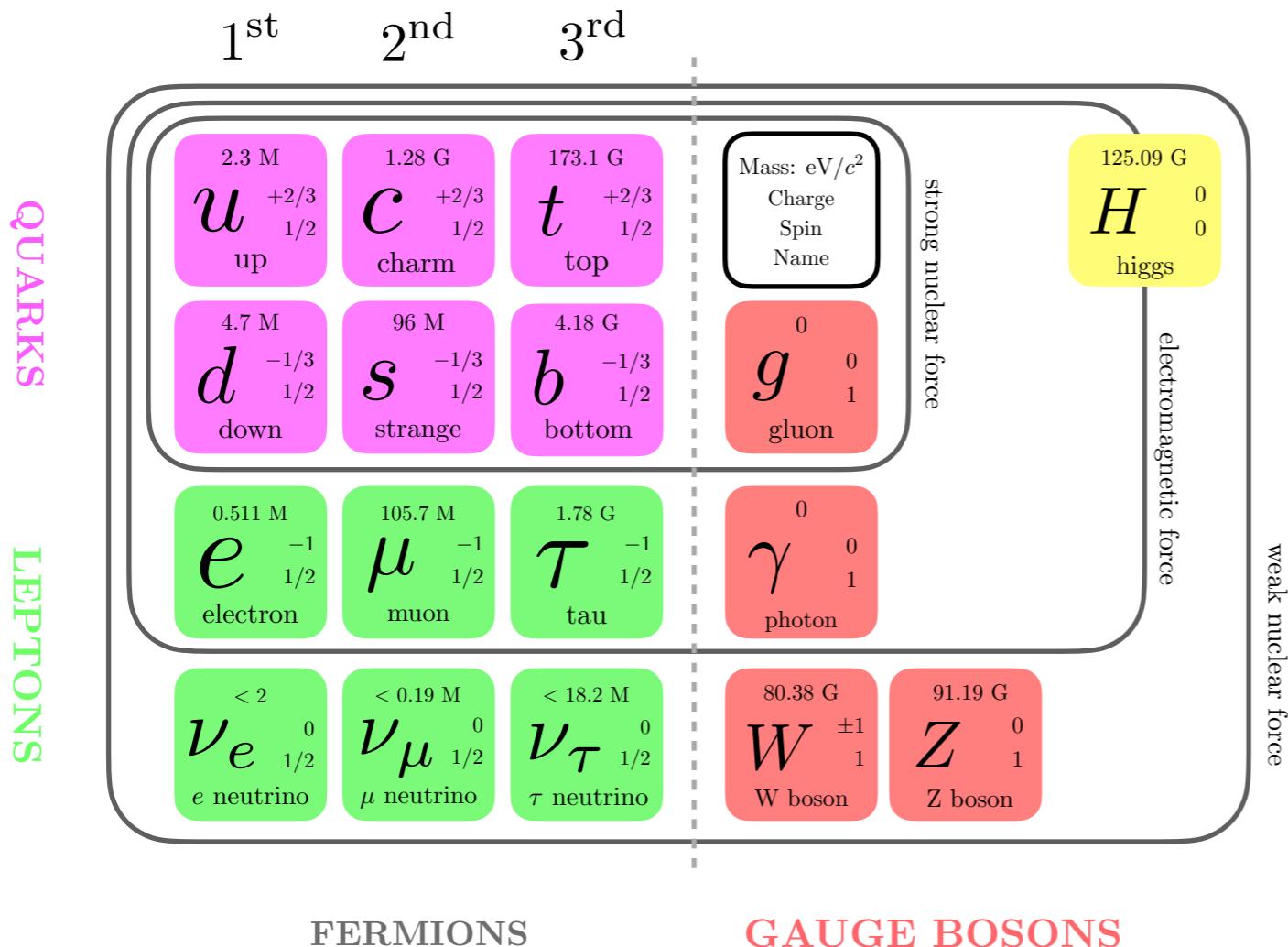
$$\alpha = \frac{e^2}{4\pi}$$



*Relativistic effects, QED, ...*

# The Standard Model of Particle Physics

The Standard Model is a remarkably *simple\** theory



\* *simple = An anomaly-free relativistic quantum gauge field theory, invariant under the gauge group  $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$  which spontaneously breaks via a scalar field to  $SU(3)_C \otimes U(1)_Q$*

# Quantum ElectroDynamics

---

Theory of electron-photon interactions

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

*Quantum Electrodynamics*

# Quantum ElectroDynamics

Theory of electron-photon interactions

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

*Quantum Electrodynamics*

*electron-positron field*



$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

*photon field*

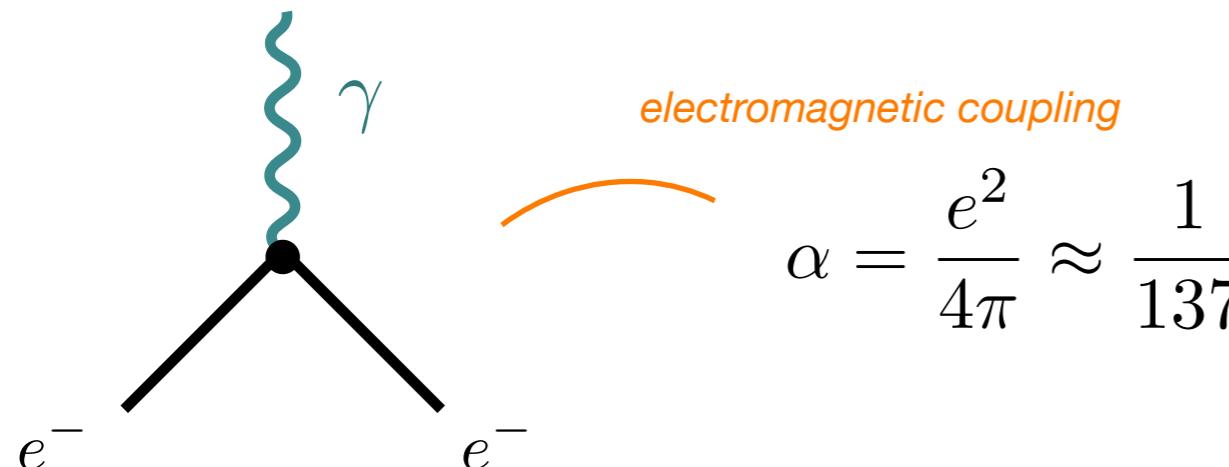
$$D_\mu = \partial_\mu + ieA_\mu$$

# Quantum ElectroDynamics

Theory of electron-photon interactions

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

*Quantum Electrodynamics* electron-positron field



$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

photon field

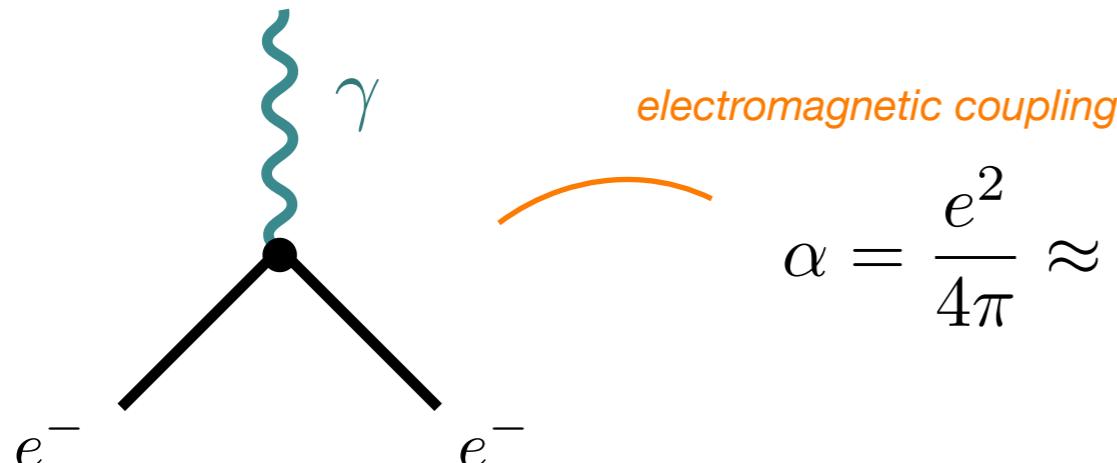
$$D_\mu = \partial_\mu + ieA_\mu$$

# Quantum ElectroDynamics

Theory of electron-photon interactions

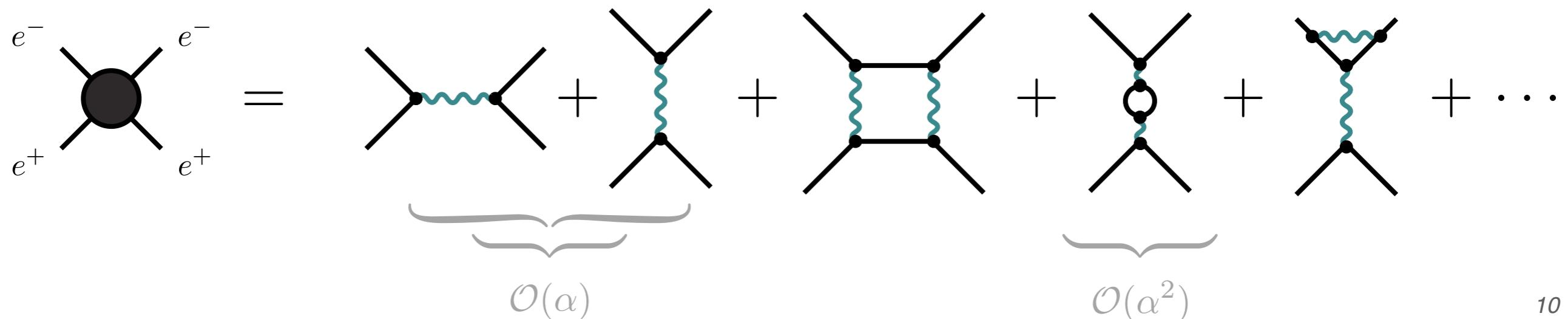
$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

*Quantum Electrodynamics*



$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

Can compute observables ***perturbatively*** in electric coupling!

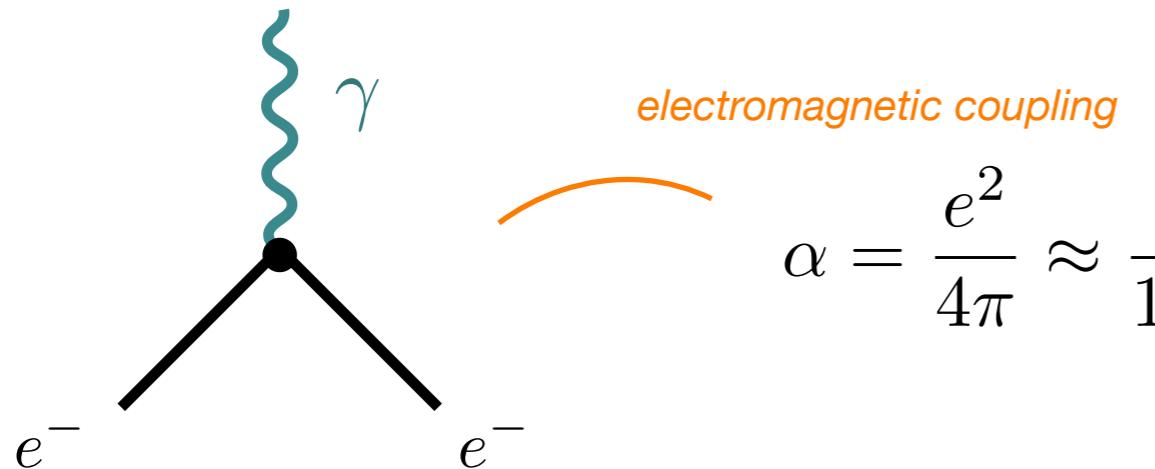


# Quantum ElectroDynamics

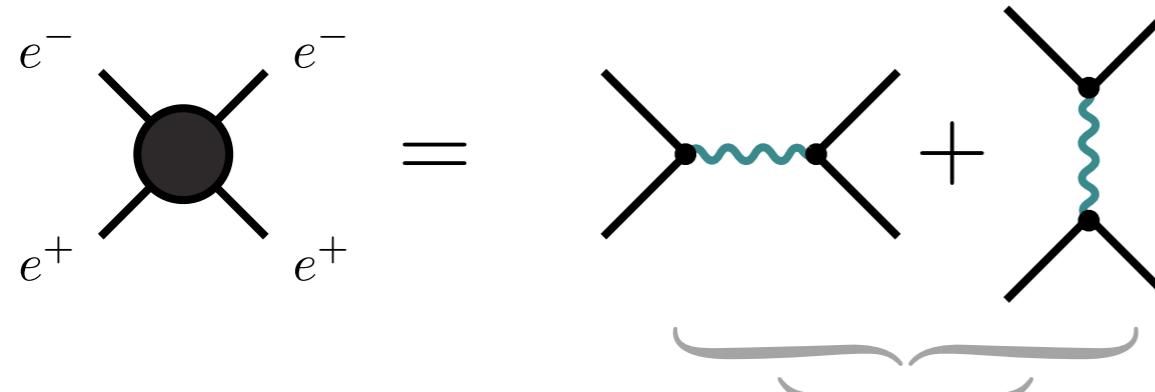
Theory of electron-photon interactions

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

*Quantum Electrodynamics*

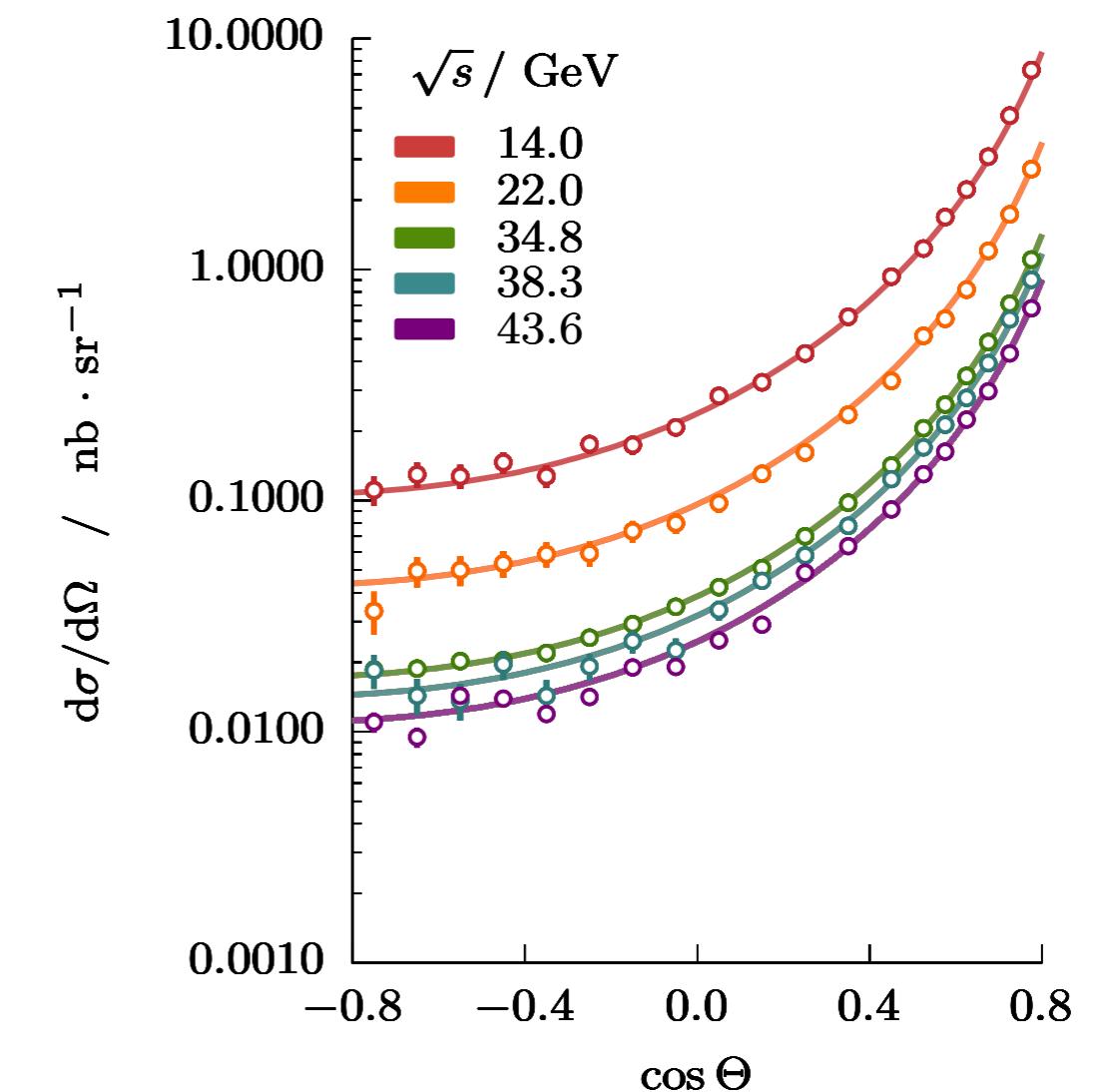


Can compute observables ***perturbatively*** in  $e^- e^-$



$\mathcal{O}(\alpha)$

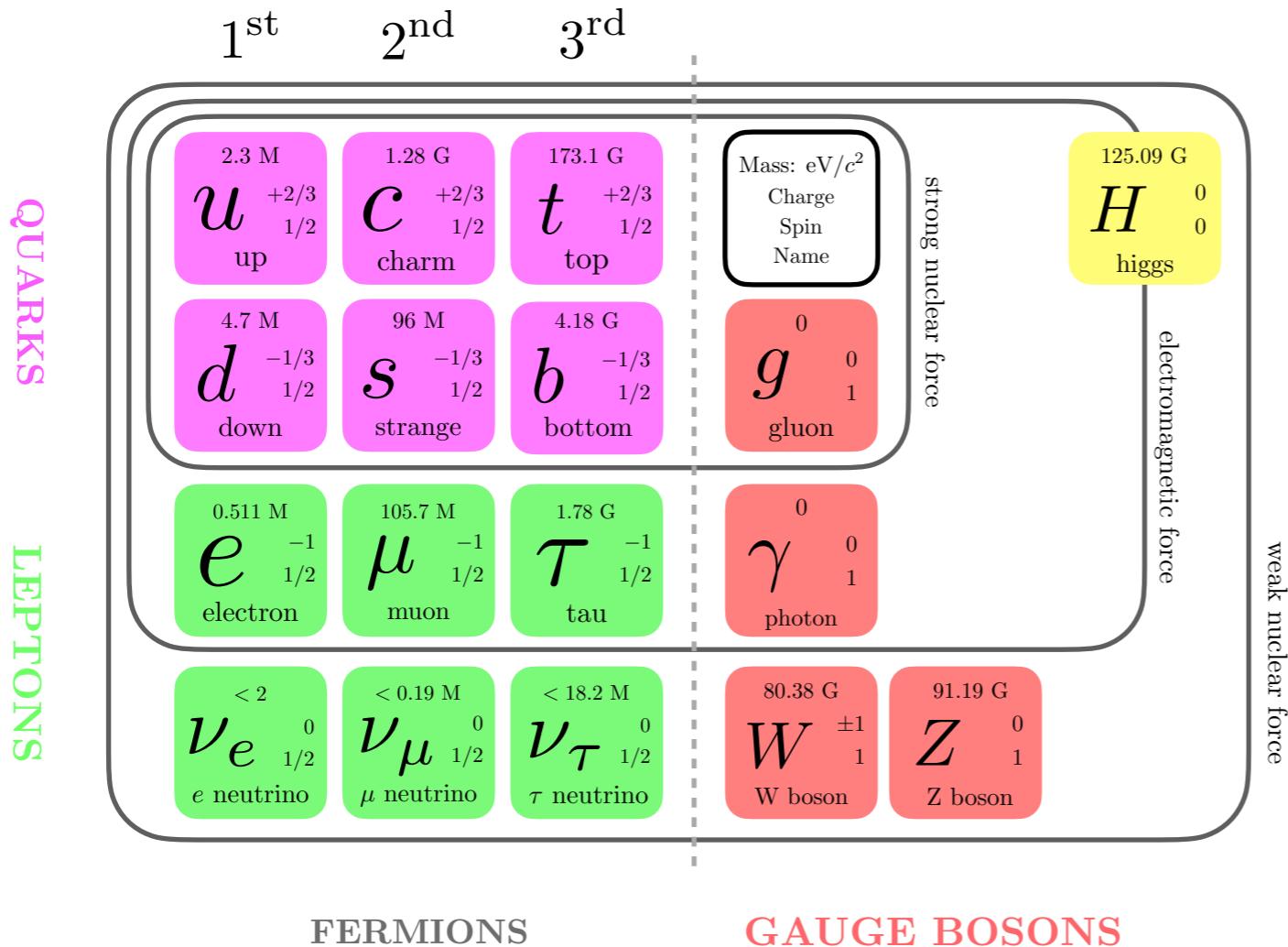
$e^- e^+ \rightarrow e^- e^+$



$\mathcal{O}(\alpha^2)$

# The Standard Model of Particle Physics

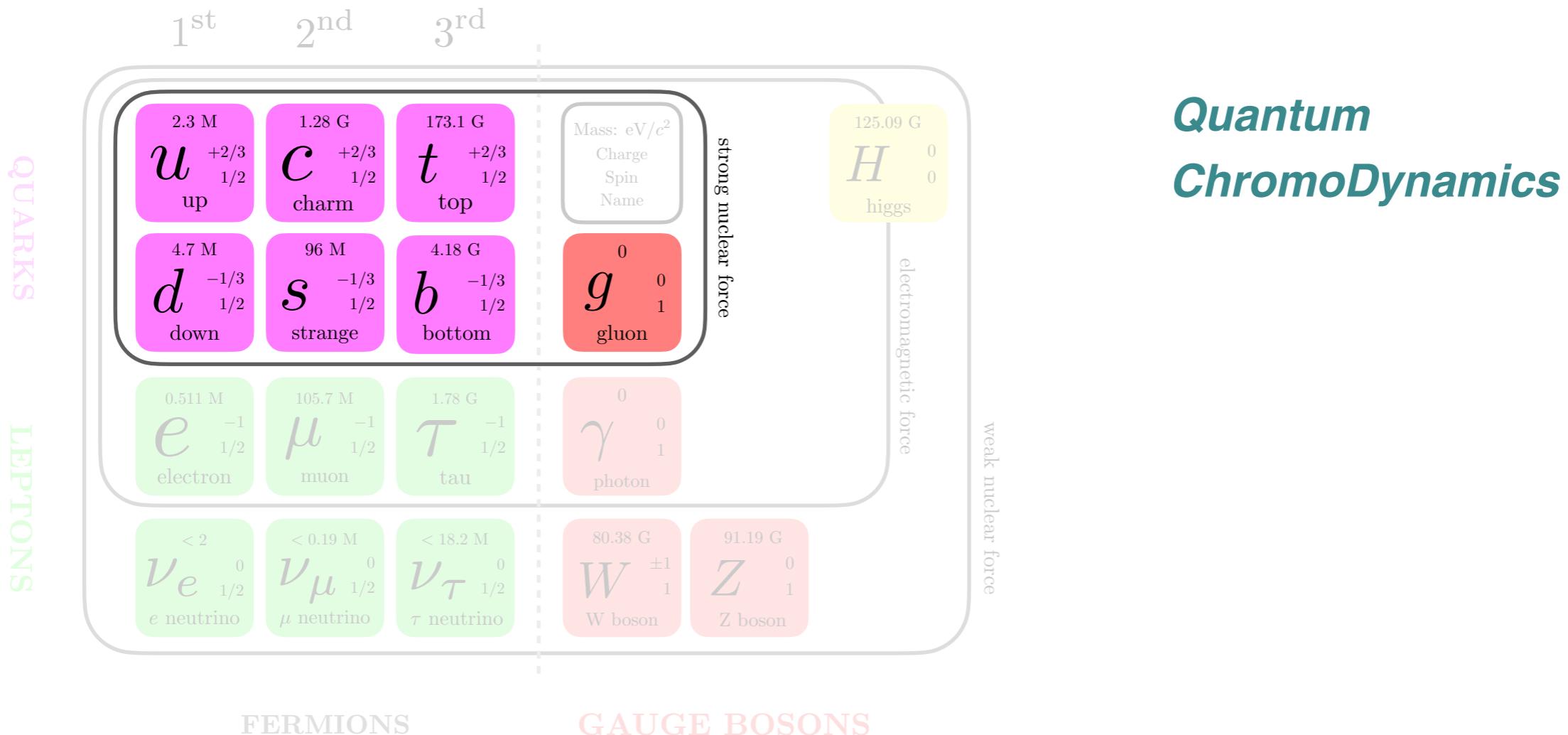
The Standard Model is a remarkably *simple\** theory



\* *simple = An anomaly-free relativistic quantum gauge field theory, invariant under the gauge group  $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$  which spontaneously breaks via a scalar field to  $SU(3)_C \otimes U(1)_Q$*

# The Standard Model of Particle Physics

The Standard Model is a remarkably *simple\** theory



\* simple = An anomaly-free **relativistic quantum gauge field theory**, invariant under the gauge group  $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$  which spontaneously breaks via a scalar field to  $SU(3)_C \otimes U(1)_Q$

# Quantum ChromoDynamics (QCD)

---

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

*Quantum Chromodynamics*

# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i \not{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

*Quantum Chromodynamics*

 *quark fields*

$$\psi_f = \begin{pmatrix} \psi_f \\ \psi_f \\ \psi_f \end{pmatrix}$$

$$f = u, d, s, c, b, t$$

# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i \not{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

*Quantum Chromodynamics*

*quark fields*

$$\psi_f = \begin{pmatrix} \psi_f \\ \psi_f \\ \psi_f \end{pmatrix}$$

$$f = u, d, s, c, b, t$$

*gluon fields*

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig_s [\mathbf{A}_\mu, \mathbf{A}_\nu]$$

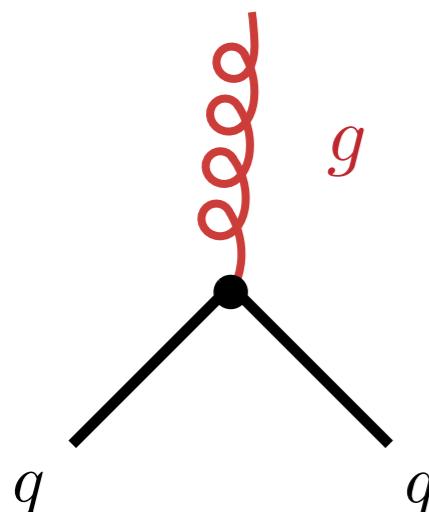
$$\mathbf{D}_\mu = \partial_\mu + ig_s \mathbf{A}_\mu$$

# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

*Quantum Chromodynamics*



*strong coupling*

$$\alpha_s = \frac{g_s^2}{4\pi}$$

*quark fields*

$$\psi_f = \begin{pmatrix} \psi_f \\ \psi_f \\ \psi_f \end{pmatrix}$$

$$f = u, d, s, c, b, t$$

*gluon fields*

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig_s [\mathbf{A}_\mu, \mathbf{A}_\nu]$$

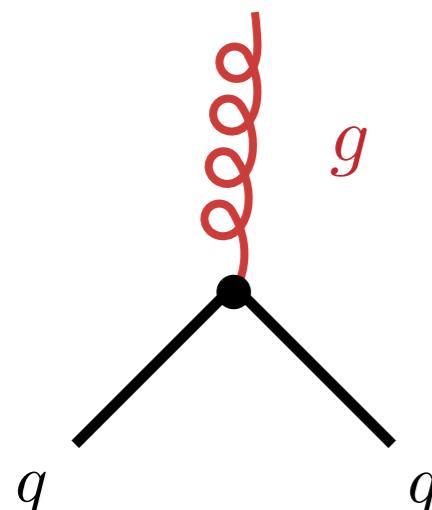
$$\mathbf{D}_\mu = \partial_\mu + ig_s \mathbf{A}_\mu$$

# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

Quantum Chromodynamics

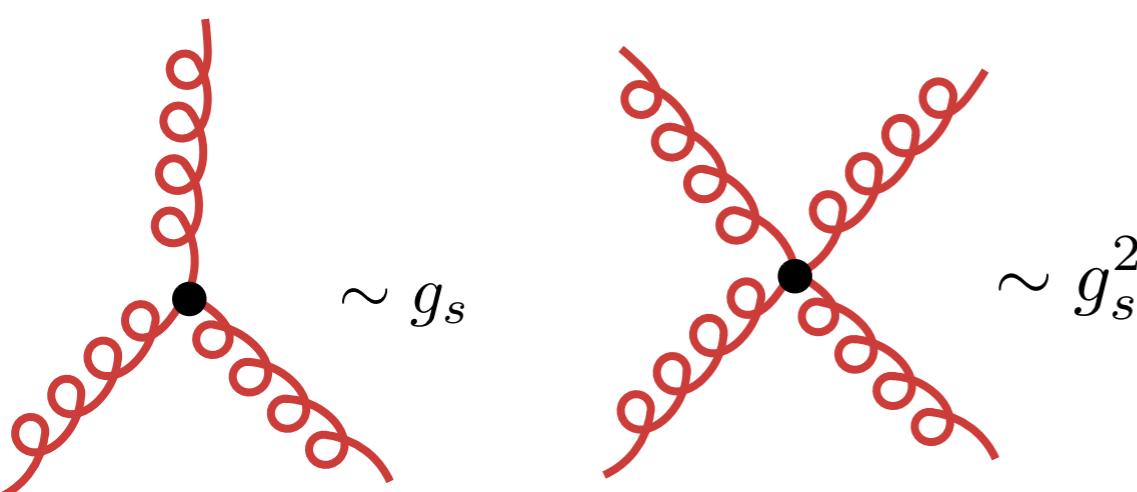


strong coupling

$$\alpha_s = \frac{g_s^2}{4\pi}$$

$$\psi_f = \begin{pmatrix} \psi_f \\ \psi_f \\ \psi_f \end{pmatrix}$$

$$f = u, d, s, c, b, t$$



gluon fields

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig_s [\mathbf{A}_\mu, \mathbf{A}_\nu]$$

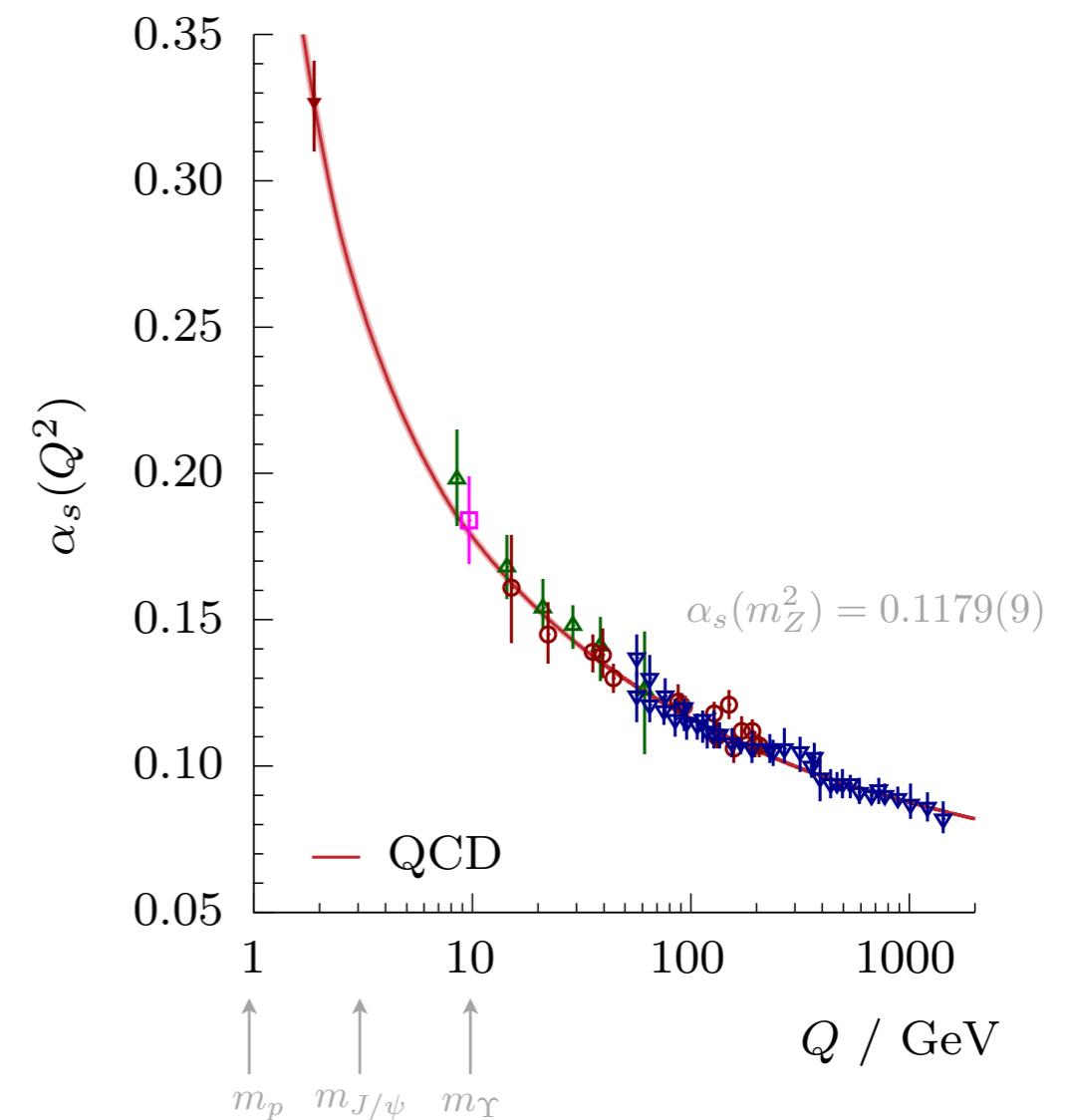
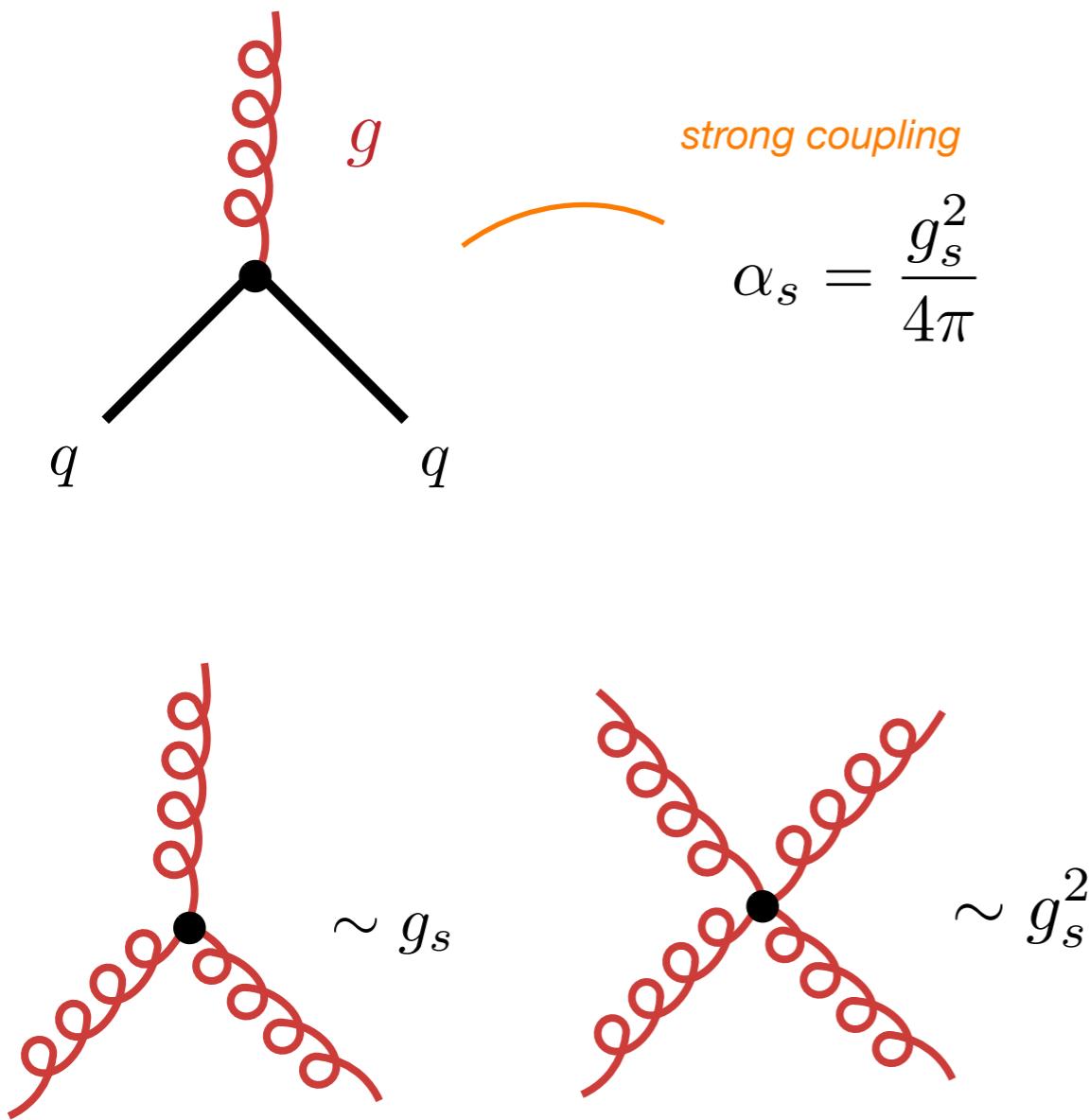
$$\mathbf{D}_\mu = \partial_\mu + ig_s \mathbf{A}_\mu$$

# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

*Quantum Chromodynamics*

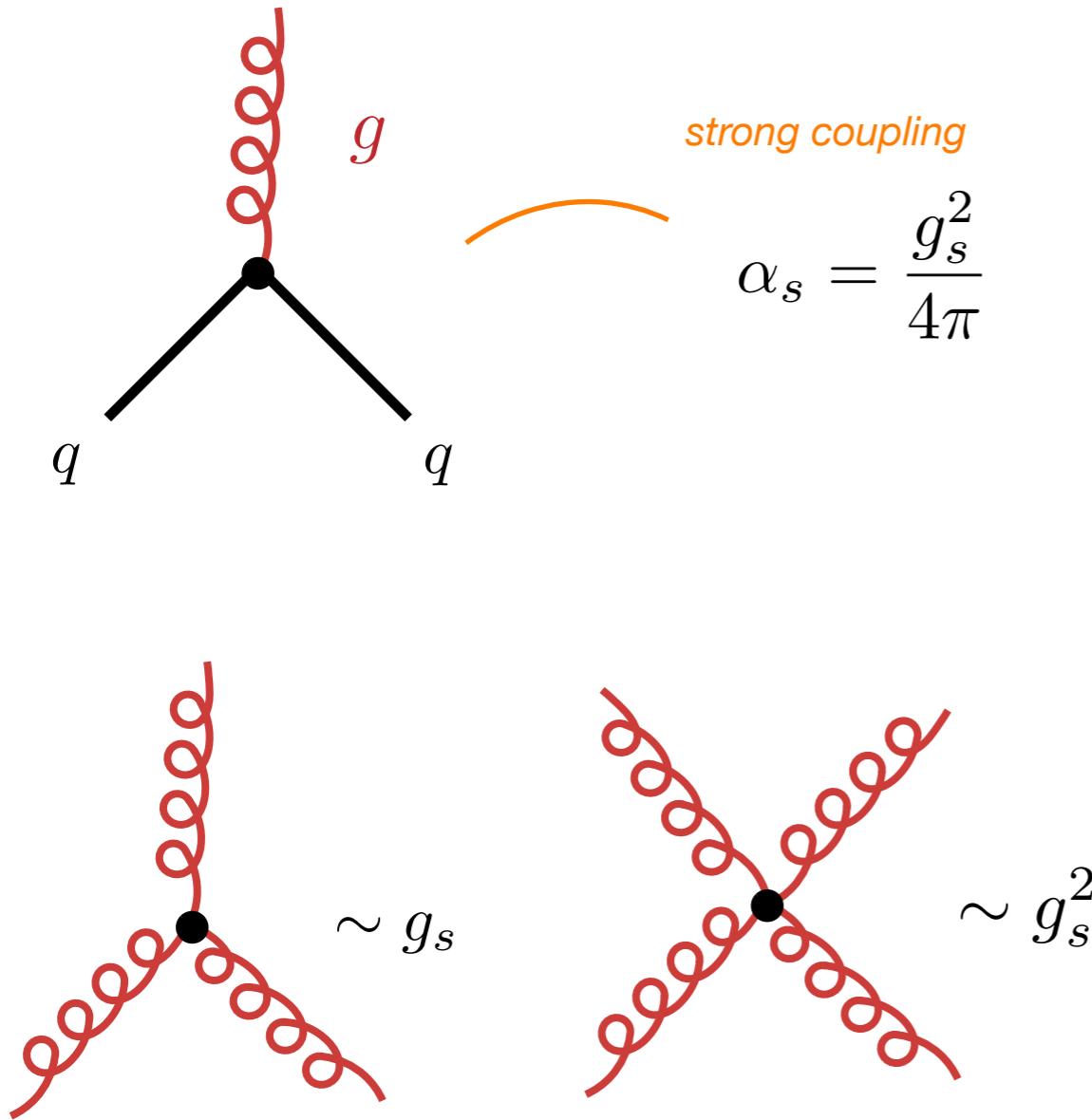


# Quantum ChromoDynamics (QCD)

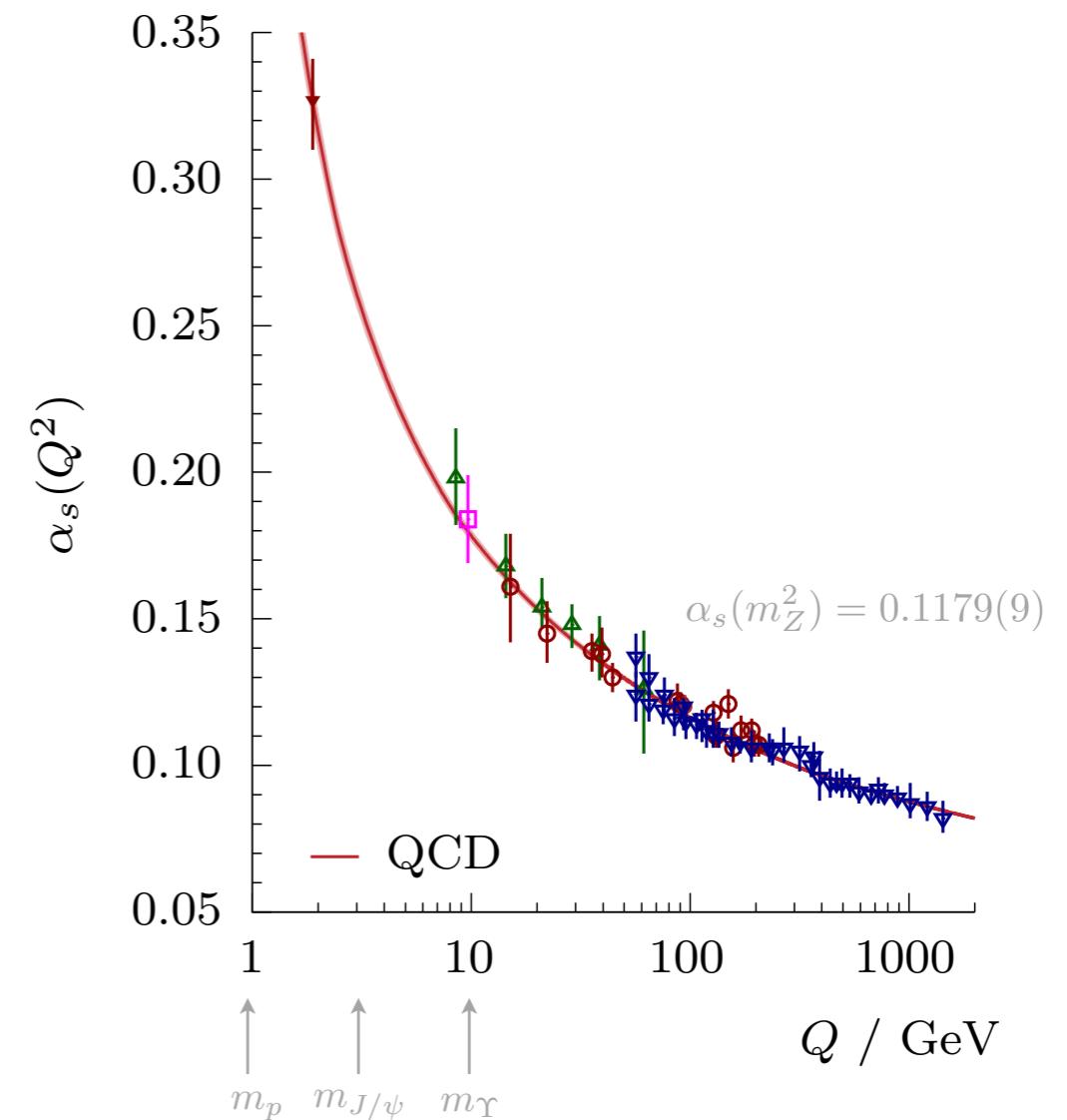
Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

*Quantum Chromodynamics*



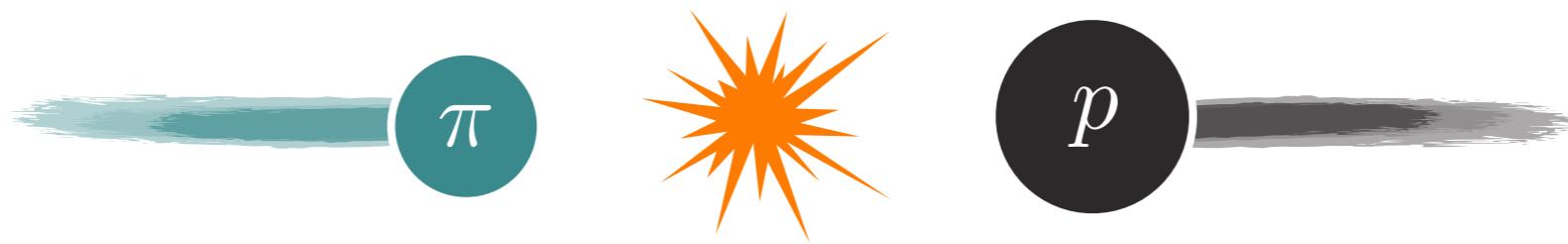
***Don't observe free quarks, observe composite hadrons***



# Hadron Spectroscopy

The strong interaction spectrum, composed of **hadrons**, contains many interesting features

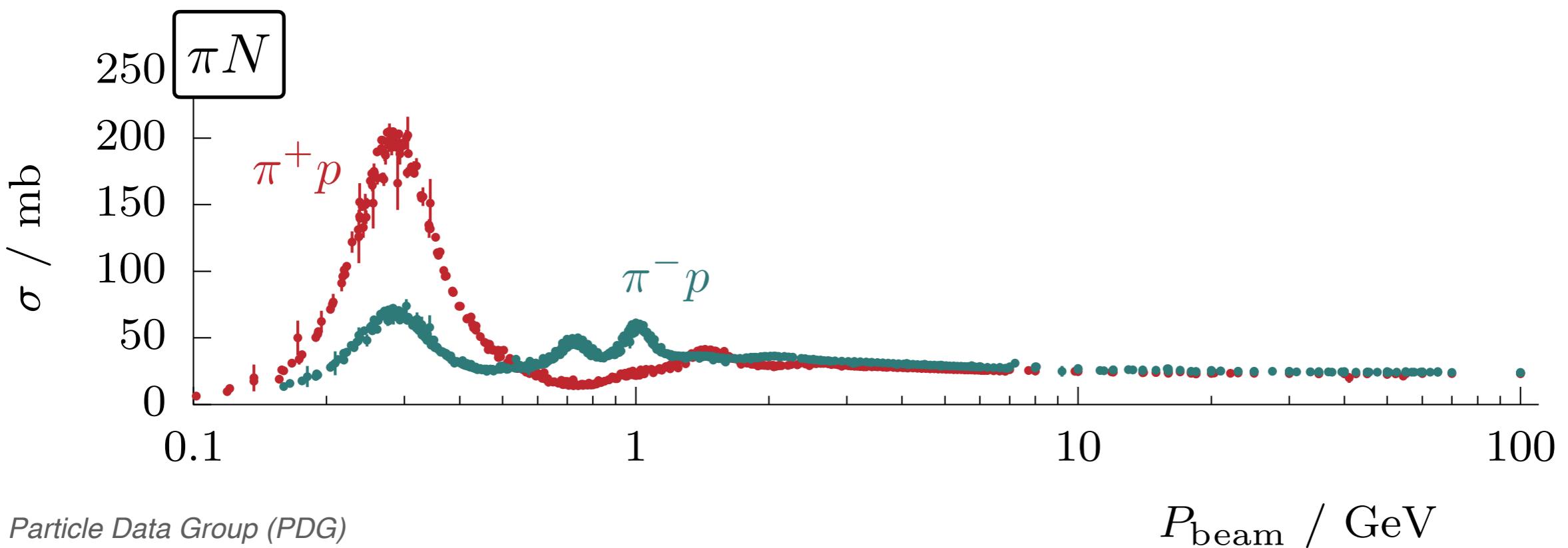
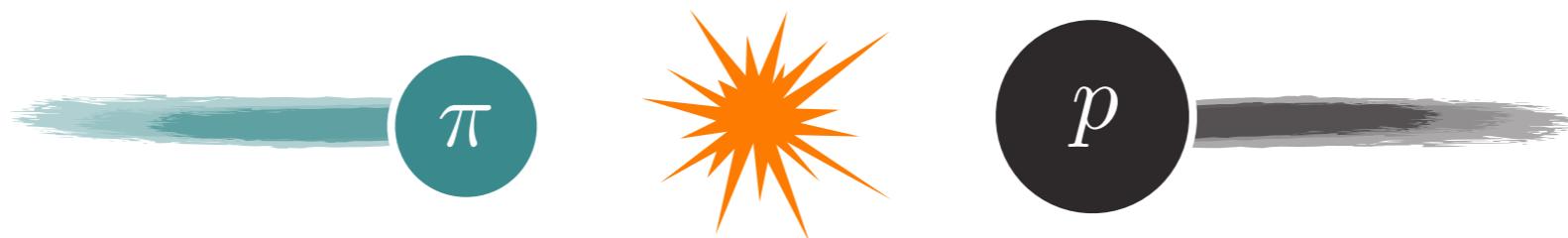
e.g.  $\pi N$  spectrum



# Hadron Spectroscopy

The strong interaction spectrum, composed of **hadrons**, contains many interesting features

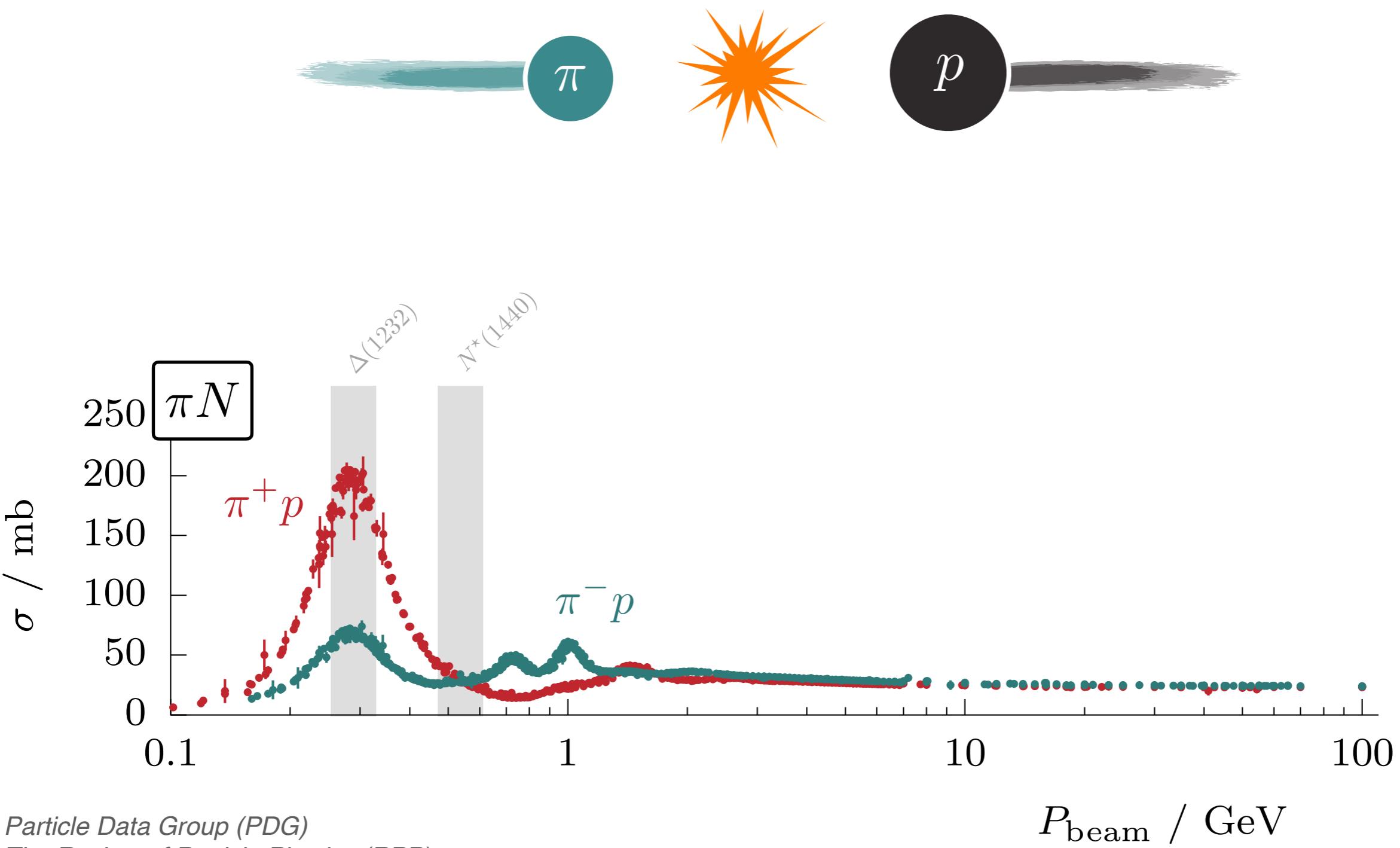
e.g.  $\pi N$  spectrum



# Hadron Spectroscopy

The strong interaction spectrum, composed of **hadrons**, contains many interesting features

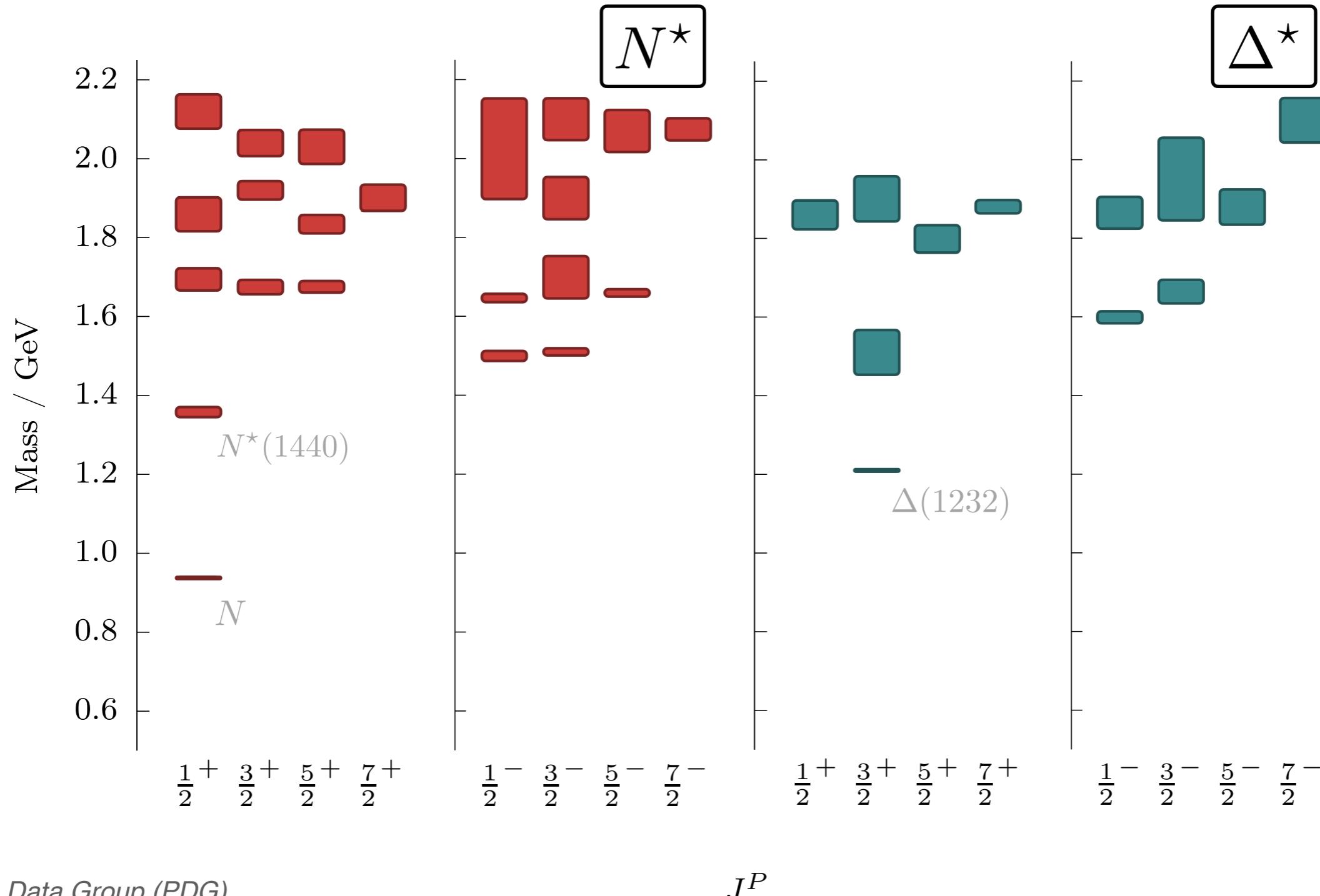
e.g.  $\pi N$  spectrum



# Hadron Spectroscopy

The strong interaction spectrum, composed of **hadrons**, contains many interesting features

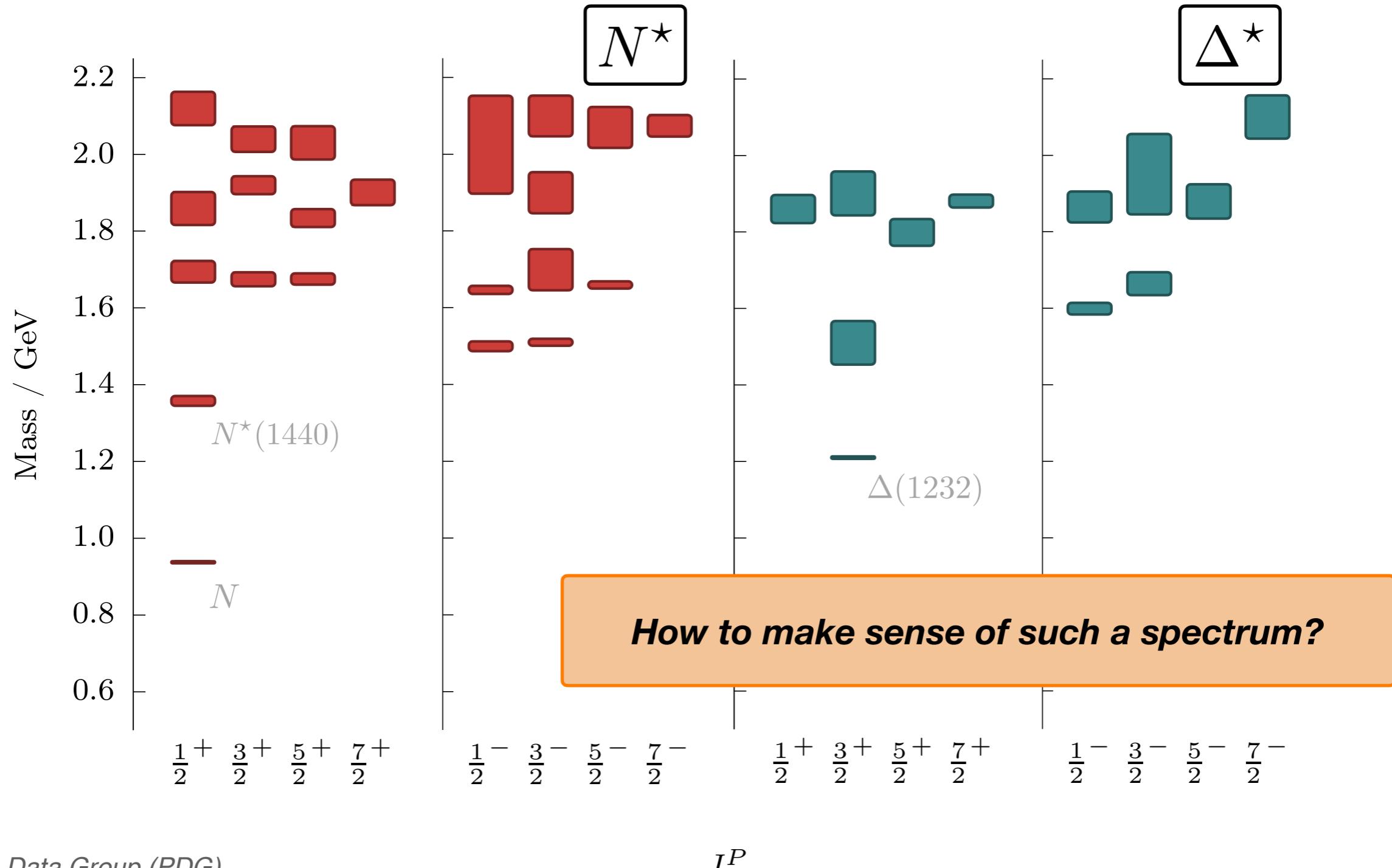
e.g.  $\pi N$  spectrum



# Hadron Spectroscopy

The strong interaction spectrum, composed of **hadrons**, contains many interesting features

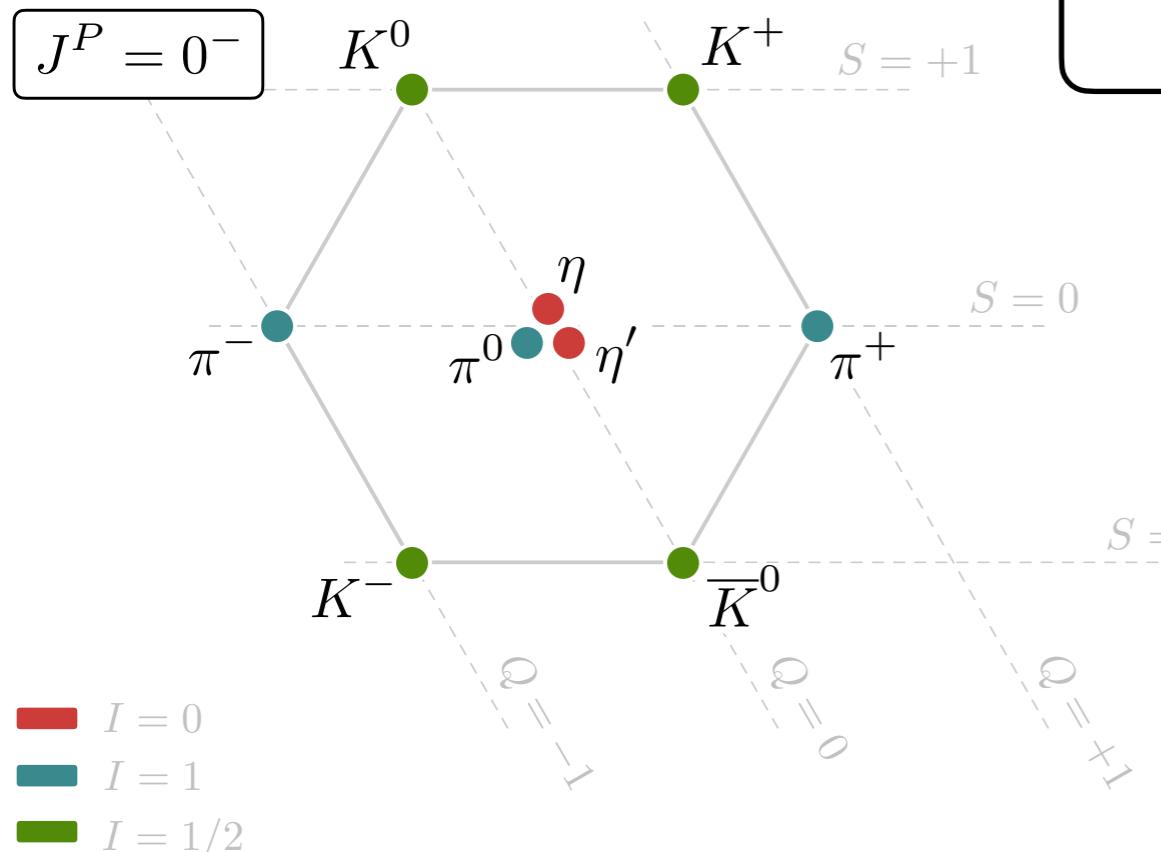
e.g.  $\pi N$  spectrum



# The Hadron Spectrum

Hadrons are classified by their **conserved quantum numbers**

- *spin ( $J$ )*
- *parity ( $P$ )*
- *charge-conjugation ( $C$ )*
- *isospin ( $I$ )*
- *strangeness ( $S$ )*
- ...



**Hadrons**

**Mesons**

$$J = 0, 1, 2, \dots$$

$$I \leq 1$$

$$|S| \leq 1$$

$$\pi, K, \eta, \dots$$

**Baryons**

$$J = 1/2, 3/2, 5/2, \dots$$

$$I \leq 3/2$$

$$|S| \leq 3$$

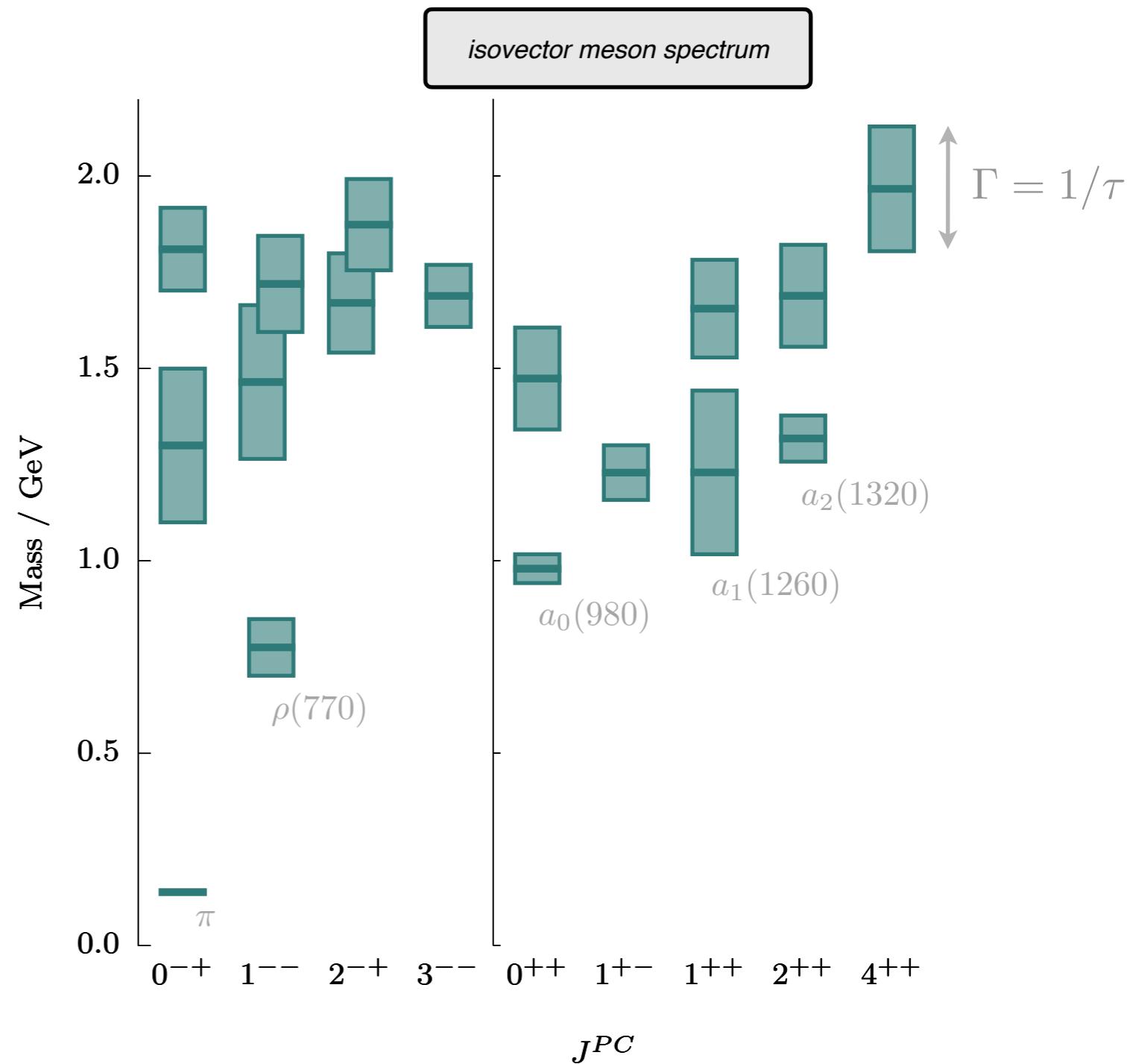
$$p, n, \Delta, \dots$$

# The Hadron Spectrum

Hadrons are classified by their **conserved quantum numbers**

- Introduce quarks as building blocks: **constituent quark model**

e.g. light isovector mesons

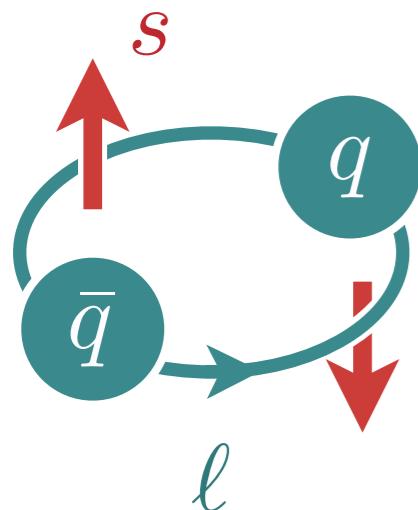


# The Hadron Spectrum

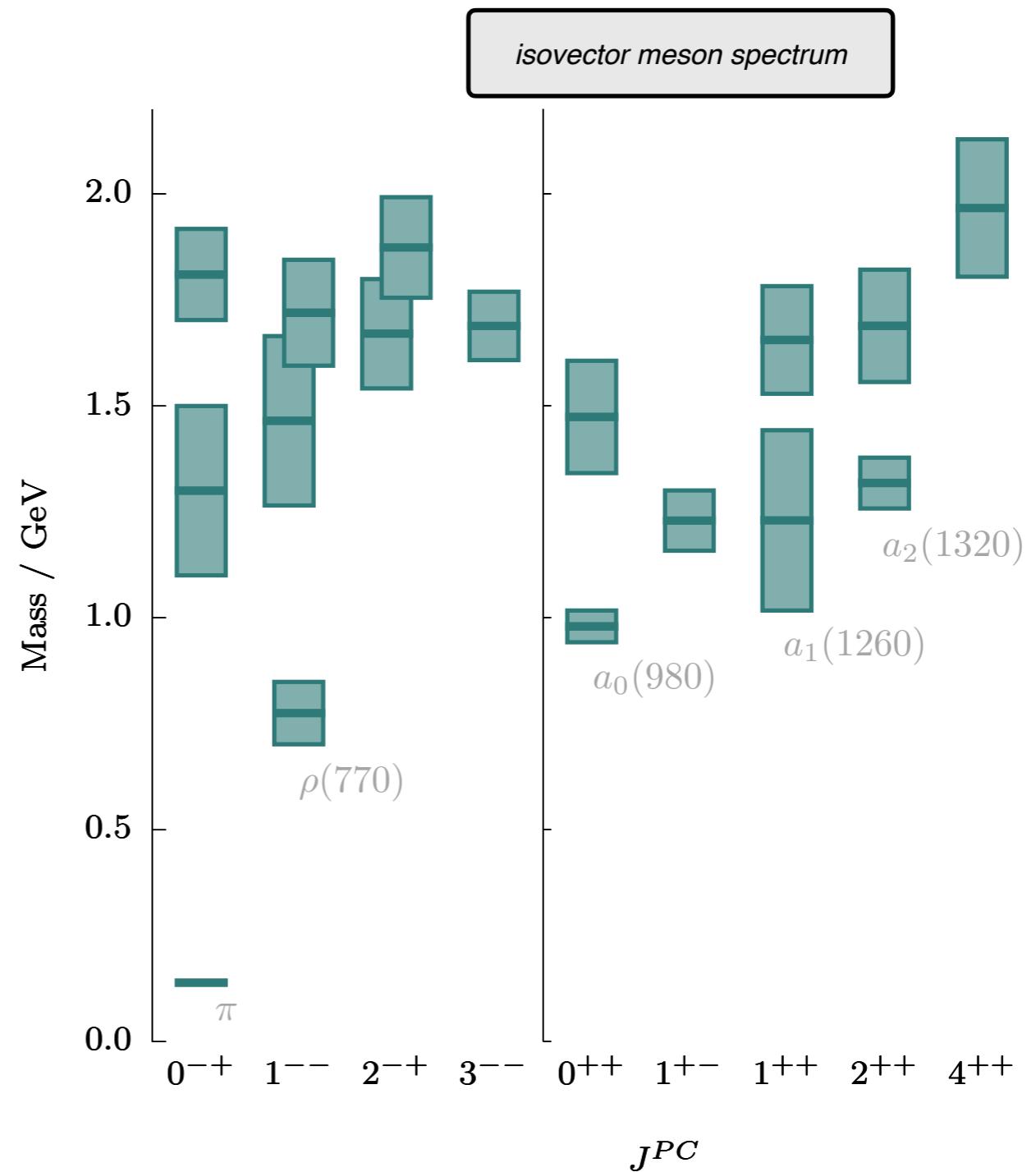
Hadrons are classified by their **conserved quantum numbers**

- Introduce quarks as building blocks: **constituent quark model**

e.g. light isovector mesons



$$q\bar{q} \quad (n^{2s+1}\ell_J)$$

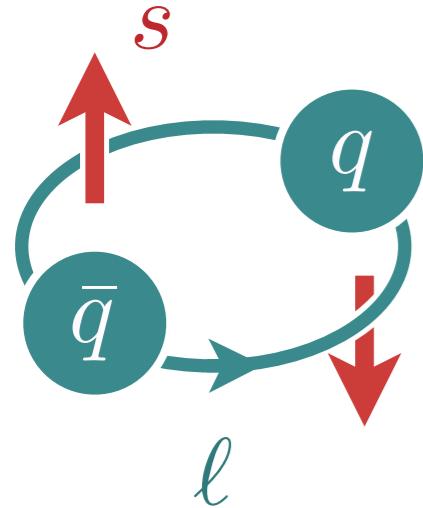


# The Hadron Spectrum

Hadrons are classified by their **conserved quantum numbers**

- Introduce quarks as building blocks: **constituent quark model**

e.g. light isovector mesons



$q\bar{q}$  ( $n^{2s+1}\ell_J$ )

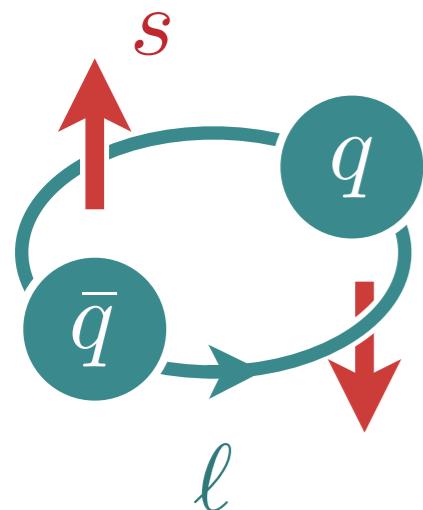
$$|n^{2s+1}\ell_J, m_J\rangle_{q\bar{q}} = \sum_{m_\ell, m_s} \langle \ell m_\ell; sm_s | J m_J \rangle \sum_{\sigma, \bar{\sigma}} \langle \frac{1}{2}\sigma; \frac{1}{2}\bar{\sigma} | sm_s \rangle \\ \times \int \frac{d^3 k}{(2\pi)^3} \varphi_{n,\ell}(k) Y_{\ell m_\ell}(\hat{k}) |q_\sigma(\mathbf{k}) \bar{q}_{\bar{\sigma}}(-\mathbf{k})\rangle$$

# The Hadron Spectrum

Hadrons are classified by their **conserved quantum numbers**

- Introduce quarks as building blocks: **constituent quark model**

e.g. light isovector mesons



$q\bar{q}$  ( $n^{2s+1}\ell_J$ )

$$|n^{2s+1}\ell_J, m_J\rangle_{q\bar{q}} = \sum_{m_\ell, m_s} \langle \ell m_\ell; sm_s | J m_J \rangle \sum_{\sigma, \bar{\sigma}} \langle \frac{1}{2}\sigma; \frac{1}{2}\bar{\sigma} | sm_s \rangle \\ \times \int \frac{d^3k}{(2\pi)^3} \varphi_{n,\ell}(k) Y_{\ell m_\ell}(\hat{k}) |q_\sigma(k)\bar{q}_{\bar{\sigma}}(-k)\rangle$$

Quantum Numbers

$$\ell = 0, 1, 2, \dots$$

$$s = \begin{cases} 0 & \text{if antialigned} \\ 1 & \text{if aligned} \end{cases}$$

$$C = (-1)^{\ell+s}$$

$$|\ell - s| \leq J \leq |\ell + s|$$

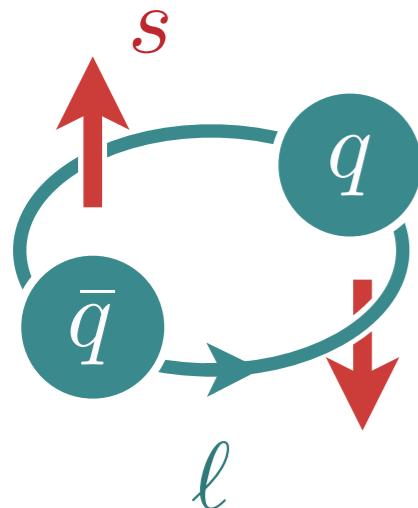
$$P = (-1)^{\ell+1}$$

# The Hadron Spectrum

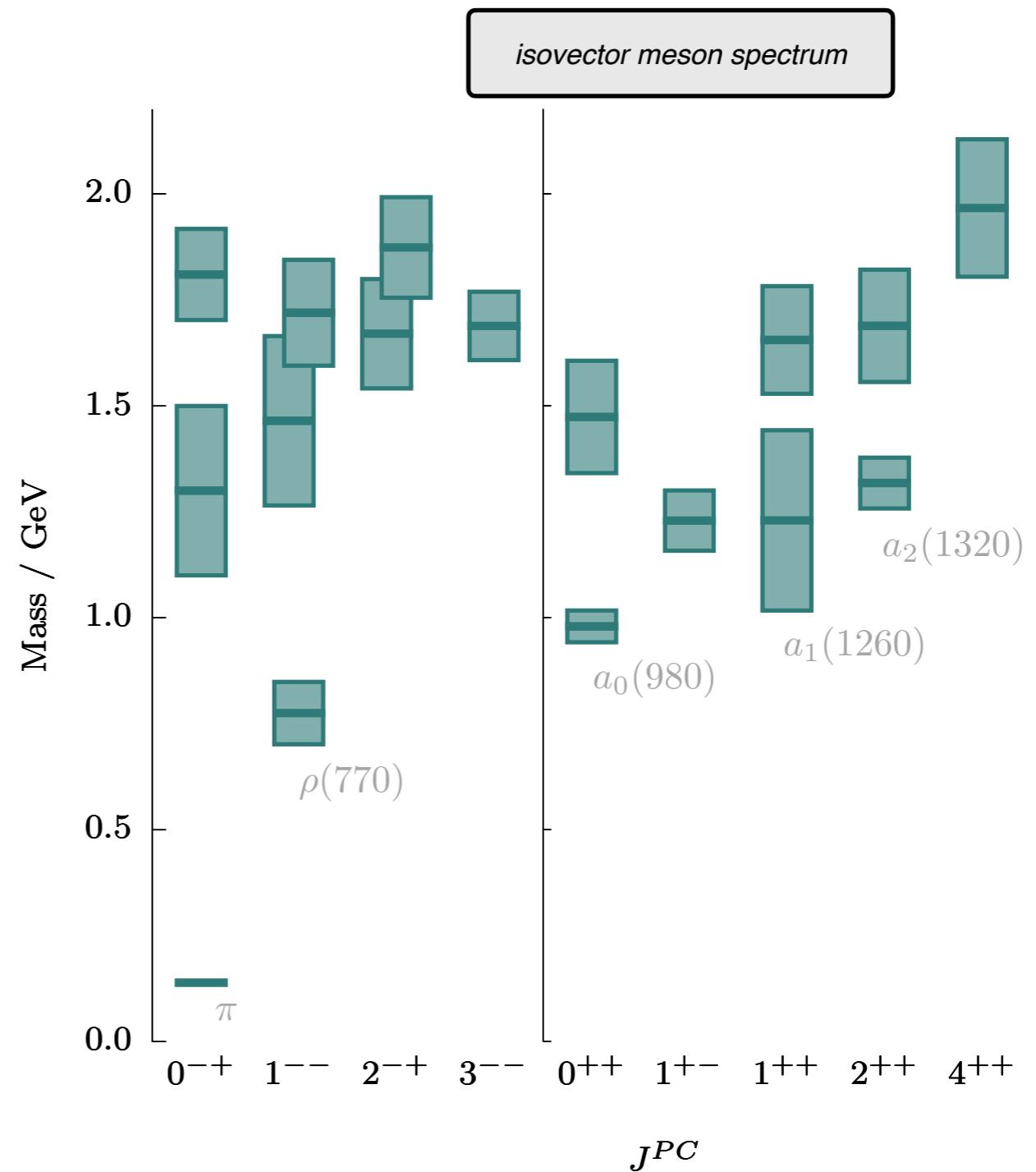
Hadrons are classified by their **conserved quantum numbers**

- Introduce quarks as building blocks: **constituent quark model**

e.g. light isovector mesons



$$q\bar{q} \ (n^{2s+1}\ell_J)$$

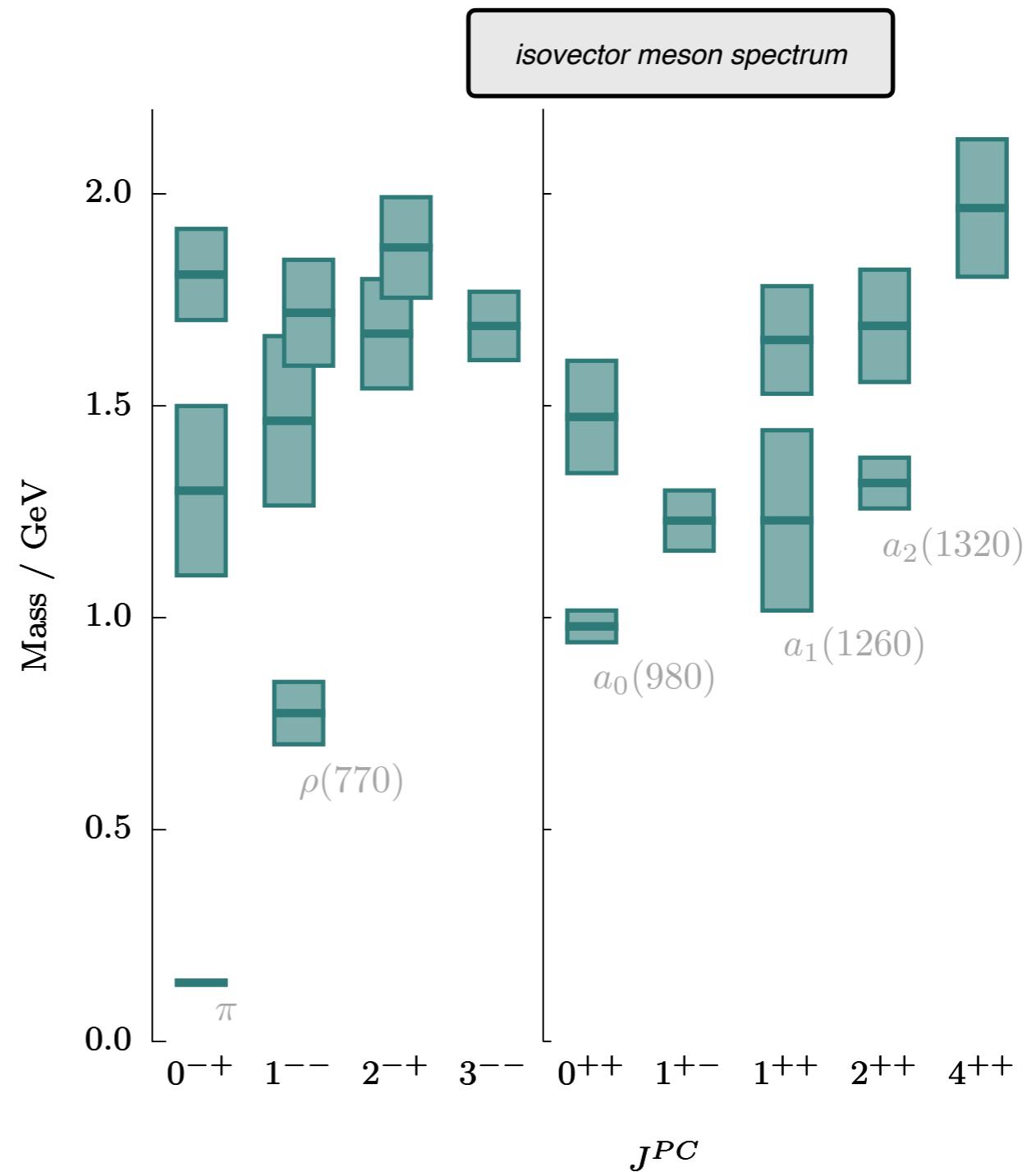
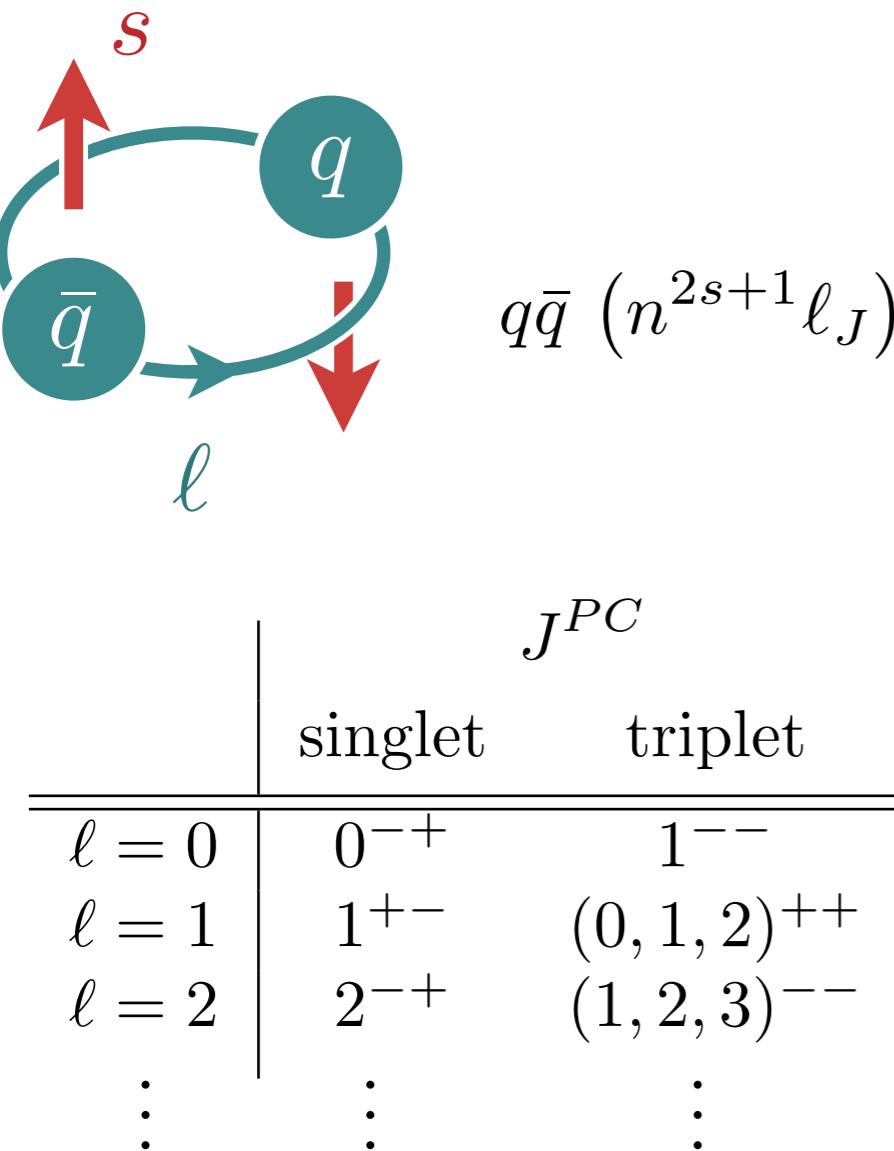


# The Hadron Spectrum

Hadrons are classified by their **conserved quantum numbers**

- Introduce quarks as building blocks: **constituent quark model**

e.g. light isovector mesons

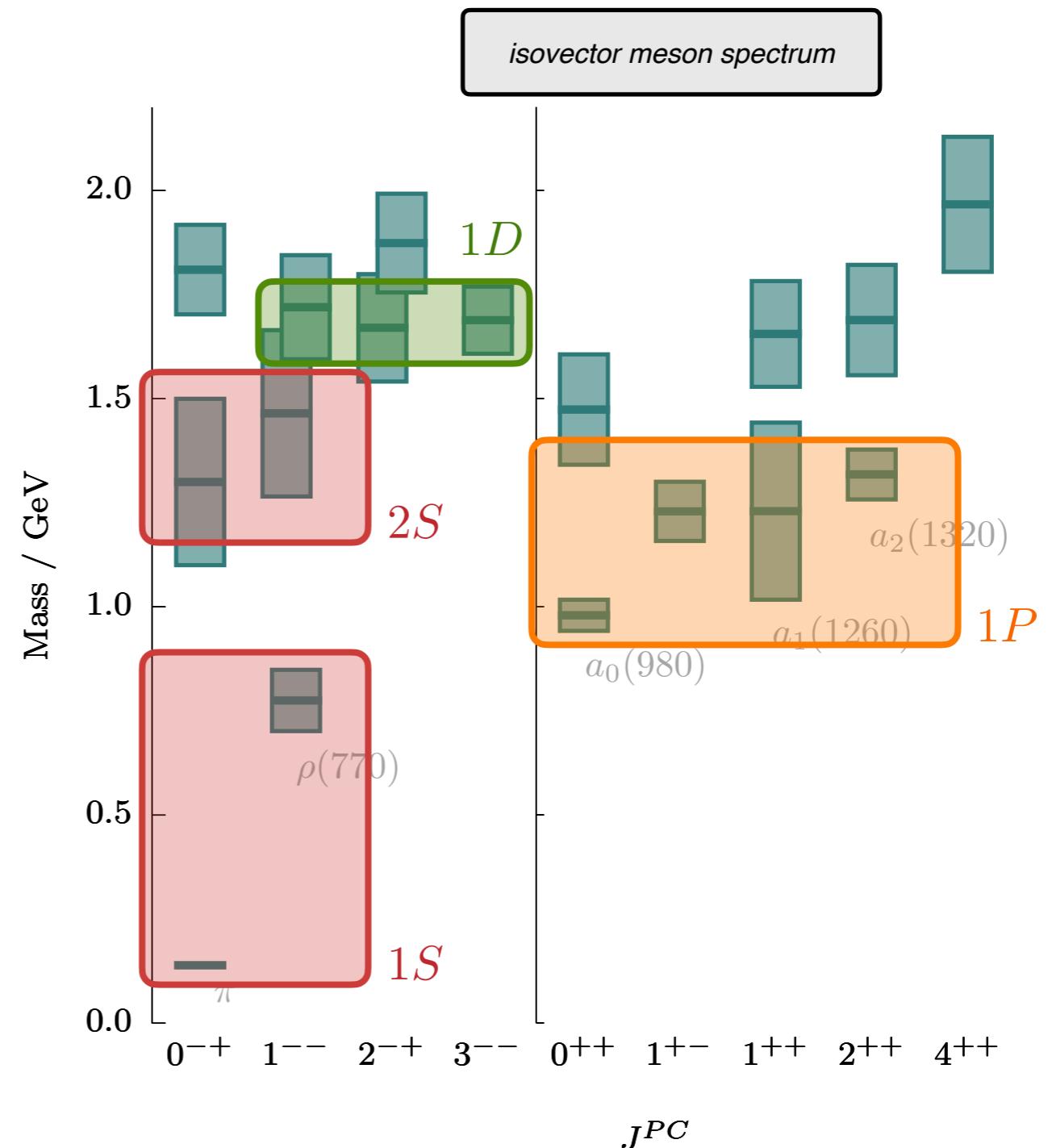
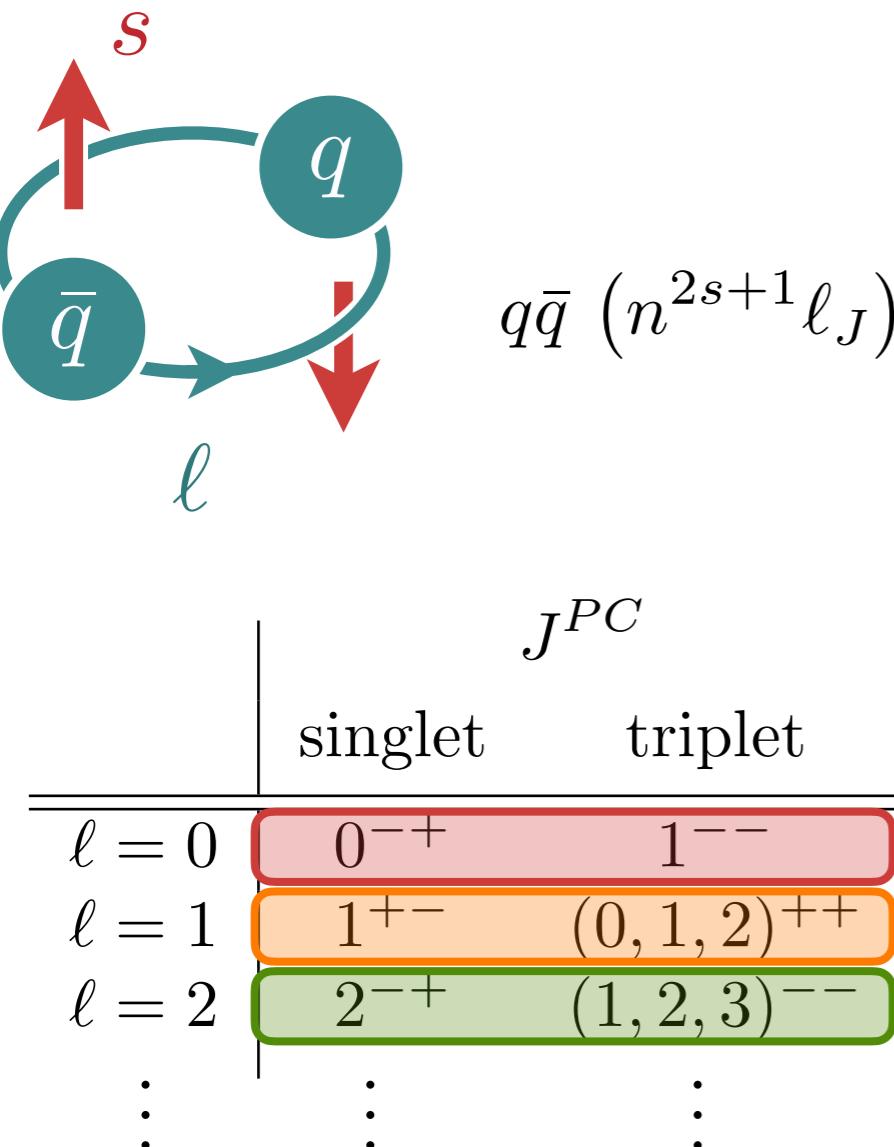


# The Hadron Spectrum

Hadrons are classified by their **conserved quantum numbers**

- Introduce quarks as building blocks: **constituent quark model**

e.g. light isovector mesons

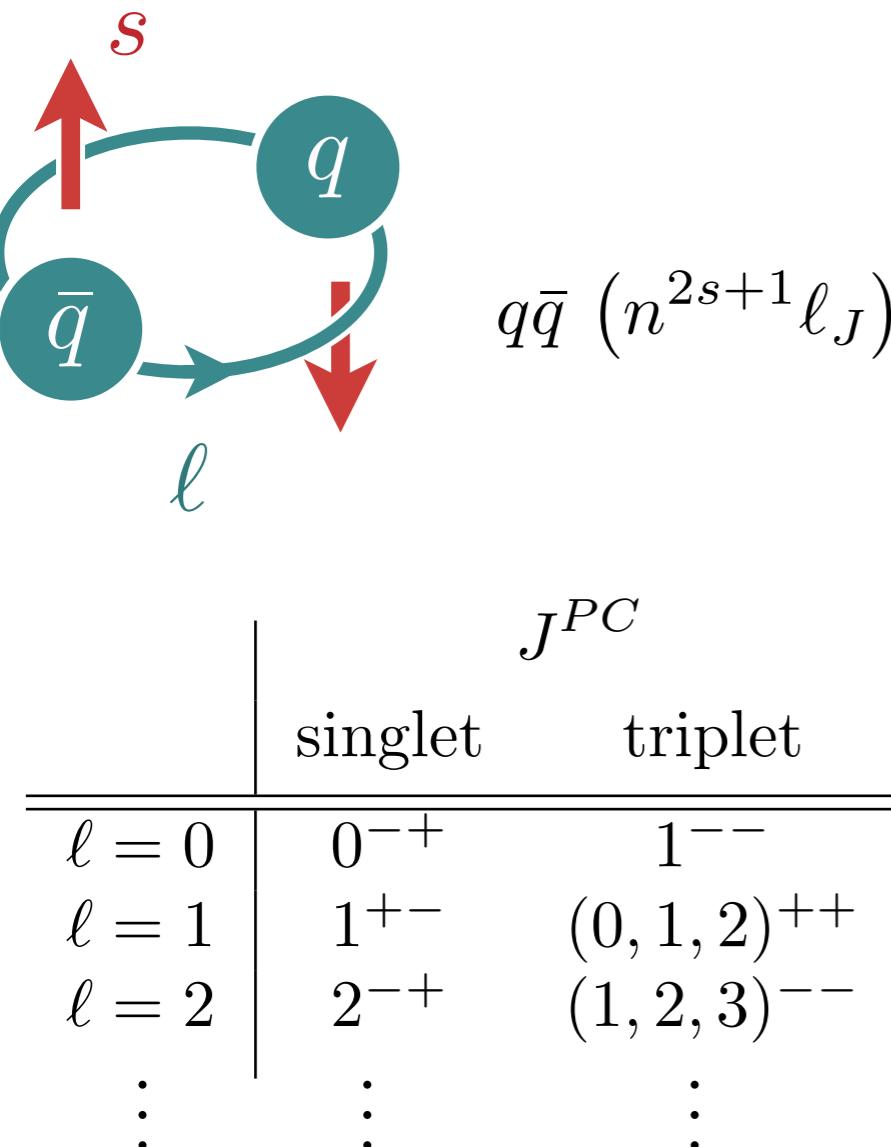


# The Hadron Spectrum

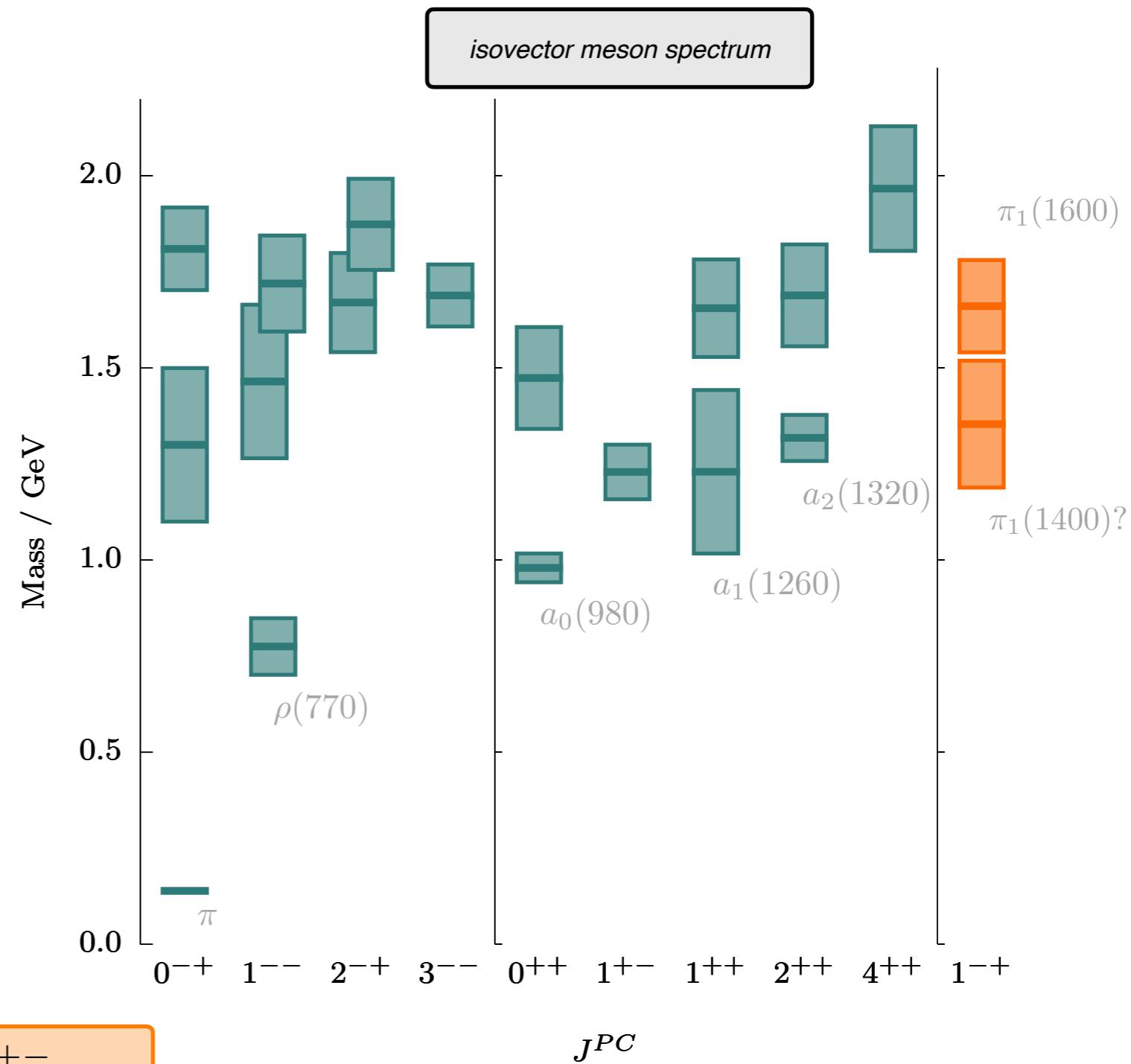
Hadrons are classified by their ***conserved quantum numbers***

- Introduce quarks as building blocks: ***constituent quark model***

e.g. light isovector mesons

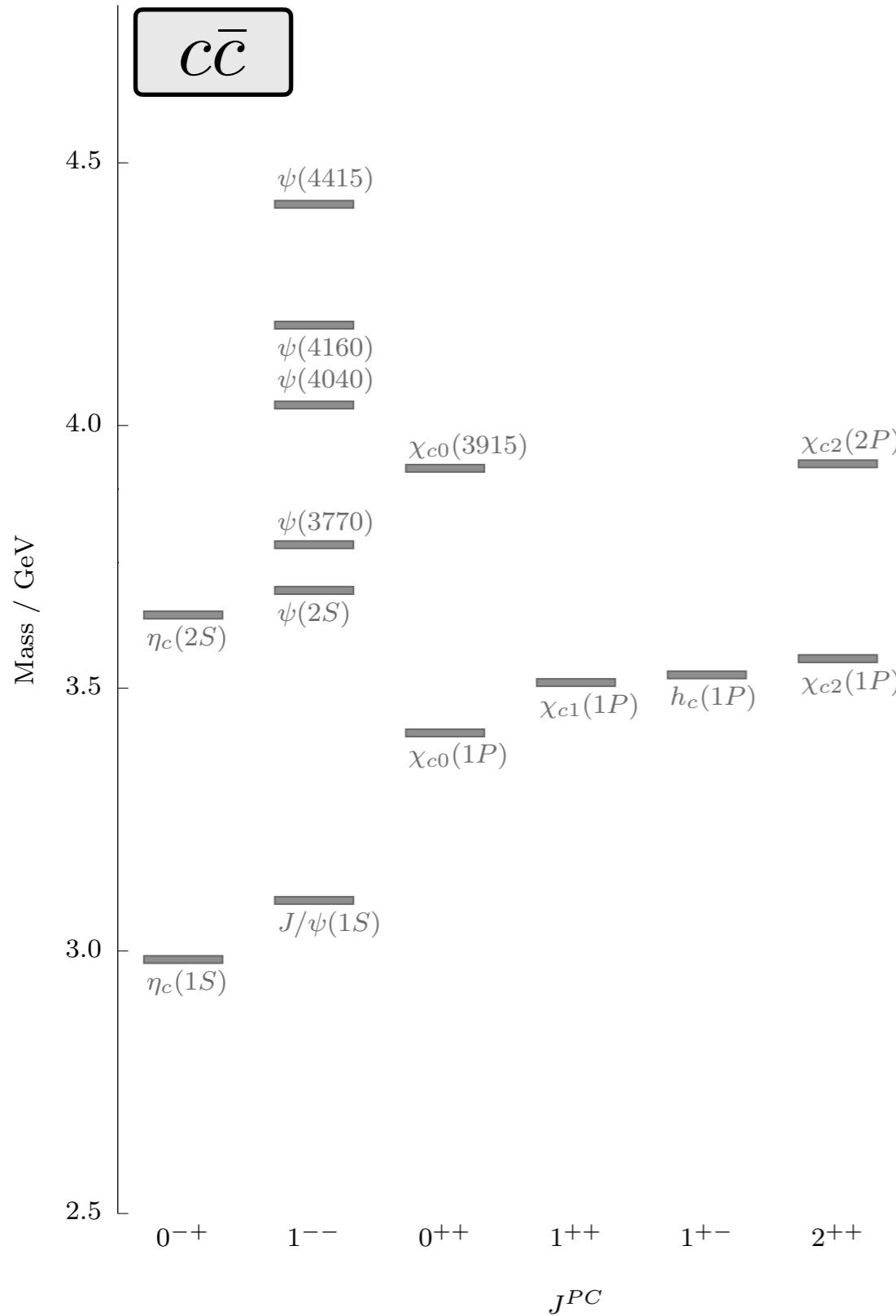


**Forbidden quantum numbers:**  $0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \dots$



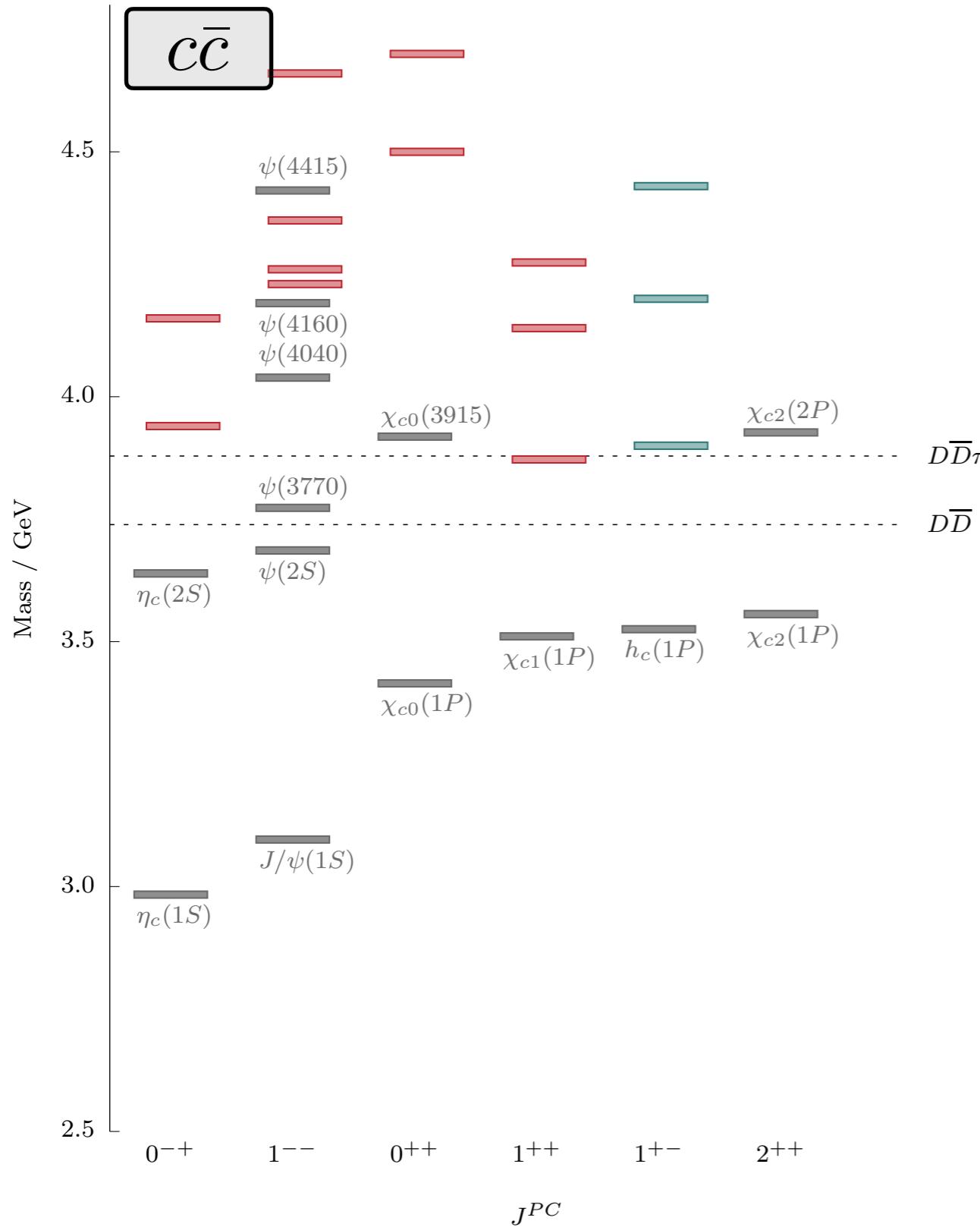
# A New Hadron Spectroscopy

Modern experiments have been finding new states which don't fit the conventional quark models



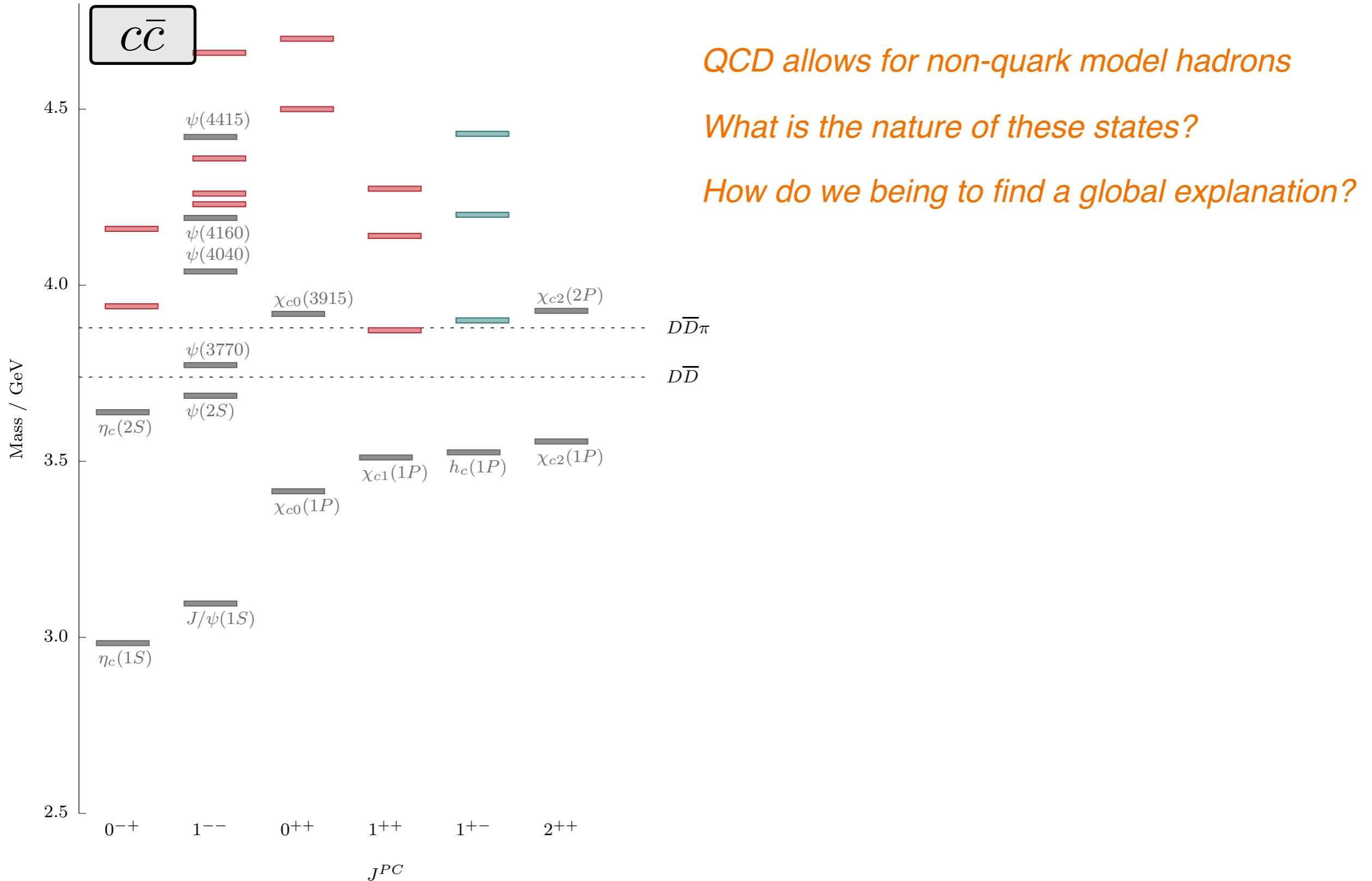
# A New Hadron Spectroscopy

Modern experiments have been finding new states which don't fit the conventional quark models



# A New Hadron Spectroscopy

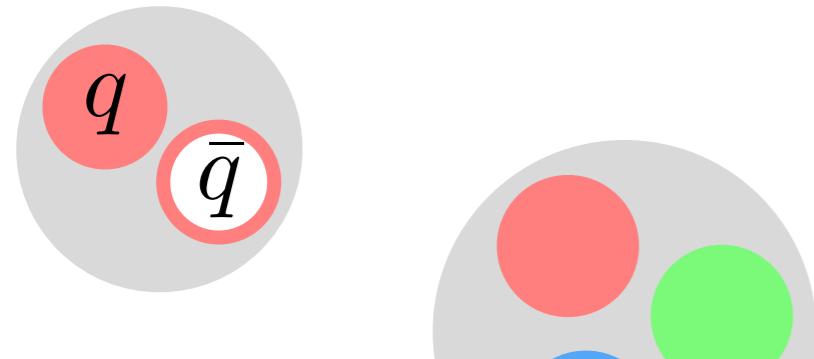
Modern experiments have been finding new states which don't fit the conventional quark models



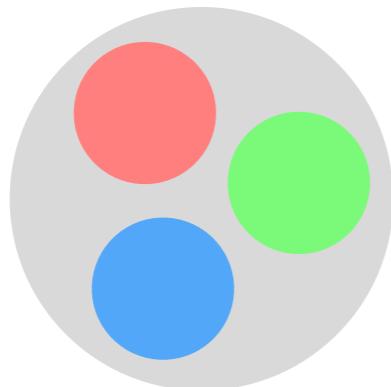
# Hadrons and QCD

---

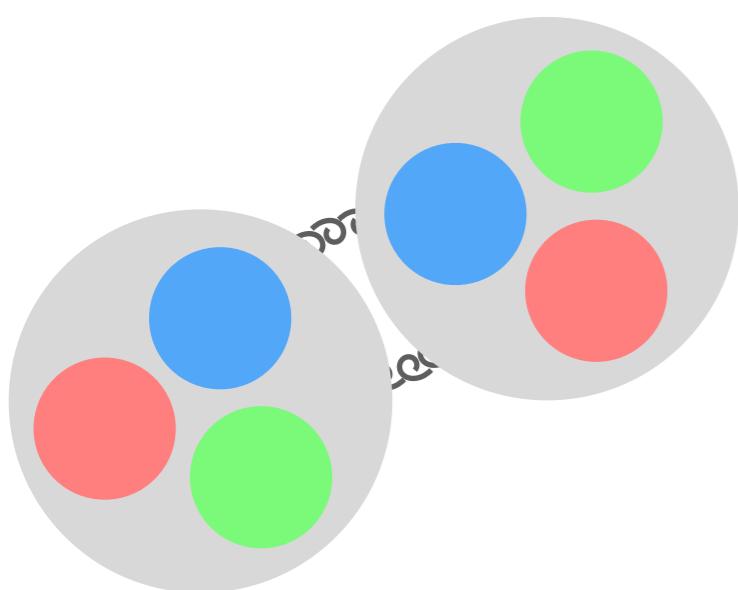
QCD allows for more exotic states – Do we understand QCD?



Meson



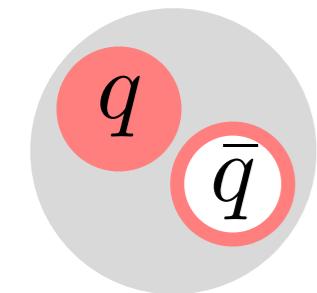
Baryon



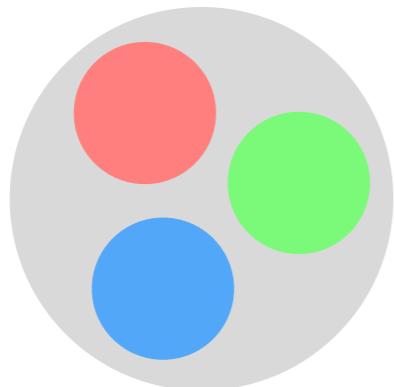
Baryonic Molecule  
(a.k.a Nuclei)

# Hadrons and QCD

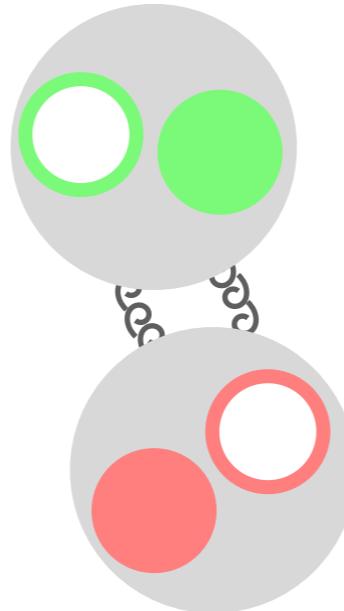
QCD allows for more exotic states – Do we understand QCD?



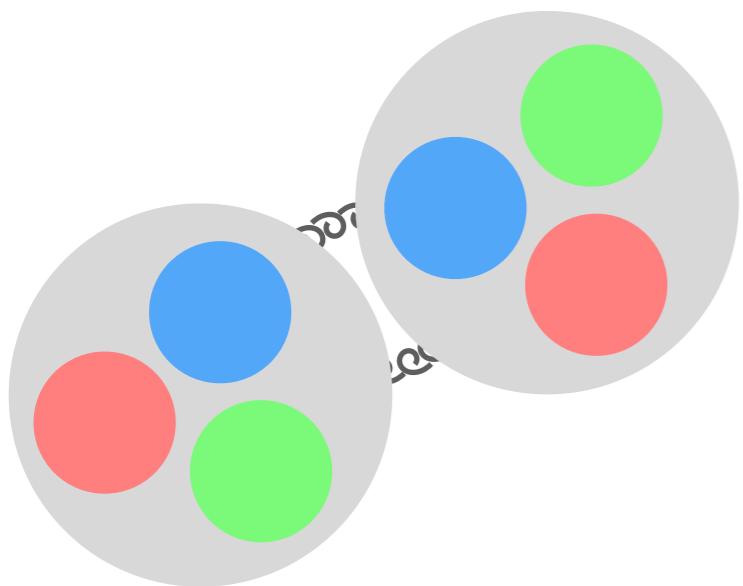
Meson



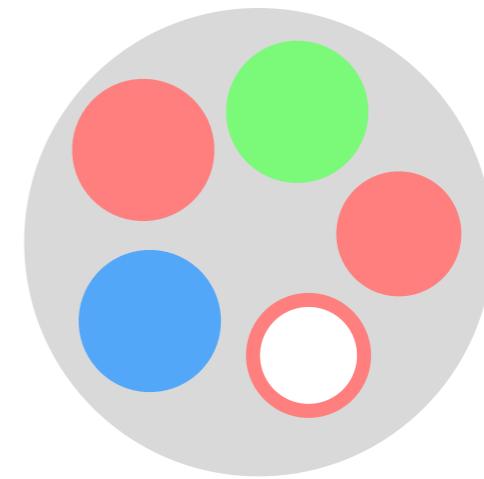
Baryon



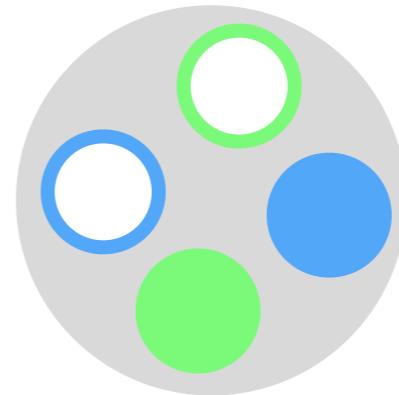
Mesonic Molecule



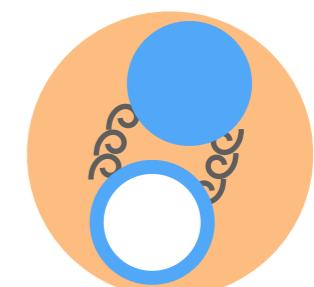
Baryonic Molecule  
(a.k.a Nuclei)



Pentaquark



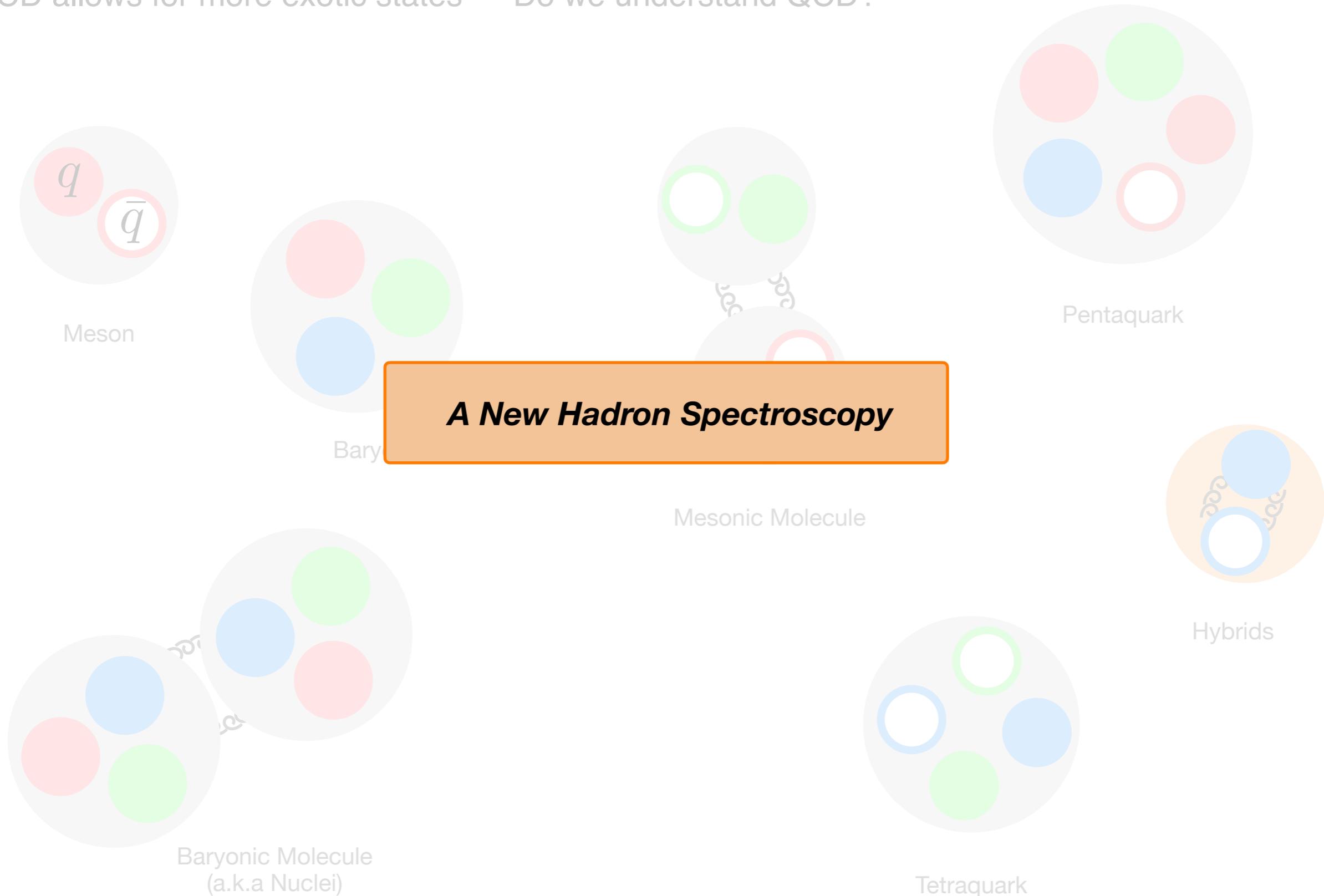
Tetraquark



Hybrids

# Hadrons and QCD

QCD allows for more exotic states — Do we understand QCD?

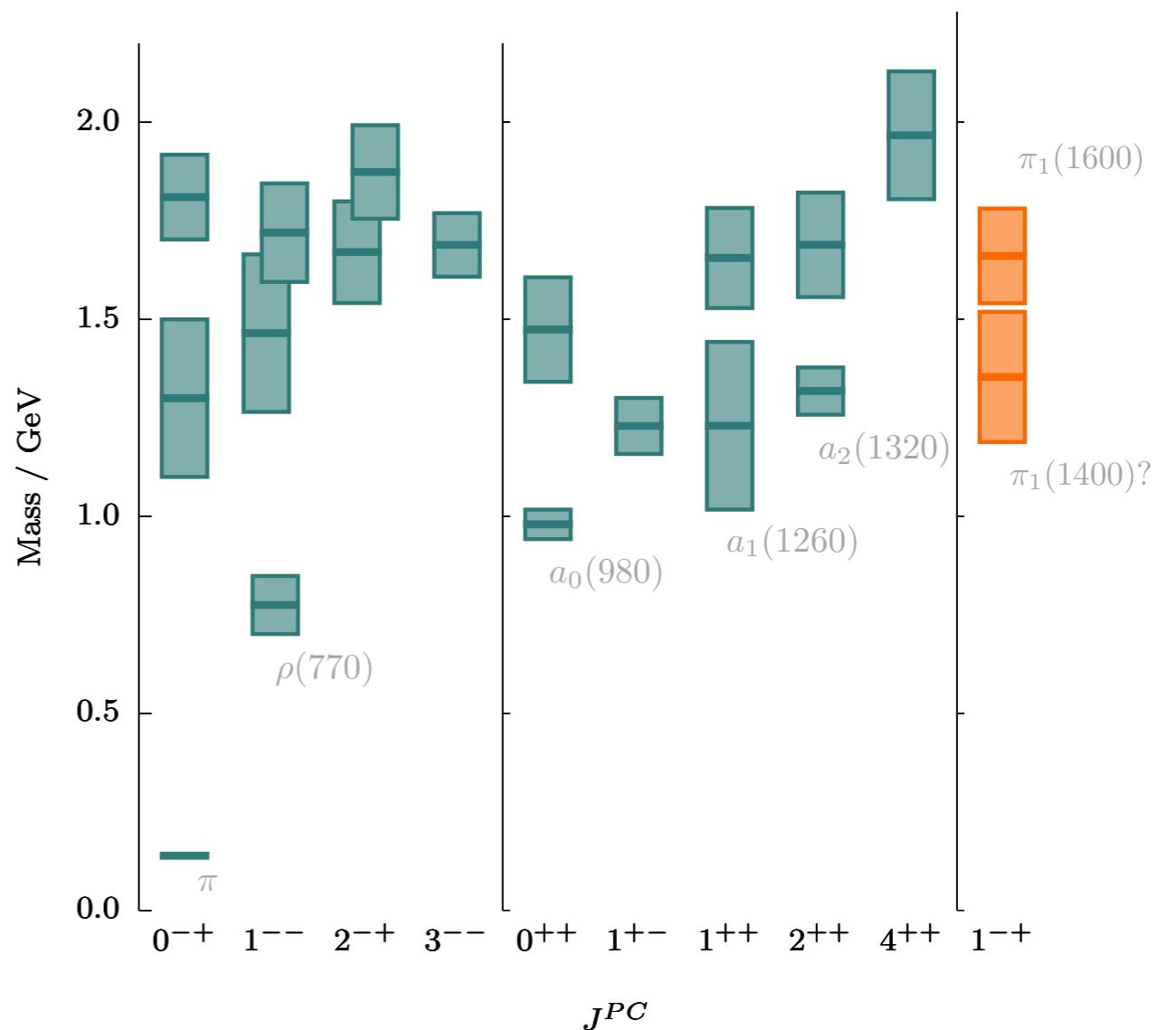
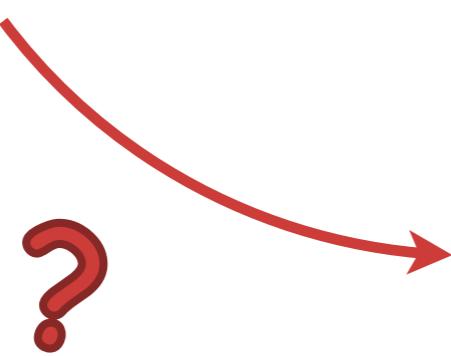


# Hadrons and QCD

QCD allows for more exotic states – Do we understand QCD?

- How to connect QCD to hadrons?
- Need to understand how to quantify what the hadrons are in nature
- Need to find non-perturbative approach to access these hadrons *rigorously*

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i \not{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$



# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

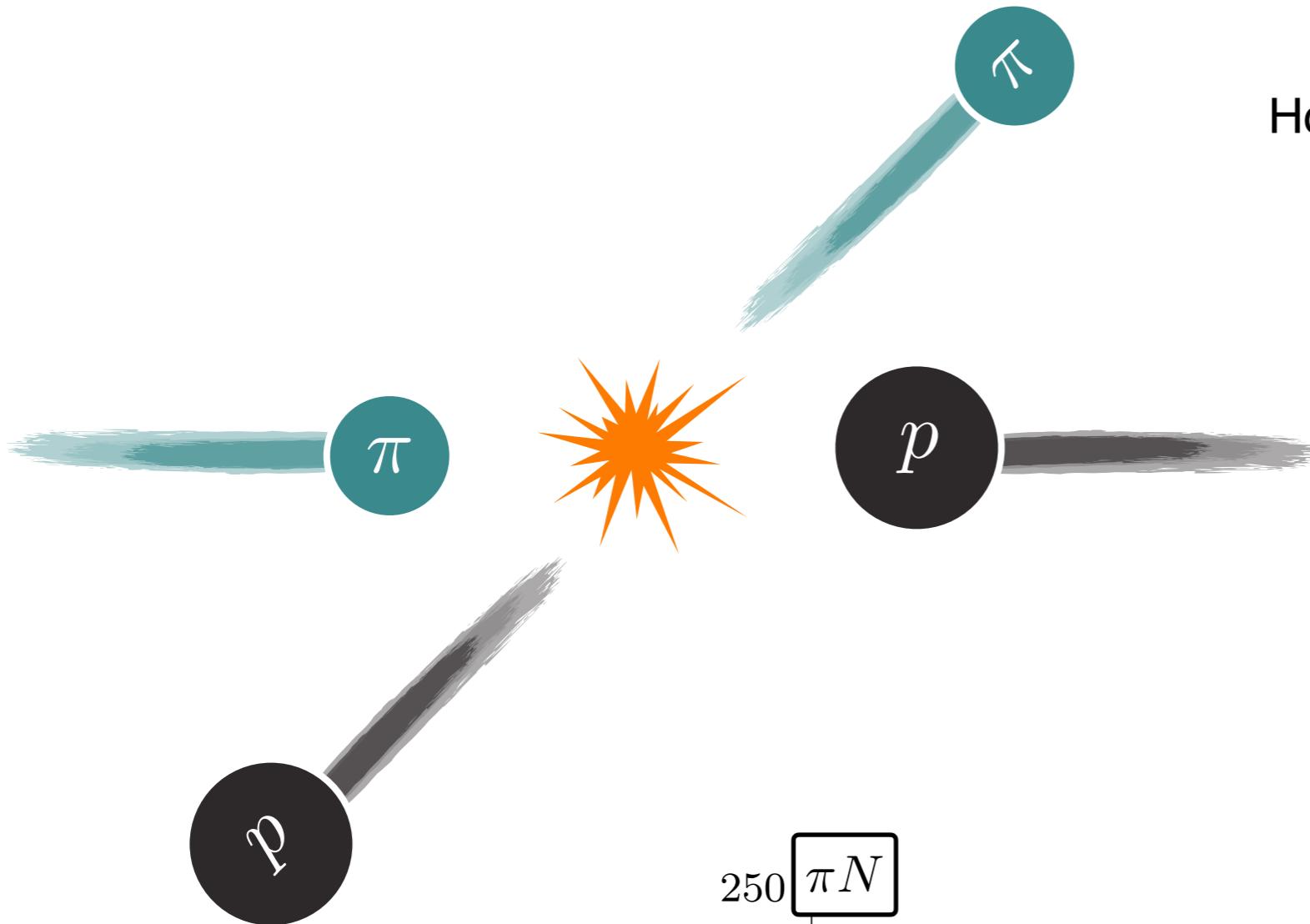
## Lattice QCD & Hadron Spectroscopy

Lattice QCD

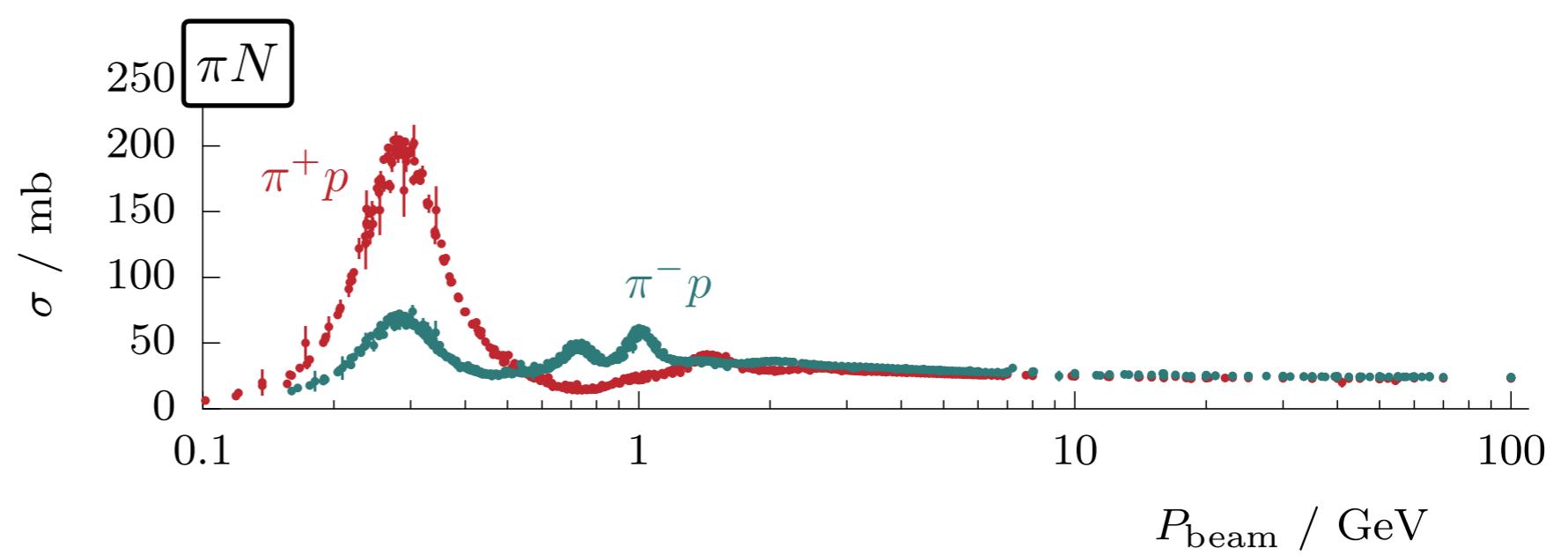
Lüscher & the Finite-Volume

# Accessing the Hadron Spectrum

All our knowledge of the hadrons comes from *scattering experiments*



How to quantify such a process?



# Accessing the Hadron Spectrum

All our knowledge of the hadrons comes from **scattering experiments**

- In quantum mechanics, want matrix elements of scattering operator — the **S-matrix**

$$S_{\alpha\beta}(E) = \langle \beta, \text{out} | S | \alpha, \text{in} \rangle$$

Outgoing particles  
Incoming particles  
Probability amplitude

$$\text{Prob}(\alpha \rightarrow \beta) = |S_{\alpha\beta}|^2$$

$$S = T \exp \left( -i \int_{-\infty}^{\infty} dt V(t) \right)$$

Interaction potential

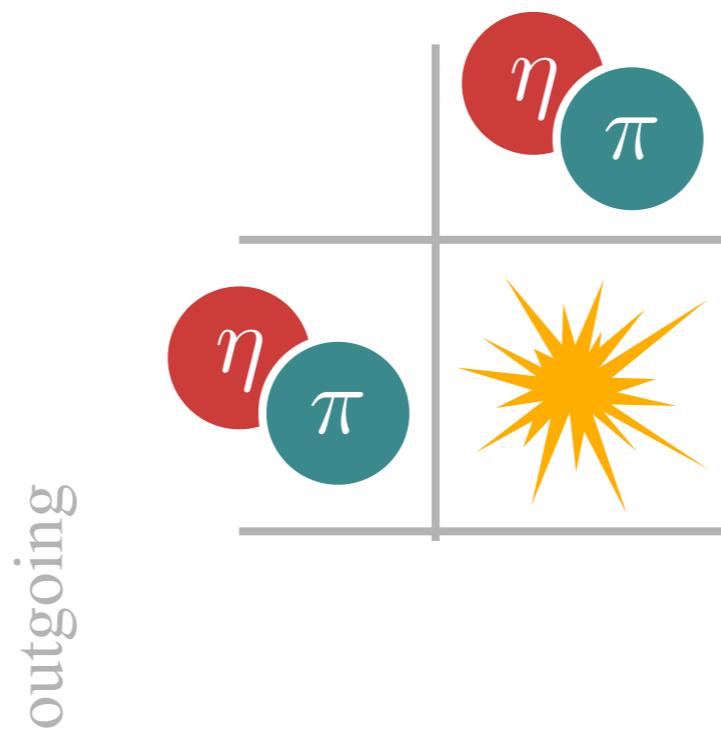
# Accessing the Hadron Spectrum

All our knowledge of the hadrons comes from **scattering experiments**

- In quantum mechanics, want matrix elements of scattering operator — the **S-matrix**
- Goal: fill in elements of this matrix!

$$S_{\alpha\beta}(E) = \langle \beta, \text{out} | S | \alpha, \text{in} \rangle$$

incoming

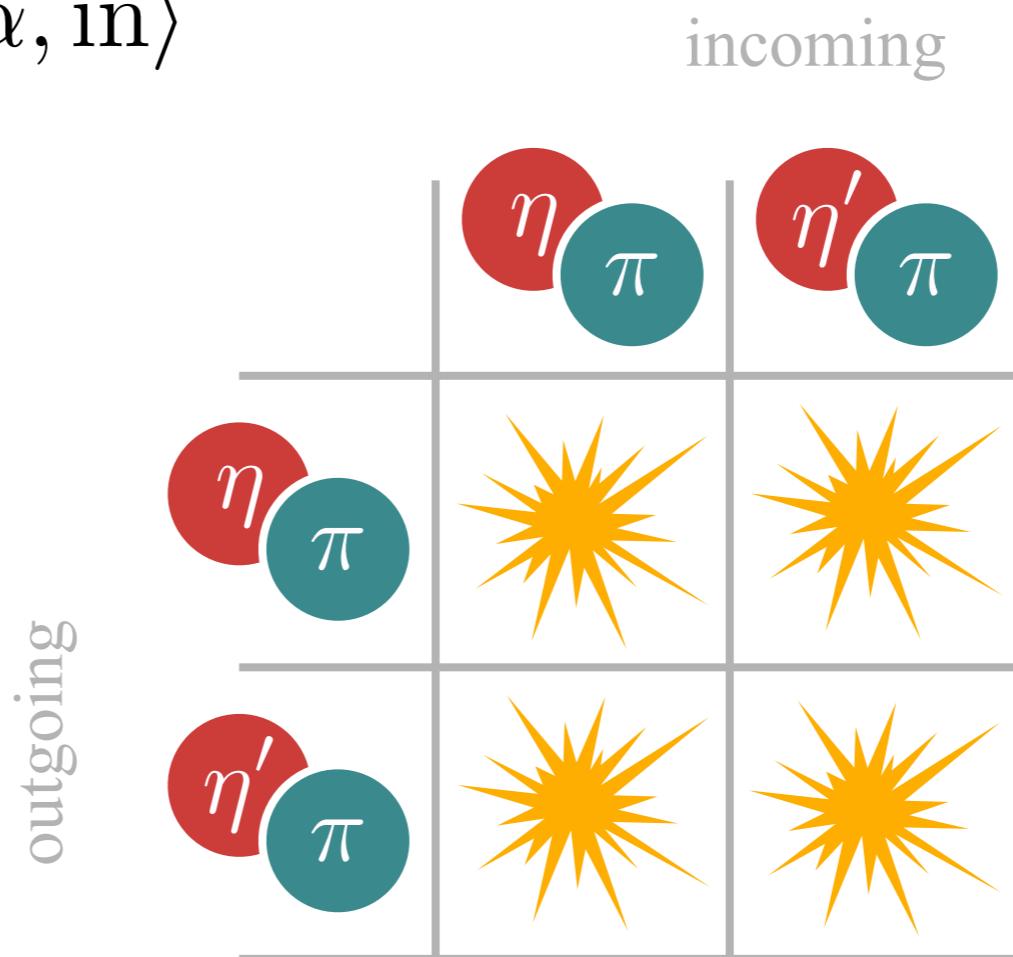


# Accessing the Hadron Spectrum

All our knowledge of the hadrons comes from **scattering experiments**

- In quantum mechanics, want matrix elements of scattering operator — the **S-matrix**
- Goal: fill in elements of this matrix!

$$S_{\alpha\beta}(E) = \langle \beta, \text{out} | S | \alpha, \text{in} \rangle$$

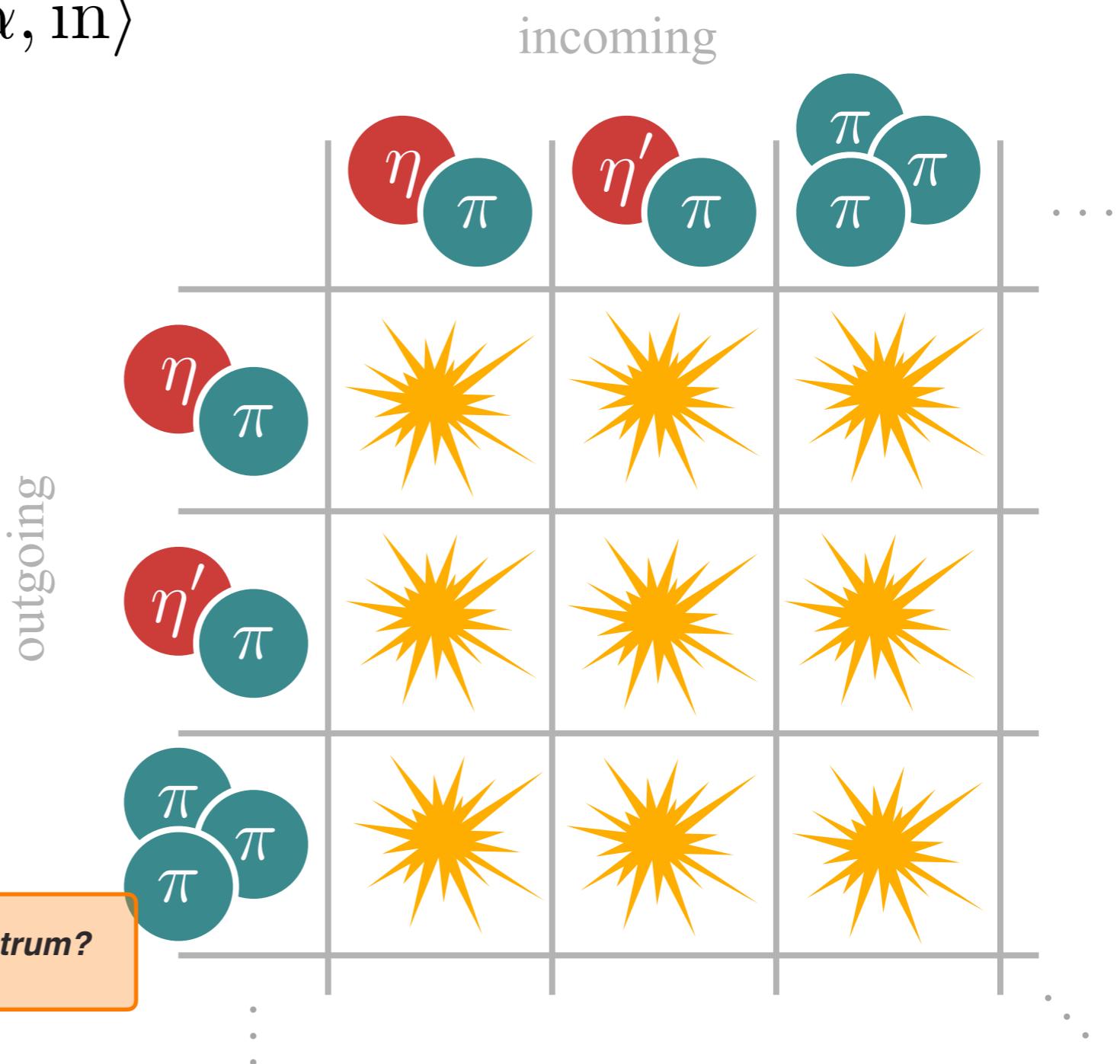


# Accessing the Hadron Spectrum

All our knowledge of the hadrons comes from **scattering experiments**

- In quantum mechanics, want matrix elements of scattering operator — the **S-matrix**
- Goal: fill in elements of this matrix!

$$S_{\alpha\beta}(E) = \langle \beta, \text{out} | S | \alpha, \text{in} \rangle$$

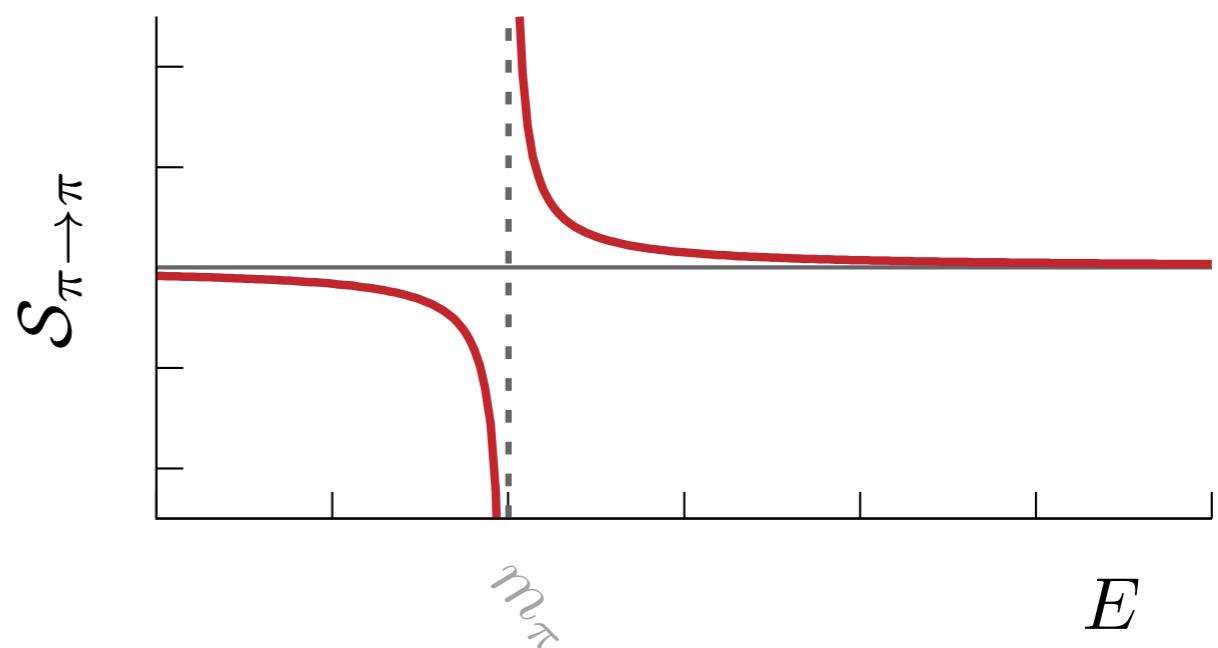


# Amplitudes & Particles

Key idea: Particles are associated with *pole singularities* of amplitudes

e.g. amplitude for a free propagating pion

$$S_{\pi \rightarrow \pi}(E) \sim \frac{1}{E^2 - m_\pi^2}$$



# Amplitudes & Particles

Key idea: Particles are associated with **pole singularities** of amplitudes

- For unstable particles, **resonances**, poles are located in **complex energy plane**

e.g pion-pion scattering producing the rho-meson

$$\mathcal{S}_{\pi\pi \rightarrow \rho \rightarrow \pi\pi}(E) \sim \frac{1}{E^2 - m_\rho^2 + im_\rho\Gamma_\rho}$$



# Amplitudes & Particles

Key idea: Particles are associated with **pole singularities** of amplitudes

- For unstable particles, **resonances**, poles are located in **complex energy plane**

e.g pion-pion scattering producing the rho-meson

$$\mathcal{S}_{\pi\pi \rightarrow \rho \rightarrow \pi\pi}(E) \sim \frac{1}{E^2 - m_\rho^2 + im_\rho\Gamma_\rho}$$



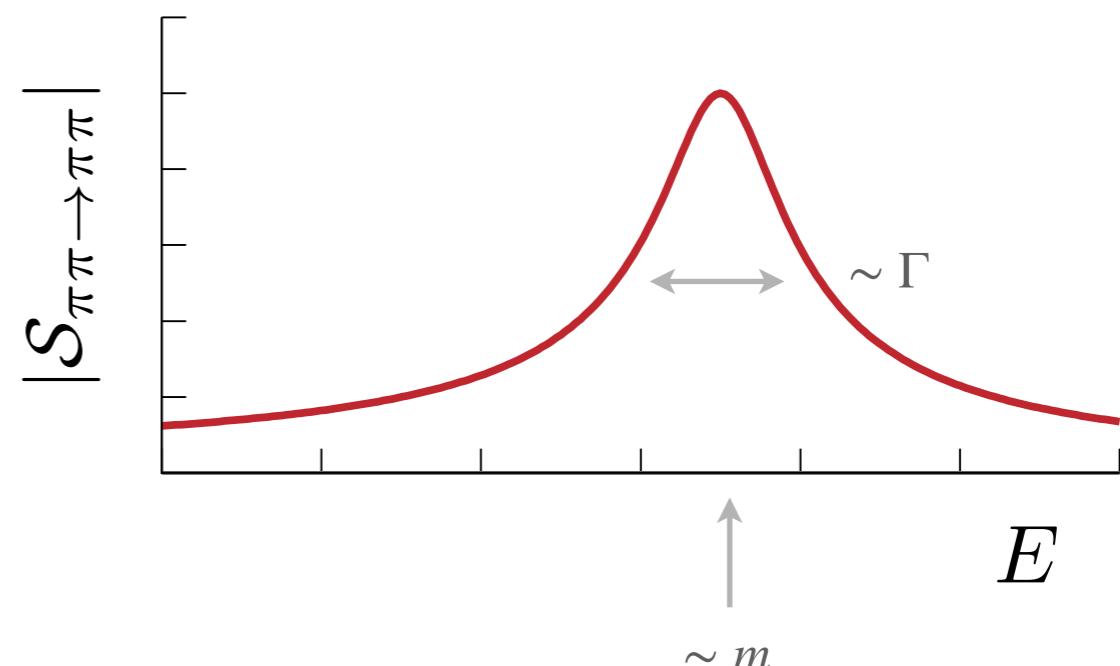
# Amplitudes & Particles

Key idea: Particles are associated with **pole singularities** of amplitudes

- For unstable particles, **resonances**, poles are located in **complex energy plane**

e.g pion-pion scattering producing the rho-meson

$$S_{\pi\pi \rightarrow \rho \rightarrow \pi\pi}(E) \sim \frac{1}{E^2 - m_\rho^2 + im_\rho\Gamma_\rho}$$



Note: sweeping many details under-the-rug

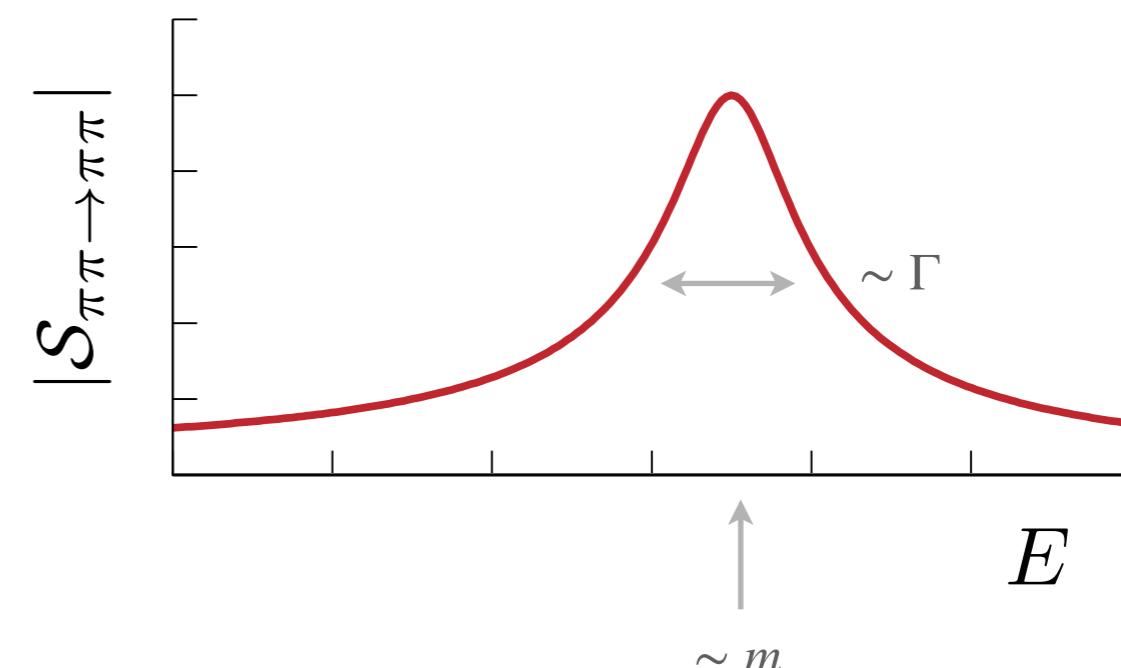
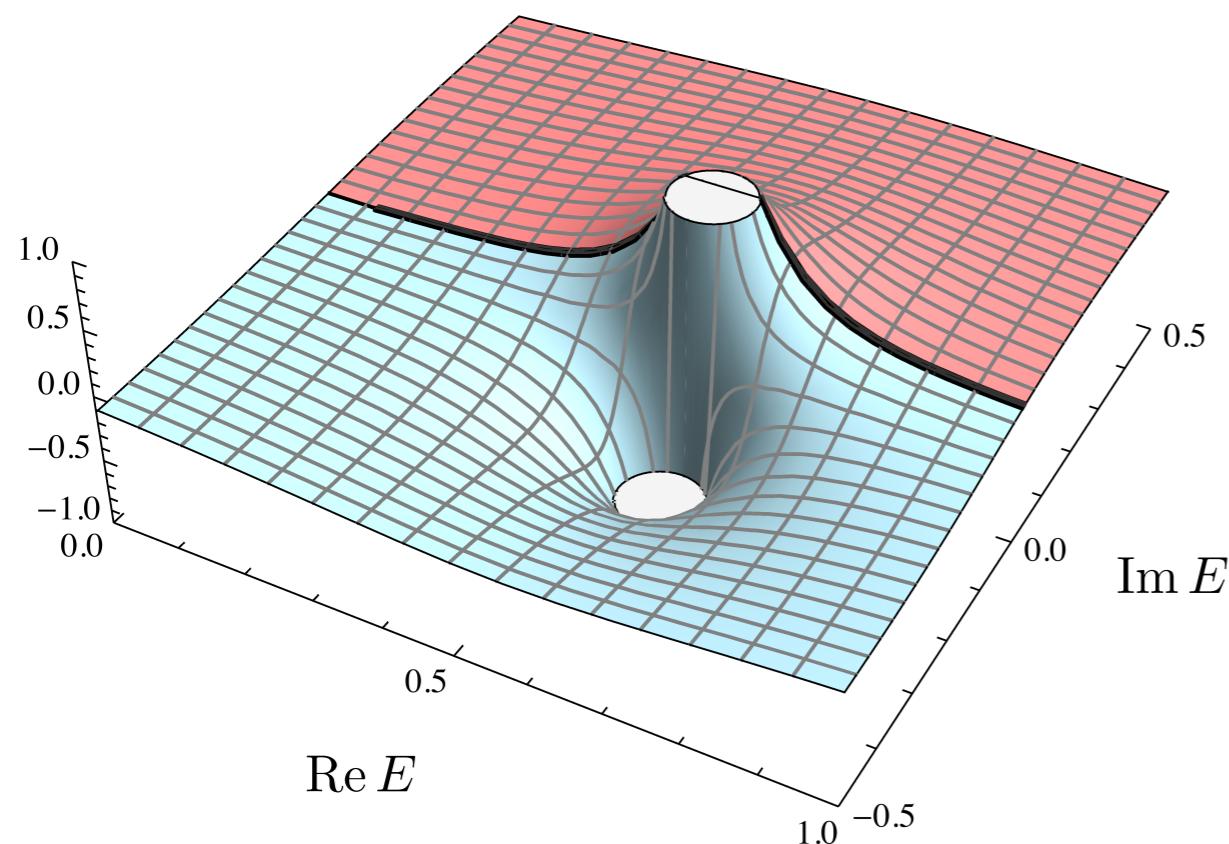
# Amplitudes & Particles

Key idea: Particles are associated with **pole singularities** of amplitudes

- For unstable particles, **resonances**, poles are located in **complex energy plane**

e.g pion-pion scattering producing the rho-meson

$$S_{\pi\pi \rightarrow \rho \rightarrow \pi\pi}(E) \sim \frac{1}{E^2 - m_\rho^2 + im_\rho \Gamma_\rho}$$



Note: sweeping many details under-the-rug

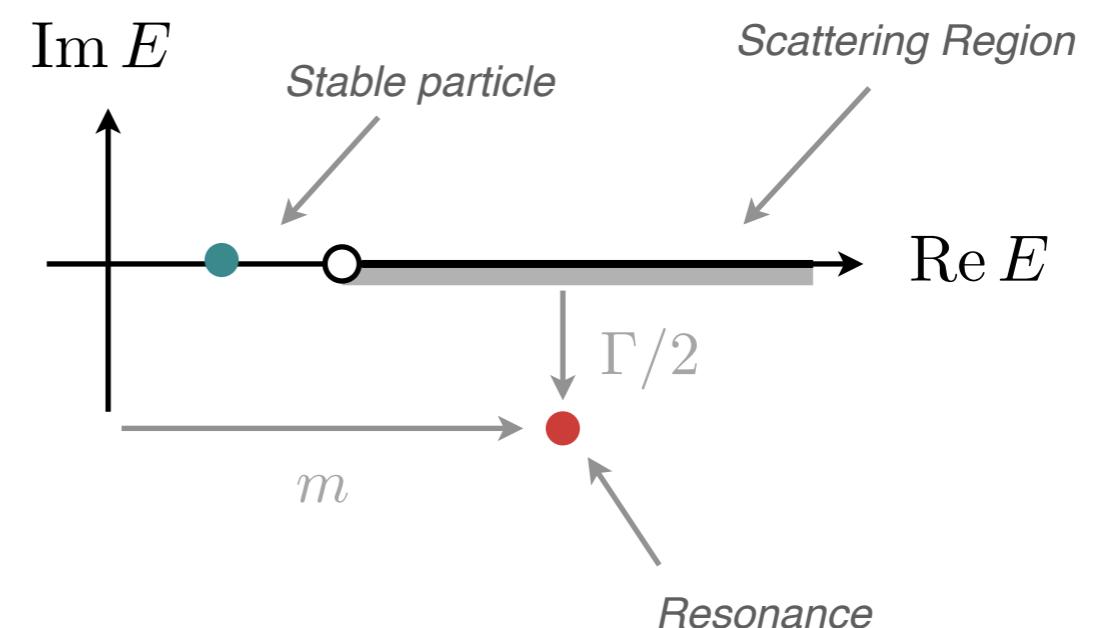
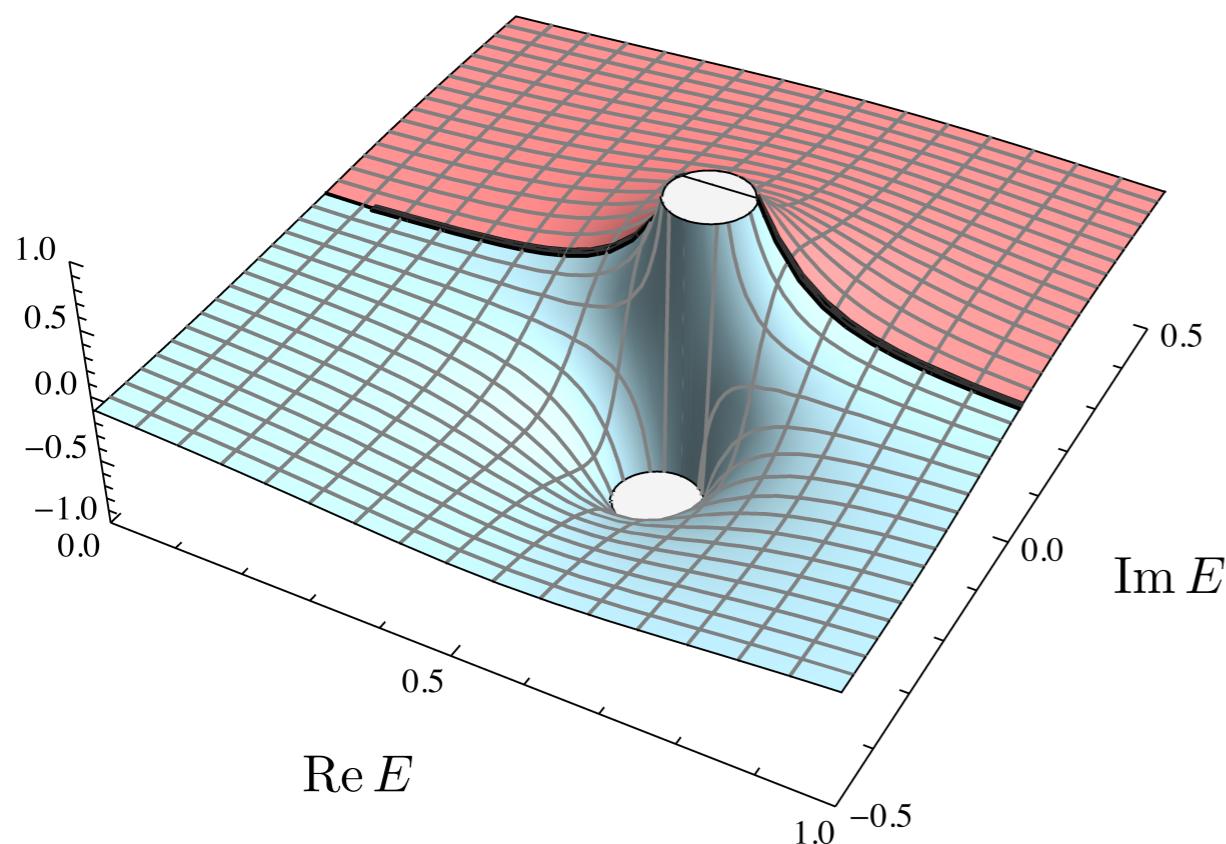
# Amplitudes & Particles

Key idea: Particles are associated with **pole singularities** of amplitudes

- For unstable particles, **resonances**, poles are located in **complex energy plane**

e.g pion-pion scattering producing the rho-meson

$$\mathcal{S}_{\pi\pi \rightarrow \rho \rightarrow \pi\pi}(E) \sim \frac{1}{E^2 - m_\rho^2 + im_\rho \Gamma_\rho}$$



Note: sweeping many details under-the-rug

# Amplitudes & Particles

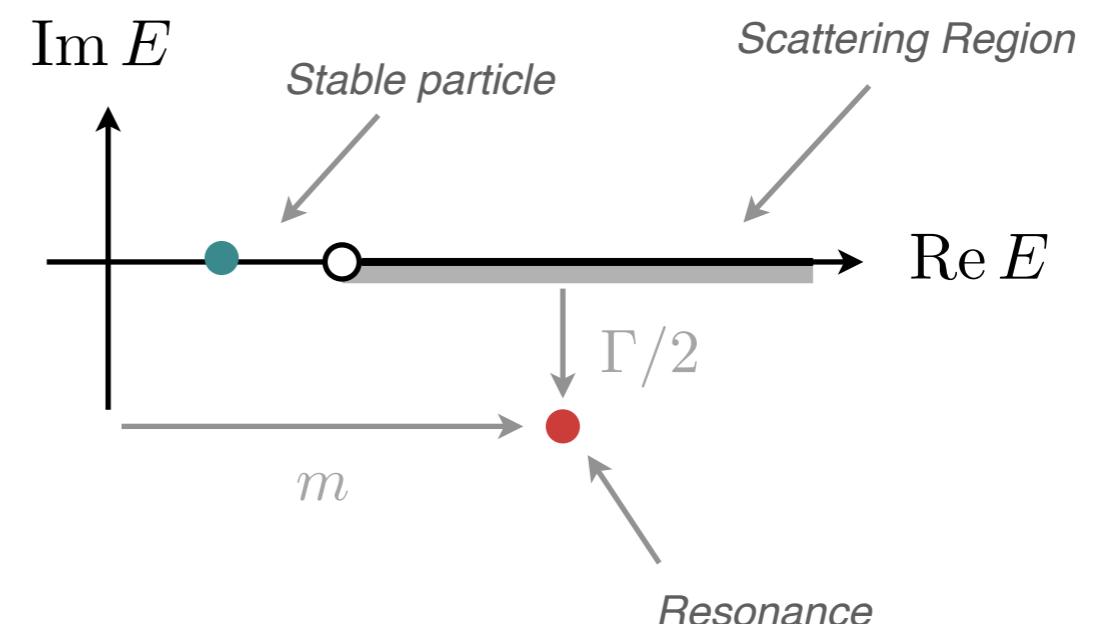
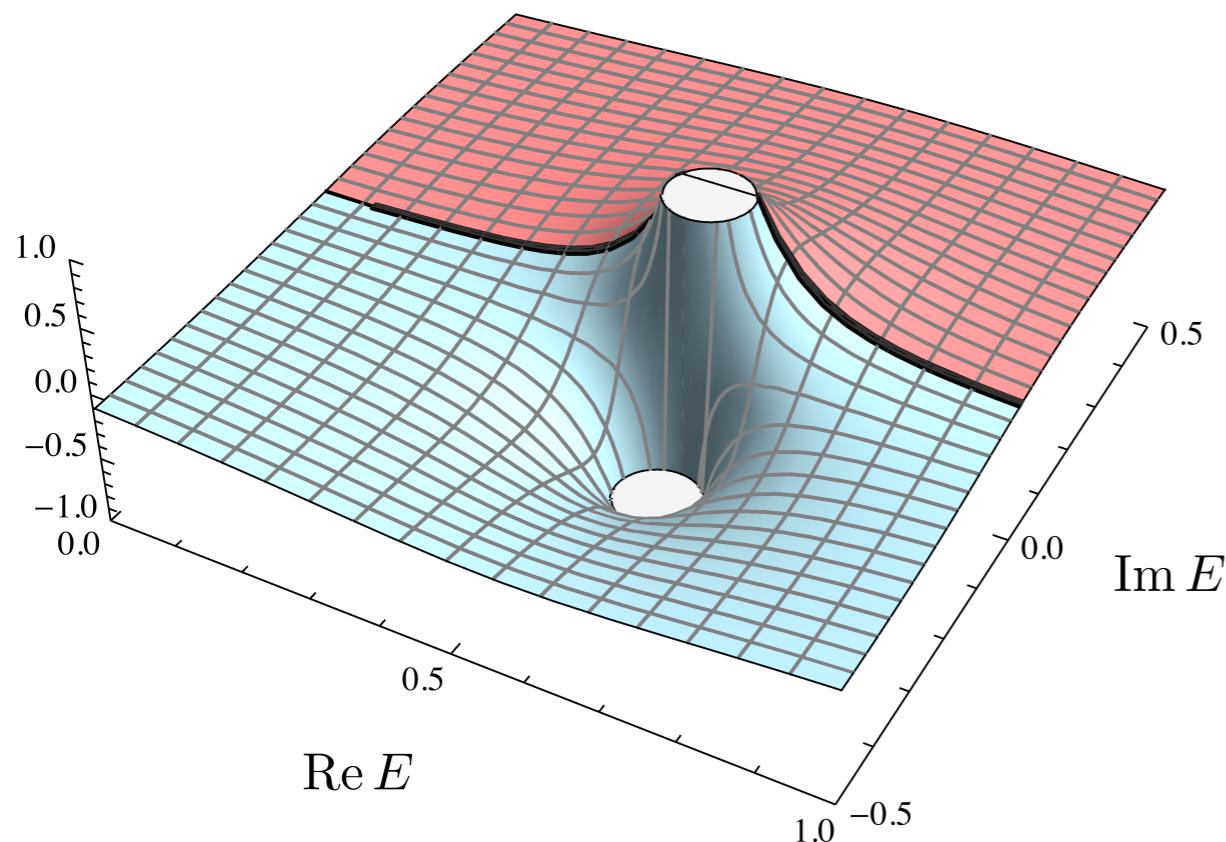
Key idea: Particles are associated with **pole singularities** of amplitudes

- For unstable particles, **resonances**, poles are located in **complex energy plane**

e.g pion-pion scattering producing the rho-meson

**Warning! More complicated processes have subtle features**

$$\mathcal{S}_{\pi\pi \rightarrow \rho \rightarrow \pi\pi}(E) \sim \frac{1}{E^2 - m_\rho^2 + im_\rho \Gamma_\rho}$$



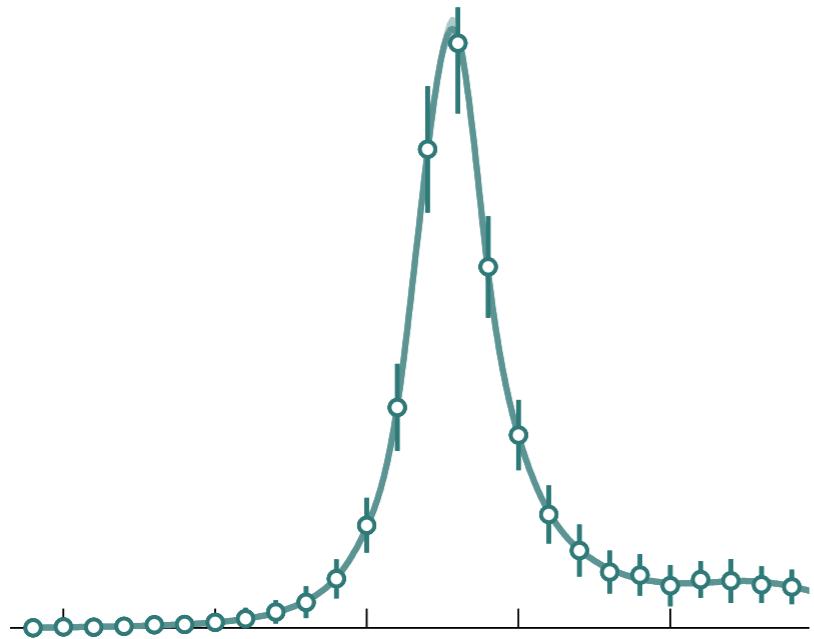
Note: sweeping many details under-the-rug

# From Data to Particles

---

Path to extract physical content from data:

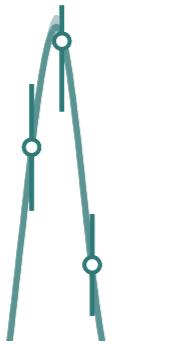
## *Scattering experiment data*



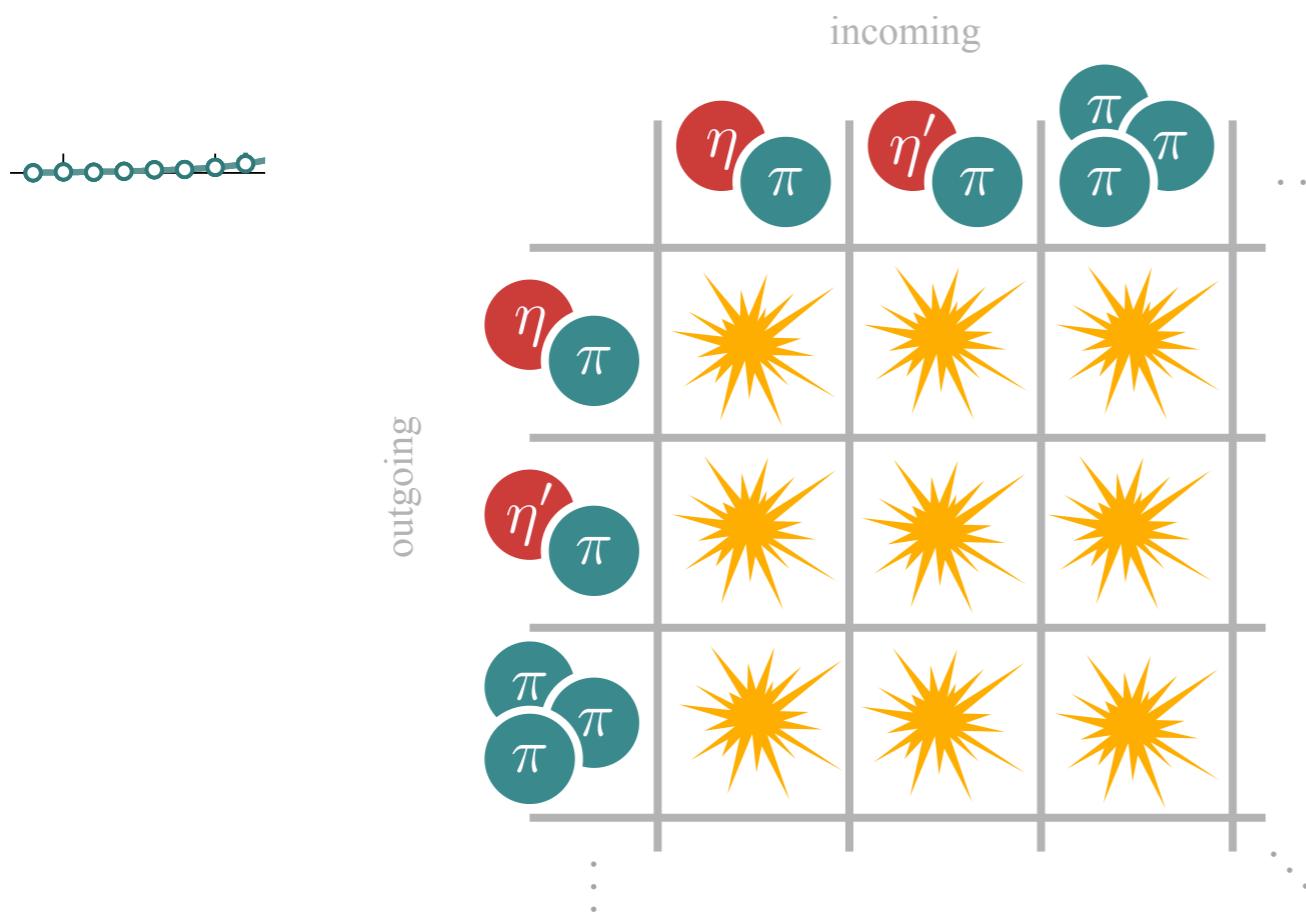
# From Data to Particles

Path to extract physical content from data:

*Scattering experiment data*



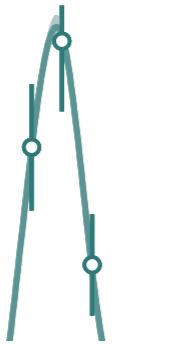
*S-matrix elements*



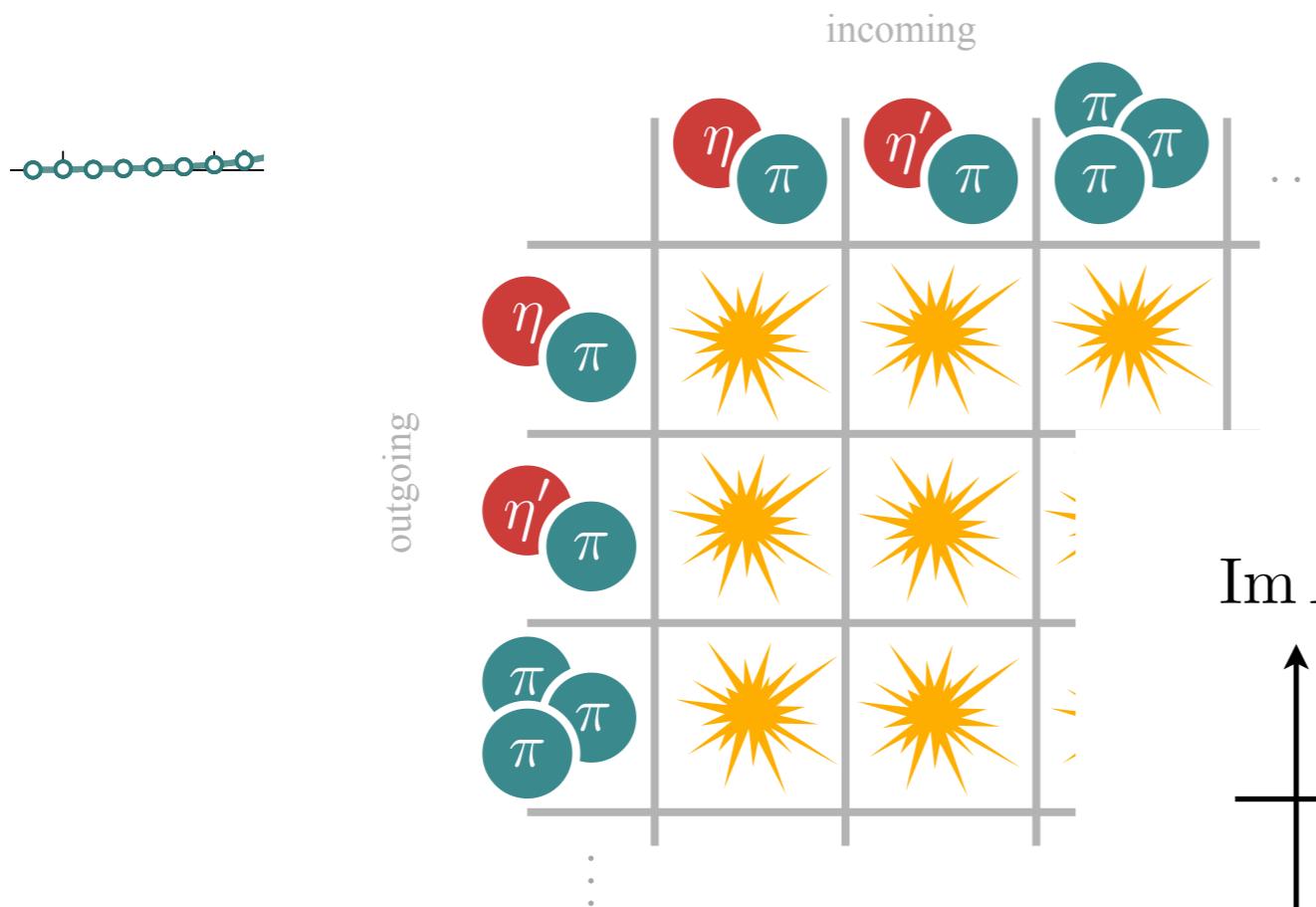
# From Data to Particles

Path to extract physical content from data:

*Scattering experiment data*



*S-matrix elements*



*Spectrum*

