Hadron Spectroscopy

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Old Dominion University & Jefferson Lab

RPI Computational Summer School

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OLD DOMINION UNIVERSITY

Jefferson Lab
Thomas Jefferson National Accelerator Facility

Outline

Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

Lattice QCD & Hadron Spectroscopy

Lattice QCD

Lüscher & the Finite-Volume

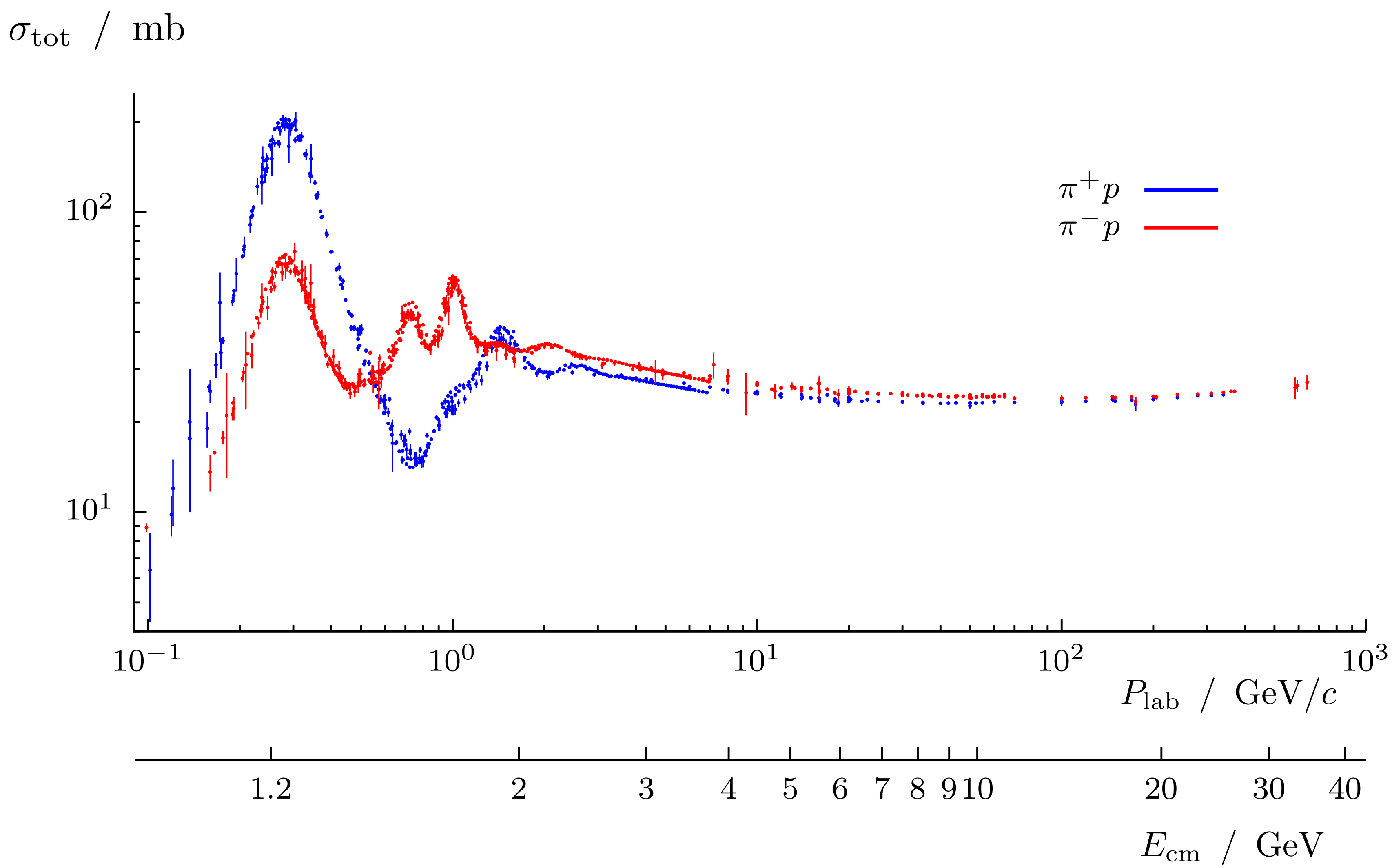
Scattering Theory & QCD Spectrum

Excited hadronic states are not stable particles…

…but are **resonances** coupling to multi-particle decay channels

…need to understand scattering amplitudes and properties

e.g., πN scattering and the excited Nucleon/Delta spectrum



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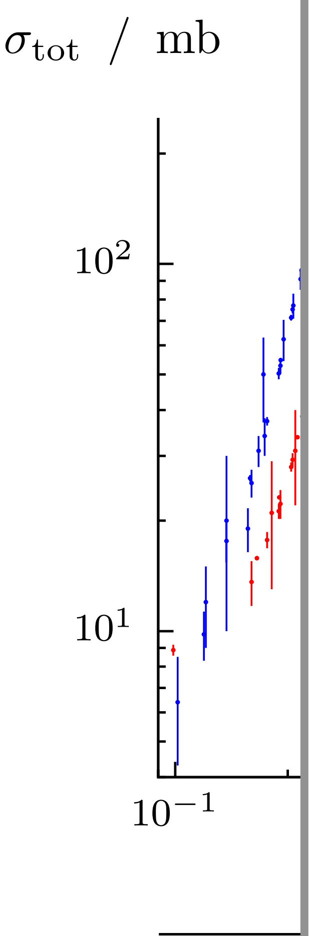
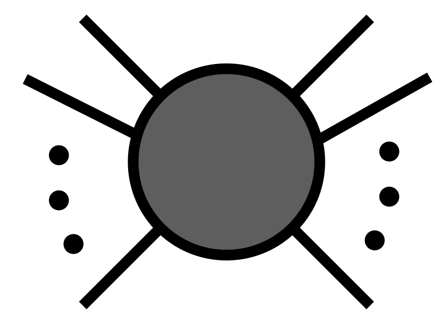


Need to rigorously define bound & resonant states within ***scattering theory***



Symmetry
Lorentz invariance, CPT, Flavor, baryon number, …

Unitarity
Probability conservation $\Longrightarrow$ The S matrix is a unitary operator

Analyticity
Causality $\Longrightarrow$ Amplitudes are boundary values of analytic functions in complex energy plane

Crossing
CPT symmetry $\Longrightarrow$ Relates particle—anti-particles in scattering processes

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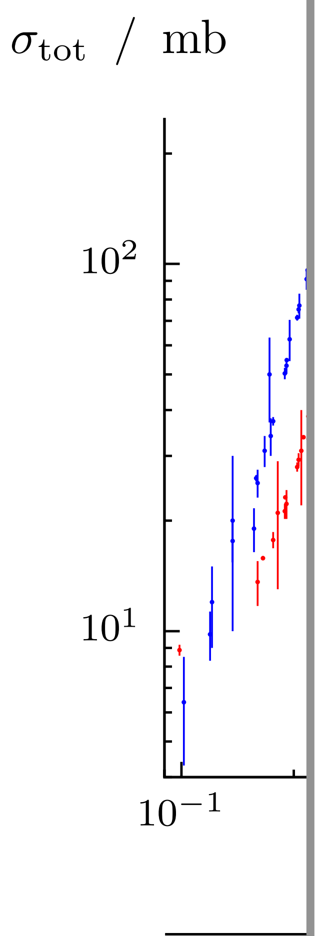

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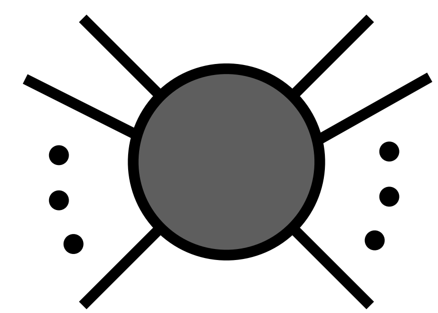

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Scattering Theory & QCD Spectrum

Amplitudes are complex functions of energy

- Poses difficulty in analyses, many parameters to fit
- ***Scattering Theory*** provides useful tools to restrict allowable amplitudes

$$\sum_{\beta} \mathrm{Prob}(\alpha \to \beta) = \sum_{\beta} |\mathcal{S}_{\alpha\beta}|^2 = 1$$

Constraint on S-matrix elements — Unitarity

Sum over all outgoing states

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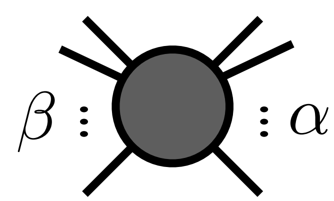
$$\mathcal{S}_{\alpha\beta} = 1 +$$



No scattering

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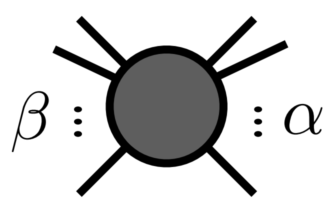
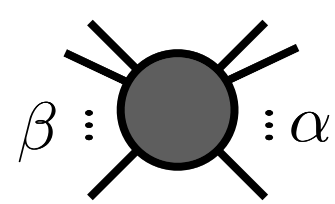
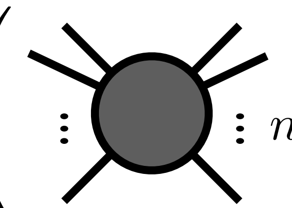


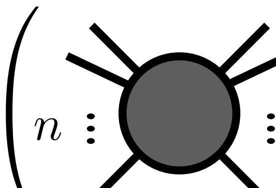
No scattering

$$\mathrm{Im}$$



$$= \sum_n \Big(\quad \Big)^* \Big(\quad \Big)$$





Relation for imaginary part of amplitudes!

Scattering Theory & QCD Spectrum

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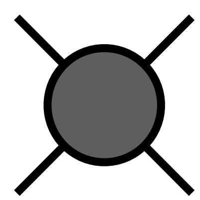
e.g. simple elastic 2-body scattering

Phase shift — real number



$$= \frac{i}{\rho} e^{i\delta} \sin\delta$$

No scattering

Kinematic factor (known)

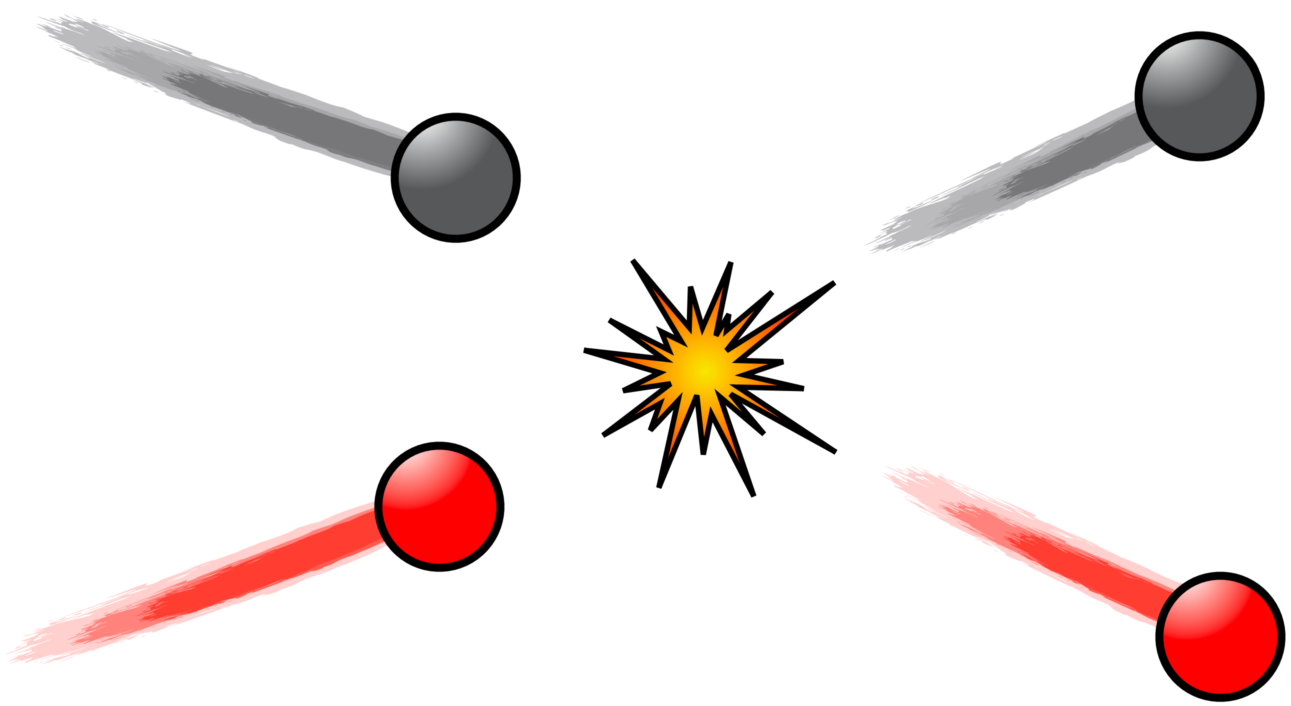
Relation for imaginary part of amplitudes!

The ***Scattering Amplitude*** $\mathcal{M}$ — Characterizes probability of an interaction to occur

S matrix *No interaction*

$$i\mathcal{M} \;\propto\; \langle \text{final} |\, S - 1 \,| \text{initial} \rangle$$

$$\sim$$

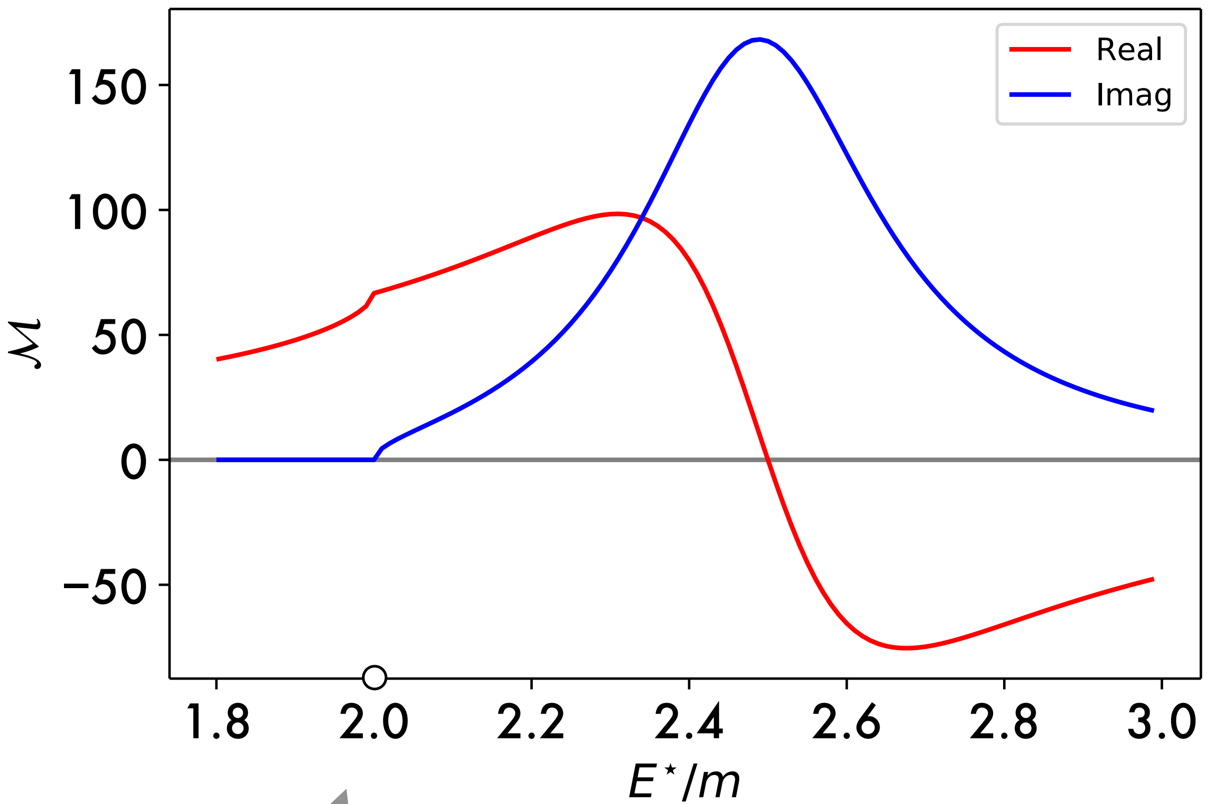


$$\mathcal{M} = \mathcal{M}(E^{\star})$$

Center-of-Momentum Energy

Scattering Theory — Amplitudes

The ***Scattering Amplitude*** $\mathcal{M}$ — Characterizes probability of an interaction to occur

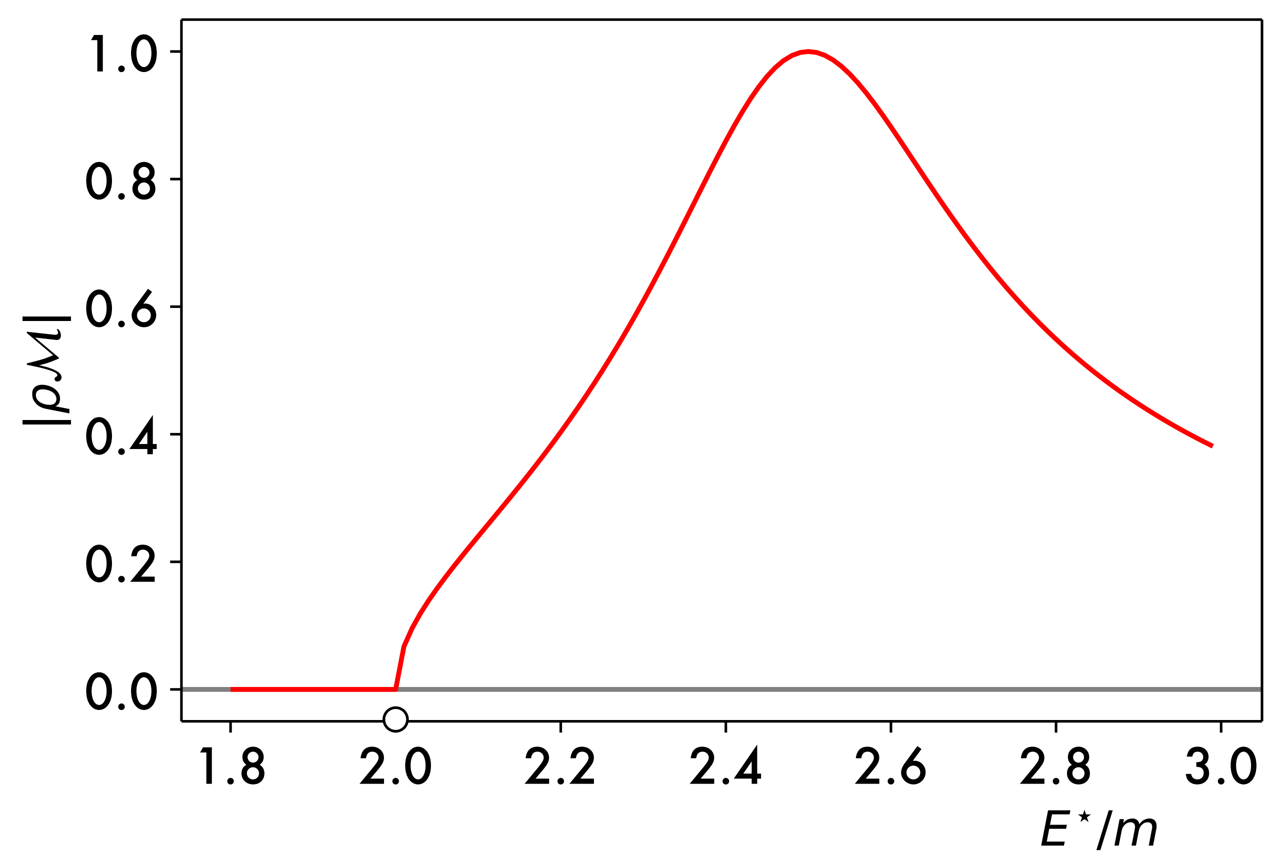


Real AND Imaginary components

Threshold Energy

$$E^\star_{\rm th} = 2m$$

$$\mathrm{Prob} \propto |\mathcal{M}|^2$$



Kinematic factor "phase space" ρ

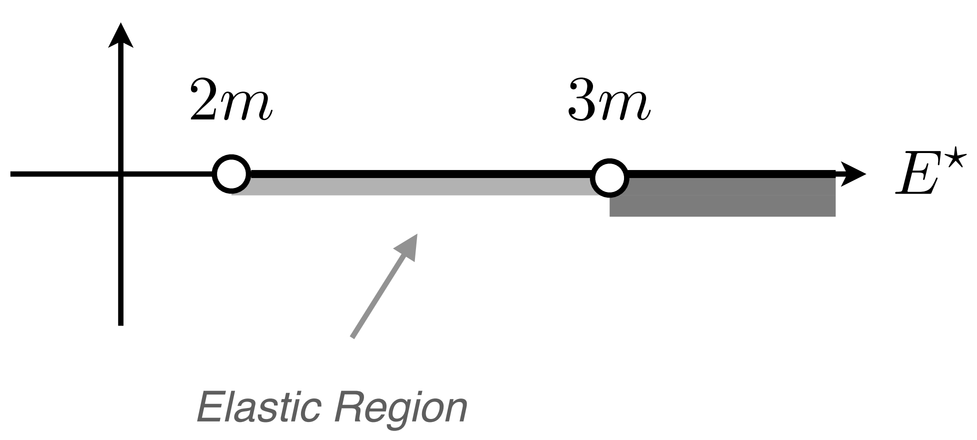
Scattering Amplitudes — Unitarity

Probability conservation $\Longrightarrow$ The *S* matrix is a unitary operator

$$\sum_f \mathrm{Prob}(i \to f) = 1 \qquad \Longrightarrow \qquad S^\dagger S = \mathbb{1}$$

After some work, can show that (in a limited energy region)

$$\mathrm{Im}\,\mathcal{M} = \rho|\mathcal{M}|^2 \qquad \text{for} \quad E^\star \geq 2m$$



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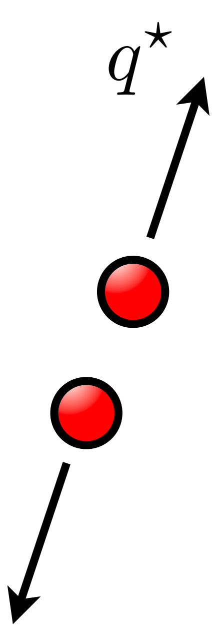
Phase space

- *Kinematic function*
- *Characterizes on-shell scattering of two-particles*

$$\rho = \frac{\xi q^\star}{8\pi E^\star}$$



Relative momentum between particles (in CM frame)

$$q^\star = \frac{1}{2}\sqrt{E^{\star\,2} - 4m^2}$$

$$\xi = \begin{cases} \frac{1}{2} & \text{identical} \\ 1 & \text{otherwise} \end{cases}$$

Python notebook — hadspec_ex0_complex.ipynb

- https://github.com/ajackura/RPISummerSchoolHadspec

hadspec_ex0_complex.ipynb — RPISummerSchoolHadspec

hadspec_ex0_complex.ipynb × | hadspec_ex1_scattering.ipynb | hadspec_ex2_luescher.ipynb

hadspec_ex0_complex.ipynb > M↓A Brief On Complex Analysis

+ Code + Markdown | Run All | Clear Outputs of All Cells | Outline ··· | Select Kernel

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A Brief On Complex Analysis

Studying scattering processes in physics inevitably leads us to require understanding fundamental concepts in *complex analysis*, such as pole singularities, branch cuts, and contour integrals. In this notebook, we review some basic features which we need in order to begin our study of scattering amplitudes.

References

I find the following references useful, you may too:

- *Fundamentals of Complex Analysis with Applications to Engineering, Science, and Mathematics* - Saff and Snider

In the context of scattering, this old (but very useful) book has a good chapter on complex analysis

- *Dispersion Relation Dynamics* - Burkhardt

This blog post has a nice discussion on moving the branch cut of the square root, some of which I have taken for these exercises

- https://flothesof.github.io/branch-cuts-with-square-roots.html

And I have taken excerpts from the following website which has some numerical exercises for Cauchy's theorem

- http://people.exeter.ac.uk/sh481/cauchy-theorem.html

```
'''
Importing useful libraries
'''
import cmath as cm                       # math library (complex)
import math as m                         # math library
import numpy as np                       # basic functions, linear algebra, etc.
import scipy.special as sp               # special functions
import numpy.random as rn                # random numbers
import matplotlib.pyplot as plt          # plotting library
from scipy import integrate              # library for integration
from mpl_toolkits import mplot3d         # for 3d plotting
from matplotlib.colors import hsv_to_rgb   # convert the color from HSV coordinates to RGB coordinates
from colorsys import hls_to_rgb          # convert the color from HLS coordinates to RGB coordinates
```

Python

Jupyter Server: local Cell 2 of 26

Scattering Amplitudes — Unitarity

Unitarity enforces some useful properties of the scattering amplitude

1. Phase shift representation

$$\mathcal{M} = |\mathcal{M}|\, e^{i\delta}$$

At a fixed energy,
amplitude determined by magnitude and phase (2 real numbers)

Impose unitarity $\mathrm{Im}\,\mathcal{M} = \rho|\mathcal{M}|^2$ …

…can show

$$|\mathcal{M}| = \frac{1}{\rho}\sin\delta$$

…such that

$$\mathcal{M} = \frac{1}{\rho} e^{i\delta}\,\sin\delta$$

At a fixed energy, 1 real parameter!
*δ — the **phase shift***

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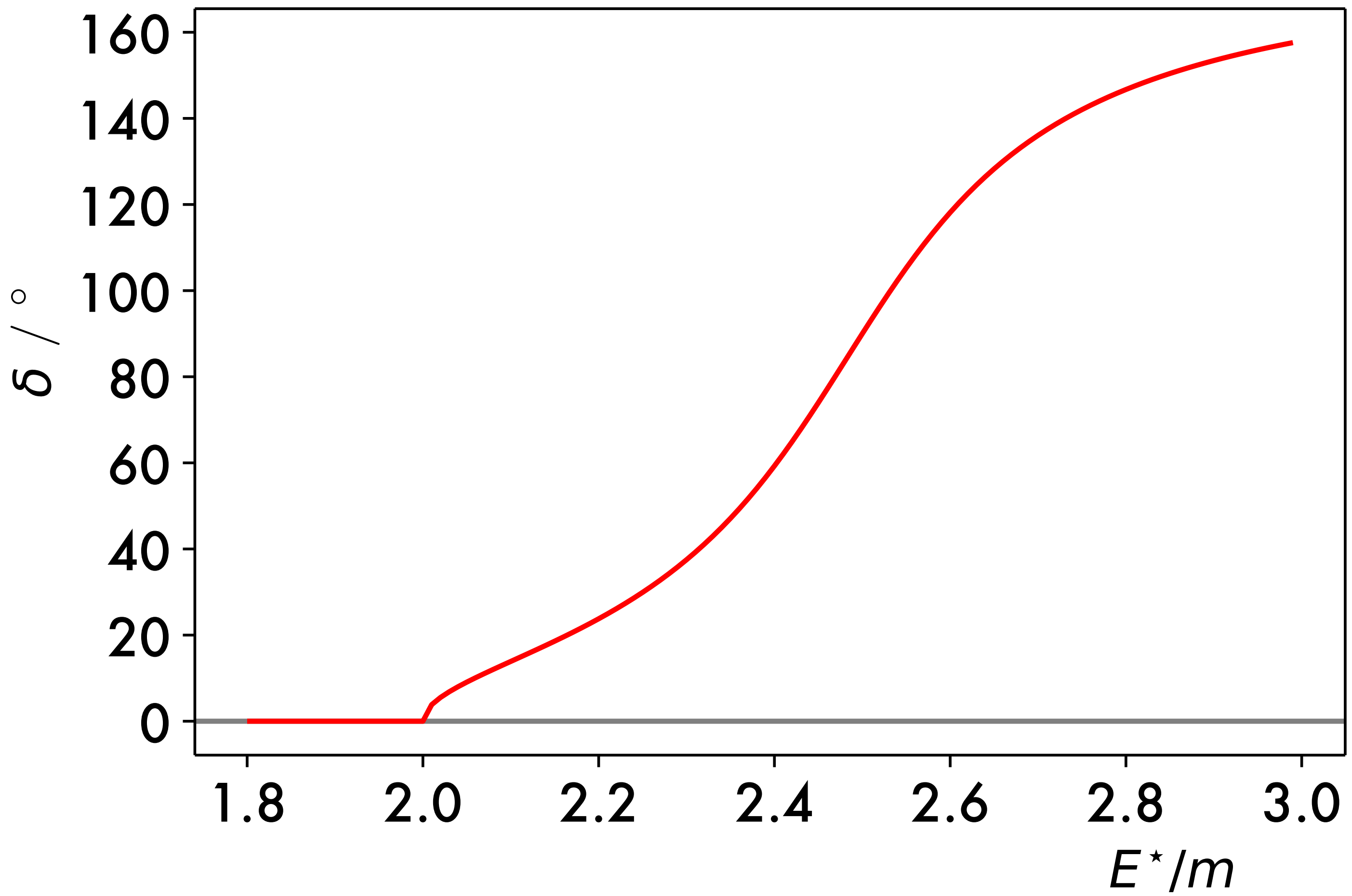


Scattering Amplitudes — Unitarity

Unitarity enforces some useful properties of the scattering amplitude

2. *K* matrix representation

$$\mathrm{Im}\,\mathcal{M} = \rho|\mathcal{M}|^2$$

$$\implies \qquad \mathrm{Im}\,\mathcal{M}^{-1} = -\rho$$

$$\implies \qquad \mathcal{M}^{-1} = \mathcal{K}^{-1} - i\rho$$

Real function
Characterizes 'short-range' forces between two particles
Not known a priori — Need to specify interaction

$$\mathcal{M} = \mathcal{K}\frac{1}{1 - i\rho\mathcal{K}}$$

Can relate to phase shift

$$\mathcal{K}^{-1} = \rho\cot\delta$$

Python notebook — hadspec_ex1_scattering.ipynb

- https://github.com/ajackura/RPISummerSchoolHadspec

Notebooks > hadspec_ex1_scattering.ipynb > M↓Scattering Amplitudes > M↓Complex Square Root

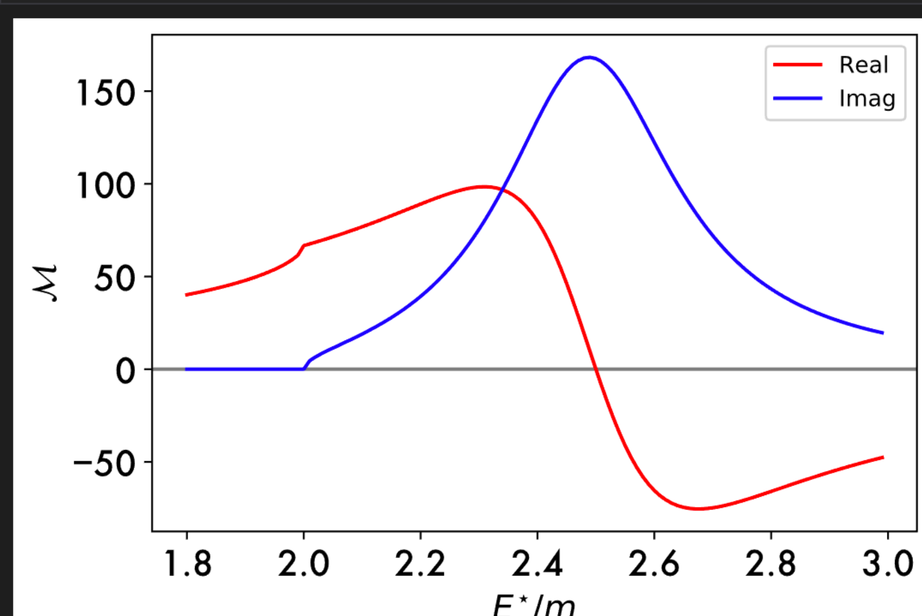
+ Code + Markdown | Run All | Clear Outputs of All Cells | Outline ···

```python
# Routine to plot amplitude
def plot_amplitude( Kmatrix ):
    eps = 1e-16                                                # shift s value slightly above real axis
    Ecm_o_m = np.arange(1.8, 3, 0.01)                          # array of Ecm / m
    amp = Amplitude( Ecm_o_m**2+1j*eps, Kmatrix )
    plt.axhline(y=0.0, color='gray', linestyle='-')
    plt.plot(Ecm_o_m, amp.real, color='red',label="Real")
    plt.plot(Ecm_o_m, amp.imag, color='blue',label="Imag")
    plt.xlabel(r'$E^{\star} / m$',size=15)
    plt.ylabel(r'$\mathcal{M}$',size=15)
    plt.xticks(fontname="Futura",fontsize=15)
    plt.yticks(fontname="Futura",fontsize=15)
    plt.legend(loc="upper right")
    plt.figure(figsize=(2,1), dpi= 100, facecolor='w', edgecolor='k')

# Sample Effective Range parameters
a = 2.0 # /m
r = 0.0 # /m

# Sample Breit-Wigner parameters
m0 = 2.5 # m
g0 = 3.0

# Plot Breit-Wigner
plot_amplitude( lambda s:Kmatrix_BreitWigner(s,m0,g0) )
```



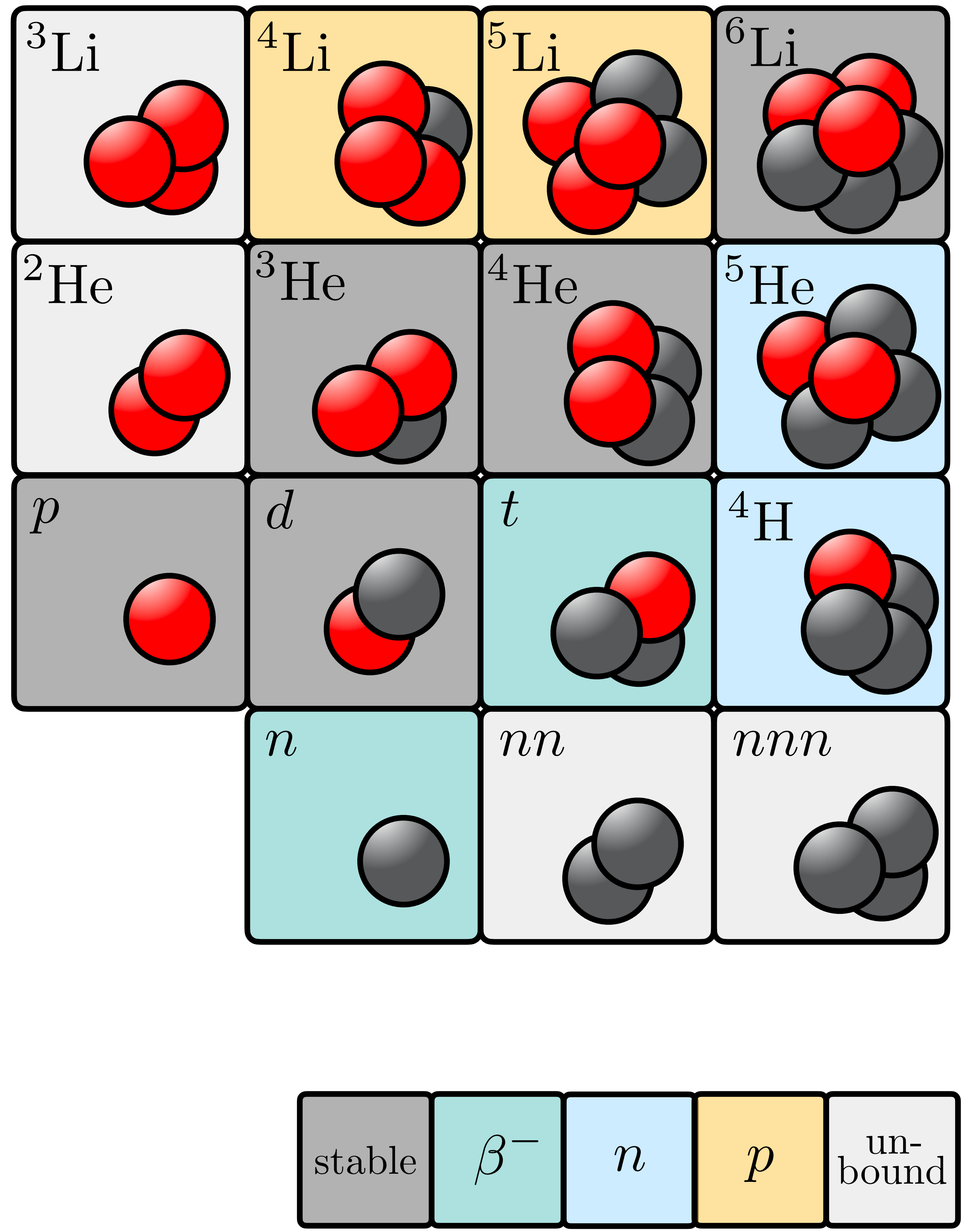
The Deuteron — connecting scattering to nuclear binding

Goal: Understand nuclear interactions from QCD

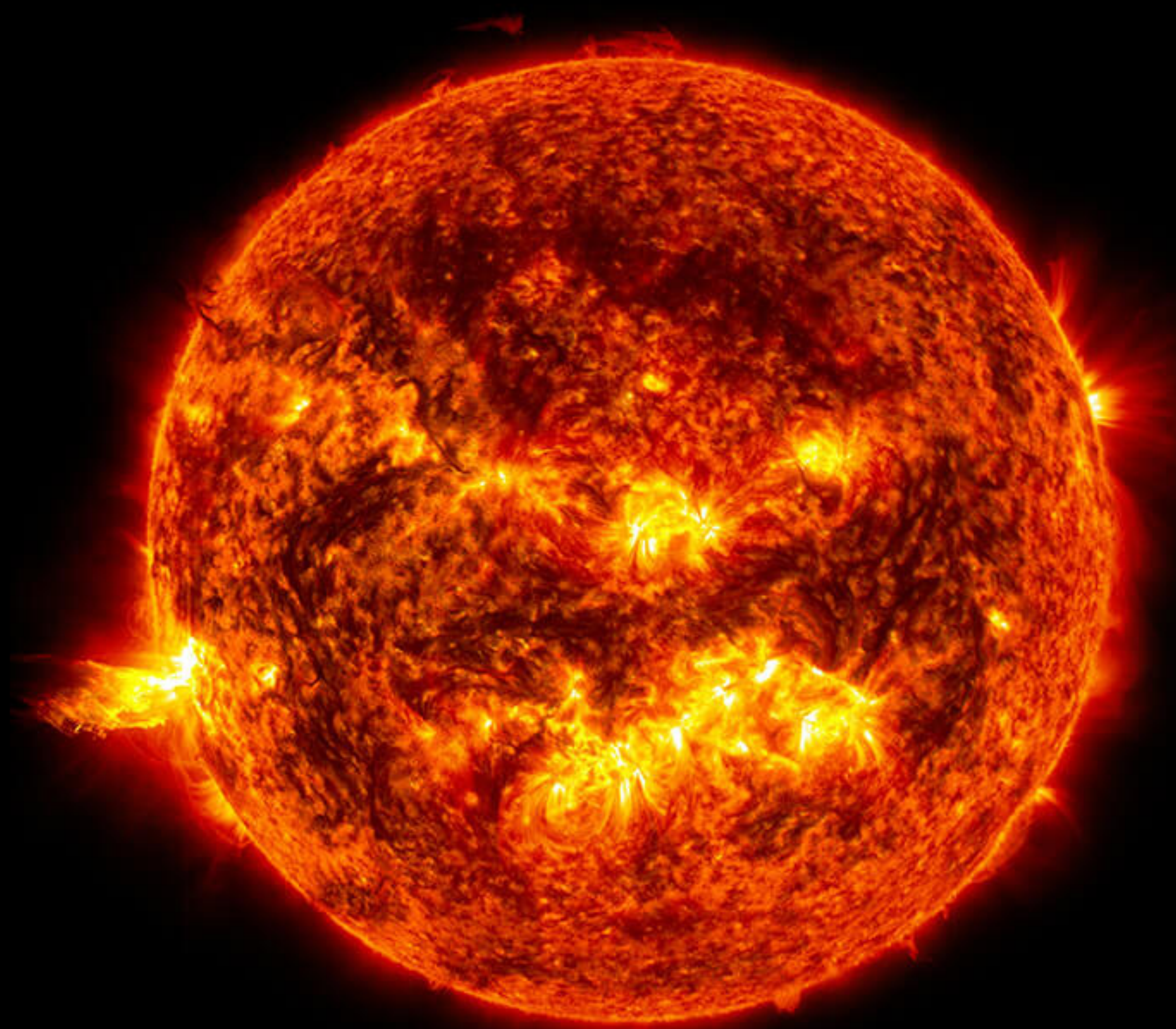
Focus on simplest interacting case:

The Deuteron

The deuteron plays a role in stellar fusion of heavier elements



Stellar Nucleosynthesis



https://spaceplace.nasa.gov/gallery-sun/en/

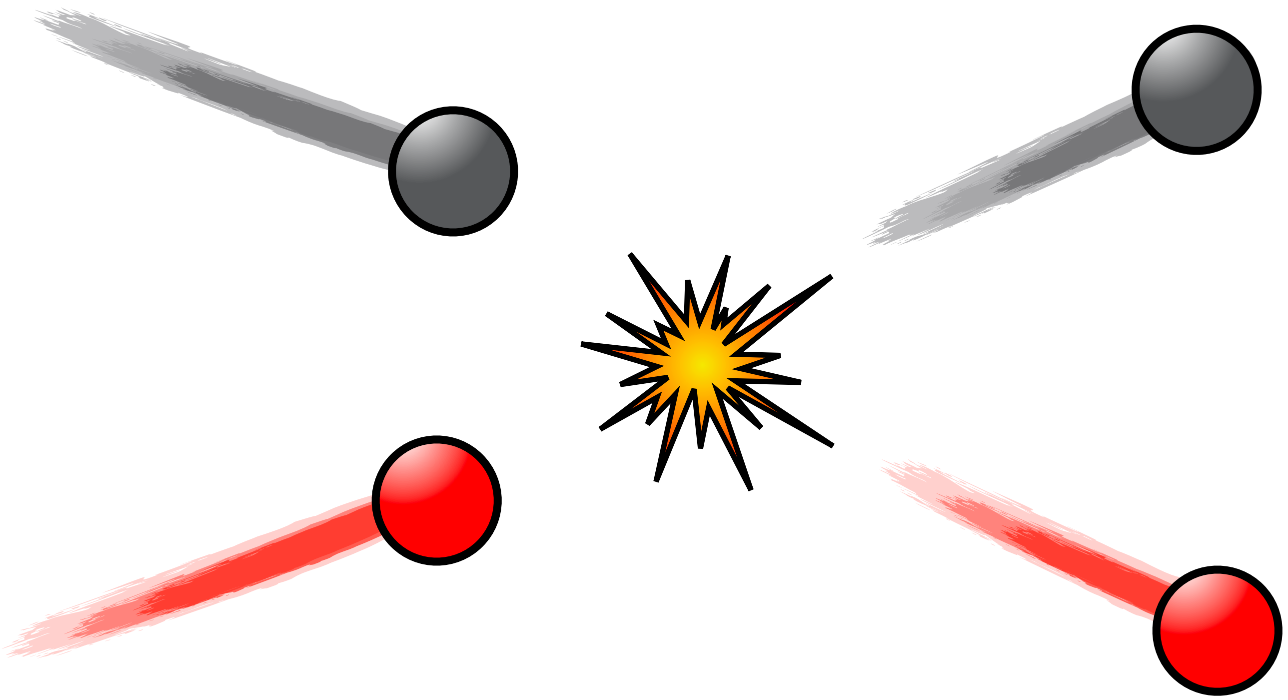
The Deuteron — Proton-Proton Fusion

The Deuteron — Proton-Proton Fusion

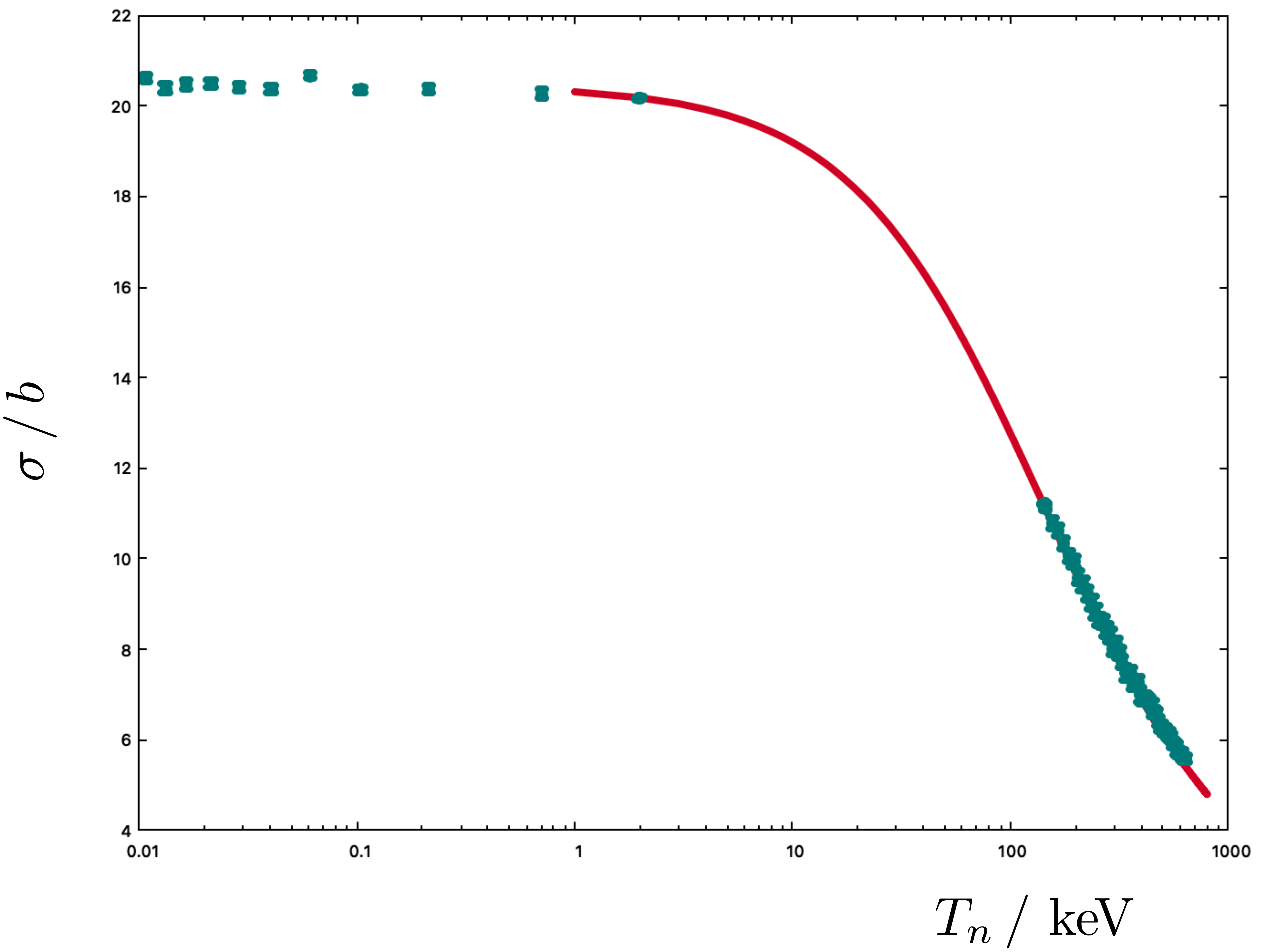
The Deuteron — Nucleon-Nucleon Scattering

In order to understand complication nuclear interactions, must understand deuteron first

Focus on proton-neutron (NN) scattering



The Deuteron — np scattering cross section



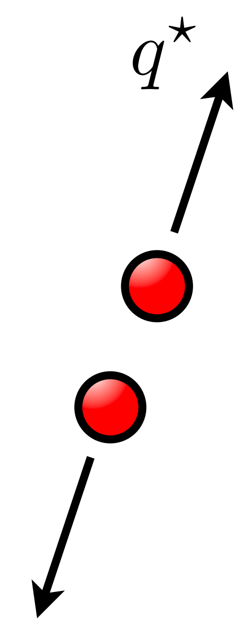
The Deuteron — Nucleon-Nucleon Scattering Amplitude

Let us consider NN scattering, assuming equal mass proton and neutron (*isospin limit*)

- Ignoring some details, the scattering amplitude has the form*

$$\mathcal{M} = \frac{1}{\mathcal{K}^{-1} - i\rho}$$

$$= \frac{8\pi E^\star/\xi}{q^\star \cot\delta - iq^\star}$$



$$m = m_N \approx 940 \text{ MeV}$$

$$q^\star = \frac{1}{2}\sqrt{E^{\star\,2} - 4m^2}$$

$$\xi = \frac{1}{2}$$

* *Technically, we focus on the* $^3S_1$ *amplitude*

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We focus on a specific representation for the *K* matrix/ phase shift

Effective Range Expansion

$$q^\star \cot\delta = -\frac{1}{a} + \frac{1}{2} r q^{\star\,2} + \mathcal{O}(q^{\star\,4})$$

a, *scattering length*

r, *effective range*

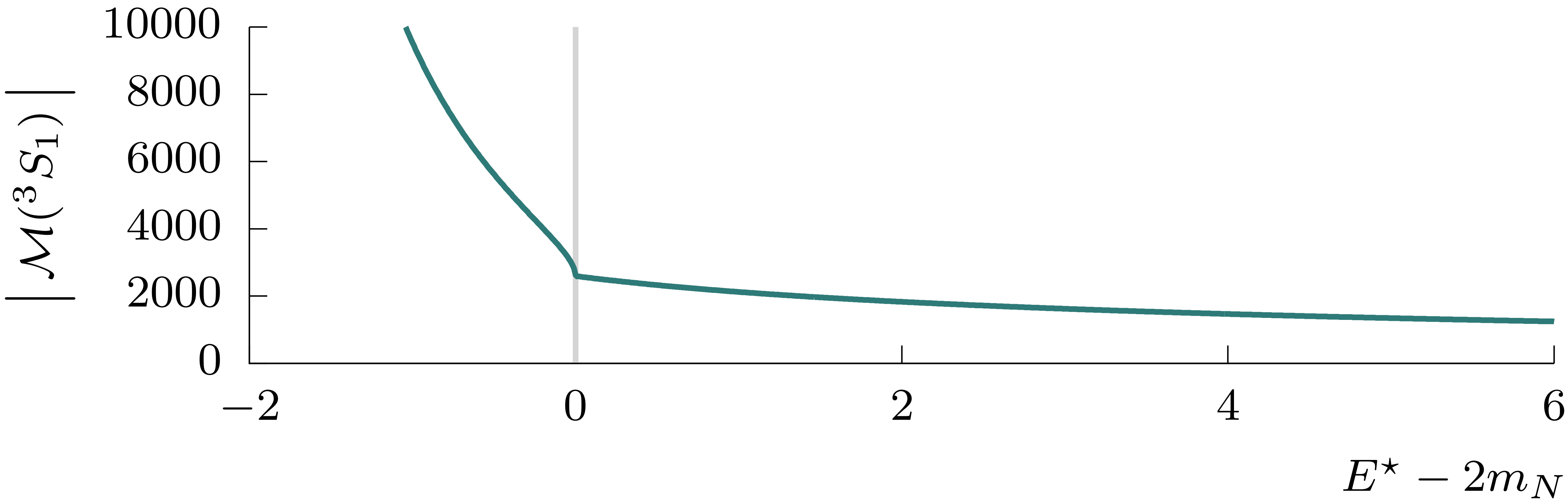
The Deuteron — Nucleon-Nucleon Scattering Amplitude

Let us consider NN scattering, assuming equal mass proton and neutron (*isospin limit*)

- After fitting data

$$m_N = 940 \text{ MeV}$$
$$a = 5.425 \text{ fm}$$
$$r = 1.749 \text{ fm}$$

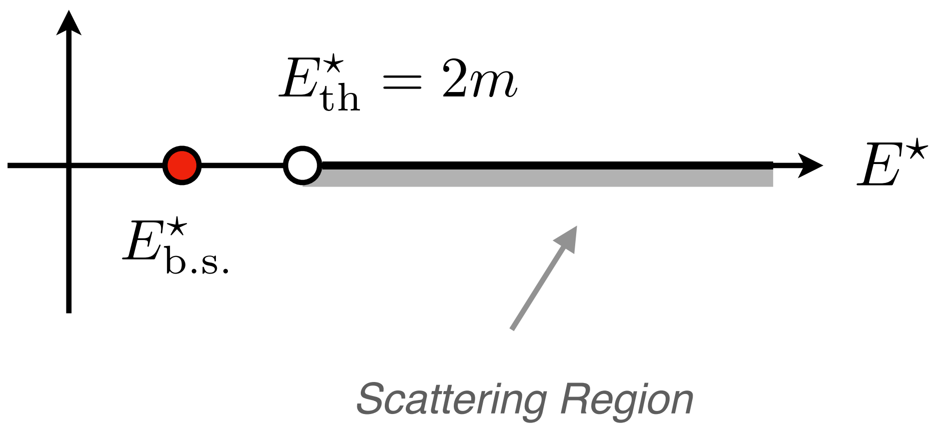
$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$



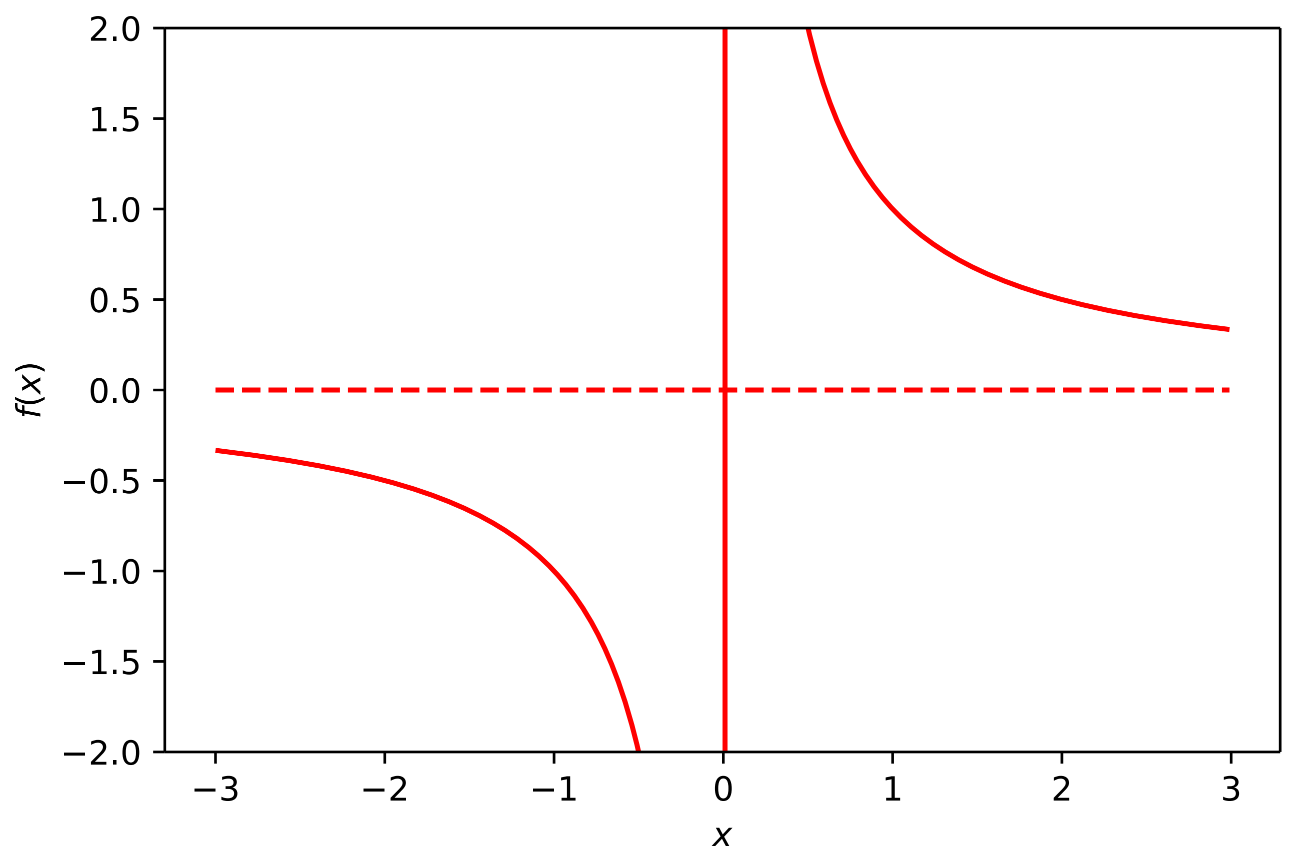
The Deuteron — Nucleon-Nucleon Scattering Amplitude

Let us consider NN scattering, assuming equal mass proton and neutron (*isospin limit*)

- Proton-Neutron scattering — attractive interactions
- Can bind to form the Deuteron (D or $^{2}\mathrm{H}$) — simplest nucleus
  - **Question**: How do bound states appear in scattering amplitudes?
  - **Answer**: They appear as *pole singularities* below threshold



$$f(x) = \frac{1}{x}$$



Bound state physics

Bound states appear as *pole singularities* below threshold

$$q^\star = i\kappa$$

Binding momentum

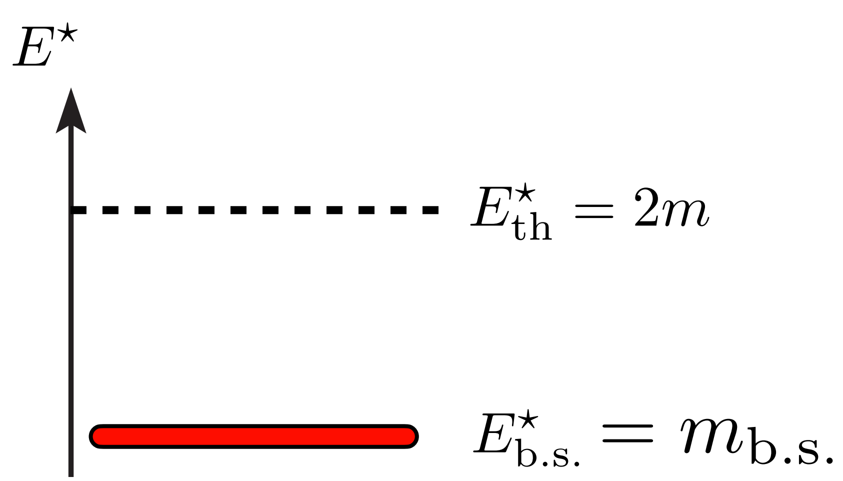
Mass of bound state

$$E^\star_{\mathrm{b.s.}} = 2\sqrt{m^2 - \kappa^2} \quad = m_{\mathrm{b.s.}}$$

$$m_{\mathrm{b.s.}} < E^\star_{\mathrm{th}}$$

Bound state mass

Binding energy



$$\Big\} \; E^\star_{\mathrm{binding}} = E^\star_{\mathrm{th}} - E^\star_{\mathrm{b.s.}}$$

$$E^\star = 2\sqrt{m^2 + q^{\star\,2}}$$

The Deuteron — Nucleon-Nucleon Scattering Amplitude

Let us consider NN scattering, assuming equal mass proton and neutron (*isospin limit*)

- Proton-Neutron scattering — attractive interactions
- Can bind to form the Deuteron (D or $^2\mathrm{H}$) — simplest nucleus
  - **Question**: Given the scattering amplitude, what is the deuteron mass?

    What is its binding energy?

    How do we get these properties from the scattering amplitude?
  - **Answer**: Compute the position of pole!

    Need scattering parameters (experimentally or theoretically)

$$m_N = 940 \text{ MeV}$$
$$a = 5.425 \text{ fm}$$
$$r = 1.749 \text{ fm}$$

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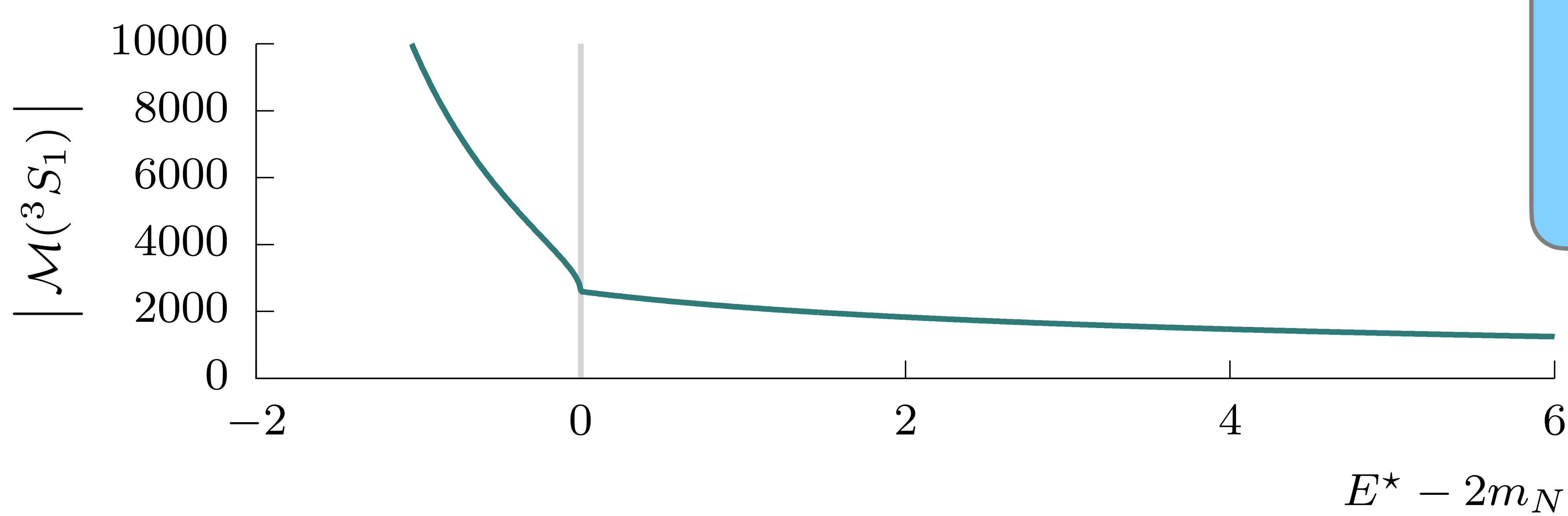

The Deuteron — Nucleon-Nucleon Scattering Amplitude

Let us find pole position

$$\mathcal{M} = \frac{8\pi E^\star/\xi}{q^\star \cot\delta - iq^\star}$$

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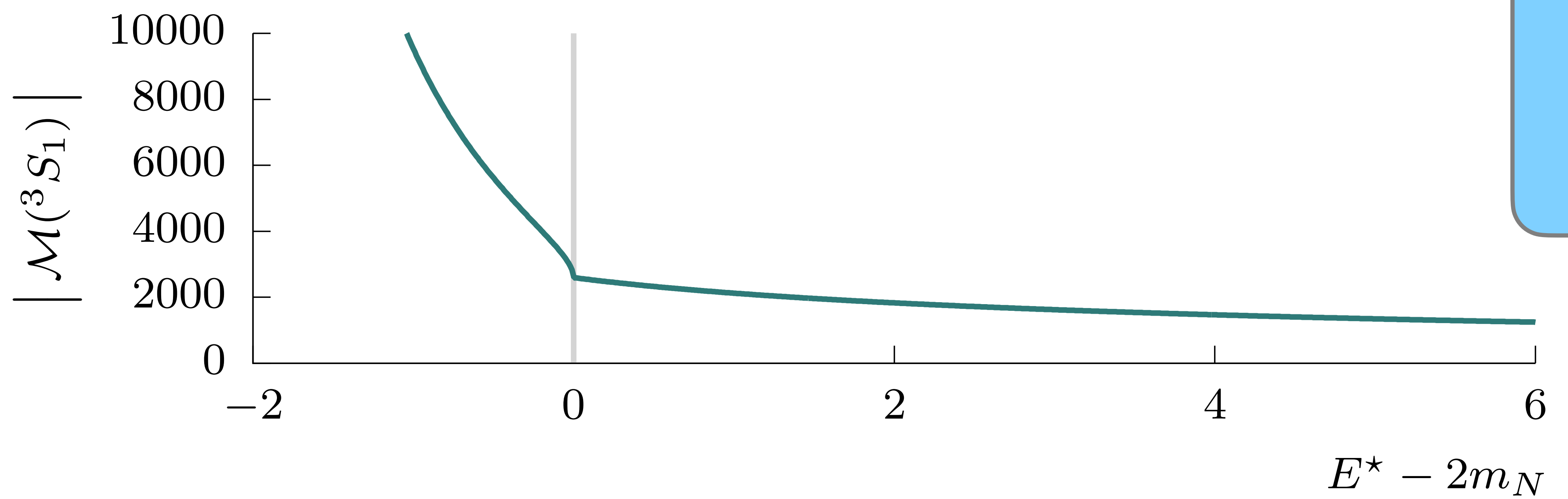

The Deuteron — Nucleon-Nucleon Scattering Amplitude

Let us find pole position

$$\mathcal{M} = \frac{8\pi E^\star/\xi}{q^\star \cot\delta - iq^\star}$$

First assume only scattering length $q^\star \cot\delta \approx -\frac{1}{a}$

$$\mathcal{M} \approx \frac{8\pi E^\star/\xi}{-\frac{1}{a} - iq^\star}$$

Pole occurs at $-\frac{1}{a} - iq^\star = 0 \implies q^\star = \frac{i}{a}$

The Deuteron — Nucleon-Nucleon Scattering Amplitude

Binding momentum of deuteron

$$\kappa = \frac{1}{a} \approx 37 \text{ MeV}$$

Mass of deuteron

$$m_{\text{b.s.}} = 2\sqrt{m^2 - \kappa^2} \approx 1878 \text{ MeV}$$

Binding energy

$$E^\star_{\text{binding}} = E^\star_{\text{th}} - E^\star_{\text{b.s.}}$$

$$\approx 2 \text{ MeV}$$

$$m_N = 940 \text{ MeV}$$
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$$r = 1.749 \text{ fm}$$

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Binding energy

$$E^{\star}_{\mathrm{binding}} = E^{\star}_{\mathrm{th}} - E^{\star}_{\mathrm{b.}}$$

$$\approx 2\ \mathrm{MeV}$$

The deuteron binding energy

C. Van Der Leun, C. Alderliesten

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https://doi.org/10.1016/0375-9474(82)90105-1

Abstract

The ^{1}H(n, γ)^{2}H γ-ray energy has been measured relative to ^{48}V and ^{144}Ce γ-rays, which are both based on the gold standard for γ-ray energies. The ensuing deut binding energy, B(^{2}H= 2224575 ± 9eV, confirms (with higher accuracy) the value from one of two conflicting recent precision measurements. This value has been used to recalculate the energies of γ-rays from thermal-neutron capture in ^{2}H, ^{1}