

# QCD Spectroscopy

## An Introduction

**Andrew W. Jackura**

University of California, Berkeley

Department of Physics

**RPI Computational Summer School**

June 19th - July 7th 2023



# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

## Lattice QCD & Hadron Spectroscopy      If time permits...

Lattice QCD

Lüscher & the Finite-Volume

# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

## Lattice QCD & Hadron Spectroscopy

Lattice QCD

Lüscher & the Finite-Volume

# Some Reviews

arXiv:1706.06223 (2017)

## Scattering processes and resonances from lattice QCD

Raúl A. Briceño,<sup>1,\*</sup> Jozef J. Dudek,<sup>1,2,†</sup> and Ross D. Young<sup>3,‡</sup>

<sup>1</sup>Thomas Jefferson National Accelerator Facility, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA

<sup>2</sup>Department of Physics, College of William and Mary, Williamsburg, Virginia 23187, USA

<sup>3</sup>Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia

(Dated: June 21, 2017)

The vast majority of hadrons observed in nature are not stable under the strong interaction, rather they are *resonances* whose existence is deduced from enhancements in the energy dependence of scattering amplitudes. The study of hadron resonances offers a window into the workings of quantum chromodynamics (QCD) in the low-energy non-perturbative region, and in addition, many probes of the limits of the electroweak sector of the Standard Model consider processes which feature hadron resonances. From a theoretical standpoint, this is a challenging field: the same dynamics that binds quarks and gluons into hadron resonances also controls their decay into lighter hadrons, so a complete approach to QCD is required. Presently, lattice QCD is the only available tool that provides the required non-perturbative evaluation of hadron observables. In this article, we review progress in the study of few-hadron reactions in which resonances and bound-states appear using lattice QCD techniques. We describe the leading approach which takes advantage of the periodic finite spatial volume used in lattice QCD calculations to extract scattering amplitudes from the discrete spectrum of QCD eigenstates in a box. We explain how from explicit lattice QCD calculations, one can rigorously garner information about a variety of resonance properties, including their masses, widths, decay couplings, and form factors. The challenges which currently limit the field are discussed along with the steps being taken to resolve them.

### CONTENTS

|   |    |  |    |
|---|----|--|----|
| I. Introduction   | 2  | 1. The importance of “multi-hadron” operators                            | 16 |
| II. Resonances, composite particles, and scattering amplitudes              | 3  | VI. Examples of resonance determination                                  | 18 |
| A. Pole singularities   | 3  | A. Elastic resonances in $\pi\pi$ scattering                             | 18 |
| B. Coupled-channel scattering   | 4  | B. Resonances in coupled-channel meson-meson scattering                  | 22 |
| C. Diagrammatic representation  | 5  | VII. Other approaches to resonance determination                         | 23 |
| III. Lattice QCD  | 5  | A. Resonances in the Lüscher formalism in the narrow-width approximation | 23 |
| IV. Scattering in a finite-volume   | 6  | B. Resonances and ‘naive’ level counting                                 | 25 |
| A. Scattering in non-relativistic quantum mechanics in one space dimension  | 7  | C. Finite volume EFT Hamiltonian approach                                | 26 |
| B. Scattering in a periodic cubic volume                                    | 7  | D. Unitarized chiral perturbation theory and chiral extrapolations       | 27 |
| C. Relating scattering amplitudes to finite-volume spectra                  | 10 | E. Other approaches  | 28 |
| 1. Dominance of the lowest partial-wave                                     | 10 | VIII. Coupling resonances to external currents                           | 29 |
| 2. Coupled-channel scattering and parameterization of scattering amplitudes | 11 | A. Determining matrix elements in lattice QCD                            | 30 |
| 3. Examples of finite-volume spectra for simple scattering amplitudes       | 11 | B. Lellouch-Lüscher formalism and its generalizations                    | 31 |
| C. Applications   | 32 | C. Applications  | 32 |
| V. Determining the finite-volume spectrum                                   | 12 | IX. Contemporary Extensions  | 33 |
| A. Variational analysis of correlation matrices                             | 12 | A. Particles with nonzero intrinsic spin                                 | 33 |
| B. Operator construction  | 15 | B. Three-particle systems  | 34 |
| X. Outlook  | 35 | C. Elastic form factors of resonances                                    | 35 |

# Some Reviews

arXiv:1706.06223 (2017)

## Scattering processes and resonances from lattice QCD

Raúl A. Briceño,<sup>1,\*</sup> Jozef J. Dudek,<sup>1,2,†</sup> and Ross D. Young<sup>3,‡</sup>

<sup>1</sup>Thomas Jefferson National Accelerator Facility, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA

<sup>2</sup>Department of Physics, College of William and Mary, Williamsburg, Virginia 23187, USA

<sup>3</sup>Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia

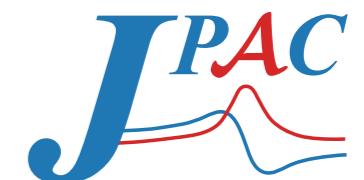
(Dated: June 21, 2017)

The vast majority of hadrons observed in nature are *bound states*, rather than *resonances* whose existence is inferred from the energy dependence of scattering amplitudes. Lattice QCD offers a window into the workings of quantum chromodynamics in the non-perturbative region, and in addition, many processes in the Standard Model consider processes where the theoretical standpoint, this is a challenging field: the conversion of gluons into hadron resonances also controls the complete approach to QCD is required. Presently, that provides the required non-perturbative evaluation of the spectrum. In this article, we review progress in the study of few-hadron scattering, where bound-states appear using lattice QCD techniques which takes advantage of the periodic finite spatial volumes to extract scattering amplitudes from the discrete box. We explain how from explicit lattice QCD calculations one can obtain information about a variety of resonance properties, decay couplings, and form factors. The challenges discussed along with the steps being taken to resolve them.

## CONTENTS

- I. Introduction
- II. Resonances, composite particles, and scattering amplitudes
  - A. Pole singularities
  - B. Coupled-channel scattering
  - C. Diagrammatic representation
- III. Lattice QCD
- IV. Scattering in a finite-volume
  - A. Scattering in non-relativistic quantum mechanics in one space dimension
  - B. Scattering in a periodic cubic volume
  - C. Relating scattering amplitudes to finite-volume spectra
    - 1. Dominance of the lowest partial-wave
    - 2. Coupled-channel scattering and parameterizations of scattering amplitudes
    - 3. Examples of finite-volume spectra for simple scattering amplitudes
- V. Determining the finite-volume spectrum
  - A. Variational analysis of correlation matrices
  - B. Operator construction

arXiv:2112.13436 (2021)



## Novel approaches in Hadron Spectroscopy

Miguel Albaladejo<sup>a,b</sup>, Łukasz Bibrzycki<sup>c</sup>, Sebastian M. Dawid<sup>d,e</sup>, César Fernández-Ramírez<sup>f,g</sup>, Sergi González-Solís<sup>d,e,h</sup>, Astrid N. Hiller Blin<sup>a</sup>, Andrew W. Jackura<sup>a,i</sup>, Vincent Mathieu<sup>j,k</sup>, Mikhail Mikhasenko<sup>l,m</sup>, Victor I. Mokeev<sup>a</sup>, Emilie Passemard<sup>a,d,e</sup>, Alessandro Pilloni<sup>n,o,\*</sup>, Arkaitz Rodas<sup>a,p</sup>, Jorge A. Silva-Castro<sup>f</sup>, Wyatt A. Smith<sup>d</sup>, Adam P. Szczepaniak<sup>a,d,e</sup>, Daniel Winney<sup>d,e,q,r</sup>,

(Joint Physics Analysis Center)

<sup>a</sup>Theory Center and Physics Division, Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA

<sup>b</sup>Instituto de Física Corpuscular (IFIC), Centro Mixto CSIC-Universidad de Valencia, E-46071 Valencia, Spain

<sup>c</sup>Pedagogical University of Krakow, 30-084 Kraków, Poland

<sup>d</sup>Department of Physics, Indiana University, Bloomington, IN 47405, USA

<sup>e</sup>Center for Exploration of Energy and Matter, Indiana University, Bloomington, IN 47403, USA

<sup>f</sup>Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Ciudad de México 04510, Mexico

<sup>g</sup>Departamento de Física Interdisciplinar, Universidad Nacional de Educación a Distancia (UNED), Madrid E-28040, Spain

<sup>h</sup>Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

<sup>i</sup>Department of Physics, Old Dominion University, Norfolk, VA 23529, USA

<sup>j</sup>Departament de Física Quàntica i Astrofísica and Institut de Ciències del Cosmos, Universitat de Barcelona, E-08028, Spain

<sup>k</sup>Departamento de Física Teórica, Universidad Complutense de Madrid and IPARCOS, E-28040 Madrid, Spain

<sup>l</sup>ORIGINS Excellence Cluster, 80939 Munich, Germany

<sup>m</sup>Ludwig-Maximilian University of Munich, Germany

<sup>n</sup>Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra, Università degli Studi di Messina, I-98122 Messina, Italy

<sup>o</sup>INFN Sezione di Catania, I-95123 Catania, Italy

<sup>q</sup>Guangdong Provincial Key Laboratory of Nuclear Science, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, China

<sup>r</sup>Guangdong-Hong Kong Joint Laboratory of Quantum Matter, Southern Nuclear Science Computing Center, South China Normal University, Guangzhou 510006, China

# Python Notebooks

I have created a few Jupyter notebooks to accompany the lectures

- All notebooks use **python**, and the modules that come with the [Anaconda package manager](#)
- Includes introductory and advanced exercises
- <https://github.com/ajackura/RPISummerSchoolHadspec>

The screenshot shows a Jupyter Notebook interface with a dark theme. The left sidebar (EXPLORER) lists three notebooks: 'hadspec\_ex0\_complex.ipynb' (selected), 'hadspec\_ex1\_scattering.ipynb', and 'hadspec\_ex2\_luescher.ipynb'. Below these are 'LICENSE' and 'README.md'. The main area displays the content of 'hadspec\_ex0\_complex.ipynb'. The title 'A Brief On Complex Analysis' is shown, along with a section on 'References' and a list of useful books and links. The code cell contains Python code for plotting scattering amplitudes, and the resulting plot shows the real (red line) and imaginary (blue line) parts of the amplitude as a function of  $E^*/m$ .

```
# Routine to plot amplitude
def plot_amplitude( Kmatrix ):
    eps = 1e-16
    Ecm_o_m = np.arange(1.8, 3, 0.01)
    amp = Amplitude( Ecm_o_m**2+1j*eps, Kmatrix )
    plt.axhline(y=0.0, color='gray', linestyle='--')
    plt.plot(Ecm_o_m, amp.real, color='red', label="Real")
    plt.plot(Ecm_o_m, amp.imag, color='blue', label="Imag")
    plt.xlabel(r'$E^*/m$', size=15)
    plt.ylabel(r'$\mathcal{M}$', size=15)
    plt.xticks(fontname="Futura", fontsize=15)
    plt.yticks(fontname="Futura", fontsize=15)
    plt.legend(loc="upper right")
    plt.figure(figsize=(2,1), dpi= 100, facecolor='w', edgecolor='k')

# Sample Effective Range parameters
a = 2.0 # /m
r = 0.0 # /m

# Sample Breit-Wigner parameters
m0 = 2.5 # m
g0 = 3.0

# Plot Breit-Wigner
plot_amplitude( lambda s:Kmatrix_BreitWigner(s,m0,g0) )
```

...  
Importing useful libraries  
...  
import cmath as cm  
import math as m  
import numpy as np  
import scipy.special as sp  
import numpy.random as rn  
import matplotlib.pyplot as plt  
from scipy import integrate  
from mpl\_toolkits import mplot3d  
from matplotlib.colors import hsv\_to\_rgb  
from colorsys import hls\_to\_rgb

# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

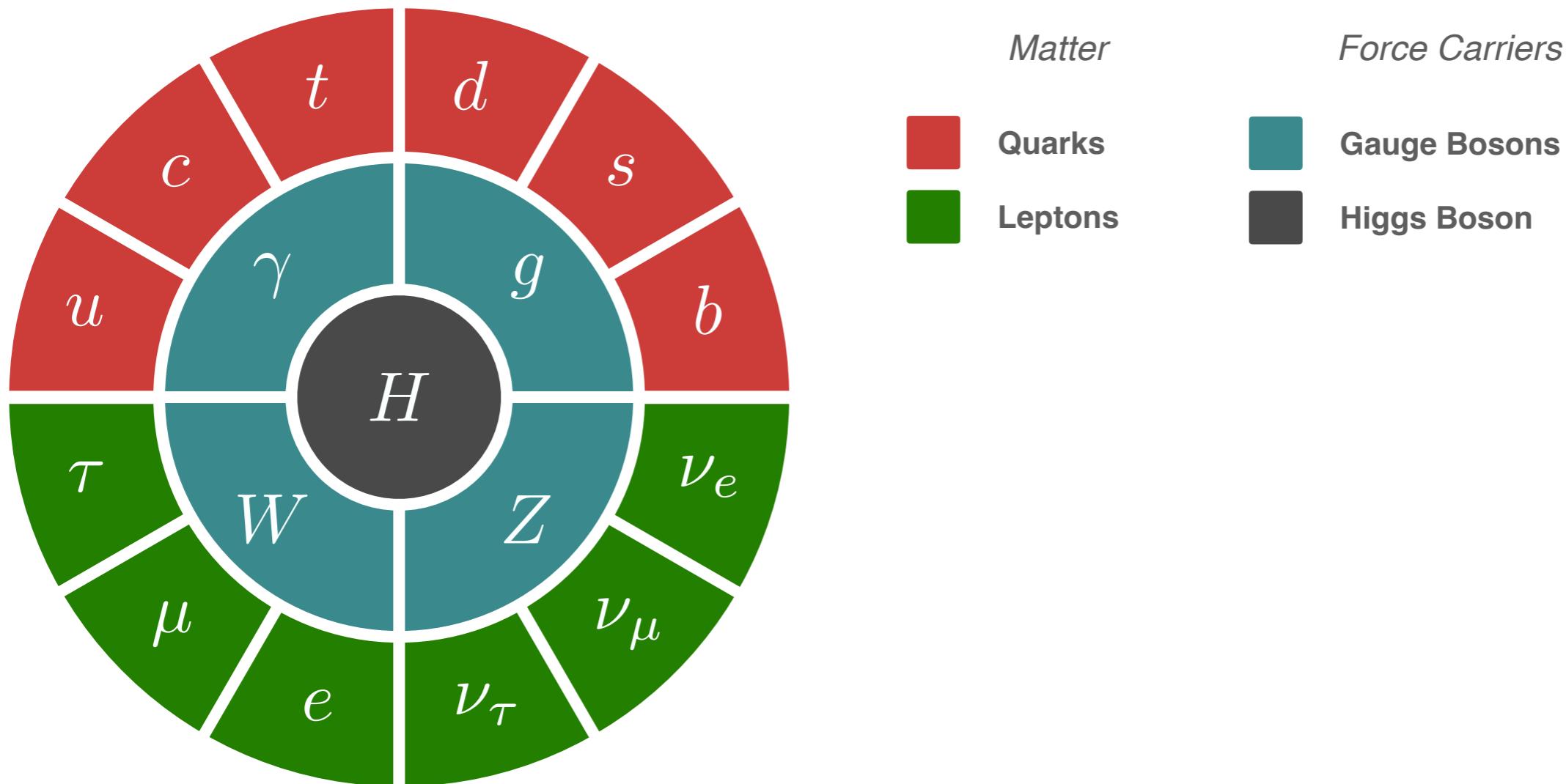
## Lattice QCD & Hadron Spectroscopy

Lattice QCD

Lüscher & the Finite-Volume

# The Standard Model of Particle Physics

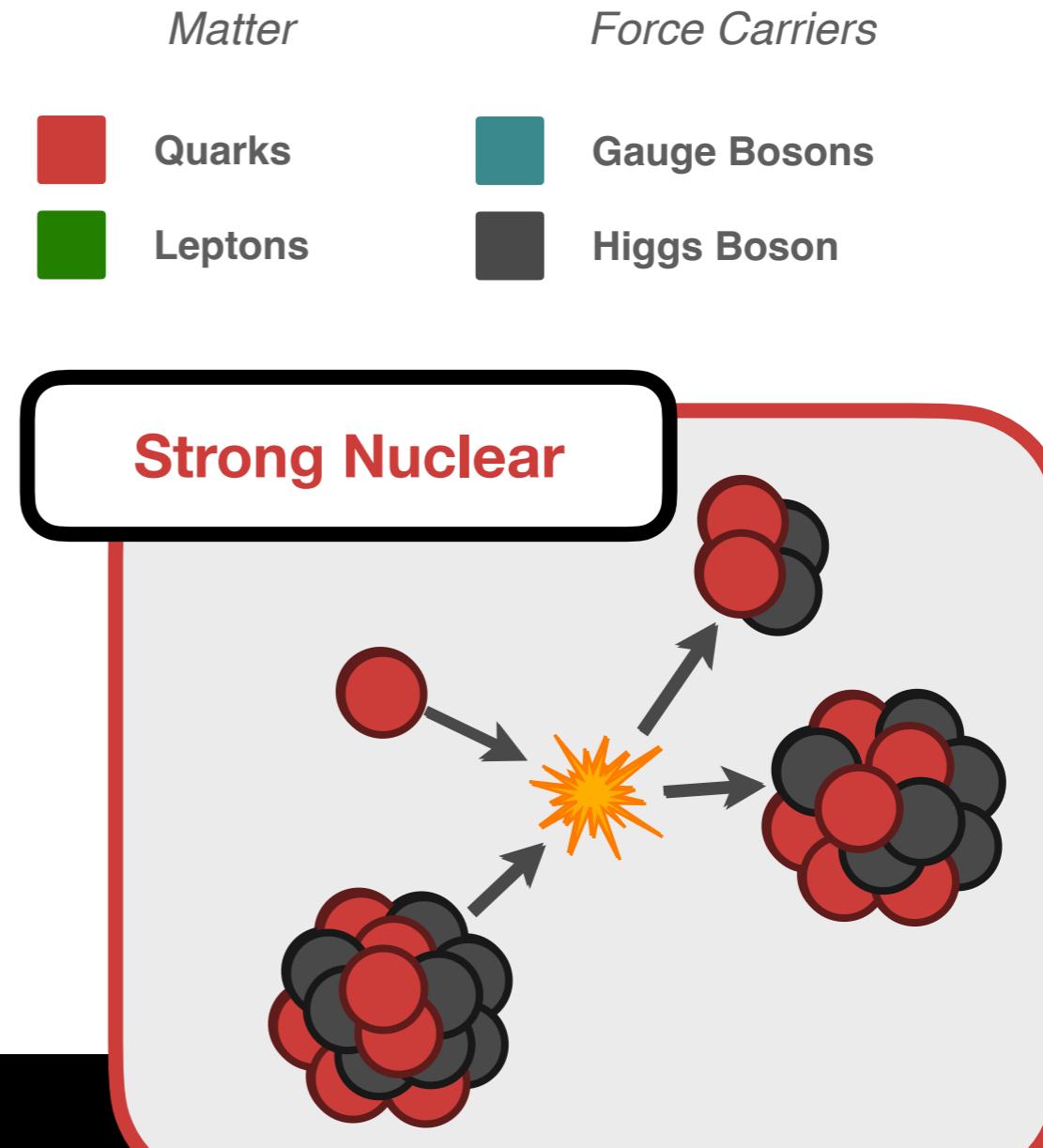
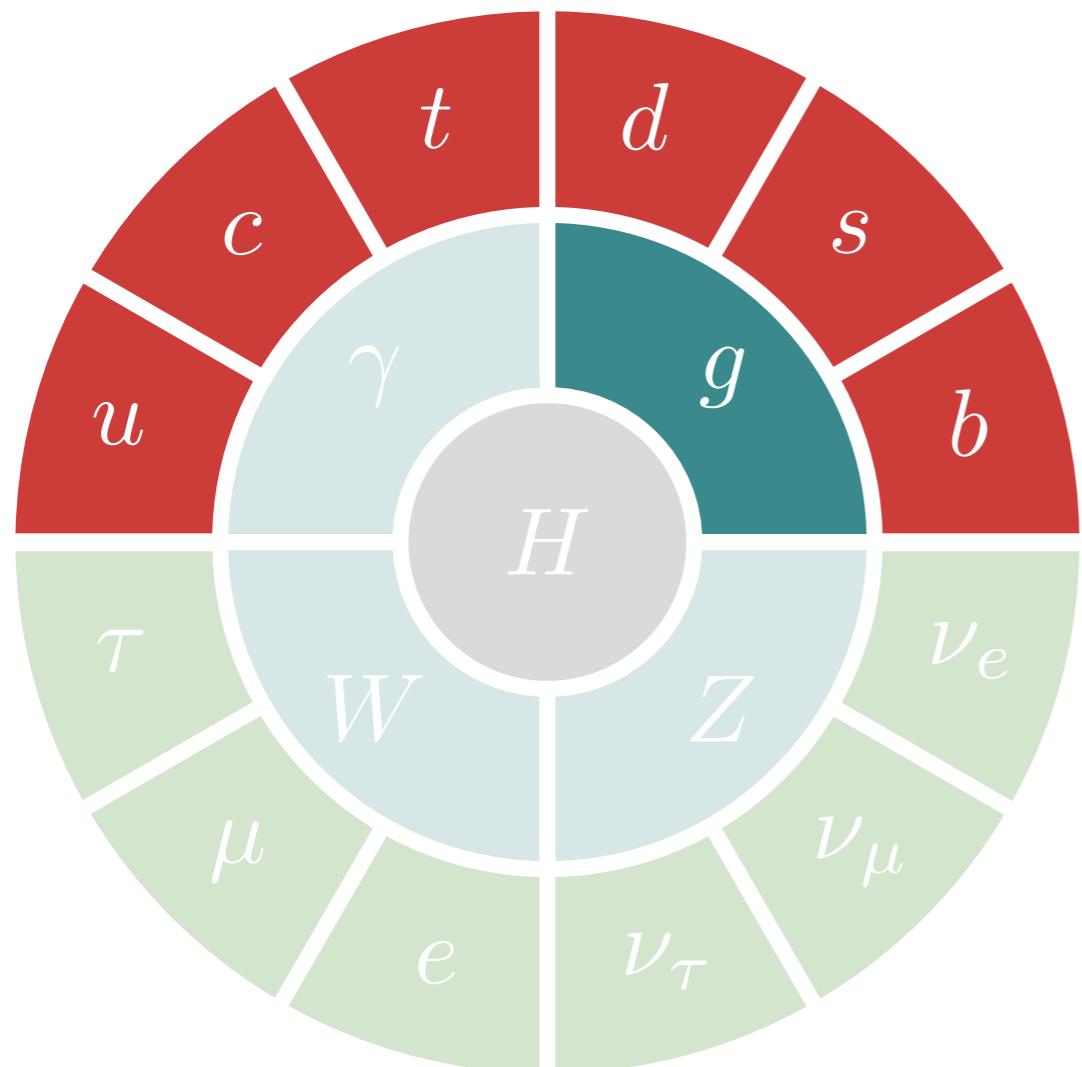
Nature can be described by a remarkably *simple\** theory



\* simple = An anomaly-free renormalizable relativistic quantum gauge field theory, invariant under the gauge group  $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$  which spontaneously breaks via a scalar field to  $SU(3)_C \otimes U(1)_Q$

# The Standard Model of Particle Physics

Nature can be described by a remarkably *simple\** theory



\* simple = An a

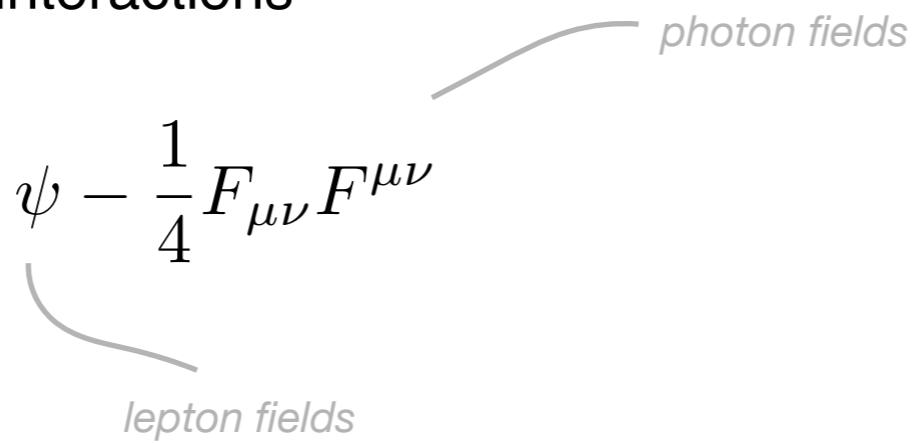
**Quantum ChromoDynamics (QCD)**

invariant under the gauge group  $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$   
which spontaneously breaks via a scalar field to  $SU(3)_C \otimes U(1)_Q$

# Quantum ElectroDynamics (QED)

Theory of electron-photon interactions

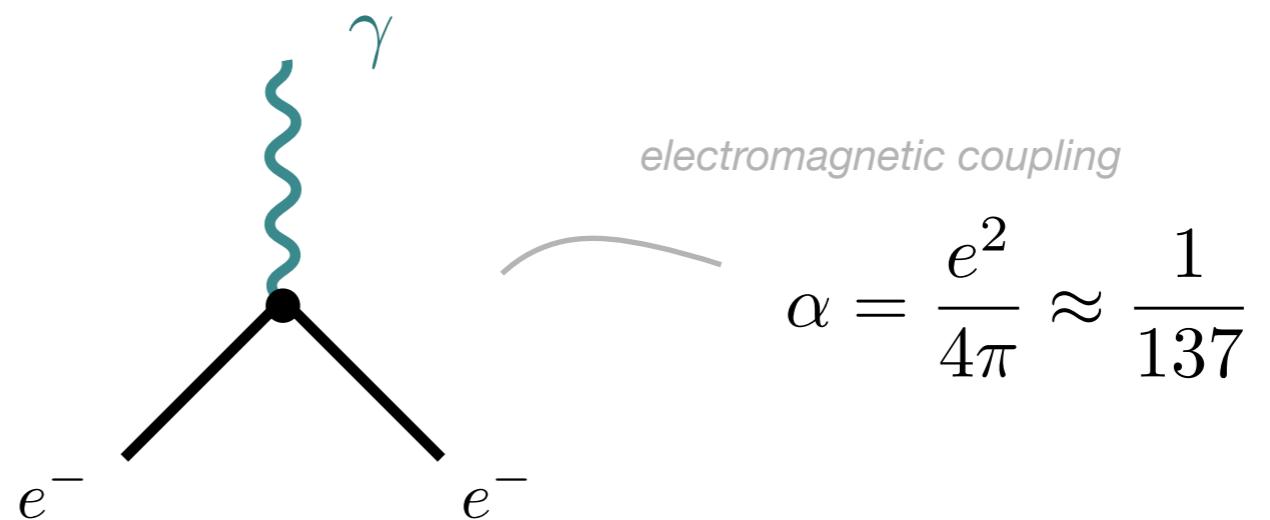
$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$



# Quantum ElectroDynamics (QED)

Theory of electron-photon interactions

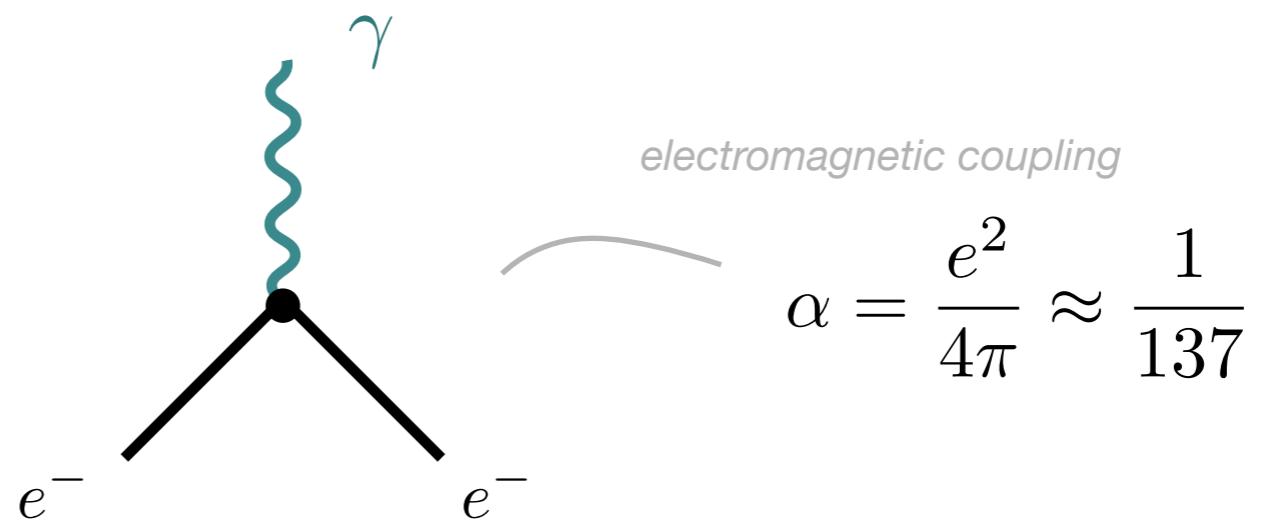
$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$



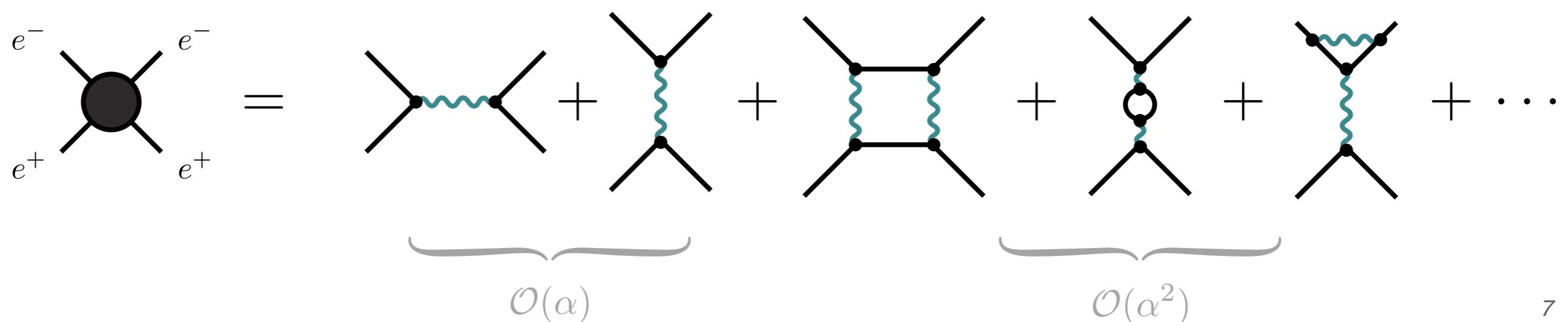
# Quantum ElectroDynamics (QED)

Theory of electron-photon interactions

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$



Can compute observables *perturbatively* in electric coupling

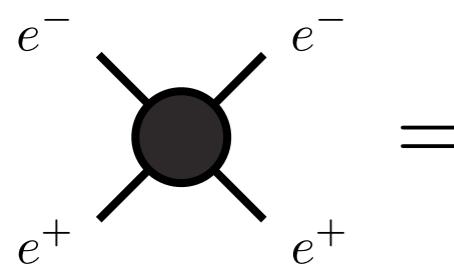


# Quantum ElectroDynamics (QED)

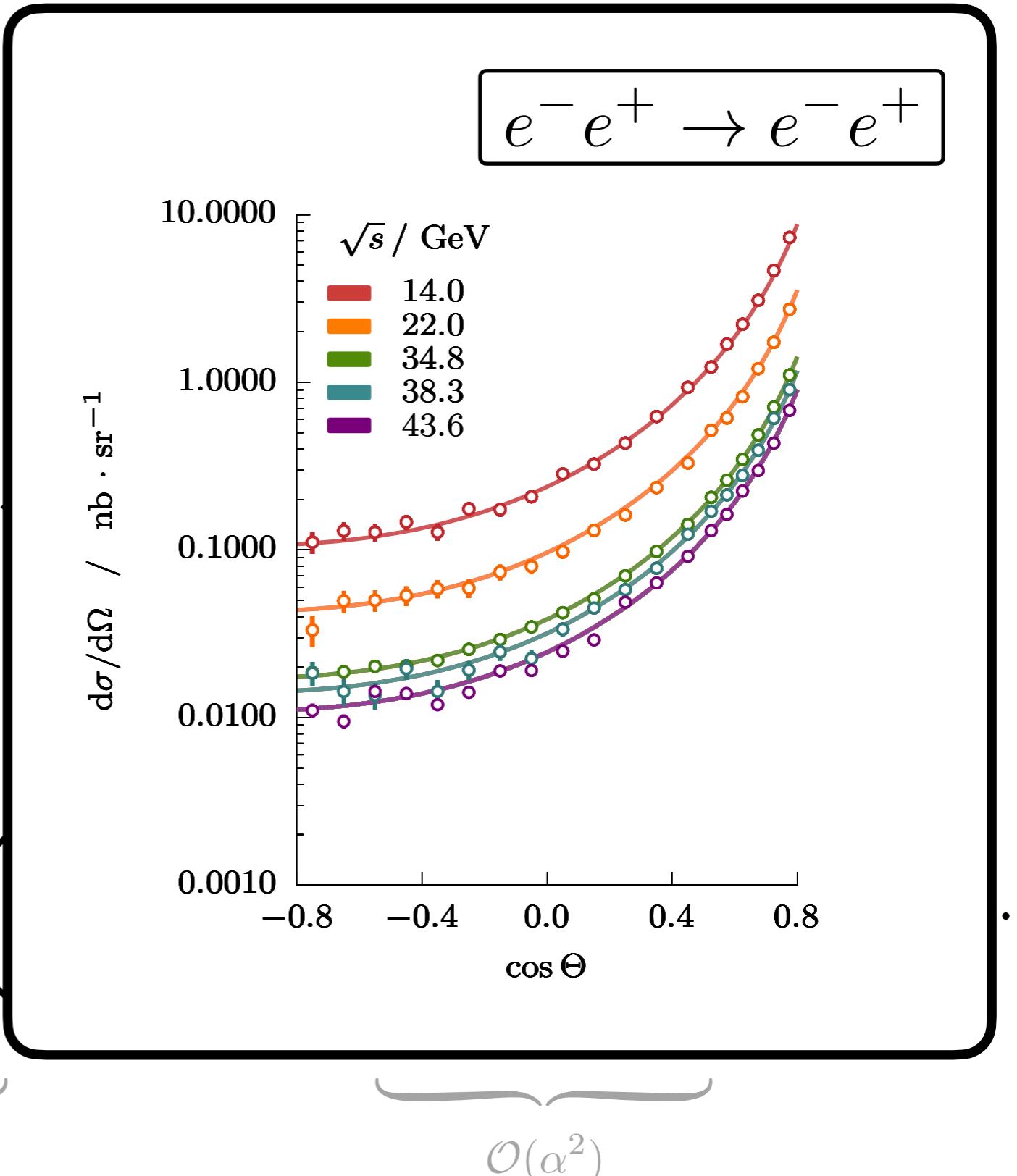
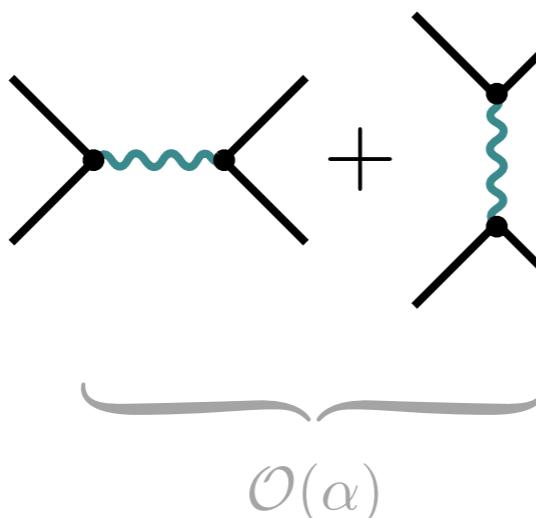
Theory of electron-photon interactions

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i \not{D} - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Can compute observables *perturbatively* in



=



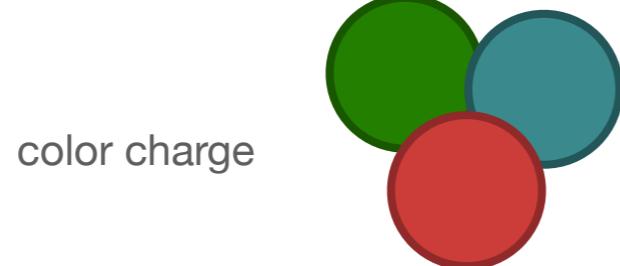
# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

quark fields

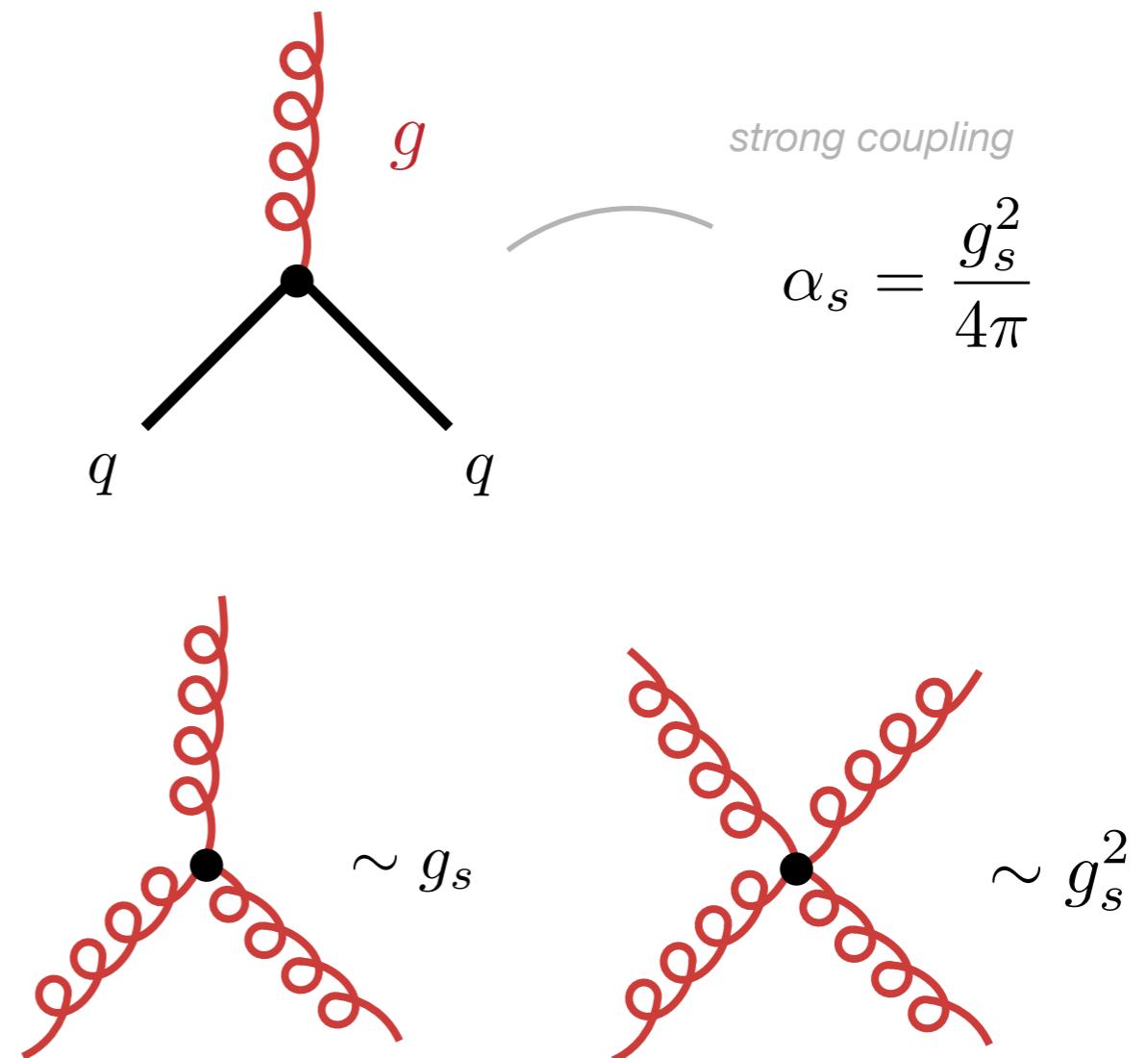
gluon fields



# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

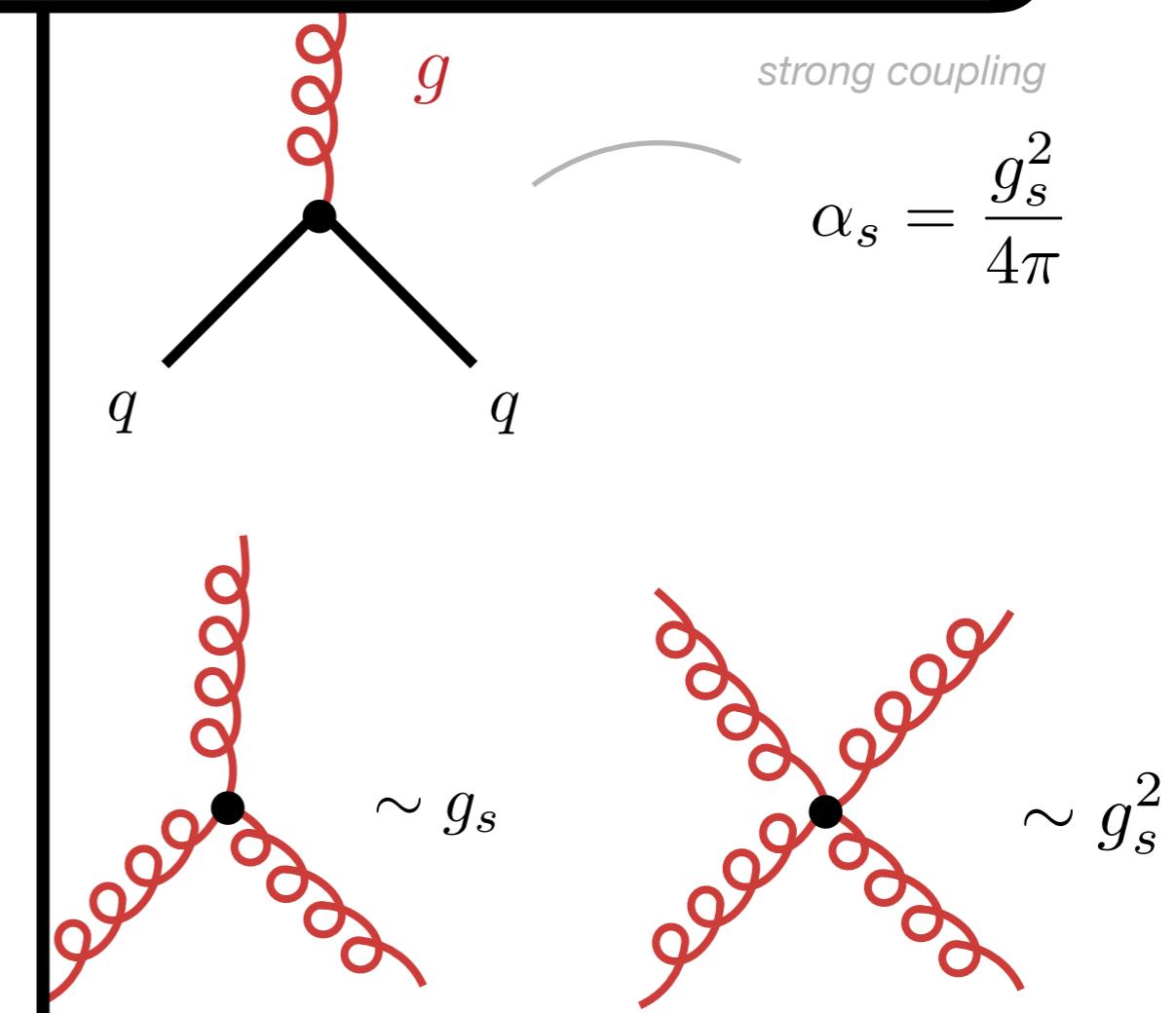
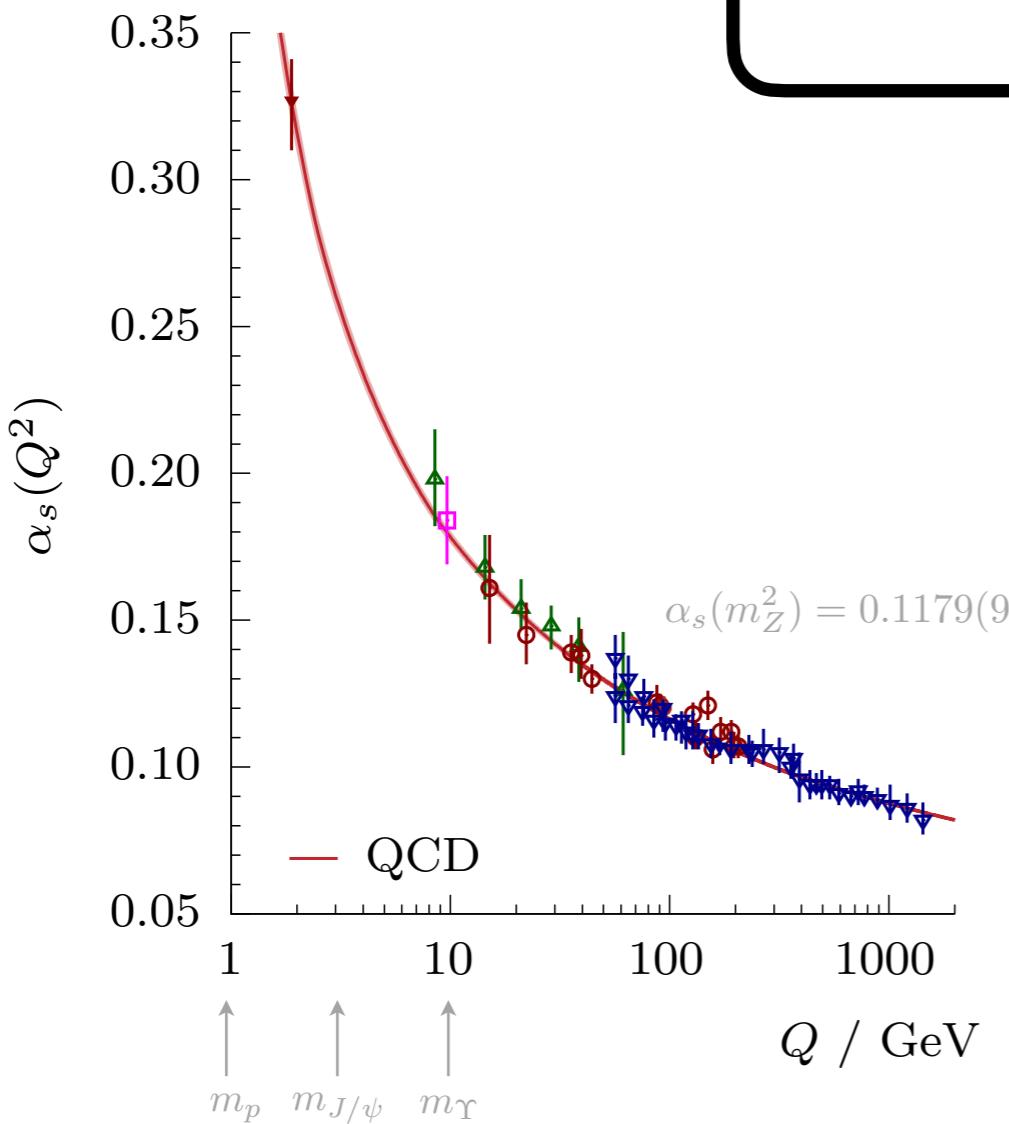


# Quantum ChromoDynamics (QCD)

Theory of quark-gluon interactions

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\cancel{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$

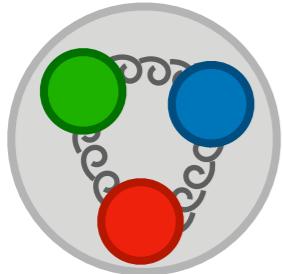
At low-energy we don't observe free quarks...  
...we observe composite **hadrons**



# Quantum ChromoDynamics (QCD)

Hadrons are the *color-neutral* bound states of quarks & gluons (**confinement**)

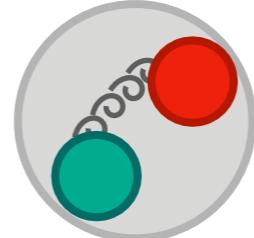
**Baryons (fermions)**



$$J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$$

$p, n, \Delta, \dots$

**Mesons (bosons)**



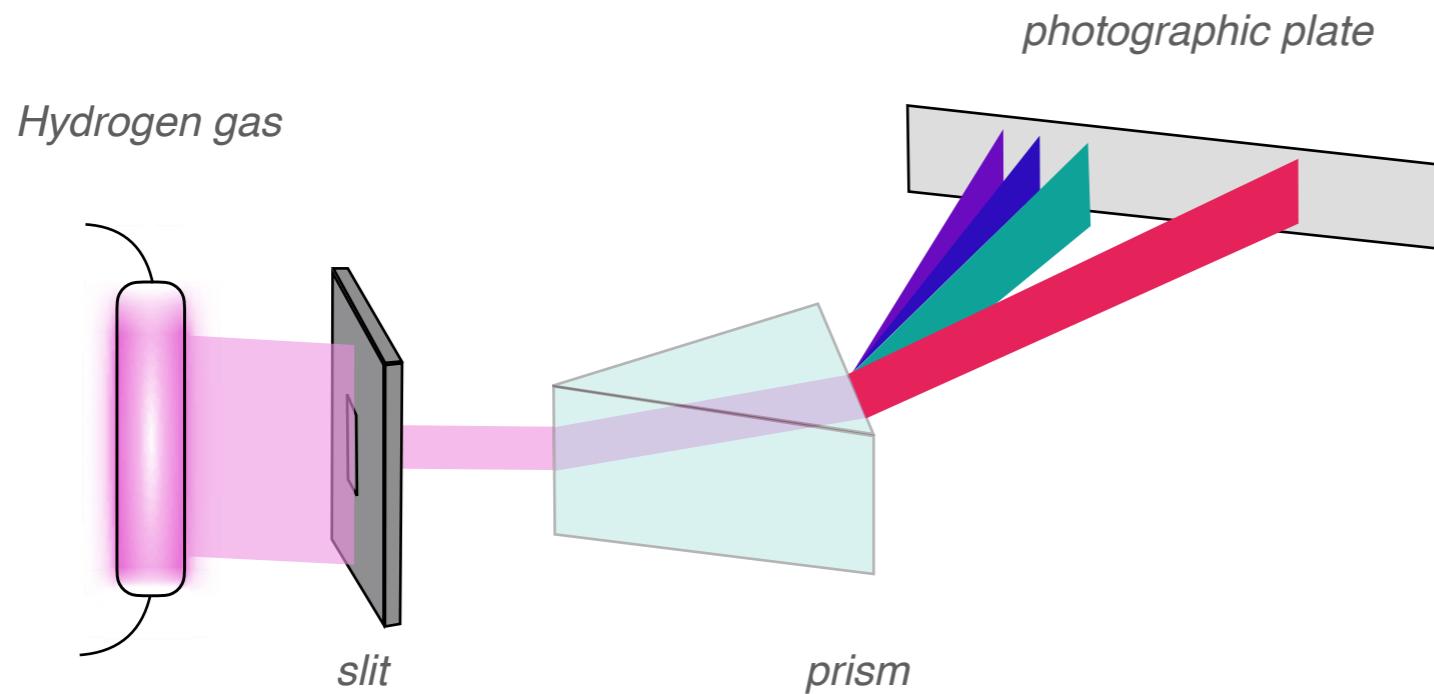
$$J = 0, 1, 2, \dots$$

$\pi, K, \eta, \dots$

# Spectroscopy

Spectroscopy provides vital information towards understanding physical phenomena

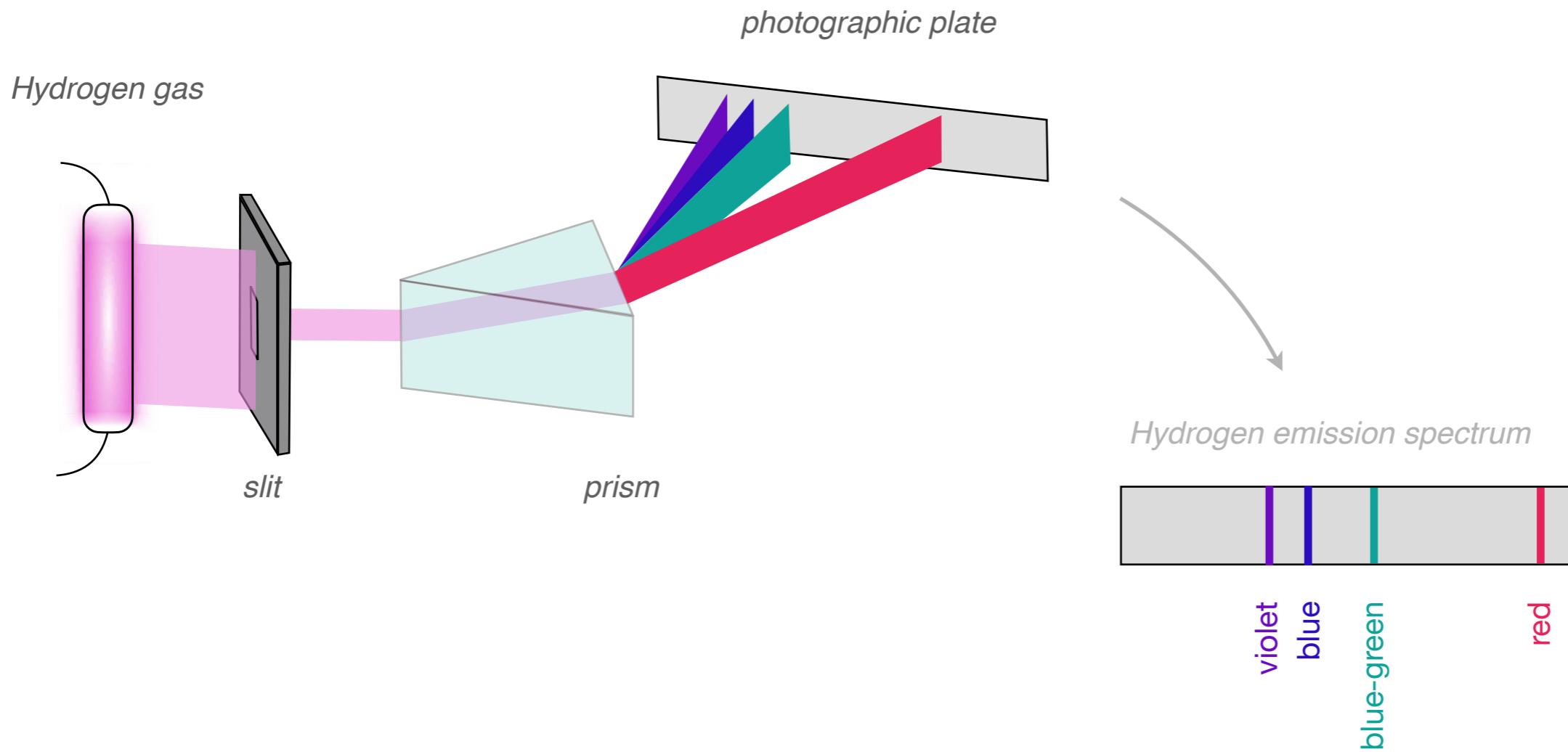
- Example – the Hydrogen atom led to the discovery of Quantum Mechanics



# Spectroscopy

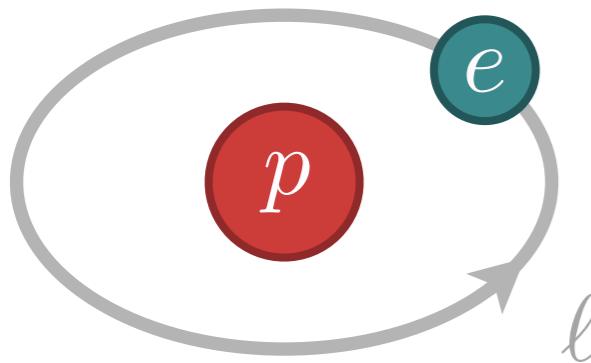
Spectroscopy provides vital information towards understanding physical phenomena

- Example – the Hydrogen atom led to the discovery of Quantum Mechanics



Spectroscopy provides vital information towards understanding physical phenomena

- Example – the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for Quantum ElectroDynamics (QED)



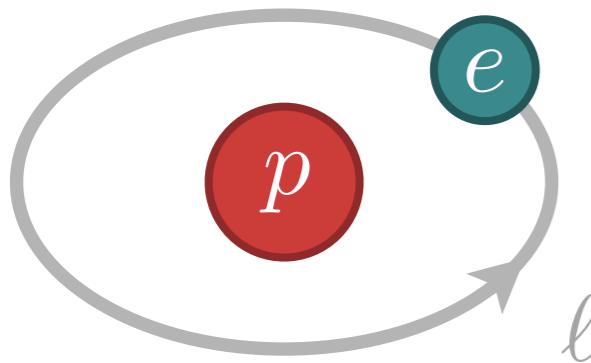
$$\hat{H} |\psi\rangle = E_n |\psi\rangle \quad \textit{Schrödinger equation}$$

# Spectroscopy

$$\hbar = c = 1$$

Spectroscopy provides vital information towards understanding physical phenomena

- Example – the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for Quantum ElectroDynamics (QED)

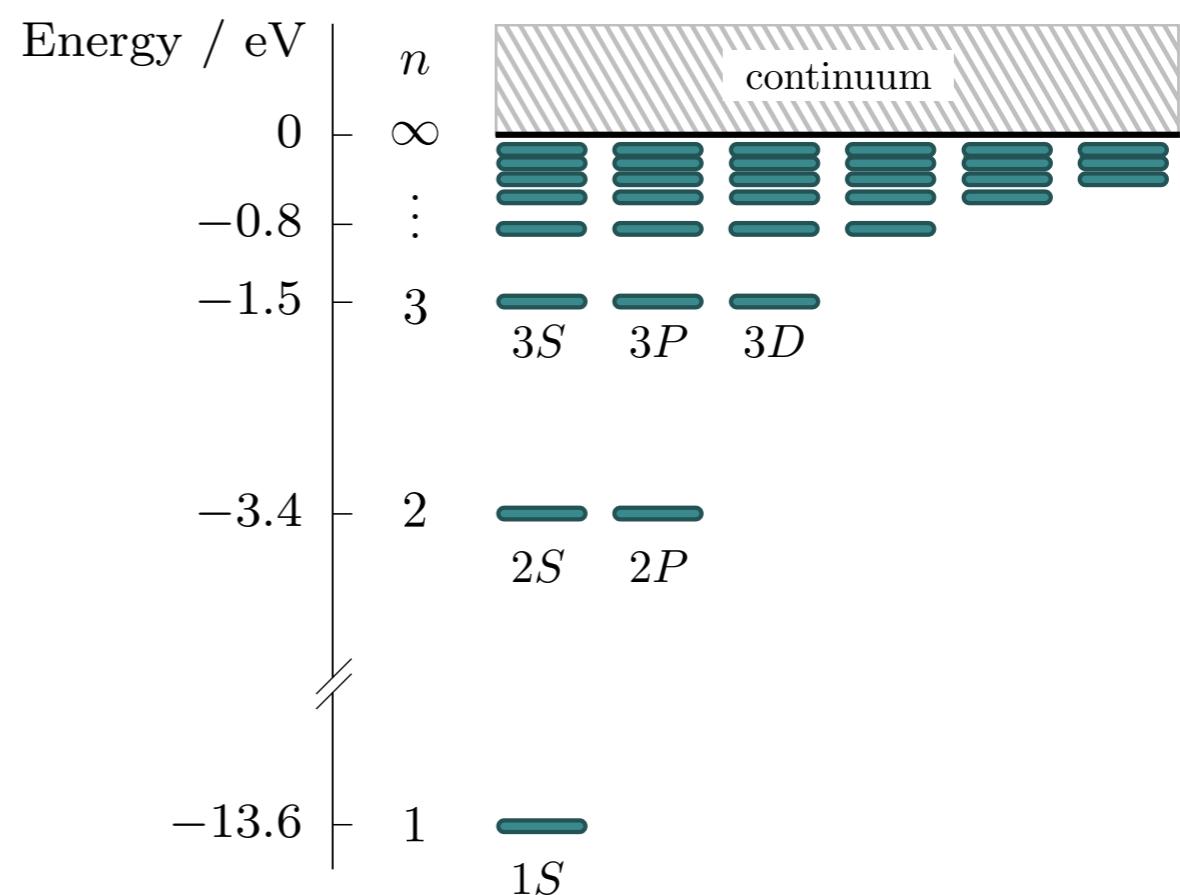


$$\hat{H} |\psi\rangle = E_n |\psi\rangle$$

spectrum

$$E_n = -\frac{m\alpha^2}{2n}$$

$$\alpha = \frac{e^2}{4\pi}$$



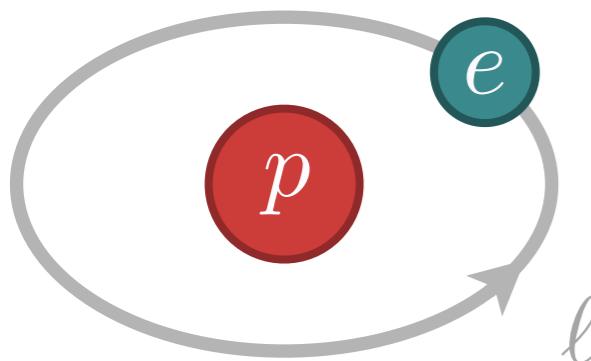
$$\ell = 0(S), 1(P), 2(D), \dots$$

# Spectroscopy

$$\hbar = c = 1$$

Spectroscopy provides vital information towards understanding physical phenomena

- Example – the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for Quantum ElectroDynamics (QED)

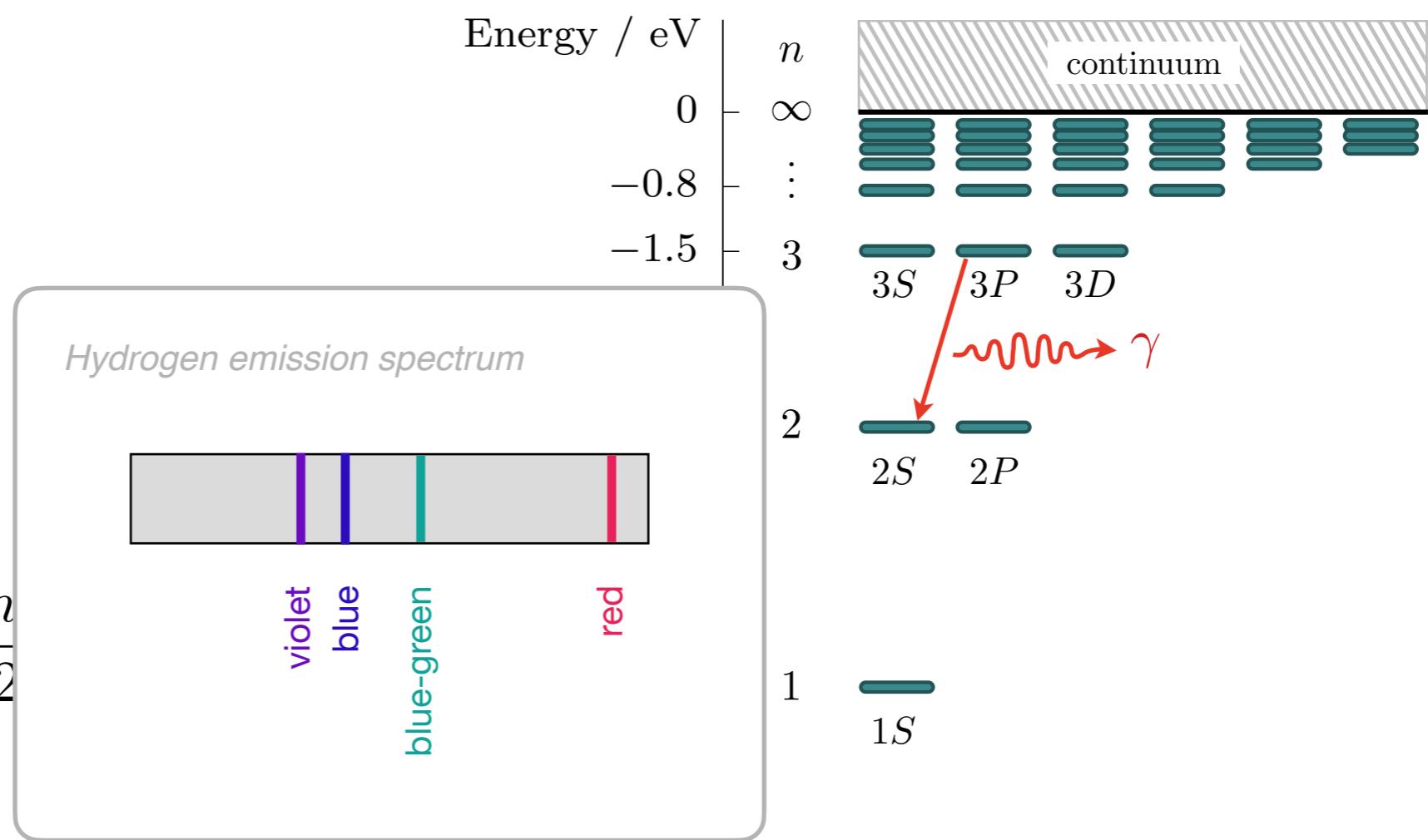


$$\hat{H} |\psi\rangle = E_n |\psi\rangle$$

spectrum

$$E_n = -\frac{m}{2}$$

$$\alpha = \frac{e^2}{4\pi}$$

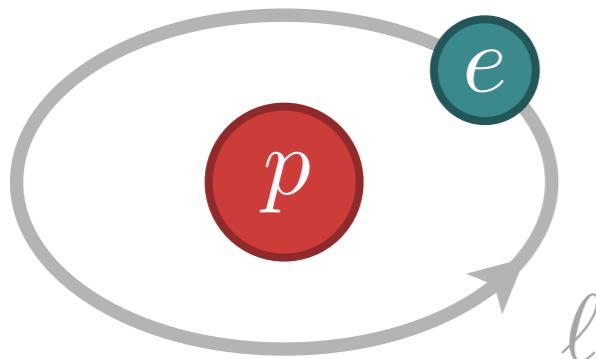


# Spectroscopy

$$\hbar = c = 1$$

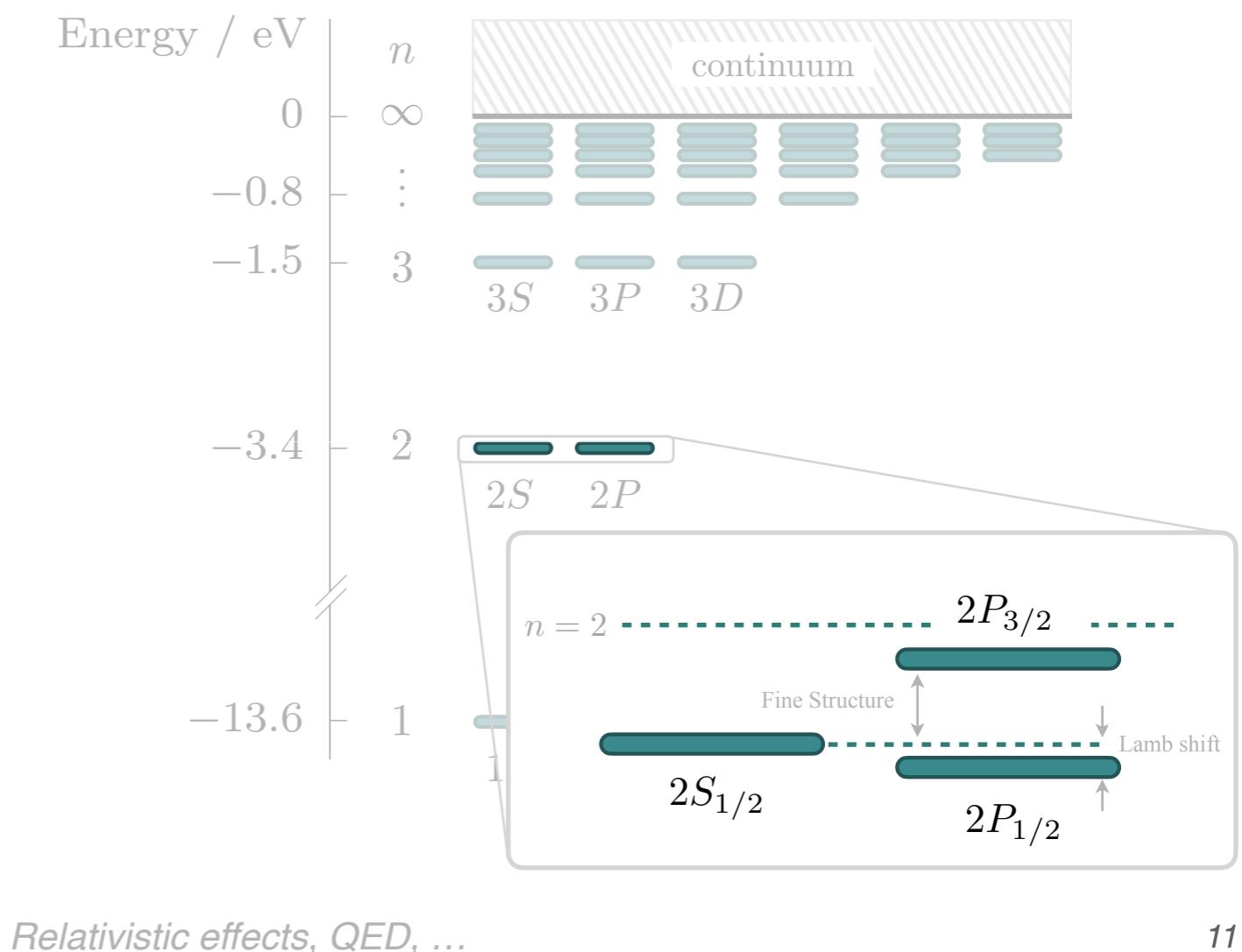
Spectroscopy provides vital information towards understanding physical phenomena

- Example – the Hydrogen atom led to the discovery of Quantum Mechanics
- Precision spectroscopy paved the way for Quantum ElectroDynamics (QED)



$$\hat{H} |\psi\rangle = E_n |\psi\rangle$$

$$E_n = -\frac{m\alpha^2}{2n} + \mathcal{O}(\alpha^4)$$
$$\alpha = \frac{e^2}{4\pi}$$

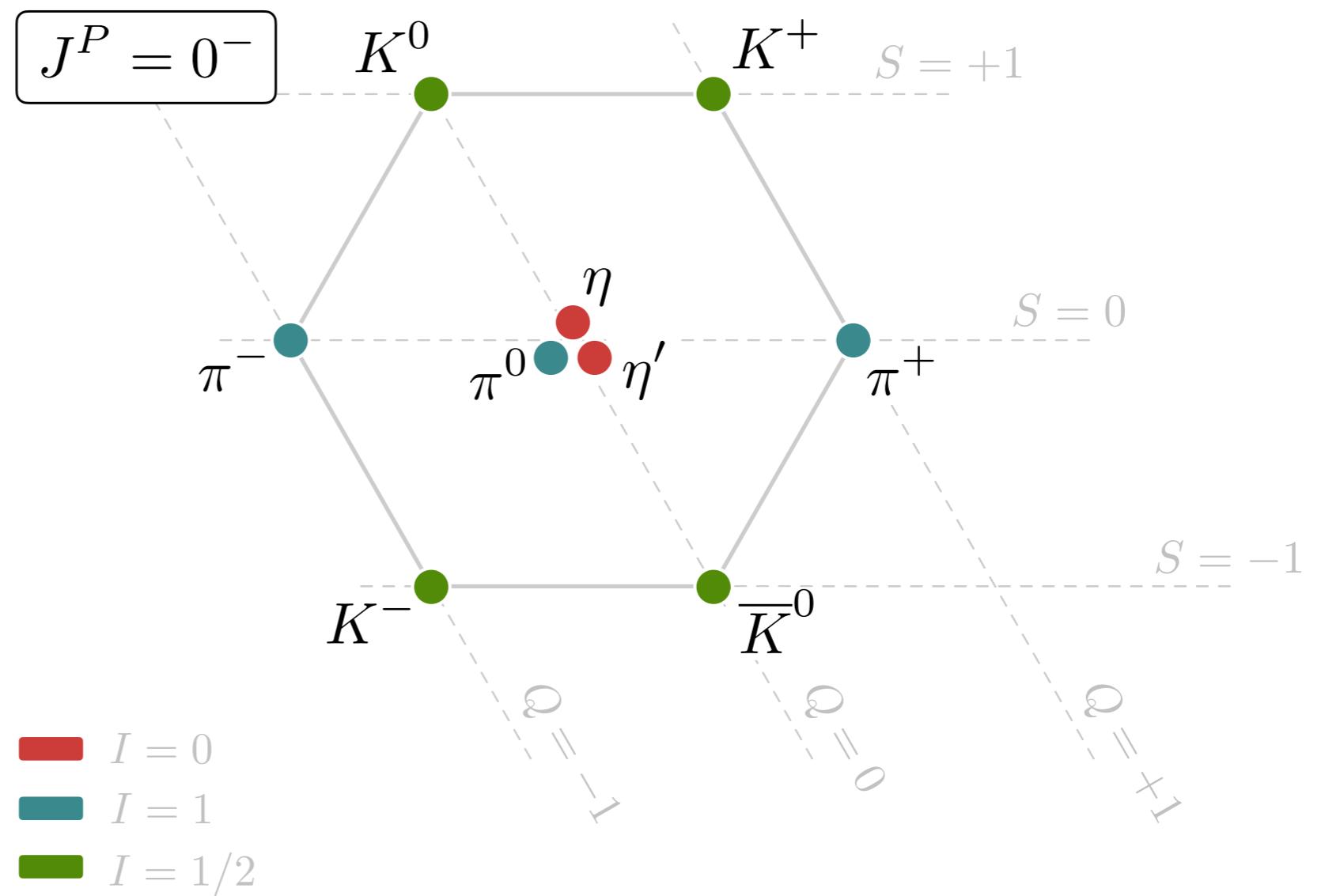


Relativistic effects, QED, ...

# QCD Spectroscopy

Hadrons are classified by their ***conserved quantum numbers***

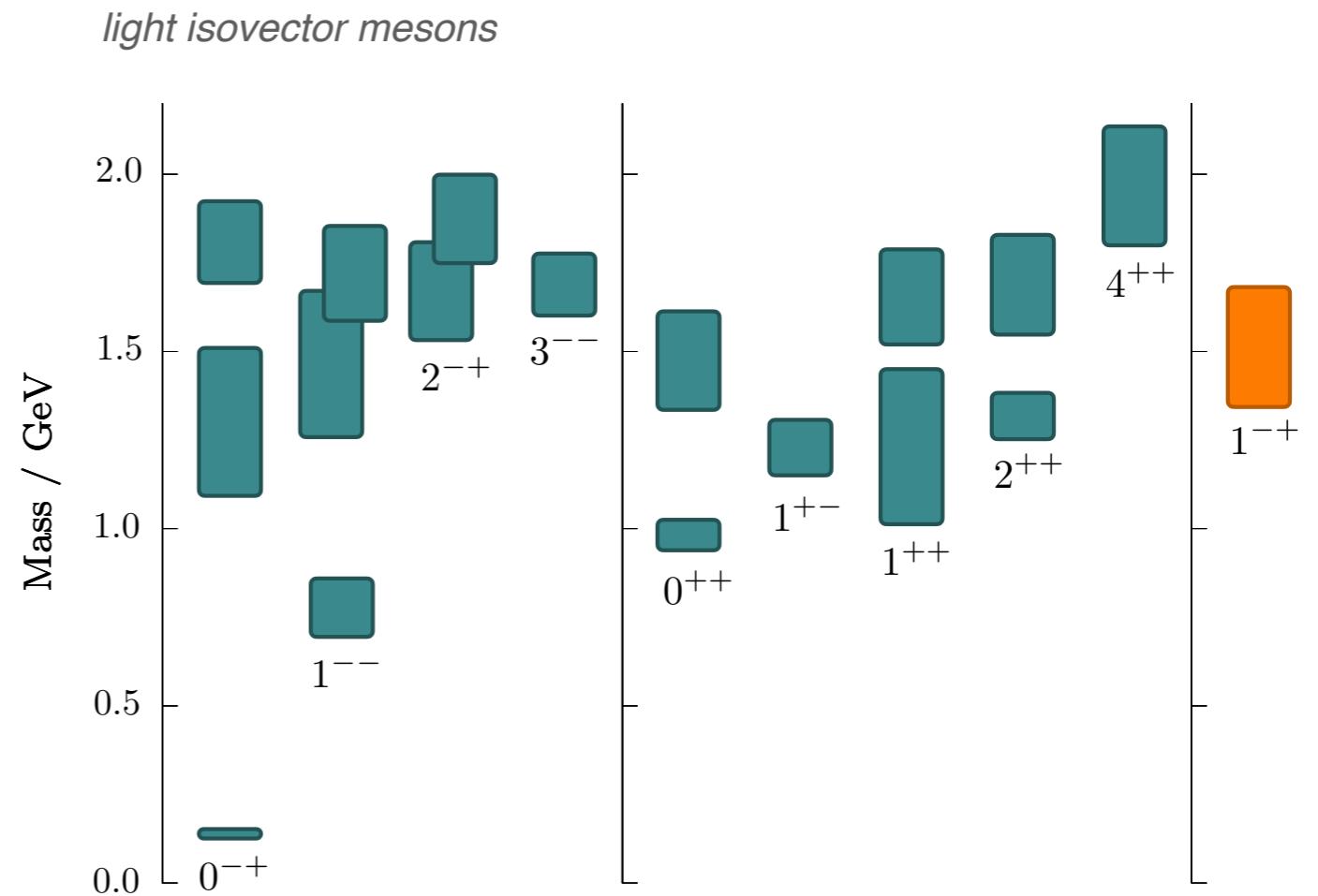
- spin ( $J$ )
- parity ( $P$ )
- charge-conjugation ( $C$ )
- isospin ( $I$ )
- strangeness ( $S$ )
- ...



# QCD Spectroscopy

Hadrons are classified by their *conserved quantum numbers*

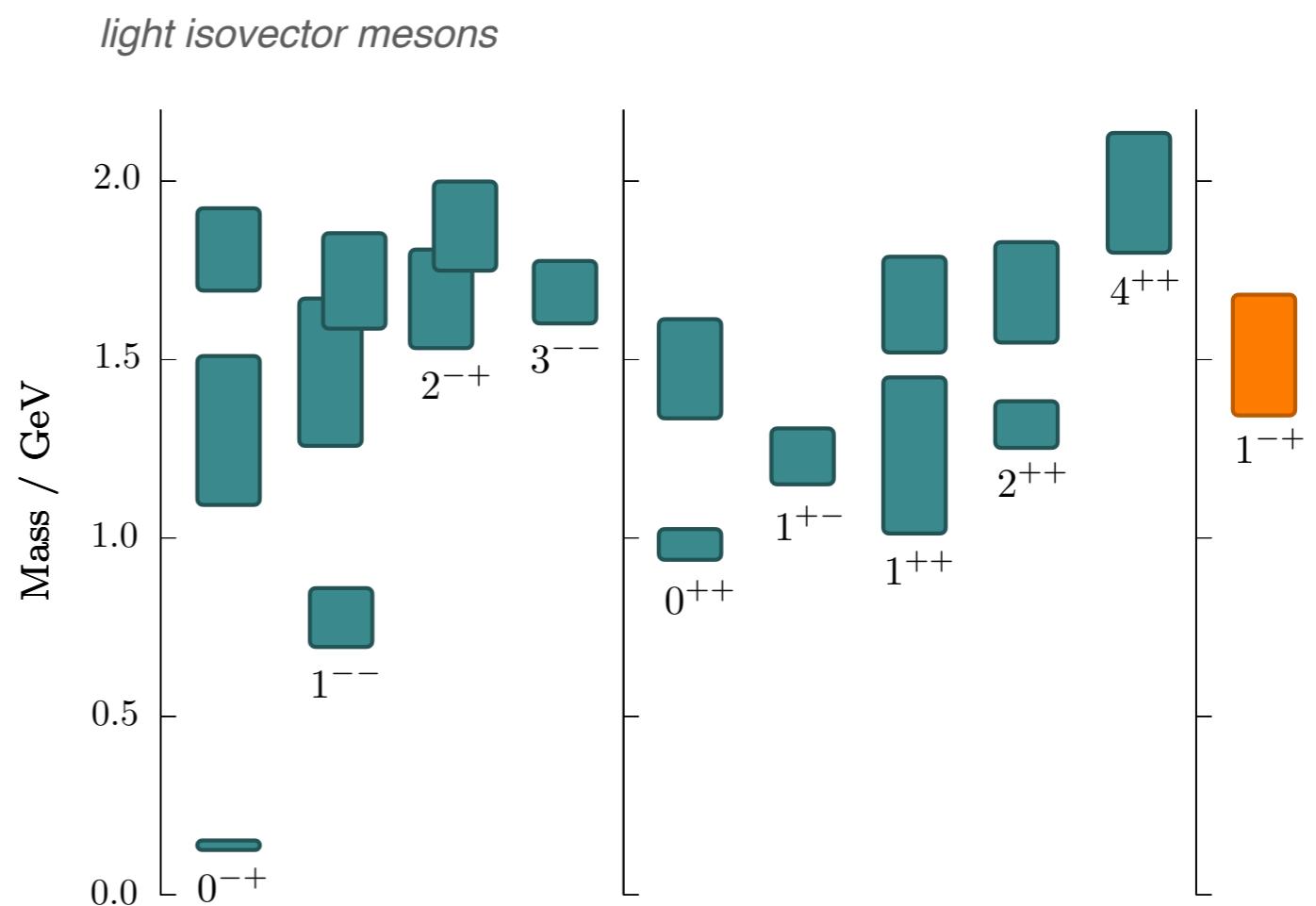
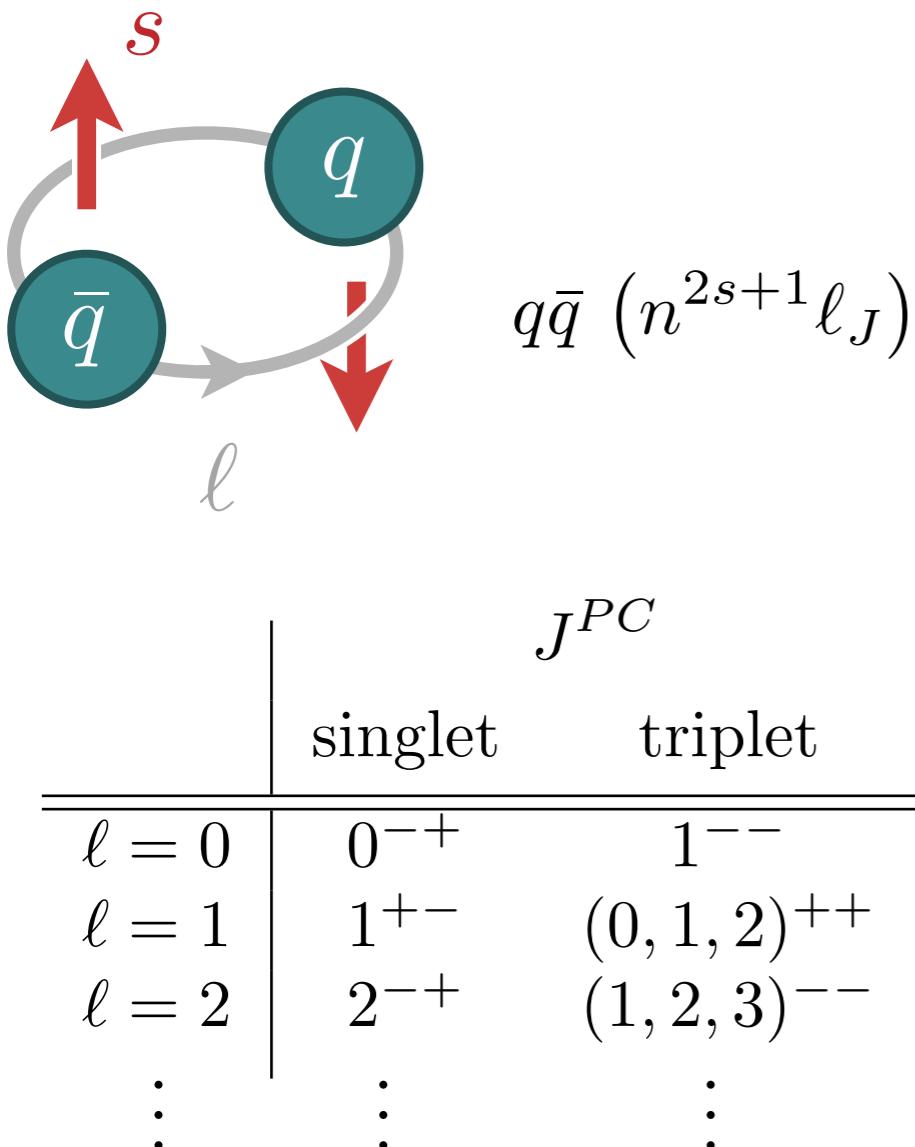
- Constituent quark model gives gross structure



# QCD Spectroscopy

Hadrons are classified by their **conserved quantum numbers**

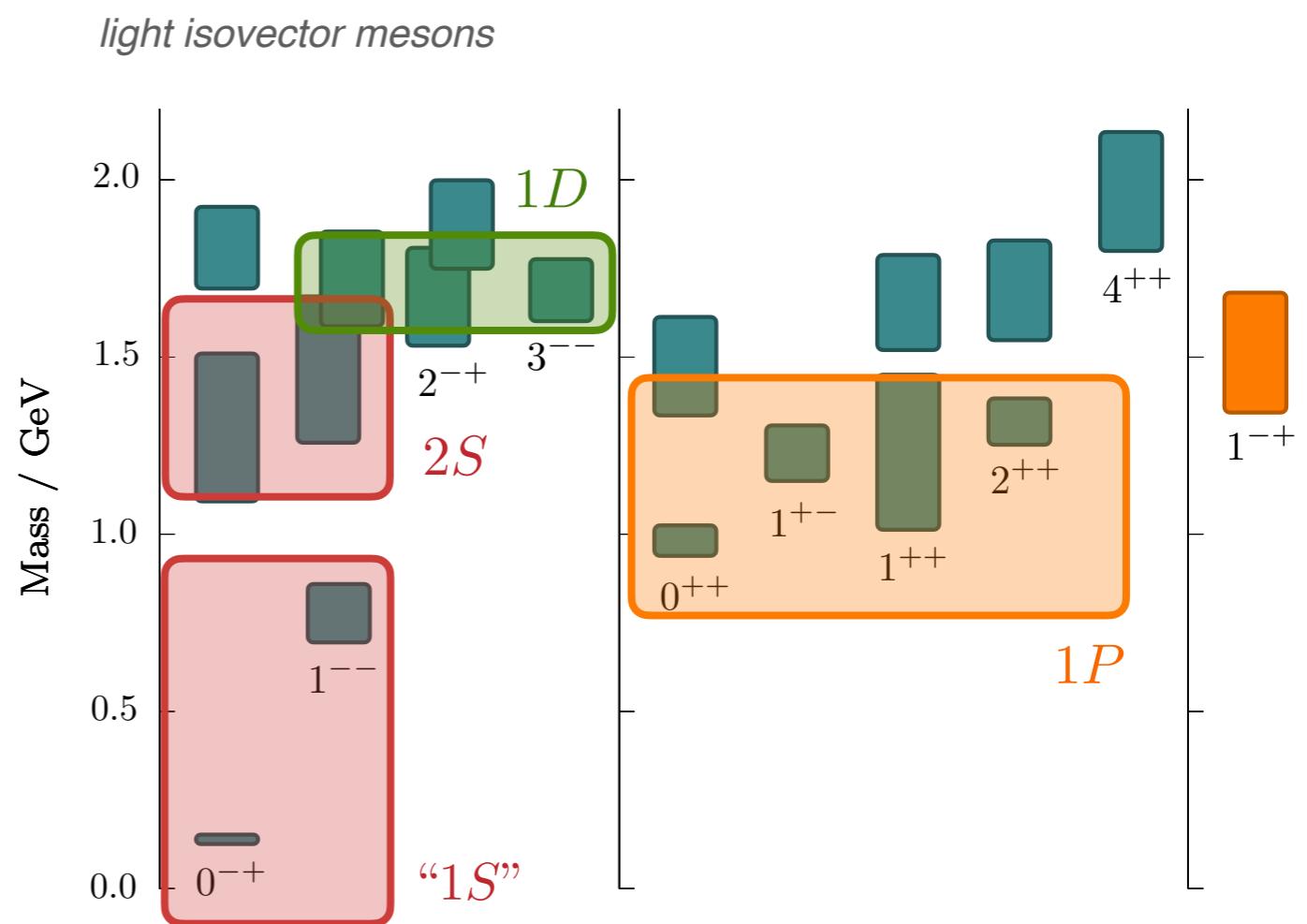
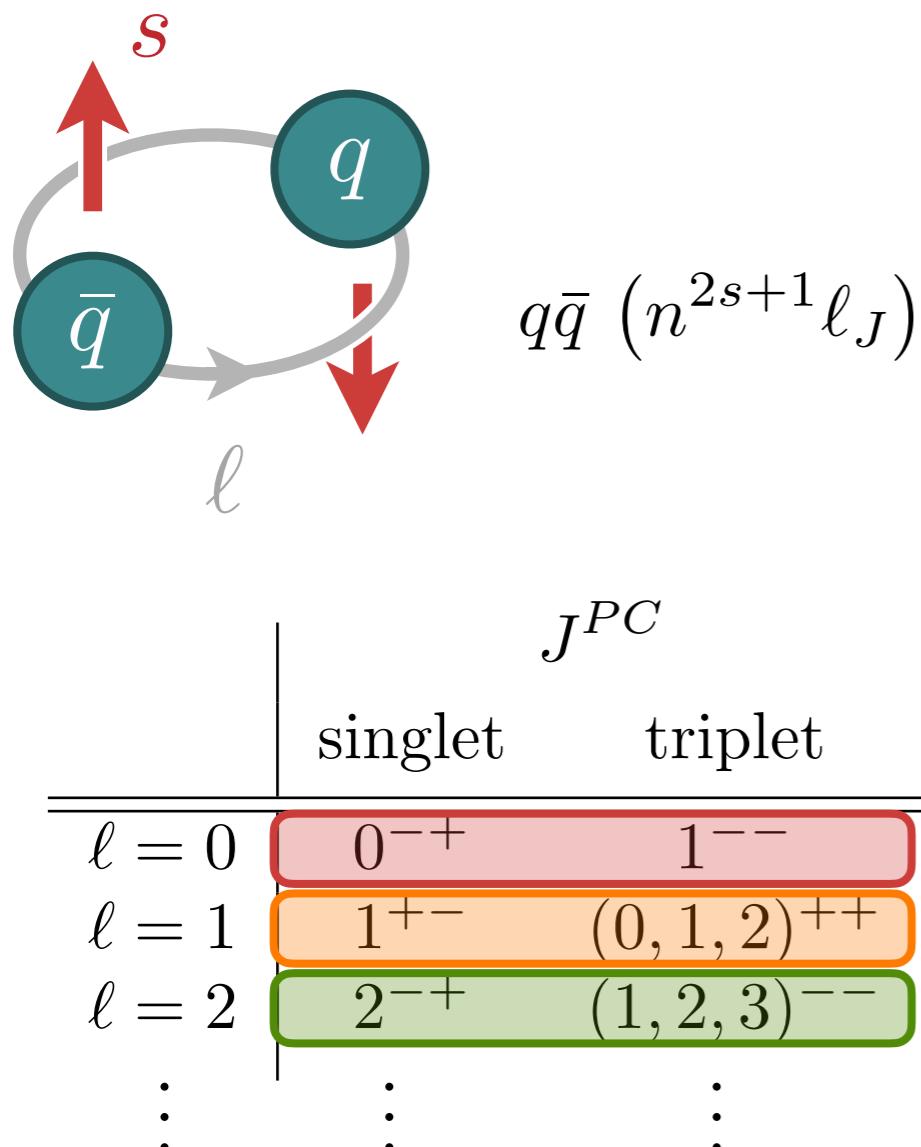
- Constituent quark model gives gross structure



# QCD Spectroscopy

Hadrons are classified by their **conserved quantum numbers**

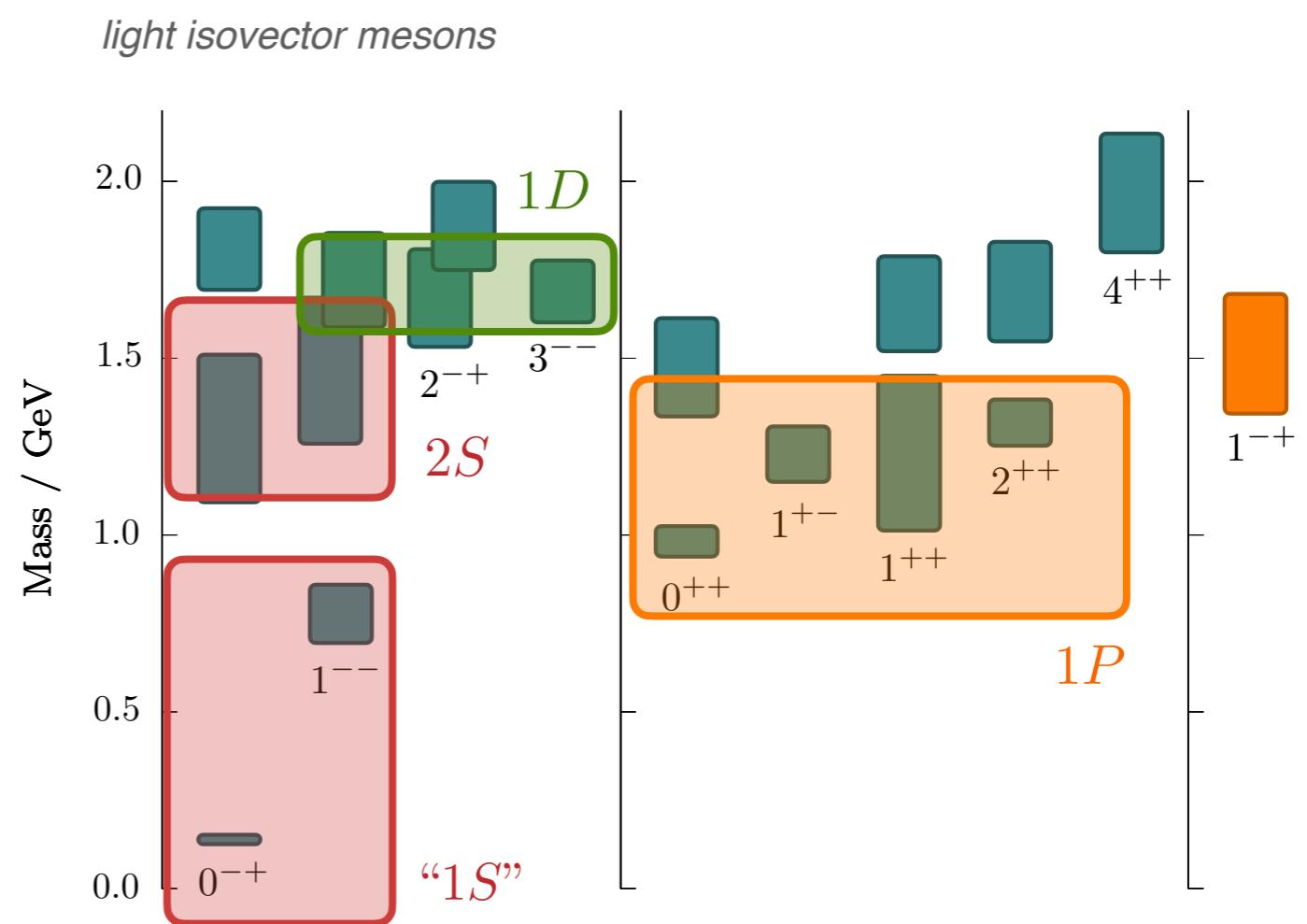
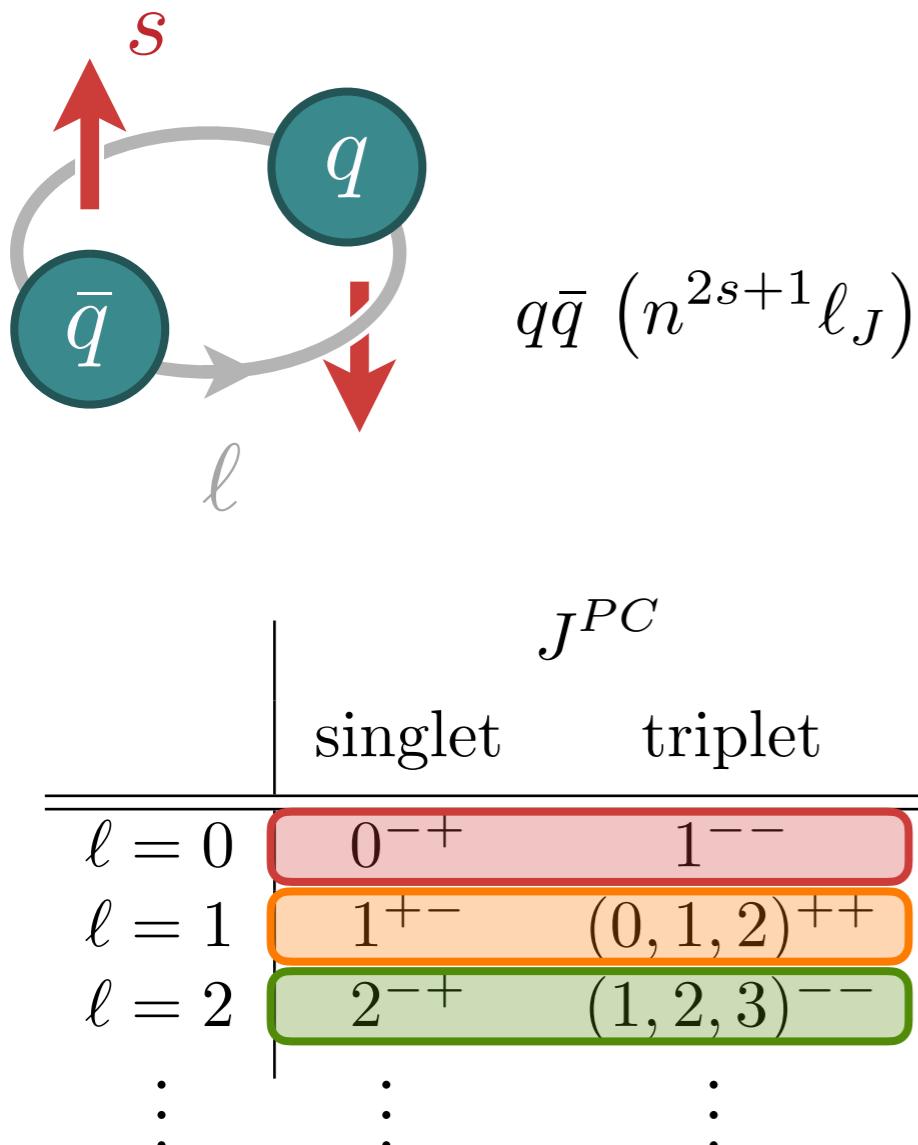
- Constituent quark model gives gross structure



# QCD Spectroscopy

Hadrons are classified by their ***conserved quantum numbers***

- Constituent quark model gives gross structure

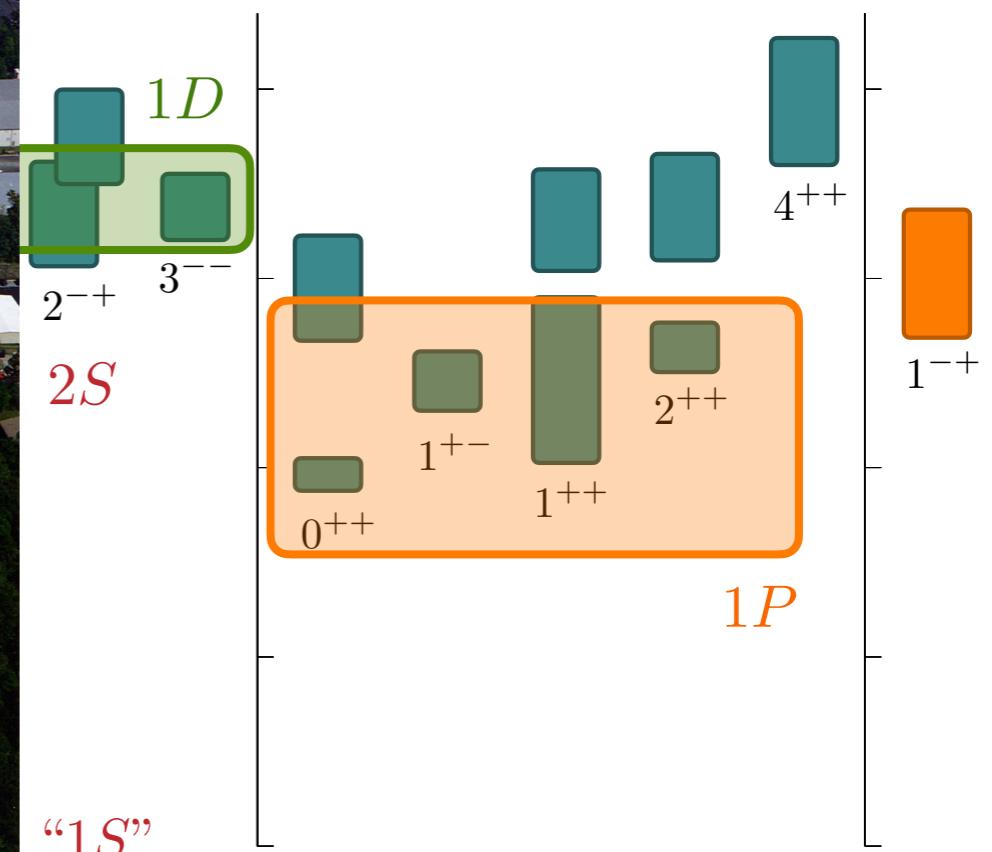
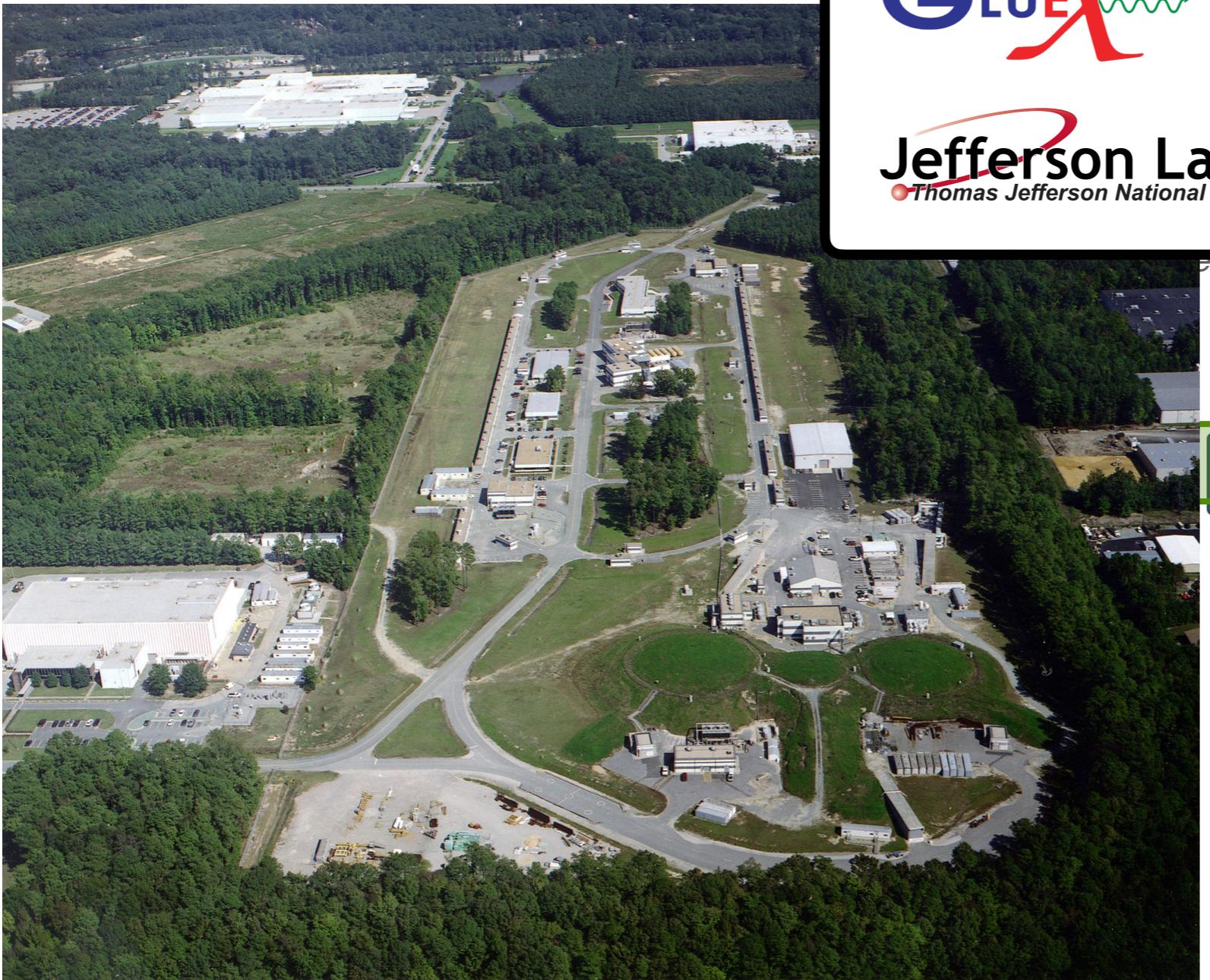


**Forbidden quantum numbers :**  $0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \dots$

# QCD Spectroscopy

Hadrons are classified by their ***conserved quantum numbers***

- Constituent quark model gives gross structure

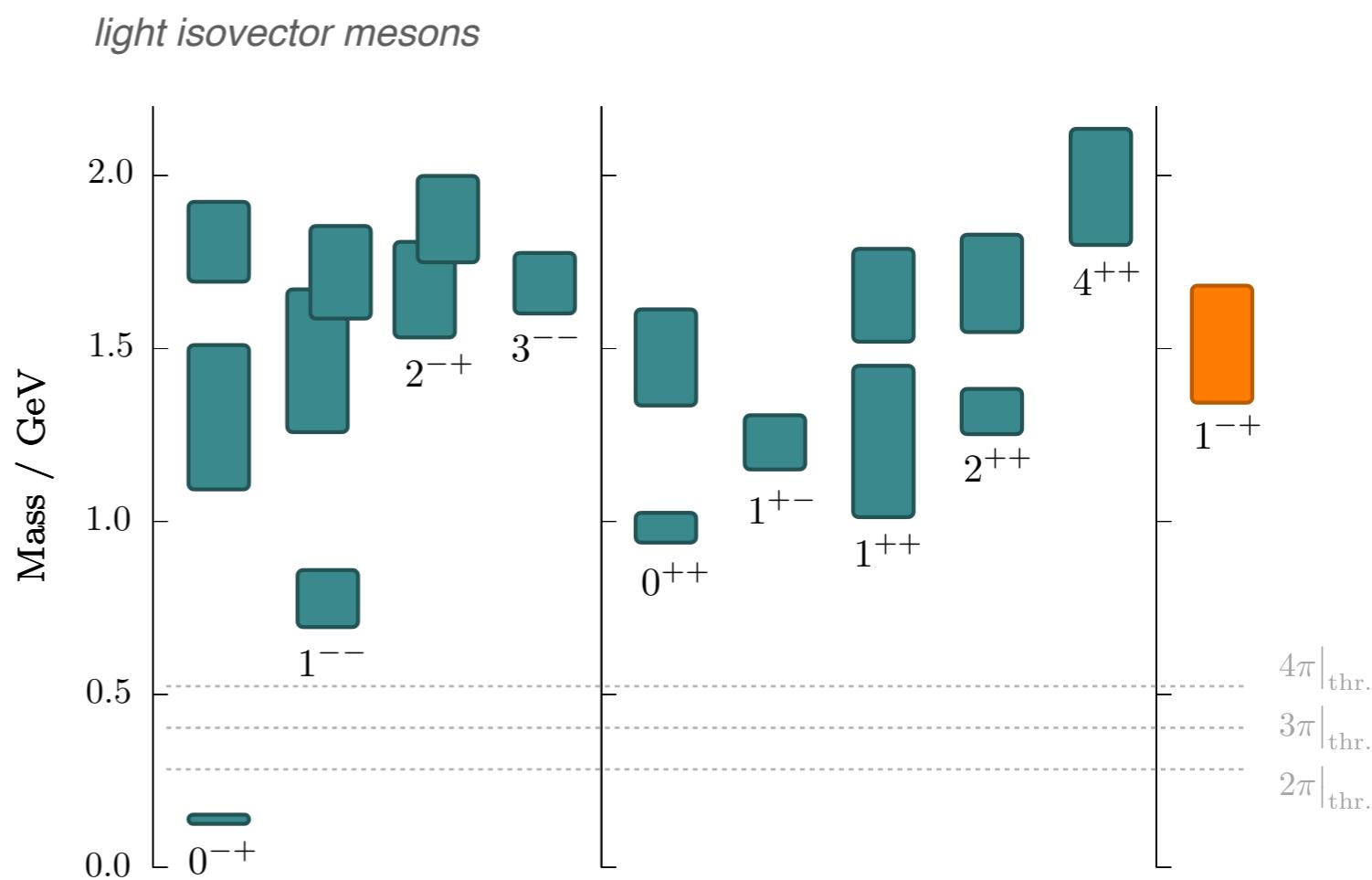
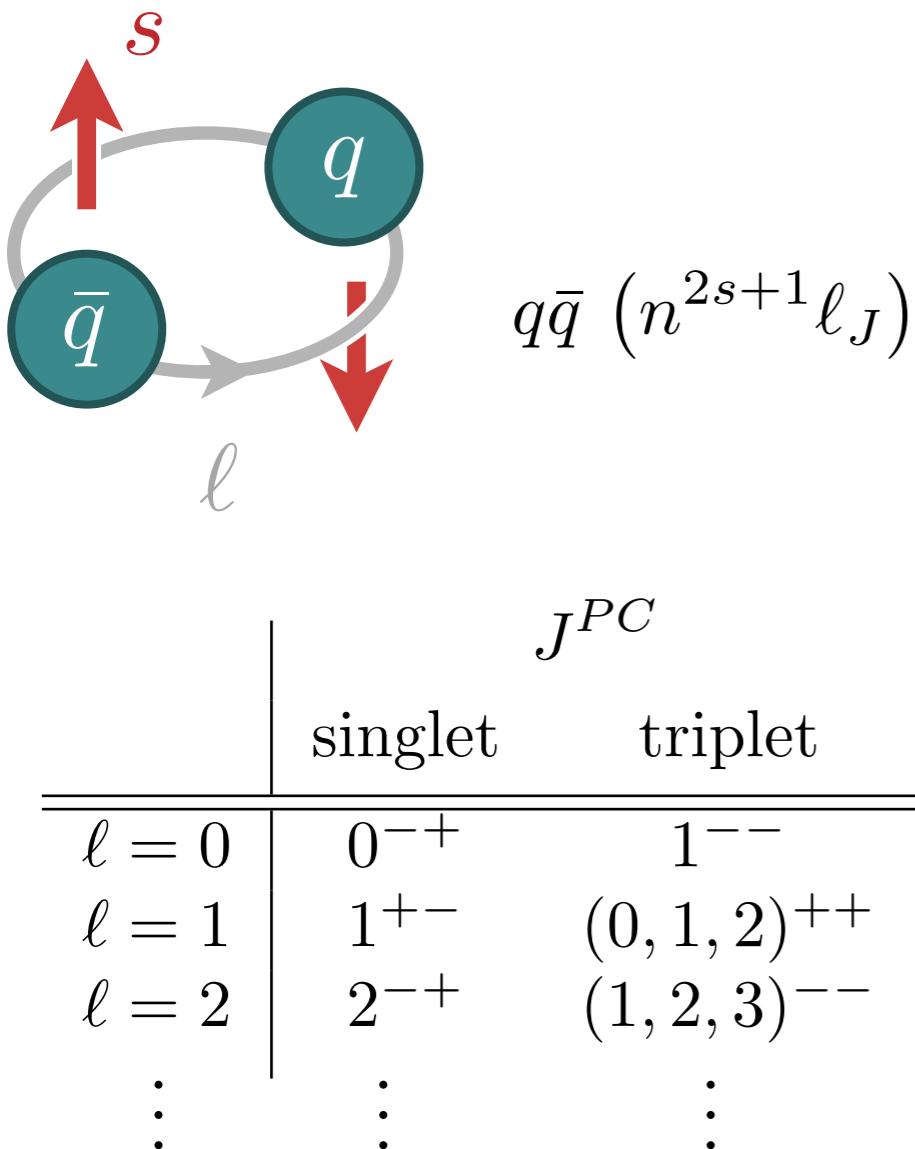


*Forbidden quantum numbers :*  $0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \dots$

# QCD Spectroscopy

Hadrons are classified by their **conserved quantum numbers**

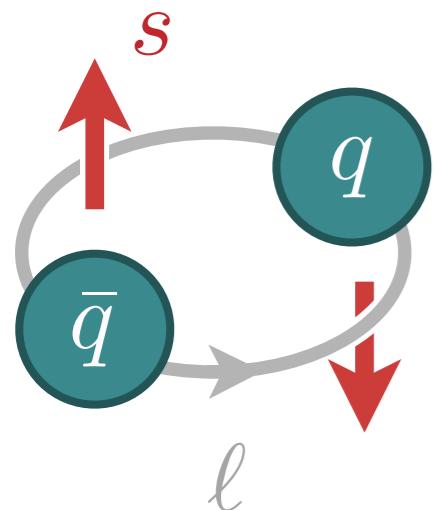
- Constituent quark model gives gross structure



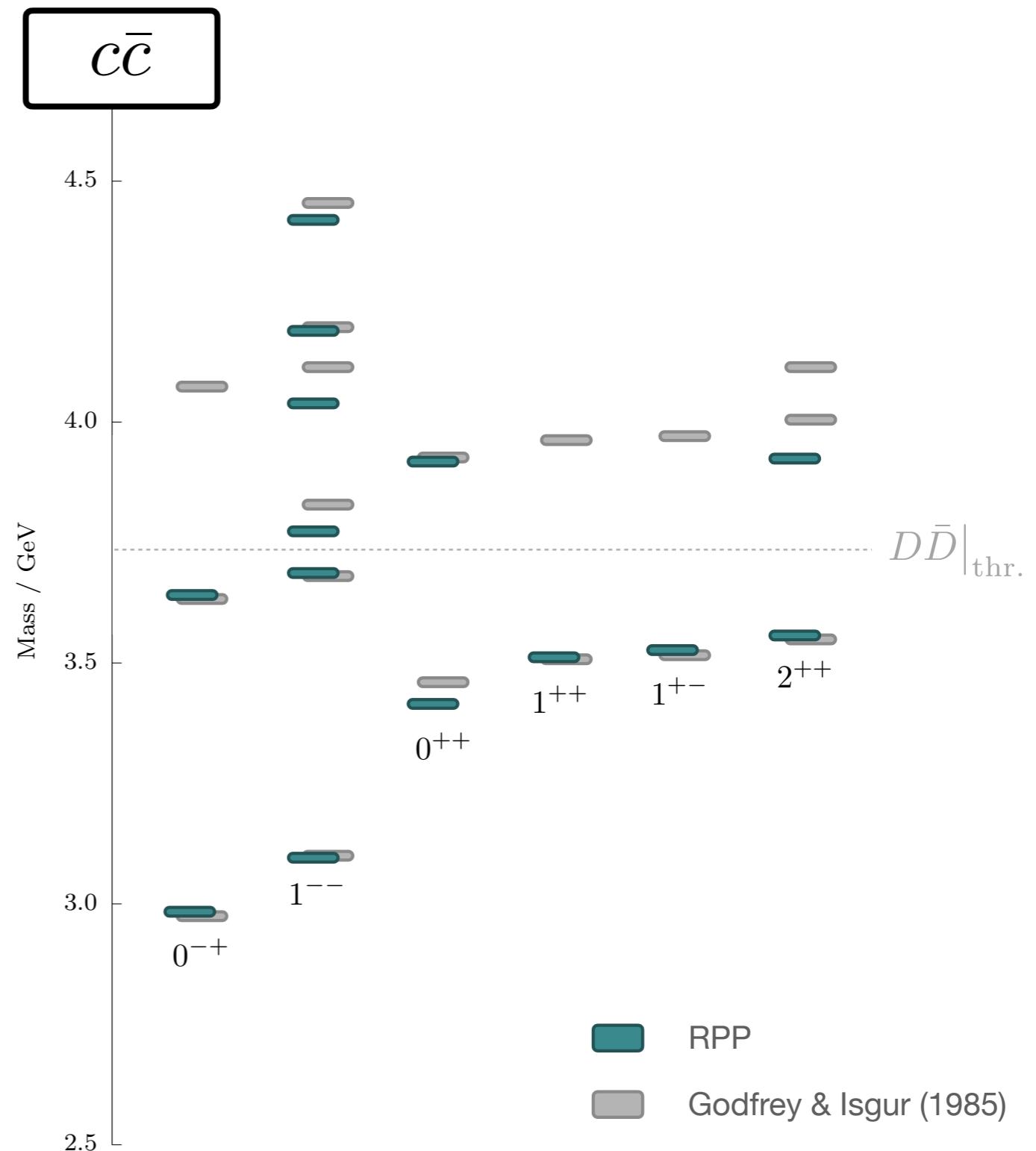
# QCD Spectroscopy

Hadrons are classified by their *conserved quantum numbers*

- Constituent quark model gives gross structure



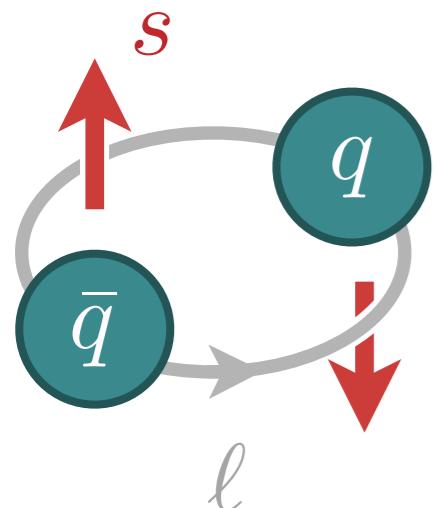
$$q\bar{q} \left( n^{2s+1} \ell_J \right)$$



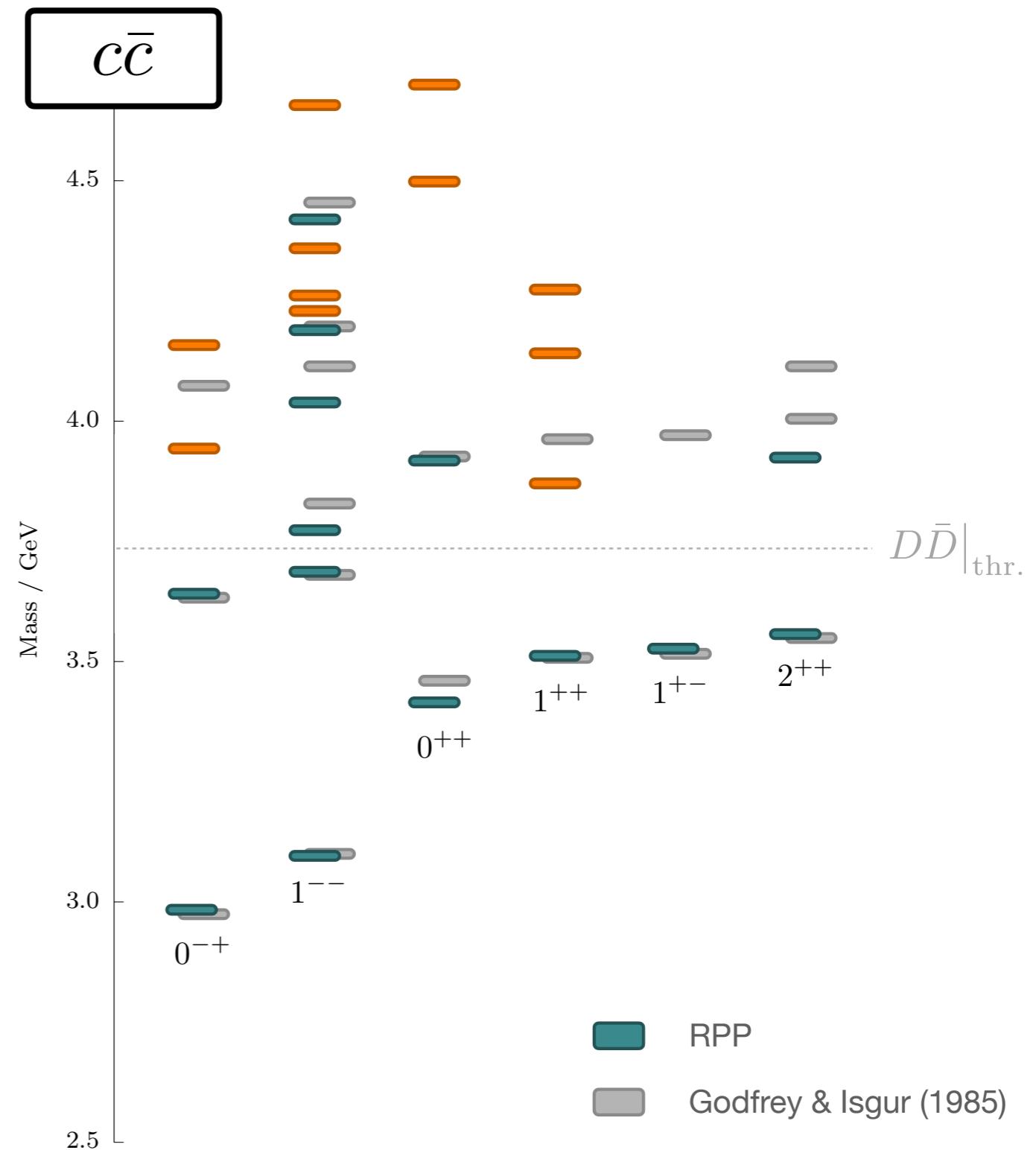
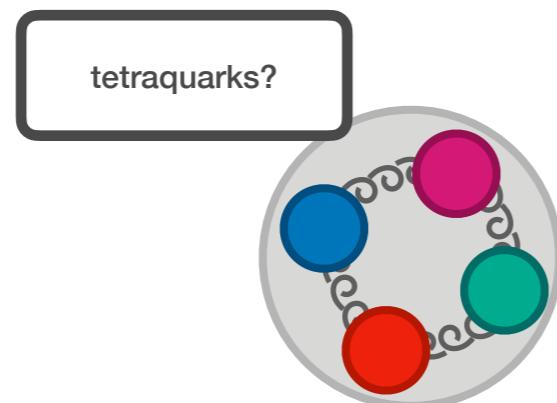
# QCD Spectroscopy

Hadrons are classified by their *conserved quantum numbers*

- Constituent quark model gives gross structure



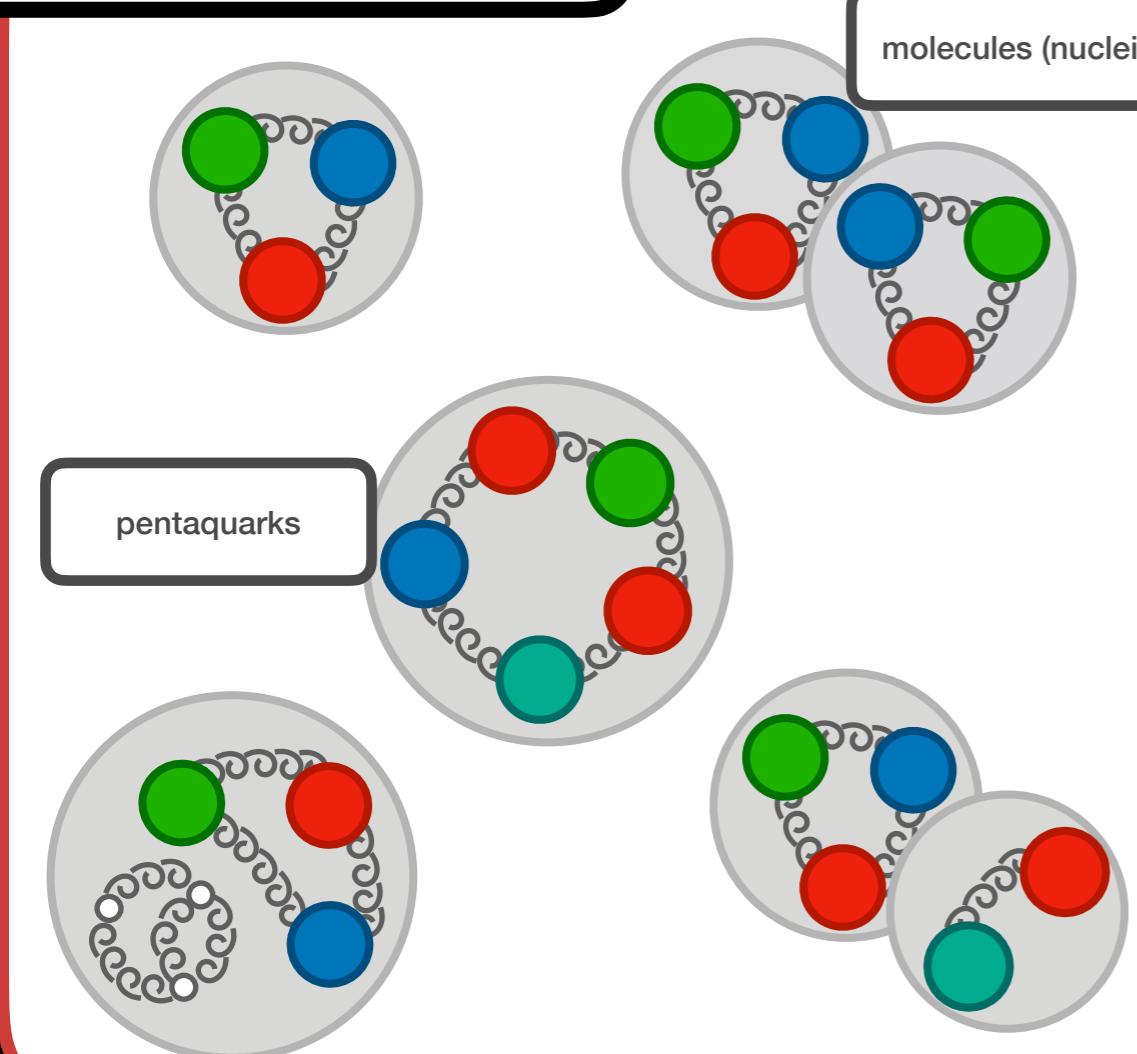
$$q\bar{q} \ (n^{2s+1}\ell_J)$$



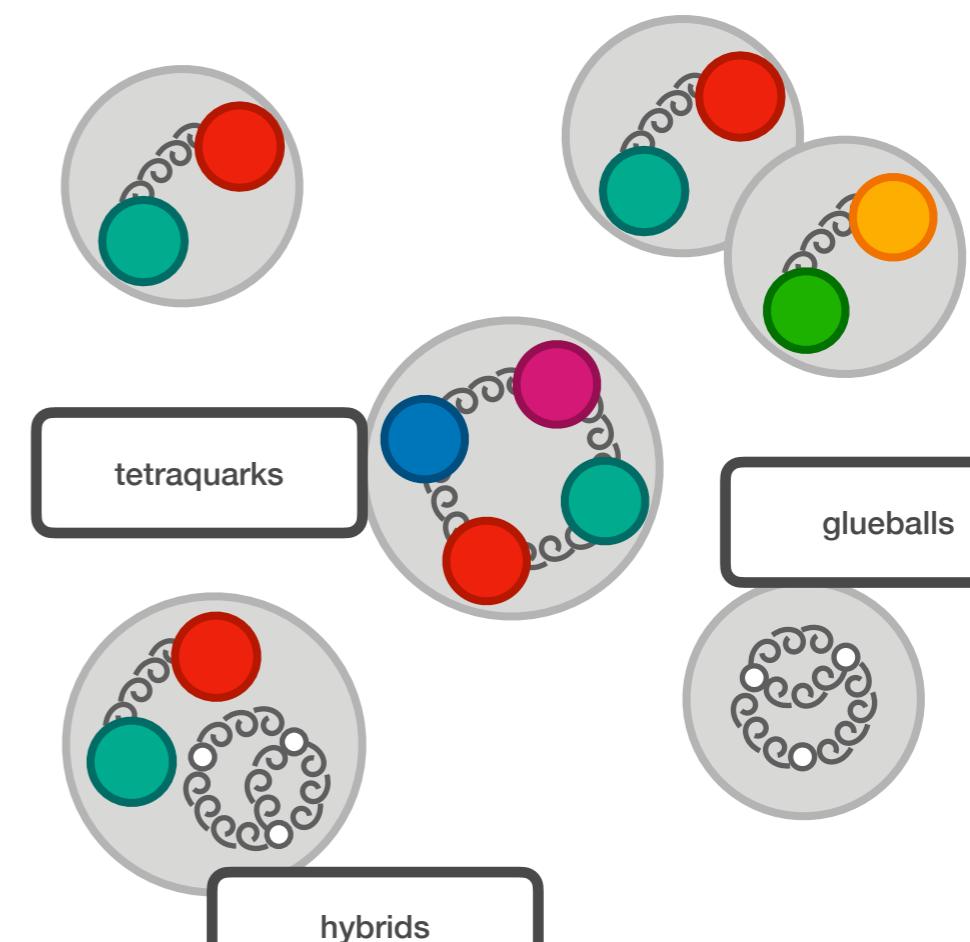
# A new QCD Spectroscopy

Can we understand these hadrons from QCD?

## Baryons (fermions)

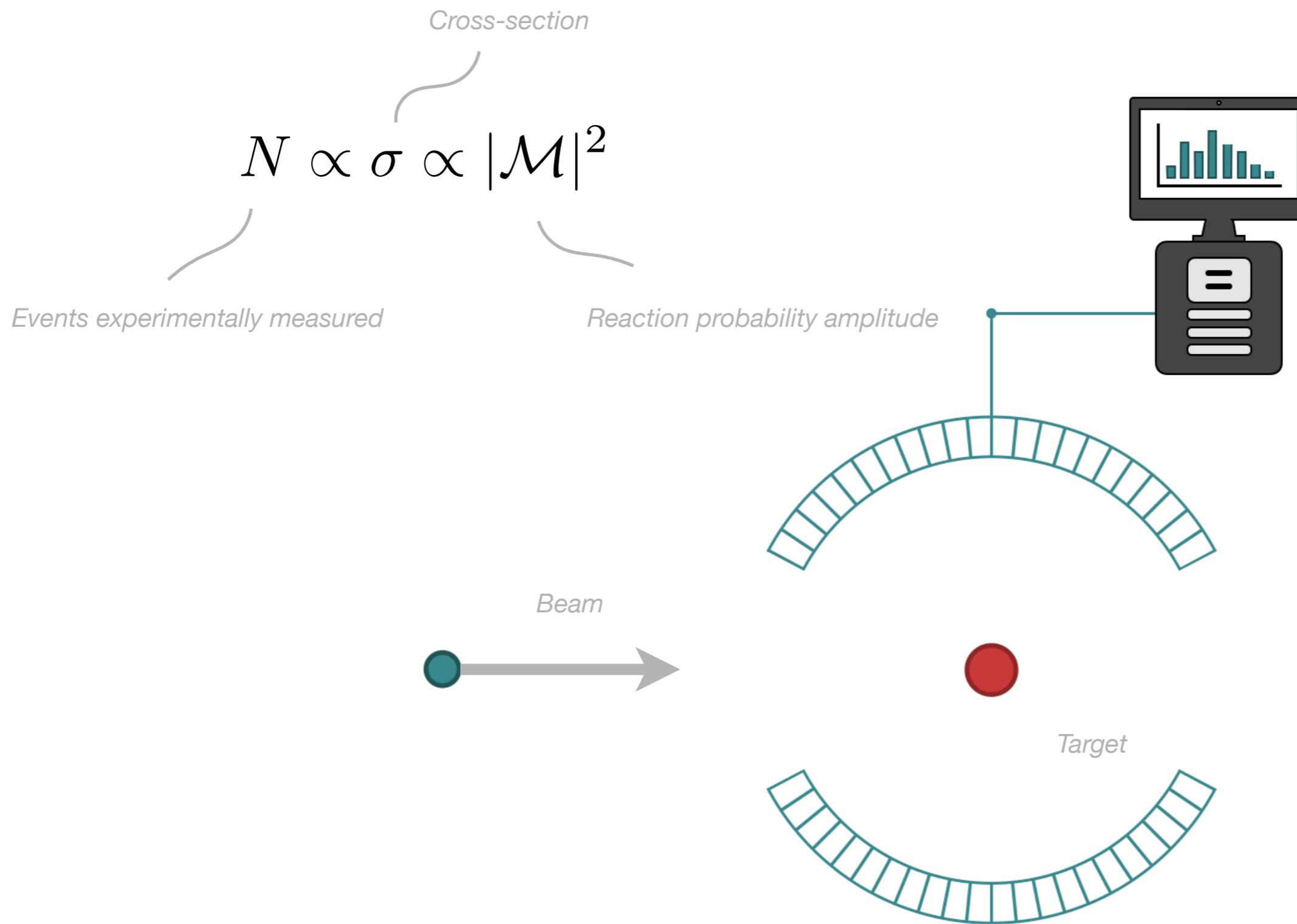


## Mesons (bosons)



# QCD Spectroscopy

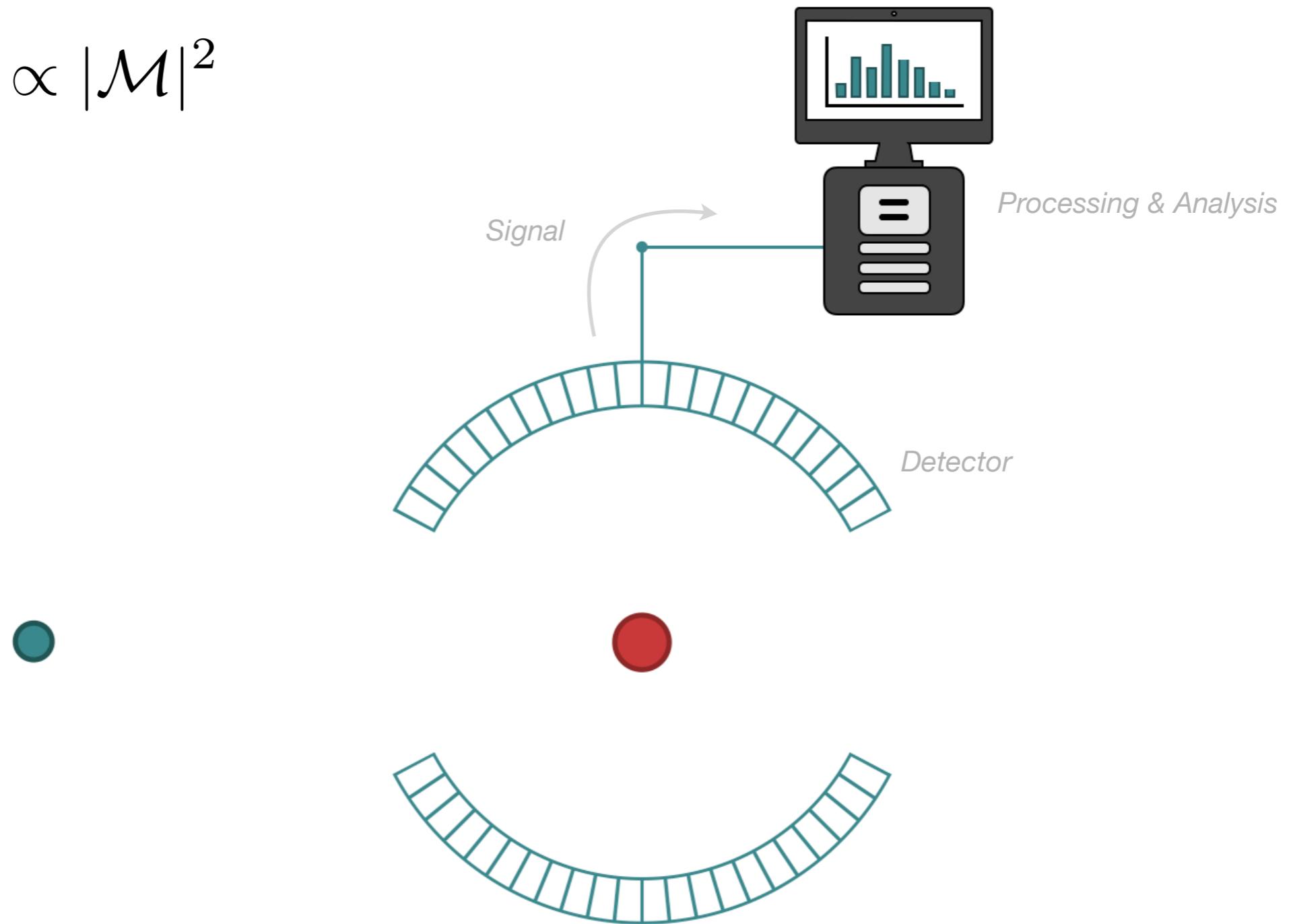
We observe strongly interacting hadrons, through reactions in accelerators/colliders



# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

$$N \propto \sigma \propto |\mathcal{M}|^2$$



# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

- If the interaction is sufficiently attractive, particles can form a **resonance**



# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

- If the interaction is sufficiently attractive, particles can form a **resonance**

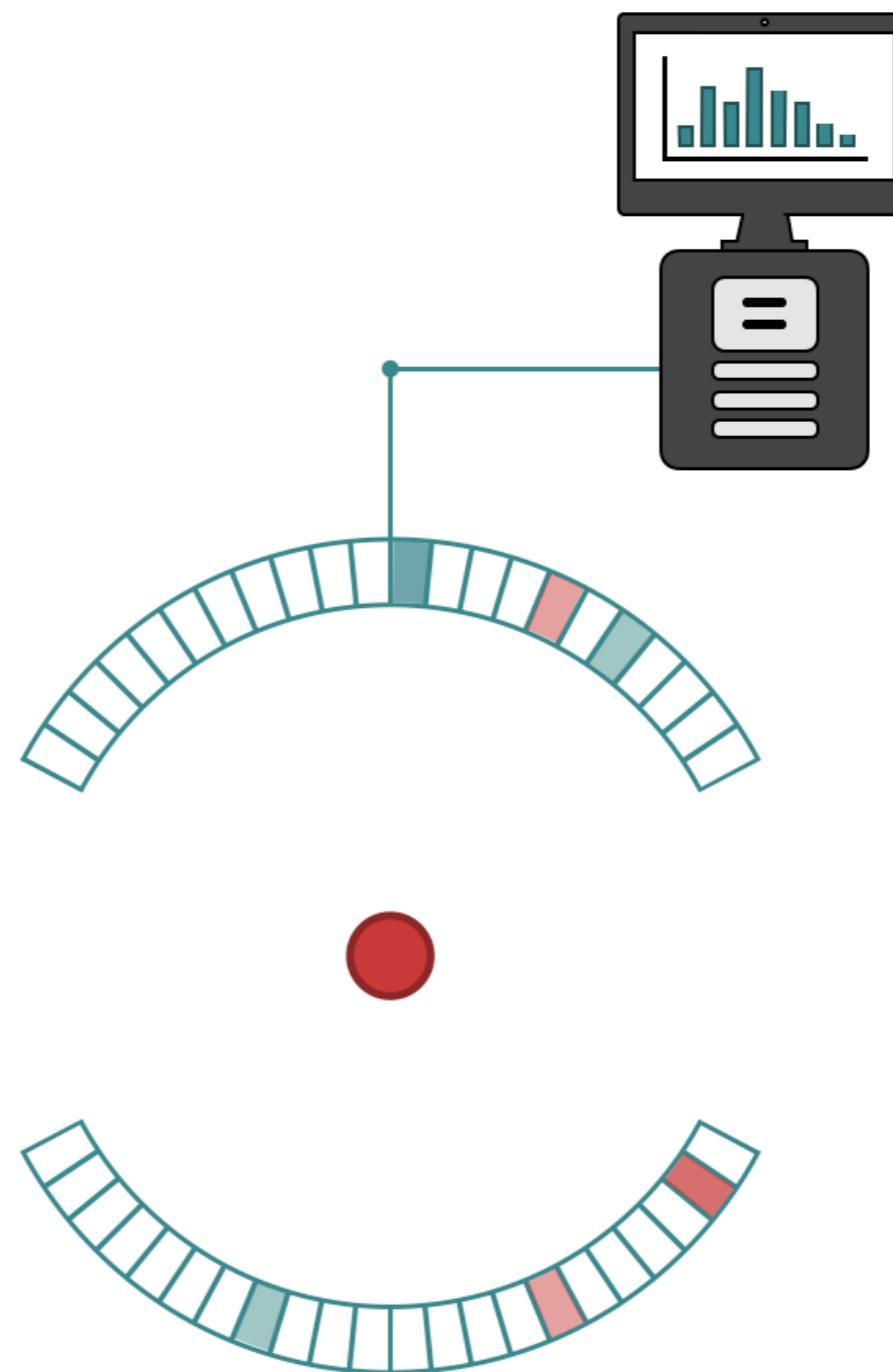
## Resonance

Unstable hadron which decays via strong nuclear interaction

Has a mass *and* finite lifetime

$$\tau \sim 10^{-23} \text{ sec}$$

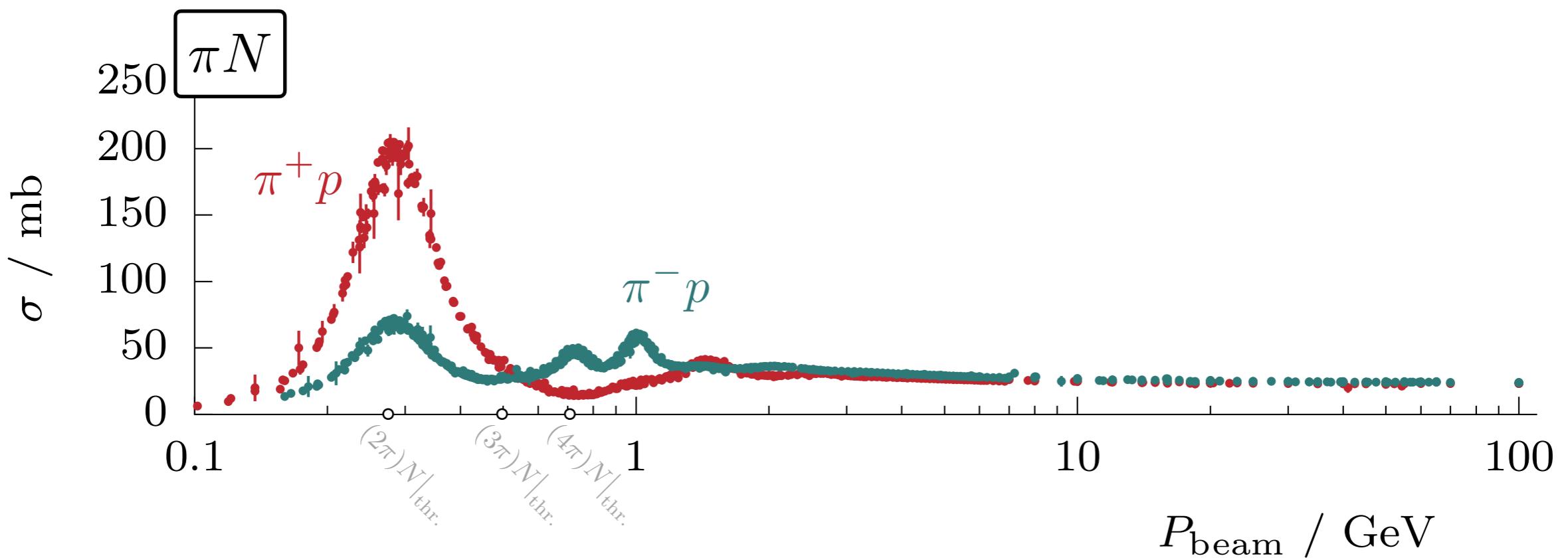
***Most hadrons are resonances***



# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

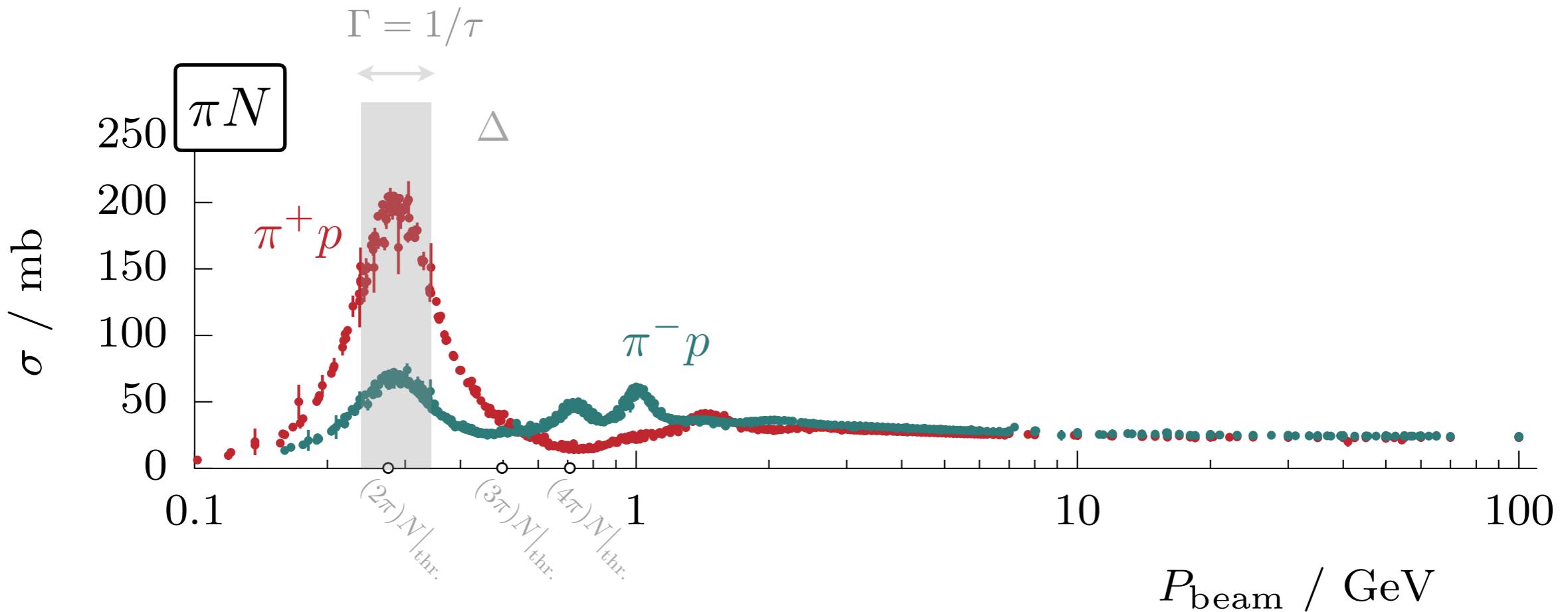
- If the interaction is sufficiently attractive, particles can form a **resonance**
- Resonances can appear as enhancements in the cross-section / amplitude



# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

- If the interaction is sufficiently attractive, particles can form a **resonance**
- Resonances can appear as enhancements in the cross-section / amplitude

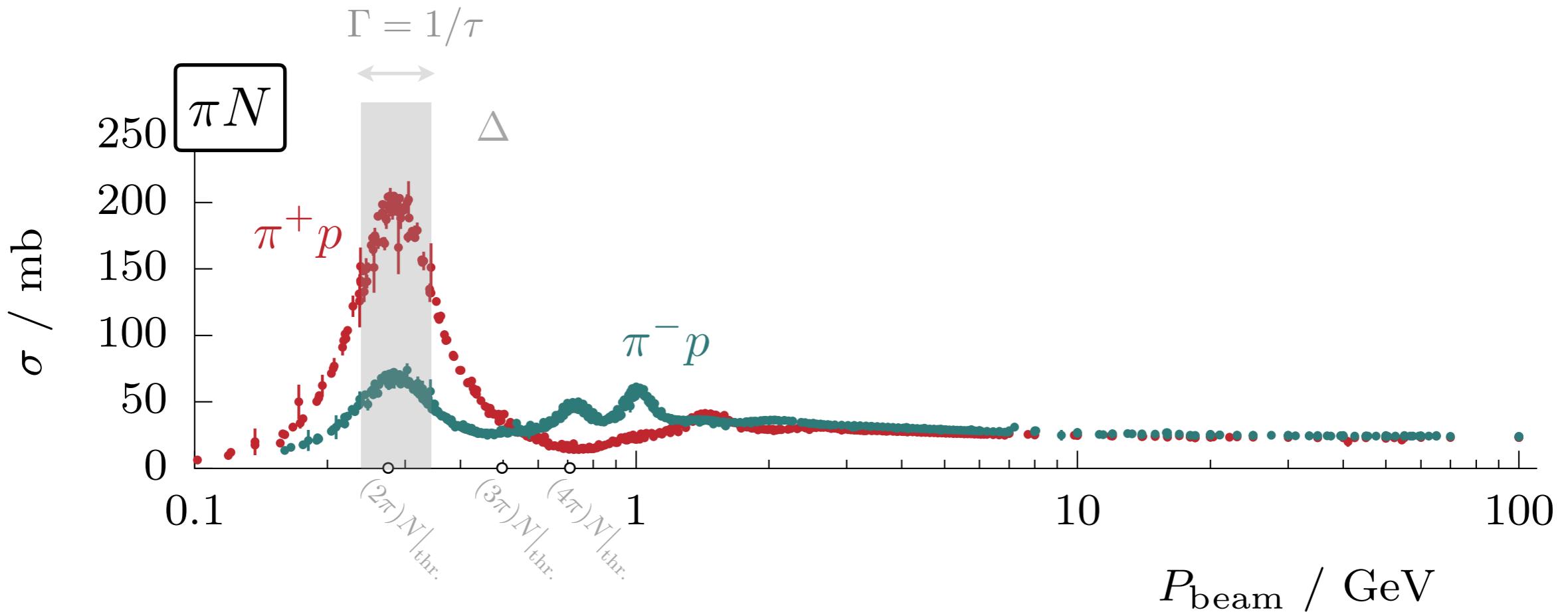


# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

- If the interaction is sufficiently attractive, particles can form a **resonance**
- Resonances can appear as enhancements in the cross-section / amplitude

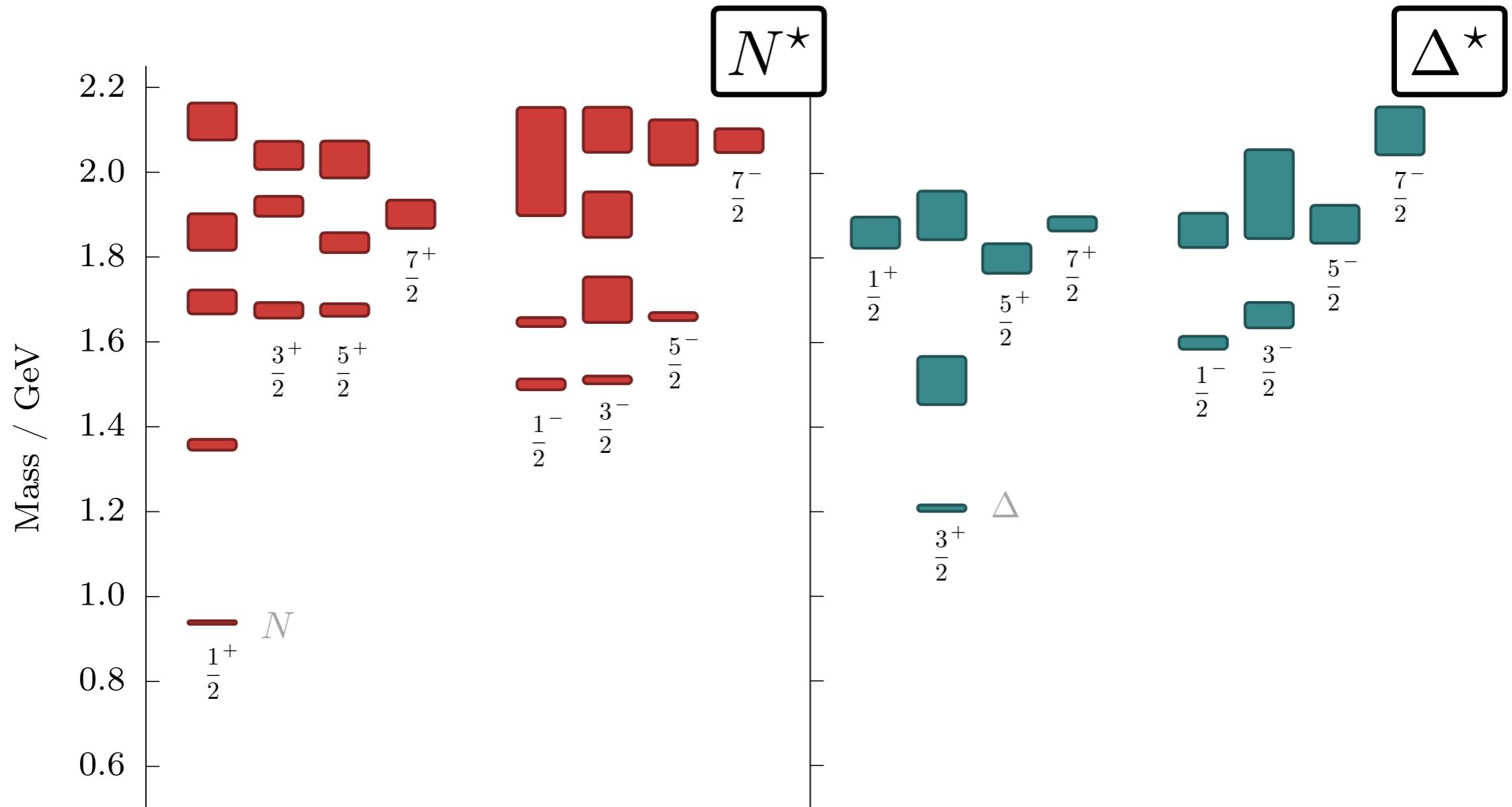
To get properties of resonances...  
...study structure of reaction amplitude  $\mathcal{M}$



# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

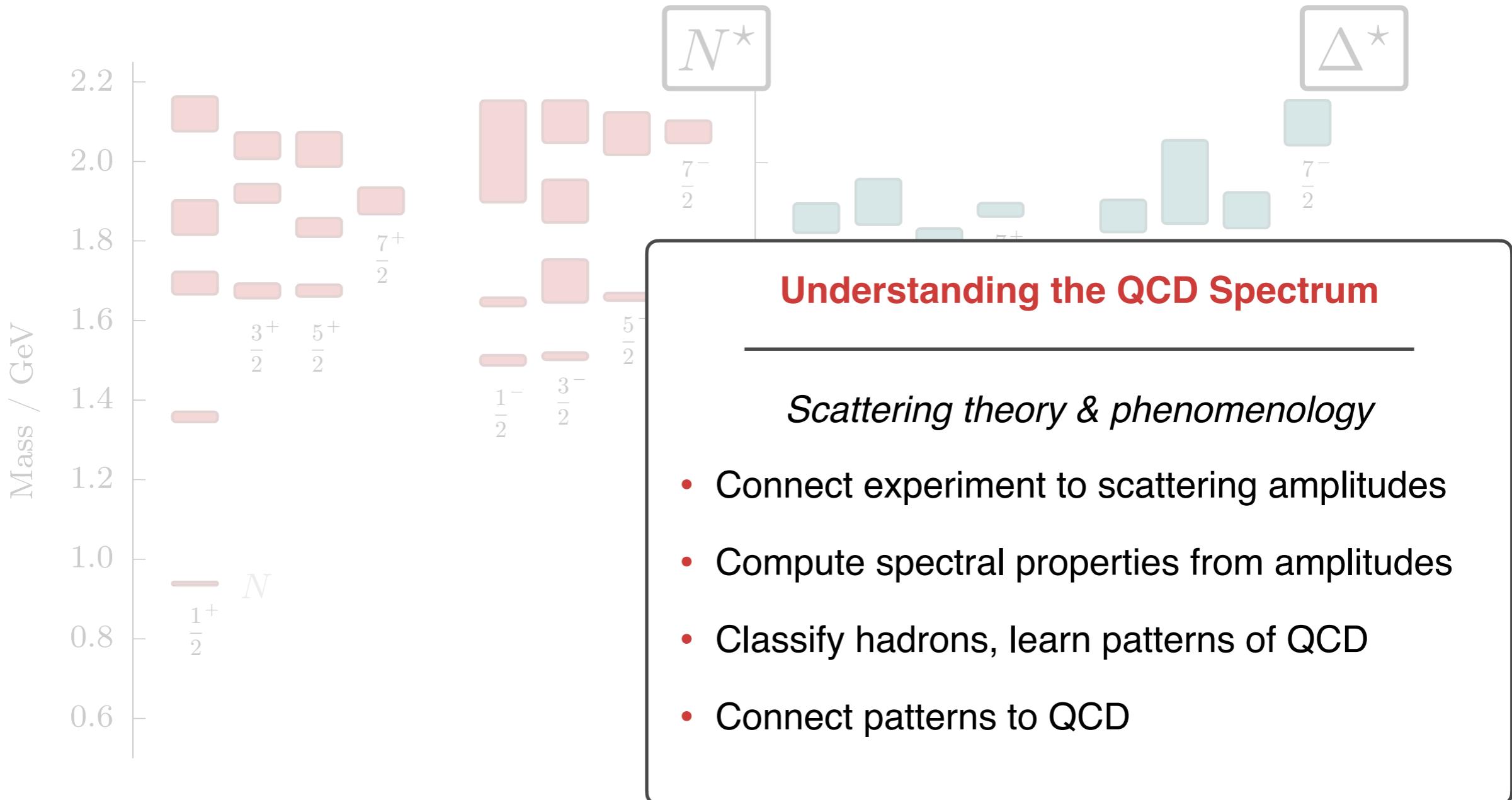
- If the interaction is sufficiently attractive, particles can form a **resonance**
- Resonances can appear as enhancements in the cross-section / amplitude



# QCD Spectroscopy

We observe strongly interacting hadrons, through reactions in accelerators/colliders

- If the interaction is sufficiently attractive, particles can form a **resonance**
- Resonances can appear as enhancements in the cross-section / amplitude



# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

## Lattice QCD & Hadron Spectroscopy

Lattice QCD

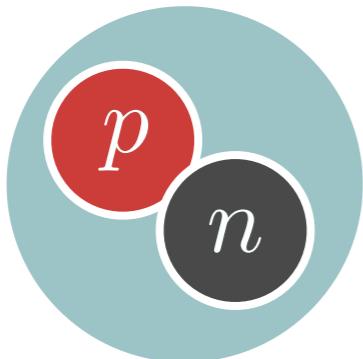
Lüscher & the Finite-Volume

## Case study — *the deuteron*

---

As an example of connecting scattering physics to hadron properties, consider ***the deuteron***

- Simplest nucleus, bound state of ***proton*** and ***neutron***
- “Hydrogen atom of nuclear physics” — study nature of two-body nuclear forces

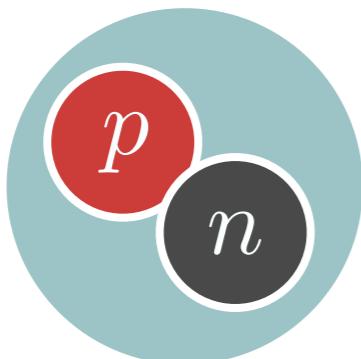


*Deuteron*

## Case study — *the deuteron*

As an example of connecting scattering physics to hadron properties, consider ***the deuteron***

- Simplest nucleus, bound state of ***proton*** and ***neutron***
- “Hydrogen atom of nuclear physics” — study nature of two-body nuclear forces



*Deuteron*

### Spectral properties checklist

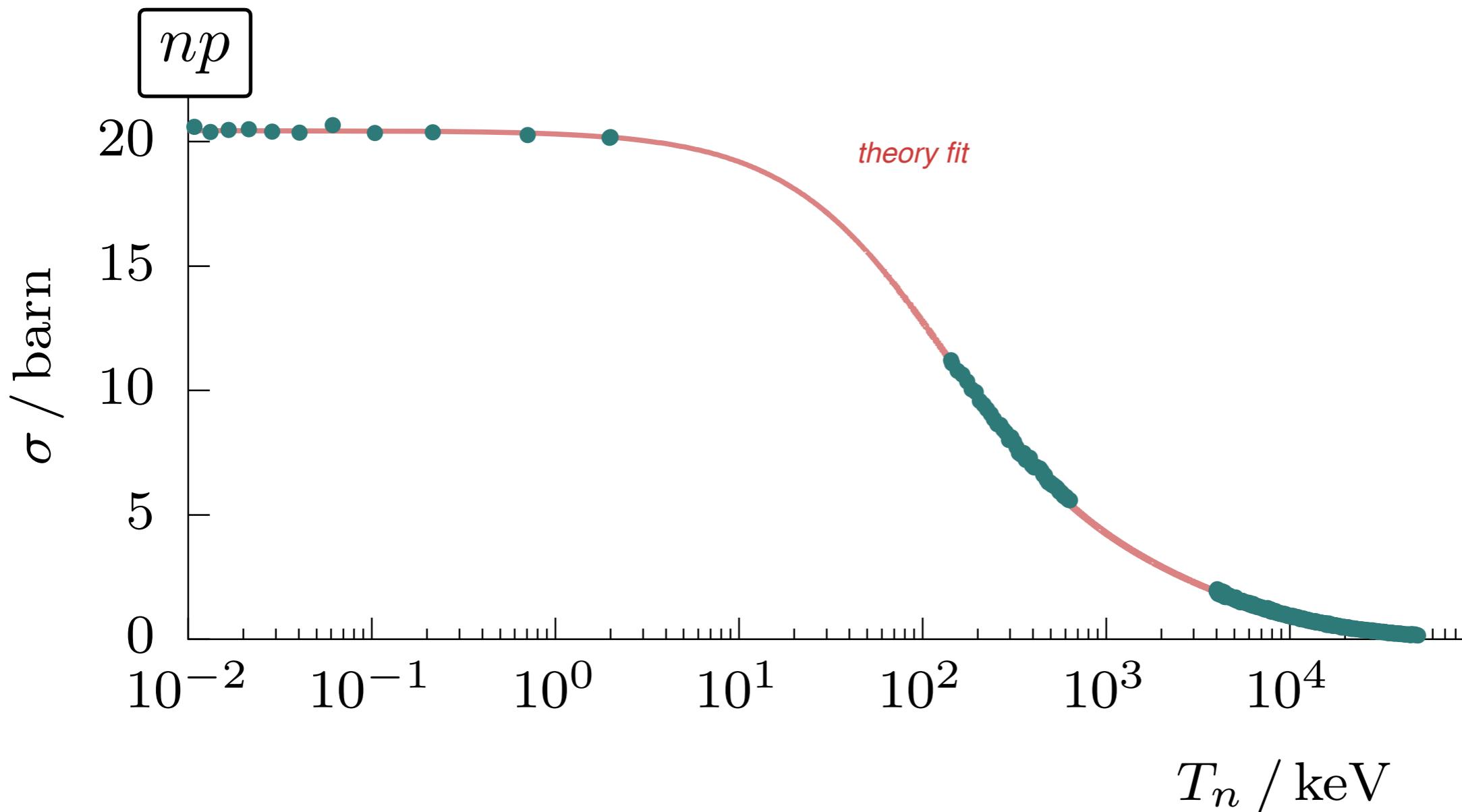
- Mass
- Lifetime
- Spin
- Parity
- Radius
- Charge

...

## Case study – *the deuteron*

As an example of connecting scattering physics to hadron properties, consider ***the deuteron***

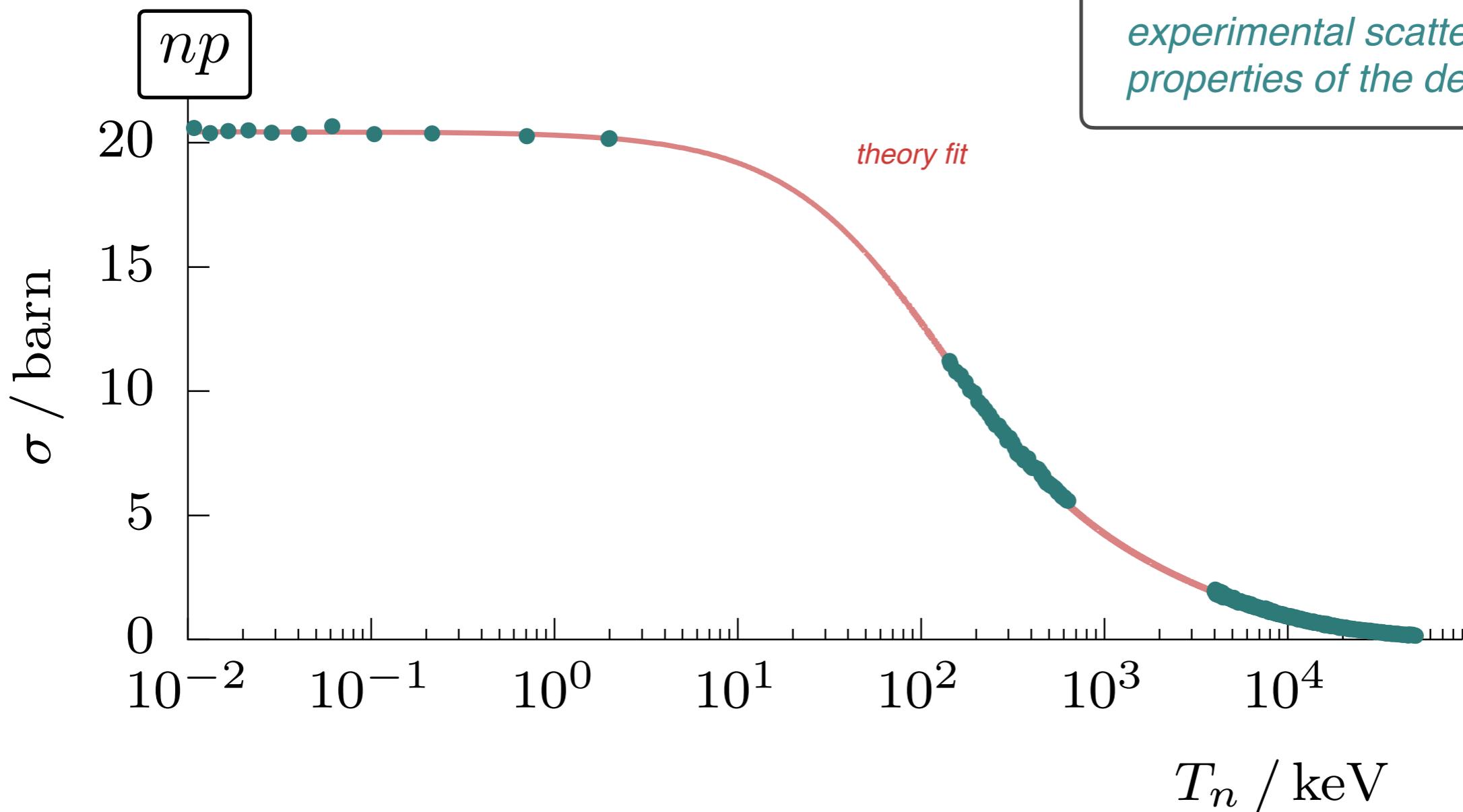
- Simplest nucleus, bound state of ***proton*** and ***neutron***
- “Hydrogen atom of nuclear physics” — study nature of two-body nuclear forces
- Examine low-energy ***neutron-proton elastic scattering***



## Case study – *the deuteron*

As an example of connecting scattering physics to hadron properties, consider ***the deuteron***

- Simplest nucleus, bound state of ***proton*** and ***neutron***
- “Hydrogen atom of nuclear physics” — study nature of two-body nuclear forces
- Examine low-energy ***neutron-proton elastic scattering***



## Kinematics – Energy & Momentum

$$\hbar = c = 1$$

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon

$$N_{\text{Nucleon}} = \begin{pmatrix} p \\ n \end{pmatrix}$$

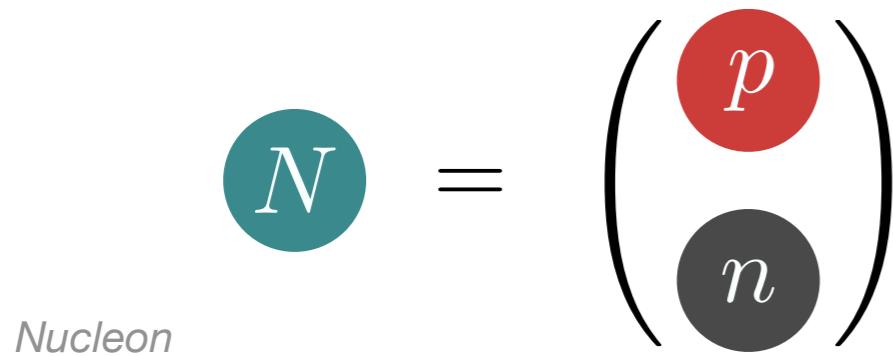
$$m_p \approx m_n$$

# Kinematics – Energy & Momentum

$$\hbar = c = 1$$

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon



$$m_p \approx m_n$$

Nucleon mass

*First approximation – Isospin limit*

$$m_p = m_n \equiv m$$

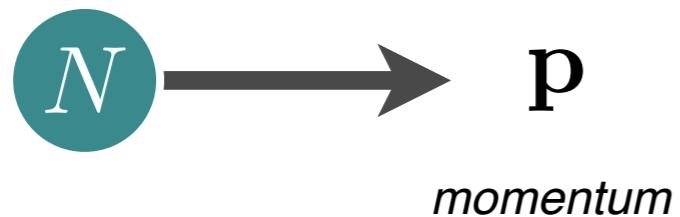
$$\approx 940 \text{ MeV}$$



## Kinematics – Energy & Momentum

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon
- Relativistic nucleon kinematics



*energy*

$$\omega_p = \sqrt{m^2 + p^2}$$

## Kinematics – Energy & Momentum

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon
- Relativistic nucleon kinematics

$$N \longrightarrow p$$

*momentum*

*energy*

$$\omega_p = \sqrt{m^2 + \mathbf{p}^2}$$

$$= m + \frac{\mathbf{p}^2}{2m} + \mathcal{O}(\mathbf{p}^4)$$

*non-relativistic limit,  $p/m \ll 1$*

## Kinematics – Energy & Momentum

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon
- Relativistic nucleon kinematics

$$N \longrightarrow p$$

*momentum*

*energy*

$$\omega_p = \sqrt{m^2 + \mathbf{p}^2}$$

$$= m + \frac{\mathbf{p}^2}{2m} + \mathcal{O}(\mathbf{p}^4)$$

*non-relativistic limit,  $p/m \ll 1$*

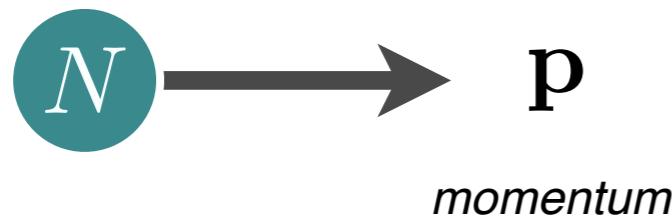
*kinetic energy*

$$T_p = \omega_p - m$$

# Kinematics – Energy & Momentum

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon
- Relativistic nucleon kinematics



*energy*

$$\omega_p = \sqrt{m^2 + p^2}$$

$$= m + \frac{p^2}{2m} + \mathcal{O}(p^4)$$

*non-relativistic limit,  $p/m \ll 1$*

*kinetic energy*

$$T_p = \omega_p - m$$

## Exercise

*Prove the non-relativistic expansion*

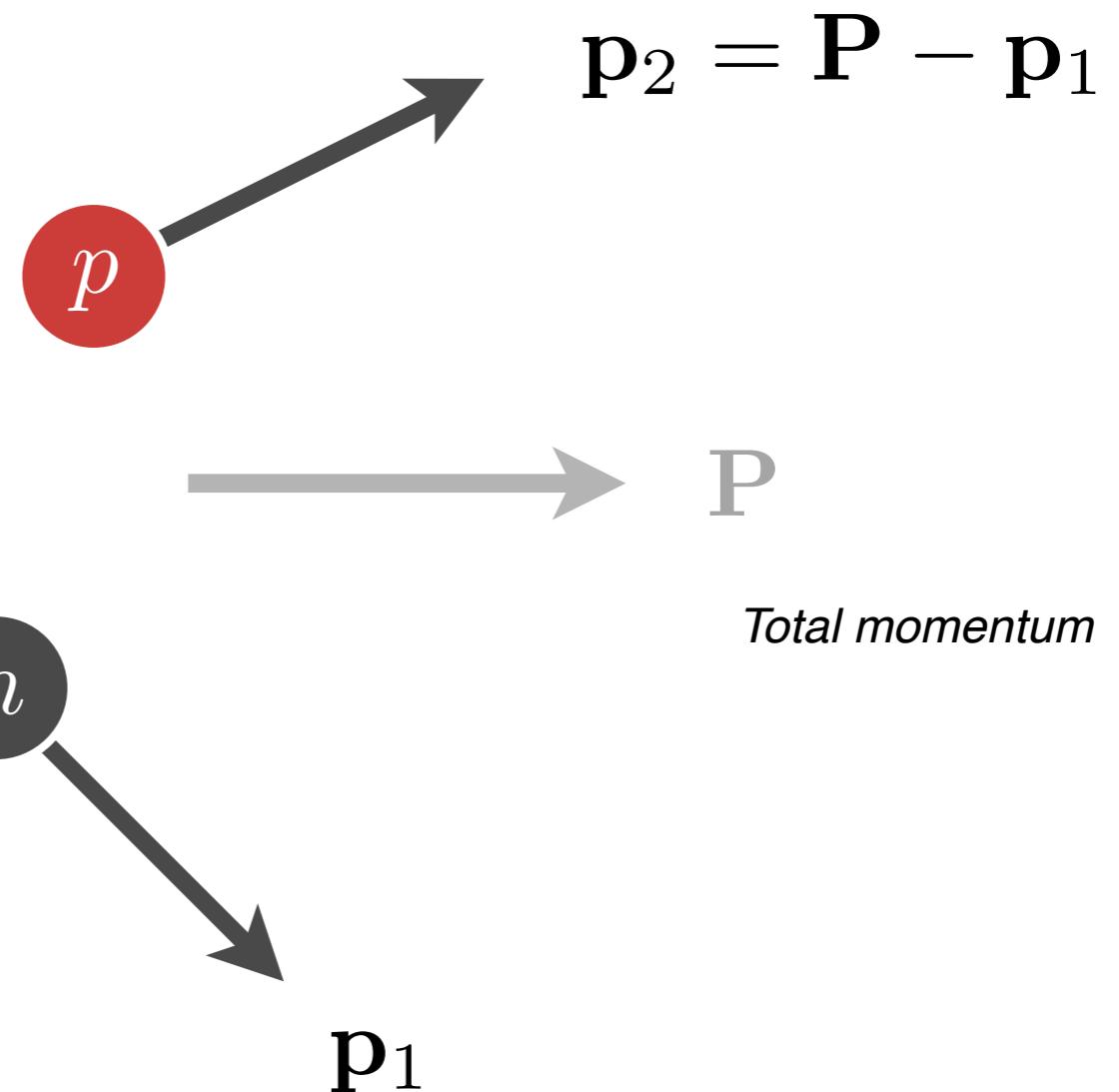
## Kinematics – Energy & Momentum

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal – different isospin projections of Nucleon
- Relativistic nucleon kinematics
- Two-nucleon kinematics

*Total energy*

$$E = \sqrt{m^2 + \mathbf{p}_1^2} + \sqrt{m^2 + \mathbf{p}_2^2}$$



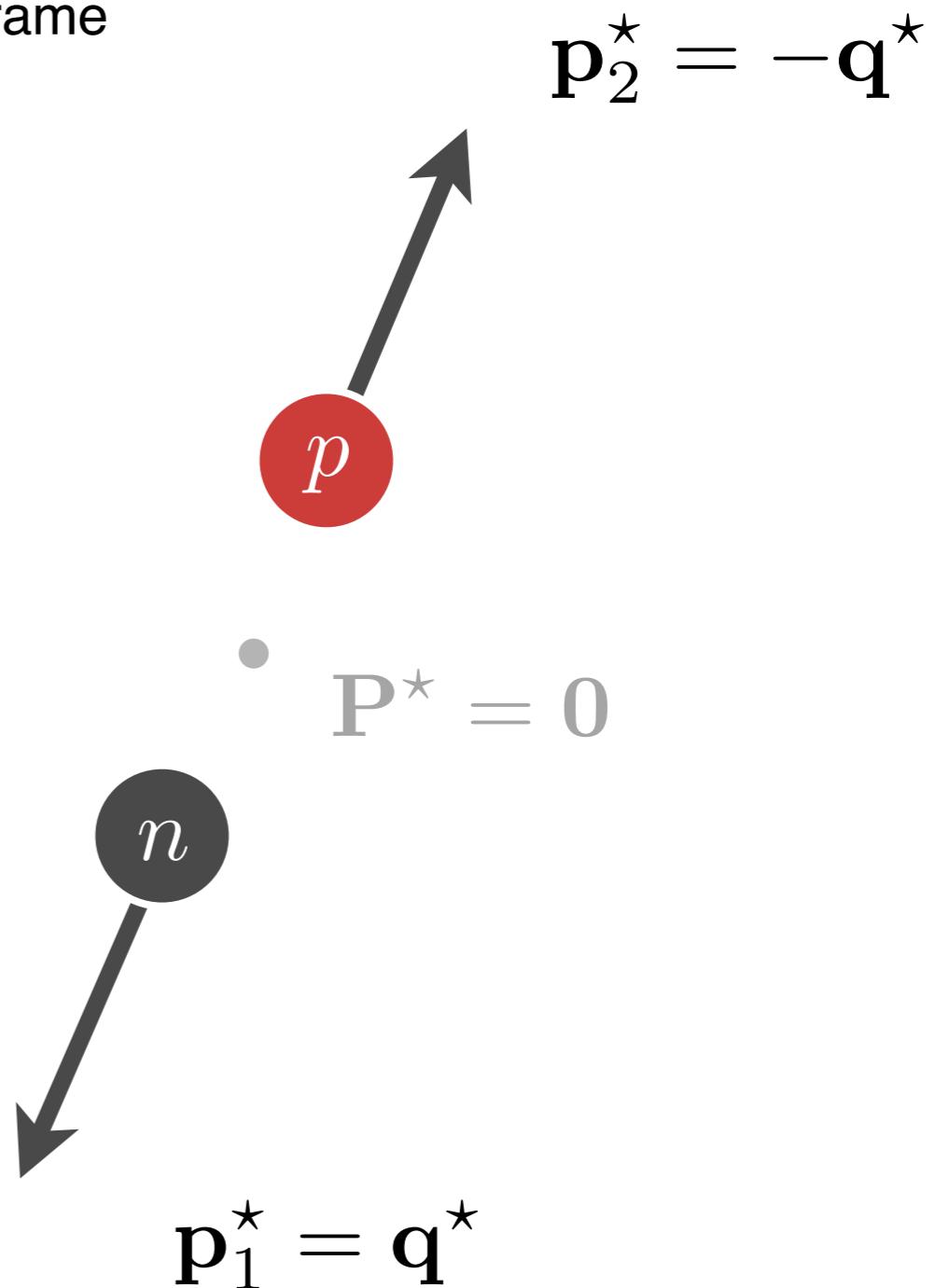
## Kinematics – Energy & Momentum

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon
- Relativistic nucleon kinematics
- Two-nucleon kinematics — choose *center-of-momentum* frame

Total CM energy

$$E^* = 2\sqrt{m^2 + q^{*2}}$$



## Kinematics – Energy & Momentum

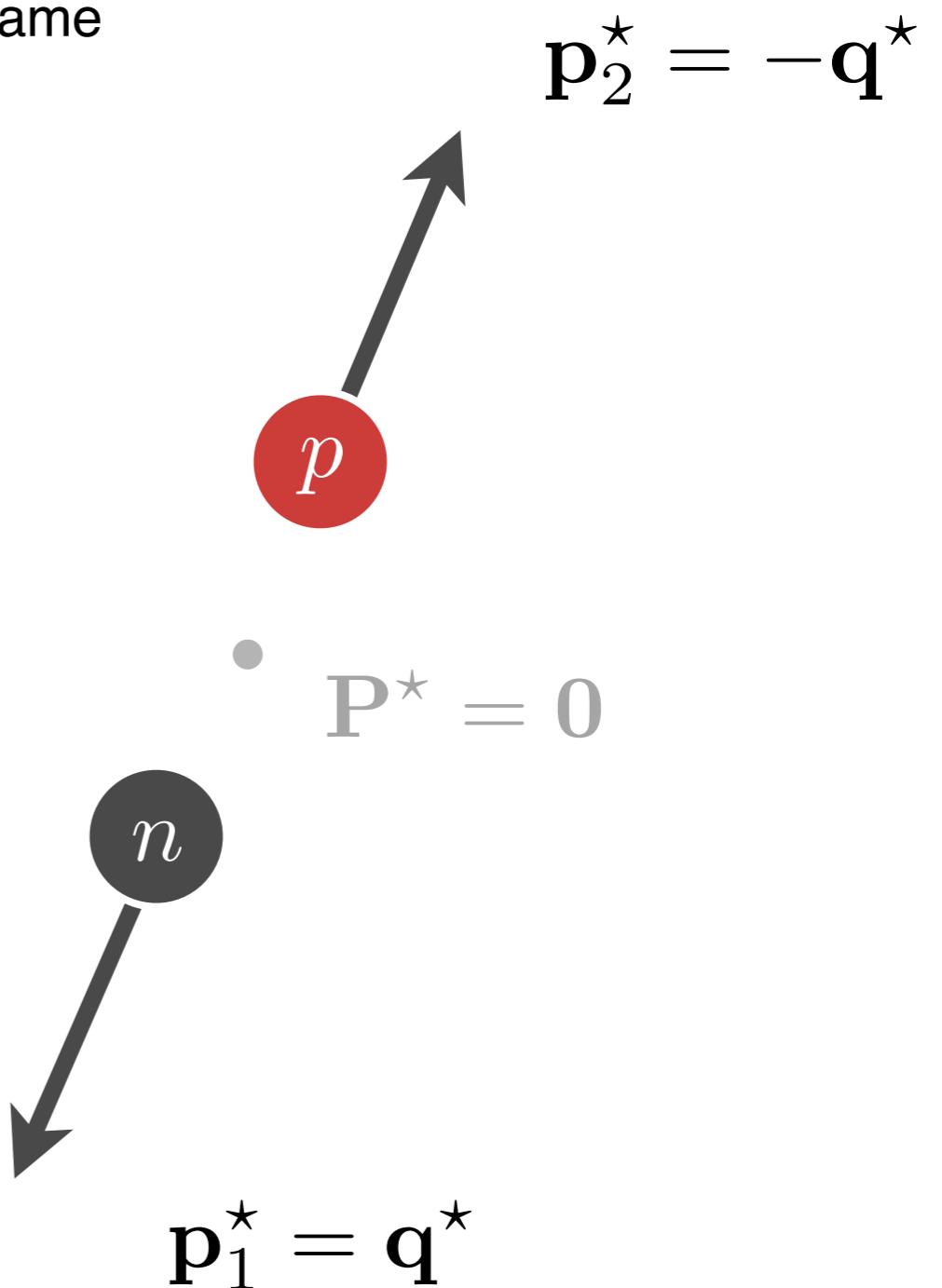
Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon
- Relativistic nucleon kinematics
- Two-nucleon kinematics — choose *center-of-momentum* frame

Total CM energy

$$E^* = 2\sqrt{m^2 + q^{*2}}$$

$$\implies q^* = \frac{1}{2}\sqrt{E^{*2} - 4m^2}$$



# Kinematics – Energy & Momentum

Let us define the kinematics of protons and neutrons

- proton and neutron masses are nearly equal — different isospin projections of Nucleon
- Relativistic nucleon kinematics
- Two-nucleon kinematics — choose *center-of-momentum* frame

Total CM energy

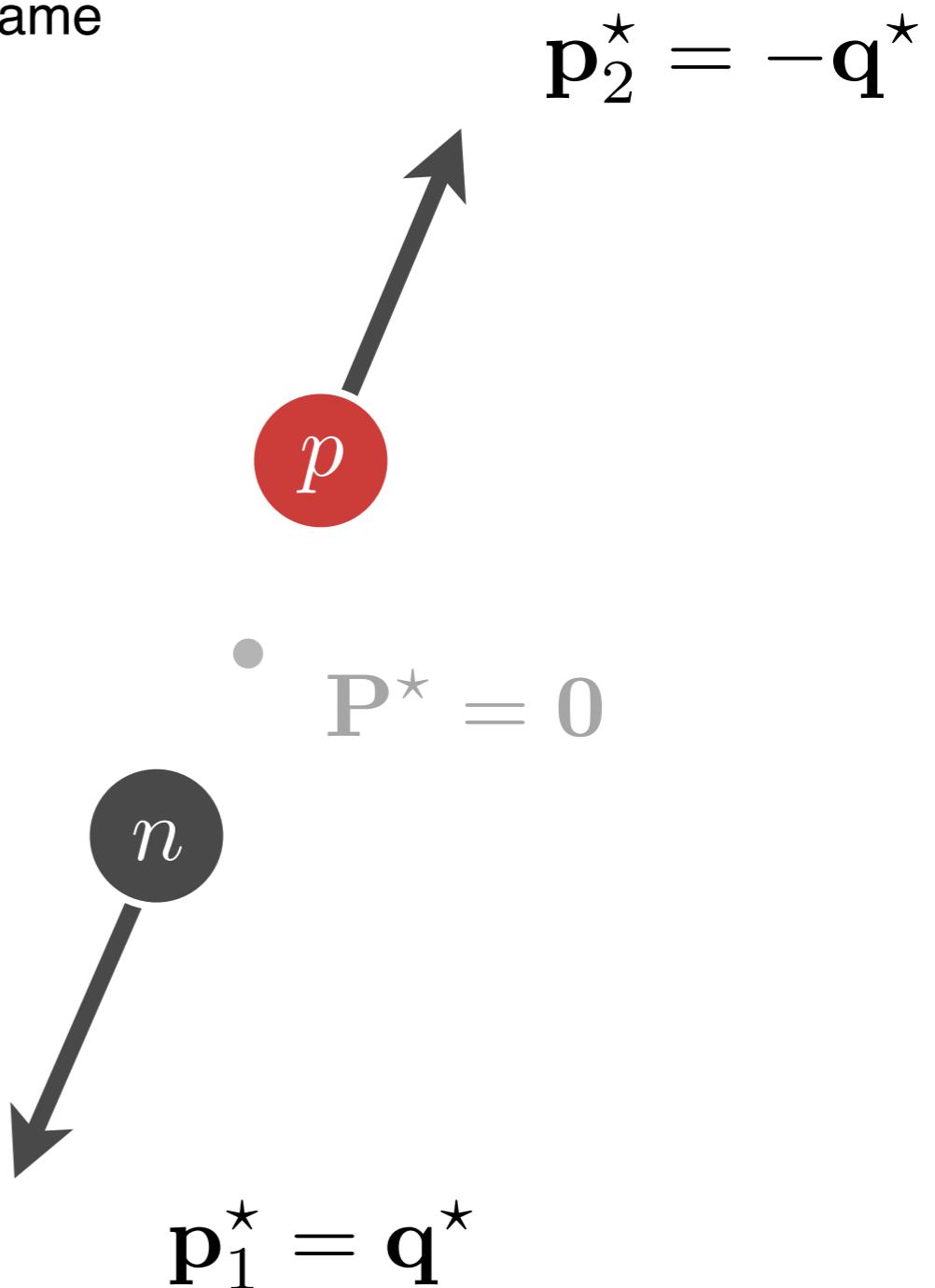
$$E^* = 2\sqrt{m^2 + q^{*2}}$$

$$\implies q^* = \frac{1}{2}\sqrt{E^{*2} - 4m^2}$$

## Exercise

(a) Show that  $q^* = \sqrt{E^{*2}/4 - m^2}$

(b) Plot  $q^*$  as a function of  $E^*$



## Kinematics – Phase Space

---

Two-nucleon energy is restricted by energy- and momentum-conservation

- Kinematic *phase space* — available space of kinematic configurations

# Kinematics – Phase Space

Two-nucleon energy is restricted by energy- and momentum-conservation

- Kinematic *phase space* – available space of kinematic configurations

Lorentz invariant phase space

$$d\Phi_2(P \rightarrow p_1 + p_2) = (2\pi)^4 \delta(E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) \delta^{(3)}(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) \frac{d^3 \mathbf{p}_1}{(2\pi)^3 2\omega_{\mathbf{p}_1}} \frac{d^3 \mathbf{p}_2}{(2\pi)^3 2\omega_{\mathbf{p}_2}}$$

Energy conservation

$$E = \omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2}$$

Momentum conservation

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$

Particle 2 distribution

Particle 1 distribution

# Kinematics – Phase Space

Two-nucleon energy is restricted by energy- and momentum-conservation

- Kinematic *phase space* – available space of kinematic configurations

*Lorentz invariant phase space*

$$d\Phi_2(P \rightarrow p_1 + p_2) = (2\pi)^4 \delta(E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) \delta^{(3)}(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) \frac{d^3 \mathbf{p}_1}{(2\pi)^3 2\omega_{\mathbf{p}_1}} \frac{d^3 \mathbf{p}_2}{(2\pi)^3 2\omega_{\mathbf{p}_2}}$$

= ...

$$= 2\rho \Theta(E^* - 2m) \frac{d\Omega_{\mathbf{q}}^*}{4\pi}$$



*Orientation (not fixed)*

*Energy greater than threshold*  
 $E^* \geq 2m$

$$E_{\text{thr.}}^* = 2m \iff q_{\text{thr.}}^* = 0$$

$$m \approx 940 \text{ MeV}$$

$$E_{\text{thr.}}^* \approx 1880 \text{ MeV}$$

## Kinematics – Phase Space

Two-nucleon energy is restricted by energy- and momentum-conservation

- Kinematic *phase space* – available space of kinematic configurations

*Lorentz invariant phase space*

$$d\Phi_2(P \rightarrow p_1 + p_2) = (2\pi)^4 \delta(E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) \delta^{(3)}(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) \frac{d^3 \mathbf{p}_1}{(2\pi)^3 2\omega_{\mathbf{p}_1}} \frac{d^3 \mathbf{p}_2}{(2\pi)^3 2\omega_{\mathbf{p}_2}}$$

= ...

$$= 2\rho \Theta(E^\star - 2m) \frac{d\Omega_{\mathbf{q}}^\star}{4\pi}$$

*Two-body kinematic phase space factor*

$$\rho = \frac{q^\star}{8\pi E^\star}$$

# Kinematics – Phase Space

Two-nucleon energy is restricted by energy- and momentum-conservation

- Kinematic *phase space* – available space of kinematic configurations

*Lorentz invariant phase space*

$$d\Phi_2(P \rightarrow p_1 + p_2) = (2\pi)^4 \delta(E - \omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_2}) \delta^{(3)}(\mathbf{P} - \mathbf{p}_1 - \mathbf{p}_2) \frac{d^3 \mathbf{p}_1}{(2\pi)^3 2\omega_{\mathbf{p}_1}} \frac{d^3 \mathbf{p}_2}{(2\pi)^3 2\omega_{\mathbf{p}_2}}$$

= ...

$$= 2\rho \Theta(E^* - 2m) \frac{d\Omega_{\mathbf{q}}^*}{4\pi}$$

*Two-body kinematic phase space factor*

## Exercise

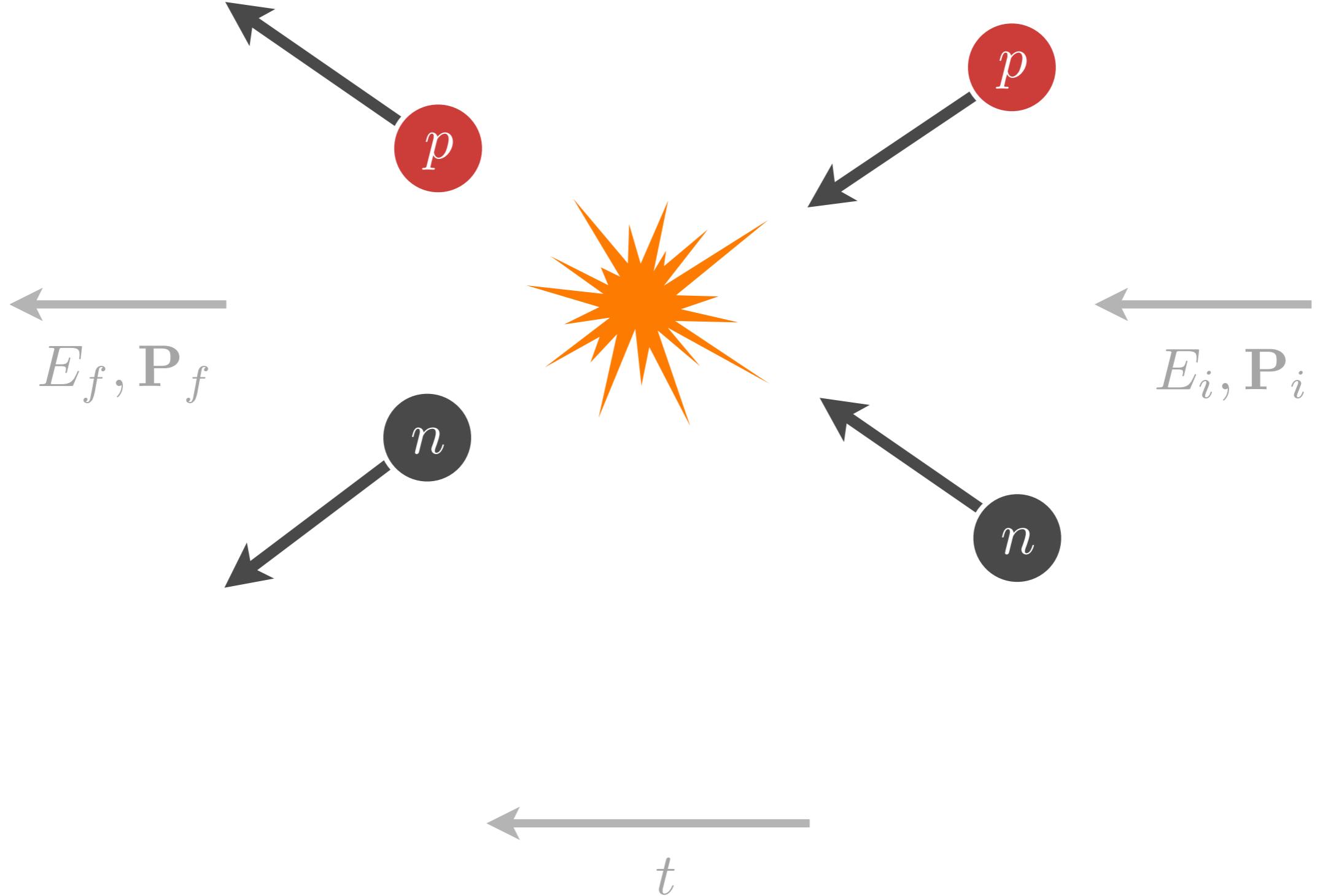
Plot  $\rho$  as a function of  $E^*$

\* Derive  $\rho$

$$\rho = \frac{q^*}{8\pi E^*}$$

# Kinematics – Binary Reactions

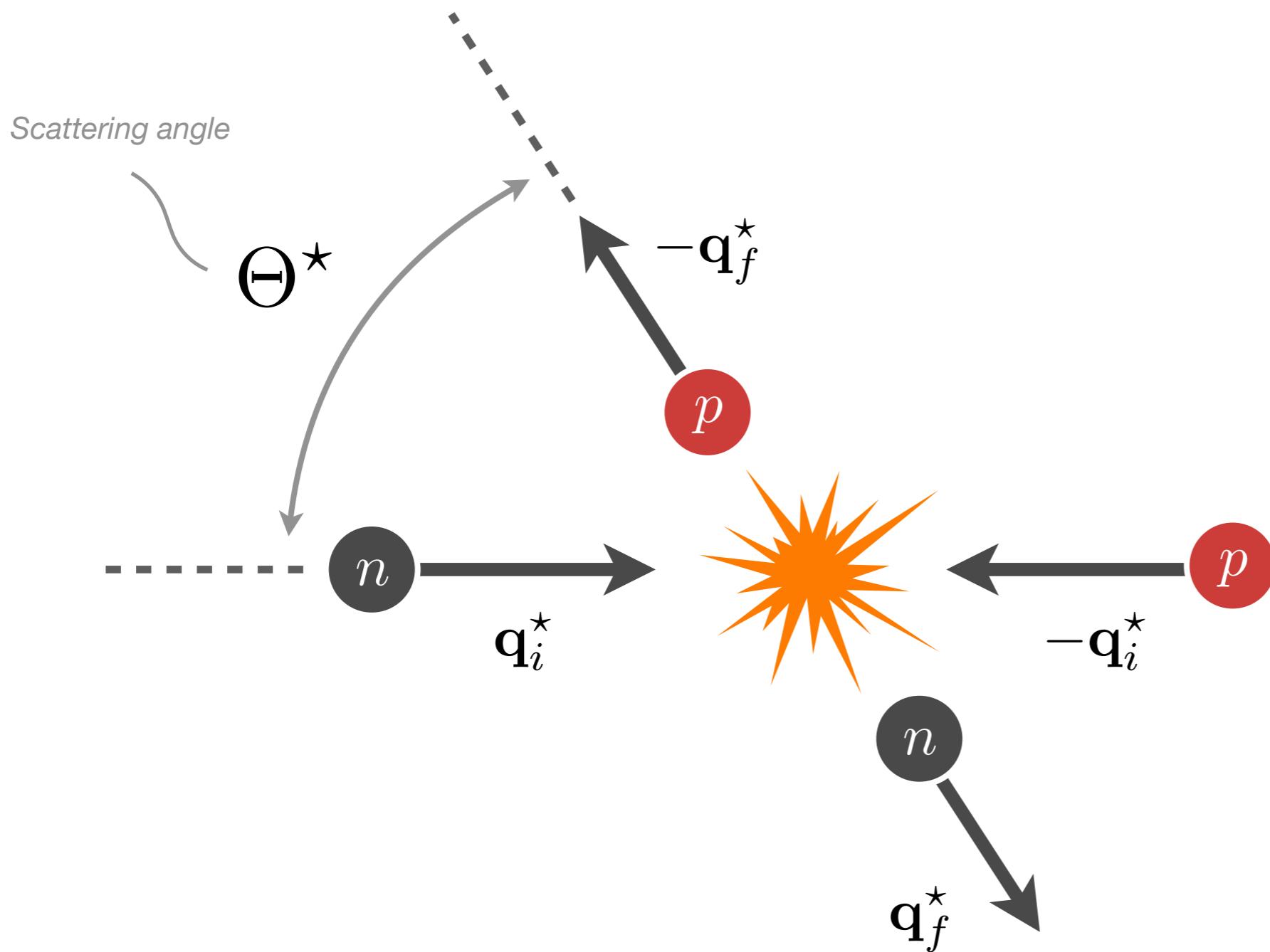
*np*-elastic scattering



# Kinematics – Binary Reactions

*np*-elastic scattering

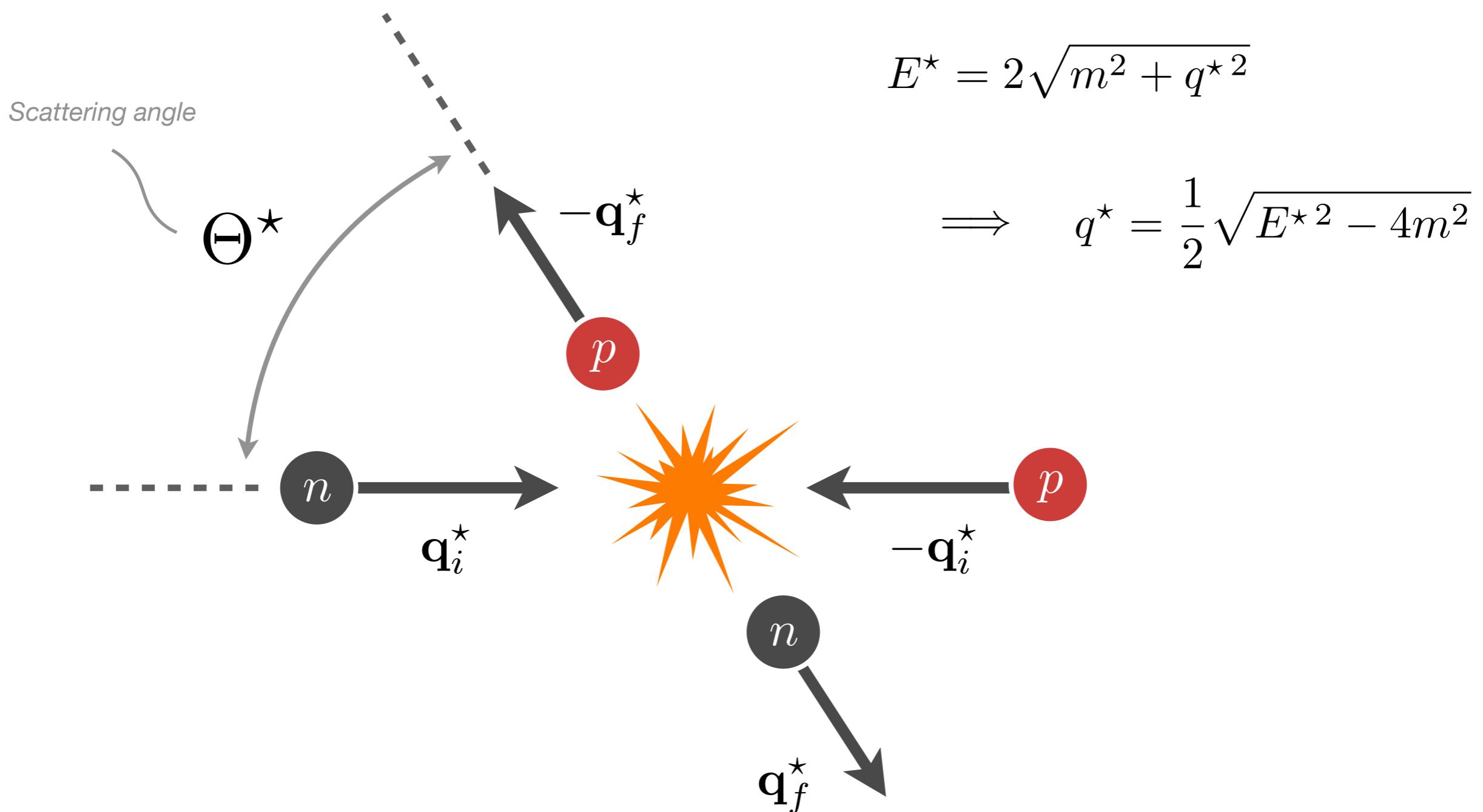
- Total energy-momentum is *conserved* –  $E_i = E_f \equiv E$ ,  $\mathbf{P}_i = \mathbf{P}_f \equiv \mathbf{P}$
- Analyze in *center-of-momentum* (CM) frame –  $\mathbf{P}^* = \mathbf{0}$
- Elastic, equal mass scattering –  $q_i^* = q_f^* \equiv q^*$



# Kinematics – Binary Reactions

*np*-elastic scattering

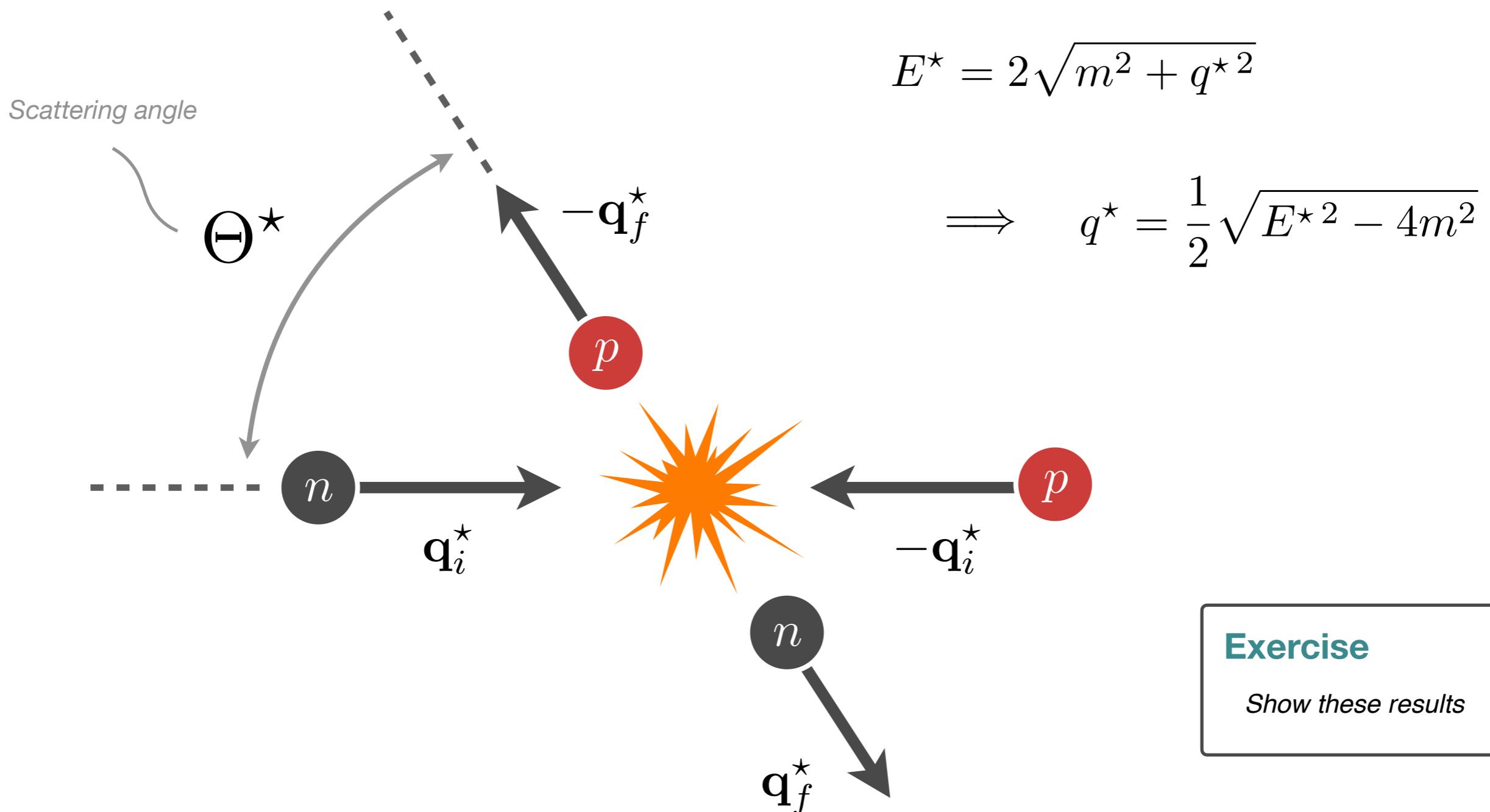
- Total energy-momentum is *conserved* –  $E_i = E_f \equiv E$ ,  $\mathbf{P}_i = \mathbf{P}_f \equiv \mathbf{P}$
- Analyze in *center-of-momentum* (CM) frame –  $\mathbf{P}^* = \mathbf{0}$
- Elastic, equal mass scattering –  $q_i^* = q_f^* \equiv q^*$



# Kinematics – Binary Reactions

*np*-elastic scattering

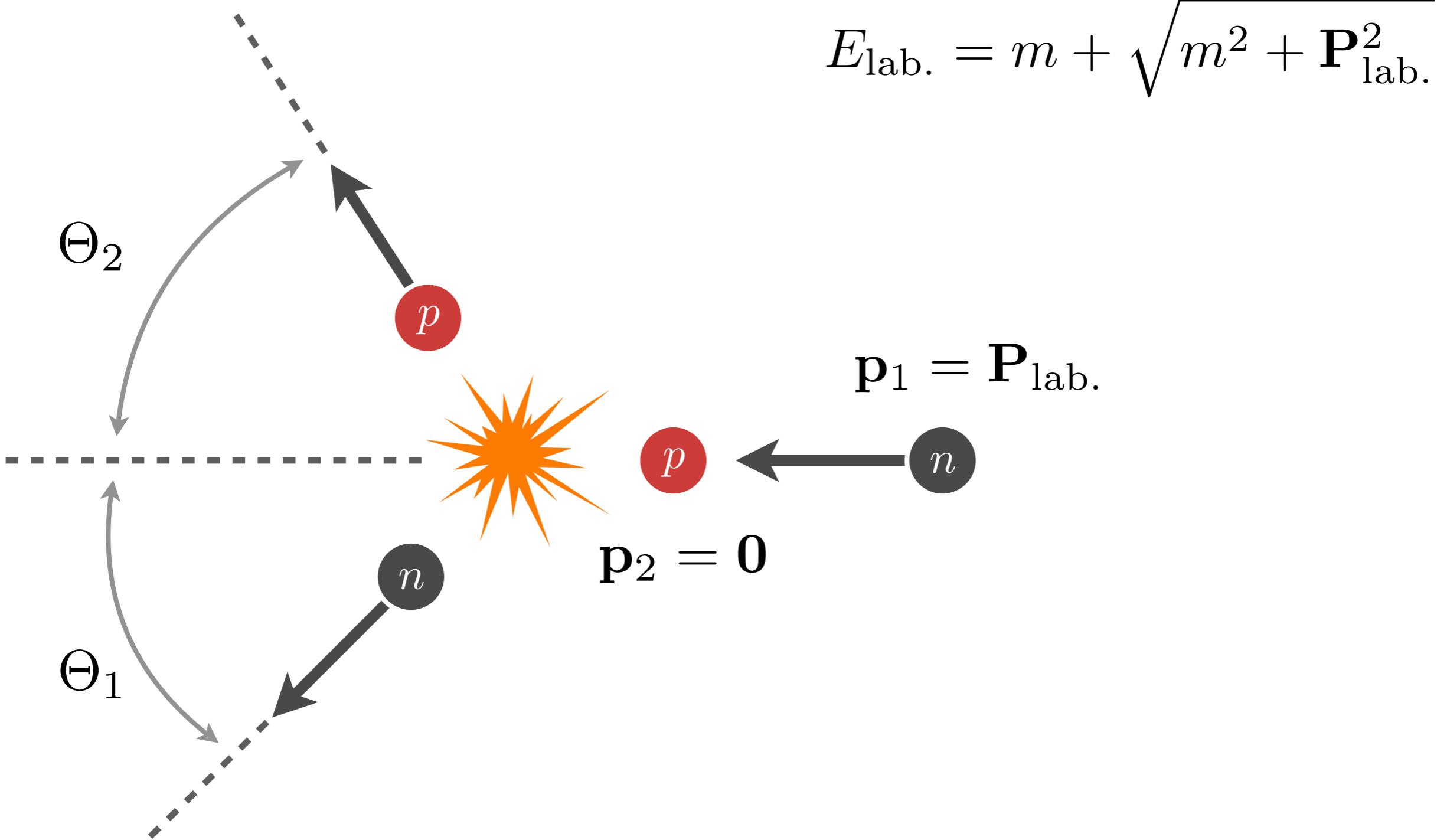
- Total energy-momentum is *conserved* –  $E_i = E_f \equiv E$ ,  $\mathbf{P}_i = \mathbf{P}_f \equiv \mathbf{P}$
- Analyze in *center-of-momentum* (CM) frame –  $\mathbf{P}^* = \mathbf{0}$
- Elastic, equal mass scattering –  $q_i^* = q_f^* \equiv q^*$



## Kinematics – Binary Reactions

*np*-elastic scattering

- Experiments often use *fixed-target* of *lab frame*

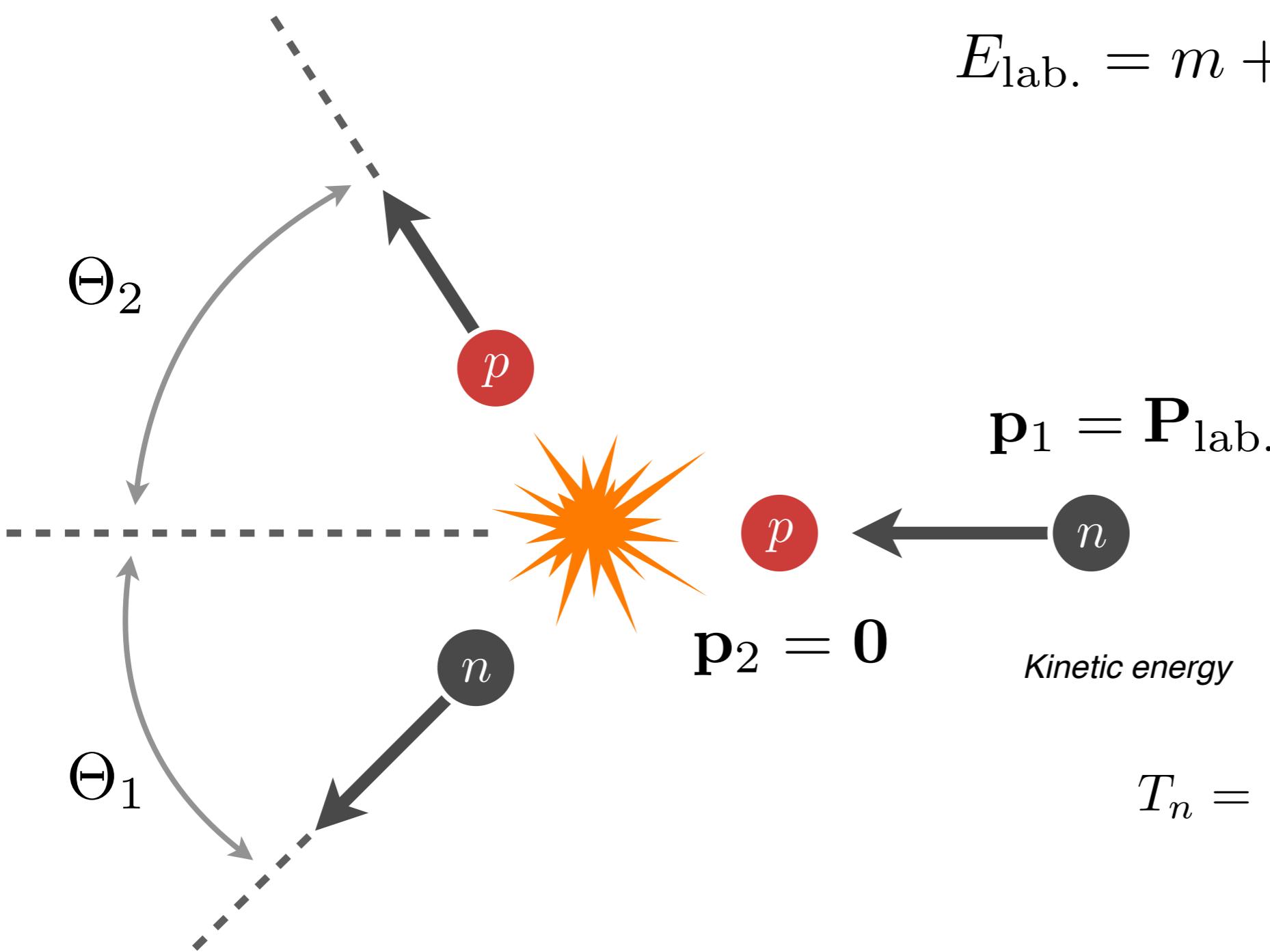


$$E_{\text{lab.}} = m + \sqrt{m^2 + \mathbf{P}_{\text{lab.}}^2}$$

## Kinematics – Binary Reactions

*np*-elastic scattering

- Experiments often use *fixed-target* of *lab frame*



$$E_{\text{lab.}} = m + \sqrt{m^2 + \mathbf{P}_{\text{lab.}}^2}$$

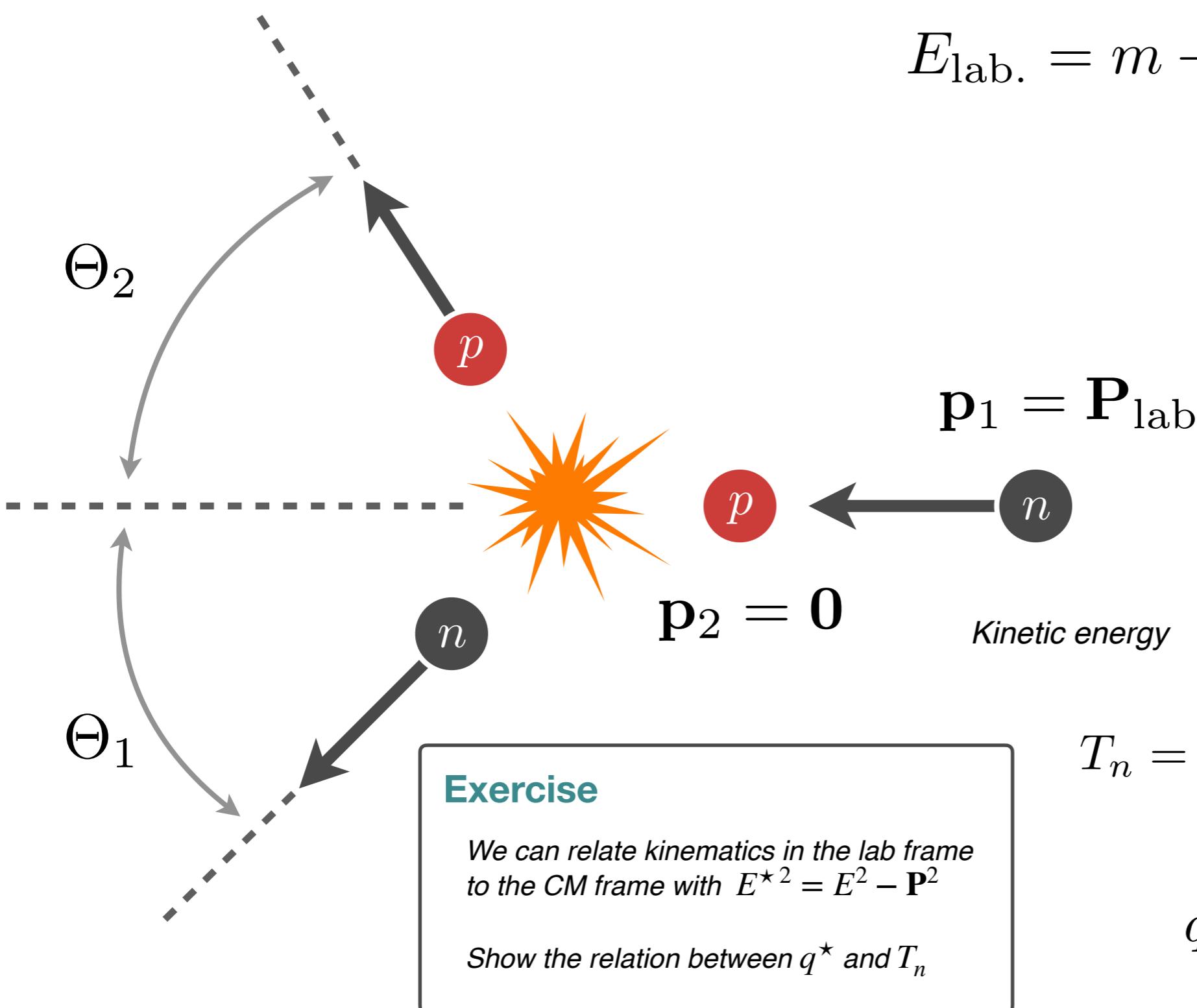
$$T_n = \sqrt{m^2 + \mathbf{P}_{\text{lab.}}^2} - m$$

$$q^{\star 2} = \frac{1}{2}mT_n$$

# Kinematics – Binary Reactions

*np*-elastic scattering

- Experiments often use *fixed-target* of *lab frame*



$$E_{\text{lab.}} = m + \sqrt{m^2 + \mathbf{P}_{\text{lab.}}^2}$$

$$T_n = \sqrt{m^2 + \mathbf{P}_{\text{lab.}}^2} - m$$

$$q^{\star 2} = \frac{1}{2}mT_n$$

# Scattering Theory – Amplitudes

How do we dynamically describe reactions?

- In quantum mechanics, reaction dynamics encoded in *scattering probability amplitudes*

$$|\psi_f\rangle = \mathcal{S} |\psi_i\rangle \sim_{\text{Initial state}} \begin{array}{c} \text{Final state} \\ \curvearrowleft \\ \text{Energy dependent amplitude } \mathcal{S} = \mathcal{S}(E^*, \Theta^*) \end{array}$$

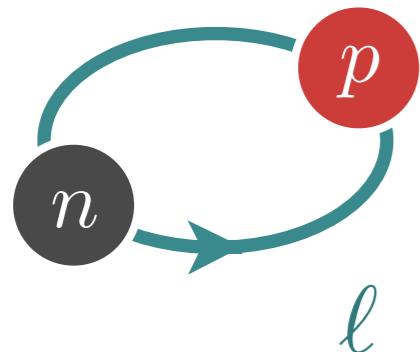
*Probability for initial state to evolve to final state*

$$\text{Prob}(i \rightarrow f) = |\mathcal{S}|^2$$

# Scattering Theory – Amplitudes

How do we dynamically describe reactions?

- In quantum mechanics, reaction dynamics encoded in *scattering probability amplitudes*
- Rotational symmetry — Break up interaction into angular momentum orbitals



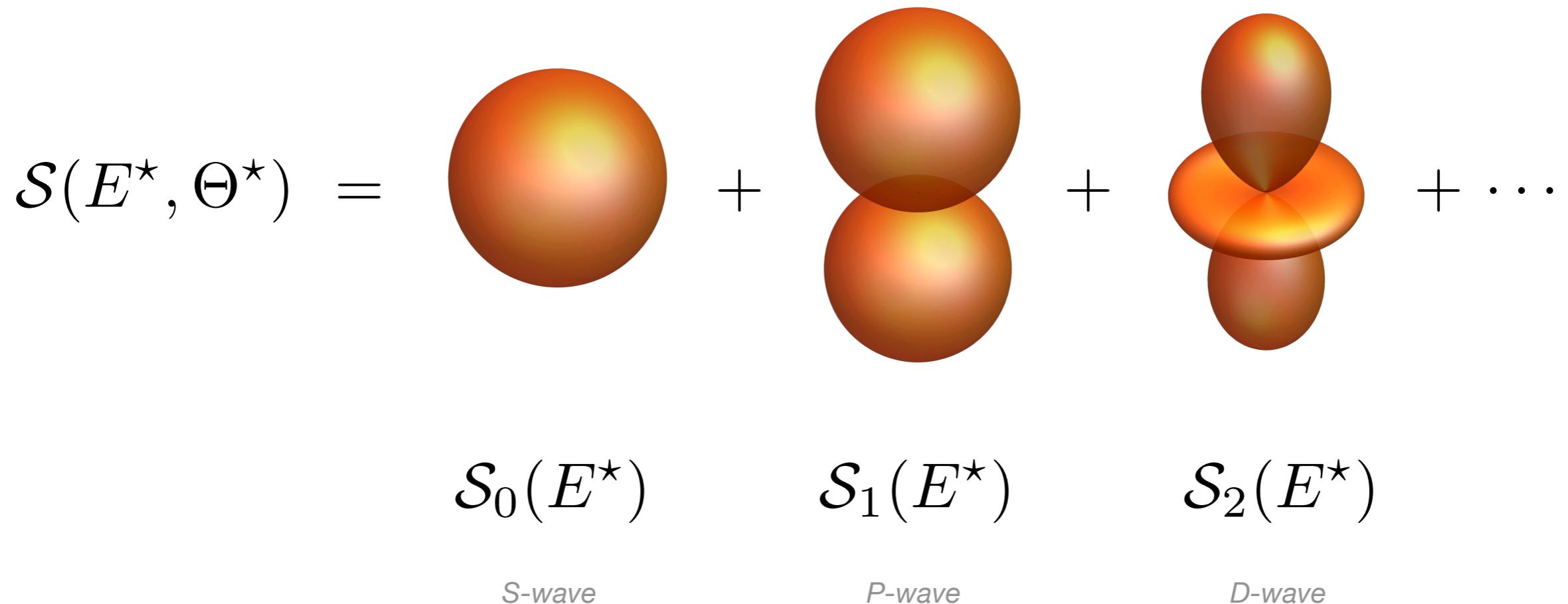
*Angular momentum quantized*

$$\ell = 0, 1, 2, \dots$$

# Scattering Theory – Amplitudes

How do we dynamically describe reactions?

- In quantum mechanics, reaction dynamics encoded in *scattering probability amplitudes*
- Rotational symmetry – Break up interaction into angular momentum orbitals

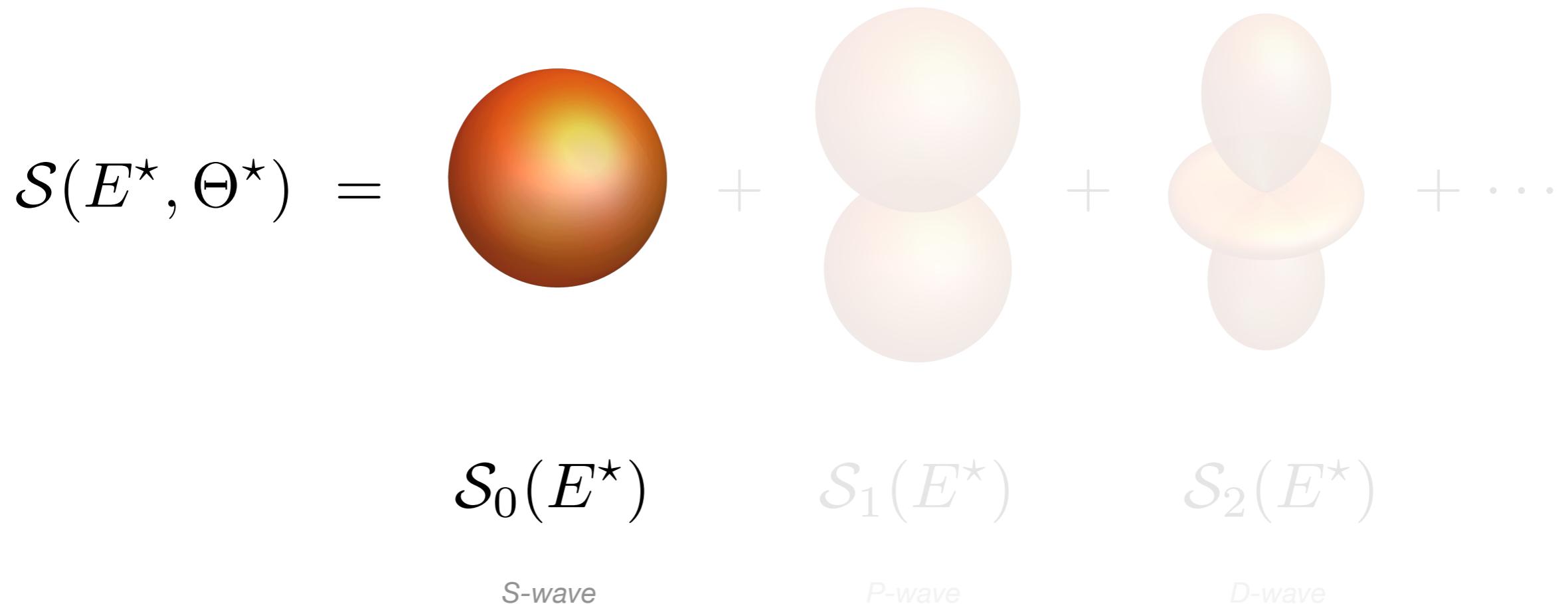


$$\mathcal{S}(E^*, \Theta^*) = \sum_{\ell} (2\ell + 1) \mathcal{S}_{\ell}(E^*) P_{\ell}(\cos \Theta^*)$$

# Scattering Theory – Amplitudes

How do we dynamically describe reactions?

- In quantum mechanics, reaction dynamics encoded in *scattering probability amplitudes*
- Rotational symmetry – Break up interaction into angular momentum orbitals



*At low energies, isotropic interaction dominates*

*Ignore subscript*  $\mathcal{S}_0 \rightarrow \mathcal{S}$

$$\mathcal{S}(E^*, \Theta^*) = \sum_{\ell} (2\ell + 1) \mathcal{S}_{\ell}(E^*) P_{\ell}(\cos \Theta^*)$$

# Scattering Theory – Amplitudes

How do we dynamically describe reactions?

- In quantum mechanics, reaction dynamics encoded in *scattering probability amplitudes*
- Rotational symmetry – Break up interaction into orbitals
- Remove trivial case where particles do not interact – define **scattering amplitude**

$$S = 1 + 2i \rho \mathcal{M}$$

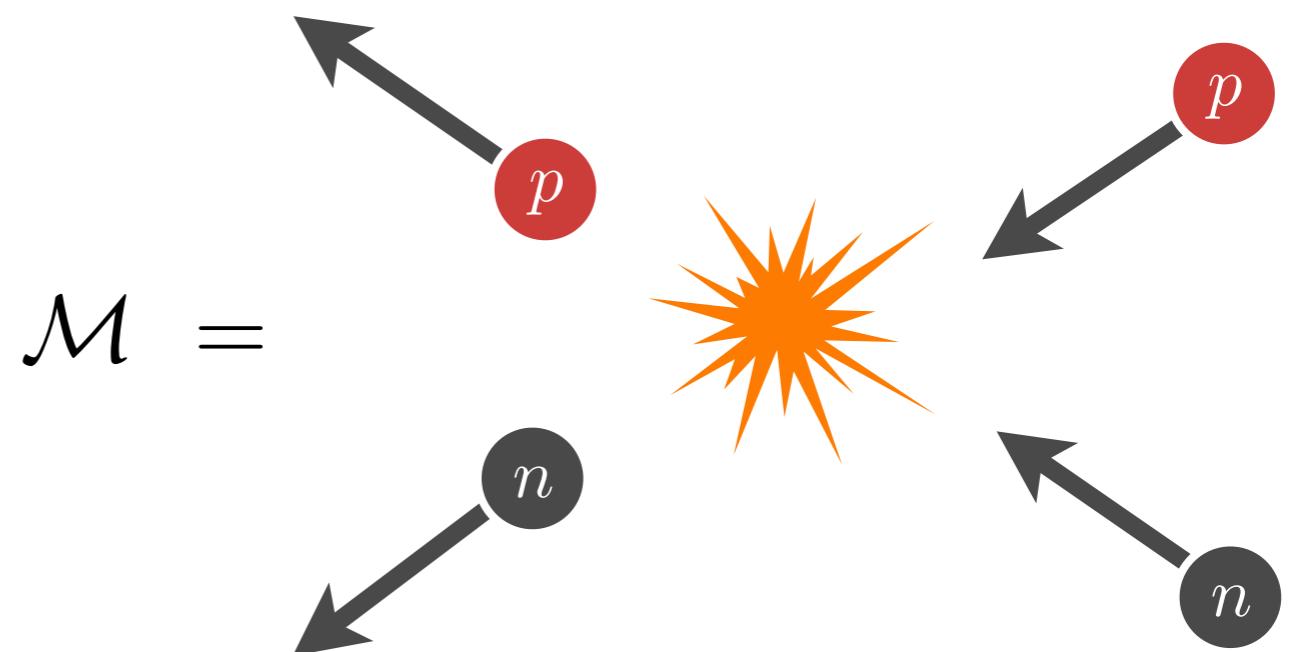
*No scattering*      *Available phase space*

Scattering amplitude – encodes ALL dynamics

$$\rho = \frac{q^*}{8\pi E^*}$$

*Relation to cross-section*

$$\sigma \propto |\mathcal{M}|^2$$



## Scattering Theory – Unitarity

Conservation of probability imposes constraints on the scattering amplitude

$$\sum_f \text{Prob}(i \rightarrow f) = 1 \quad \textcolor{red}{\text{Probability must be conserved!}}$$

*Elastic scattering – only one final state*

$$\implies |\mathcal{S}|^2 = \mathcal{S}^* \mathcal{S} = 1$$

$$\text{Prob}(i \rightarrow f) = |\mathcal{S}|^2$$

## Scattering Theory – Unitarity

Conservation of probability imposes constraints on the scattering amplitude

$$\sum_f \text{Prob}(i \rightarrow f) = 1 \quad \textcolor{red}{\text{Probability must be conserved!}}$$

*Elastic scattering – only one final state*

$$\implies |\mathcal{S}|^2 = \mathcal{S}^* \mathcal{S} = 1$$

*Unitarity condition for scattering amplitude*

$$\mathcal{S} = 1 + 2i \rho \mathcal{M}$$

$$\implies \text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

$$\text{Prob}(i \rightarrow f) = |\mathcal{S}|^2$$

# Scattering Theory – Unitarity

Conservation of probability imposes constraints on the scattering amplitude

$$\sum_f \text{Prob}(i \rightarrow f) = 1 \quad \textcolor{red}{\text{Probability must be conserved!}}$$

*Elastic scattering – only one final state*

$$\implies |\mathcal{S}|^2 = \mathcal{S}^* \mathcal{S} = 1$$

*Unitarity condition for scattering amplitude*

$$\mathcal{S} = 1 + 2i \rho \mathcal{M}$$

$$\implies \text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

$$\text{Prob}(i \rightarrow f) = |\mathcal{S}|^2$$

## Exercise

*Derive the unitarity condition for the scattering amplitude*

# Scattering Theory – Unitarity

Conservation of probability imposes constraints on the scattering amplitude

- At a fixed energy, amplitude needs *two* real numbers

$$\mathcal{M} = \text{Re } \mathcal{M} + i \text{Im } \mathcal{M}$$

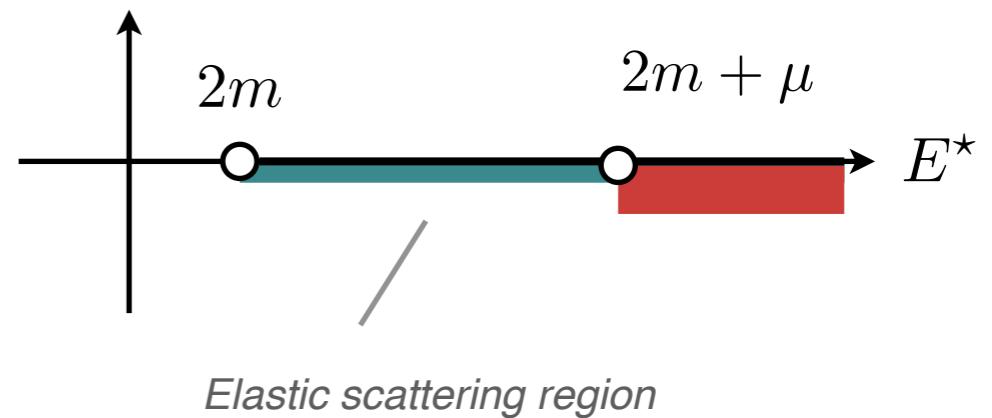
*Unitarity relates real and imaginary parts*

$$\text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

For  $2m \leq E^* < 2m + \mu$

*Available phase space*

$$\rho = \frac{q^*}{8\pi E^*}$$



## Scattering Theory – Unitarity

Conservation of probability imposes constraints on the scattering amplitude

- At a fixed energy, amplitude needs *two* real numbers

$$\mathcal{M} = \text{Re } \mathcal{M} + i \text{Im } \mathcal{M}$$

*Unitarity relates real and imaginary parts*

$$\text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

*Imaginary part fixed by kinematics*

*Can rewrite as*

$$\text{Im } \mathcal{M}^{-1} = -\rho$$

## Scattering Theory – Unitarity

Conservation of probability imposes constraints on the scattering amplitude

- At a fixed energy, amplitude needs *two* real numbers

$$\mathcal{M} = \text{Re } \mathcal{M} + i \text{Im } \mathcal{M}$$

*Unitarity relates real and imaginary parts*

$$\text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

*Imaginary part fixed by kinematics*

*Can rewrite as*

$$\text{Im } \mathcal{M}^{-1} = -\rho$$

*Only need one real number to completely specify scattering*

$$\implies \mathcal{M}^{-1} = \mathcal{K}^{-1} - i\rho$$

*Real function – encodes ALL dynamics!*

# Scattering Theory – Unitarity

Conservation of probability imposes constraints on the scattering amplitude

- At a fixed energy, amplitude needs *two* real numbers

$$\mathcal{M} = \text{Re } \mathcal{M} + i \text{Im } \mathcal{M}$$

*Unitarity relates real and imaginary parts*

$$\text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

*Imaginary part fixed by kinematics*

*Can rewrite as*

$$\text{Im } \mathcal{M}^{-1} = -\rho$$

## Exercise

Show that  $\text{Im } \mathcal{M}^{-1} = -\rho$

*Only need one real number to completely specify scattering*

$$\implies \mathcal{M}^{-1} = \mathcal{K}^{-1} - i\rho$$

*Real function – encodes ALL dynamics!*

## Scattering Theory – Unitarity

Unitarity implies only one real dynamical function needed to describe scattering

- Theoretically, want to compute  $K$  matrix
- Can infer  $K$  matrix by fitting experimental data

$$\mathcal{M}^{-1} = \mathcal{K}^{-1} - i\rho \quad \text{---} \quad \begin{matrix} \text{Know this function} \\ \text{Want to determine this function} \end{matrix}$$

A curly brace is positioned under the term  $\mathcal{K}^{-1}$ , indicating that it is the function we know.

## Scattering Theory – Phase shifts

---

Alternatively, can cast amplitude in terms of phase shifts

- Unitarity is starting point

$$\text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

*Use polar representation*

$$\mathcal{M} = |\mathcal{M}| e^{i\delta}$$

## Scattering Theory – Phase shifts

Alternatively, can cast amplitude in terms of phase shifts

- Unitarity is starting point

$$\text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

*Use polar representation*

$$\mathcal{M} = |\mathcal{M}| e^{i\delta}$$

*Impose unitarity, can show*

$$\implies \mathcal{M} = \frac{1}{\rho} e^{i\delta} \sin \delta \quad \text{Phase shift, a real function of energy}$$

## Scattering Theory – Phase shifts

Alternatively, can cast amplitude in terms of phase shifts

- Unitarity is starting point

$$\text{Im } \mathcal{M} = \rho |\mathcal{M}|^2$$

*Use polar representation*

$$\mathcal{M} = |\mathcal{M}| e^{i\delta}$$

*Impose unitarity, can show*

$$\implies \mathcal{M} = \frac{1}{\rho} e^{i\delta} \sin \delta \quad \text{Phase shift, a real function of energy}$$

### Exercise

(a) Derive the phase shift representation

(b) Show that  $\mathcal{K}^{-1} = \rho \cot \delta$

# Scattering Theory – Effective Range Expansion

Can use either  $K$  matrix or phase shifts to describe reaction (often use both)

- They are real functions, can Taylor expand in real variables – assign physical meaning

$$\mathcal{K}^{-1} = \rho \cot \delta$$

$$\propto q^* \cot \delta = -\frac{1}{a} + \frac{1}{2} r q^{*2} + \mathcal{O}(q^{*4})$$

*Effective range expansion*



*Effective range – “range of interaction”*

*Scattering length – “strength of interaction”*

$$\rho = \frac{q^*}{8\pi E^*}$$

# Scattering Theory – Effective Range Expansion

Can use either  $K$  matrix or phase shifts to describe reaction (often use both)

- They are real functions, can Taylor expand in real variables — assign physical meaning

$$\mathcal{K}^{-1} = \rho \cot \delta$$

$$\propto q^* \cot \delta = -\frac{1}{a} + \frac{1}{2} r q^{*2} + \mathcal{O}(q^{*4})$$

*Effective range expansion*



*Effective range — “range of interaction”*

*Scattering length — “strength of interaction”*

*Amplitude at threshold is constant, fixed by strength of interaction*

$$\mathcal{M}_{\text{thr.}} \equiv \mathcal{M}(E_{\text{thr.}}^*) = -16\pi ma$$

$$\rho = \frac{q^*}{8\pi E^*}$$

# Scattering Theory – Effective Range Expansion

Can use either  $K$  matrix or phase shifts to describe reaction (often use both)

- They are real functions, can Taylor expand in real variables — assign physical meaning

$$\mathcal{K}^{-1} = \rho \cot \delta$$

$$\propto q^* \cot \delta = -\frac{1}{a} + \frac{1}{2} r q^{*2} + \mathcal{O}(q^{*4})$$

*Effective range expansion*



*Effective range — “range of interaction”*

*Scattering length — “strength of interaction”*

*Amplitude at threshold is constant, fixed by strength of interaction*

$$\rho = \frac{q^*}{8\pi E^*}$$

$$\mathcal{M}_{\text{thr.}} \equiv$$

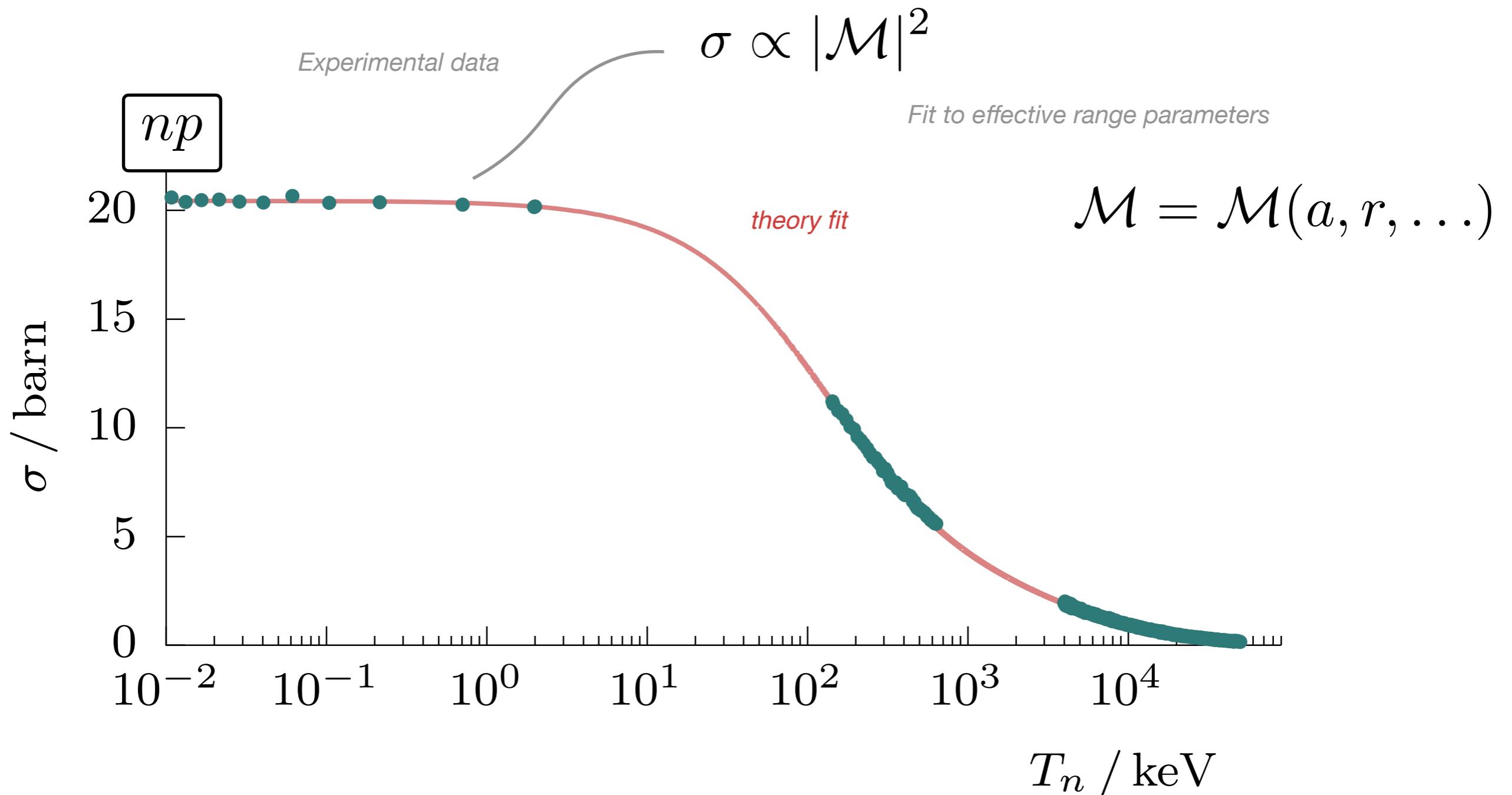
## Exercise

Show that at threshold the scattering amplitude is constant and fixed by the scattering length

# Scattering Theory – Effective Range Expansion

Can use either  $K$  matrix or phase shifts to describe reaction (often use both)

- They are real functions, can Taylor expand in real variables — assign physical meaning
- These are fit parameters — can learn about the interaction



# Spectroscopy — Bound states

Once we have the amplitude, we can now search for the **spectrum**

- Bound states are enhancements in the amplitude — they are pole singularities

**Scattering  $\iff$  Bound States**

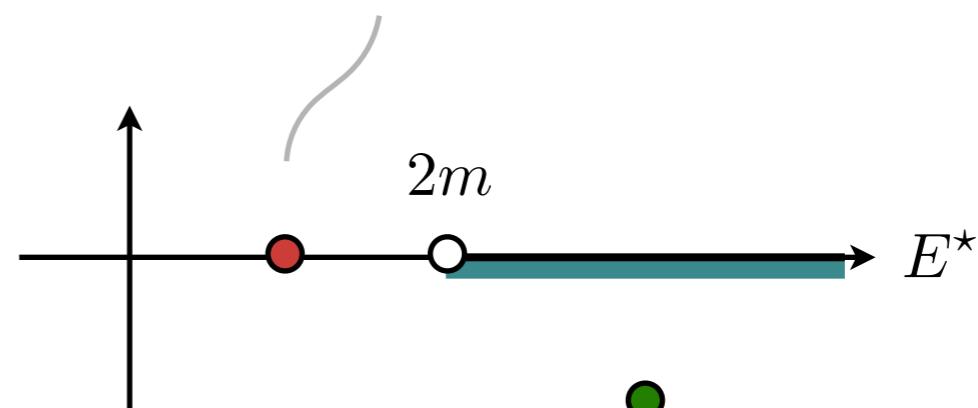
$$\mathcal{M} \propto \frac{1}{E^* - M - i\Gamma/2}$$

*Hadron Mass*

$$\tau = \frac{1}{\Gamma}$$

*Hadron Lifetime*

*Stable bound states ( $\tau \rightarrow \infty$ )*



*Unstable states (resonances)*

# Spectroscopy — Bound states

Once we have the amplitude, we can now search for the **spectrum**

- Bound states are enhancements in the amplitude — they are pole singularities

**Scattering  $\iff$  Bound States**

$$\mathcal{M} \propto \frac{1}{E^* - M - i\Gamma/2}$$

*Hadron Mass*

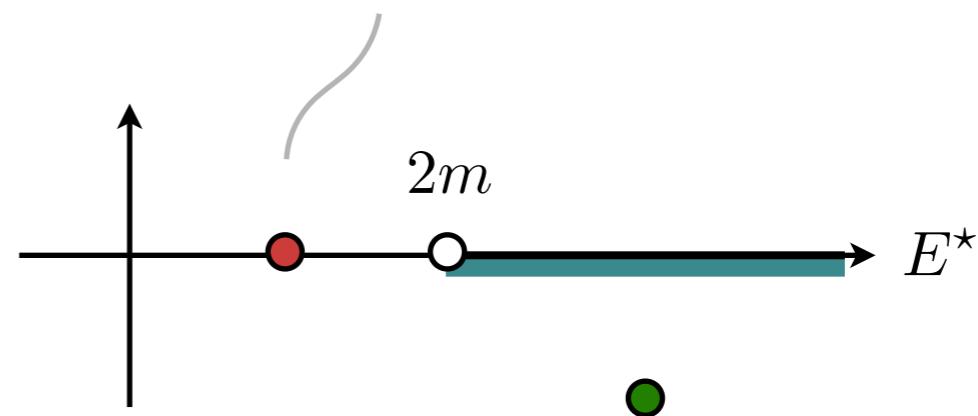
$$\tau = \frac{1}{\Gamma}$$

*Hadron Lifetime*

*Stable bound states ( $\tau \rightarrow \infty$ )*

*Search for zeroes of denominator*

$$\mathcal{M}^{-1}(E_b^*) = 0$$



*Unstable states (resonances)*

## Spectroscopy — Bound states

---

Once we have the amplitude, we can now search for the **spectrum**

- Bound states are enhancements in the amplitude — they are pole singularities

$$\mathcal{M} = \frac{8\pi E^*}{q^* \cot \delta - iq^*}$$

## Spectroscopy — Bound states

---

Once we have the amplitude, we can now search for the **spectrum**

- Bound states are enhancements in the amplitude — they are pole singularities

$$\mathcal{M} = \frac{8\pi E^*}{q^* \cot \delta - iq^*}$$

$$\mathcal{M}^{-1} = 0 \implies q^* \cot \delta = iq^*$$

## Spectroscopy — Bound states

Once we have the amplitude, we can now search for the **spectrum**

- Bound states are enhancements in the amplitude — they are pole singularities

$$\mathcal{M} = \frac{8\pi E^*}{q^* \cot \delta - iq^*}$$

$$\mathcal{M}^{-1} = 0 \implies q^* \cot \delta = iq^*$$

Assume leading order effective range,  $a > 0$

$$q^* \cot \delta = -\frac{1}{a} \implies q^* = \frac{i}{a} \equiv i\kappa$$



Binding momentum

$$\begin{aligned} E^* &= 2\sqrt{m^2 + q^{*2}} \\ &= 2\sqrt{m^2 - \kappa^2} \end{aligned}$$

## Spectroscopy — Bound states

Once we have the amplitude, we can now search for the **spectrum**

- Bound states are enhancements in the amplitude — they are pole singularities

$$\mathcal{M} = \frac{8\pi E^*}{q^* \cot \delta - iq^*}$$

$$\mathcal{M}^{-1} = 0 \implies q^* \cot \delta = iq^*$$

Compare to

$$\mathcal{M} \propto \frac{1}{E^* - M - i\Gamma/2}$$

$$= i\kappa$$

$$\implies M = 2\sqrt{m^2 - \frac{1}{a^2}} \quad \Gamma = 0$$

Binding momentum

## Spectroscopy — Bound states

Once we have the amplitude, we can now search for the **spectrum**

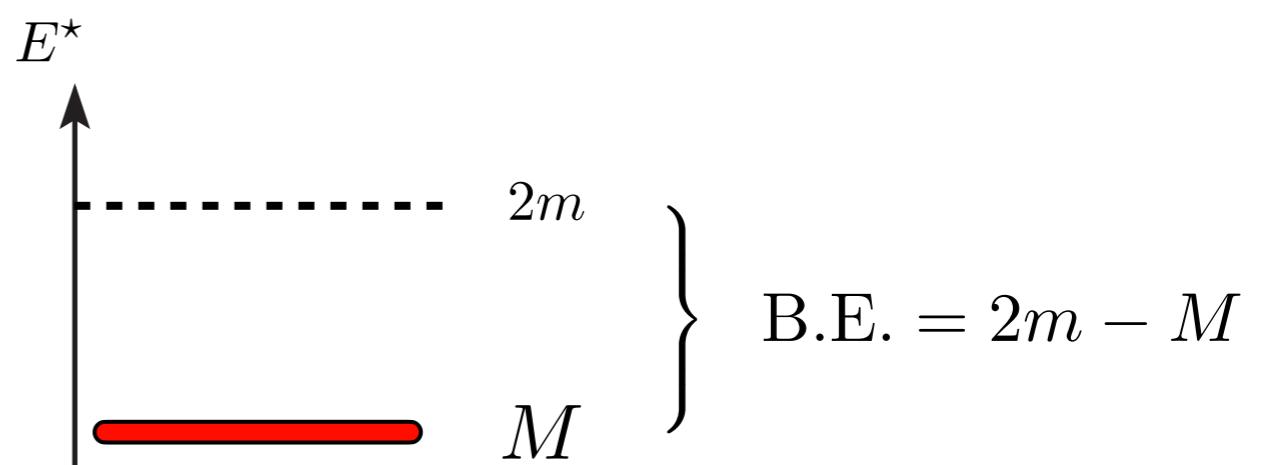
- Bound states are enhancements in the amplitude — they are pole singularities

*For  $a > 0$ , find  $M < 2m$  — Stable bound state*

$$M = 2\sqrt{m^2 - \frac{1}{a^2}}$$

$$\Gamma = 0$$

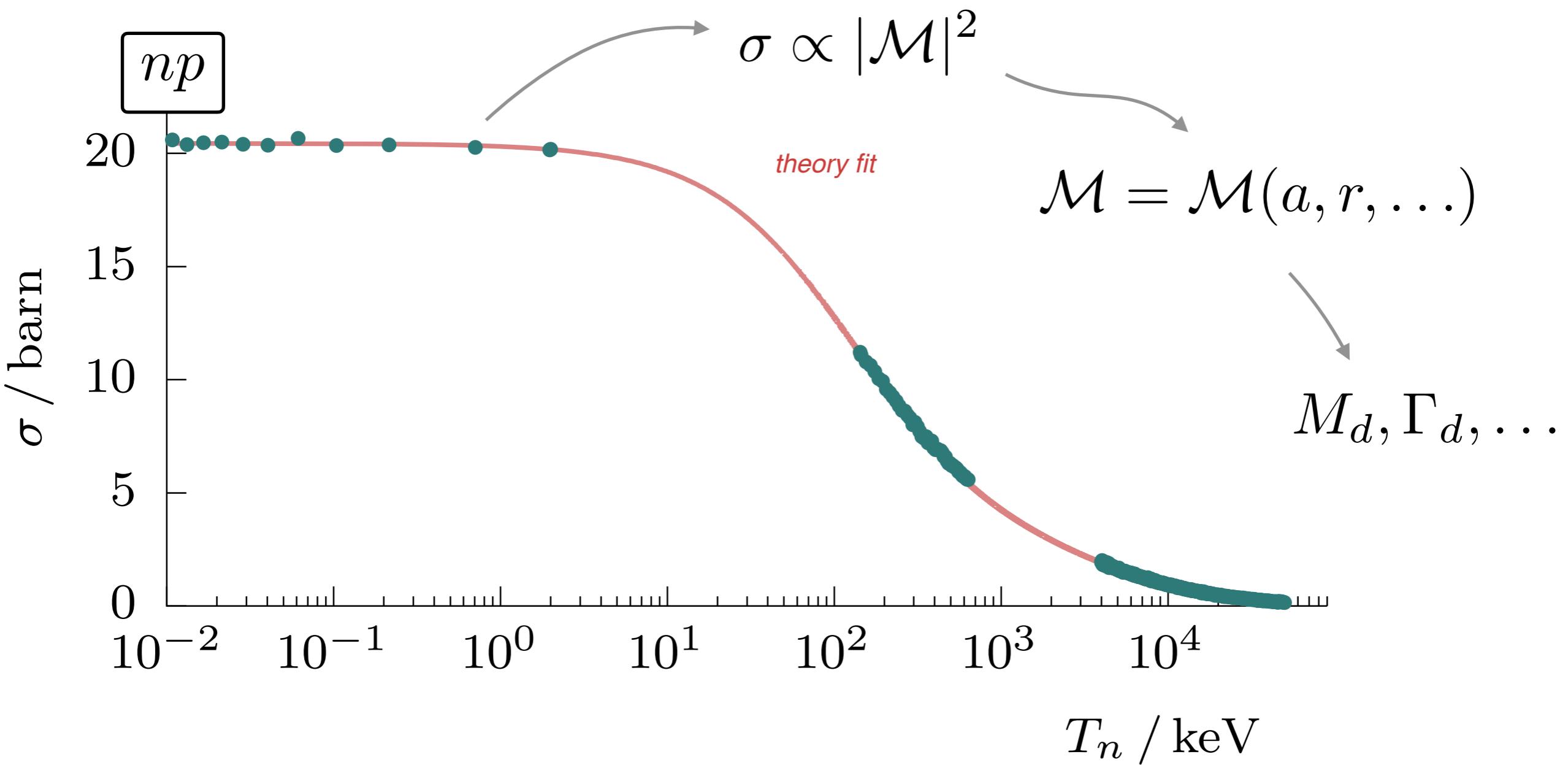
*Binding energy*



# Neutron-Proton Scattering – Spin Effects

Goal — analyze low-energy neutron-proton scattering data

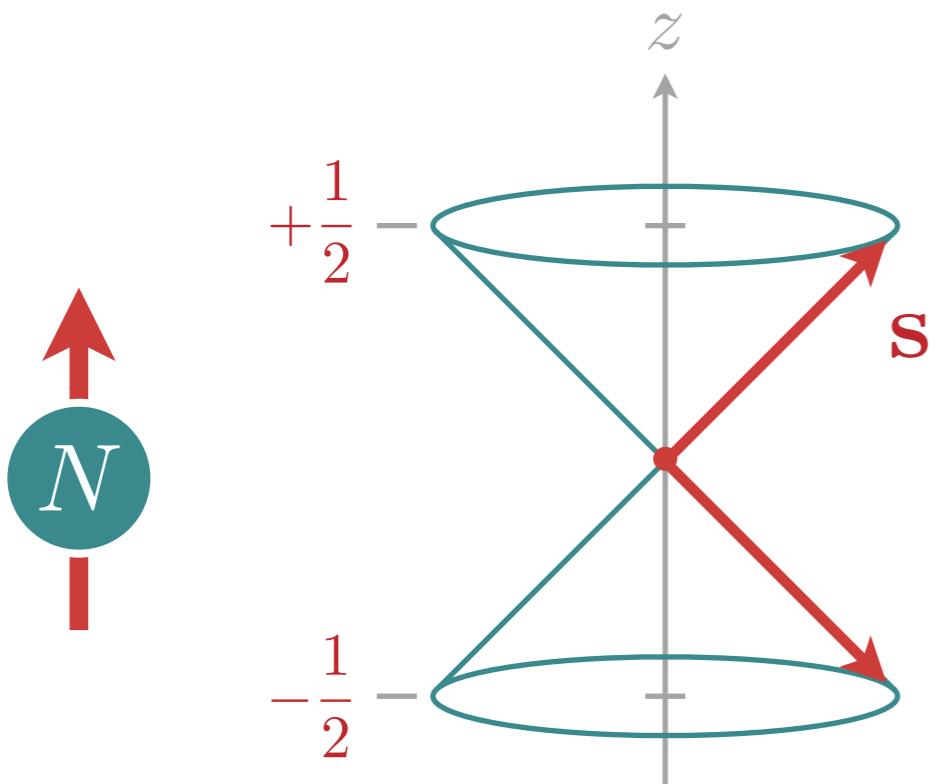
- Have a kinematic description of reaction
- Know how to parameterize amplitude in terms of physical parameters
- Can fit amplitude with unknown parameters to data
- Once we have scattering parameters, can determine bound state spectrum



# Neutron-Proton Scattering – Spin Effects

So far, assumed nucleon was structure-less

- Nucleons have intrinsic *spin* angular momentum – spin-1/2 fermions
- The strong interactions can (and does) depend on the spin degrees-of-freedom
- Assume still in low-energy regime (no angular dependence)



# Neutron-Proton Scattering – Spin Effects

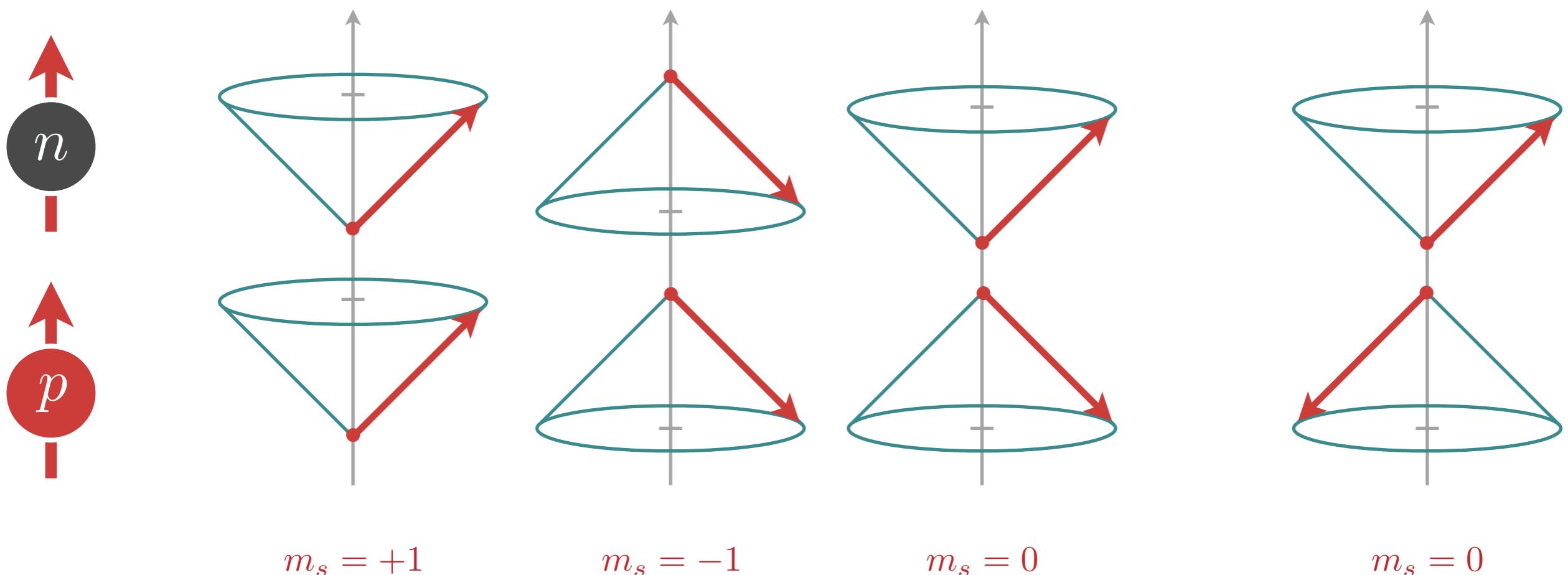
So far, assumed nucleon was structure-less

- Nucleons have intrinsic *spin* angular momentum – spin-1/2 fermions
- The strong interactions can (and does) depend on the spin degrees-of-freedom
- Assume still in low-energy regime (no angular dependence)

*Total spin of neutron-proton system*

$$m_s = s_n + s_p$$

$$\Rightarrow s = 0 \text{ or } 1$$



# Neutron-Proton Scattering – Spin Effects

So far, assumed nucleon was structure-less

- Nucleons have intrinsic *spin* angular momentum – spin-1/2 fermions
- The strong interactions can (and does) depend on the spin degrees-of-freedom
- Assume still in low-energy regime (no angular dependence)
- np scattering can occur in the singlet or triplet state
- Total cross section is weighted sum of singlet and triplet cross sections

$$\sigma = \frac{1}{4} \sigma_s + \frac{3}{4} \sigma_t$$

*Singlet ( $s = 0$ )*      *Triplet ( $s = 1$ )*

Total number of spin states = 4

No spin-orbit interactions

$$\ell = 0, s = 0 \text{ or } 1 \implies J = s$$

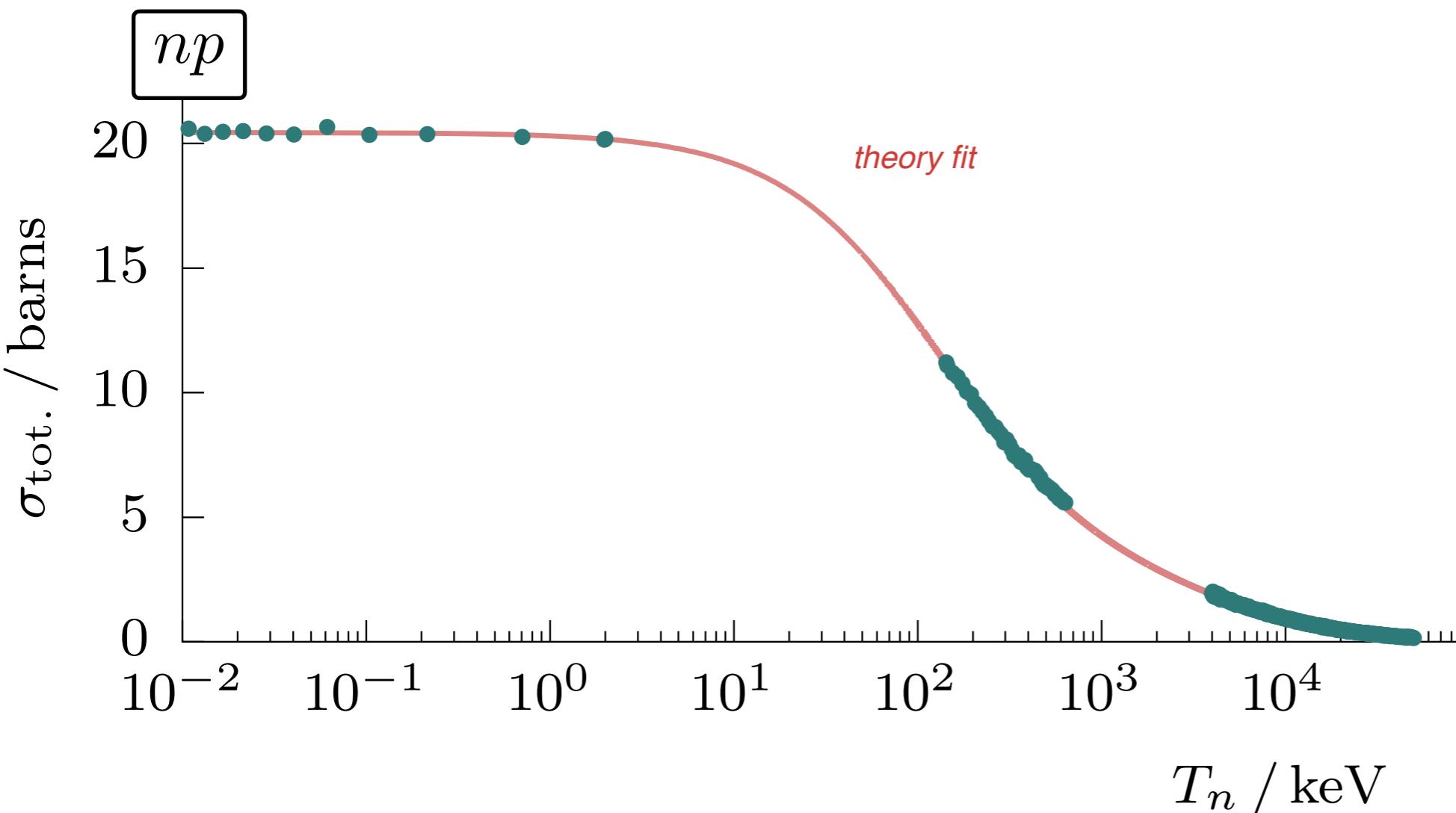
## Neutron-Proton Scattering – Cross-section

$$\sigma_x = \frac{|\mathcal{M}_x|^2}{16\pi E^{\star 2}}$$

Have enough tools to analyze data

$$\sigma = \frac{1}{4} \sigma_s + \frac{3}{4} \sigma_t$$

$$\mathcal{M}_x = \frac{8\pi E^{\star}}{q^{\star} \cot \delta_x(q^{\star}) - iq^{\star}}$$



# Neutron-Proton Scattering – Cross-section

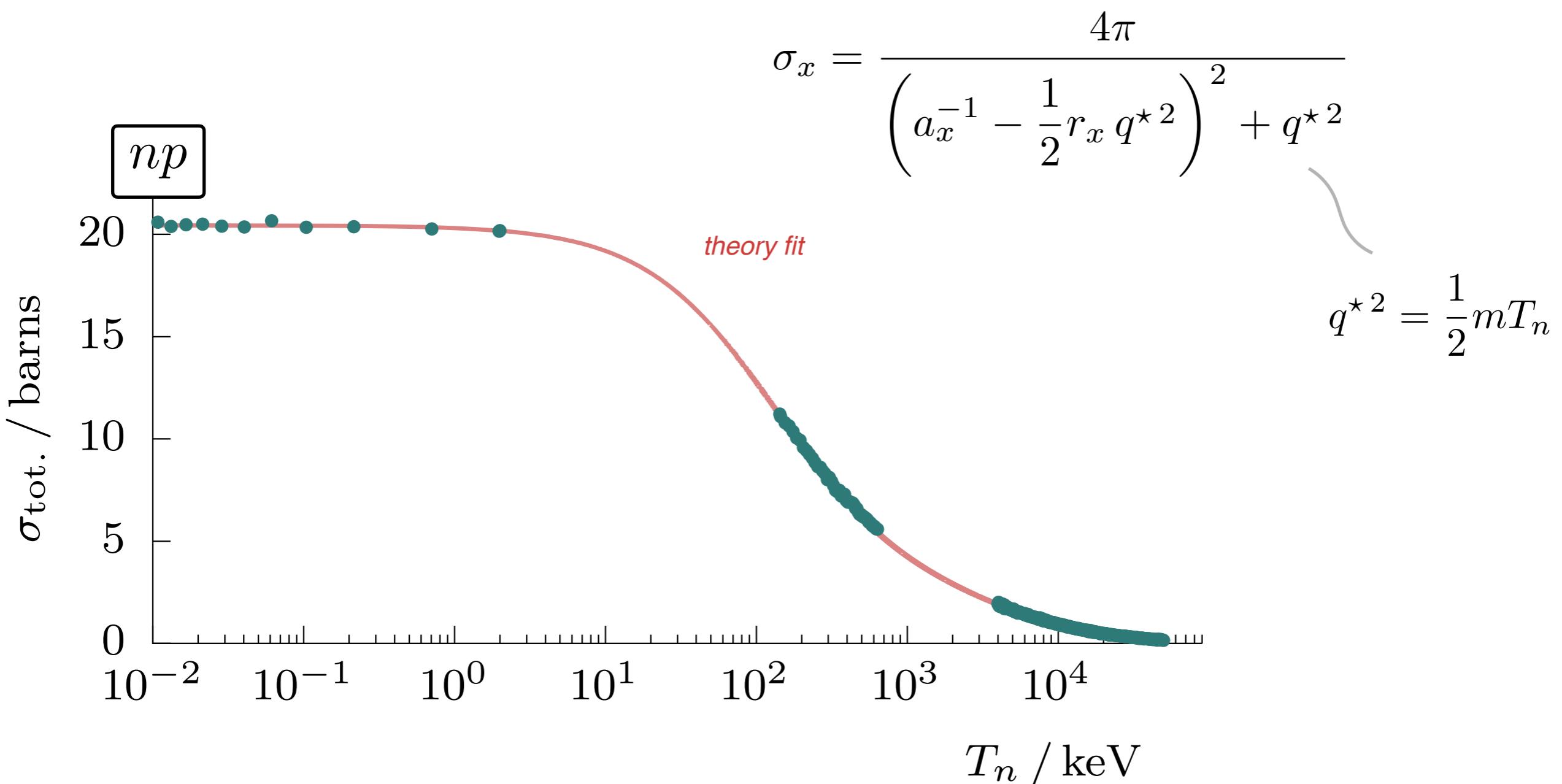
$$\sigma_x = \frac{|\mathcal{M}_x|^2}{16\pi E^{\star 2}}$$

Have enough tools to analyze data

$$\sigma = \frac{1}{4} \sigma_s + \frac{3}{4} \sigma_t$$

$$\mathcal{M}_x = \frac{8\pi E^{\star}}{q^{\star} \cot \delta_x(q^{\star}) - iq^{\star}}$$

*Low-energy shape-independent approximation*



$$\sigma_x = \frac{4\pi}{\left(a_x^{-1} - \frac{1}{2}r_x q^{\star 2}\right)^2 + q^{\star 2}}$$

$$q^{\star 2} = \frac{1}{2}mT_n$$

# Neutron-Proton Scattering – Cross-section

$m \approx 940 \text{ MeV}$

Have enough tools to analyze data

$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$

$$1 \text{ b} = 100 \text{ fm}^2 (= 10^{-28} \text{ m}^2)$$

$$\chi^2 = \sum \left( \frac{\sigma_{\text{exp.}} - \sigma_{\text{th}}}{\Delta \sigma_{\text{exp.}}} \right)^2$$

*My simple fit*

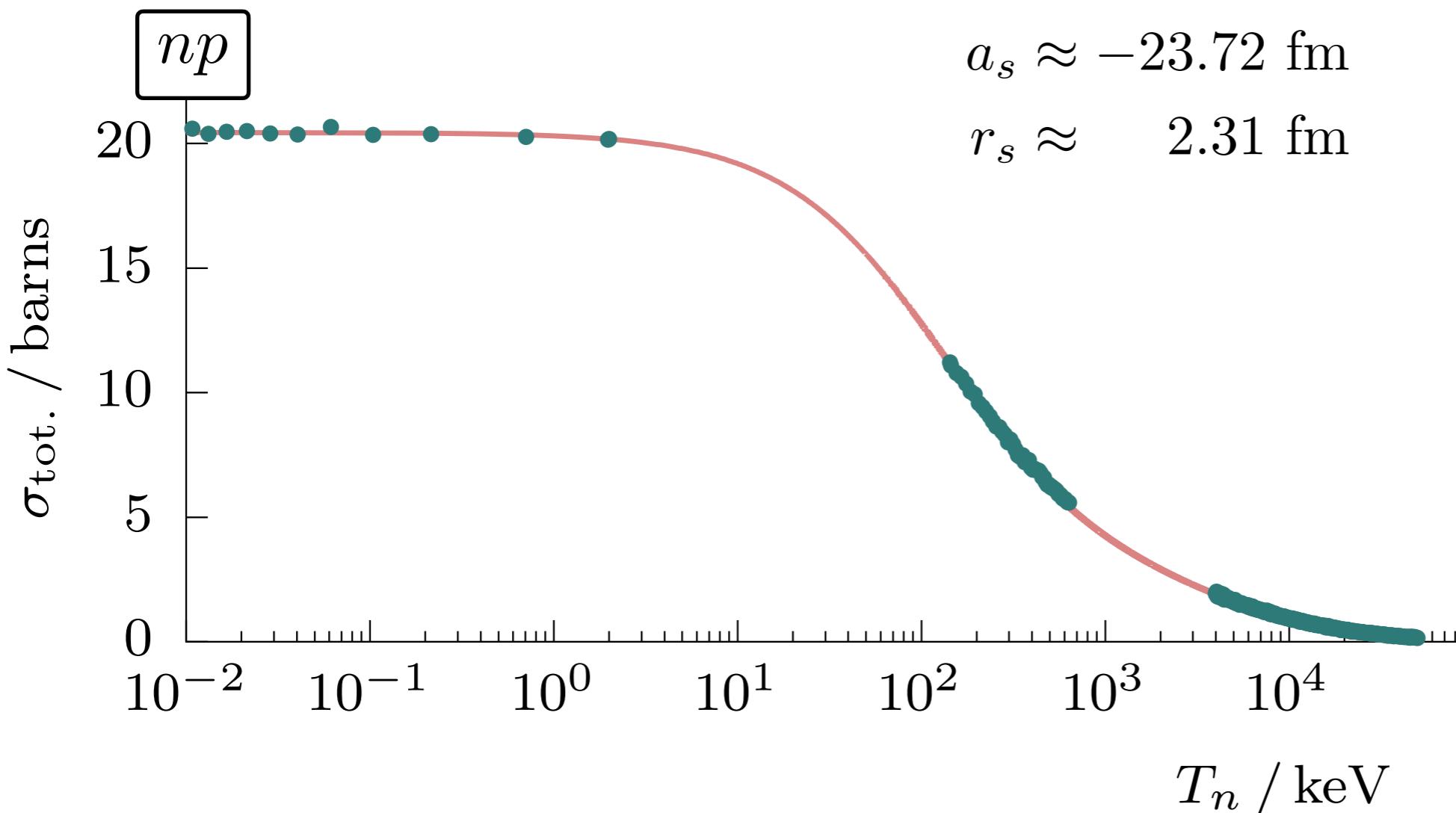
$$\chi^2/\nu = 472.9/(486 - 4) \approx 0.98$$

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

$$a_s \approx -23.72 \text{ fm}$$

$$r_s \approx 2.31 \text{ fm}$$



# Neutron-Proton Scattering – Cross-section

$m \approx 940$  MeV

Have enough tools to analyze data

$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$

$$\chi^2 = \sum \left( \frac{\sigma_{\text{exp.}} - \sigma_{\text{th}}}{\Delta\sigma_{\text{exp.}}} \right)^2$$

*My simple fit*

$$1 \text{ b} = 100 \text{ fm}^2 (= 10^{-28} \text{ m}^2)$$

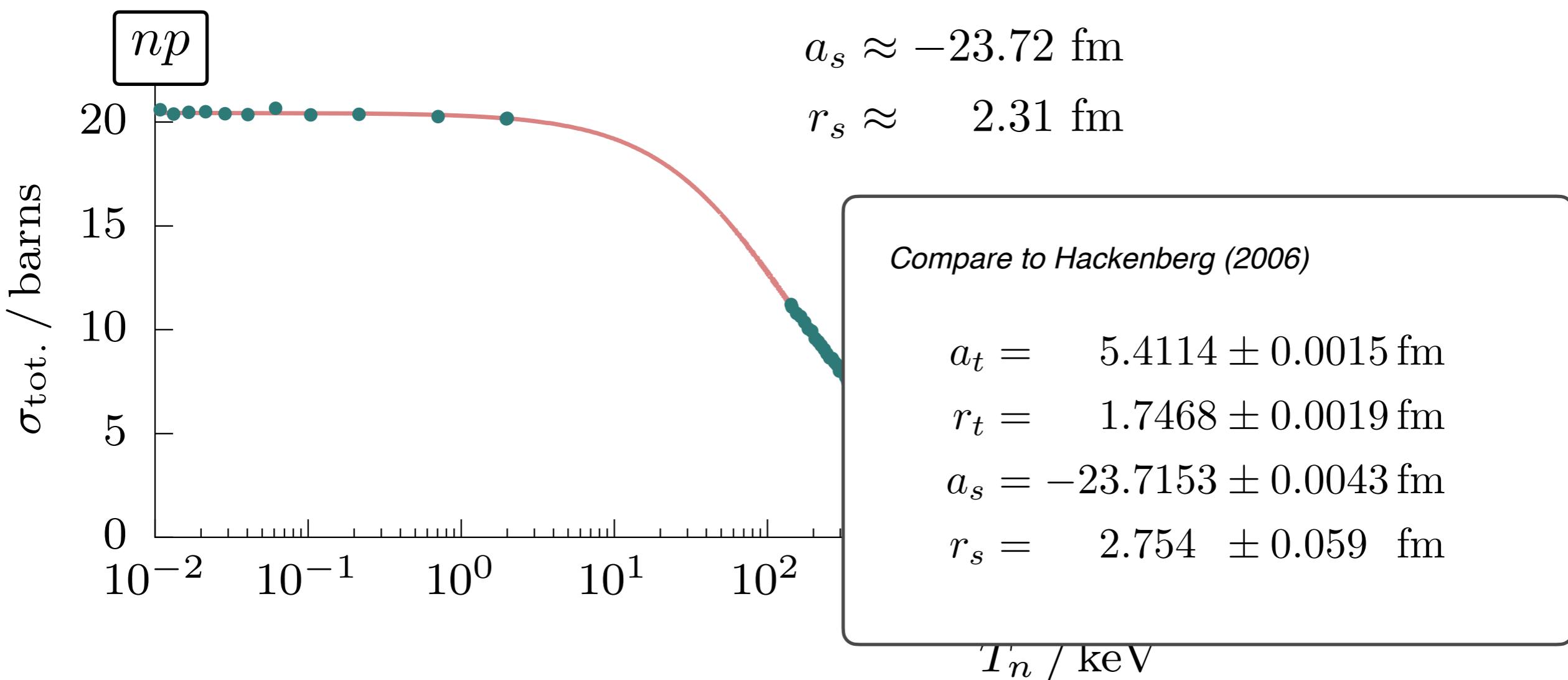
$$\chi^2/\nu = 472.9/(486 - 4) \approx 0.98$$

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

$$a_s \approx -23.72 \text{ fm}$$

$$r_s \approx 2.31 \text{ fm}$$



# Neutron-Proton Scattering – Cross-section

$m \approx 940 \text{ MeV}$

Have enough tools to analyze data

$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$

$$1 \text{ b} = 100 \text{ fm}^2 (= 10^{-28} \text{ m}^2)$$

$$\chi^2 = \sum \left( \frac{\sigma_{\text{exp.}} - \sigma_{\text{th}}}{\Delta \sigma_{\text{exp.}}} \right)^2$$

*My simple fit*

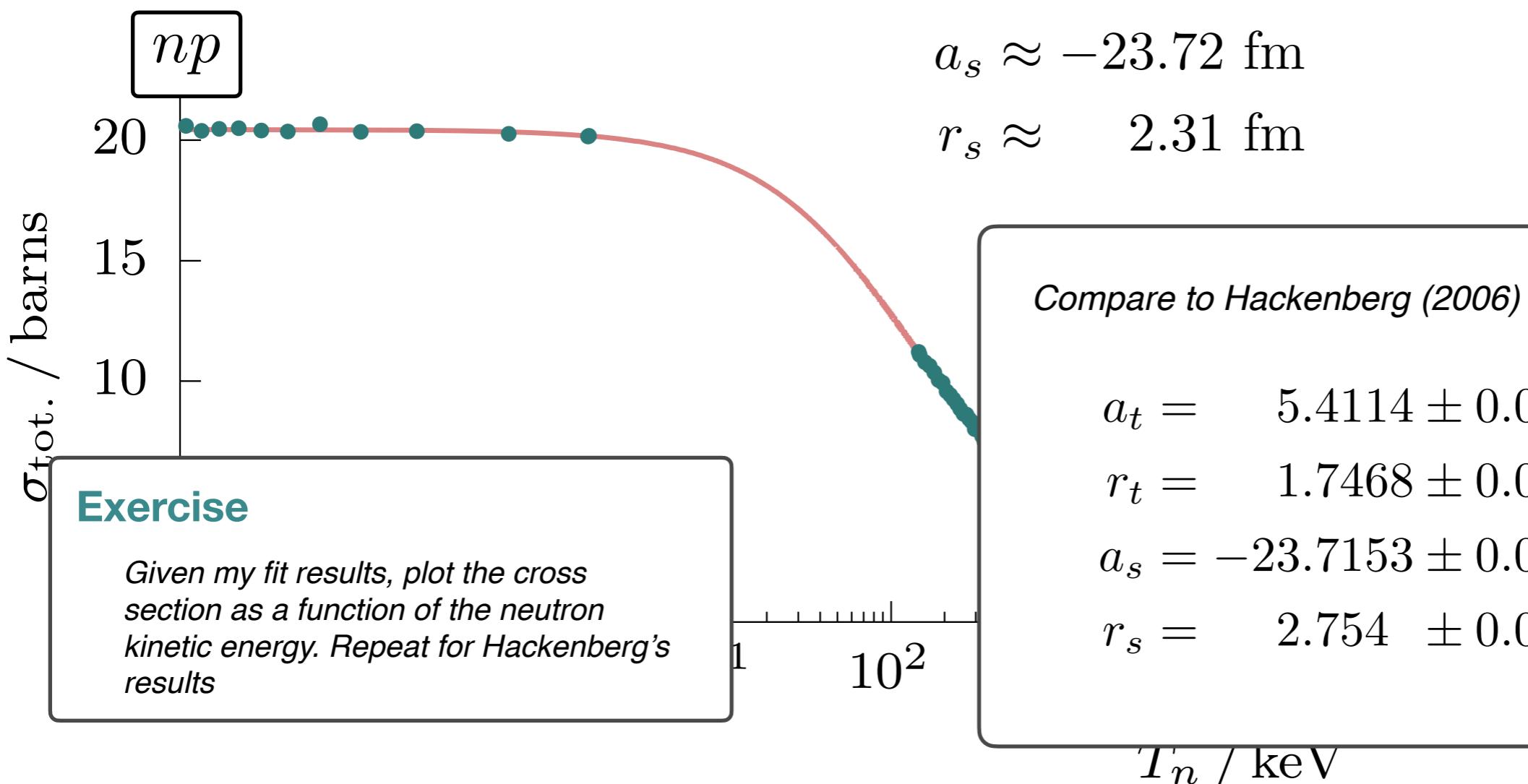
$$\chi^2/\nu = 472.9/(486 - 4) \approx 0.98$$

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

$$a_s \approx -23.72 \text{ fm}$$

$$r_s \approx 2.31 \text{ fm}$$



# Neutron-Proton Scattering – the Deuteron

$$m \approx 940 \text{ MeV}$$

Examine bound-state spectrum

- Focus on spin-triplet results

$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$

*My simple fit*

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

# Neutron-Proton Scattering – the Deuteron

$m \approx 940 \text{ MeV}$

Examine bound-state spectrum

- Focus on spin-triplet results

$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$

*My simple fit*

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

*Search for poles of amplitude*

$$\mathcal{M}^{-1} = 0 \quad \implies \quad -\frac{1}{a} + \frac{1}{2} r {q^*}^2 - i q^* = 0$$

# Neutron-Proton Scattering – the Deuteron

$m \approx 940 \text{ MeV}$

Examine bound-state spectrum

- Focus on spin-triplet results

$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$

*My simple fit*

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

*Search for poles of amplitude*

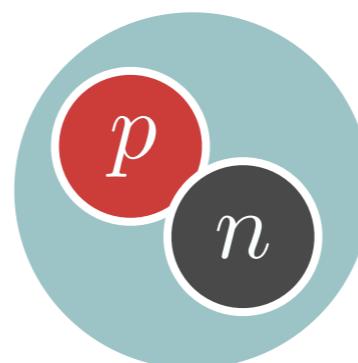
$$\mathcal{M}^{-1} = 0 \quad \implies \quad -\frac{1}{a} + \frac{1}{2} r q^{\star 2} - i q^{\star} = 0$$

*Find a shallow bound state (with my numbers) – the deuteron*

$$M_d \approx 1878 \text{ MeV}$$

$$\Gamma_d = 0$$

$$\text{B.E.} = 2.1 \text{ MeV}$$



# Neutron-Proton Scattering – the Deuteron

$m \approx 940 \text{ MeV}$

Examine bound-state spectrum

- Focus on spin-triplet results

$$1 \text{ fm} \approx \frac{1}{200 \text{ MeV}}$$

*My simple fit*

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

*Search for poles of amplitude*

$$\mathcal{M}^{-1} = 0 \quad \implies \quad -\frac{1}{a} + \frac{1}{2} r {q^*}^2 - i q^* = 0$$

*Find a shallow bound state (with my numbers) – the deuteron*

$$M_d \approx 1878 \text{ MeV}$$

$$\Gamma_d = 0$$

$$\text{B.E.} \approx 2.1 \text{ MeV}$$

*Compare to best analyses*

$$M_d \approx 1875.6 \text{ MeV}$$

$$\text{B.E.} \approx 2.224 \text{ MeV}$$

# Neutron-Proton Scattering – the Deuteron

$m \approx 940 \text{ MeV}$

Examine bound-state spectrum

- Focus on spin-triplet results

*My simple fit*

$$a_t \approx 5.41 \text{ fm}$$

$$r_t \approx 1.62 \text{ fm}$$

*Search for poles of amplitude*

$$\mathcal{M}^{-1} = 0 \quad \implies \quad -\frac{1}{a} + \frac{1}{2} r q^{\star 2} - i q^{\star} = 0$$

*Find a shallow bound state (with my numbers) – the deuteron*

$$M_d \approx 1878 \text{ MeV}$$

$$\Gamma_d = 0$$

$$\text{B.E.} \approx 2.1 \text{ MeV}$$

## Exercise

*Given my fit results, compute the deuteron mass. Repeat the exercise for Hackenberg's results*

*Compare to best analyses*

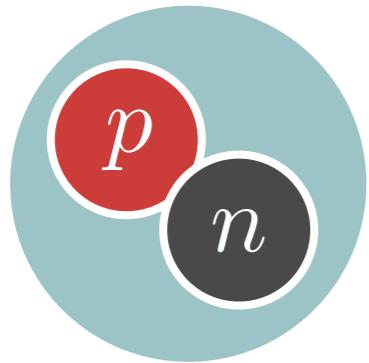
$$M_d \approx 1875.6 \text{ MeV}$$

$$\text{B.E.} \approx 2.224 \text{ MeV}$$

# Neutron-Proton Scattering – the Deuteron

Bound state spectrum determined from scattering analysis

*The deuteron is a shallow bound state  
of triplet S-wave np scattering*



$$M_d \approx 1875.6 \text{ MeV}$$

$$\text{B.E.} \approx 2.224 \text{ MeV}$$

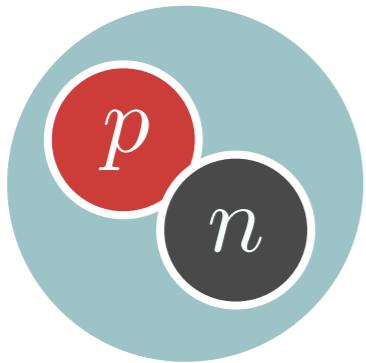
$$\Gamma_d = 0$$

$$J^P = 1^+$$

# Neutron-Proton Scattering – the Deuteron

Bound state spectrum determined from scattering analysis

*The deuteron is a shallow bound state  
of triplet S-wave np scattering*



$$M_d \approx 1875.6 \text{ MeV}$$

$$\text{B.E.} \approx 2.224 \text{ MeV}$$

$$\Gamma_d = 0$$

$$J^P = 1^+$$

## Spectral properties checklist

- Mass
- Lifetime
- Spin
- Parity
- Charge

...

# Outline

---

## Hadrons, Quarks, & QCD

Hadron Spectroscopy

The Quark Model

Quantum Chromodynamics

## Scattering Theory & the Hadron Spectrum

Scattering Amplitudes

Bound & Resonant States

## Lattice QCD & Hadron Spectroscopy

Lattice QCD

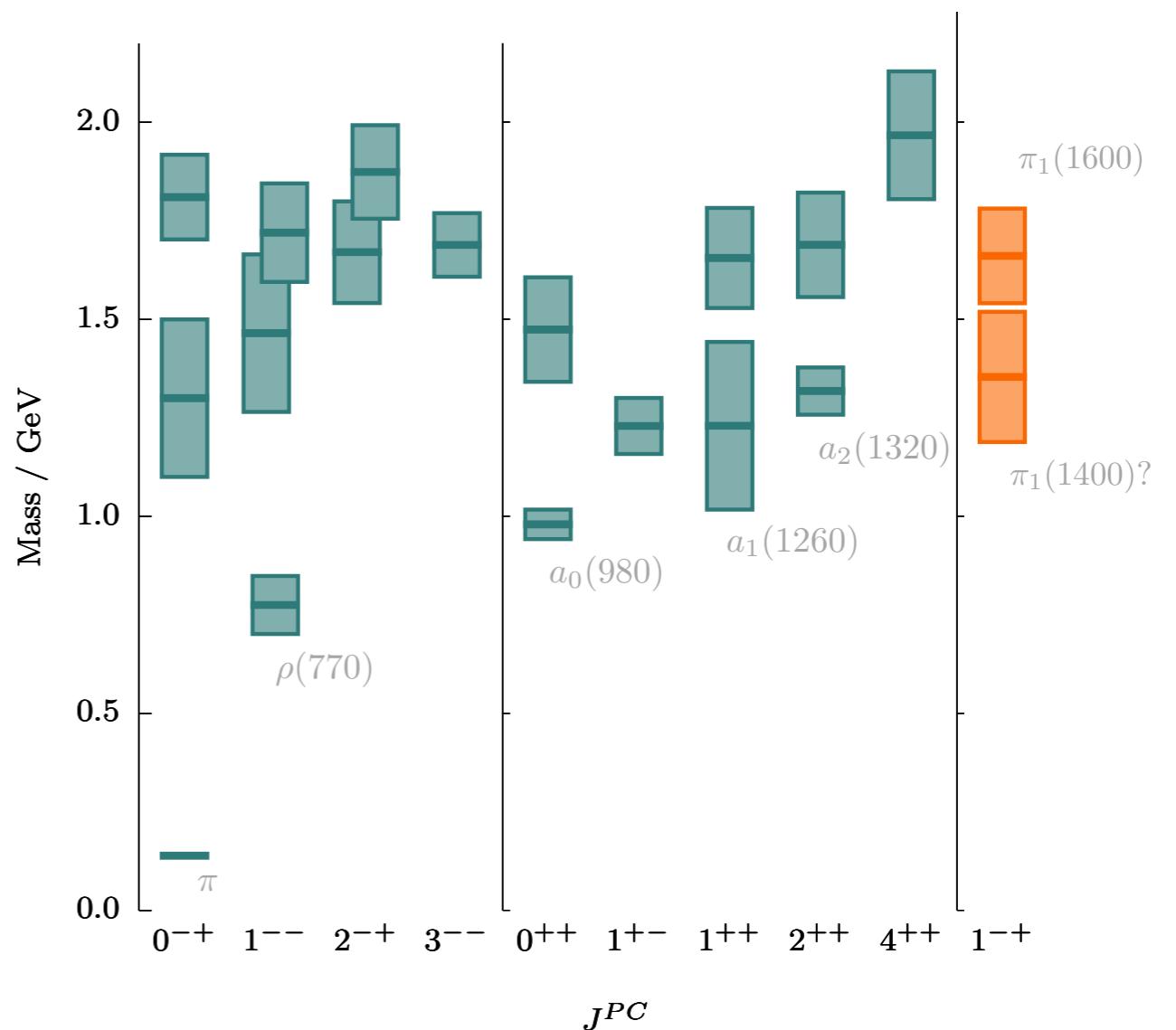
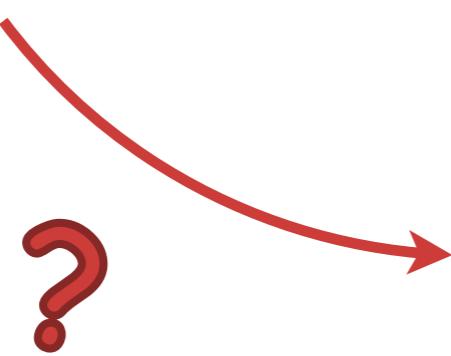
Lüscher & the Finite-Volume

# Hadrons and QCD

QCD allows for more exotic states – Do we understand QCD?

- How to connect QCD to hadrons?
- Need to understand how to quantify what the hadrons are in nature
- Need to find non-perturbative approach to access these hadrons *rigorously*

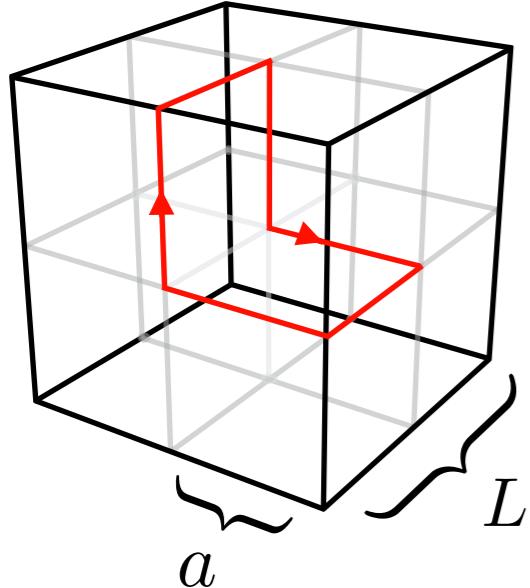
$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i \not{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$



# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling



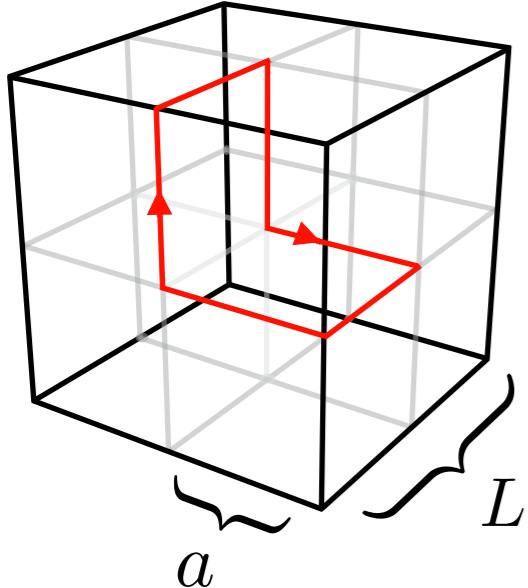
$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_\mu)}$$

- *Euclidean spacetime,  $t \rightarrow -i\tau$*
- *Finite volume,  $L$*
- *Discrete spacetime,  $a$*
- *Heavier than physical quark mass,  $m > m_{\text{phys.}}$*

# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling

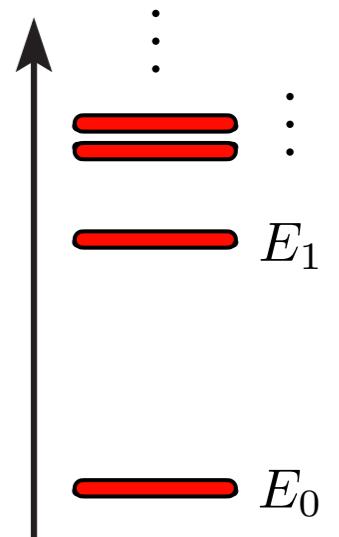


$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_\mu)}$$

- Euclidean spacetime,  $t \rightarrow -i\tau$
- Finite volume,  $L$
- Discrete spacetime,  $a$
- Heavier than physical quark mass,  $m$

Correlation functions yield discrete spectrum

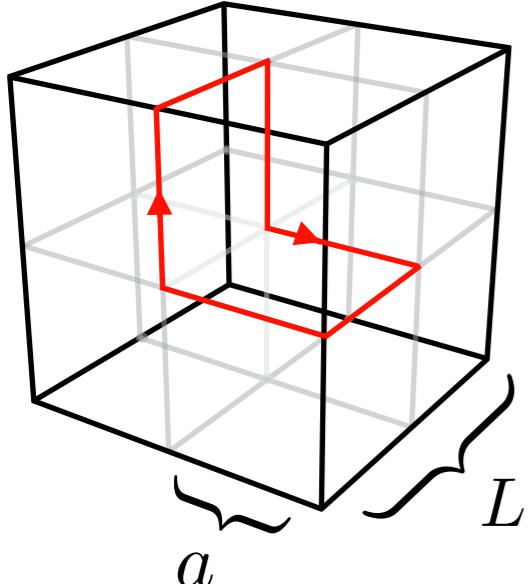
$$\langle \mathcal{O}(\tau) \mathcal{O}^\dagger(0) \rangle = \sum_{\mathfrak{n}} |\langle 0 | \mathcal{O} | \mathfrak{n} \rangle|^2 e^{-E_{\mathfrak{n}} \tau}$$



# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling



$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_\mu)}$$

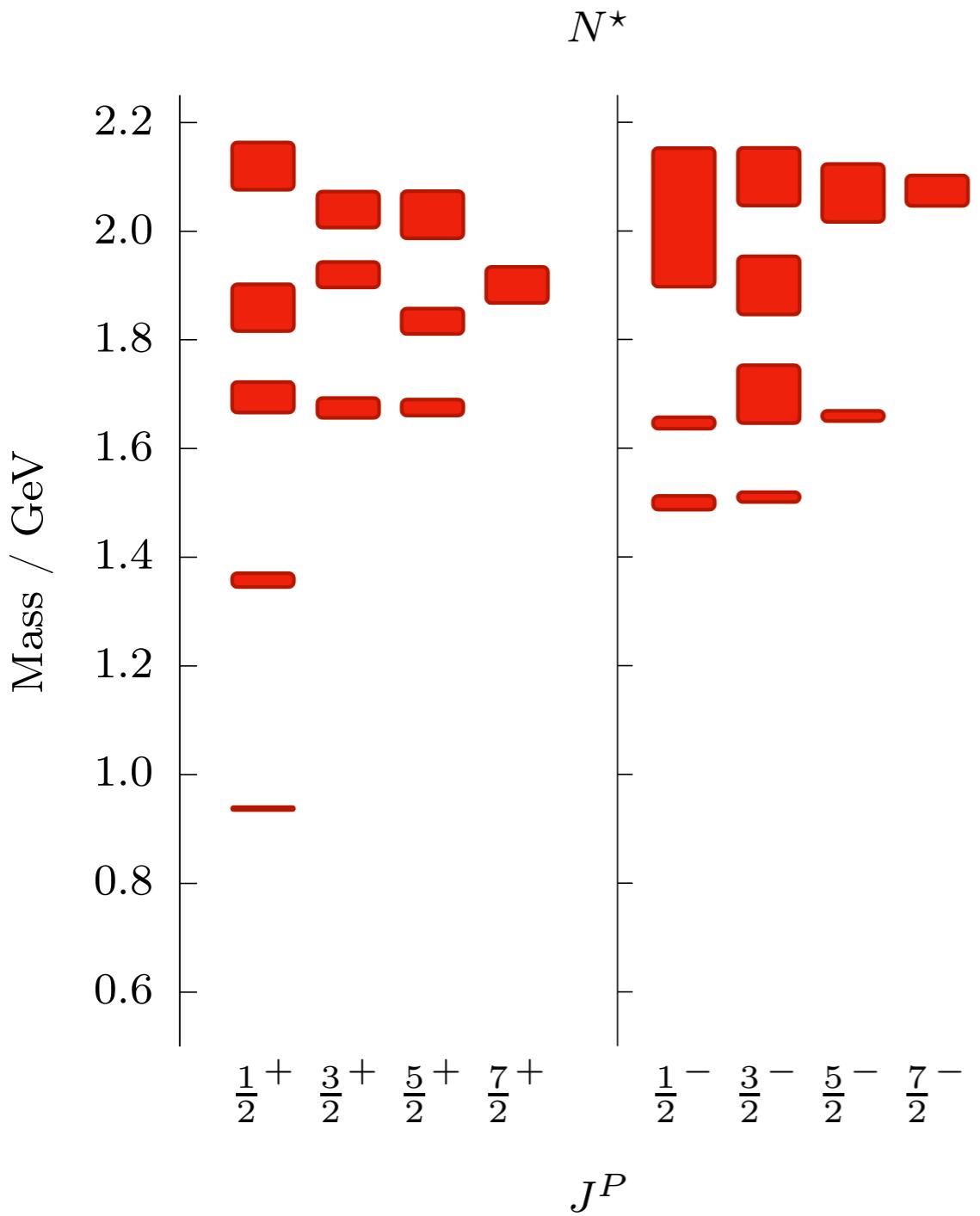
- Euclidean spacetime,  $t$
- Finite volume,  $L$
- Discrete spacetime,  $a$
- Heavier than physical quarks



# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

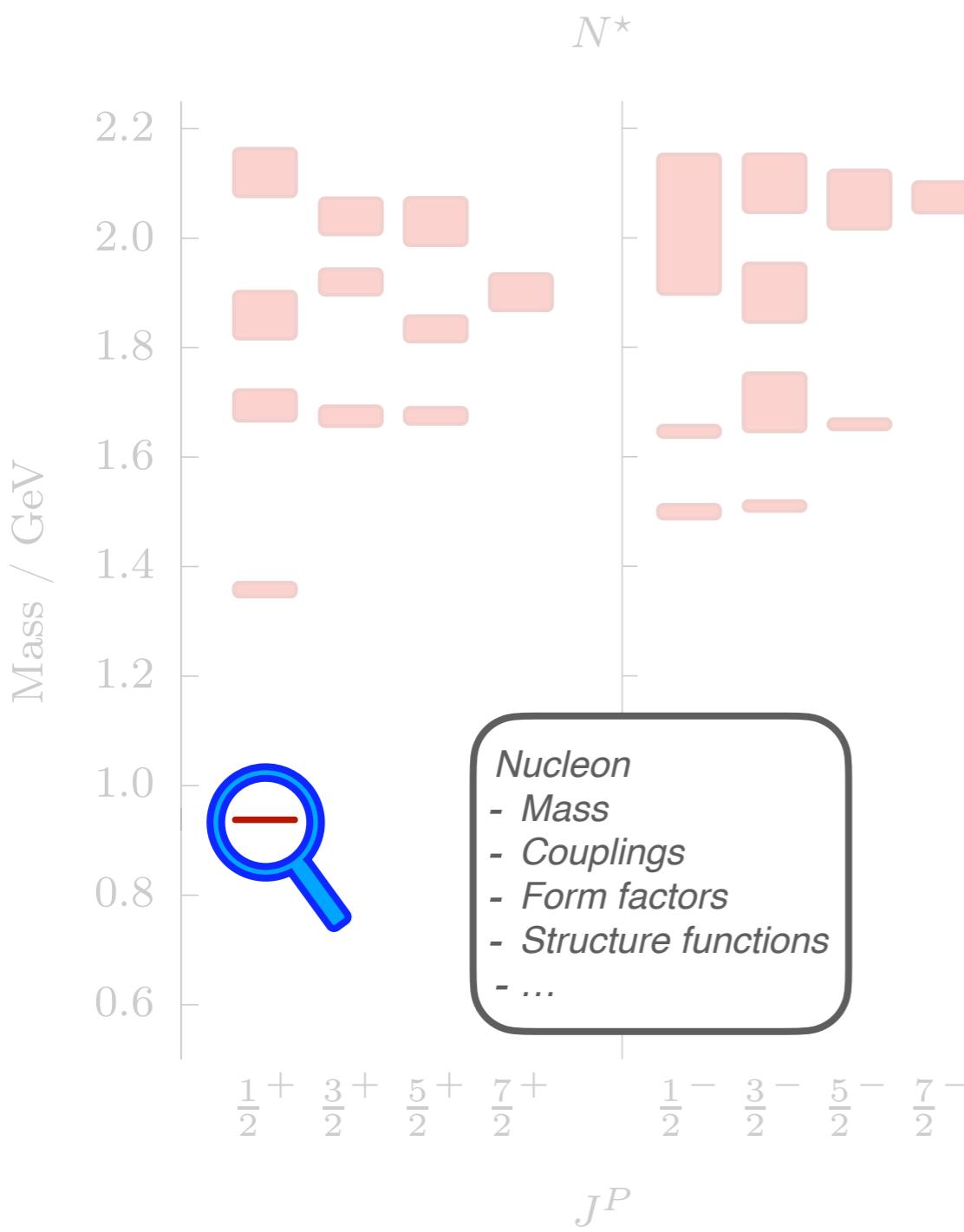
- Numerically evaluate QCD path integral via Monte Carlo sampling



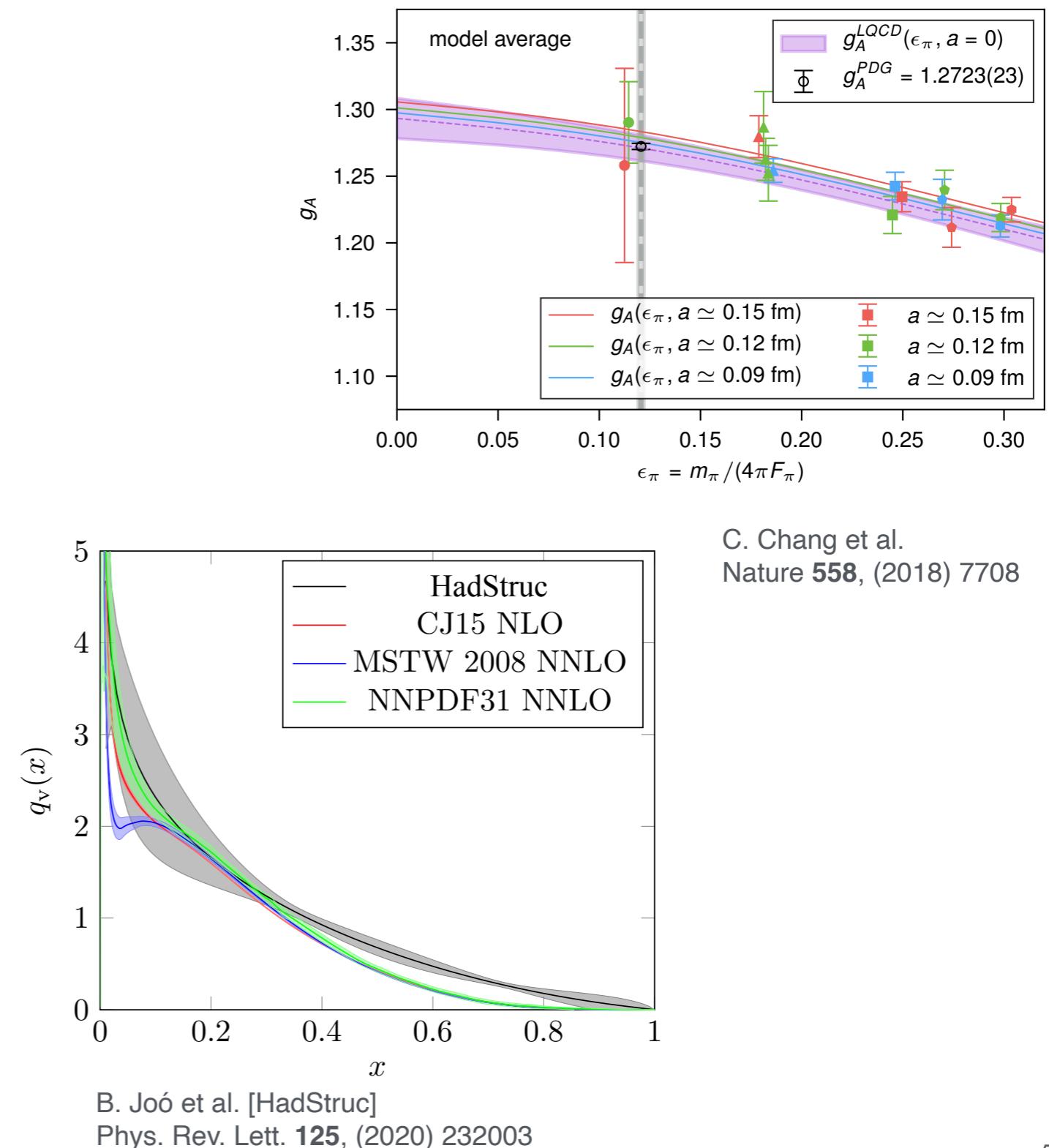
# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling



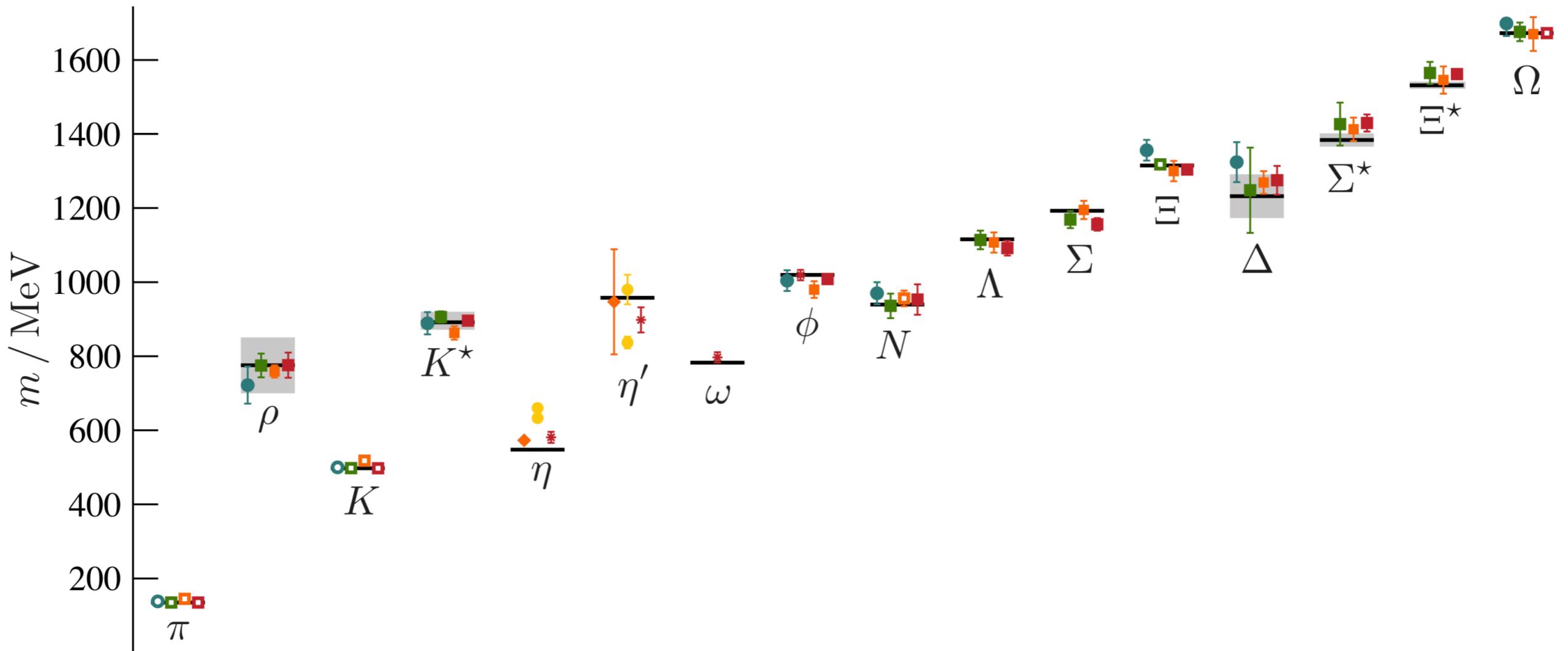
PDG listings



# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling

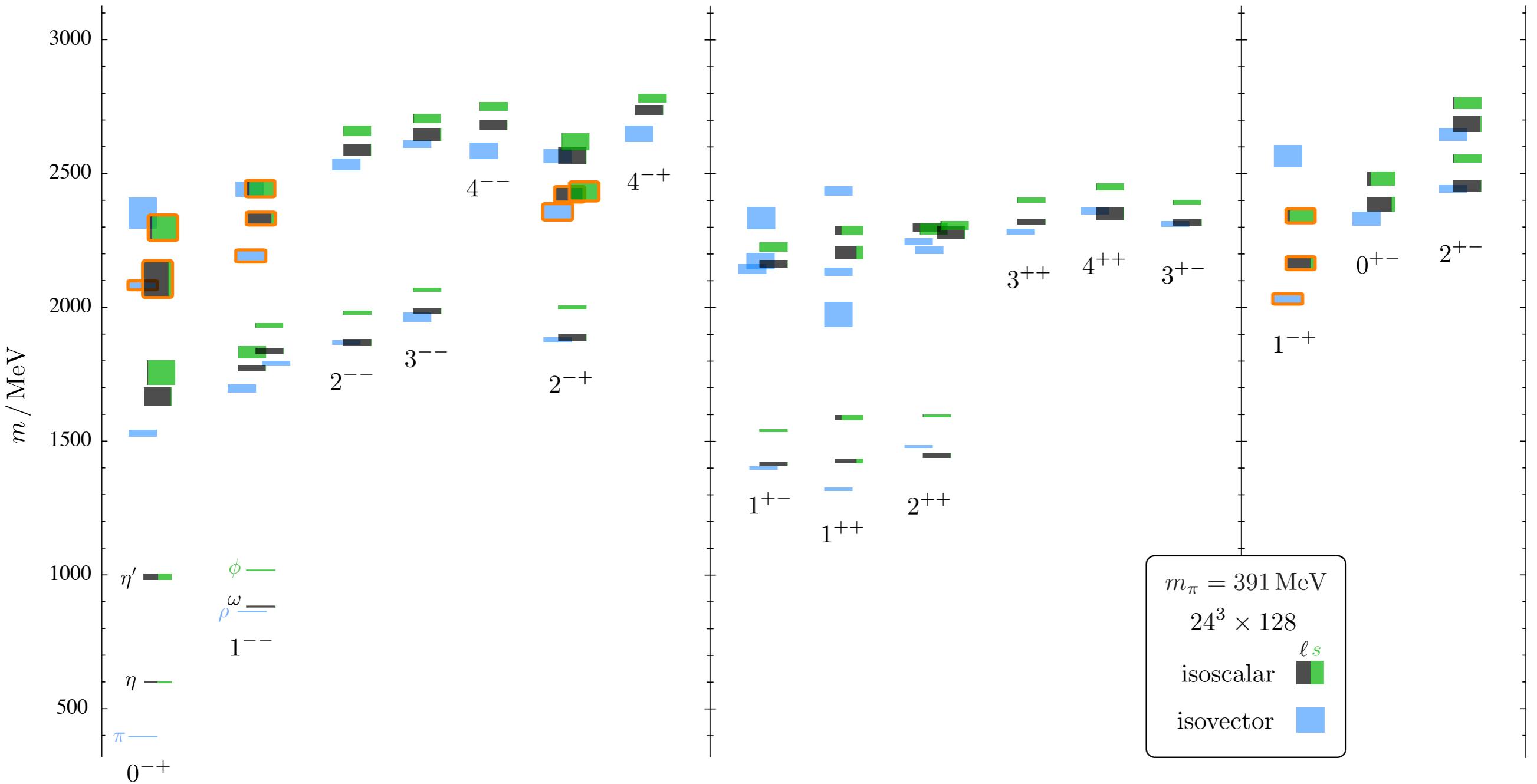


summary compiled by Andreas Kronfeld  
Ann. Rev. Nucl. Part. Sci 62 265 (2012)

# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

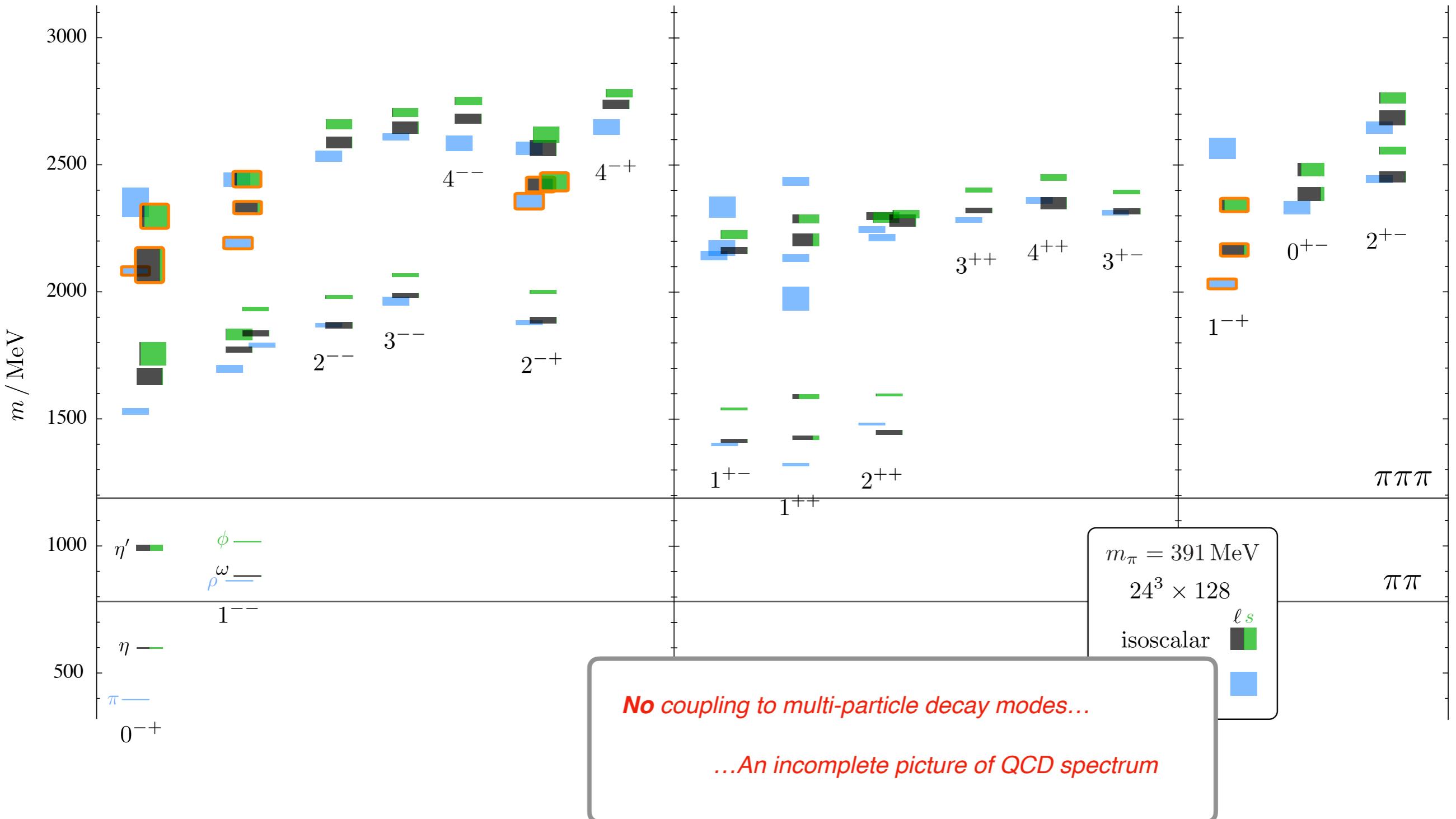
- Numerically evaluate QCD path integral via Monte Carlo sampling



# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

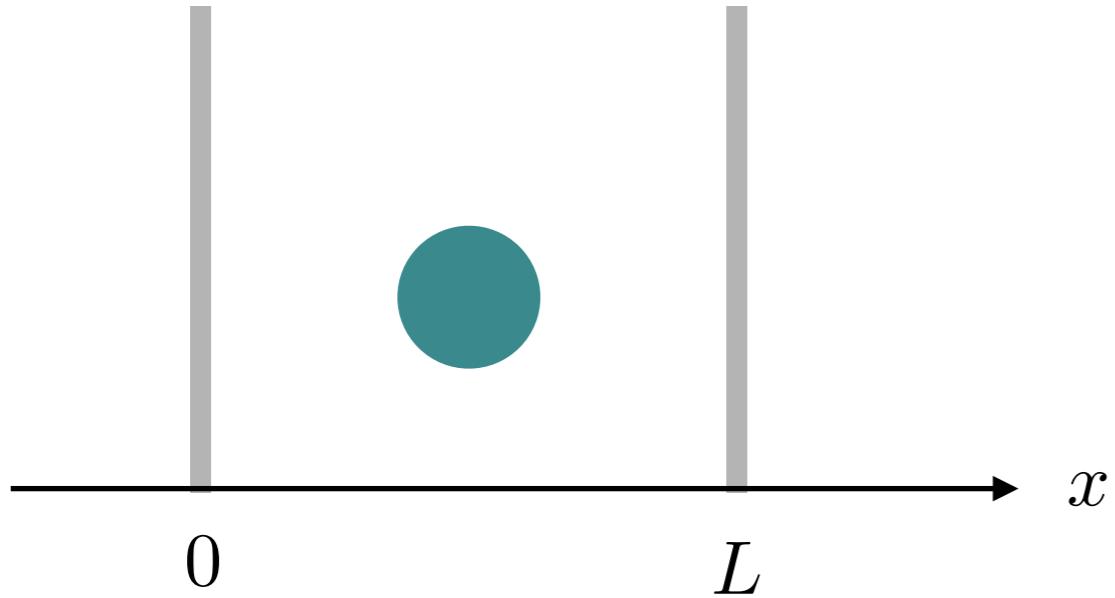
- Numerically evaluate QCD path integral via Monte Carlo sampling



# Particles in a box

Consider a particle, confined in a 1d box

*Schrödinger equation*

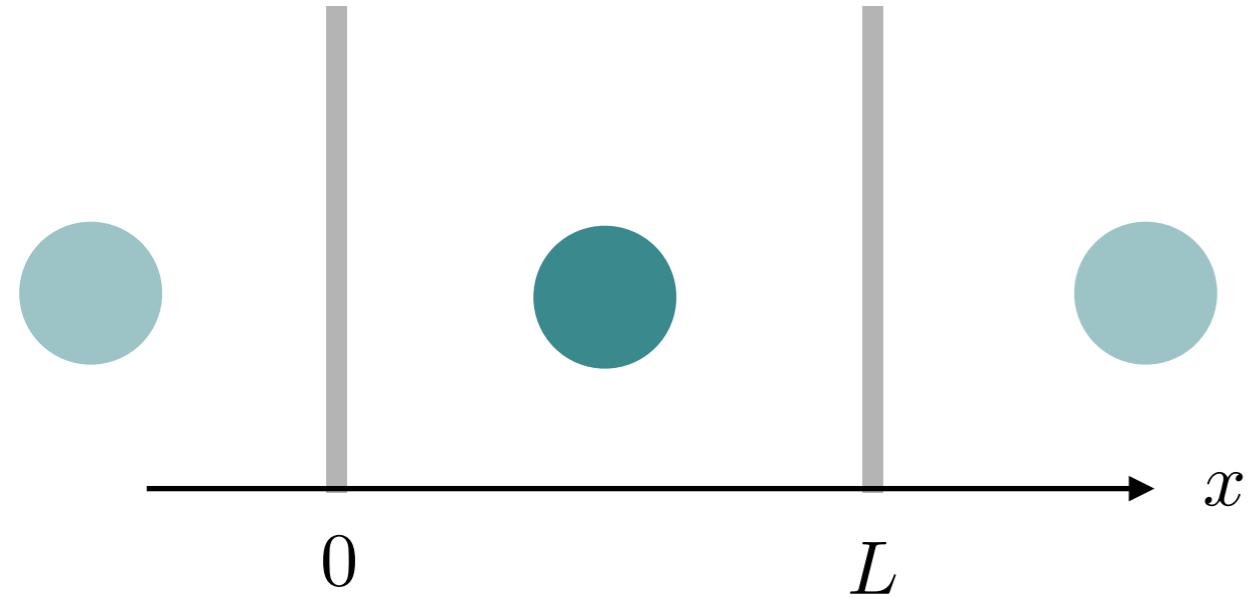


$$-\frac{1}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

# Particles in a box

Consider a particle, confined in a 1d box

*Schrödinger equation*



$$-\frac{1}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

*Periodic boundary conditions*

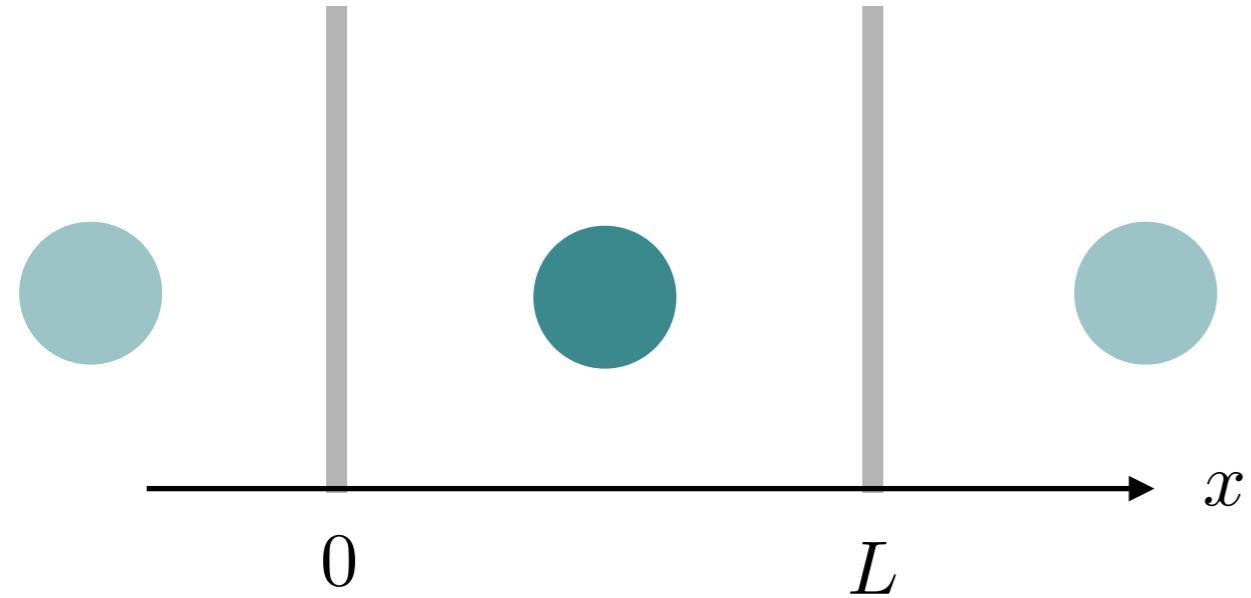
$$\psi(0) = \psi(L)$$

$$\frac{d\psi(0)}{dx} = \frac{d\psi(L)}{dx}$$

# Particles in a box

Consider a particle, confined in a 1d box

*Schrödinger equation*



$$-\frac{1}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

*Periodic boundary conditions*

$$\psi(0) = \psi(L)$$

$$\frac{d\psi(0)}{dx} = \frac{d\psi(L)}{dx}$$

*Periodicity condition*

$$\psi(x + L) = \psi(x)$$

*Wave functions*

$$\psi_n(x) \sim e^{ip_n x}$$

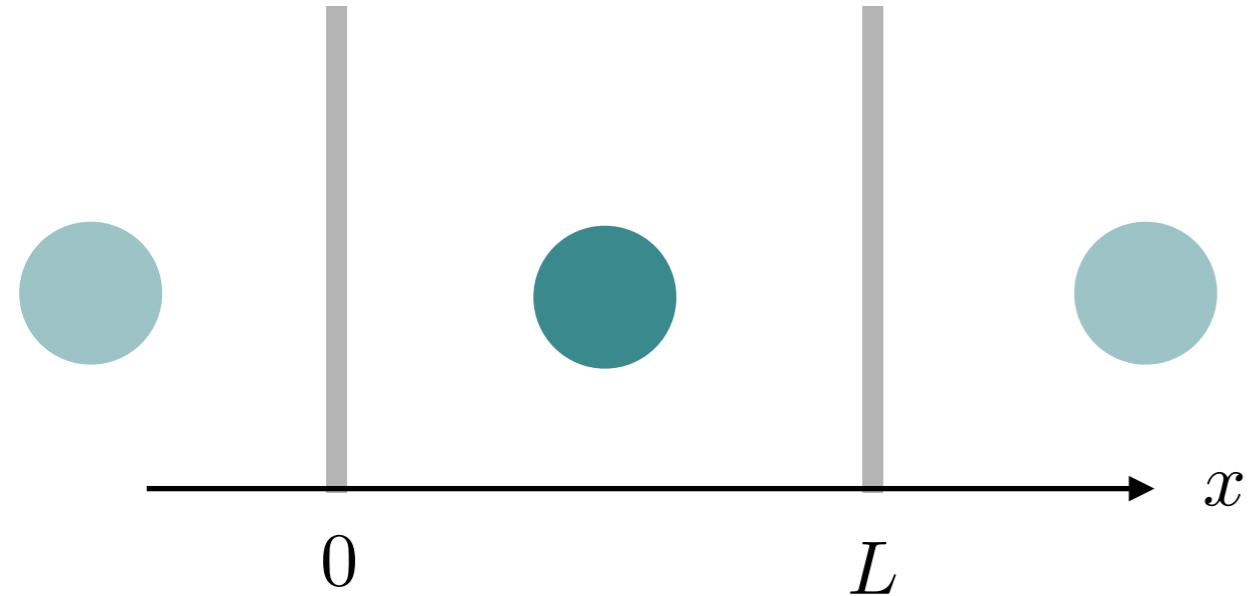
*Momentum quantization*

$$p_n = \frac{2\pi}{L} n , \quad n = 0, \pm 1, \pm 2, \dots \implies E_n = \frac{p_n^2}{2m} = \frac{2\pi^2}{mL^2} n^2$$

# Particles in a box

Consider a particle, confined in a 1d box

*Schrödinger equation*



$$-\frac{1}{2m} \frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

*Periodic boundary conditions*

$$\psi(0) = \psi(L)$$

$$\frac{d\psi(0)}{dx} = \frac{d\psi(L)}{dx}$$

*Periodicity condition*

$$\psi(x + L) = \psi(x)$$

*Wave functions*

$$\psi_n(x) \sim e^{ip_n x}$$

*Momentum quantization*

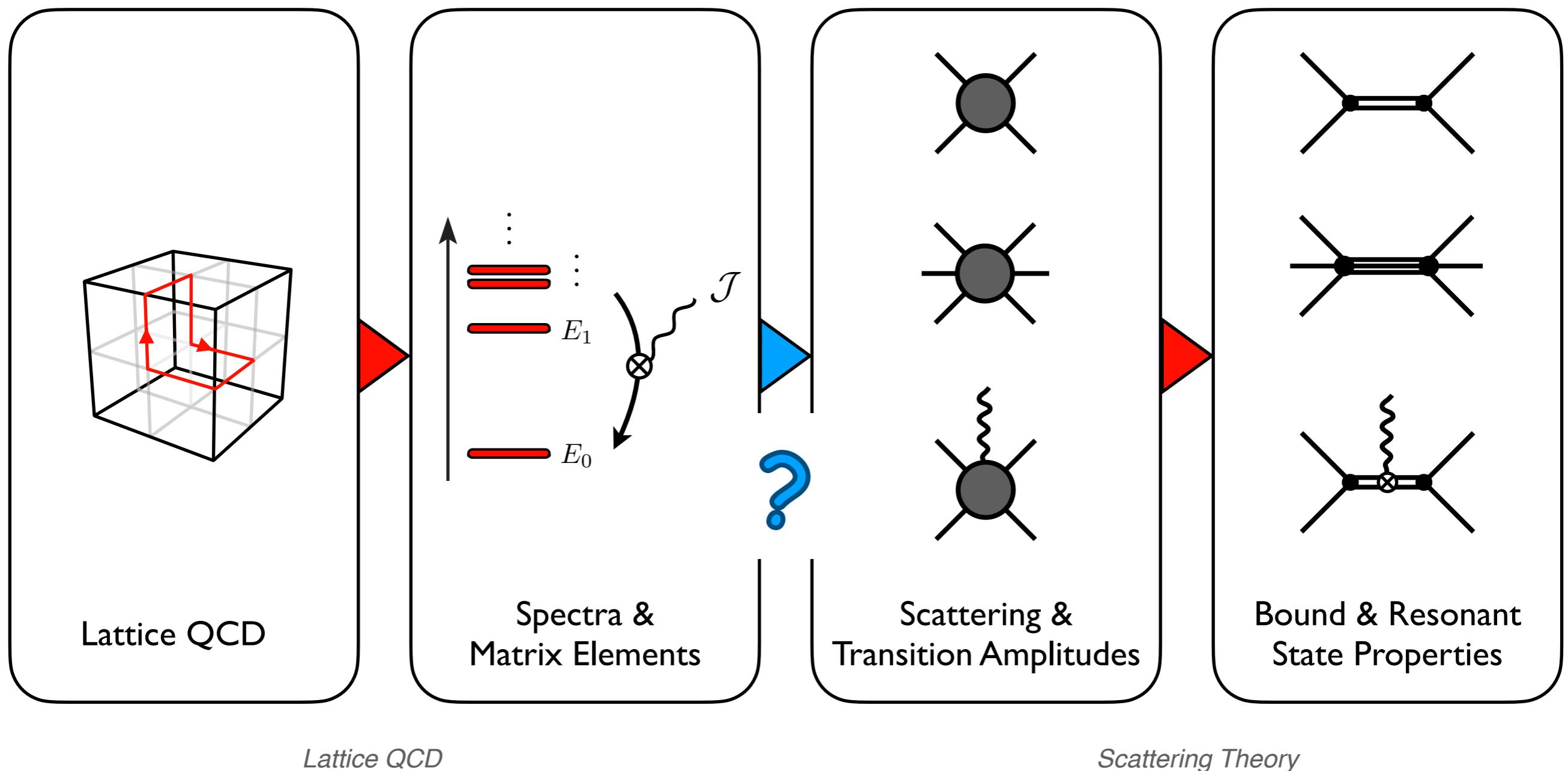
$$p_n = \frac{2\pi}{L} n , \quad n = 0, \pm 1, \pm 2, \dots$$

$$\Rightarrow E_n = \frac{p_n^2}{2m} = \frac{2\pi^2}{mL^2} n^2$$

# Connecting Scattering Physics to QCD

Path to few-body physics from QCD

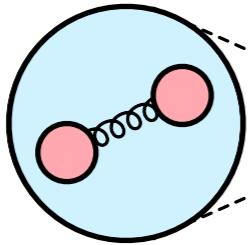
- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD, Scattering Theory, & Effective Field Theory*



# Connecting Scattering Physics to QCD

Use generic Effective Field Theory to generate connection

*Quarks and gluons  
at low energy*

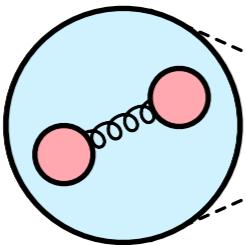


Hadron d.o.f., e.g.  $\pi, K, \dots$

# Connecting Scattering Physics to QCD

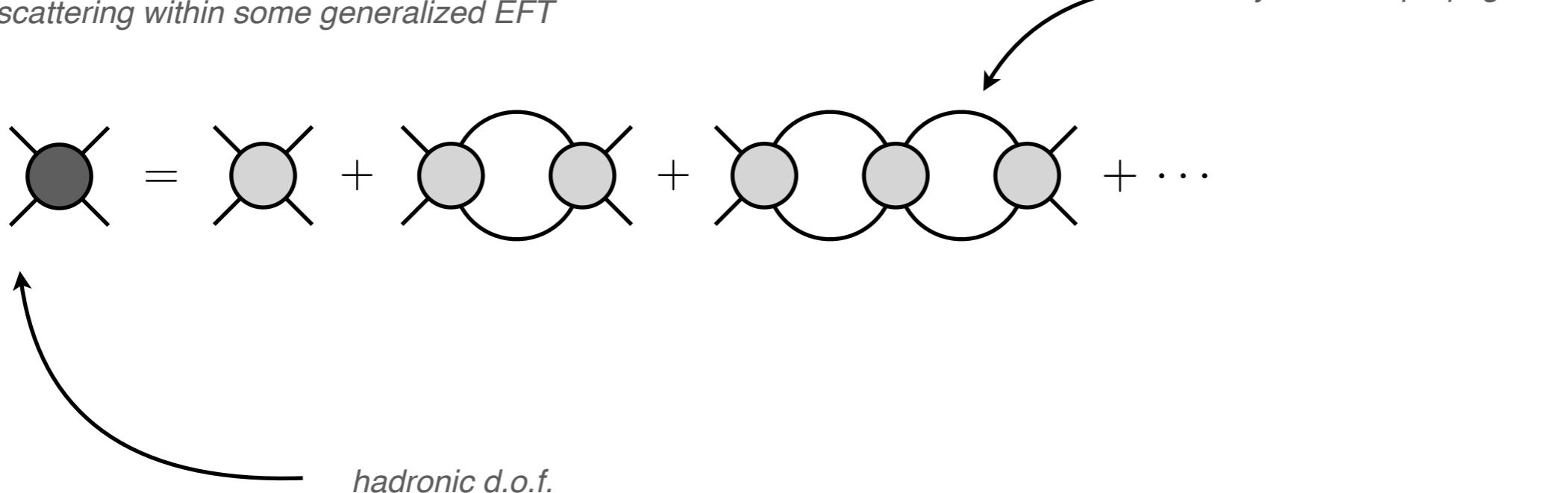
Use generic Effective Field Theory to generate connection

*Quarks and gluons  
at low energy*



Hadron d.o.f., e.g.  $\pi, K, \dots$

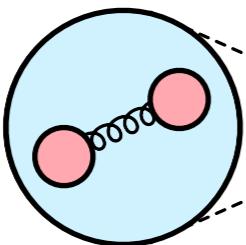
*e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT*



# Connecting Scattering Physics to QCD

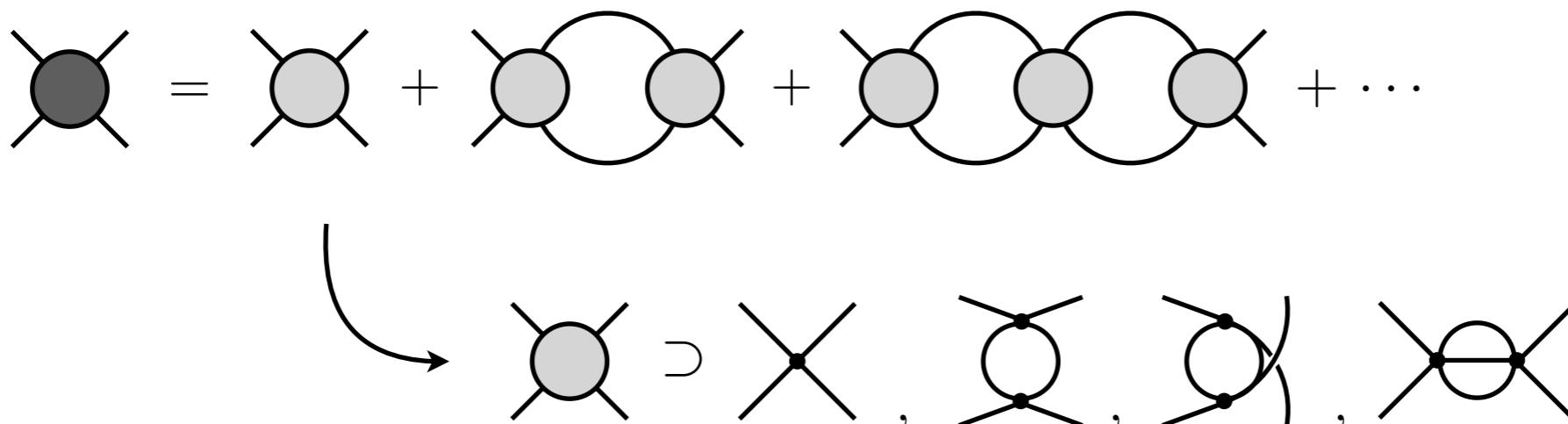
Use generic Effective Field Theory to generate connection

*Quarks and gluons  
at low energy*



Hadron d.o.f., e.g.  $\pi, K, \dots$

*e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT*



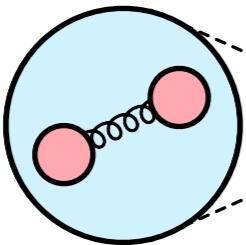
*Bethe-Salpeter kernels*

*All 2PI diagrams - left hand cuts & higher multi-particle thresholds*

# Connecting Scattering Physics to QCD

Use generic Effective Field Theory to generate connection

*Quarks and gluons  
at low energy*



Hadron d.o.f., e.g.  $\pi, K, \dots$

*e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT*

$$\text{[Two-point function diagram]} = \text{[One-loop diagram]} + \text{[Two-loop diagram]} + \text{[Three-loop diagram]} + \dots$$

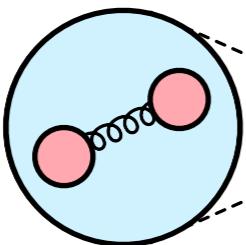
$$\text{[Two-loop diagram]} = \text{[One-loop diagram with PV]} + \text{[Two-loop diagram with red vertical line]}$$

$$\rho = \frac{q}{8\pi E} \sim \sqrt{s - s_{\text{th}}}$$

# Connecting Scattering Physics to QCD

Use generic Effective Field Theory to generate connection

*Quarks and gluons  
at low energy*

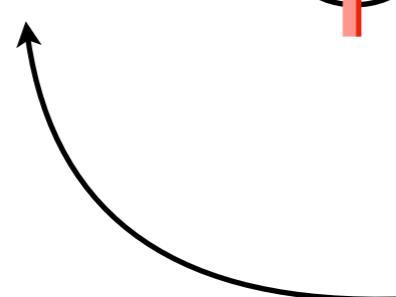


Hadron d.o.f., e.g.  $\pi, K, \dots$

*e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT*

$$\text{Diagram of a vertex} = \text{Diagram of a vertex} + \text{Diagram of a vertex with one loop} + \text{Diagram of a vertex with two loops} + \dots$$

$$= \text{Diagram of a vertex} + \text{Diagram of a vertex with one loop} + \text{Diagram of a vertex with two loops} + \dots$$

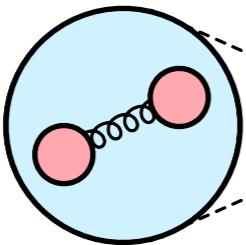


*K-matrix – All short-distance physics which cannot go on-shell  
– Unknown! – theory specific*

# Connecting Scattering Physics to QCD

Use generic Effective Field Theory to generate connection

*Quarks and gluons  
at low energy*



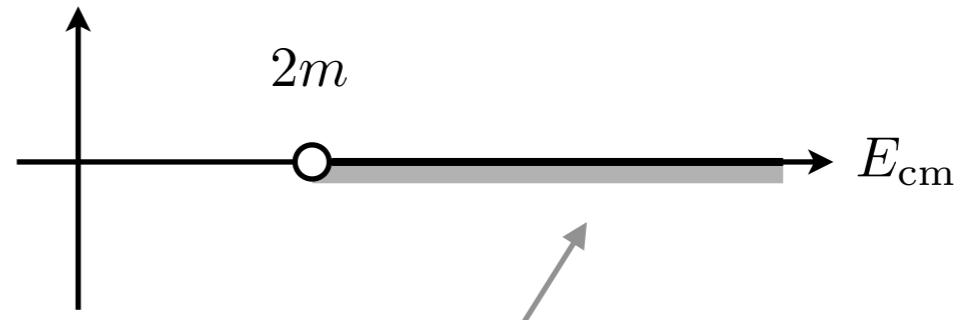
Hadron d.o.f., e.g.  $\pi, K, \dots$

*e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT*

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$
$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

*On-shell representation of scattering amplitude*

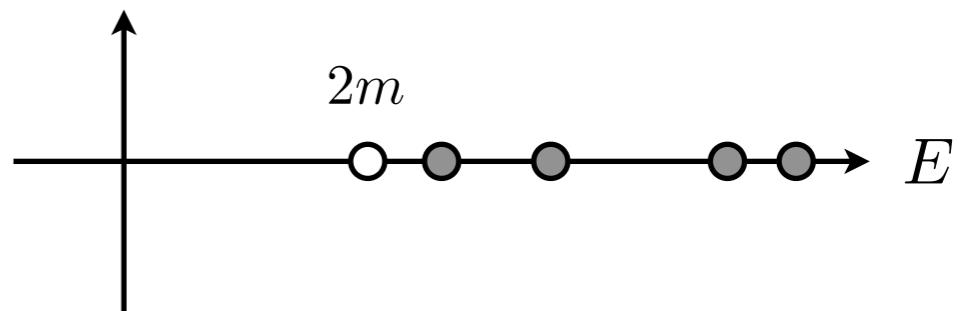
$$\mathcal{M} = \mathcal{K} \frac{1}{1 - i\rho\mathcal{K}}$$



# Connecting Scattering Physics to QCD

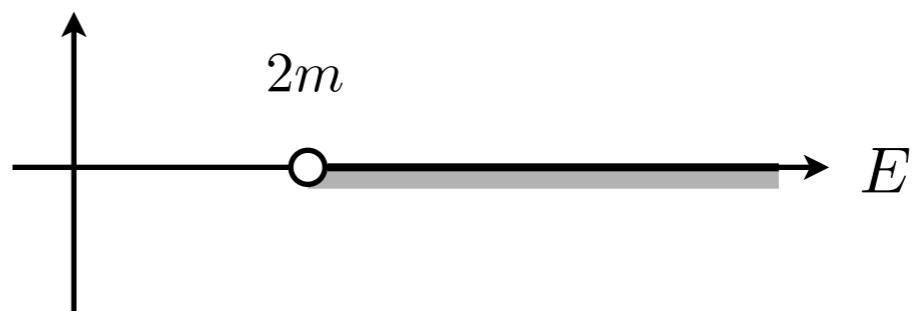
Q: How do we connect a finite-volume spectrum computed from QCD...

$$\int_L d^4x e^{iP \cdot x} \langle \mathcal{O}(t) \mathcal{O}^\dagger(0) \rangle = \sum_{\mathfrak{n}} \frac{i |\langle 0 | \mathcal{O} | \mathfrak{n} \rangle|^2}{E - E_{\mathfrak{n}}}$$



...to infinite-volume scattering amplitudes?

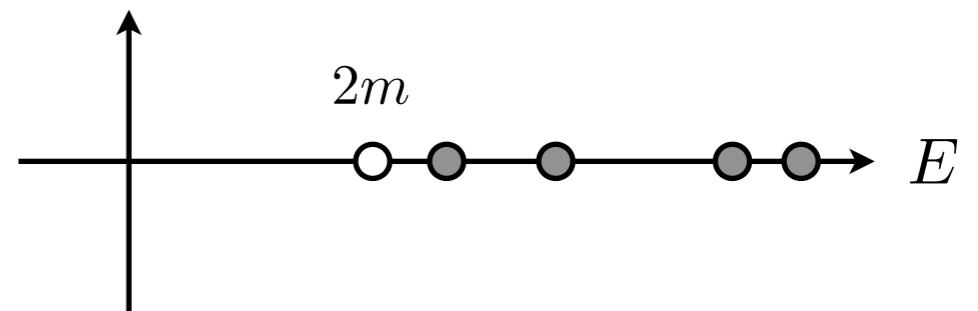
$$\mathcal{M} = \mathcal{K} \frac{1}{1 - i\rho\mathcal{K}}$$



# Connecting Scattering Physics to QCD

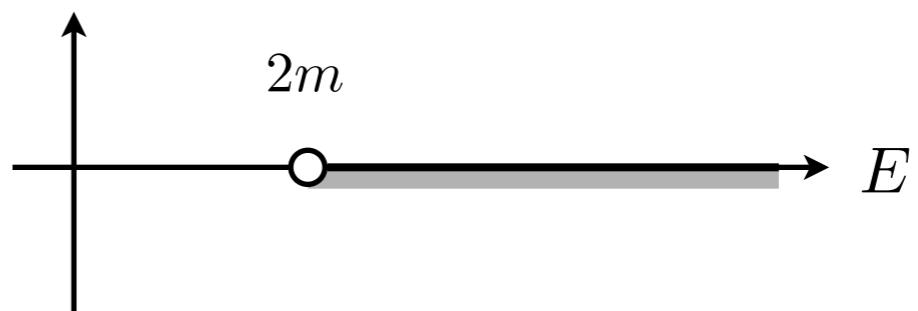
**Q:** How do we connect a finite-volume spectrum computed from QCD...

$$\int_L d^4x e^{iP \cdot x} \langle \mathcal{O}(t) \mathcal{O}^\dagger(0) \rangle = \sum_n \frac{i |\langle 0 | \mathcal{O} | n \rangle|^2}{E - E_n}$$



...to infinite-volume scattering amplitudes?

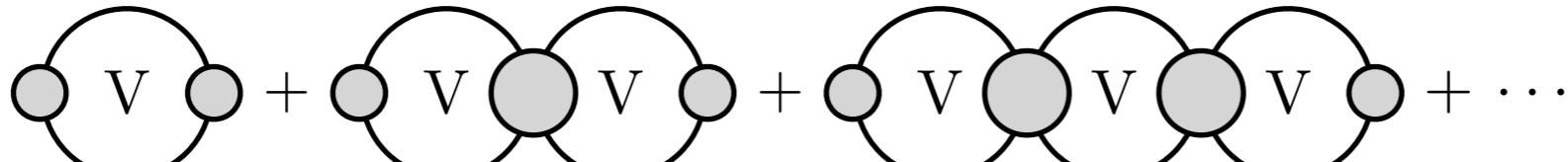
$$\mathcal{M} = \mathcal{K} \frac{1}{1 - i\rho\mathcal{K}}$$



**A:** Correct analytic structure of finite-volume correlators

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$C_L(P) = \text{Diagram } 1 + \text{Diagram } 2 + \text{Diagram } 3 + \dots$$


# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$C_L(P) = \text{V} + \text{V} \text{V} \text{V} + \text{V} \text{V} \text{V} \text{V} + \dots$$

$$\text{V} = \infty + \left[ \text{V} - \infty \right]$$

$$= \infty + \text{V}$$



$F_L$

*Geometric function – characterizes finite-volume distortions*

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$C_L(P) = \text{Diagram } 1 + \text{Diagram } 2 + \text{Diagram } 3 + \dots$$
$$= C(P) + \text{Diagram } 1' + \text{Diagram } 2' + \text{Diagram } 3' + \dots$$

The diagrams represent the two-point correlator  $C_L(P)$  and its QCD counterpart  $C(P)$ . They are shown as horizontal chains of circles. The first diagram in each row is a single circle labeled 'V'. Subsequent diagrams show multiple circles connected by horizontal lines, with the number of circles increasing in each term. In the top row, the circles are light gray, while in the bottom row, they are dark gray. Red vertical lines connect the centers of the circles in the bottom row diagrams.

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$\begin{aligned} C_L(P) &= \text{Diagram } 1 + \text{Diagram } 2 + \text{Diagram } 3 + \dots \\ &= C(P) + \text{Diagram } 1' + \text{Diagram } 2' + \text{Diagram } 3' + \dots \\ &= C(P) + i\mathcal{A} \frac{i}{\mathcal{M}^{-1} + F_L} F_L i\mathcal{A} \end{aligned}$$

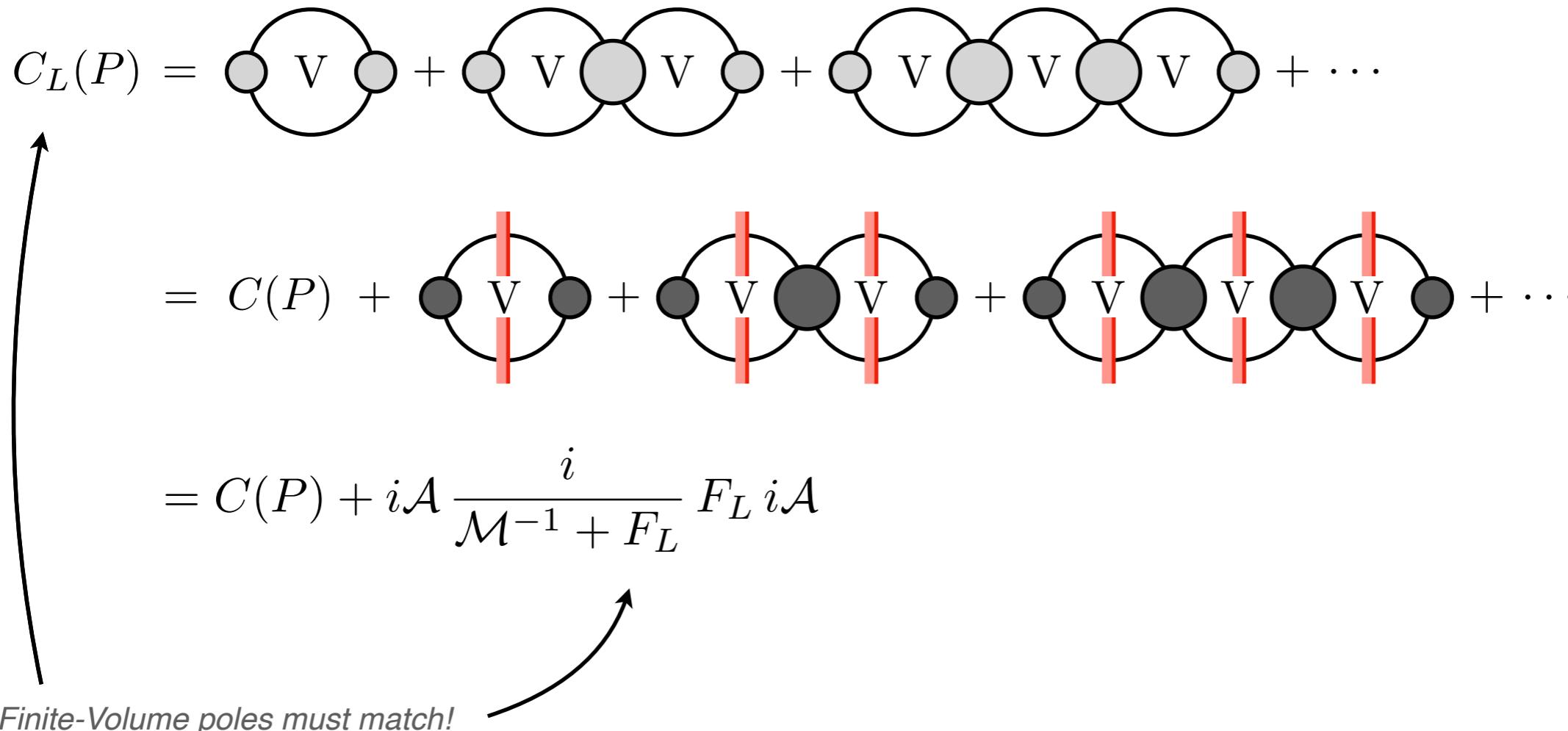
The diagrams consist of a sequence of circles connected by horizontal lines. The first row shows three diagrams where each circle contains a central 'V'. The second row shows three diagrams where each circle contains a central 'V' and has a vertical red line passing through its center. The third row shows the expression for the two-point correlator.

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$\begin{aligned} C_L(P) &= \text{Diagram } 1 + \text{Diagram } 2 + \text{Diagram } 3 + \dots \\ &= C(P) + \text{Diagram } 1' + \text{Diagram } 2' + \text{Diagram } 3' + \dots \\ &= C(P) + i\mathcal{A} \frac{i}{\mathcal{M}^{-1} + F_L} F_L i\mathcal{A} \end{aligned}$$

*Finite-Volume poles must match!*

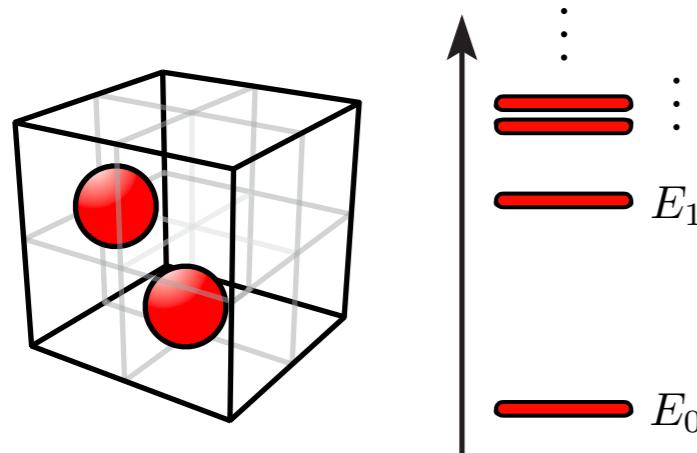


$$\det (\mathcal{M}^{-1} + F_L)_{E=E_n} = 0$$

Lüscher quantization condition

# Few-Body Physics & QCD

Employing scattering theory and EFTs to all-orders connects lattice QCD to scattering observables

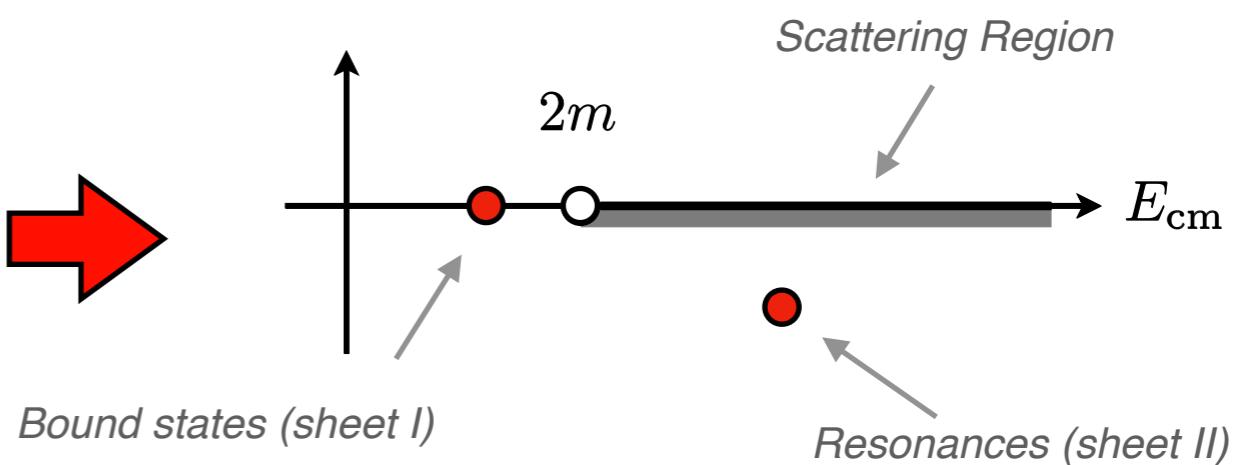


$$\rightarrow \det (\mathcal{M}^{-1} + F_L)_{E=E_n} = 0$$

$$\rightarrow \mathcal{M} = \mathcal{K} \frac{1}{1 - i\rho\mathcal{K}}$$

M. Lüscher  
Commun.Math.Phys. **105**, 153 (1986)  
Nucl.Phys. **B354**, 531 (1991)

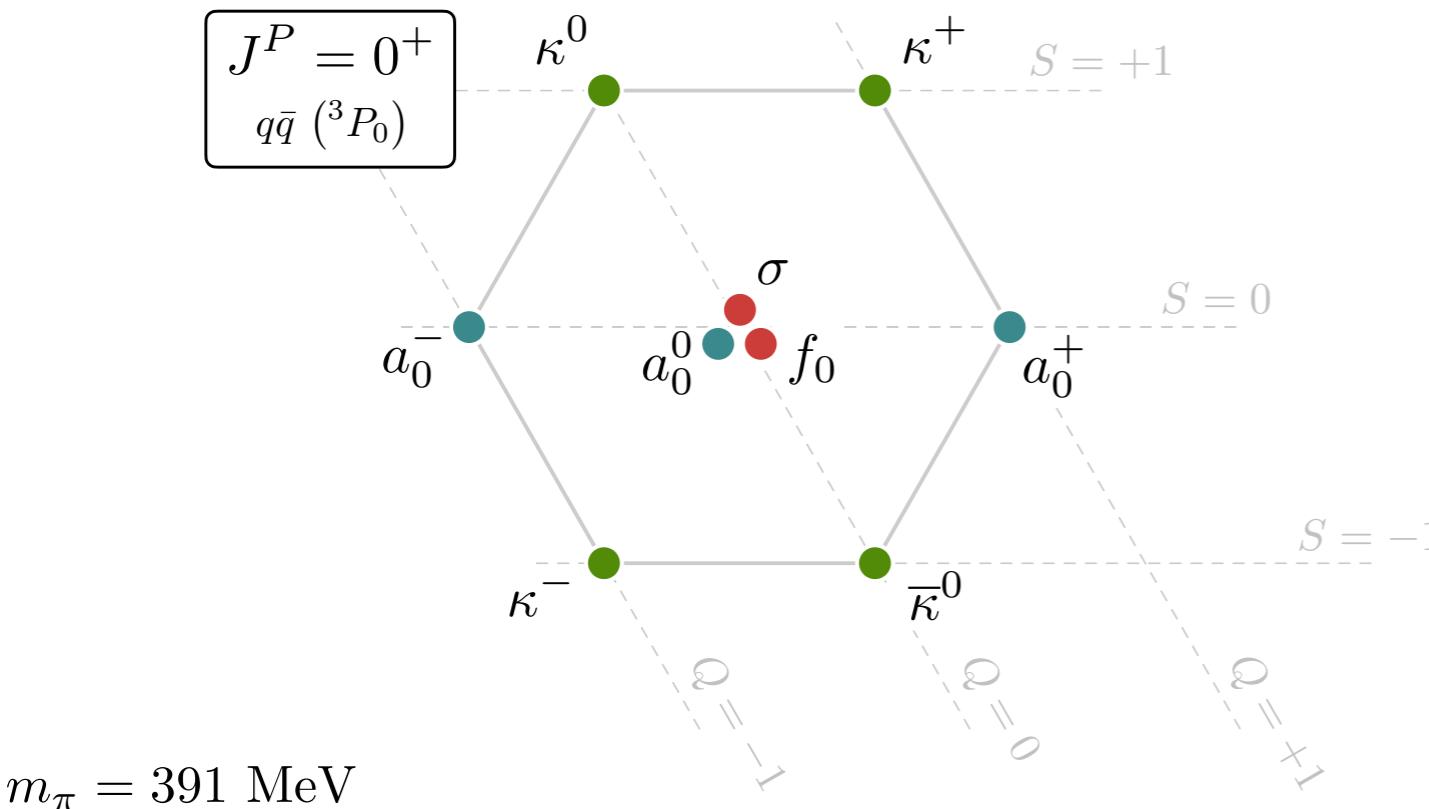
Many others...



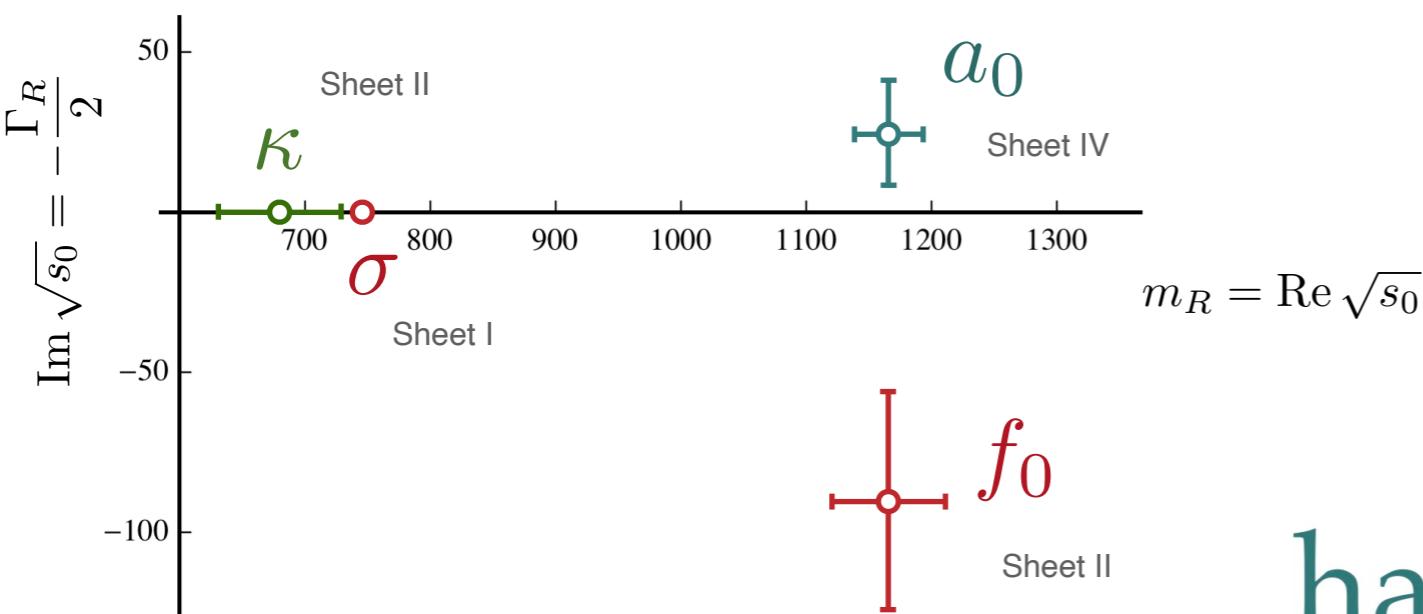
# Connecting Resonances to QCD

Can we directly connect resonances to QCD?

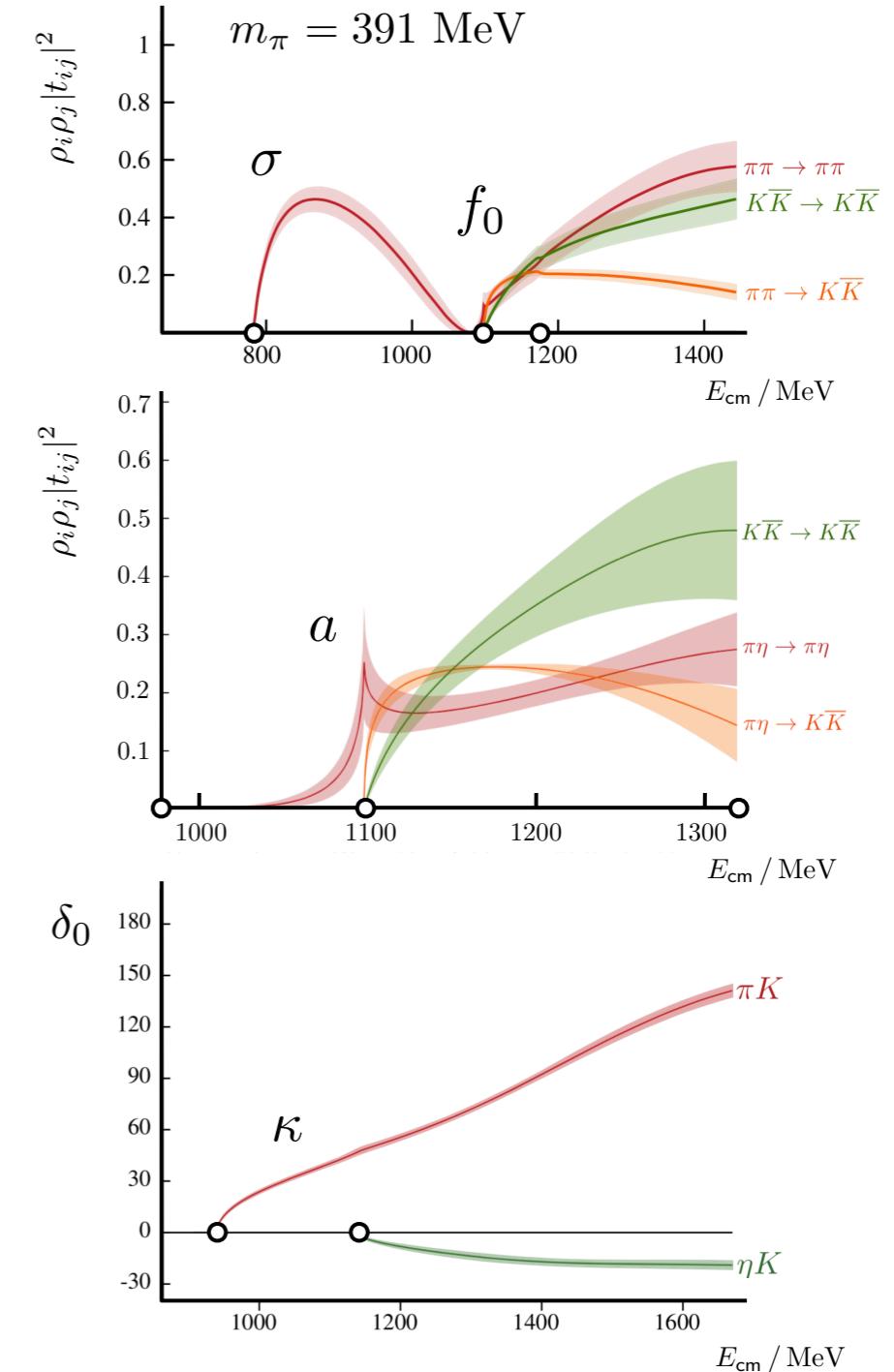
- Yes! Lattice QCD offers pathway



$$m_\pi = 391 \text{ MeV}$$



**had spec**



R.A. Briceño et al. [HadSpec]  
Phys. Rev. **D97**, 054513 (2018)

J.J. Dudek et al. [HadSpec]  
Phys. Rev. **D93**, 094506 (2016)

J.J. Dudek et al. [HadSpec]  
Phys. Rev. Lett. **113**, 182001 (2014)

# Few-Body Physics & QCD

Path to few-body physics from QCD

- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD, Scattering Theory, & Effective Field Theory*

