Technical Report

ProbOnto 2.5
Ontology and Knowledge Base of Probability Distributions

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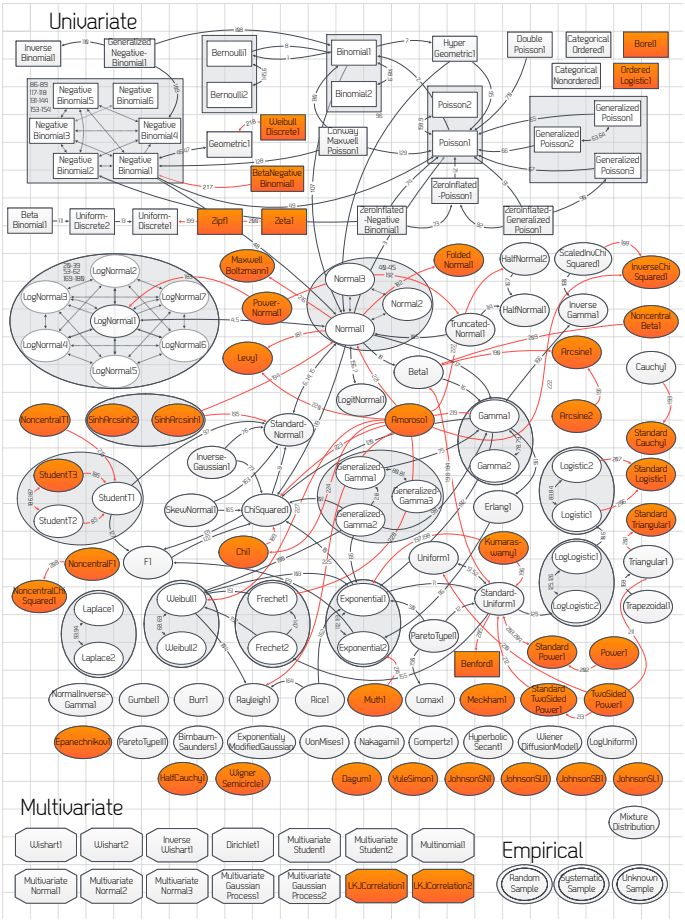

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Chapter 1

Overview

1.1 Summary of bugs/typos

- Bugs in Figure 2.3
  - $StandardUniform1$ is related with $LogLogistic2$ not $LogLogistic1$.
  - $Erlang1$ is related with $Exponential2$ not $Exponential1$.
- The parameter definition in StudentT2 should read '$\mu \in R$' instead of '$\mu in R$'.
- in few relationships the Latex was incorrect, e.g. it was '... rightarrow ...' instead of '... \rightarrow ...'.

1.2 Release history

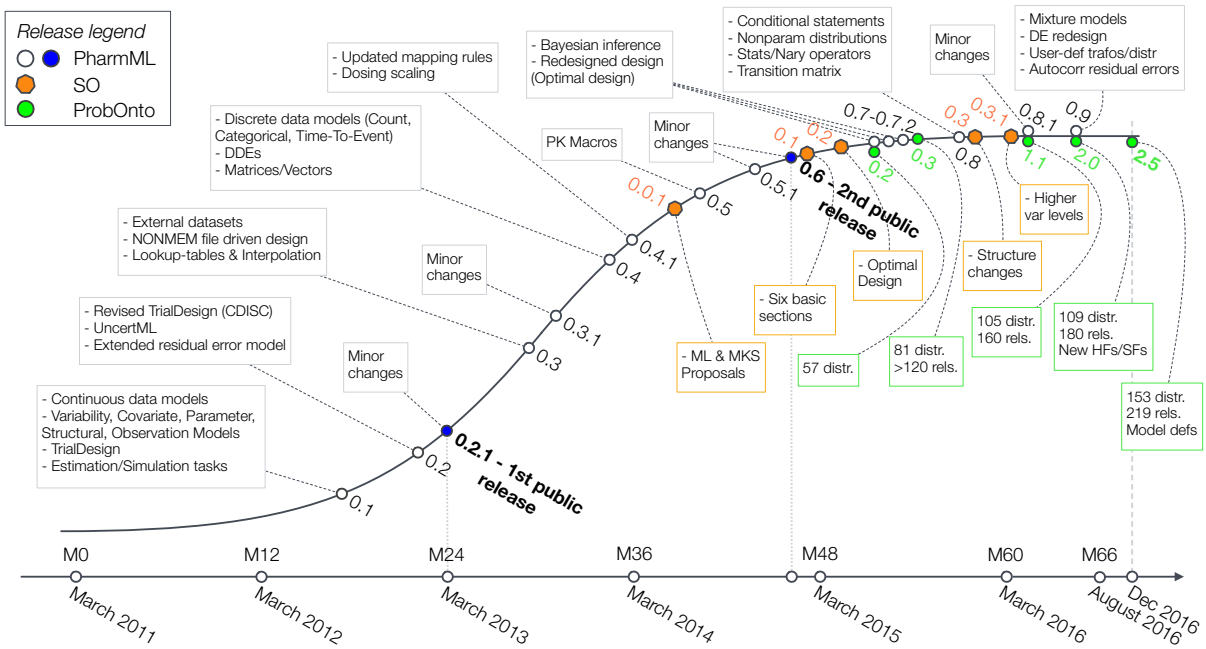

Figure 1.1: Development timeline of exchange formats and the ontology in overview.

1.3 Summary of extensions

ProbOnto element	**Version $\leq$ 2.0**	**Version 2.5**
Univariate distributions Section 5.1	94 distributions	NEW distributions: Amoroso, Arcsine 1, Arcsine 2, Asymmetric Laplace, Benford, Beta-Negative Binomial, Borel, Chi, Dagum, Epanechnikov, Folded Normal, Half-Cauchy, Inverse Chi-Square (STAN), Johnson SB, Johnson SL, Johnson SU, Johnson SN, Kumaraswamy, Lévy, Makeham, Maxwell-Boltzmann, Muth, Noncentral beta, Noncentral Chi-Squared, Noncentral F, Noncentral T, Power, Power Normal, Sinh-Arcsinh 1, Sinh-Arcsinh 2, Standard Cauchy, Standard Logistic, Standard Power, Standard Triangular, Standard Two-Sided Power, Student T 3 (STAN), Two-Sided Power, Weibull Discrete, Wigner Semicircle, Yule-Simon, Zeta, Zipf
Multivariate distributions Section 5.3	12 distributions	NEW 2 correlation matrix distributions: LKJ Correlation 1 (STAN), LKJ Correlation 2 (STAN)
All distributions Section A.2	no model definition	NEW model definitions available for many distributions added
Univariate relationships Table 2.3, Section 2.7	180 relationships	NEW 48 new entries

Table 1.1: Overview of major differences between versions 2.5 and 2.0

1.4 Development in numbers

Date	**Version**	**# Distributions**	**# Relationships**
2 July 2015	0.2.1	53	–
4 September 2015	0.2.4	70	100
11 October 2015	0.3	81	120
April 2016	1.0	81	130
14 March 2016	1.1	101	160
20 July 2016	2.0	106	180
13 January 2017	2.5	150	228

Table 1.3: ProbOnto development in terms of supported parametric distributions and relationships.

Chapter 2

Introduction

Background and motivation ProbOnto was first designed to facilitate the encoding of nonlinear mixed effect models and their annotation in version 0.7 of the Pharmacometrics Markup Language (PharmML) [Swat et al., 2015] developed by the DDMoRe consortium [Harnisch et al., 2013]. In the meantime we published a short application note, [Swat et al., 2016].

When encoding probabilistic uncertainties using a parametric distribution, its name and parameters are usually sufficient to specify it unambiguously as in most cases such parameter set are unique. But, because for a number of cases two or more parameterisations exist, one needs to be precise which parameters are used when referring to a distribution, otherwise one might end up with a wrong model (see for an example Figure 2.1). For this purpose, an external standard reference is very useful as it requires less effort to declare the required distribution in an exchange format or programming language.

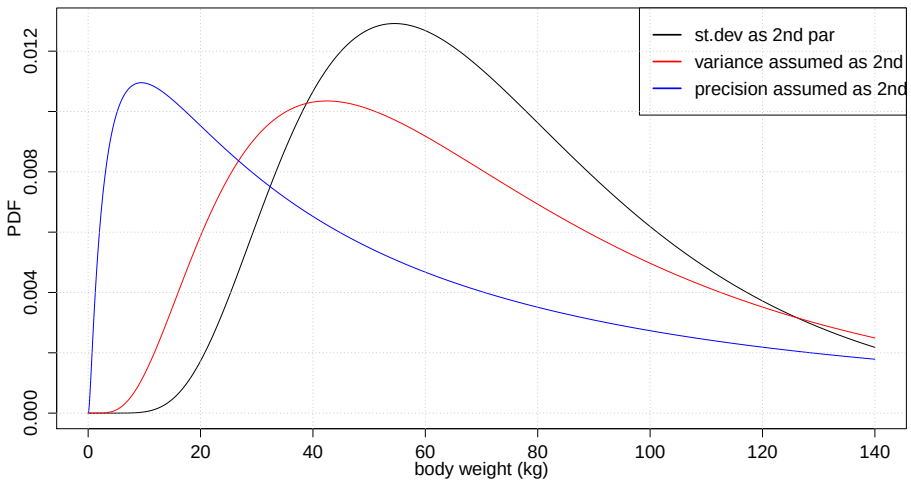

Figure 2.1: Illustration of possible model misspecification when using incorrect parameterisation. (1) The black curve corresponds to a log-normal distribution of body weight, $\mathcal{LN}(\mu{=}\log(70), \sigma{=}0.5)$, the intended parameterisation. (2) Here it was mistakenly assumed that 0.5 corresponds to the variance and calculation of standard deviation as required by the R function *dlnorm* gives $\sigma = \sqrt{0.5} = 0.707$ (red). (3) Here the modeller assumed that the 2nd input value corresponds to the precision and calculated the standard deviation as $\sigma = 1/\sqrt{0.5} = 1.41$ (blue). Small numerical differences in σ on the log-scale, result in significant differences on the natural scale.

Until now, we have been relying on UncertML, a high quality reference resource for encapsulating probabilistic uncertainties, [UncertML Team, 2014], [Williams et al., 2008]. It facilitates the encoding of a range of continuous and discrete uni/multi-variate probability distributions. However, from the perspective of PharmML, it has some limitations as described in section 2.2.

Status Quo The initial motivation for *ProbOnto* was to create an ontology of probability distributions purely for annotation purposes. Many resources are available online and in printed format but no suitable ontology exists so far[1]. Similarly, the databases of distributions available online come with analog issues.

[1]For example, the Statistics Ontology, STATO, `http://stato-ontology.org/`, provides for most distributions merely a link to an external reference/definition. No parameters or related functions and quantities are defined in the ontology making their annotation impossible. Other ontologies, we have analysed number of them featured in the BioPortal, [Noy et al., 2009], suffer

The largest and most comprehensive known to us collection of probability distributions, the UUPDE by [Marichev and Trott, 2013]. With up to 60 properties for each of its 500 probability distributions it is an invaluable resource for parametric distributions. Unfortunately, it comes with univariate cases only, still lacks a number of distributions and doesn't provide references or parameter names making its use in our context impracticable[2].

Idea It turns out that *ProbOnto* can be very helpful in designing a flexible alternative for UncertML with many additional features. It can be used in PharmML or other target tools/languages *both* as an ontological resource for annotation purposes and as a knowledge base, providing the means to specify a wide range of distributions, distribution related functions and quantities.

In fact, such a solution is indispensable in the face of requirements posed by the models we would like to encode, both currently and in the foreseeable future.

2.1 Ontology versus Knowledge Base

The following definitions are based on a post by Jeen Broekstra found on the *dataversity* discussion forum[3].

Ontology is a formal representation of a domain of knowledge. It is an abstract entity defining the vocabulary for a domain and the relations between concepts. However, an ontology doesn't specify how that knowledge is stored (as physical file, in a database, or in some other form), and how the knowledge can be accessed.

Knowledge base is a physical artifact. It is a database, a repository of information that can be accessed and manipulated in some predefined fashion.

The information stored in a knowledge base is modelled according to rules and relationships defined in an ontological model.

2.2 Limitations in UncertML application in PharmML

Although very useful to a certain extent, there are limitations in the design and scope of UncertML making the encoding of statistical models as used in pharmacometrics cumbersome (e.g. especially when a required distribution is not available). Here some known limitations:

- it doesn't support the assignment of expressions for distribution parameters or the specification of block references, which is required if the parameter in question is defined elsewhere in the model.
- it doesn't cover many distributions used in Pharmacometrics, e.g.
  - discrete ones such as Generalized Poisson or Zero-inflated Poisson, see section 4.1.3
  - alternative parameterisations for distributions such as Negative Binomial, Log-Normal etc.
  - multivariate continuous distributions such as Inverse-Wishart.
- UncertML is a reference resource for distributions but does not provide mechanisms to retrieve programmatically related functions and quantities.
- every extension requires changes in the already complex XML schema.

Other minor issues:

- Negative Binomial (NB1) comes in an unusual formulation (as opposed to just different parameterisation) which is rarely seen in the literature. It is different from the formulation supported by R, Matlab [MathWorks, 2015] and winBUGS and might go unnoticed to an unexperienced user, leading to false results. A detailed discussion can be found in the appendix A.3.
- `<degreesOfFreedom>` parameter element of the Wishart distribution doesn't support referencing a variable (required for Bayesian inference)

from equivalent limitations as they are designed in a similar way.

[2]Another example is Distributome, `http://www.distributome.org/`, which comes with an impressive and well-referenced collection of 90+ distributions but doesn't contain many of relevant for us types and/or parameterisations and is limited to univariate parametric ones.

[3]`http://answers.semanticweb.com/questions/21500/what-is-the-difference-between-knowledge-base-and-ontology`

- the implementation of `<MultivariateNormalDistribution>` requires the specification of the `dimension` attribute of the covariance matrix – although this can be estimated it requires unnecessary calculations when translating models to PharmML.
- version 3.0 which we currently use is not yet released publicly, the UncertML website is not updated and 3.0 documentation is not available.

UncertML extension A seemingly easy solution would be to extend UncertML, but to do so it would require the introduction of major extensions and changes to the XML schema. Only the support of the most important missing features would *de facto* require a rewrite of the entire standard, as UncertML doesn't possess the structure to encode algebraic expressions. This would most certainly result in a different implementation of the mathematical notation, compared to PharmML, resulting in inconsistent, layered and/or overlapping schemas, difficult to handle and to validate.

Suggested way forward ProbOnto offers an alternative solution avoiding all the limitations listed above, while providing a number of additional features and means to build in a very flexible probability distribution support in MDL, PharmML and other languages/tools within DDMoRe and beyond.

2.3 ProbOnto Features

The Knowledge Base had been build based on the UncertML standard collection of 28 parametric distributions. Following its implementation we extended it for models used in pharmacometrics. We have added a number of most essential discrete data models, all distributions and/or their parameterisations used in Monolix and winBUGS, the majority of distributions supported by the STAN, Matlab Statistical Toolbox and many relationships between distributions.

In the following list we summarise ProbOnto 2.5 features (January 2017)

- General
  - 150 parametric distributions and alternative parameterisations
    - 135 discrete & continuous univariates
    - 1 mixture univariate
    - 14 discrete & continuous multivariates
  - 3 empirical distributions, *aka* samples
  - 228 relationships and re-parameterisation formulas.
  - Allows for easy encoding of distributions and related functions in target tools/languages thanks to its generic format which doesn't enforce specific implementation.
  - For the use of ProbOnto in PharmML a simple schema has been designed providing generic encoding support for all distributions and their features, see section 2.4.
  - Collection of supported distributions is easily extendable.
  - All mathematical functions and quantities are available in Latex and additionally for functions R-code is provided.
- ProbOnto as Ontology
  - Can be used to annotate statistical models based on supported probability distributions, e.g. their name, parameters, truncation bounds, their defining functions and quantities.
- ProbOnto as Knowledge Base
  - Provides, for each distribution, either PDF or PMF and in many cases also other distribution related functions such as CDF, hazard and survival functions – the level of coverage depends on the particular distribution,
  - Related quantities such as mean, median, mode, variance,
  - Support/range definition, relationships to other distributions,
  - Model definition for most distributions and references.

The distribution collection and their features are based on the project by [UncertML Team, 2014], probability distribution pages of the English Wikipedia[4], [Forbes et al., 2011], [Leemis and Mcqueston, 2008], [Johnson et al., 2005], [Song and Chen, 2011] and [Weisstein, 2007] to name the main sources.

[4]See the list of distributions on Wikipedia at `https://en.wikipedia.org/wiki/List_of_probability_distributions`

2.4 Working with ProbOnto

The subsequent chapters come with a number of examples of ProbOnto use but it is worth highlighting two basic implementation rules

- The name of a distribution, encoded in the `<ProbOnto name="...">` tag, where instead of the dots one of the 'Code names' assigned to each distribution in ProbOnto must be used.
- The same holds for the parameters of a distribution, encoded in the `<Parameter name="...">` tag. Note that the order of parameters doesn't matter.

To remain consistent with the nomenclature used so far in PharmML and MDL which was based on the UncertML vocabulary, the majority of parameter *code names* is identical to those used in UncertML. For new distributions and their parameters we have defined the most common names used in the literature. The *code names* are collected in the tables 5.3, 5.7 and 5.9 of the appendix.

2.4.1 Basic example

The implementation of the negative binomial model illustrates how this works. There are many parameterisations for this distribution but the version with Poisson intensity, λ, and over-dispersion, τ, as parameters, code name *NegativeBinomial2*, is used here for demonstration

```
<Distribution>
    <ProbOnto name="NegativeBinomial1">
        <Parameter name="rate">
            <ct:Assign>
                <ct:SymbRef blkIdRef="pm1" symbIdRef="rabbit"/>
            </ct:Assign>
        </Parameter>
        <Parameter name="overdispersion">
            <ct:Assign>
                <ct:SymbRef blkIdRef="pm2" symbIdRef="piggy"/>
            </ct:Assign>
        </Parameter>
    </ProbOnto>
</Distribution>
```

According to the rules, the names of the distributions and their parameters must be the code names defined by ProbOnto, see table 5.7. The user can then assign any symbols to the parameters, with *rabbit* for *rate*, defined in parameter model `pm1` and *piggy* for *overdispersion*, defined in parameter model `pm2`.

2.4.2 PharmML elements supporting ProbOnto

ProbOnto has been designed to be applicable in any target tool and comes without any limitations or requirements with respect to its use in a given software. For use within PharmML we provide an XML schema which can be easily updated, e.g. when new distributions are added to the knowledge base. After adding a new distribution to the knowledge base, such extensions of the schema consist merely in adding new values to the `name` attributes.

The following elements support ProbOnto distribution encoding

- `<ProbOnto>` tag with the `name` attribute for the distribution code names with following child elements
  - `<Parameter>` with the `name` attribute for the parameter code names. It can be assigned any expression.
  - `<LowerTruncationBound>` and `<UpperTruncationBound>` to indicate the truncation bounds for univariate distributions with attribute `type` which can be either *closed* or *open*.
  - `<MixtureComponent>` with the `name` attribute for the code name of mixture component.

2.5 Empirical and non-parametric distributions

When formulating models it is often not possible to specify a probability distribution using a parametric one because its defining function is not known. Instead

- **empirical** – defined by a set of values
- **non-parametric**[5] – defined by e.g. intervals

can be considered as constituting a distribution model.

[5]Support for non-parametric distributions will be provided in a future release.

2.5.1 Empirical distributions

The support for empirical distributions is based on UncertML [UncertML Team, 2014], there called *Samples*. Sample are sets of realisations obtained from a known or unknown distribution. We distinguish between

Random sample – a set of independent realisations, x_i, drawn from a probability distribution $p(x)$ (or alternatively a population where every member has an equal chance of being drawn, but is randomly selected). The sample will typically be obtained using some form of simulation algorithm for base distributions using a random number generator of some sort.

Systematic sample – a set of realisations, x_i, taken from a probability distribution $p(x)$ (or alternatively a population where selection is used preferentially sample from certain targets). The sample will typically be obtained using some form of deterministic algorithm such as the unscented sampling methods for Gaussian random variables.

Unknown sample – an unknown sample is a set of independent realisations, x_i, drawn from some distribution $p(x)$ although without any known properties. It is useful in situations where we are not sure how the sample was obtained and whether it is truly random.

with an optional *weight* – to encode the weighting given to each realisation.

In short, the samples are used to describe a variable that can take one of several explicit discrete value vectors $a_i, b_i, \ldots z_i, i = 0, 1, ..., N$ and where a weight $0 < p_i < 1$ can optionally be assigned to each value (pair/triple/quadruple/.../n-tuple). To specify a empirical distribution we need to declare

- sets/columns with explicit values (at least one vector is required) – $\{a_i\}, \ldots, \{z_i\}$ with $i = 0, 1, ..., N$
- (optionally) weights, $\{p_i\}$ with $i = 0, 1, ..., N$

See section 4.5 for use examples of samples and their encoding in PharmML.

2.6 Annotation of models with ProbOnto ontology

2.6.1 Implementation in PharmML

The following code shows the typical Poisson model implementation

```
<PMF>
    <Distribution>
        <ProbOnto id="X1" name="Poisson1">
            <Parameter id="X2" name="rate">
                <ct:Assign>
                    <ct:SymbRef symbIdRef="lambda"/>
                </ct:Assign>
            </Parameter>
        </ProbOnto>
    </Distribution>
</PMF>
```

2.6.2 Annotation of PharmML

Notice that in the above code snippet the elements defining the distribution used, `<ProbOnto>`, and its parameter, `<Parameter>` are given identifiers, `id="X1"` and `id="X2"`, respectively. This allows us to annotate these elements so that we can make explicit their intended interpretation. Using the PharmML metadata annotation schema, we would record such interpretation using the property *has-interpreted-type*. In what follows, 'ps' abbreviates the namespace for this schema and 'probonto' abbreviates the namespace for the ProbOnto ontology, part of the stack of ontologies used in PharmML annotation.

The distribution element is interpreted as an instantiation of the Poisson distribution.

X1 ps:has-interpreted-type probonto:0000111.

The parameter element is interpreted as an instantiation of the parameter element, λ, of the Poisson distribution.

X2 ps:has-interpreted-type probonto:0000114.

In ProbOnto, *0000111* and *0000114* are the identifiers for the Poisson distribution and its (unique) parameter, respectively. The two statements above encode the interpretation of the PharmML code defining the distribution. Such statements can in principle be generated automatically after processing the PharmML code.

Annotating the actual PharmML model and the element to which the distribution applies would involve more or, a variation upon, the above to the effect that the ProbOnto distribution is identified.

2.6.3 Background Information is Contained in ProbOnto

Given the annotation of the PharmML code linking to ProbOnto, we can then use ProbOnto to make explicit all the information that is packed into these two very terse annotation statements.

Underlying accessible knowledge about Poisson distribution

We thus have access to the following regarding the distribution contained in the PharmML code (as much as is contained in the ProbOnto definition of the Poisson distribution):

name	Poisson (ID: 0000111)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

Additionally, we can obtain from ProbOnto the following type of information.

Underlying accessible knowledge about the related functions

PMF

$$\frac{\lambda^k}{k!} e^{-\lambda}$$

PMF in R

```
lambda^k/factorial(k) * exp(-lambda)
```

CDF

$$\frac{\Gamma(\lfloor k+1 \rfloor, \lambda)}{\lfloor k \rfloor !}$$

CDF in R

```
Igamma(floor(k+1), lambda, lower=F) / factorial(floor(k))
```

using *Igamma* function from the *zipfR* package, `http://cran.r-project.org/web/packages/zipfR/zipfR.pdf`.

Underlying accessible knowledge about the (rate) parameter

name	Poisson intensity (ID: 0000114)
type	scalar
symbol	λ
definition	$\lambda \in R, \lambda > 0$

The amount of information that may be encoded in ProbOnto is extensible. Thus, via a very simple and direct mechanism of annotation that amounts to linking a distribution and its parameter(s) in a piece of PharmML code, we can inherit and obtain all the background relevant information. This knowledge can be used either for our understanding and the validation of our PharmML encoding or, with adequate software support, for processing by tools.

Currently, such extensive software support is not available but is part of the development path for PharmML and its implementation of ProbOnto.

2.7 Relationships

Relationships which connect different distributions is what makes an ontology and knowledge base so useful. Information of this type can provide a modeller with hints about alternative models applicable to a particular dataset/experiment, Figure 2.3. As an example, consider the normal distribution in its *Normal1* parameterisation connected to multiple distributions, both discrete and continuous ones, see Figure 2.2. Another interesting subset of these relationships are re-parameterisations within a given distribution family, described in detail in chapter 3. We distinguish the following relationship types and their combinations:

- Approximation/limiting – type of relationships following when a parameter approaches a certain value or infinity.
- Re-parameterisation – relationship between distributions of the same family.
- Special case – when one distribution is a special case of another with a broader parameter space.
- Transformation – when a variate with one distribution is a function of (an)other variate(s) with different distributions.

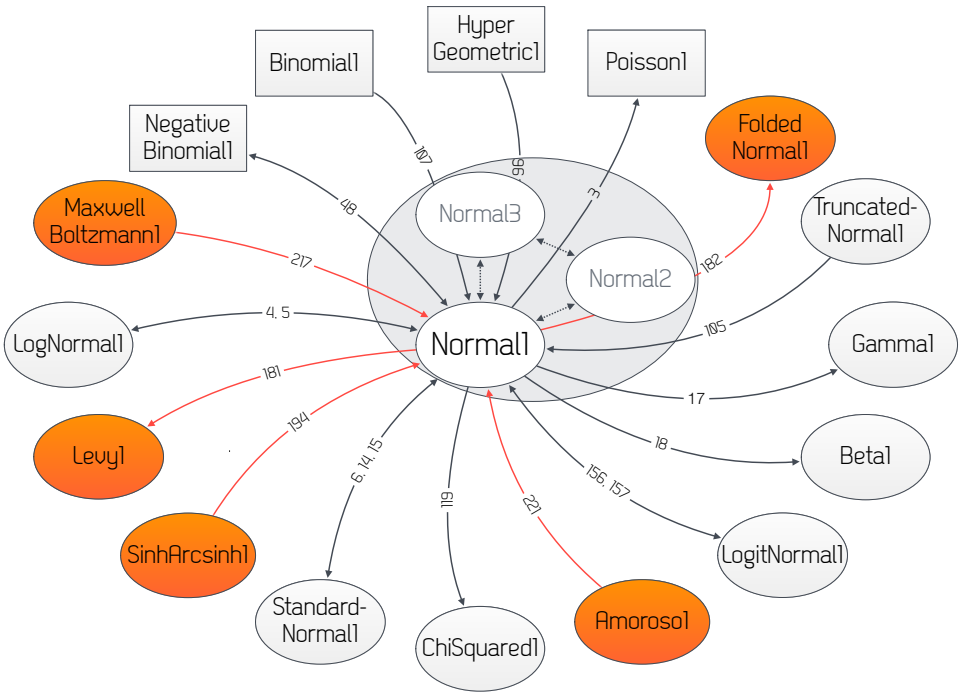

Figure 2.2: Relationships of the Normal1 distribution which are a subset of the network shown in Figure 2.3. Distributions and relationships in orange are new in this version.

2.8 Programmatic access

2.8.1 Ontology Lookup Service API

OLS-API can be used to access ontologies. The documentation below provides examples specific to ProbOnto using the OLS API.

Retrieve ProbOnto
Example request:

```
GET /api/ontologies/{ontology_id}
$ curl 'http://www.ebi.ac.uk/ols/beta/api/ontologies/probonto' -i -H 'Accept: application/json'
```

Listing probonto terms
Example request:

```
GET /api/ontologies/{ontology_id}/terms
$ curl 'http://www.ebi.ac.uk/ols/beta/api/ontologies/probonto/terms' -i -H 'Accept: application/json'
```

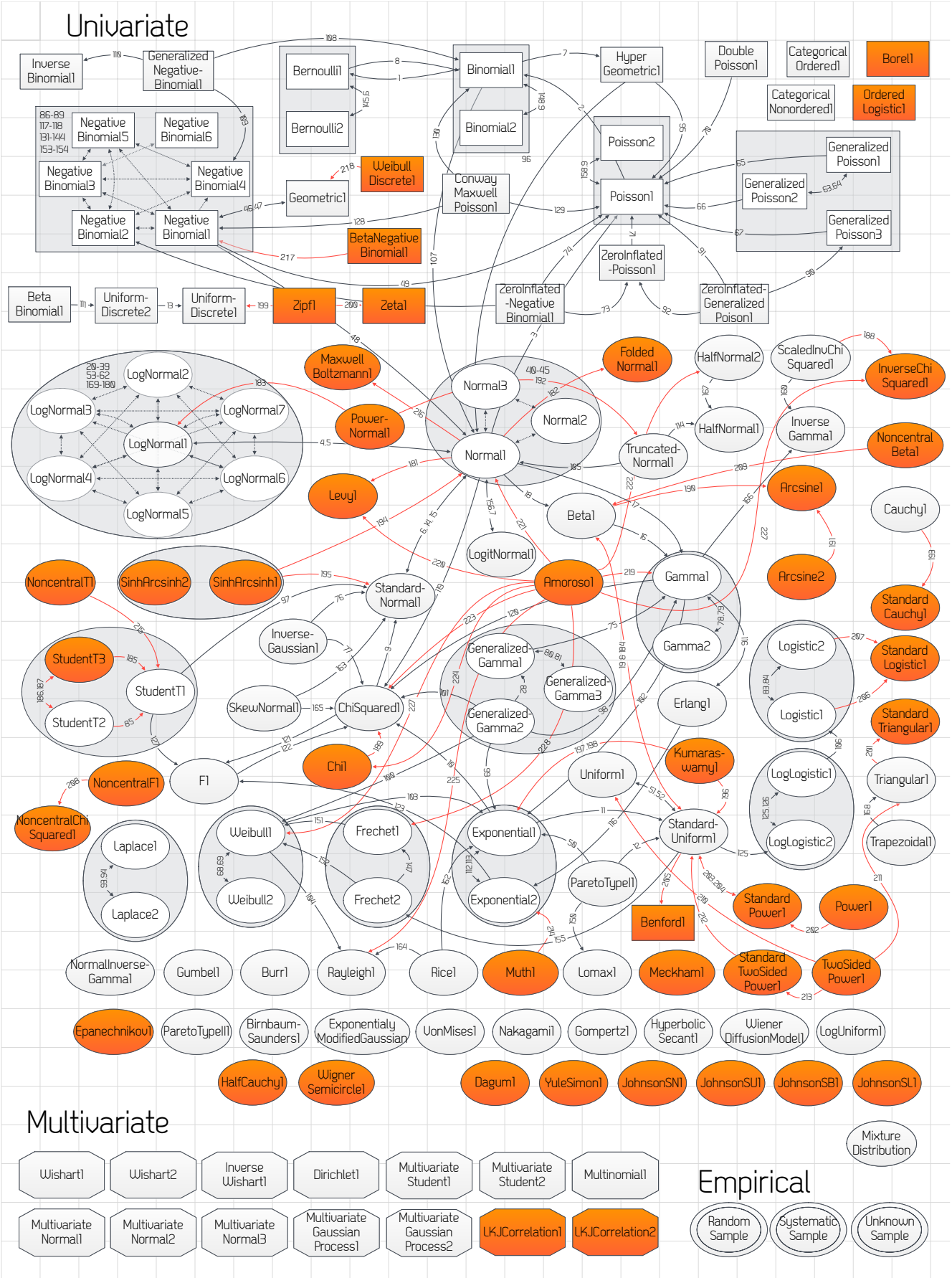

Figure 2.3: ProbOnto 2.5 collection. Distributions and relationships in orange are new in this version.

Retrieve a term: probability distribution (`http://www.probonto.org/core#0000020`)
Example request:

```
GET /api/ontologies/{ontology_id}/terms/{iri}
$ curl 'http://www.ebi.ac.uk/ols/beta/api/ontologies/probonto/terms/http%253A%252F%252F...
www.probonto.org%252Fcore%25230000020' -i -H 'Accept: application/json'
```

Retrieve an individual: CDF of Poisson 1 (`http://www.probonto.org/kb#0000043`)
Example request:

```
GET /api/ontologies/{ontology}/individuals/{iri}
$ curl 'http://www.ebi.ac.uk/ols/beta/api/ontologies/probonto/individuals/http%253A%252F%252F...
www.probonto.org%252Fkb%25230000043' -i -H 'Accept: application/json'
```

2.9 Independency of ProbOnto

It is worth stressing that the ProbOnto ontology and knowledge base are fully independent from PharmML. ProbOnto also doesn't enforce specific implementation from tool designers – this allows it to be used across the DDMoRe target tools, languages and beyond.

For the purpose of the use in PharmML we have developed a schema which is tailored towards NLME models as used in pharmacometrics and is otherwise entirely optional. See chapter 4 for examples illustrating ProbOnto use in PharmML.

Chapter 3

Alternative parameterisations

The existence of alternative parameterisations is apparent for number of reasons, such as model type, application area, available data and/or target tool – e.g. BUGS using precision, τ, rather then standard deviation or variance for a normal/log-normal and a number of other distributions. ProbOnto contains, wherever available, re-parameterisations formulas of distributions coming in different forms. This can be seen as an improvement of the inter-operability support between target tools supporting different parameter sets for a given probability density/mass function.

The needs for the support of re-parameterisation, e.g. between BUGS and R has been discussed before, [LeBauer et al., 2013]. The authors state that "[...] R and BUGS languages use different representations of common probability density functions, creating a potential for errors to occur in the implementation or interpretation of analyses that use both languages". While this paper addresses only two tools and five distributions, we aim to cover a much larger scope, with respect to both number of tools and parameterisations, see Figure 2.3.

All alternative parameterisations supported in ProbOnto are given in this chapter. We provide for clarity the defining probability density/mass functions and the according formulas required to go from one form to another.

3.1 Negative binomial distribution

The available parameterisations are

- NegativeBinomial1(r, p) with r – *number of successes* and p – *success probability*

$$P(k; r, p) = \binom{k+r-1}{k} p^r (1-p)^k, \quad k \in \{0, 1, 2, \dots\} \text{ number of failures}$$

- NegativeBinomial4(r, p) with r – *number of failures* and p – *success probability*

$$P(k; r, p) = \binom{k+r-1}{k} (1-p)^r p^k, \quad k \in \{0, 1, 2, \dots\} \text{ number of successes}$$

- NegativeBinomial2(λ, τ), [Plan et al., 2009], with λ – *mean* and τ – *over-dispersion*,

$$P(k; \lambda, \tau) = \frac{\Gamma(k + \frac{1}{\tau})}{k!\, \Gamma(\frac{1}{\tau})} \Big(\frac{1}{1+\tau\lambda}\Big)^{\frac{1}{\tau}} \Big(\frac{\lambda}{\frac{1}{\tau}+\lambda}\Big)^k, \quad k \in \{0, 1, 2, \dots\}$$

- NegativeBinomial3(μ, ϕ), [STAN Development Team, 2015b],

$$P(k; \mu, \phi) = \binom{k+\phi-1}{k} \Big(\frac{\phi}{\mu+\phi}\Big)^{\phi} \Big(\frac{\mu}{\mu+\phi}\Big)^k, \quad k \in \{0, 1, 2, \dots\}$$

- NegativeBinomial5(α, β), [STAN Development Team, 2015b],

$$P(k; \alpha, \beta) = \binom{k+\alpha-1}{\alpha-1} \Big(\frac{\beta}{\beta+1}\Big)^{\alpha} \Big(\frac{1}{\beta+1}\Big)^k, \quad k \in \{0, 1, 2, \dots\}$$

- NegativeBinomial6(η, ϕ), [STAN Development Team, 2015b],

$$P(k; \eta, \phi) = \binom{k+\phi-1}{k} \Big(\frac{\exp(\eta)}{\exp(\eta)+\phi}\Big)^k \Big(\frac{\phi}{\phi+\exp(\eta)}\Big)^{\phi}, \quad k \in \{0, 1, 2, \dots\}$$

with NB2 being used in typical pharmacometric discrete data models, [Plan et al., 2009, Trocóniz et al., 2009].

Re-parameterisation formulas

- **NB1**$(r,\, p) \to$ **NB2**$(\lambda,\, \tau)$: $\tau = 1/r; \quad \lambda = \frac{r(1-p)}{p}$
  NB2$(\lambda,\, \tau) \to$ **NB1**$(r,\, p)$: $r = 1/\tau; \quad p = \frac{1}{1+\tau\lambda}$
- **NB1**$(r,\, p) \to$ **NB3**$(\mu,\, \phi)$: $\phi = r; \quad \mu = \frac{r(1-p)}{p}$
  NB3$(\mu,\, \phi) \to$ **NB1**$(r,\, p)$: $r = \phi; \quad p = \frac{r}{\mu+r}$
- **NB1**$(r,\, p) \leftrightarrow$ **NB4**$(r,\, p)$: $p \to 1-p$
- **NB1**$(r,\, p) \to$ **NB5**$(\alpha,\, \beta)$: $\alpha = r; \quad \beta = \frac{p}{1-p}$
  NB5$(\alpha,\, \beta) \to$ **NB1**$(r,\, p)$: $r = \alpha; \quad p = \frac{\beta}{1+\beta}$
- **NB2**$(\lambda,\, \tau) \to$ **NB3**$(\mu,\, \phi)$: $\mu = \lambda; \quad \phi = 1/\tau$
  NB3$(\mu,\, \phi) \to$ **NB2**$(\lambda,\, \tau)$: $\lambda = \mu; \quad \tau = 1/\phi$
- **NB2**$(\lambda,\, \tau) \to$ **NB4**$(r,\, p)$: $r = 1/\tau; \quad p = \frac{\tau\lambda}{1+\tau\lambda}$
  NB4$(r,\, p) \to$ **NB2**$(\lambda,\, \tau)$: $\tau = 1/r; \quad \lambda = \frac{rp}{1-p}$
- **NB2**$(\lambda,\, \tau) \to$ **NB5**$(\alpha,\, \beta)$: $\alpha = 1/\tau; \quad \beta = 1/(\tau\lambda)$
  NB5$(\alpha,\, \beta) \to$ **NB2**$(\lambda,\, \tau)$: $\lambda = \alpha/\beta; \quad \tau = 1/\alpha$
- **NB3**$(\mu,\, \phi) \to$ **NB4**$(r,\, p)$: $r = \phi; \quad p = \mu/(\phi+\mu)$
  NB4$(r,\, p) \to$ **NB3**$(\mu,\, \phi)$: $\phi = r; \quad \mu = \frac{rp}{1-p}$
- **NB3**$(\mu,\, \phi) \to$ **NB5**$(\alpha,\, \beta)$: $\alpha = \phi; \quad \beta = \phi/\mu$
  NB5$(\alpha,\, \beta) \to$ **NB3**$(\mu,\, \phi)$: $\mu = \alpha/\beta; \quad \phi = \alpha$
- **NB3**$(\mu,\, \phi) \to$ **NB6**$(\eta,\, \phi)$: $\eta = \log(\mu)$
  NB6$(\eta,\, \phi) \to$ **NB3**$(\mu,\, \phi)$: $\mu = \exp(\eta)$
- **NB4**$(r,\, p) \to$ **NB5**$(\alpha,\, \beta)$: $\alpha = r; \quad \beta = (1-p)/p$
  NB5$(\alpha,\, \beta) \to$ **NB4**$(r,\, p)$: $r = \alpha; \quad p = 1/(1+\beta)$

3.2 Normal distribution

Available parameterisations are

- Normal1$(\mu,\, \sigma)$ with μ – *mean* and σ – *standard deviation*,

$$P(x; \boldsymbol{\mu}, \boldsymbol{\sigma}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}\Big]$$

- Normal2$(\mu,\, v)$ with μ – *mean* and v – *variance*,

$$P(x; \boldsymbol{\mu}, \boldsymbol{v}) = \frac{1}{\sqrt{v}\sqrt{2\pi}} \exp\Big[-\frac{(x-\mu)^2}{2v}\Big]$$

- Normal3$(\mu,\, \tau)$ with μ – *mean* and τ – *precision* $(\tau = 1/\sigma^2)$

$$P(x; \boldsymbol{\mu}, \boldsymbol{\tau}) = \sqrt{\frac{\tau}{2\pi}}\Big[-\frac{\tau}{2}(x-\mu)^2\Big].$$

Re-parameterisation formulas

In this case the formulas are very simple but are given here for the completeness

- **N1**$(\mu,\, \sigma) \to$ **N2**$(\mu,\, v)$: $v = \sigma^2$
  N2$(\mu,\, v) \to$ **N1**$(\mu,\, \sigma)$: $\sigma = \sqrt{v}$
- **N1**$(\mu,\, \sigma) \to$ **N3**$(\mu,\, \tau)$: $\tau = 1/\sigma^2$
  N3$(\mu,\, \tau) \to$ **N1**$(\mu,\, \sigma)$: $\sigma = 1/\sqrt{\tau}$
- **N2**$(\mu,\, v) \to$ **N3**$(\mu,\, \tau)$: $\tau = 1/v$
  N3$(\mu,\, \tau) \to$ **N2**$(\mu,\, v)$: $v = 1/\tau$.

3.3 Log-normal distribution

The log-normal distribution is special in that not only different parameter sets exist but also because they are defined either on the natural or logarithmic scale. Interestingly, in one case the parameters are defined on two different scales, called LN3, see Figure 3.1, for an overview.

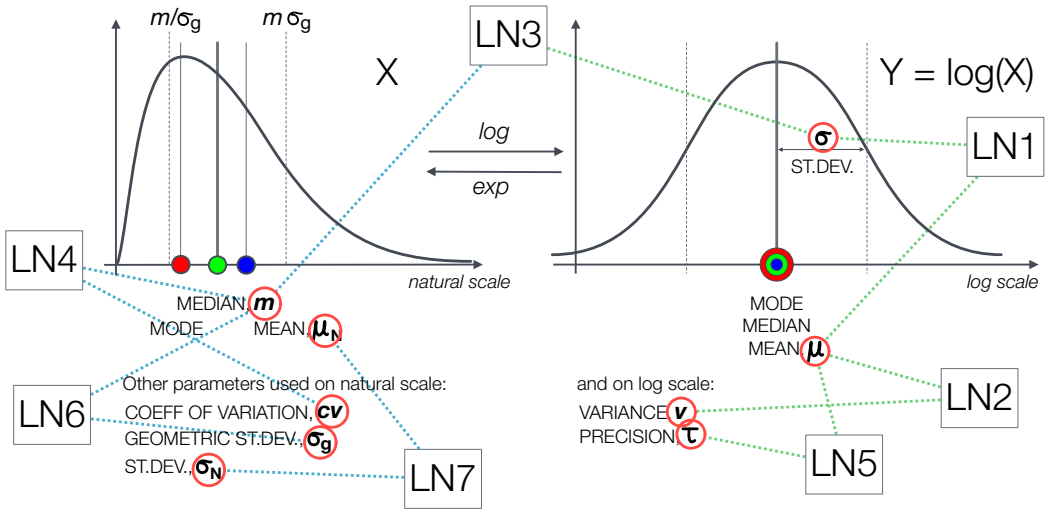

Figure 3.1: Schematic representation of the lognormaly distributed data on the natural (left) and logarithmic scale (right), see Figure 3.2 for real-life data example. **Bold** symbols stand for quantities commonly used to parameterise a log-normally distributed variable.

Available parameterisations are

- LogNormal1(μ, σ) with *mean*, μ, and *standard deviation*, σ, both on the log-scale,

$$P(x; \boldsymbol{\mu}, \boldsymbol{\sigma}) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[\frac{-(\log x - \mu)^2}{2\sigma^2}\right]$$

- LogNormal2(μ, v) with *mean*, μ, and *variance*, v, both on the log-scale,

$$P(x; \boldsymbol{\mu}, \boldsymbol{v}) = \frac{1}{x\sqrt{v}\sqrt{2\pi}} \exp\left[\frac{-(\log x - \mu)^2}{2v}\right]$$

- LogNormal3(m, σ) with *median*, m, on the natural scale and *standard deviation*, σ, on the log-scale,

$$P(x; \boldsymbol{m}, \boldsymbol{\sigma}) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[\frac{-[\log(x/m)]^2}{2\sigma^2}\right]$$

- LogNormal4(m, cv) with *median*, m, and *coefficient of variation*, cv, both on the natural scale,

$$P(x; \boldsymbol{m}, \boldsymbol{cv}) = \frac{1}{x\sqrt{\log(cv^2+1)}\sqrt{2\pi}} \exp\left[\frac{-[\log(x/m)]^2}{2\log(cv^2+1)}\right]$$

- LogNormal5(μ, τ) with *mean*, μ, and *precision*, τ, both on the log-scale,

$$P(x; \boldsymbol{\mu}, \boldsymbol{\tau}) = \sqrt{\frac{\tau}{2\pi}}\frac{1}{x} \exp\left[-\frac{\tau}{2}(\log x - \mu)^2\right]$$

- LogNormal6(m, σ_g) with *median*, m, and *geometric st. deviation*, σ_g, both on the natural scale,

$$P(x; \boldsymbol{m}, \boldsymbol{\sigma_g}) = \frac{1}{x\log(\sigma_g)\sqrt{2\pi}} \exp\left[\frac{-[\log(x/m)]^2}{2\log^2(\sigma_g)}\right]$$

- NEW LogNormal7(μ_N, σ_N) with *mean*, μ_N, and *st. deviation*, σ_N, both on the natural scale,

$$P(x; \boldsymbol{\mu_N}, \boldsymbol{\sigma_N}) = \frac{1}{x\sqrt{2\pi\log\left(1+\sigma_N^2/\mu_N^2\right)}} \exp\left(\frac{-\left[\log(x) - \log\left(\frac{\mu_N}{\sqrt{1+\sigma_N^2/\mu_N^2}}\right)\right]^2}{2\log\left(1+\sigma_N^2/\mu_N^2\right)}\right).$$

Figure 3.1 shows the schematic representation of lognormaly distributed data on the natural (left) and logarithmic scale (right), and indicates, on bold font, the available parameters characterising the distribution. On the other hand Figure 3.2 shows a real-life data example. We have used the basal insulin data in diabetic patients [Rudenski et al., 1991].

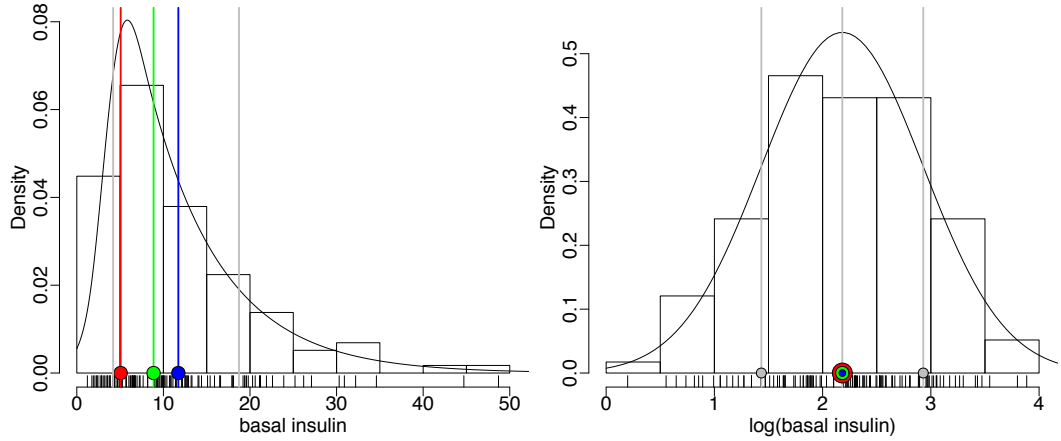

Figure 3.2: Representation of the lognormaly distributed basal insulin data in diabetic patients [Rudenski et al., 1991] on the natural scale (left) and on the logarithmic scale after *log*–transformation (right), colour code as in Figure 3.1. The density estimation for the data on the natural scale and its plotting was performed using the R package *logspline*, [Kooperberg, 2013]. The mode (red) on the natural scale doesn't tally with the maximum of the density curve as it should because it the latter is an approximation.

Re-parameterisation formulas

The recalculation between given parameterisations is error prone and should, whenever required, be taken over by the converters. The following equations might be useful when providing such translation support between target tools. For example when translating a model implemented for Monolix/NONMEM, which use either LN1 or LN2, with winBUGS as the target tool, which uses only LN5.

- LN1 relationships
  - $\mathbf{LN1}(\mu,\, \sigma) \rightarrow \mathbf{LN2}(\mu,\, v)\ :\ \sigma \rightarrow v = \sigma^2$
    $\mathbf{LN2}(\mu,\, v) \rightarrow \mathbf{LN1}(\mu,\, \sigma)\ :\ \sigma = \sqrt{v}$
  - $\mathbf{LN1}(\mu,\, \sigma) \rightarrow \mathbf{LN3}(m,\, \sigma)\ :\ m = \exp(\mu)$
    $\mathbf{LN3}(m,\, \sigma) \rightarrow \mathbf{LN1}(\mu,\, \sigma)\ :\ \mu = \log(m)$
  - $\mathbf{LN1}(\mu,\, \sigma) \rightarrow \mathbf{LN4}(m,\, cv)\ :\ m = \exp(\mu);\quad cv = \sqrt{\exp(\sigma^2) - 1}$
    $\mathbf{LN4}(m,\, cv) \rightarrow \mathbf{LN1}(\mu,\, \sigma)\ :\ \mu = \log(m);\quad \sigma = \sqrt{\log(cv^2 + 1)}$
  - $\mathbf{LN1}(\mu,\, \sigma) \rightarrow \mathbf{LN5}(\mu,\, \tau)\ :\ \tau = 1/\sigma^2$
    $\mathbf{LN5}(\mu,\, \tau) \rightarrow \mathbf{LN1}(\mu,\, \sigma)\ :\ \sigma = 1/\sqrt{\tau}$
  - $\mathbf{LN1}(\mu,\, \sigma) \rightarrow \mathbf{LN6}(m,\, \sigma_g)\ :\ m = \exp(\mu);\quad \sigma_g = \exp(\sigma)$
    $\mathbf{LN6}(m,\, \sigma_g) \rightarrow \mathbf{LN1}(\mu,\, \sigma)\ :\ \mu = \log(m);\quad \sigma = \log(\sigma_g)$
  - $\mathbf{LN1}(\mu,\, \sigma) \rightarrow \mathbf{LN7}(\mu_N,\, \sigma_N)\ :\ \mu_N = \exp\left(\mu + \frac{1}{2}\sigma^2\right);\quad \sigma_N = \exp\left(\mu + \frac{1}{2}\sigma^2\right)\sqrt{\exp(\sigma^2) - 1}$
    $\mathbf{LN7}(\mu_N,\, \sigma_N) \rightarrow \mathbf{LN1}(\mu,\, \sigma)\ :\ \mu = \log\left(\mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}\right);\quad \sigma = \sqrt{\log(1 + \sigma_N^2/\mu_N^2)}$
- remaining LN2 relationships
  - $\mathbf{LN2}(\mu,\, v) \rightarrow \mathbf{LN3}(m,\, \sigma)\ :\ m = \exp(\mu);\quad \sigma = \sqrt{v}$
    $\mathbf{LN3}(m,\, \sigma) \rightarrow \mathbf{LN2}(\mu,\, v)\ :\ \mu = \log(m);\quad v = \sigma^2$
  - $\mathbf{LN2}(\mu,\, v) \rightarrow \mathbf{LN4}(m,\, cv)\ :\ m = \exp(\mu);\quad cv = \sqrt{\exp(v) - 1}$
    $\mathbf{LN4}(m,\, cv) \rightarrow \mathbf{LN2}(\mu,\, v)\ :\ \mu = \log(m);\quad cv \rightarrow v = \log(cv^2 + 1)$
  - $\mathbf{LN2}(\mu,\, v) \rightarrow \mathbf{LN5}(\mu,\, \tau)\ :\ \tau = 1/v$
    $\mathbf{LN5}(\mu,\, \tau) \rightarrow \mathbf{LN2}(\mu,\, v)\ :\ v = 1/\tau$

- **LN2**$(\mu,\, v) \rightarrow$ **LN6**$(m,\, \sigma_g)$: $\mu \rightarrow m = \exp(\mu); \quad v \rightarrow \sigma_g = \exp(\sqrt{v})$
  LN6$(m,\, \sigma_g) \rightarrow$ **LN2**$(\mu,\, v)$: $m \rightarrow \mu = \log(m); \quad \sigma_g \rightarrow v = \log(\sigma_g^2)$
- **LN2**$(\mu,\, v) \rightarrow$ **LN7**$(\mu_N,\, \sigma_N)$: $\mu_N = \exp(\mu + v/2); \quad \sigma_N = \exp(\mu + v/2)\sqrt{\exp(v) - 1}$
  LN7$(\mu_N,\, \sigma_N) \rightarrow$ **LN2**$(\mu,\, v)$: $\mu = \log\left(\mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}\right); \quad v = \log(1 + \sigma_N^2/\mu_N^2)$

- remaining LN3 relationships
  - **LN3**$(m,\, \sigma) \rightarrow$ **LN4**$(m,\, cv)$: $cv = \sqrt{\exp(\sigma^2) - 1}$
    LN4$(m,\, cv) \rightarrow$ **LN3**$(m,\, \sigma)$: $\sigma = \sqrt{\log(cv^2 + 1)}$
  - **LN3**$(m,\, \sigma) \rightarrow$ **LN5**$(\mu,\, \tau)$: $\mu = \log(m); \quad \tau = 1/\sigma^2$
    LN5$(\mu,\, \tau) \rightarrow$ **LN3**$(m,\, \sigma)$: $m = \exp(\mu); \quad \sigma = 1/\sqrt{\tau}$
  - **LN3**$(m,\, \sigma) \rightarrow$ **LN6**$(m,\, \sigma_g)$: $\sigma_g = \exp(\sigma)$
    LN6$(m,\, \sigma_g) \rightarrow$ **LN3**$(m,\, \sigma)$: $\sigma = \log(\sigma_g)$
  - **LN3**$(m,\, \sigma) \rightarrow$ **LN7**$(\mu_N\, ,\sigma_N)$: $\mu_N = m\ \exp(\sigma^2/2); \quad \sigma_N = m\ \exp(\sigma^2/2)\sqrt{\exp(\sigma^2) - 1}$
    LN7$(\mu_N,\, \sigma_N) \rightarrow$ **LN3**$(m,\, \sigma)$: $m = \mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}; \quad \sigma = \sqrt{\log(1 + \sigma_N^2/\mu_N^2)}$

- remaining LN4 relationships
  - **LN4**$(m,\, cv) \rightarrow$ **LN5**$(\mu,\, \tau)$: $\mu = \log(m); \quad \tau = 1/\log(cv^2 + 1)$
    LN5$(\mu,\, \tau) \rightarrow$ **LN4**$(m,\, cv)$: $m = \exp(\mu); \quad cv = \sqrt{\exp(1/\tau) - 1}$
  - **LN4**$(m,\, cv) \rightarrow$ **LN6**$(m,\, \sigma_g)$: $\sigma_g = \exp\big(\sqrt{\log(cv^2 + 1)}\big)$
    LN6$(m,\, \sigma_g) \rightarrow$ **LN4**$(m,\, cv)$: $cv = \sqrt{\exp\big(\log^2(\sigma_g)\big) - 1}$
  - **LN4**$(m,\, cv) \rightarrow$ **LN7**$(\mu_N,\, \sigma_N)$: $\mu_N = m\sqrt{cv^2 + 1}; \quad \sigma_N = m\ cv\ \sqrt{cv^2 + 1}$
    LN7$(\mu_N,\, \sigma_N) \rightarrow$ **LN4**$(m,\, cv)$: $m = \mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}; \quad cv = \sigma_N/\mu_N$

- remaining LN5 relationships
  - **LN5**$(\mu,\, \tau) \rightarrow$ **LN6**$(m,\, \sigma_g)$: $m = \exp(\mu); \quad \sigma_g = \exp(1/\sqrt{\tau})$
    LN6$(m,\, \sigma_g) \rightarrow$ **LN5**$(\mu,\, \tau)$: $\mu = \log(m); \quad \tau = 1/\log^2(\sigma_g)$
  - **LN5**$(\mu,\, \tau) \rightarrow$ **LN7**$(\mu_N,\, \sigma_N)$: $\mu_N = \exp\big(\mu + 1/(2\tau)\big); \quad \sigma_N = \exp\big(\mu + 1/(2\tau)\big)\sqrt{\exp\big(1/\tau\big) - 1}$
    LN7$(\mu_N,\, \sigma_N) \rightarrow$ **LN5**$(\mu,\, \tau)$: $m = \log\left(\mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}\right); \quad \tau = 1/\log\big(1 + \sigma_N^2/\mu_N^2\big)$

- remaining LN6 relationships
  - **LN6**$(m,\, \sigma_g) \rightarrow$ **LN7**$(\mu_N,\, \sigma_N)$: $\mu_N = m\exp\big(\frac{1}{2}\log^2(\sigma_g)\big); \quad \sigma_N = m\exp\big(\frac{1}{2}\log^2(\sigma_g)\big)\sqrt{\exp\big[\log^2(\sigma_g)\big] - 1}$
    LN7$(\mu_N,\, \sigma_N) \rightarrow$ **LN6**$(m,\, \sigma_g)$: $m = \mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}; \quad \sigma_g = \exp\left(\sqrt{\log(1 + \sigma_N^2/\mu_N^2)}\right)$

The proof of the majority of the formulas is straightforward taking into account the definition of the parameters in question. The relationship between σ or τ (on the log scale) and cv (on the natural scale), essential for the re-calculation formulas involving LN4 parameterisation, is a bit more tricky to see. The proof starts with the known relationships for the mean, $mean$, and variance, var, on the natural scale. Then the square of the coefficient of variation, cv, on the natural scale reads

$$cv^2 = \frac{var}{mean^2} = \frac{(e^{\sigma^2} - 1)\, e^{2\mu+\sigma^2}}{[e^{(\mu+1/2\sigma^2)}]^2} = (e^{\sigma^2} - 1) \Leftrightarrow cv = \sqrt{e^{\sigma^2} - 1} \quad \& \quad \sigma = \sqrt{\log(cv^2 + 1)}.$$

To see then the relationships between cv and σ_g on the natural scale one has to use the formula $\sigma_g = exp(\sigma)$ and we get the expected results

$$cv = \sqrt{\exp\big(\log^2(\sigma_g)\big) - 1} \quad \& \quad \sigma_g = \exp\big(\sqrt{\log(cv^2 + 1)}\big).$$

3.3.1 Quantities of the log-normal distribution

In the most cases (apart from LN1 which is the by far most common form) the reference sources list only the PDF and in some cases CDF and/or mean and variances for each parameterisation. The following table tries to fill the gap listing number of relevant quantities for each form both for the natural and log-normal scale.

Log-normal distribution on the natural scale (NS)	Quantity	Normal distribution on the log-transformed scale (LS)
	$LN1(\boldsymbol{\mu}, \boldsymbol{\sigma})$	
$e^{\mu+\frac{1}{2}\sigma^2}$	Mean	$\boxed{\boldsymbol{\mu}}$
$e^{2\mu+\sigma^2}[e^{\sigma^2}-1]$	Variance	σ^2
$e^{\mu+\frac{1}{2}\sigma^2}\sqrt{e^{\sigma^2}-1}$	Standard deviation	$\boxed{\boldsymbol{\sigma}}$
$e^{\mu-\sigma^2}$	Mode	μ
e^{μ}	Median	μ
$\sqrt{e^{\sigma^2}-1}$	Coefficient of variation	σ/μ
	$LN2(\boldsymbol{\mu}, \boldsymbol{v})$	
$e^{\mu+\frac{1}{2}v}$	Mean	$\boxed{\boldsymbol{\mu}}$
$e^{2\mu+v}[e^{v}-1]$	Variance	$\boxed{\boldsymbol{v}}$
$e^{\mu+\frac{1}{2}v}\sqrt{e^{v}-1}$	Standard deviation	$\sqrt{v}$
$e^{\mu-v}$	Mode	μ
e^{μ}	Median	μ
$\sqrt{e^{v}-1}$	Coefficient of variation	$\sqrt{v}/\mu$
	$LN3(\boldsymbol{m}, \boldsymbol{\sigma})$	
$m\,e^{\frac{1}{2}\sigma^2}$	Mean	$\log(m)$
$m^2e^{\sigma^2}[e^{\sigma^2}-1]$	Variance	σ^2
$m\sqrt{e^{\sigma^2}(e^{\sigma^2}-1)}$	Standard deviation	$\boxed{\boldsymbol{\sigma}}$
m/e^{σ^2}	Mode	$\log(m)$
$\boxed{\boldsymbol{m}}$	Median	$\log(m)$
$\sqrt{e^{\sigma^2}-1}$	Coefficient of variation	$\sigma/\log(m)$
	$LN4(\boldsymbol{m}, \boldsymbol{cv})$	
$m\sqrt{cv^2+1}$	Mean	$\log(m)$
$m^2\,cv^2\,(cv^2+1)$	Variance	$\log(cv^2+1)$
$m\,cv\sqrt{cv^2+1}$	Standard deviation	$\sqrt{\log(cv^2+1)}$
$m/(cv^2+1)$	Mode	$\log(m)$
$\boxed{\boldsymbol{m}}$	Median	$\log(m)$
$\boxed{\boldsymbol{cv}}$	Coefficient of variation	$\sqrt{\log(cv^2+1)}/\log(m)$
	$LN5(\boldsymbol{\mu}, \boldsymbol{\tau})$	
$e^{\mu+\frac{1}{2\tau}}$	Mean	$\boxed{\boldsymbol{\mu}}$
$e^{2\mu+\frac{1}{\tau}}[e^{\frac{1}{\tau}}-1]$	Variance	$1/\tau$
$e^{\mu+\frac{1}{2}\frac{1}{\tau}}\sqrt{e^{\frac{1}{\tau}}-1}$	Standard deviation	$\sqrt{1/\tau}$
$e^{\mu-\frac{1}{\tau}}$	Mode	μ
e^{μ}	Median	μ
$\sqrt{e^{\frac{1}{\tau}}-1}$	Coefficient of variation	$\sqrt{1/\tau}/\mu$
$1/\left(e^{2\mu+\frac{1}{\tau}}[e^{\frac{1}{\tau}}-1]\right)$	Precision	$\boxed{\boldsymbol{\tau}}$
	$LN6(\boldsymbol{m}, \boldsymbol{\sigma_g})$	
$m\,e^{\frac{1}{2}\log^2(\sigma_g)}$	Mean	$\log(m)$
$m^2e^{\log^2(\sigma_g)}[e^{\log^2(\sigma_g)}-1]$	Variance	$\log^2(\sigma_g)$
$m\sqrt{e^{\log^2(\sigma_g)}[e^{\log^2(\sigma_g)}-1]}$	Standard deviation	$\log(\sigma_g)$
$m/e^{\log^2(\sigma_g)}$	Mode	$\log(m)$
$\boxed{\boldsymbol{m}}$	Median	$\log(m)$
$\sqrt{\exp(\log^2(\sigma_g))-1}$	Coefficient of variation	$\log(\sigma_g)/\log(m)$
	$LN7(\boldsymbol{\mu_N}, \boldsymbol{\sigma_N})$	
$\boxed{\boldsymbol{\mu_N}}$	Mean	$\log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right)$
σ_N^2	Variance	$\log(1+\sigma_N^2/\mu_N^2)$
$\boxed{\boldsymbol{\sigma_N}}$	Standard deviation	$\sqrt{\log(1+\sigma_N^2/\mu_N^2)}$
$\mu_N/(1+\sigma_N^2/\mu_N^2)^{3/2}$	Mode	$\log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right)$
$\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}$	Median	$\log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right)$
σ_N/μ_N	Coefficient of variation	$\sqrt{\log(1+\sigma_N^2/\mu_N^2)}/\log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right)$

Table 3.1: The available parameterisations for the log-normal distribution and their characterising quantities as functions of the respective parameters. See Figure 3.1 and section 3.3 for the meaning of the symbols.

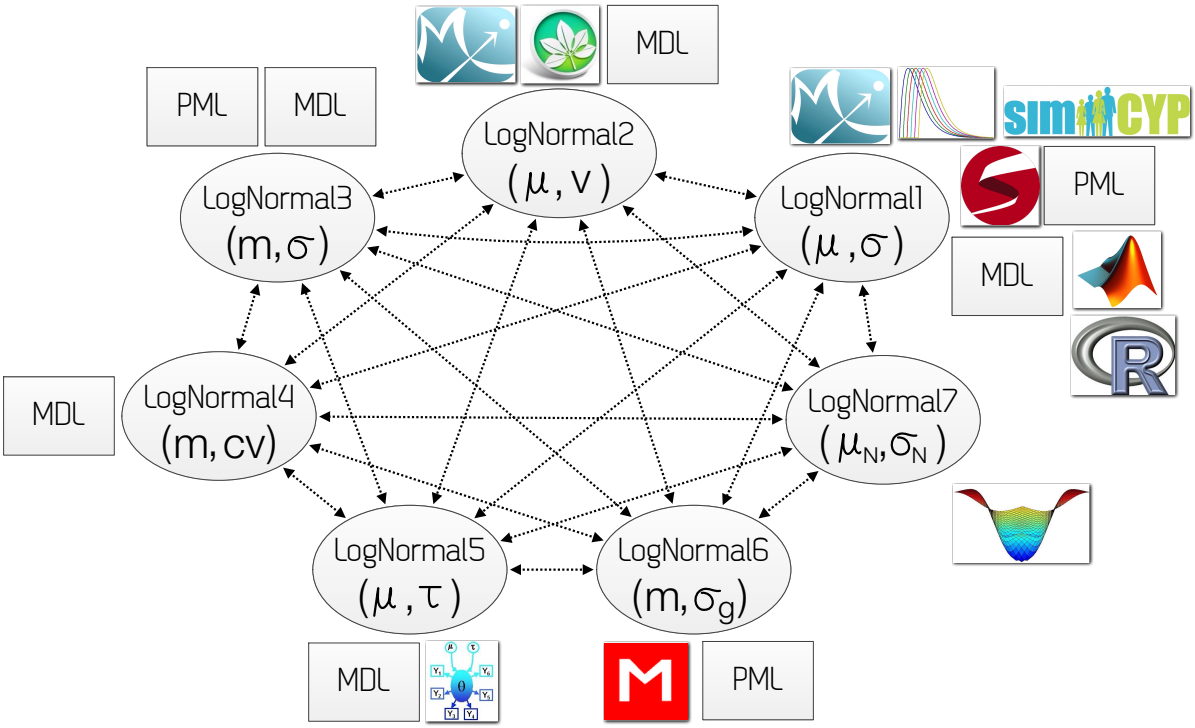

Figure 3.3: Re-parameterisation relationships implemented in ProbOnto and their support in target tools.

3.4 Bernoulli distribution

- Bernoulli1(p)

$$\binom{n}{k} p^k (1-p)^{n-k}$$

- Bernoulli2(α)

$$\binom{n}{k} \left(\text{logit}^{-1}(\alpha)\right)^k \left(1 - \text{logit}^{-1}(\alpha)\right)^{n-k}$$

Re-parameterisation formulas

- **Bernoulli1**(p) $\rightarrow$ **Bernoulli2**(α) : $\alpha = \log(p/(1-p))$
  Bernoulli2(α) $\rightarrow$ **Bernoulli1**(p) : $p = \exp(\alpha)/(1+\exp(\alpha))$

3.5 Binomial distribution

- Binomial1(p)

$$\begin{cases} q = (1-p) & \text{for } k = 0 \\ p & \text{for } k = 1 \end{cases}$$

- Binomial2(α)

$$\begin{cases} \text{logit}^{-1}(\alpha) & \text{for } k = 1 \\ 1 - \text{logit}^{-1}(\alpha) & \text{for } k = 0 \end{cases}$$

Re-parameterisation formulas

- **Binomial1**(p) $\rightarrow$ **Binomial2**(α) : $\alpha = \log(p/(1-p))$
  Binomial2(α) $\rightarrow$ **Binomial1**(p) : $p = \exp(\alpha)/(1+\exp(\alpha))$

3.6 Exponential distribution

- Exponential1(λ)

$$\lambda e^{-\lambda x}$$

- Exponential2(β)

$$1/\beta e^{-x/\beta}$$

Re-parameterisation formulas

- **Exponential1**(λ) $\rightarrow$ **Exponential2**(β) : $\beta = 1/\lambda$
  Exponential2(β) $\rightarrow$ **Exponential1**(λ) : $\lambda = 1/\beta$

3.7 Gamma distribution

- Gamma1(k, θ)

$$\frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$$

- Gamma2(r, μ)

$$\frac{\mu^r x^{r-1} e^{-\mu x}}{\Gamma(r)}$$

Re-parameterisation formulas

- **Gamma1**(k, θ) $\rightarrow$ **Gamma2**(r, μ) : $r = k; \quad \mu = 1/\theta$
  Gamma2(r, μ) $\rightarrow$ **Gamma1**(k, θ) : $k = r; \quad \theta = 1/\mu$

3.8 Generalized Gamma distribution

- GeneralizedGamma1(a, d, p)

$$\frac{p/a^d}{\Gamma(d/p)} x^{d-1} e^{-(x/a)^p}$$

- GeneralizedGamma2(a, b, c, k)

$$\frac{k(x-a)^{kc-1}}{b^{kc}\Gamma(c)} \exp\Big[-\Big(\frac{x-a}{b}\Big)^k\Big]$$

- GeneralizedGamma3(r, μ, β)

$$\frac{\beta}{\Gamma(r)} \mu^{\beta r} x^{\beta r - 1} \exp[-(\mu x)^\beta]$$

Re-parameterisation formulas

- **GeneralizedGamma1**(a, d, p) $\rightarrow$ **GeneralizedGamma3**(r, μ, β) : $r = d/p; \quad \beta = p; \quad \mu = 1/a$
  GeneralizedGamma3(r, μ, β) $\rightarrow$ **GeneralizedGamma1**(a, d, p) : $a = 1/\mu; \quad d = \beta r; \quad p = \beta$
  GeneralizedGamma2(a, b, c, k) $\rightarrow$ **GeneralizedGamma1**(a, d, p) : $a = 0; \quad d = kc$ and rename $k \rightarrow p, b \rightarrow a$

3.9 Generalized Poisson distribution

- GeneralizedPoisson1(θ, δ)

$$\frac{\theta(\theta + \delta k)^{k-1} \times e^{-\theta - \delta k}}{k!}$$

- GeneralizedPoisson2(μ, δ)

$$\frac{\mu(1-\delta)[\mu(1-\delta) + \delta k]^{k-1}}{k!} e^{-[\mu(1-\delta)+\delta k]}$$

Re-parameterisation formulas

- **GeneralizedPoisson1**(θ, δ) $\rightarrow$ **GeneralizedPoisson2**(μ, δ) : $\mu = \theta/(1-\delta)$
  GeneralizedPoisson2(μ, δ) $\rightarrow$ **GeneralizedPoisson1**(θ, δ) : $\theta = \mu(1-\delta)$

3.10 Laplace distribution

- Laplace1(μ, b)

$$\frac{1}{2\,b} \exp\left(-\frac{|x-\mu|}{b}\right)$$

- Laplace2(μ, τ)

$$\frac{\tau}{2} \exp\left(-\tau|x-\mu|\right)$$

Re-parameterisation formulas

- **Laplace1**(μ, b) $\rightarrow$ **Laplace2**(μ, τ) : $\tau = 1/b$
  Laplace2(μ, τ) $\rightarrow$ **Laplace1**(μ, b) : $b = 1/\tau$

3.11 Logistic distribution

- Logistic1(μ, s)

$$\frac{\exp\left[-\dfrac{x-\mu}{s}\right]}{s\left(1 + \exp\left[-\dfrac{x-\mu}{s}\right]\right)^2}$$

- Logistic2(μ, τ)

$$\frac{\tau \exp[-\tau(x-\mu)]}{\left(1 + e^{-\tau(x-\mu)}\right)^2}$$

Re-parameterisation formulas

- **Logistic1**(μ, s) $\rightarrow$ **Logistic2**(μ, τ) : $\tau = 1/s$
  Logistic2(μ, τ) $\rightarrow$ **Logistic1**(μ, s) : $s = 1/\tau$

3.12 Log-Logistic distribution

- LogLogistic1(α, β)

$$\frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{(1+(x/\alpha)^{\beta})^2}$$

- LogLogistic2(λ, κ)

$$\frac{\lambda\kappa(\lambda x)^{\kappa-1}}{(1+(\lambda x)^{\kappa})^2}$$

Re-parameterisation formulas

- **LogLogistic1**(α, β) $\rightarrow$ **LogLogistic2**(λ, κ) : $\lambda = 1/\alpha; \quad \kappa = \beta$
  LogLogistic2(λ, κ) $\rightarrow$ **LogLogistic1**(α, β) : $\alpha = 1/\lambda; \quad \beta = \kappa$

3.13 Poisson distribution

- Poisson1(λ)

$$\frac{\lambda^k}{k!}e^{-\lambda}$$

- Poisson1(α)

$$\frac{1}{k!}\exp\big(k\alpha - \exp(\alpha)\big)$$

Re-parameterisation formulas

- **Poisson1**(λ) $\rightarrow$ **Poisson2**(α) : $\alpha = \log(\lambda)$
  Poisson2(α) $\rightarrow$ **Poisson1**(λ) : $\lambda = \exp(\alpha)$

3.14 Student T distribution

- StudentT2(μ, τ, k)

$$\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\sqrt{\frac{\tau}{k\pi}}\left[1+\frac{\tau}{k}(x-\mu)^2\right]^{-\frac{k+1}{2}}$$

- StudentT3(ν, μ, σ)

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}\sigma}\left(1+\frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{\nu+1}{2}}$$

Re-parameterisation formulas

- **StudentT2**(μ, τ, k) $\rightarrow$ **StudentT3**(ν, μ, σ) : $k = \nu; \quad \tau = 1/\sigma^2$
  StudentT3(ν, μ, σ) $\rightarrow$ **StudentT2**(μ, τ, k) : $\nu = k; \quad \sigma = 1/\sqrt{\tau}$

3.15 Weibull distribution

- Weibull1(λ, k)

$$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

- Weibull2(λ, v)

$$v\lambda\, x^{v-1} e^{-\lambda x^v}$$

Re-parameterisation formulas

- **Weibull1**(λ_{WB1}, k) $\rightarrow$ **Weibull2**(λ_{WB2}, v) : $k = v; \quad \lambda_{WB2} = 1/\lambda_{WB1}^k$
  Weibull2(λ_{WB2}, v) $\rightarrow$ **Weibull1**(λ_{WB1}, k) : $v = k; \quad \lambda_{WB1} = \lambda_{WB2}^{-1/v}$

Chapter 4

Use Examples

4.1 Count data models

4.1.1 Zero-inflated Poisson – ZIP

ProbOnto simplifies the encoding of many discrete models significantly. In UncertML unavailable distributions such as Generalized Poisson (GP), Zero-inflated Poisson (ZIP) and other models frequently used in pharmacometrics, see for example [Plan et al., 2009] and [Trocóniz et al., 2009], are now available and very easy to encode. This example illustrates that.

The essential elements of the model is the PMF, here provided in the log-transformed form,

$$\begin{cases} \log(P(Y=k)) = \log(1-p0) - \lambda + k\log(\lambda) - \text{factln}(k) & \text{if } k > 0 \\ \log(P(Y=k)) = \log(p0 + (1-p0)\exp(-\lambda)) & \text{otherwise} \end{cases}$$

and the definition of model parameters: the Poisson intensity, λ, and the probability of extra zeros, $p0$. In the case of explicitly encoded PMF the model becomes lengthly. This comes always with a risk of encoding bugs/typos.

Explicitly encoded PMF The complex conditional definition of this model within the `<PMF>` element reads

```
<mdef:ObservationModel blkId="om1">
    <mdef:Discrete>
        <mdef:CountData>
            <mdef:CountVariable symbId="y"/>
            <mdef:NumberCounts symbId="k"/>

            <mdef:PMF transform="log">
                <ct:Assign>
                    <math:Piecewise>
                        <!-- !!! 50 lines of code skipped here !!! -->
                        <!-- for encoding of the conditional PMF: -->
                        <!-- if (k > 0): log(1-p0)-lambda+k*log(lambda)-factln(k) -->
                        <!-- else: log(p0+(1-p0)*exp(-lambda)) -->
                    </math:Piecewise>
                </ct:Assign>
            </mdef:PMF>
        </mdef:CountData>
    </mdef:Discrete>
</mdef:ObservationModel>
```

Note also that the explicit encoding of PMF's can be used if the modeller wishes to encode a model not featured in ProbOnto. It therefore remains as an option in PharmML for any user-defined or other distribution.

Model encoded using ProbOnto In contrast to the first option, encoding of models supported by ProbOnto becomes very easy as only the code names for the distribution and its parameters need to be specified as the following code shows

```
<ObservationModel blkId="om1A">
    <Discrete>
        <CountData>
            <CountVariable symbId="y"/>
```

```
            <PMF transform="log">
                <Distribution>
                    <po:ProbOnto name="ZeroInflatedPoisson1">
                        <po:Parameter name="rate">
                            <ct:Assign>
                                <ct:SymbRef blkIdRef="pm1" symbIdRef="Lambda"/>
                            </ct:Assign>
                        </po:Parameter>
                        <po:Parameter name="probabilityOfZero">
                            <ct:Assign>
                                <ct:SymbRef blkIdRef="pm1" symbIdRef="P0"/>
                            </ct:Assign>
                        </po:Parameter>
                    </po:ProbOnto>
                </Distribution>
            </PMF>
        </CountData>
    </Discrete>
</ObservationModel>
```

4.1.2 Poisson with mixtures – PMIX

Rather then creating specific model types for e.g. mixture of Poisson models, we use a generic mixture model. For example for the mixture of two Poisson models

$$P(y_{ij} = k; \pi, \lambda_1, \lambda_2) \quad = \quad \pi \frac{e^{-\lambda_1}\lambda_1^k}{k!} + (1-\pi)\frac{e^{-\lambda_2}\lambda_2^k}{k!}$$

with $\lambda_1, \lambda_2 > 0$ and $\pi \in [0, 1]$, the *MixtureDistribution* can be used as the following code shows

```
        <PMF>
            <Distribution>
                <po:ProbOnto name="MixtureDistribution">
                    <!-- mixing weight -->
                    <po:Parameter name="weight">
                        <ct:Assign>
                            <ct:SymbRef symbIdRef="pi1"/>
                        </ct:Assign>
                    </po:Parameter>
                    <!-- lambda1 - Poisson intensity -->
                    <po:MixtureComponent name="Poisson1">
                        <po:Parameter name="rate">
                            <ct:Assign>
                                <ct:SymbRef symbIdRef="lambda1"/>
                            </ct:Assign>
                        </po:Parameter>
                    </po:MixtureComponent>
                    <!-- lambda2 - Poisson intensity -->
                    <po:MixtureComponent name="Poisson1">
                        <po:Parameter name="rate">
                            <ct:Assign>
                                <ct:SymbRef symbIdRef="lambda2"/>
                            </ct:Assign>
                        </po:Parameter>
                    </po:MixtureComponent>
                </po:ProbOnto>
            </Distribution>
        </PMF>
```

The `<MixtureComponent>` elements hold the mixtures in question, here Poisson with λ_1 and Poisson λ_2, and `<Parameter>`, π, the mixture probability or *weight*. The solution has the advantage to be extendable to any number of mixing components, m, [Forbes et al., 2011]. The parameter π becomes a vector, π_i with $\pi_i \in [0, 1], i = 1, \ldots, m$, and can be encoded as such using the `<Vector>` element.

Discussion The use of a generic *MixtureDistribution* seems well justified in this case. However, because reference literature exists for general Mixed Poisson regression models, [Wang et al., 1996], the introduction of such specialised mixture distribution could be considered for ProbOnto in future as well.

4.1.3 Other count data models

Growing number of count data related distributions means better coverage of this important type of models, see for example [Paule et al., 2012] and [Plan, 2014]. The models supported so far include

- Conway-Maxwell-Poisson (λ, ν)
- Double Poisson (μ, ϕ)
- Generalized Negative Binomial (θ, β, m)
- Generalized Poisson
  - Generalized Poisson 1 (θ, δ)
  - Generalized Poisson 2 (μ, δ)
  - Generalized Poisson 3 (μ, α)
- Inverse Binomial (k, p)
- Negative Binomial
  - Negative Binomial 1$(r,\ p)$
  - Negative Binomial 2 (λ, τ)
  - Negative Binomial 3 (μ, k)
  - Negative Binomial 4 (r, p)
  - Negative Binomial 5 (α, β)
  - Negative Binomial 6 (η, ϕ)
- Poisson (λ)
- Zero-Inflated Negative Binomial $(\lambda, \tau, p0)$
- Zero-Inflated Generalized Poisson $(\mu, \alpha, p0)$
- Zero-Inflated Poisson (λ, π)

As can be seen from the listing, two models come with multiple parameterisations. This aspect is described in detail in section 3. In short, their presence, with where available re-parameterisation formulas, allows for

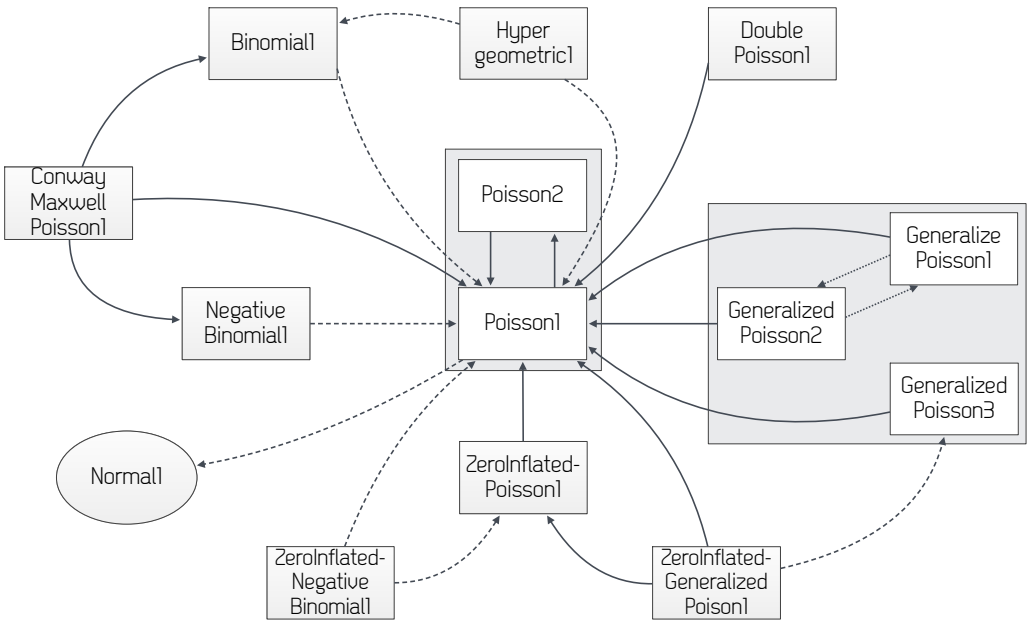

Figure 4.1: Count data models and the relationships between models stored in ProbOnto which are a subset of the network shown in Figure 2.3.

accounting for and dealing with varying support in target tools. The re-parameterisation formulas will be supported by the libPharmML making model translation to the target tool an easy task. Their relationships are visualised in Figure 4.1 which are a subset of the network shown in Figure 2.3.

4.2 Categorical data models

For categorical models we always first list in PharmML all categories, see below element `<ListOfCategories>`... `</ListOfCategories>` which informs the user/target tool about the number and identifiers of the categories in question. The probabilities vector $p_i, i = 1, \ldots, k$ is the only parameter and the user has two options. One can declare either

- explicitly the probabilities for all k categories or
- the $k-1$ probabilities, with the last probability, p_k, known assuming $\Sigma_i p_i = 1$.

Note that the order of the categories in `<ListOfCategories>` and the `<Parameter>` (where the probability vector, p_i, is encoded) elements, must be preserved. ☛

4.2.1 Nominal categorical model

Consider the following *Observation model* for nominal categorical data, [Swat et al., 2014]:

- Type of observed variable – discrete/categorical
- Category variable: y
- Set of categories: $\{1, 2, 3\}$
- Probabilities for category '1' and '2'

$$p1 := P(y = 1) = a1/(a1 + a2 + a3)$$
$$p2 := P(y = 2) = a2/(a1 + a2 + a3)$$

The following code shows how to implement the model starting with the general information: parameters involved and equations for the probabilities

```
<ObservationModel blkId="om1">

    <Discrete>
        <CategoricalData ordered="no">
            <!-- can alternatively be defined as individual parameters with IIV etc.-->
            <PopulationParameter symbId="a1"/>
            <PopulationParameter symbId="a2"/>
            <PopulationParameter symbId="a3"/>

            <PopulationParameter symbId="p1">
                <ct:Assign>
                    <!-- omitted formula p1 = a1/(a1+a2+a3) -->
                </ct:Assign>
            </PopulationParameter>
            <PopulationParameter symbId="p2">
                <ct:Assign>
                    <!-- omitted formula p2 = a2/(a1+a2+a3) -->
                </ct:Assign>
            </PopulationParameter>
```

then listing the categories and specifying the category variable

```
            <ListOfCategories>
                <Category symbId="cat1"/>
                <Category symbId="cat2"/>
                <Category symbId="cat3"/>
            </ListOfCategories>
            <CategoryVariable symbId="y"/>
```

and eventually defining the unordered categorical distribution, *CategoricalNonordered* in ProbOnto, with the parameter

- `categoryProb` – event probabilities vector, $p_1, \ldots, p_k$

with the number of categories, here equal $k = 3$, which can be inferred from the length of the `<ListOfCategories>`. The PMF reads then

```
                <PMF>
                    <Distribution>
                        <po:ProbOnto name="CategoricalNonordered1">
                            <!-- category probabilities - a vector of length 2 (=k-1) -->
                            <po:Parameter name="categoryProb">
                                <ct:Assign>
                                    <ct:Vector>
                                        <ct:VectorElements>
                                            <ct:SymbRef symbIdRef="p1"/>
                                            <ct:SymbRef symbIdRef="p2"/>
                                        </ct:VectorElements>
                                    </ct:Vector>
                                </ct:Assign>
                            </po:Parameter>
                        </po:ProbOnto>
                    </Distribution>
                </PMF>
            </CategoricalData>
        </Discrete>
    </ObservationModel>
```

Given that there are k categories, by default the specification of $k-1$ probabilities is sufficient assuming $\Sigma p_i = 1$.

4.3 Truncated distributions

Truncation bounds can be set using `<LowerTruncationBound>` and `<UpperTruncationBound>` elements which accept any expression, such as $X \sim \mathcal{N}(\mu, \sigma, lower = \mu - 1.96\,\sigma, upper = \mu + 1.96\,\sigma)$ which in PharmML reads

```
<IndividualParameter symbId="pTruncated">
    <Distribution>
        <po:ProbOnto name="Normal1">
            <po:Parameter name="mean">
                <ct:Assign>
                    <ct:SymbRef symbIdRef="mu"/>
                </ct:Assign>
            </po:Parameter>
            <po:Parameter name="stdev">
                <ct:Assign>
                    <ct:SymbRef symbIdRef="sigma"/>
                </ct:Assign>
            </po:Parameter>
            <po:LowerTruncationBound>
                <ct:Assign>
                    <math:Binop op="minus">
                        <ct:SymbRef symbIdRef="mu"/>
                        <math:Binop op="times">
                            <ct:Real>1.96</ct:Real>
                            <ct:SymbRef symbIdRef="sigma"/>
                        </math:Binop>
                    </math:Binop>
                </ct:Assign>
            </po:LowerTruncationBound>
            <po:UpperTruncationBound>
                <ct:Assign>
                    <math:Binop op="plus">
                        <ct:SymbRef symbIdRef="mu"/>
                        <math:Binop op="times">
                            <ct:Real>1.96</ct:Real>
                            <ct:SymbRef symbIdRef="sigma"/>
                        </math:Binop>
                    </math:Binop>
                </ct:Assign>
            </po:UpperTruncationBound>
        </po:ProbOnto>
    </Distribution>
</IndividualParameter>
```

The following Figure 4.2 illustrates few examples of truncated normal distribution on the interval [-2.5,1]

Note, that the normal distribution has been selected for demonstration purposes how to use the truncation bounds. ProbOnto features in its collection the $TruncatedNormal1(\mu, \sigma, a, b)$ distribution, which could be used instead.

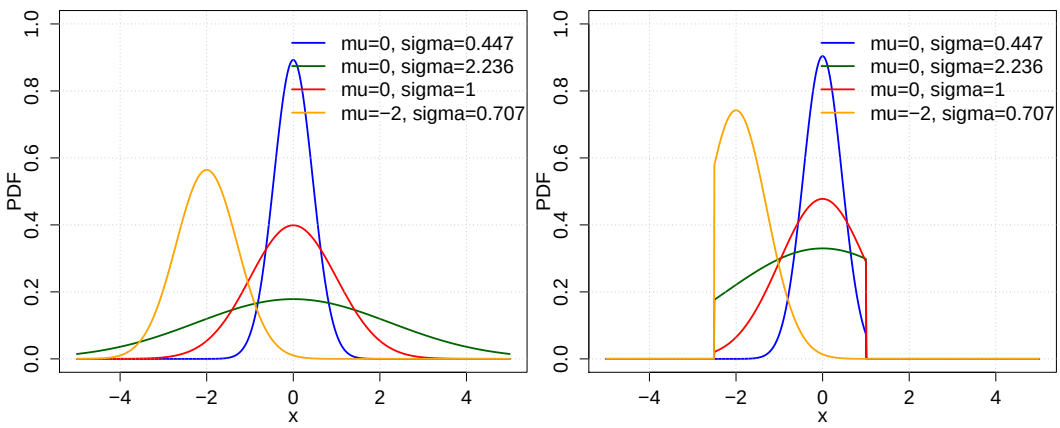

Figure 4.2: Truncated normal distribution, N1. The truncated density plots on the right hand side were performed using the *truncnorm* function of the *dtruncnorm* R-package.

4.4 Complete PharmML model

At last, we provide a complete PharmML coded model – the Poisson model for count data defined as follows

- Type of observed variable – discrete/count
- Category variable: Y
- PMF

$$P(y_{ij} = k; \lambda) \quad = \quad \frac{e^{-\lambda}\lambda^k}{k!}$$

Its implementation reads

```
<?xml version="1.0" encoding="UTF-8"?>
<PharmML xmlns:xsi="http://www.w3.org/2001/XMLSchema-instance"
    xmlns="http://www.pharmml.org/pharmml/0.9/PharmML"
    xsi:schemaLocation="http://www.pharmml.org/pharmml/0.9/PharmML
    http://www.pharmml.org/pharmml/0.9/PharmML"
    xmlns:ct="http://www.pharmml.org/pharmml/0.9/CommonTypes"
    xmlns:po="http://www.pharmml.org/probonto/ProbOnto"
    xmlns:mdef="http://www.pharmml.org/pharmml/0.9/ModelDefinition"
    implementedBy="MJS" writtenVersion="0.9" id="i1">

    <ct:Name>Poisson_model</ct:Name>

    <mdef:ModelDefinition>
        <mdef:ParameterModel blkId="pm1">
            <mdef:Parameter symbId="lambda"/>
        </mdef:ParameterModel>
        <mdef:ObservationModel blkId="om1">
            <mdef:Discrete>
                <mdef:CountData>
                    <mdef:CountVariable symbId="Y"/>
                    <mdef:PMF>
                        <mdef:Distribution>
                            <po:ProbOnto name="Poisson1">
                                <po:Parameter name="rate">
                                    <ct:Assign>
                                        <ct:SymbRef blkIdRef="pm1" symbIdRef="lambda"/>
                                    </ct:Assign>
                                </po:Parameter>
                            </po:ProbOnto>
                        </mdef:Distribution>
                    </mdef:PMF>
                </mdef:CountData>
            </mdef:Discrete>
        </mdef:ObservationModel>
    </mdef:ModelDefinition>
</PharmML>
```

In PharmML, such model can be extended by a declaration of an explicit or dataset sourced trail design and modelling steps, such as simulation, estimation or optimal trial design – for brevity these elements have been omitted. For full examples visit `ddmore.eu` or `pharmml.org`.

4.5 Samples – empirical distributions support

Encoding of empirical distributions in PharmML requires the introduction of a new attribute, `type`, new code names declared via the `name` attribute and two new elements. The following list summarises the extensions

- new elements
  - `<Realisation>` – allowing assignment of a symbol identifier used later for mapping with data columns in the dataset
  - `<Weight>` – samples weight element allowing assignment of a symbol identifier mapped to the weight in the dataset
- new `type` (optional) attribute with values *univariate* and *multivariate*, on the `<ProbOnto>` element – to be used with samples to simply the handling of empirical distributions
- new distribution code names *RandomSample, SystematicSample, UnknownSample*

Note, that we keep the UncertML naming convention based on which the empirical distributions are defined in PharmML [UncertML Team, 2014].

4.5.1 Two modes of operation

There are two ways samples can be stored and referred to in PharmML, see Table 4.2:

- Option 1: Data stored together with the corresponding parameter
- Option 2: Data stored in `<TrialDesign>`

Both encoding result in exact the same model and the mapping between data columns and parameters is essential to assure complete model definition (if only one column is given, mapping is optional, see Table 4.2 alternative **Option 1** version). For the initial discussion consider the following dataset

POP_K
0.10
0.23
0.3

Option 1: Data stored together with parameter

```
<ParameterModel blkId="pm1">

    <PopulationParameter symbId="samples_POP_K"/>
    <PopulationParameter symbId="POP_K">
        <Distribution>
            <po:ProbOnto type="univariate" name="RandomSample">
                <po:Realisation>
                    <ct:Assign>
                        <ct:SymbRef symbIdRef="samples_POP_K"/>
                    </ct:Assign>
                </po:Realisation>

                <po:ColumnMapping>
                    <ds:ColumnRef columnIdRef="pop_K"/>
                    <ct:SymbRef symbIdRef="samples_POP_K"/>
                </po:ColumnMapping>
                <ds:DataSet>
                    <ds:Definition>
                        <ds:Column columnId="pop_K" valueType="real" columnNum="1"/>
                    </ds:Definition>
                    <ds:Table>
                        <ds:Row><ct:Real>0.10</ct:Real></ds:Row>
                        <!-- other values omitted -->
                    </ds:Table>
                </ds:DataSet>
            </po:ProbOnto>
        </Distribution>
    </PopulationParameter>
```

```
        <!-- ALTERNATIVE - short version -->
        <PopulationParameter symbId="POP_K">
            <Distribution>
                <po:ProbOnto name="RandomSample">
                    <ds:DataSet>
                        <ds:Table>
                            <ds:Row><ct:Real>0.10</ct:Real></ds:Row>
                            <!-- other values omitted -->
                        </ds:Table>
                    </ds:DataSet>
                </po:ProbOnto>
            </Distribution>
        </PopulationParameter>

    </ParameterModel>
```

Option 2: Data stored in `<TrialDesign>`

```
<ParameterModel blkId="pm1">
    <PopulationParameter symbId="samples_POP_K"/>
    <PopulationParameter symbId="POP_K">
        <Distribution>
            <po:ProbOnto name="RandomSample">
                <po:Realisation>
                    <ct:Assign>
                        <ct:SymbRef symbIdRef="samples_POP_K"/>
                    </ct:Assign>
                </po:Realisation>
            </po:ProbOnto>
        </Distribution>
    </PopulationParameter>
</ParameterModel>

<TrialDesign xmlns="http://www.pharmml.org/pharmml/0.9/TrialDesign">
    <ExternalDataSet oid="RdataSet1">
        <ColumnMapping>
            <ds:ColumnRef columnIdRef="pop_K"/>
            <ct:SymbRef blkIdRef="pm1" symbIdRef="samples_POP_K"/>
        </ColumnMapping>
        <ds:DataSet>
            <ds:Definition>
                <ds:Column columnId="pop_K" valueType="real" columnNum="1"/>
            </ds:Definition>
            <ds:Table>
                <ds:Row><ct:Real>0.10</ct:Real></ds:Row>
                <!-- other values omitted -->
            </ds:Table>
        </ds:DataSet>
    </ExternalDataSet>
</TrialDesign>
```

Table 4.2: Two ways to encode samples. **Option 1**: data stored together with parameter declaration in `<ModelDefinition>`. Two version are shown: (top) with full data/model mapping and dataset column declaration, (bottom) short form when only one dataset is required – mapping and column declaration are redundant. **Option 2**: data stored in the `<TrialDesign>` – mapping is mandatory even for single column.

4.5.2 Mapping without `weight`

Option 1

(See the use case *bayesianHierarchical/example3423.xml* using this option.)

Sometimes the `weight` is not defined and the assumption is that all values are equally probable with $p = 1/n$ where n is equal the row number, see Table 4.3.

POP_V
5.050013
5.064166
5.078318
...

Table 4.3: A dataset represented using equiprobable samples. Because no `weight` is defined, the assumption is that all values are equiprobable with $p = 1/n$ where n is equal the row number.

```
<PopulationParameter symbId="POP_V">
    <Distribution>
        <po:ProbOnto name="RandomSample">
            <ds:DataSet>
                <ds:Table>
                    <ds:Row><ct:Real>5.050013</ct:Real></ds:Row>
                    <ds:Row><ct:Real>5.064166</ct:Real></ds:Row>
                    <ds:Row><ct:Real>5.078318</ct:Real></ds:Row>
                </ds:Table>
                <!-- <ds:ExternalFile oid="sxtData">
            <ds:path>POP_V.csv</ds:path>
        </ds:ExternalFile>-->
            </ds:DataSet>
        </po:ProbOnto>
    </Distribution>
</PopulationParameter>
```

Here the short version was used which is applicable when only one data column in required. Providing mapping and column definition is in such cases redundant, unless the data is stored in `<TrialDesign>`, see below.

Option 2

Figure 4.3 shows how to store and map dataset with samples declared in `<TrialDesign>` to a parameter defined in `<ParameterModel>`, here POP_V. In this case column mapping and column declaration are required.

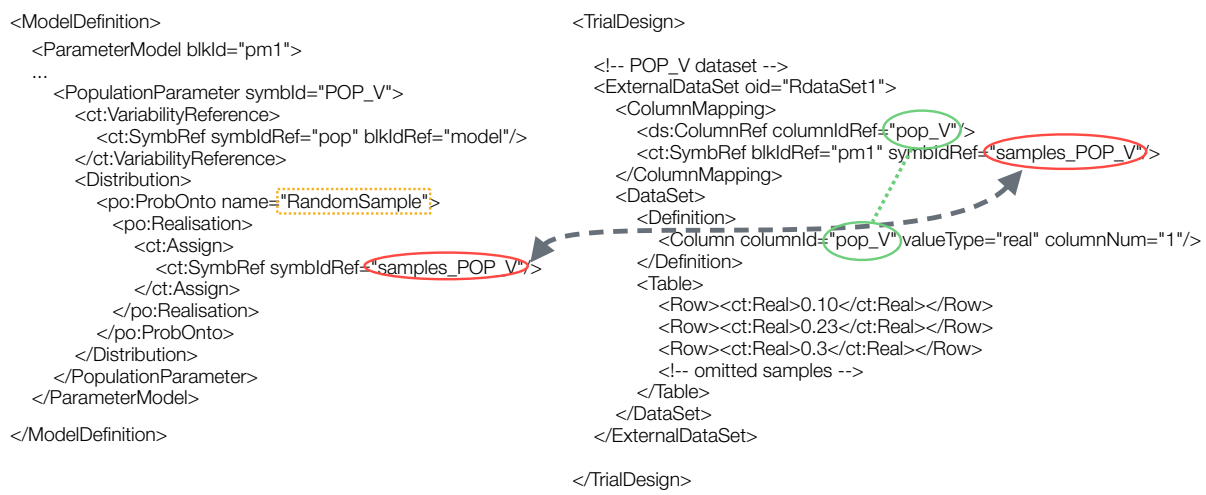

Figure 4.3: Mapping of samples without weights – explained graphically.

4.5.3 Mapping with `weight`

Option 1

(See the use case *bayesianHierarchical/example3423.xml* using this option.)

In the case when single variables are defined with `weight`, p.

P	POP_K
0.25	0.10
0.25	0.23
0.5	0.3

Table 4.4: This data set represents sample use case with `weight` parameter, P.

Here the according samples are encoded directly with the parameter of interest. The mappings are straightforward; the realisation, `samples_POP_K`, and weight, `p`, are mapped to the columns `pop_K` and `P`, respectively. The samples can be encoded either inline, using `<Table>`, or in external datasets referenced within the `<ExternalFile>` element.

```
<PopulationParameter symbId="samples_POP_K"/>
<PopulationParameter symbId="p"/>
<PopulationParameter symbId="POP_K">
    <Distribution>
        <po:ProbOnto name="RandomSample">
            <po:Realisation>
                <ct:Assign>
                    <ct:SymbRef symbIdRef="samples_POP_K"/>
                </ct:Assign>
            </po:Realisation>
            <po:Weight>
                <ct:Assign>
                    <ct:SymbRef symbIdRef="p"/>
                </ct:Assign>
            </po:Weight>

            <po:ColumnMapping xmlns="http://www.pharmml.org/pharmml/0.9/TrialDesign">
                <ds:ColumnRef columnIdRef="P"/>
                <ct:SymbRef symbIdRef="p"/>
            </po:ColumnMapping>
            <po:ColumnMapping>
                <ds:ColumnRef columnIdRef="pop_K"/>
                <ct:SymbRef symbIdRef="samples_POP_K"/>
            </po:ColumnMapping>
            <ds:DataSet>
                <ds:Definition>
                    <ds:Column columnId="P" valueType="real" columnNum="1"/>
                    <ds:Column columnId="pop_K" valueType="real" columnNum="2"/>
                </ds:Definition>
                <ds:Table>
                    <ds:Row><ct:Real>0.25</ct:Real><ct:Real>0.10</ct:Real></ds:Row>
                    <ds:Row><ct:Real>0.25</ct:Real><ct:Real>0.23</ct:Real></ds:Row>
                    <ds:Row><ct:Real>0.5</ct:Real><ct:Real>0.3</ct:Real></ds:Row>
                </ds:Table>
                <!-- <ds:ExternalFile oid="sxtData">
                    <ds:path>POP_K_sample.csv</ds:path>
                </ds:ExternalFile>-->
            </ds:DataSet>
        </po:ProbOnto>
    </Distribution>
</PopulationParameter>
```

Option 2

This is an example for a model with two parameters POP_K and POP_K drawn from a random sample with probability p, see Table 4.5 and Figure 4.4 for a graphical explanation,

- `RandomSample` is used with
- `weight` p declared as `<PopulationParameter>` applies to each row.

POP_V	POP_K	p
8	0.10	0.25
14	0.23	0.25
32	0.3	0.5

Table 4.5: Test data set represented using samples. Weight, p, applies to each pair of values.

PharmML implementation of the parameter model with distributions defined using *RandomSample* with *weight* is as following

```
<PopulationParameter symbId="samples_POP_K"/>
<PopulationParameter symbId="p"/>
<PopulationParameter symbId="POP_K">
    <ct:VariabilityReference>
        <ct:SymbRef symbIdRef="pop" blkIdRef="vm1"/>
```

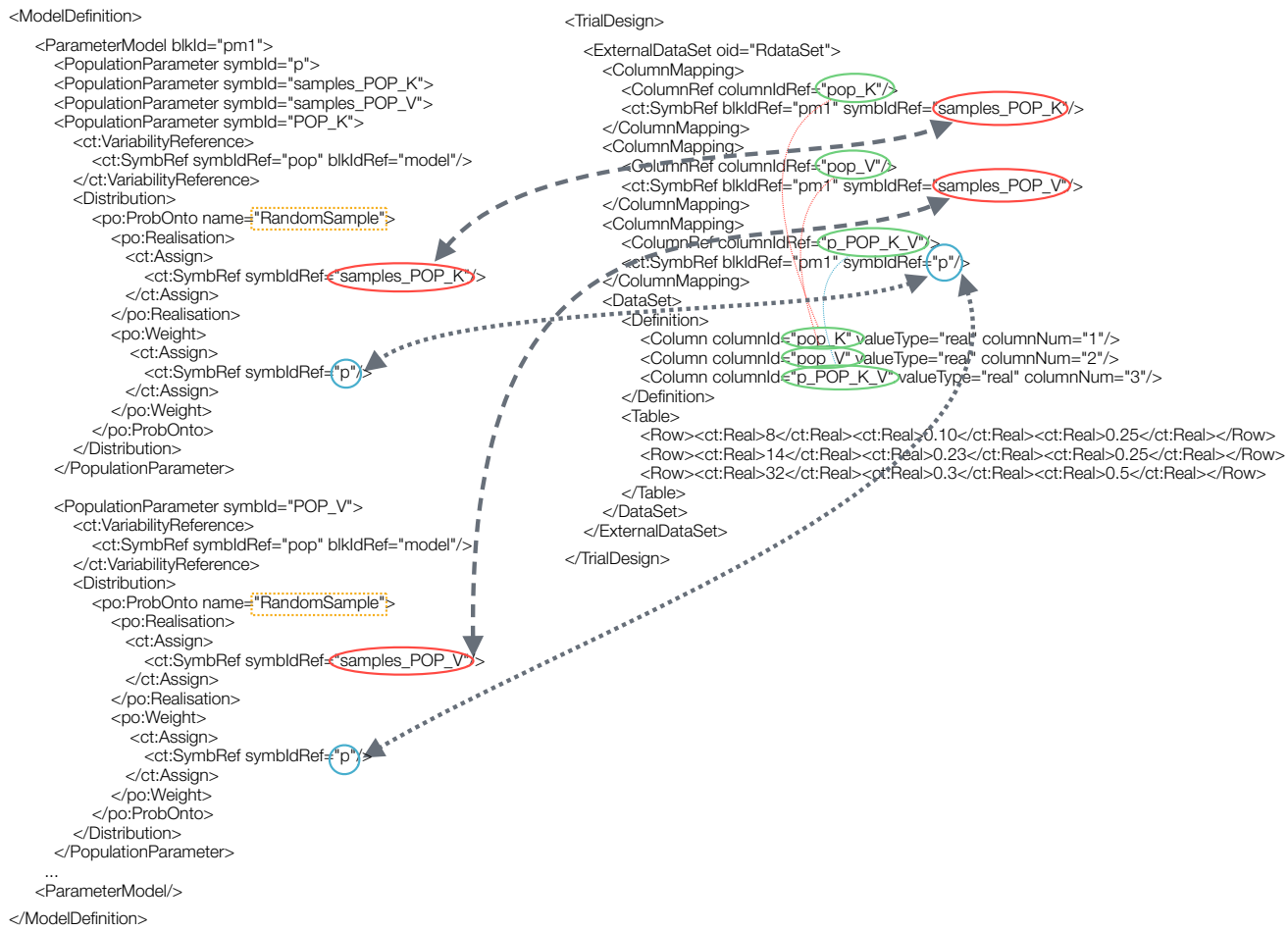

Figure 4.4: Mapping of samples with weights (option 2) – explained graphically.

```
            </ct:VariabilityReference>
            <Distribution>
                <po:ProbOnto name="RandomSample">
                    <po:Realisation>
                        <ct:Assign>
                            <ct:SymbRef symbIdRef="samples_POP_K"/>
                        </ct:Assign>
                    </po:Realisation>
                    <po:Weight>
                        <ct:Assign>
                            <ct:SymbRef symbIdRef="p"/>
                        </ct:Assign>
                    </po:Weight>
                </po:ProbOnto>
            </Distribution>
        </PopulationParameter>

        <PopulationParameter symbId="samples_POP_V"/>
        <PopulationParameter symbId="POP_V">
            <ct:VariabilityReference>
                <ct:SymbRef symbIdRef="pop" blkIdRef="vm1"/>
            </ct:VariabilityReference>
            <Distribution>
                <po:ProbOnto name="RandomSample">
                    <po:Realisation>
                        <ct:Assign>
                            <ct:SymbRef symbIdRef="samples_POP_V"/>
                        </ct:Assign>
                    </po:Realisation>
                    <po:Weight>
                        <ct:Assign>
                            <ct:SymbRef symbIdRef="p"/>
                        </ct:Assign>
                    </po:Weight>
                </po:ProbOnto>
```

```
            </Distribution>
        </PopulationParameter>
```

Then the dataset is defined in the `<TrialDesign>` as shown below with the proper column mapping.

```
<TrialDesign xmlns="http://www.pharmml.org/pharmml/0.9/TrialDesign">
    <ExternalDataSet oid="RdataSet3">
        <ColumnMapping>
            <ds:ColumnRef columnIdRef="pop_K"/>
            <ct:SymbRef blkIdRef="pm1" symbIdRef="samples_POP_K"/>
        </ColumnMapping>
        <ColumnMapping>
            <ds:ColumnRef columnIdRef="pop_V"/>
            <ct:SymbRef blkIdRef="pm1" symbIdRef="samples_POP_V"/>
        </ColumnMapping>
        <ColumnMapping>
            <ds:ColumnRef columnIdRef="p_pop_V_K"/>
            <ct:SymbRef blkIdRef="pm1" symbIdRef="p"/>
        </ColumnMapping>
        <ds:DataSet>
            <ds:Definition>
                <ds:Column columnId="pop_K" valueType="real" columnNum="1"/>
                <ds:Column columnId="pop_V" valueType="real" columnNum="2"/>
                <ds:Column columnId="p_pop_V_K" valueType="real" columnNum="3"/>
            </ds:Definition>
            <ds:Table>
                <ds:Row><ct:Real>8</ct:Real><ct:Real>0.10</ct:Real><ct:Real>0.25</ct:Real></ds:Row>
                <ds:Row><ct:Real>14</ct:Real><ct:Real>0.23</ct:Real><ct:Real>0.25</ct:Real></ds:Row>
                <ds:Row><ct:Real>32</ct:Real><ct:Real>0.3</ct:Real><ct:Real>0.5</ct:Real></ds:Row>
            </ds:Table>
            <!-- <ds:ExternalFile oid="sxtData">
                    <ds:path>POP_V_K_sample.csv</ds:path>
                  </ds:ExternalFile>-->
        </ds:DataSet>
    </ExternalDataSet>
</TrialDesign>
```

4.5.4 Mapping using vectors

(See the use case *bayesianHierarchical/example3421dep_NM.xml* using this option.)

In the case the modeller wants to encode a parameter in vector form, here POP_K and POP_V as vector elements of POP_K_V (see use cases in the *bayesianHierarchical* example folder) there exist a possibility to map the vector to sample elements, see Figure 4.5 for a graphical explanation or the code snippet below.

First POP_K_V is encoded as the following snippet shows

```
        <PopulationParameter symbId="samples_POP_K_V"/>
        <PopulationParameter symbId="p_POP_K_V"/>
        <PopulationParameter symbId="POP_K_V">
            <ct:VariabilityReference>
                <ct:SymbRef symbIdRef="pop" blkIdRef="vm1"/>
            </ct:VariabilityReference>
            <Distribution>
                <po:ProbOnto type="multivariate" name="RandomSample">
                    <po:Realisation>
                        <ct:Assign>
                            <ct:SymbRef symbIdRef="samples_POP_K_V"/>
                        </ct:Assign>
                    </po:Realisation>
                    <po:Weight>
                        <ct:Assign>
                            <ct:SymbRef symbIdRef="p_POP_K_V"/>
                        </ct:Assign>
                    </po:Weight>
                </po:ProbOnto>
            </Distribution>
        </PopulationParameter>
```

with `<Realisation>` assigned symbol id *samples_POP_K_V*. Its elements are then mapped in trial design section to *pop_K* and *pop_V* columns using `<ColumnMapping>`

```
    <ExternalDataSet oid="RdataSet4">
        <ColumnMapping>
            <ColumnRef xmlns="http://www.pharmml.org/pharmml/0.9/Dataset" columnIdRef="pop_K"/>
            <ct:Assign>
```

```
<ModelDefinition>
  <ParameterModel blkId="pm1">

    <PopulationParameter symbId="samples_POP_K_V">
    <PopulationParameter symbId="p_POP_K_V">
    <PopulationParameter symbId="POP_K_V">
      <ct:VariabilityReference>
        <ct:SymbRef symbIdRef="pop" blkIdRef="model"/>
      </ct:VariabilityReference>
      <Distribution>
        <po:ProbOnto type="multivariate" name="RandomSample">
          <po:Realisation>
            <ct:Assign>
              <ct:SymbRef symbIdRef="samples_POP_V"/>
            </ct:Assign>
          </po:Realisation>
          <po:Weight>
            <ct:Assign>
              <ct:SymbRef symbIdRef="p_POP_K_V"/>
            </ct:Assign>
          </po:Weight>
        </po:ProbOnto>
      </Distribution>
    </PopulationParameter>

    <PopulationParameter symbId="POP_K">
      <ct:Assign>
        <Uniop op="exp">
          <ct:VectorSelector>
            <ct:SymbRef symbIdRef="POP_K_V"/>
            <ct:Cell>
              <ct:Int>1</ct:Int>
            </ct:Cell>
          </ct:VectorSelector>
        </Uniop>
      </ct:Assign>
    </PopulationParameter>

    <PopulationParameter symbId="POP_V">
      ... omitted, analog as above
    </PopulationParameter>

  </ParameterModel>
</ModelDefinition>
```

```
<TrialDesign>
  <ExternalDataSet oid="RdataSet">
    <ColumnMapping>
      <ColumnRef columnIdRef="samples_POP_K"/>
      <ct:Assign>
        <ct:VectorSelector>
          <ct:SymbRef symbIdRef="samples_POP_K_V" blkIdRef="pm1"/>
          <ct:Cell>
            <ct:Int>1</ct:Int>
          </ct:Cell>
        </ct:VectorSelector>
      </ct:Assign>
    </ColumnMapping>
    <ColumnMapping>
      <ColumnRef columnIdRef="samples_POP_V"/>
      <ct:Assign>
        <ct:VectorSelector>
          <ct:SymbRef symbIdRef="samples_POP_K_V" blkIdRef="pm1"/>
          <ct:Cell>
            <ct:Int>2</ct:Int>
          </ct:Cell>
        </ct:VectorSelector>
      </ct:Assign>
    </ColumnMapping>
    <ColumnMapping>
      <ColumnRef columnIdRef="p_POP_K_V"/>
      <ct:SymbRef blkIdRef="pm1" symbIdRef="p_POP_K_V"/>
    </ColumnMapping>
    <DataSet>
      <Definition>
        <Column columnId="samples_POP_K" valueType="real" columnNum="1"/>
        <Column columnId="samples_POP_V" valueType="real" columnNum="2"/>
        <Column columnId="p_POP_K_V" valueType="real" columnNum="3"/>
      </Definition>
      <Table>
        <Row><ct:Real>2.006</ct:Real><ct:Real>5.050</ct:Real><ct:Real>0.033</ct:Real></Row>
        <Row><ct:Real>2.045</ct:Real><ct:Real>5.050</ct:Real><ct:Real>0.100</ct:Real></Row>
        <Row><ct:Real>2.084</ct:Real><ct:Real>5.050</ct:Real><ct:Real>0.100</ct:Real></Row>
        <Row><ct:Real>2.123</ct:Real><ct:Real>5.050</ct:Real><ct:Real>0.200</ct:Real></Row>
        <!-- omitted rows -->
      </Table>
    </DataSet>
  </ExternalDataSet>
</TrialDesign
```

Figure 4.5: Mapping of samples with vectors and weights – explained graphically.

```
                <ct:VectorSelector>
                    <ct:SymbRef symbIdRef="samples_POP_K_V" blkIdRef="pm1"/>
                    <ct:Cell>
                        <ct:Int>1</ct:Int>
                    </ct:Cell>
                </ct:VectorSelector>
            </ct:Assign>
        </ColumnMapping>
        <ColumnMapping>
            <ColumnRef xmlns="http://www.pharmml.org/pharmml/0.9/Dataset" columnIdRef="pop_V"/>
            <ct:Assign>
                <ct:VectorSelector>
                    <ct:SymbRef symbIdRef="samples_POP_K_V" blkIdRef="pm1"/>
                    <ct:Cell>
                        <ct:Int>2</ct:Int>
                    </ct:Cell>
                </ct:VectorSelector>
            </ct:Assign>
        </ColumnMapping>
        <ColumnMapping>
            <ColumnRef xmlns="http://www.pharmml.org/pharmml/0.9/Dataset" columnIdRef="p_POP_K_V"/>
            <ct:SymbRef symbIdRef="p_POP_K_V" blkIdRef="pm1"/>
        </ColumnMapping>
        <DataSet xmlns="http://www.pharmml.org/pharmml/0.9/Dataset">
            <Definition>
                <Column columnId="pop_K" valueType="real" columnNum="1"/>
                <Column columnId="pop_V" valueType="real" columnNum="2"/>
                <Column columnId="p_pop_K_V" valueType="real" columnNum="3"/>
            </Definition>
            <Table>
                <Row><ct:Real>2.006510</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.033</ct:Real></Row>
                <Row><ct:Real>2.045465</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.100</ct:Real></Row>
                <Row><ct:Real>2.084421</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.100</ct:Real></Row>
                <Row><ct:Real>2.123377</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.200</ct:Real></Row>
                <!-- rows skipped -->
            </Table>
        </DataSet>
```

```
    </ExternalDataSet>
```

What is new in this version is the ability to map a particular vector element. Here the essential part is repeated to underline how the mapping of column *pop_V* and vector element *samples_POP_K_V*[2] works.

```
        <ColumnMapping>
            <ColumnRef columnIdRef="pop_V"/>
            <ct:Assign>
                <ct:VectorSelector>
                    <ct:SymbRef symbIdRef="samples_POP_K_V" blkIdRef="pm1"/>
                    <ct:Cell>
                        <ct:Int>2</ct:Int>
                    </ct:Cell>
                </ct:VectorSelector>
            </ct:Assign>
        </ColumnMapping>
    <ColumnMapping>
```

See also Figure 4.5 which shows the mappings graphically.

4.5.5 Samples in SO

The concept of encoding samples of empirical distributions, either as inline tables of external dataset, is very useful also in the SO context, [Terranova et al., 2016]. For example the posterior distribution of individual parameters estimates is captured using an empirical distribution, *RandomSample*. Here the values for three estimated parameters are encoded inline.

```
<PosteriorDistribution>
    <Distribution>
        <po:ProbOnto name="RandomSample">
            <ds:DataSet>
                <ds:Definition>
                    <ds:Column columnId="K" columnType="popParameter" valueType="real" columnNum="1"/>
                    <ds:Column columnId="V" columnType="popParameter" valueType="real" columnNum="2"/>
                    <ds:Column columnId="CL" columnType="popParameter" valueType="real" columnNum="3"/>
                </ds:Definition>
                <ds:Table>
                    <ds:Row><ct:Real>2.006510</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.033333</ct:Real></ds:Row>
                    <ds:Row><ct:Real>2.045465</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.100000</ct:Real></ds:Row>
                    <ds:Row><ct:Real>2.084421</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.100000</ct:Real></ds:Row>
                    <ds:Row><ct:Real>2.123377</ct:Real><ct:Real>5.050013</ct:Real><ct:Real>0.200000</ct:Real></ds:Row>
                    <ds:Row><ct:Real>2.162333</ct:Real><ct:Real>5.064166</ct:Real><ct:Real>0.100000</ct:Real></ds:Row>
                    <ds:Row><ct:Real>2.201288</ct:Real><ct:Real>5.064166</ct:Real><ct:Real>0.066667</ct:Real></ds:Row>
                    <!-- other sample rows skipped -->
                </ds:Table>
                <!--
                        <ds:ExternalFile oid="extDataId">
                            <ds:path>samples_KVCL.csv</ds:path>
                        </ds:ExternalFile>-->
            </ds:DataSet>
        </po:ProbOnto>
    </Distribution>
</PosteriorDistribution>
```

The alternative using external datasets is commented out. Note also that no mapping is required in this case, the symbol assigned to the attribute `columnId` identifies the parameter.

4.6 Distributions with default parameter values or without parameters

Distributions such as `Arcsine1`, `Benford1`, `Epanechnikov1`, `HalfCauchy1`, `HyperbolicSecant1`, `StandardCauchy1`, `StandardLogistic1`, `StandardNormal1`, `StandardTriangular1` and `StandardUniform1` are defined with default parameter values.

In PharmML, the specification of these parameters, using the `<Parameter>` tag, is therefore not necessary. E.g. the standard normal distribution, in mathematical notation commonly written as $\mathcal{N}(0,1)$ is simply encoded as

```
                <Distribution>
                    <po:ProbOnto name="StandardNormal1"/>
                </Distribution>
```

Chapter 5

Distributions in ProbOnto

This document contains overview tables of the supported univariate and multivariate distributions, the tool coverage and the code names.

All DDMoRe target tools distributions are covered, i.e. those used in (Monolix [Lixoft Team, 2014], NONMEM [Beal et al., 2009] and winBUGS [Spiegelhalter et al., 2003]), all distributions featured in STAN, [STAN Development Team, 2015b], and many others useful in pharmacometrics and beyond (Matlab Statistical Toolbox and R).

5.1 Univariate distributions – tool coverage and code names

5.1.1 Tool coverage – discrete distributions

ProbOnto 2.5	Parame- -ters	**UncertML 3.0**	**BUGS 1.4**	**Monolix 4.3**	**NONMEM 7.3**	**STAN 2.10.0**
Benford NEW	–	–	–	–	–	–
Bernoulli 1	p	y	y	–	–	y
Bernoulli 2	α	–	–	–	–	y
Beta-Binomial	n, α, β	–	–	–	–	y
Beta-Negative Binomial1 NEW	α, β, n	–	–	–	–	–
Binomial 1	n, p	y	y	–	–	y
Binomial 2	n, α	–	–	–	–	y
Borel NEW	μ	–	–	–	–	–
Categorical ordered	$p_1, ..., p_k$	y	y	–	–	–
Categorical nonordered	$p_1, ..., p_k$	y	y	–	–	y
Conway-Maxwell-Poisson1	λ, ν	–	–	–	–	–
Double Poisson1	μ, ϕ	–	–	–	–	–
Generalized Negative Binomial	θ, β, m	–	–	–	–	–
Generalized Poisson 1	θ, δ	–	–	–	–	–
Generalized Poisson 2	μ, δ	–	–	–	–	–
Generalized Poisson 3	μ, α	–	–	–	–	–
Geometric	p	y	–	–	–	–
Hypergeometric	N, K, n	y	–	–	–	y
Inverse Binomial	k, p	–	–	–	–	–
Negative Binomial 1	r, p	–	y	–	–	–
Negative Binomial 2	λ, τ	–	–	–	–	–
Negative Binomial 3	μ, ϕ	–	–	–	–	y
Negative Binomial 4	r, p	y	–	–	–	–
Negative Binomial 5	α, β	–	–	–	–	y
Negative Binomial 6	η, ϕ	–	–	–	–	y
Ordered Logistic	$c_1, ..., c_{K-1}, \eta$	–	–	–	–	y
Poisson 1	λ	y	y	–	–	y
Poisson 2	α	–	–	–	–	y
Uniform Discrete 1	a, b, n	–	–	–	–	–
Uniform Discrete 2	$a = 0, n$	–	–	–	–	–
Weibull Discrete NEW	$a = 0, n$	–	–	–	–	–
Zero-Inflated Negative Binomial	$\lambda, \tau, p0$	–	–	–	–	–
Zero-Inflated Generalized Poisson	$\mu, \alpha, p0$	–	–	–	–	–
Zero-Inflated Poisson	λ, π	–	–	–	–	–
Zeta NEW	α	–	–	–	–	–
Zipf NEW	α, n	–	–	–	–	–

Table 5.1: Univariate discrete distributions supported in ProbOnto with overview of tool support.

5.1.2 Tool coverage – continuous distributions

ProbOnto 2.5	Parame- -ters	**UncertML 3.0**	**BUGS 1.4**	**Monolix 4.3**	**NONMEM 7.3**	**STAN 2.10.0**
Amoroso NEW	a, θ, α, β	–	–	–	–	–
Arcsine 1 NEW	–	–	–	–	–	–

Arcsine 2 NEW	a, b	–	–	–	–	–
Asymmetric Laplace NEW	μ, σ	–	–	–	–	–
Beta	α, β	y	y	y	–	y
Birnbaum-Saunders	n, α, γ	–	–	–	–	–
Burr	α, c, k	–	–	–	–	–
Cauchy	x_0, γ	y	–	–	–	y
Chi NEW	k	–	–	–	–	–
Chi-Squared	k	y	y	y	–	y
Dagum NEW	p, a, b	–	–	–	–	–
Epanechnikov1 NEW	–	–	–	–	–	–
Erlang	b, c	–	–	–	–	–
Exponential 1	λ	y	y	y	–	y
Exponential 2	β	–	–	–	–	–
Exponentially Modified Gaussian	μ, σ, λ	–	–	–	–	y
ExtremeValue 1 NEW	μ, σ	–	–	–	–	–
F (aka Fisher-Snedecor)	n_1, n_2	y	–	y	–	–
Folded Normal 1 NEW	μ, σ	–	–	–	–	–
Fréchet 1	α, σ	–	–	–	–	y
Fréchet 2	α, σ, m	–	–	–	–	–
Gamma 1	k, θ	y	–	y	–	–
Gamma 2	r, μ	–	y	–	–	y
Generalized Gamma 1	$a,\ d,\ p$	–	–	–	–	–
Generalized Gamma 2	$a,\ b,\ c,\ k$	–	–	–	–	–
Generalized Gamma 3	$r,\ \mu,\ \beta$	–	y	–	–	–
Gompertz	η, b	–	–	y	–	–
Gumbel (aka Extreme Value)	μ, β	–	–	y	–	y
Half-Cauchy NEW	–	–	–	–	–	–
Half-Normal 1	θ	–	–	–	–	–
Half-Normal 2	μ, σ	–	–	–	–	–
Hyperbolic Secant	–	–	–	–	–	–
Inverse Chi-Square1 NEW	ν	–	–	–	–	y
Inverse-Gamma	α, β	y	–	–	–	y
Inverse-Gaussian (aka Wald)	λ, μ	–	–	–	–	–
JohnsonSB1 NEW	$\gamma, \delta, \mu, \sigma$	–	–	–	–	–
JohnsonSL1 NEW	$\gamma, \delta, \mu, \sigma$	–	–	–	–	–
JohnsonSN1 NEW	$\gamma, \delta, \mu, \sigma$	–	–	–	–	–
JohnsonSU1 NEW	$\gamma, \delta, \mu, \sigma$	–	–	–	–	–
Kumaraswamy NEW	a, b	–	–	–	–	–
Laplace 1(aka Double-exponential)	μ, b	y	–	–	–	y
Laplace 2	μ, τ	–	y	–	–	–
Lévy NEW	μ, c	–	–	–	–	–
Logistic 1	μ, s	y	–	–	–	y
Logistic 2	μ, τ	–	y	–	–	–
Logit-Normal 1	μ, σ	–	–	–	–	–
Log-Logistic 1 (aka Fisk)	α, β	y	–	–	–	–
Log-Logistic 2	λ, κ	–	–	–	–	–
Log-Normal 1	μ, σ	y	–	y	–	y
Log-Normal 2	$\mu,\ v$	y	–	y	–	–
Log-Normal 3	$m,\ \sigma$	–	–	–	–	–
Log-Normal 4	$m,\ cv$	–	–	–	–	–

Log-Normal 5	μ, τ	–	y	–	–	–
Log-Normal 6	m, σ_g	–	–	–	–	–
Log-Normal 7	μ_N, σ_N	–	–	–	–	–
Log-Uniform	min, max	–	–	–	–	–
Lomax	λ, α	–	–	–	–	–
Makeham New	δ, κ, γ	–	–	–	–	–
Maxwell-Boltzmann New	a	–	–	–	–	–
Muth New	β	–	–	–	–	–
Nakagami	m, Ω	–	–	–	–	–
Noncentral beta New	λ, α, β	–	–	–	–	–
Noncentral chi-squared New	n, δ	–	–	–	–	–
Noncentral F New	λ, m, n	–	–	–	–	–
Noncentral T New	δ, n	–	–	–	–	–
Normal 1	μ, σ	y	–	y	y	y
Normal 2	μ, v	y	–	y	–	–
Normal 3	μ, τ	–	y	–	–	–
Normal-inverse-gamma	$\mu, \lambda, \alpha, \beta$	y	–	–	–	–
Pareto Type I	x_m, α	y	y	–	–	y
Pareto Type II	μ, λ, α	–	–	–	–	y
Power New	α, β	–	–	–	–	–
Power Normal New	λ, μ, σ	–	–	–	–	–
Rayleigh	σ	–	–	y	–	y
Rice	ν, σ	–	–	–	–	–
Scaled Inverse Chi-Square	ν, σ	–	–	–	–	y
Sinh-Arcsinh 1 New	μ, σ, ν, τ	–	–	–	–	–
Sinh-Arcsinh 2 New	$\mu, \sigma, \epsilon, \delta$	–	–	–	–	–
Skew-Normal	μ, σ, α	–	–	–	–	y
Standard Cauchy New	–	–	–	–	–	–
Standard Logistic New	–	–	–	–	–	–
Standard Normal	–	y	–	y	y	–
Standard Power	β	–	–	–	–	–
Standard Triangular New	–	–	–	–	–	–
Standard Two-Sided Power New	m, n	–	–	–	–	–
Standard Uniform	–	–	–	y	y	–
Student's T 1	ν	y	–	y	–	–
Student's T 2	μ, τ, k	–	y	–	–	–
Student's T 3 New	ν, μ, σ	–	–	–	–	y
Two-Sided Power New	a, b, m, n	–	–	–	–	–
Triangular	a, b, c	–	–	–	–	–
Trapezoidal	a, b, c, d	–	–	–	–	–
Truncated Normal	μ, σ, a, b	–	–	–	–	–
Uniform	$a,\ b$	y	y	y	–	y
Von Mises	μ, κ	–	–	–	–	y
Weibull 1	λ, k	y	–	y	–	y
Weibull 2	λ, v	–	y	–	–	–
Wiener Diffusion Model	$\alpha, \beta, \delta, \tau$	–	–	–	–	y
Wigner Semicircle New	R	–	–	–	–	–

Table 5.2: Univariate distributions supported in ProbOnto with overview of tool support.

5.1.3 Code names

Distribution	**Parameters**			
Code name	Symbol	Code name	Symbol	Code name
Discrete Univariate				
Benford1 NEW	–	–	–	–
Bernoulli1	p	probability	–	–
Bernoulli2	α	logitProbability	–	–
BetaBinomial1	n	numberOfTrials	α	alpha
			β	beta
BetaNegativeBinomial1 NEW	α	alpha	β	beta
	–	–	n	numberOfSuccesses
Binomial1	n	numberOfTrials	p	probability
Binomial2	n	numberOfTrials	α	logitProbability
Borel1 NEW	μ	shape	–	–
CategoricalOrdered1	$p_1, \ldots, p_k$	categoryProb	–	–
CategoricalNonordered1	$p_1, \ldots, p_k$	categoryProb	–	–
ConwayMaxwellPoisson1	λ	rate	ν	rateOfDecay
DoublePoisson1	μ	rate	ϕ	dispersion
GeneralizedNegativeBinomial1	θ	theta	β	beta
			m	m
GeneralizedPoisson1	θ	rate	δ	dispersion
GeneralizedPoisson2	μ	mean	δ	dispersion
GeneralizedPoisson3	μ	mean	α	dispersion
Geometric1	p	probability	–	–
Hypergeometric1	N	populationSize	K	numberOfSuccesses
			n	numberOfTrials
InverseBinomial1	k	index	p	probability
NegativeBinomial1	r	numberOfSuccesses	p	probability
NegativeBinomial2	λ	rate	τ	overdispersion
NegativeBinomial3	μ	mean	ϕ	index
NegativeBinomial4	r	numberOfFailures	p	probability
NegativeBinomial5	α	shape	β	inverseScale
NegativeBinomial6	η	logMean	ϕ	dispersion
OrderedLogistic1	$c_1, \ldots, c_{K-1}$	cutpoints	η	predictor
Poisson1	λ	rate	–	–
Poisson2	α	logRate	–	–
UniformDiscrete1	a	minimum	b	maximum
			n	numberOfValues
UniformDiscrete2	$a = 0$	minimum	n	numberOfValues
WeibullDiscrete1 NEW	p	shape1	β	shape2
ZeroInflatedNegativeBinomial1	λ	rate	τ	overdispersion
			$p0$	probabilityOfZero
ZeroInflatedGeneralizedPoisson1	μ	mean	α	dispersion
			$p0$	probabilityOfZero
ZeroInflatedPoisson1	λ	rate	π	probabilityOfZero
Zeta1 NEW	α	shape	–	–
Zipf1 NEW	α	shape	n	numberOfElements
Continuous Univariate				
Amoroso1 NEW	a	location	θ	scale
	α	shape1	β	shape2
Arcsine1 NEW	–	–	–	–
Arcsine2 NEW	a	lowerBound	b	upperBound
AsymmetricLaplace1 NEW	μ	location	σ	scale
Beta1	α	alpha	β	beta
BirnbaumSaunders1	α	scale	γ	shape
Burr1	α	scale	c	shape1
			k	shape2
Cauchy1	x_0	location	γ	scale
Chi1 NEW	k	degreesOfFreedom	–	–
ChiSquared1	k	degreesOfFreedom	–	–

Dagum1 NEW	p	shape1	a	shape2
	–	–	b	scale
Epanechnikov1 NEW	–	–	–	–
Erlang1	b	scale	c	shape
Exponential1	λ	rate	–	–
Exponential2	β	mean	–	–
ExponentiallyModifiedGaussian1	μ	mean	σ	stdev
			λ	rate
ExtremeValue1 NEW	μ	location	σ	scale
F1	n_1	numerator	n_2	denominator
FoldedNormal1 NEW	μ	location	σ	scale
Frechet1	α	shape	σ	scale
Frechet2	α	shape	σ	scale
			m	locationOfMinimum
Gamma1	k	shape	θ	scale
Gamma2	r	shape	μ	rate
GeneralizedGamma1	a	scale	d	shape1
			p	shape2
GeneralizedGamma2	a	location	b	scale
	c	shape1	k	shape2
GeneralizedGamma3	r	scale	μ	shape1
			β	shape2
Gompertz1	η	shape	b	scale
Gumbel1	μ	location	β	scale
HalfCauchy1 NEW	–	–	–	–
HalfNormal1	θ	inverseScale	–	–
HalfNormal2	μ	location	σ	scale
HyperbolicSecant1	–	–	–	–
InverseChiSquare1 NEW	ν	degreesOfFreedom	–	–
InverseGamma1	α	shape	β	scale
InverseGaussian1	λ	shape	μ	mean
JohnsonSB1 NEW	γ	shape1	δ	shape2
	μ	location	σ	scale
JohnsonSL1 NEW	γ	shape1	δ	shape2
	μ	location	σ	scale
JohnsonSN1 NEW	γ	shape1	δ	shape2
	μ	location	σ	scale
JohnsonSU1 NEW	γ	shape1	δ	shape2
	μ	location	σ	scale
Kumaraswamy1 NEW	a	shape1	b	shape2
Laplace1	μ	location	b	scale
Laplace2	μ	location	τ	tau
Levy NEW	μ	location	c	scale
Logistic1	μ	location	s	scale
Logistic2	μ	location	τ	inverseScale
LogitNormal1	μ	location	σ	scale
LogLogistic1	α	scale	β	shape
LogLogistic2	λ	scale	κ	shape
LogNormal1	μ	meanLog	σ	stdevLog
LogNormal2	μ	meanLog	v	varLog
LogNormal3	m	median	σ	stdevLog
LogNormal4	m	median	cv	coefVar
LogNormal5	μ	meanLog	τ	precision
LogNormal6	m	median	σ_g	geomStdev
LogNormal7	μ_N	mean	σ_N	stdev
LogUniform1	min	minimum	max	maximum
Lomax1	λ	scale	α	shape
Makeham1 NEW	δ	delta	κ	kappa
			γ	gamma
MaxwellBoltzmann1 NEW	a	scale	–	–

Muth1 NEW	β	shape	–	–
Nakagami1	m	shape	Ω	spread
NoncentralBeta1 NEW	λ	noncentrality	α	shape1
			β	shape2
NoncentralChiSquared1 NEW	δ	noncentrality	n	degreesOfFreedom
NoncentralF1 NEW	λ	noncentrality	m	degreesOfFreedom1
			n	degreesOfFreedom2
NoncentralT1 NEW	δ	noncentrality	n	degreesOfFreedom
Normal1	μ	mean	σ	stdev
Normal2	μ	mean	v	var
Normal3	μ	mean	τ	precision
NormalInverseGamma1	μ	mean	λ	lambda
	α	alpha	β	beta
ParetoTypeI1	x_m	scale	α	shape
ParetoTypeII1	μ	location	λ	scale
			α	tailIndex
Power1 NEW	α	scale	β	shape
PowerNormal1 NEW	λ	shape	μ	location
			σ	scale
Rayleigh1	σ	scale	–	–
Rice1	ν	noncentrality	σ	scale
ScaledInverseChiSquare1	ν	degreesOfFreedom	σ	scale
SinhArcsinh1 NEW	μ	location	σ	scale
	ν	leftHandTail	τ	rightHandTail
SinhArcsinh2 NEW	μ	location	σ	scale
	ϵ	skewness	δ	kurtosis
SkewNormal1	μ	location	σ	scale
			α	shape
StandardCauchy1 NEW	–	–	–	–
StandardLogistic1 NEW	–	–	–	–
StandardNormal1	–	–	–	–
StandardPower1	β	shape	–	–
StandardTriangular1 NEW	–	–	–	–
StandardTwoSidedPower1 NEW	m	location	n	shape
StandardUniform1	–	–	–	–
StudentT1	ν	degreesOfFreedom	–	–
StudentT2	μ	mean	τ	scale
			ν	degreesOfFreedom
StudentT3 NEW	ν	degreesOfFreedom	μ	location
			σ	scale
Trapezoidal1	a	lowerBound	b	levelStart
	c	levelEnd	d	upperBound
Triangular1	a	lowerLimit	b	upperLimit
			c	shape
TruncatedNormal1	μ	mean	σ	stdev
	a	lowerBound	b	upperBound
TwoSidedPower1 NEW	a	lowerBound	b	upperBound
	m	location	n	shape
Uniform1	a	minimum	b	maximum
VonMises1	μ	location	κ	concentration
Weibull1	λ	scale	k	shape
Weibull2	λ	lambda	v	shape
WienerDiffusionModel1	α	boundSeparation	β	initialBias
	δ	driftRate	τ	nondecisionTime
WignerSemicircle1 NEW	R	radius	–	–

Table 5.3: Code names for distribution and parameter names of the univariate distributions.

5.2 Mixture distribution

ProbOnto 2.5	Parameters	**UncertML** 3.0	**WinBUGS** 1.4	**Monolix** 4.3	**NONMEM** 7.3	**STAN** 2.10.0
`MixtureDistribution1`	$\pi_1, \ldots, \pi_k$	y	–	–	–	–

Table 5.4: Mixture distribution with overview of tool support.

Distribution Code name	Parameters Symbol	Code name
`MixtureDistribution1`	$\pi_1, \ldots, \pi_k$	`weight`

Table 5.5: Code names for mixture distribution – defined for univariate distributions only.

5.3 Multivariate distributions – tool coverage and code names

5.3.1 Tool coverage

ProbOnto 2.5	Parameters	**UncertML** 3.0	**WinBUGS** 1.4	**Monolix** 4.3	**NONMEM** 7.3	**STAN** 2.10.0
Discrete Multivariate						
Multinomial	$n, p_1, \ldots, p_k$	y	y	–	–	y
Continuous Multivariate						
Dirichlet	$\alpha_1, \ldots, \alpha_K$	y	y	–	–	y
Inverse-Wishart	Ψ, ν	–	–	–	y	y
LKJ Correlation 1 New	η	–	–	–	–	y
LKJ Correlation 2 New	η	–	–	–	–	y
Multivariate Gaussian Process 1	Σ, w	–	–	–	–	y
Multivariate Gaussian Process 2	L, w	–	–	–	–	y
Multivariate Normal 1	μ, Σ	y	–	–	–	y
Multivariate Normal 2	μ, T	–	y	–	–	y
Multivariate Normal 3	μ, L	–	–	–	–	y
Multivariate (Student)-T1	μ, Σ, ν	y	–	–	–	y
Multivariate (Student)-T2	μ, T, k	–	y	–	–	–
Wishart 1	V, n	y	–	–	–	y
Wishart 2	R, k	–	y	–	–	–

Table 5.6: Multivariate distributions with overview of tool support.

5.3.2 Code names

Distribution Code name	**Parameters** Symbol	Code name	Symbol	Code name
Discrete Multivariate				
`Multinomial1`	n	`numberOfTrials`	$p_1, \ldots, p_k$	`probabilityOfSuccess`
Continuous Multivariate				
`Dirichlet1`	$\alpha_1, \ldots, \alpha_K$	`concentration`	–	–
`InverseWishart1`	Ψ	`scaleMatrix`	ν	`degreesOfFreedom`
`LKJCorrelation1` New	η	`shape`	–	–
`LKJCorrelation2` New	η	`shape`	–	–
`MultivariateGaussianProcess1`	Σ	`kernelMatrix`	w	`inverseScale`

`MultivariateGaussianProcess2`	L	`choleskyFactor`	w	`inverseScale`
`MultivariateNormal1`	μ	`mean`	Σ	`covarianceMatrix`
`MultivariateNormal2`	μ	`mean`	T	`precisionMatrix`
`MultivariateNormal3`	μ	`mean`	L	`choleskyFactor`
`MultivariateStudentT1`	μ	`mean`	Σ	`covarianceMatrix`
			ν	`degreesOfFreedom`
`MultivariateStudentT2`	μ	`mean`	T	`precisionMatrix`
			k	`degreesOfFreedom`
`Wishart1`	V	`scaleMatrix`	n	`degreesOfFreedom`
`Wishart2`	R	`inverseScaleMatrix`	k	`degreesOfFreedom`

Table 5.7: Code names for the multivariate distributions and their parameters.

5.4 Empirical distributions – tool coverage and code names

ProbOnto 2.5	Weight	**UncertML** 3.0	**WinBUGS** 1.4	**Monolix** 4.3	**NONMEM** 7.3	**STAN** 2.10.0
Random Sample	*weight*	y	–	–	y	–
Systematic Sample	*weight*	y	–	–	–	–
Unknown Sample	*weight*	y	–	–	–	–

Table 5.8: Empirical distributions with overview of tool support.

Distribution	Weight	
Code name	Symbol	Code name
`RandomSample1`	`weight`	weight
`SystematicSample1`	`weight`	weight
`UnknownSample1`	`weight`	weight

Table 5.9: Empirical distributions.

Appendix A

ProbOnto Knowledge Base

This appendix provides detailed summary of the most important features for supported univariate and multivariate distributions. After the symbol definitions we list for each distribution

- its type
- support definition
- parameters
- defining functions (with R code)
- quantities such as mean, median, mode and variance
- relationships and re-parameterisations
- (for univariate distributions only) PDF/PMF and CDF plots based on the R code stored in the KB.

A.1 Symbols

Some of the symbols used in definitions of the functions and quantities listed in the subsequent sections are collected here with references

- Beta function, $B(x, y)$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

`mathworld.wolfram.com/BetaFunction.html`
`en.wikipedia.org/wiki/Beta_function`

- Regularized incomplete Beta function, $I_p(a, b)$, $I_{1-p}(a, b)$

$$I_x(a, b) = \frac{B(x; a, b)}{B(a, b)}$$

with

$$B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$$

then *incomplete beta function.*
`mathworld.wolfram.com/RegularizedBetaFunction.html`
`en.wikipedia.org/wiki/Beta_function#Incomplete_beta_function`

- Error function, erf

$$\mathrm{erf}(z) = \frac{2}{\pi} \int_0^z e^{-t^2} dt$$

`mathworld.wolfram.com/Erf.html`
`en.wikipedia.org/wiki/Error_function`

- Euler constant, γ_E

$$\gamma_E = \lim_{n\to\infty} \Big(-\log(n) \sum_{k=1}^{n} \frac{1}{k} \Big) = \int_1^\infty \Big(\frac{1}{\lfloor x \rfloor} - \frac{1}{x} \Big) dx$$

`mathworld.wolfram.com/Euler-MascheroniConstant.html`
`en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant`

- Floor function, $\lfloor x \rfloor$

$$\mathrm{floor}(x) = \lfloor x \rfloor \text{ is the largest integer not greater than } x$$

`mathworld.wolfram.com/FloorFunction.html`
`en.wikipedia.org/wiki/Floor_and_ceiling_functions`

- Gamma function, Γ

$$\Gamma(n) = (n-1)!, \text{ for } n - \text{ positive integer}$$

`mathworld.wolfram.com/GammaFunction.html`
`en.wikipedia.org/wiki/Gamma_function`

- Lower incomplete gamma function, $\gamma(s, x)$

$$\gamma(s, x) = \int_0^x t^{s-1}\, e^{-t}\, \mathrm{d}t$$

`mathworld.wolfram.com/IncompleteGammaFunction.html`
`https://en.wikipedia.org/wiki/Incomplete_gamma_function#Lower_incomplete_Gamma_function`
`https://cran.r-project.org/web/packages/zipfR/zipfR.pdf`

- Upper incomplete gamma function, $\Gamma(s, x)$

$$\Gamma(s, x) = \int_x^\infty t^{s-1}\, e^{-t}\, \mathrm{d}t$$

mathworld.wolfram.com/IncompleteGammaFunction.html
https://en.wikipedia.org/wiki/Incomplete_gamma_function#Upper_incomplete_Gamma_function
https://cran.r-project.org/web/packages/zipfR/zipfR.pdf

- Multivariate Gamma function, Γ_p

$$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\left[a + (1-j)/2\right]$$

en.wikipedia.org/wiki/Multivariate_gamma_function

- Iverson bracket, $[x = i]$

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true;} \\ 0 & \text{otherwise.} \end{cases}$$

mathworld.wolfram.com/IversonBracket.html
en.wikipedia.org/wiki/Iverson_bracket

- Pochhammer symbol, $(x)_n$

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

http://mathworld.wolfram.com/PochhammerSymbol.html
https://en.wikipedia.org/wiki/Pochhammer_symbol

- Generalized Hypergeometric function, ${}_pF_q$

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}$$

en.wikipedia.org/wiki/Generalized_hypergeometric_function

- Hypergeometric function, ${}_2F_1$

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

mathworld.wolfram.com/HypergeometricFunction.html
en.wikipedia.org/wiki/Hypergeometric_function

- Riemann's Zeta function, $\zeta(a)$

$$\zeta(\alpha) = \sum_{i=1}^{\infty} (1/i)^\alpha$$

https://en.wikipedia.org/wiki/Riemann_zeta_function
http://mathworld.wolfram.com/RiemannZetaFunction.html

A.2 Distributions – properties and relationships

Amoroso1

name	Amoroso 1 (ID: 0001082)
type	continuous
variate	x, scalar
support	$x \geq a$ if $\theta > 0, x \leq a$ if $\theta < 0$

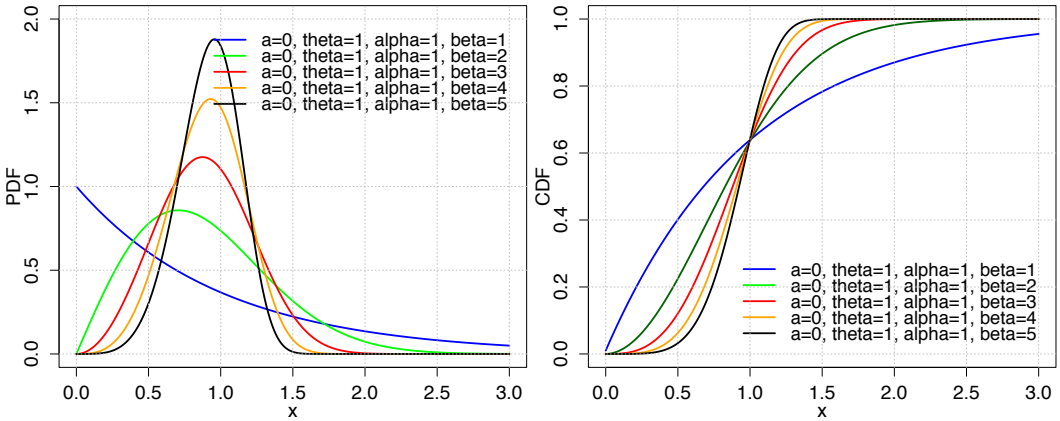

Figure A.1: Amoroso1 distribution plotted using the provided R code.

Model

The Amoroso distribution was originally developed to model lifetimes. It occurs as the Weibullization of the standard gamma distribution and, with integer, in extreme value statistics. A useful and important property of the Amoroso distribution is that many common and interesting probability distributions occur as special cases or limits.

Parameter: location

name	location parameter
type	scalar
symbol	a
definition	$a \in R$

Parameter: scale

name	scale parameter
type	scalar
symbol	θ
definition	$\theta \in R$

Parameter: shape1

name	shape parameter
type	scalar
symbol	α
definition	$\alpha \in R, \alpha > 0$

Parameter: shape2

name	shape paremeter
type	scalar
symbol	β
definition	$\beta \in R$

Functions

PDF

$$\frac{1}{\Gamma(\alpha)}\left|\frac{\beta}{\theta}\right|\left(\frac{x-a}{\theta}\right)^{\alpha\beta-1}\exp\left[-\left(\frac{x-a}{\theta}\right)^{\beta}\right]$$

PDF in R

```
1/gamma(alpha) * abs(beta/theta) * ((x-a)/theta)^(alpha*beta-1) * exp(-((x-a)/theta)^beta)
```

Characteristics

Relationships

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Rayleigh1(\sigma)$
- Relationship type: Special case
- Relationship definition: Rayleigh1(σ) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=0,\theta=\sqrt{2\sigma^2},\alpha=1,\beta=2$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to InverseChiSquare1(k)$
- Relationship type: Special case
- Relationship definition: InverseChiSquare1(k) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=0,\theta=1/2,\alpha=k/2,\beta=-1$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Weibull1(\lambda,k)$
- Relationship type: Special case
- Relationship definition: Weibull1(λ,k) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=0,\theta=\lambda,\alpha=1$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Frechet1(\alpha,\sigma)$
- Relationship type: Special case
- Relationship definition: Frechet1(α,σ) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=0,\theta=\sigma,\alpha=1,\beta\to-\beta$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Gamma1(k,\theta)$
- Relationship type: Special case
- Relationship definition: Gamma1 distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=0,\beta=1$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Levy1(\mu,c)$
- Relationship type: Special case
- Relationship definition: Levy1(μ,c) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $\theta=c/2,\alpha=1/2,\beta=-1$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Normal1(\mu,\sigma)$
- Relationship type: Limiting
- Relationship definition: Normal1(μ,σ) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=\mu-\sigma\sqrt{\alpha},\theta=\sigma/\sqrt{\alpha},\beta=1$ and $\alpha\to\infty$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to HalfNormal2(\mu,\sigma)$
- Relationship type: Special case
- Relationship definition: HalfNormal2(μ,σ) distribution is a special case of the Amoroso1(a,θ,α,β) distribution for $\mu=0$ and when $a=0,\theta=\sqrt{2\sigma^2},\alpha=1/2,\beta=2$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to ChiSquared1(k)$
- Relationship type: Special case
- Relationship definition: ChiSquared1(k) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=0,\theta=2,\alpha=k/2,\beta=1$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Chi1(k)$

- Relationship type: Special case
- Relationship definition: Chi1(k) distribution is a special case of the Amoroso1(a, θ, α, β) distribution when $a = 0, \theta = \sqrt{2}, \alpha = k/2, \beta = 2$
- Relationship reference(s): [Crooks, 2010]

References and sources

[Crooks, 2010]

UncertML mapping:

no UncertML mapping found.

Arcsine1

name	Arcsine 1 (ID: 0001096)
type	continuous
variate	x, scalar
support	$x \in (0, 1)$

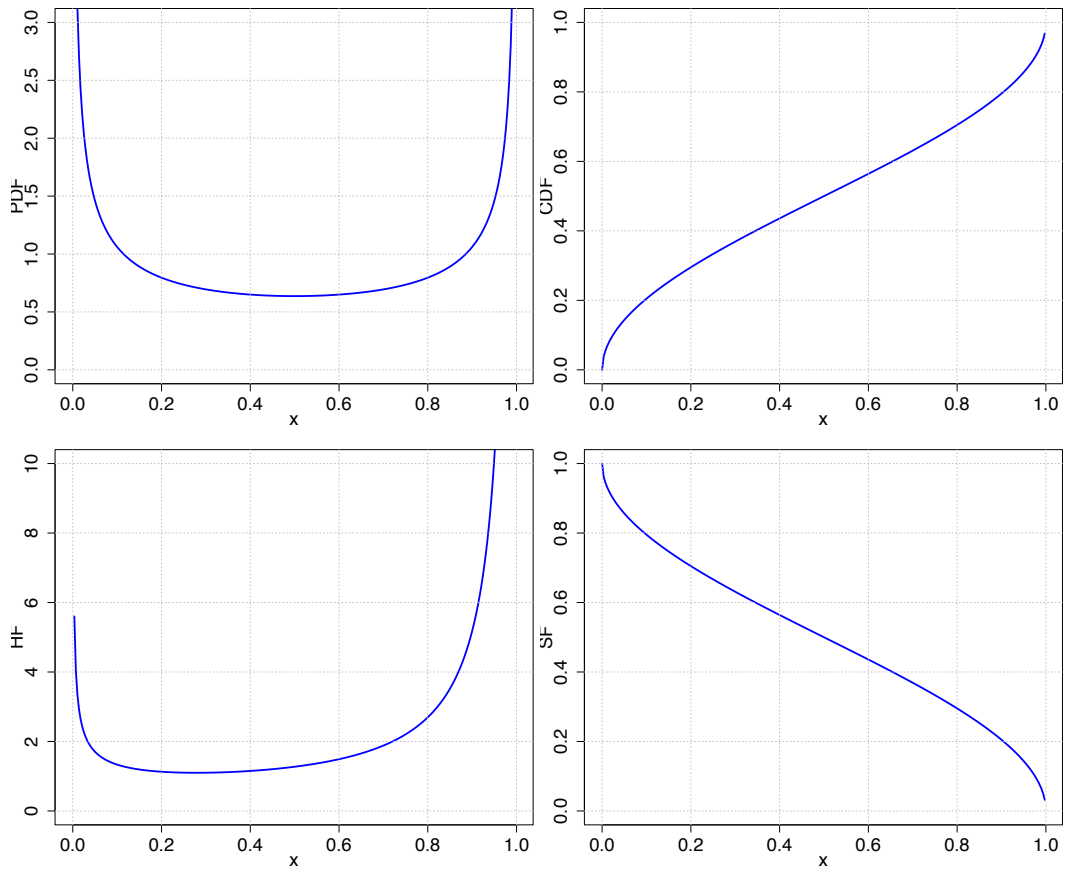

Figure A.2: Arcsine1 distribution plotted using the provided R code.

Model

The arcsine distribution models the proportion of time that Brownian motion is positive.

Functions

PDF

$$\frac{1}{\pi\sqrt{x(1-x)}}$$

PDF in R

```
1/pi/sqrt(x*(1-x))
```

CDF

$$\frac{2}{\pi}\arcsin\left(\sqrt{x}\right)$$

CDF in R

```
2/pi*arcsin(sqrt(x))
```

HF

$$\frac{2}{\sqrt{x(1-x)}(\pi - 2\arcsin(2x-1))}$$

HF in R

```
2/(sqrt(x*(1-x))*(pi - 2 * asin(2*x -1)))
```

SF

$$\frac{\pi - 2\arcsin(2x-1)}{2\pi}$$

SF in R

```
(pi - 2 * asin(2*x -1)) / (2*pi)
```

Characteristics

Mean

$$\frac{1}{2}$$

Median

$$\frac{1}{2}$$

Mode

$$x \in \{0, 1\}$$

Variance

$$\frac{1}{8}$$

Relationships

- Relationship pair: $Beta1(\alpha, \beta) \rightarrow Arcsine1$
- Relationship type: Special case
- Relationship definition: $\alpha = 1/2, \beta = 1/2$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/UDR.html`
- Relationship pair: $Arcsine2(a, b) \rightarrow Arcsine1$
- Relationship type: Special case
- Relationship definition: $a = 0, b = 1$
- Relationship reference(s): ProbOnto spec

References and sources

`https://en.wikipedia.org/wiki/Arcsine_distribution`

UncertML mapping:

no UncertML mapping found.

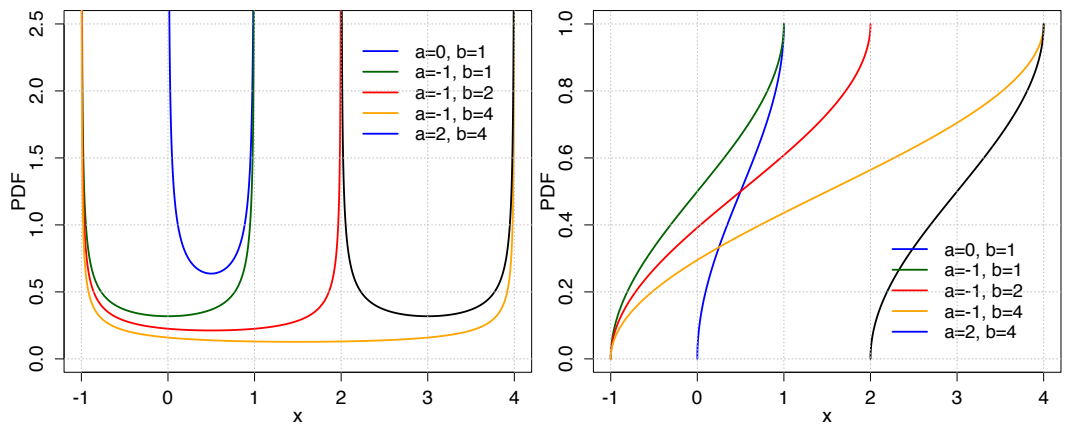

Figure A.3: Arcsine2 distribution plotted using the provided R code.

Arcsine2

name	Arcsine 2 (ID: 0001105)
type	continuous
variate	x, scalar
support	$x \in (a, b)$

Model

The arcsine distribution models the proportion of time that Brownian motion is positive.

Parameter: lowerBound

name	lower bound
type	scalar
symbol	a
definition	$a \in R$

Parameter: upperBound

name	upper bound
type	scalar
symbol	b
definition	$b \in R, b > a$

Functions

PDF

$$\frac{1}{\pi\sqrt{(x-a)(b-x)}}$$

PDF in R

```
1/pi/sqrt((x-a)*(b-x))
```

CDF

$$\frac{2}{\pi}\arcsin\left(\sqrt{\frac{x-a}{b-a}}\right)$$

CDF in R

```
2/pi*asin(sqrt((x-a)/(b-a)))
```

Characteristics

Mean

$$(a+b)/2$$

Median

$$(a+b)/2$$

Mode

$$x \in \{a, b\}$$

Variance

$$(b-a)^2/8$$

Relationships

- Relationship pair: $Arcsine2(a,b) \rightarrow Arcsine1$
- Relationship type: Special case
- Relationship definition: $a = 0, b = 1$
- Relationship reference(s): ProbOnto spec

References and sources

`http://doc.openturns.org/openturns-1.6/sphinx/user_manual/_generated/openturns.Arcsine.html`
`http://www.boost.org/doc/libs/1_58_0/libs/math/doc/html/math_toolkit/dist_ref/dists/arcine_dist.html`

UncertML mapping:

no UncertML mapping found.

AsymmetricLaplace1

name	Asymmetric Laplace 1 (ID: 0001120)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

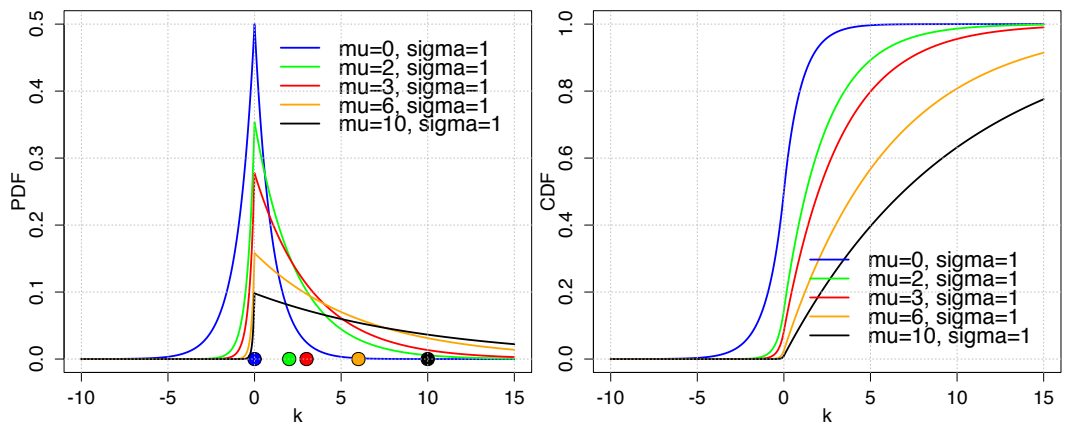

Figure A.4: AsymmetricLaplace1 distribution plotted using the provided R code.

Parameter: location

name	location parameter
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale parameter
type	scalar
symbol	σ
definition	$\sigma \geq 0$

Functions

PDF

$$\frac{1}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases}\exp(\frac{1}{\sigma\kappa}x) & \text{if } x<0\\ \exp\left(-\frac{\kappa}{\sigma}x\right) & \text{if } x\geq 0\end{cases} \text{with } \kappa = \frac{2\sigma}{\sqrt{4\sigma^2+\mu^2}+\mu}$$

PDF in R

```
kappa = function(mu,sigma){ 2*sigma/(sqrt(4*sigma^2+mu^2)+mu) }
PDF1 = function(x,mu,sigma) {
  1/sigma*kappa(mu,sigma)/(1+kappa(mu,sigma)^2)*exp(1/(sigma*kappa(mu,sigma))*x ) }
PDF2 = function(x,mu,sigma) {
  1/sigma*kappa(mu,sigma)/(1+kappa(mu,sigma)^2)*exp( -kappa(mu,sigma)/sigma *x) }
```

CDF

$$\begin{cases}\frac{\kappa^2}{1+\kappa^2}\exp\left(\frac{1}{\sigma\kappa}x\right) & \text{if } x\leq 0\\ 1-\frac{1}{1+\kappa^2}\exp\left(-\frac{\kappa}{\sigma}x\right) & \text{if } x>0\end{cases}$$

CDF in R

```
kappa = function(mu,sigma){ 2*sigma/(sqrt(4*sigma^2+mu^2)+mu) }
CDF1 = function(x,mu,sigma) {
  kappa(mu,sigma)^2/(1+kappa(mu,sigma)^2)*exp(1/(sigma*kappa(mu,sigma))*x )}
CDF2 = function(x,mu,sigma) {
  1-1/(1+kappa(mu,sigma)^2)*exp(-kappa(mu,sigma)/sigma*x)}
```

Characteristics

Mean

$$\mu$$

Variance

$$\mu^2+2\sigma^2$$

Relationships

References and sources

[Kozubowski and Podgórski, 1999]

UncertML mapping:

no UncertML mapping found.

Benford1

name	Benford 1 (ID: 0001134)
type	discrete
variate	k, scalar
support	$k \in \{1, 2, ..., 9\}$

Model

"Benford's law, also called the first-digit law, is an observation about the frequency distribution of leading digits in many real-life sets of numerical data. The law states that in many naturally occurring collections of numbers, the leading significant digit is likely to be small."

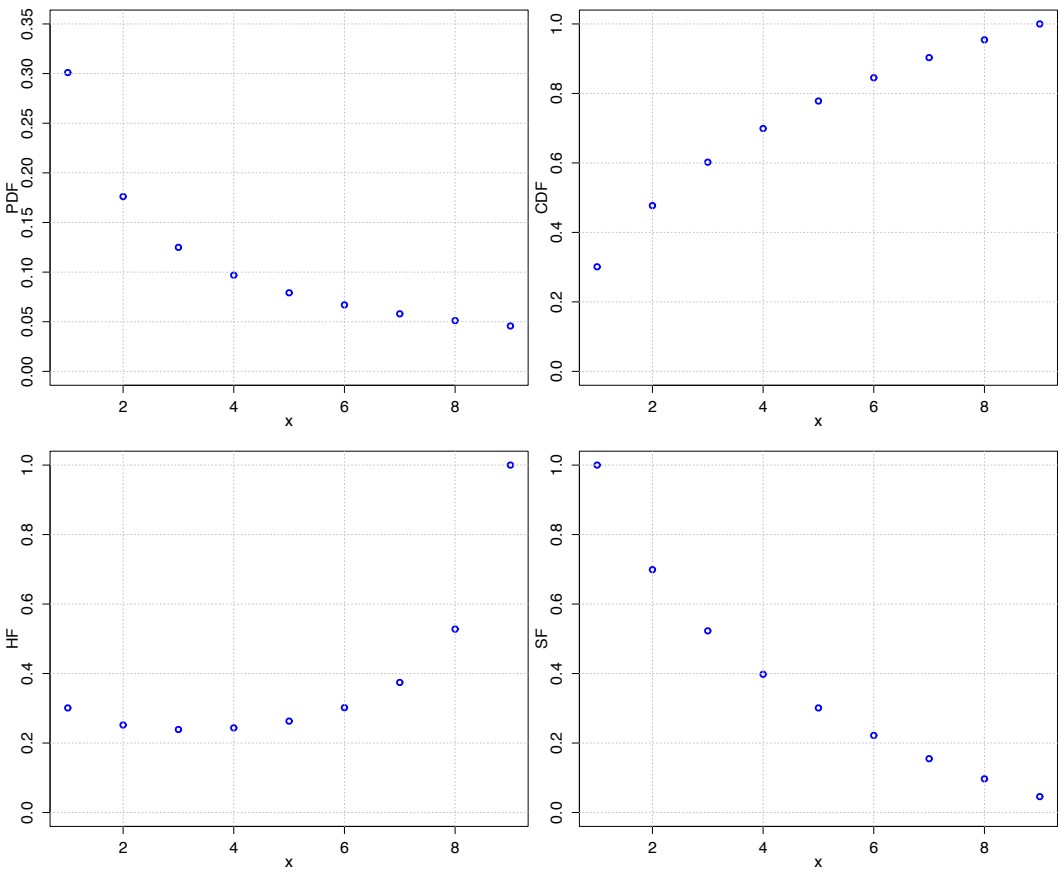

Figure A.5: Benford1 distribution plotted using the provided R code.

Functions

PMF

$$\log_{10}\left(1+\frac{1}{x}\right)$$

PMF in R

```
log10(1+1/x)
```

CDF

$$\log_{10}(1+x)$$

CDF in R

```
log10(1+x)
```

HF

$$\frac{\log_{10}\left(1+\frac{1}{x}\right)}{1-\log_{10}x}$$

HF in R

```
log10(1+1/x) / (1-log10(x))
```

SF

$$1-\log_{10}(x)$$

SF in R

```
1-log10(x)
```

Characteristics

Mean

$$\approx 3.4402$$

Median

$$3$$

Mode

$$1$$

Variance

$$\approx 6.0565$$

Relationships

- Relationship pair: $StandardUniform1 \rightarrow Benford1$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ StandardUniform1, then $Y = \lfloor 10^X \rfloor$ has the Benford1 distribution
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformBenford.pdf`

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Benford.pdf`
[Marichev and Trott, 2013]

UncertML mapping:

no UncertML mapping found.

Bernoulli1

name	Bernoulli 1 (ID: 0000000)
type	discrete
variate	k, scalar
support	$k \in \{0, 1\}$

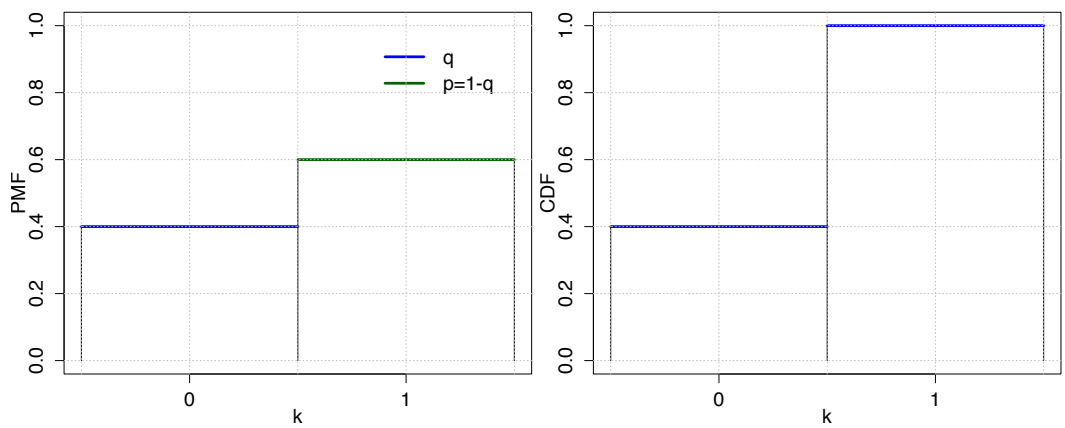

Figure A.6: Bernoulli1 distribution plotted using the provided R code.

Model

A Bernoulli trial is a probabilistic experiment that can have one of two outcomes, success ($x = 1$) or failure ($x = 0$), and in which the probability of success is p.

Parameter: probability

name	probability of success
type	scalar
symbol	p
definition	$0 < p < 1, p \in R$

Functions

PMF

$$\begin{cases} q = (1-p) & \text{for } k = 0 \\ p & \text{for } k = 1 \end{cases}$$

PMF in R

```
q=(1-p) for k=0
p for k=1
```

CDF

$$\begin{cases} 0 & \text{for } k < 0 \\ q & \text{for } 0 \leq k < 1 \\ 1 & \text{for } k \geq 1 \end{cases}$$

Characteristics

Mean

$$p$$

Median

$$\begin{cases} 0 & \text{if } q > p \\ 0.5 & \text{if } q = p \\ 1 & \text{if } q < p \end{cases}$$

Mode

$$\begin{cases} 0 & \text{if } q > p \\ 0,1 & \text{if } q = p \\ 1 & \text{if } q < p \end{cases}$$

Variance

$$p(1-p)$$

Relationships

- Relationship pair: $Bernoulli1(p) \rightarrow Binomial1(n,p)$
- Relationship type: Transformation
- Relationship definition: $\Sigma X(iid)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BernoulliBinomial.pdf`
- Relationship pair: $Bernoulli1(p) \rightarrow Bernoulli2(\alpha)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = \log(p/(1-p))$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Binomial1(n,p) \rightarrow Bernoulli1(p)$
- Relationship type: Special case
- Relationship definition: $n = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BinomialBernoulli.pdf`
- Relationship pair: $Bernoulli2(\alpha) \rightarrow Bernoulli1(p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = \exp(\alpha)/(1+\exp(\alpha))$
- Relationship reference(s): ProbOnto spec

References and sources

https://en.wikipedia.org/wiki/Bernoulli_distribution

UncertML mapping:

http://www.uncertml.org/distributions/bernoulli

Bernoulli2

name	Bernoulli 2 (ID: 0000028)
type	discrete
variate	k, scalar
support	$k \in \{0, 1\}$

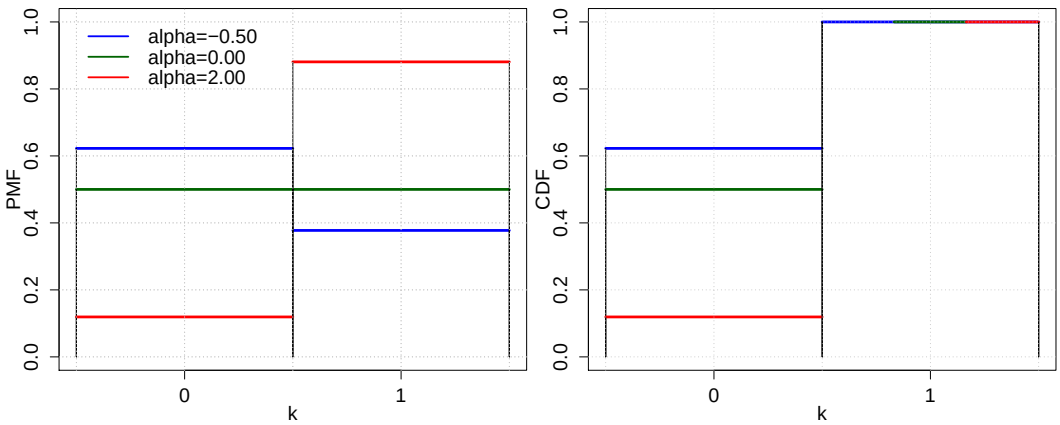

Figure A.7: Bernoulli2 distribution plotted using the provided R code.

Model

A Bernoulli trial is a probabilistic experiment that can have one of two outcomes, success ($x = 1$) or failure ($x = 0$), with the logit-transformed probability of success as parameter.

Parameter: logitProbability

name	logit of probability of success
type	scalar
symbol	α
definition	$\alpha \in R$

Functions

PMF

$$\begin{cases} logit^{-1}(\alpha) & \text{for } k = 1 \\ 1 - logit^{-1}(\alpha) & \text{for } k = 0 \end{cases}$$

PMF in R

```
invLogit = function(x) { exp(x)/(1+exp(x)) }
invLogit(alpha1) for k = 1
1-invLogit(alpha1) for k = 0
```

Characteristics

Relationships

- Relationship pair: $Bernoulli2(\alpha) \rightarrow Bernoulli1(p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = \exp(\alpha)/(1 + \exp(\alpha))$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Bernoulli1(p) \rightarrow Bernoulli2(\alpha)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = \log(p/(1 - p))$
- Relationship reference(s): ProbOnto spec

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

Beta1

name	Beta 1 (ID: 0000057)
type	continuous
variate	x, scalar
support	$x \in (0, 1)$

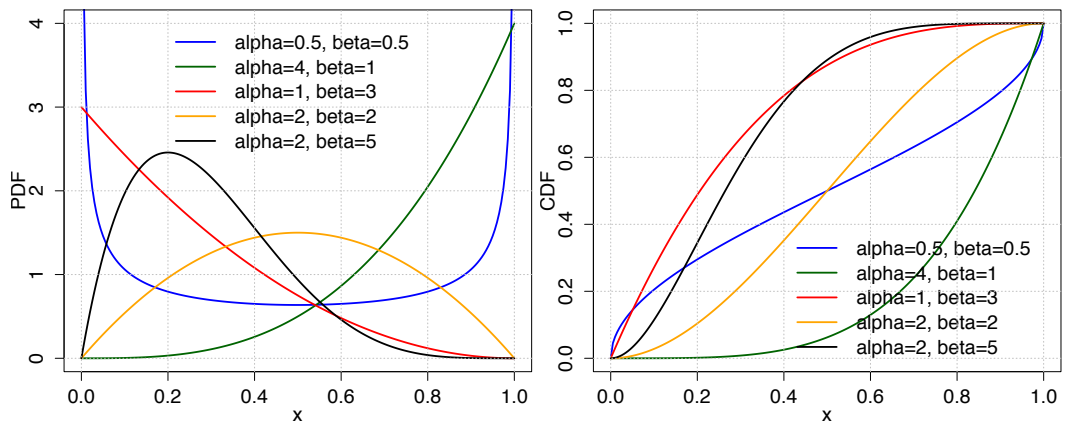

Figure A.8: Beta1 distribution plotted using the provided R code.

Model

Beta distribution is a family of continuous probability distributions defined on the interval $[0, 1]$ parametrized by two positive shape parameters, denoted by α and β, that appear as exponents of the random variable and control the shape of the distribution.

Parameter: alpha

name	shape
type	scalar
symbol	α
definition	$\alpha > 0$

Parameter: beta

name	shape
type	scalar
symbol	β
definition	$\beta > 0$

Functions

PDF

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

PDF in R

```
(x^(alpha-1)*(1-x)^(beta-1))/beta(alpha,beta)
```

CDF

$$I_x(\alpha,\beta)$$

CDF in R

```
Rbeta(x, alpha, beta)
```

HF

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1-I_x(\alpha,\beta))B(\alpha,\beta)}$$

HF in R

```
(x^(alpha-1) * (1-x)^(beta-1))/((1-Rbeta(x,alpha,beta))*beta(alpha,beta))
```

SF

$$1-I_x(\alpha,\beta)$$

SF in R

```
1 - Rbeta(x, alpha, beta)
```

Characteristics

Mean

$$\frac{\alpha}{\alpha+\beta}$$

Median

$$I_{\frac{1}{2}}^{[-1]}(\alpha,\beta)$$

Mode

$$\frac{\alpha-1}{\alpha+\beta-2}$$

Variance

$$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Relationships

- Relationship pair: $Beta1(alpha, beta) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $\alpha = \beta, \beta \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BetaNormal.pdf`

- Relationship pair: $Beta1(\alpha, \beta) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: $\alpha = 1, \beta = 1$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Beta1(\alpha, \beta) \rightarrow Arcsine1$
- Relationship type: Special case
- Relationship definition: $\alpha = 1/2, \beta = 1/2$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/UDR.html`

- Relationship pair: $Gamma1(k, \theta) \rightarrow Beta1(\alpha, \beta)$
- Relationship type: Transformation
- Relationship definition: $X1, X2 \sim Gamma1(k, \theta)$ and $Y = X1/(X1 + X2) \Rightarrow Y \sim Beta1(\alpha, \beta)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaBeta.pdf`

- Relationship pair: $NoncentralBeta1(\lambda, \alpha, \beta) \rightarrow Beta1(\alpha, \beta)$
- Relationship type: Special case
- Relationship definition: $\lambda = 0$
- Relationship reference(s): [Walck, 2007]

- Relationship pair: $StandardUniform1(0, 1) \rightarrow Beta1(alpha, beta)$
- Relationship type: Transformation
- Relationship definition: $aX_1, X_2, ..., X_n(iid) \sim StandardNormal1 \Rightarrow$ with $\alpha = r$ and $\beta = n-r+1, X_{(r)} \sim Beta(\alpha, \beta)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformBeta.pdf`

References and sources

`http://en.wikipedia.org/wiki/Beta_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/beta`

BetaBinomial1

name	Beta-binomial 1 (ID: 0000090)
type	discrete
variate	k, scalar
support	$k \in \{0, \ldots, n\}$

Model

Beta-binomial distribution is a family of discrete probability distributions on a finite support of non-negative integers arising when the probability of success in each of a fixed or known number of Bernoulli trials is either unknown or random. The beta-binomial distribution is the binomial distribution in which the probability of success at each trial is not fixed but random and follows the beta distribution.

Parameter: numberOfTrials

name	number of trials
type	scalar
symbol	n
definition	$n \in N, n > 0$

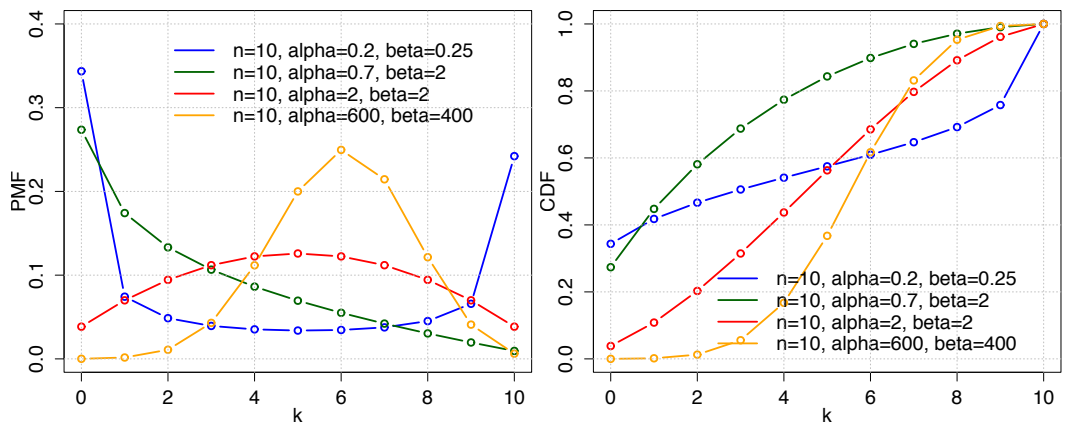

Figure A.9: BetaBinomial1 distribution plotted using the provided R code.

Parameter: alpha

name	left beta parameter
type	scalar
symbol	α
definition	$\alpha > 0$

Parameter: beta

name	right beta parameter
type	scalar
symbol	β
definition	$\beta > 0$

Functions

PMF

$$\binom{n}{k}\frac{B(k+\alpha, n-k+\beta)}{B(\alpha,\beta)}$$

PMF in R

```
choose(n,k) * beta(k+alpha,n-k+beta) / beta(alpha,beta)
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\frac{n\alpha}{\alpha+\beta}$$

Variance

$$\frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Relationships

- Relationship pair: $BetaBinomial1(n, \alpha, \beta) \rightarrow UniformDiscrete2(n)$
- Relationship type: Special case
- Relationship definition: $\alpha = 1, \beta = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BetabinomialRectangular.pdf`

References and sources

`https://en.wikipedia.org/wiki/Beta-binomial_distribution`

UncertML mapping:

no UncertML mapping found.

BetaNegativeBinomial1

name	Beta Negative Binomial1 (ID: 0001167)
type	discrete
variate	x, scalar
support	$x \in \{0, \ldots, n\}$

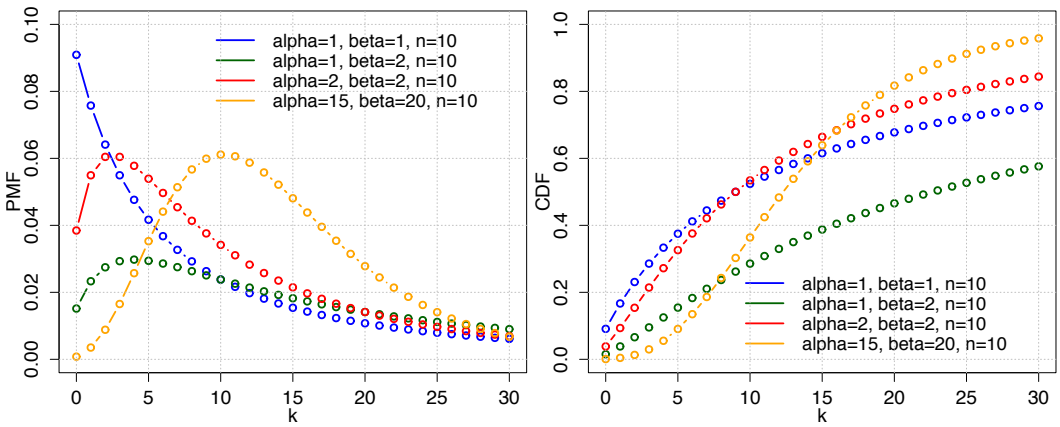

Figure A.10: BetaNegativeBinomial1 distribution plotted using the provided R code.

Model

Parameter: shape1

name	shape parameter
type	scalar
symbol	α
definition	$\alpha > 0$

Parameter: shape2

name	shape parameter
type	scalar
symbol	β
definition	$\beta > 0$

Parameter: numberOfSuccesses

name	number of successes
type	scalar
symbol	n
definition	$n \in N$

Functions

PMF

$$\frac{\binom{n-1+x}{x} B(n+\alpha, \beta+x)}{B(\alpha, \beta)}$$

PMF in R

```
choose(n-1+x,x)*beta(n+alpha,beta+x)/beta(alpha,beta)
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\begin{cases} \frac{n\beta}{\alpha-1} & \text{if } \alpha > 1 \\ \infty & \text{otherwise} \end{cases}$$

Relationships

- Relationship pair: $BetaNegativeBinomial1(\alpha, \beta, n) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ NegativeBinomial1(n,p) and $p \sim$ Beta1(α, β) then $X \sim$ NegativeBinomial1(α, β, n)
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalBetapascal.pdf`

References and sources

`https://en.wikipedia.org/wiki/Beta_negative_binomial_distribution`
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Betapascal.pdf`

UncertML mapping:

no UncertML mapping found.

Binomial1

name	Binomial 1 (ID: 0000117)
type	discrete
variate	k, scalar
support	$k \in \{0, \ldots, n\}$

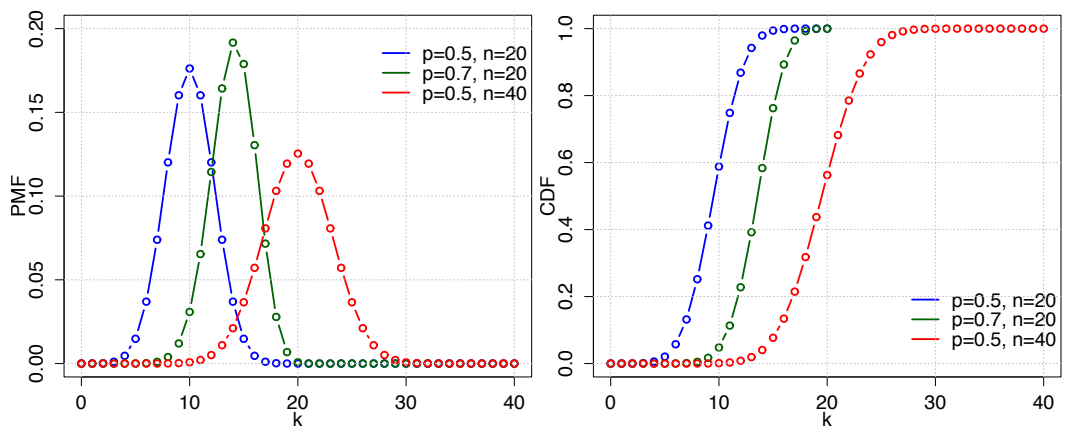

Figure A.11: Binomial1 distribution plotted using the provided R code.

Model

Binomial distribution with parameters n and p is the probability distribution of the number of successes in a sequence of n independent yes/no experiments (Bernoulli trial) each of which yields success with probability p.

Parameter: numberOfTrials

name	number of trials
type	scalar
symbol	n
definition	$n \in N, n \geq 0$

Parameter: probability

name	success probability in each trial
type	scalar
symbol	p
definition	$p \in [0, 1]$

Functions

PMF

$$\binom{n}{k} p^k (1-p)^{n-k}$$

PMF in R

```
choose(n,k) * p^k*(1-p)^(n-k)
```

CDF

$$I_{1-p}(n-k, 1+k)$$

CDF in R

```
Rbeta(1-p, n-k, 1+k)
```

Characteristics

Mean

$$np$$

Median

$$\lfloor np \rfloor \text{ or } \lceil np \rceil$$

Mode

$$\lfloor (n+1)p \rfloor \; or \; \lfloor (n+1)p \rfloor - 1$$

Variance

$$np(1-p)$$

Relationships

- Relationship pair: $Binomial1(n, p) \rightarrow Bernoulli1(p)$
- Relationship type: Special case
- Relationship definition: $n = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BinomialBernoulli.pdf`

- Relationship pair: $Binomial1(n, p) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $\lambda = np, n \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BinomialPoisson.pdf`

- Relationship pair: $Binomial1(n, p) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: For $X \sim Binomial1(n, p)$ as $n \rightarrow \infty, X$ is approximately normally distributed $Normal1(\mu, \sigma)$ with $\mu = np, \sigma = np(1-p)$.
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BinomialNormal.pdf`

- Relationship pair: $Binomial1(n, p) \rightarrow Binomial2(n, \alpha)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = \log(p/(1-p))$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $ConwayMaxwellPoisson1(\lambda, \nu) \rightarrow Binomial1(p)$
- Relationship type: Transformation
- Relationship definition: For $\nu = \infty$ the distribution of the sum is binomial with parameters n and $\lambda/(1+\lambda)$
- Relationship reference(s): [Shmueli et al., 2005]

- Relationship pair: $Hypergeometric1(N, K, n) \rightarrow Binomial1(n, p)$
- Relationship type: Limiting
- Relationship definition: $p = K/N, n = n, N \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/HypergeometricBinomial.pdf`

- Relationship pair: $Bernoulli1(p) \rightarrow Binomial1(n, p)$
- Relationship type: Transformation
- Relationship definition: $\Sigma X(iid)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BernoulliBinomial.pdf`

- Relationship pair: $GeneralizedNegativeBinomial1(\theta, \beta, m) \rightarrow Binomial1(n, p)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 0$ and set $m = n, \theta = p$
- Relationship reference(s): [Consul and Famoye, 2006]

- Relationship pair: $Binomial2(n, \alpha) \rightarrow Binomial1(n, p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = \exp(\alpha)/(1+\exp(\alpha))$
- Relationship reference(s): ProbOnto spec

References and sources

`https://en.wikipedia.org/wiki/Binomial_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/binomial`

Binomial2

name	Binomial 2 (ID: 0000149)
type	discrete
variate	k, scalar
support	$k \in \{0, \ldots, n\}$

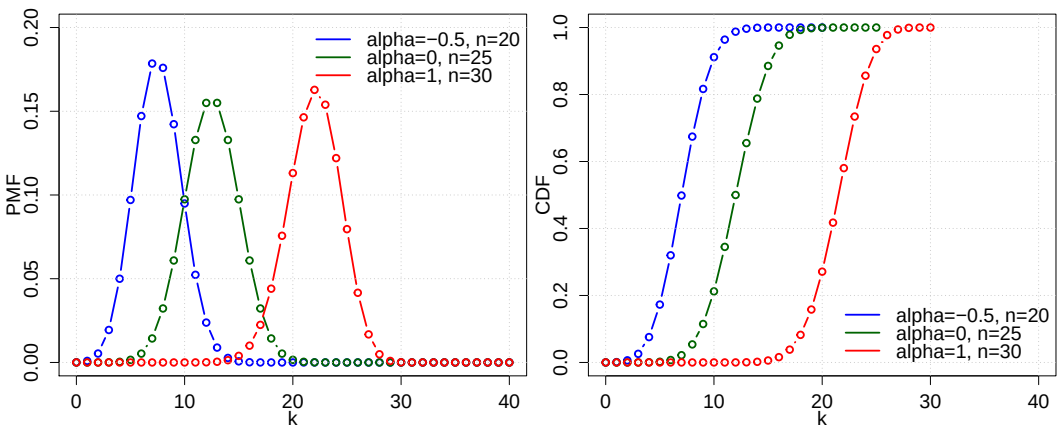

Figure A.12: Binomial2 distribution plotted using the provided R code.

Model

Binomial distribution with parameters n and α is the probability distribution of the number of successes in a sequence of n independent yes/no experiments (Bernoulli trial) each of which yields success with the chance of success parameterised on the unconstrained logistic scale.

Parameter: numberOfTrials

name	number of trials
type	scalar
symbol	n
definition	$n \in N, n \geq 0$

Parameter: logitProbability

name	logit of success probability in each trial
type	scalar
symbol	α
definition	$\alpha \in R$

Functions

PMF

$$\binom{n}{k} \left(logit^{-1}(\alpha)\right)^k \left(1 - logit^{-1}(\alpha)\right)^{n-k}$$

PMF in R

```
invLogit = function(x) { exp(x)/(1+exp(x)) }
choose(n,k)*invLogit(alpha)^k*(1-invLogit(alpha))^(n-k)
```

CDF

$$I_{1-logit^{-1}(\alpha)}(n-k, 1+k)$$

CDF in R

```
invLogit = function(x) { exp(x)/(1+exp(x)) } \\
Rbeta(1-invLogit(alpha), n-k, 1+k)
```

Characteristics

Relationships

- Relationship pair: $Binomial2(n, \alpha) \rightarrow Binomial1(n, p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = \exp(\alpha)/(1 + \exp(\alpha))$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Binomial1(n, p) \rightarrow Binomial2(n, \alpha)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = \log(p/(1 - p))$
- Relationship reference(s): ProbOnto spec

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

BirnbaumSaunders1

name	Birnbaum-Saunders 1 (ID: 0000177)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

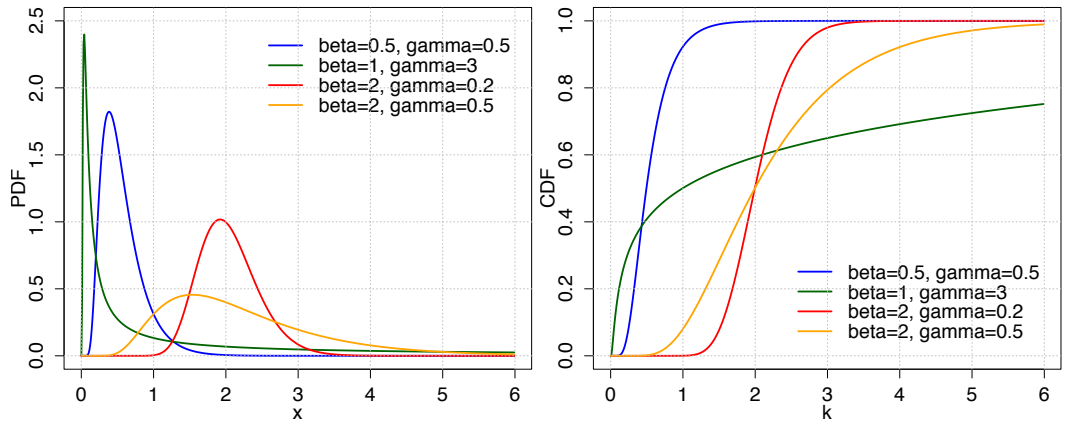

Figure A.13: BirnbaumSaunders1 distribution plotted using the provided R code.

Model

The Birnbaum-Saunders distribution, also known as the fatigue life distribution, is a probability distribution used extensively in reliability applications to model failure times.

Parameter: scale

name	scale
type	scalar
symbol	β
definition	$\beta > 0$

Parameter: shape

name	shape
type	scalar
symbol	γ
definition	$\gamma > 0$

Functions

PDF

$$\frac{1}{\sqrt{2\pi}} \exp\Big[-\frac{(\sqrt{x/\beta}-\sqrt{\beta/x})^2}{2\gamma^2}\Big]\Big[\frac{\sqrt{x/\beta}+\sqrt{\beta/x}}{2\gamma x}\Big]$$

PDF in R

```
1/(sqrt(2*pi))* exp( -(sqrt(x/beta) - sqrt(beta/x))^2 / (2*gamma^2) )
                        * ( sqrt(x/beta) + sqrt(beta/x) ) / (2*gamma*x)
```

CDF

$$\int_0^x f(x), \text{ with f the PDF}$$

CDF in R

```
cumsum(PDF*rep(stepSize,length(PDF)))
```

Characteristics

Relationships

References and sources

[MathWorks, 2015]
https://en.wikipedia.org/wiki/Birnbaum%E2%80%93Saunders_distribution

UncertML mapping:

no UncertML mapping found.

Borel1

name	Borel 1 (ID: 0001183)
type	discrete
variate	k, scalar
support	$k \in \{1, 2, \ldots, \infty\}$

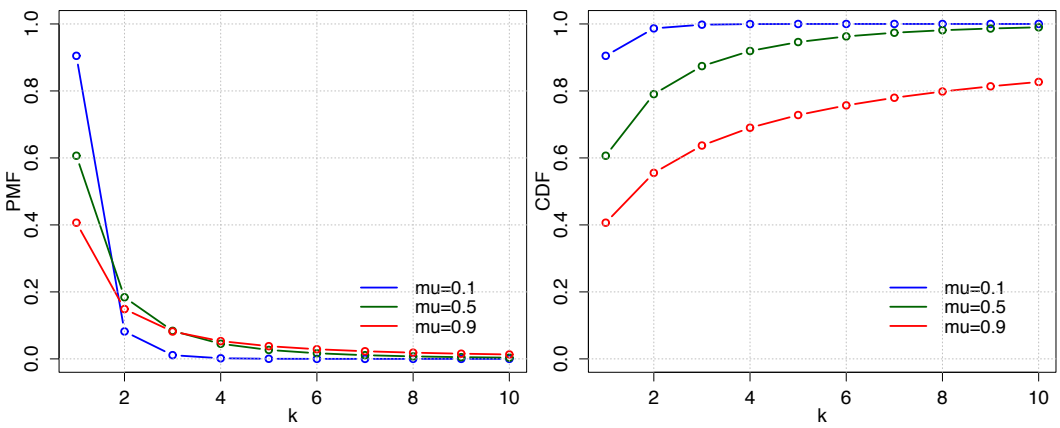

Figure A.14: Borel1 distribution plotted using the provided R code.

Model

The Borel distribution is a discrete probability distribution, arising in contexts including branching processes and queueing theory. If the number of offspring that an organism has is Poisson-distributed, and if the average number of offspring of each organism is no bigger than 1, then the descendants of each individual will ultimately become extinct. The number of descendants that an individual ultimately has in that situation is a random variable distributed according to a Borel distribution.

Parameter: shape

name	rate
type	scalar
symbol	μ
definition	$\mu \in (0, 1)$

Functions

PMF

$$\frac{e^{\mu n}(\mu n)^{n-1}}{n!}$$

PMF in R

```
e^(mu*n)*(mu*n)^(n-1) / factorial(n)
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Relationships

References and sources

https://en.wikipedia.org/wiki/Borel_distribution
http://www.itl.nist.gov/div898/software/dataplot/refman2/auxillar/btapdf.htm

UncertML mapping:

no UncertML mapping found.

Burr1

name	Burr 1 (ID: 0000204)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

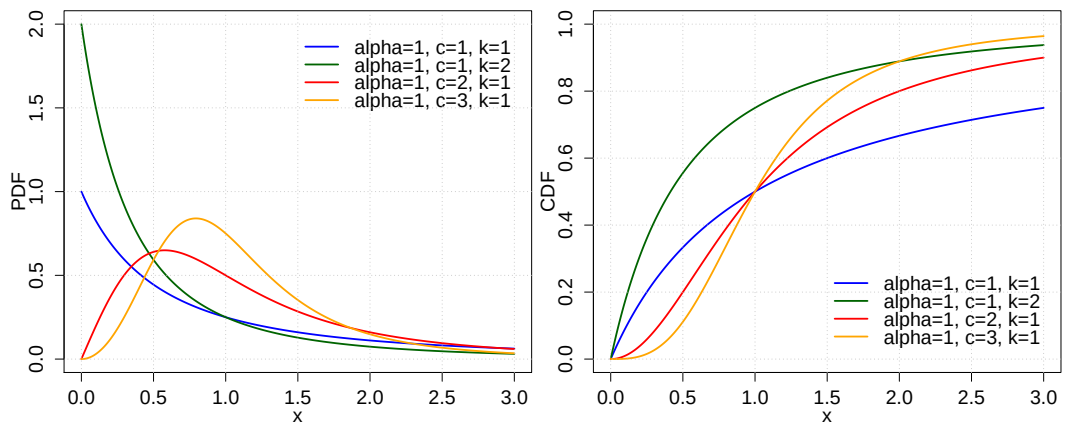

Figure A.15: Burr1 distribution plotted using the provided R code.

Model

In probability theory, statistics and econometrics, the Burr Type XII distribution or simply the Burr distribution is a continuous probability distribution for a non-negative random variable. It is also known as the Singh-Maddala distribution and is one of a number of different distributions sometimes called the generalised log-logistic distribution. It is most commonly used to model household income.

Parameter: scale

name	scale parameter
type	scalar
symbol	α
definition	$\alpha > 0$

Parameter: shape1

name	shape parameter
type	scalar
symbol	c
definition	$c > 0$

Parameter: shape2

name	shape parameter
type	scalar
symbol	k
definition	$k > 0$

Functions

PDF

$$\frac{\frac{kc}{\alpha}(x/\alpha)^{c-1}}{\left[1+(x/\alpha)^c\right]^{k+1}}$$

PDF in R

```
k*c/alpha * (x/alpha)^(c-1) / (1+(x/alpha)^c)^(k+1)
```

CDF

$$1-\frac{1}{\left[1+(x/\alpha)^c\right]^{k}}$$

CDF in R

```
1 - 1/(1+(x/alpha)^c)^k
```

HF

$$\frac{\frac{kc}{\alpha}(x/\alpha)^{c-1}}{1+(x/\alpha)^c}$$

HF in R

```
(k*c)/alpha * (x/alpha)^(c-1) / (1+(x/alpha)^c)
```

SF

$$\frac{1}{\left[1+(x/\alpha)^c\right]^{k}}$$

SF in R

```
1/(1+(x/alpha)^c)^k
```

Characteristics

Relationships

References and sources

http://nl.mathworks.com/help/stats/burr-type-xii-distribution.html

UncertML mapping:

no UncertML mapping found.

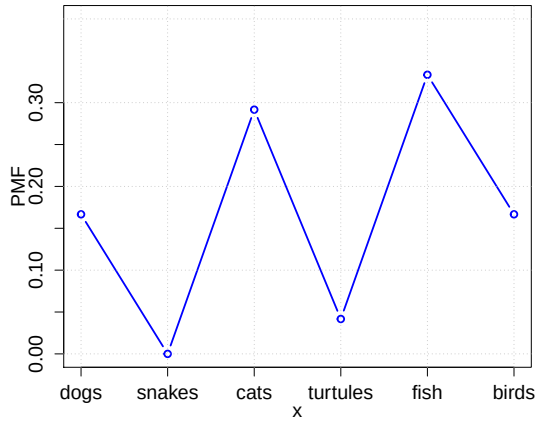

Figure A.16: CategoricalNonordered1 distribution plotted using the provided R code.

CategoricalNonordered1

name	Categorical Nonordered 1 (ID: 0000248)
type	discrete
variate	x, scalar
support	$x \in \{1, \ldots, k\}$

Parameter: categoryProb

name	category probabilities
type	vector
symbol	$p_1, \ldots, p_k$
definition	$0 \leq p_i \leq 1, \Sigma p_i = 1$

Functions

PMF

$$p(x = i) = p_i$$

CDF

$$undefined$$

Characteristics

Mean

$$E([x = i]) = p_i, \text{this is the mean of the Iverson bracket } [x = i] \text{ and not the mean of } x$$

Median

$$i \text{ such that } \sum_{j=1}^{i-1} p_j \leq 0.5 \text{ and } \sum_{j=1}^{i} p_j \geq 0.5$$

Mode

$$i \text{ such that } p_i = \max(p_1, \ldots, p_k)$$

Variance

$$Var([x = i]) = p_i(1 - p_i); \quad Cov([x = i], [x = j]) = -p_i p_j \quad (i \neq j)$$

Relationships

References and sources

http://en.wikipedia.org/wiki/Categorical_distribution

UncertML mapping:

no UncertML mapping found.

CategoricalOrdered1

name	Categorical Ordered 1 (ID: 0000224)
type	discrete
variate	x, scalar
support	$x \in \{1, \ldots, k\}$

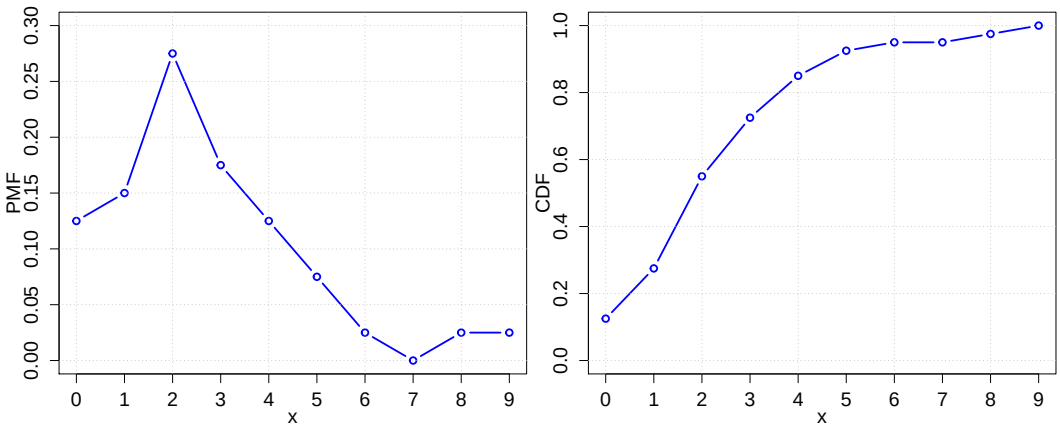

Figure A.17: CategoricalOrdered1 distribution plotted using the provided R code.

Parameter: categoryProb

name	category probabilities
type	vector
symbol	$p_1, \ldots, p_k$
definition	$0 \leq p_i \leq 1, \Sigma p_i = 1$

Functions

PMF

$$p(x = i) = p_i$$

CDF

$$\begin{cases} 0 & \text{for } x < 1 \\ \sum_{j=1}^{i} p_j & \text{for } x \in [i, i+1) \\ 1 & \text{for } x \geq k \end{cases}$$

Characteristics

Mean

$$E([x = i]) = p_i, \text{this is the mean of the Iverson bracket } [x = i] \text{ and not the mean of } x$$

Median

$$i \text{ such that } \sum_{j=1}^{i-1} p_j \leq 0.5 \text{ and } \sum_{j=1}^{i} p_j \geq 0.5$$

Mode

$$i \text{ such that } p_i = \max(p_1, \ldots, p_k)$$

Variance

$$Var([x = i]) = p_i(1 - p_i); \quad Cov([x = i], [x = j]) = -p_i p_j \quad (i \neq j)$$

Relationships

References and sources

`http://en.wikipedia.org/wiki/Categorical_distribution`

UncertML mapping:

no UncertML mapping found.

Cauchy1

name	Cauchy 1 (ID: 0000274)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

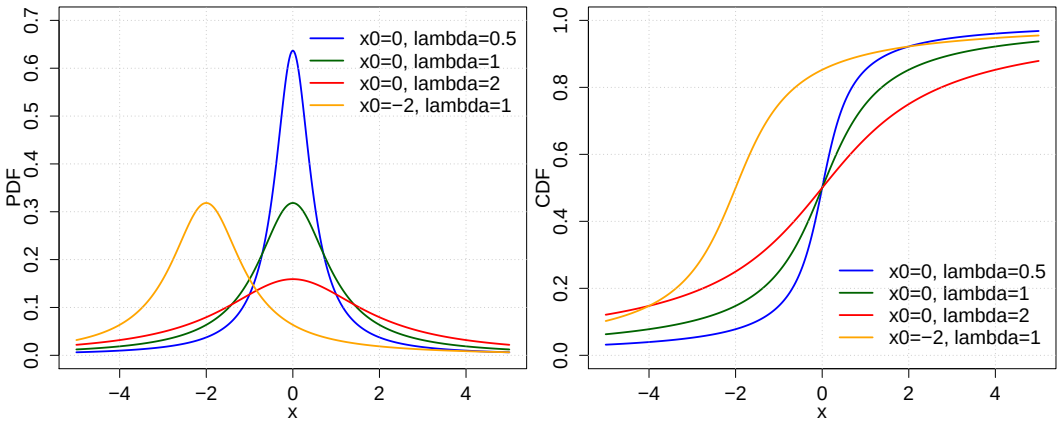

Figure A.18: Cauchy1 distribution plotted using the provided R code.

Model

The Cauchy distribution $f(x; x_0, \gamma)$ is the distribution of the X-intercept of a ray issuing from (x_0, γ) with a uniformly distributed angle. The Cauchy distribution is often used in statistics as the canonical example of a 'pathological' distribution since both its mean and its variance are undefined.

Parameter: location

name	location
type	scalar
symbol	x_0
definition	$x_0 \in R$

Parameter: scale

name	scale
type	scalar
symbol	γ
definition	$\gamma \in R$

Functions

PDF

$$\frac{1}{\pi\gamma\left[1+\left(\frac{x-x_0}{\gamma}\right)^2\right]}$$

PDF in R

```
1 / (pi*gamma*(1 + ((x-x0)^2/gamma^2)))
```

CDF

$$\frac{1}{\pi}\arctan\left(\frac{x-x_0}{\gamma}\right)+\frac{1}{2}$$

CDF in R

```
1/pi * atan((x-x0)/gamma)+1/2
```

Characteristics

Mean

$$undefined$$

Median

$$x_0$$

Mode

$$x_0$$

Variance

$$undefined$$

Relationships

- Relationship pair: $Cauchy1(x_0, \gamma) \rightarrow StandardCauchy1$
- Relationship type: Special case
- Relationship definition: $x0 = 0, \gamma = 1$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Cauchy_distribution`

References and sources

`http://en.wikipedia.org/wiki/Cauchy_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/cauchy`

Chi1

name	Chi 1 (ID: 0001243)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

Model

The chi distribution of the square root of the sum of squares of independent random variables having a standard normal distribution, or equivalently, the distribution of the Euclidean distance of the random variables from the origin.

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	k
definition	$k > 0$

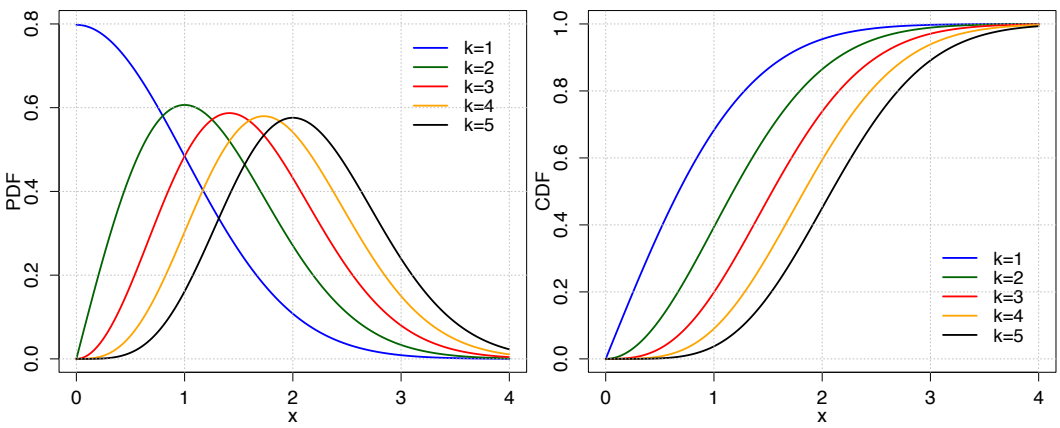

Figure A.19: Chi1 distribution plotted using the provided R code.

Functions

PDF

$$\frac{2^{1-k/2}x^{k-1}e^{-x^2/2}}{\Gamma\left(\frac{k}{2}\right)}$$

PDF in R

```
2^(1-k/2) * x^(k-1) * exp(-x^2/2) / gamma(k/2)
```

CDF

$$\frac{\gamma\left(\frac{k}{2}, \frac{x^2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}$$

CDF in R

```
Igamma(k/2,x^2/2,lower=T) / gamma(k/2)
```

Characteristics

Mean

$$\mu = \sqrt{2}\,\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}$$

Mode

$$\sqrt{k-1} \text{ for } k \geq 1$$

Variance

$$k - \mu^2$$

Relationships

- Relationship pair: $Chi1(k) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Chi1(k)$ then $X^2 \sim ChiSquared1(k)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Chi_distribution`

- Relationship pair: $Amoroso1(a, \theta, \alpha, \beta) \rightarrow Chi1(k)$
- Relationship type: Special case
- Relationship definition: Chi1(k) distribution is a special case of the Amoroso1$(a, \theta, \alpha, \beta)$ distribution when $a = 0, \theta = \sqrt{2}, \alpha = k/2, \beta = 2$
- Relationship reference(s): [Crooks, 2010]

References and sources

`https://en.wikipedia.org/wiki/Chi_distribution`
`http://doc.openturns.org/openturns-1.6/sphinx/user_manual/_generated/openturns.Chi.html`

UncertML mapping:

no UncertML mapping found.

ChiSquared1

name	Chi-squared 1 (ID: 0000299)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

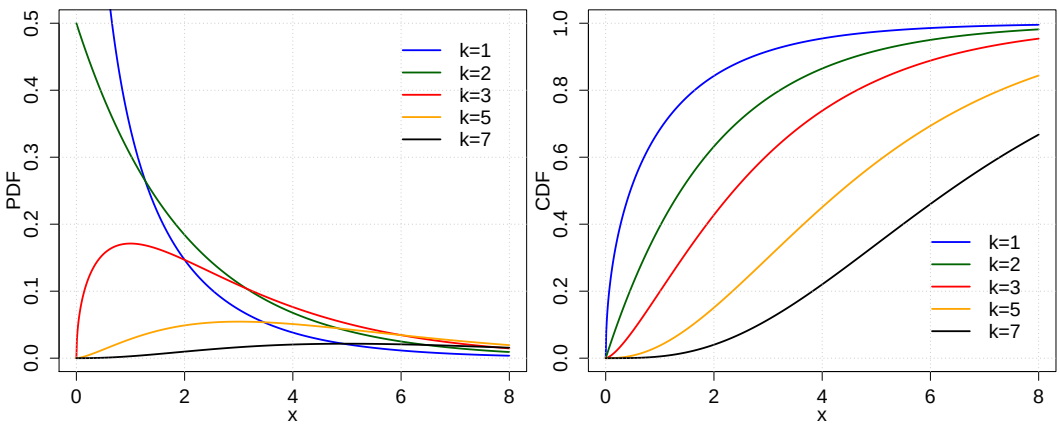

Figure A.20: ChiSquared1 distribution plotted using the provided R code.

Model

Chi-squared distribution with k degrees of freedom is the distribution of a sum of the squares of k independent standard normal random variables. It is a special case of the gamma distribution and is one of the most widely used probability distributions in inferential statistics, e.g., in hypothesis testing or in construction of confidence intervals.

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	k
definition	$k \in N$

Functions

PDF

$$\frac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}\, x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$$

PDF in R

```
1/( 2^k/2 * gamma(k/2) ) *  x^(k/2-1) * exp(-x/2)
```

CDF

$$\frac{1}{\Gamma\left(\frac{k}{2}\right)}\, \gamma\left(\frac{k}{2}, \frac{x}{2}\right)$$

CDF in R

```
1/gamma(k/2) * Igamma(k/2,x/2)
```

Characteristics

Mean

$$k$$

Median

$$\approx k\left(1-\frac{2}{9k}\right)^3$$

Mode

$$max\{k-2,0\}$$

Variance

$$2k$$

Relationships

- Relationship pair: $ChiSquared1(k) \rightarrow Exponential1(\lambda)$
- Relationship type: Special case
- Relationship definition: $k=2$ and $\lambda = 1/2$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ChisquareExponential.pdf`

- Relationship pair: $ChiSquared1(n) \rightarrow F1(n_1, n_2)$
- Relationship type: Transformation
- Relationship definition: If $X_1 \sim ChiSquared1(n_1), X_2 \sim ChiSquared1(n_2)$ are independent random variables $\Rightarrow \frac{X_1/n_1}{X_2/n_2} \sim F1(n_1, n_2)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ChisquareF.pdf`

- Relationship pair: $GeneralizedGamma2(a,b,c,k) \rightarrow ChiSquared1(k)$
- Relationship type: Reparameterisation
- Relationship definition: $a=0, b=2, c=k_{ChiSquare1}/2, k=1$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $SkewNormal1(\mu,\sigma,\alpha) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: If $X \sim SkewNormal1(\mu,\sigma,\alpha)$ then $X^2 \sim ChiSquared1(1)$
- Relationship reference(s): `http://azzalini.stat.unipd.it/SN/Intro/intro.html`

- Relationship pair: $StandardNormal1(0,1) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: ΣX^2
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Standardnormalchisquare.pdf`

- Relationship pair: $InverseGaussian1(\lambda,\mu) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: $X \sim InverseGaussian1(\lambda,\mu)$ and $Y = \lambda(X-\mu)^2/(\mu^2 X) \Rightarrow Y \sim ChiSquared1(k)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/InversegaussianChisquare.pdf`

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow ChiSquared1(n)$
- Relationship type: Transformation
- Relationship definition: If $X_i \sim N(\mu,\sigma), i=1,2,...,n$ are mutually independent and identically distributed random variables and $Y = \sum_{i=1}^{n}((X_i-\mu)/\sigma)^2 \Rightarrow Y \sim ChiSquared1(n)$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalChisquare.pdf`

- Relationship pair: $Gamma1(k,\theta) \rightarrow ChiSquared1(n)$
- Relationship type: Special case
- Relationship definition: $k_{ChiSquared1} = 2k, \theta = 2$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaChisquareT.pdf`

- Relationship pair: $F1(n_1,n_2) \rightarrow ChiSquared1(n)$
- Relationship type: Limiting
- Relationship definition: If $X \sim F1(n_1,n_2)$, the limiting distribution of $n_1 X$ as $n_2 \rightarrow \infty$ is the chi-square distribution with n_1 degrees of freedom

- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/FChisquare.pdf`
- Relationship pair: $Chi1(k) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Chi1(k)$ then $X^2 \sim ChiSquared1(k)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Chi_distribution`
- Relationship pair: $Amoroso1(a, \theta, \alpha, \beta) \rightarrow ChiSquared1(k)$
- Relationship type: Special case
- Relationship definition: ChiSquared1(k) distribution is a special case of the Amoroso1$(a, \theta, \alpha, \beta)$ distribution when $a = 0, \theta = 2, \alpha = k/2, \beta = 1$
- Relationship reference(s): [Crooks, 2010]

References and sources

`http://en.wikipedia.org/wiki/Chi-squared_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/chi-square`

ConwayMaxwellPoisson1

name	Conway-Maxwell-Poisson 1 (ID: 0000323)
type	discrete
variate	x, scalar
support	$x \in \{0, 1, 2, 3, \dots\}$

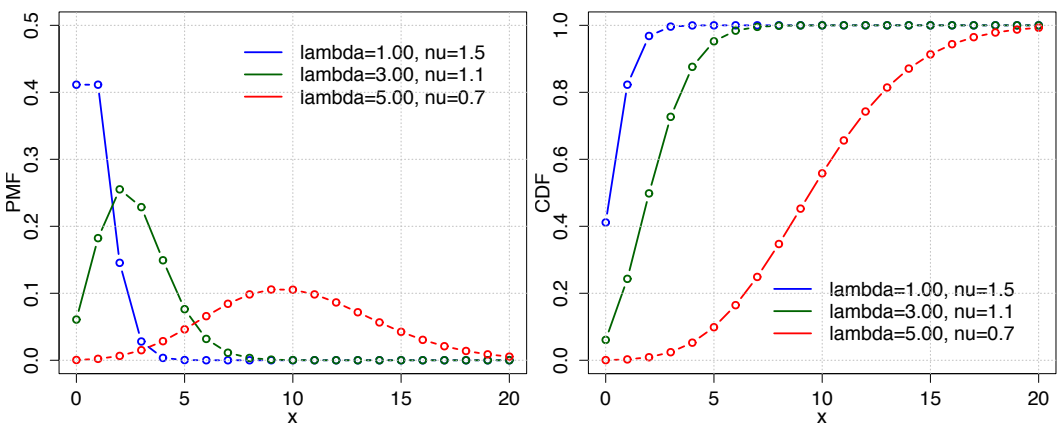

Figure A.21: ConwayMaxwellPoisson1 distribution plotted using the provided R code.

Model

The Conway-Maxwell-Poisson (CMP or COM-Poisson) distribution is a discrete probability distribution named after Richard W. Conway, William L. Maxwell, and Simeon Denis Poisson that generalises the Poisson distribution by adding a parameter to model overdispersion and underdispersion.

Parameter: rate

name	Poisson intensity
type	scalar
symbol	λ
definition	$\lambda \in R, \lambda > 0$

Parameter: rateOfDecay

name	rate of decay
type	scalar
symbol	ν
definition	$\nu \geq 0$

Functions

PMF

$$\frac{\lambda^x}{(x!)^\nu}\frac{1}{Z(\lambda,\nu)} \text{ with } Z(\lambda,\nu) = \sum_{j=0}^{\infty}\frac{\lambda^j}{(j!)^\nu}$$

PMF in R

```
lambda^x/(factorial(x))^nu*1/Z(lambda,nu,n);
 Z(lambda,nu,n): for(i in 0:n) { Z=Z+lambda^i/(factorial(i))^nu }}
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0,1,2,...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\sum_{j=0}^{\infty}\frac{j\lambda^j}{(j!)^\nu Z(\lambda,\nu)}$$

Median

No closed form

Variance

$$\sum_{j=0}^{\infty}\frac{j^2\lambda^j}{(j!)^\nu Z(\lambda,\nu)} - \text{Mean}^2$$

Relationships

- Relationship pair: $ConwayMaxwellPoisson1(\lambda,\nu) \rightarrow Poisson1(\lambda)$
- Relationship type: Transformation
- Relationship definition: For $\nu = 1$ the sum has a Poisson distribution with parameter $n\lambda$
- Relationship reference(s): [Shmueli et al., 2005]
- Relationship pair: $ConwayMaxwellPoisson1(\lambda,\nu) \rightarrow Binomial1(p)$
- Relationship type: Transformation
- Relationship definition: For $\nu = \infty$ the distribution of the sum is binomial with parameters n and $\lambda/(1+\lambda)$
- Relationship reference(s): [Shmueli et al., 2005]
- Relationship pair: $ConwayMaxwellPoisson1(\lambda,\nu) \rightarrow NegativeBinomial1(r,p)$
- Relationship type: Transformation
- Relationship definition: For $\nu = 0$ and $\lambda < 1$ the sum of Conway-Maxwell-Poisson distributed variables reduces to the sum of geometric variables, which follows a Negative Binomial distribution with parameters n and $1-\lambda$
- Relationship reference(s): [Shmueli et al., 2005]

References and sources

[Shmueli et al., 2005]
https://en.wikipedia.org/wiki/Conway%E2%80%93Maxwell%E2%80%93Poisson_distribution

UncertML mapping:

no UncertML mapping found.

Dagum1

name	Dagum 1 (ID: 0001269)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

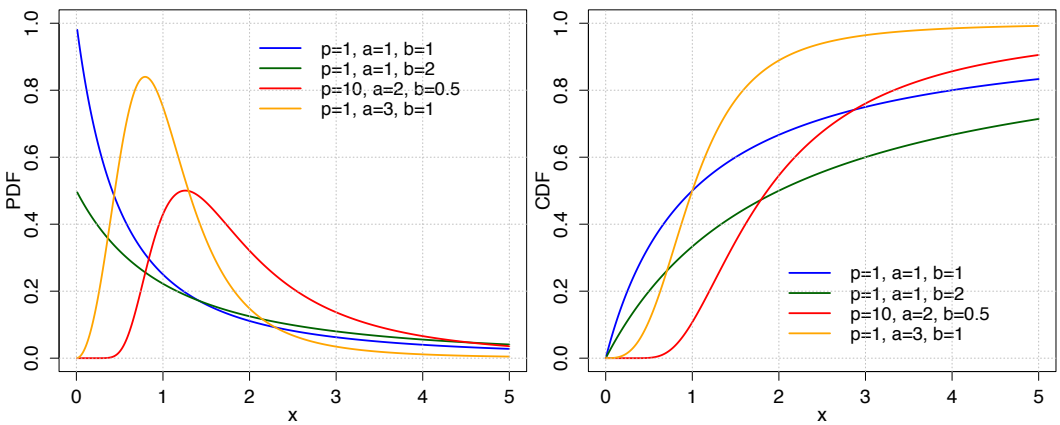

Figure A.22: Dagum1 distribution plotted using the provided R code.

Model

Parameter: shape1

name	shape parameter
type	scalar
symbol	p
definition	$p > 0$

Parameter: shape2

name	shape parameter
type	scalar
symbol	a
definition	$a > 0$

Parameter: scale

name	scale parameter
type	scalar
symbol	b
definition	$b > 0$

Functions

PDF

$$\frac{ap}{x}\left(\frac{(\frac{x}{b})^{ap}}{\left((\frac{x}{b})^{a}+1\right)^{p+1}}\right)$$

PDF in R

```
a*p/x * (x/b)^(a*p) / ((x/b)^a + 1)^(p+1)
```

CDF

$$\left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}$$

CDF in R

```
(1+(x/b)^(-a))^(-p)
```

Characteristics

Mean

$$\begin{cases} -\frac{b}{a}\frac{\Gamma\left(-\frac{1}{a}\right)\Gamma\left(\frac{1}{a}+p\right)}{\Gamma(p)} & \text{if } a>1 \\ \text{Indeterminate} & \text{otherwise} \end{cases}$$

Median

$$b\left(-1+2^{\frac{1}{p}}\right)^{-\frac{1}{a}}$$

Mode

$$b\left(\frac{ap-1}{a+1}\right)^{\frac{1}{a}}$$

Variance

$$\begin{cases} -\frac{b^2}{a^2}\left(2a\frac{\Gamma\left(-\frac{2}{a}\right)\Gamma\left(\frac{2}{a}+p\right)}{\Gamma(p)}+\left(\frac{\Gamma\left(-\frac{1}{a}\right)\Gamma\left(\frac{1}{a}+p\right)}{\Gamma(p)}\right)^2\right) & \text{if } a>2 \\ \text{Indeterminate} & \text{otherwise} \end{cases}$$

Relationships

References and sources

https://en.wikipedia.org/wiki/Dagum_distribution

UncertML mapping:

no UncertML mapping found.

Dirichlet1

name	Dirichlet 1 (ID: 0000345)
type	continuous
variate	x, vector
support	$x_1, \cdots, x_K$ where $x_i \in [0,1]$ *and* $\sum_{i=1}^{K} x_i = 1$

Model

The Dirichlet distribution is a multivariate generalisation of the beta distribution.

Parameter: concentration

name	concentration
type	vector
symbol	$\alpha_1, \cdots, \alpha_K$
definition	$\alpha_1, \cdots, \alpha_K, \alpha_i > 0$

Functions

PDF

$$\frac{1}{B(\alpha)}\prod_{i=1}^{K} x_i^{\alpha_i-1} \text{ where } B(\alpha)=\frac{\prod_{i=1}^{K}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{K}\alpha_i\right)} \text{ where } \alpha=(\alpha_1,\ldots,\alpha_K)$$

Characteristics

Mean

$$E[X_i] = \frac{\alpha_i}{\sum_k \alpha_k}$$

Mode

$$x_i = \frac{\alpha_i - 1}{\sum_{i=1}^{K} \alpha_i - K}, \quad \alpha_i > 1$$

Variance

$$Var[X_i] = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)} \quad \text{where} \quad \alpha_0 = \sum_{i=1}^{K} \alpha_i; \quad Cov[X_i, X_j] = \frac{-\alpha_i \alpha_j}{\alpha_0^2(\alpha_0 + 1)} \quad (i \neq j)$$

Relationships

References and sources

`http://en.wikipedia.org/wiki/Dirichlet_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/dirichlet`

DoublePoisson1

name	Double Poisson 1 (ID: 0000370)
type	discrete
variate	x, scalar
support	$x \in \{1, 2, 3, \ldots, n\}$

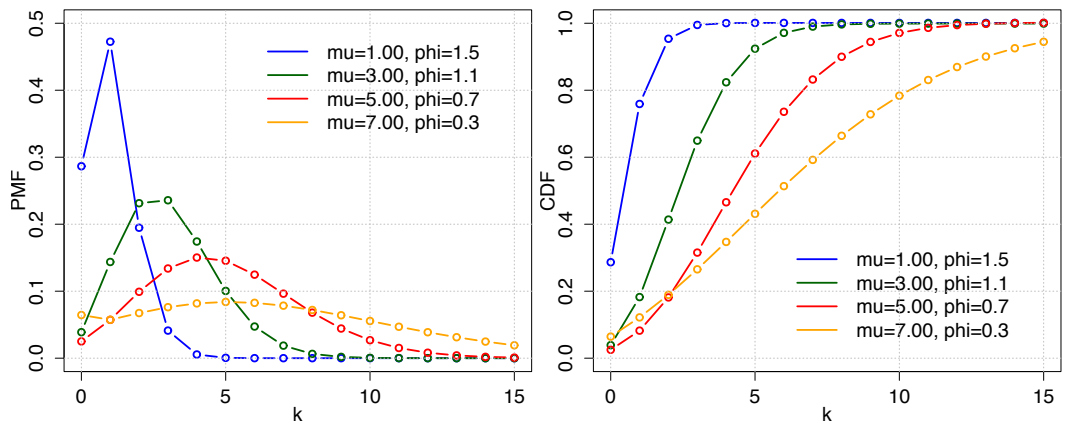

Figure A.23: DoublePoisson1 distribution plotted using the provided R code.

Model

The Double Poisson distribution is obtained as an exponential combination of two Poisson distributions.

Parameter: rate

name	Poisson intensity
type	scalar
symbol	μ
definition	$\mu \in R, \mu > 0$

Parameter: dispersion

name	dispersion
type	scalar
symbol	ϕ
definition	$\phi \in R$

Functions

PMF

$$K(\mu,\phi)\phi^{1/2}\exp(-\phi\mu)\frac{\exp(-y)y^y}{y!}\Big(\frac{e\mu}{y}\Big)^{\phi y} \text{ with } \frac{1}{K(\mu,\phi)} \approx 1 + \frac{1-\phi}{12\phi\mu}\Big(1+\frac{1}{\phi\mu}\Big)$$

PMF in R

```
K(mu,phi) * phi^(1/2) * exp(-phi*mu) * (exp(-y)*y^y)/factorial(y) * (exp(1)*mu/y)^(phi*y) \\
# with
K(mu,phi) = 1 / (1 + (1-phi)/(12*mu*phi)*(1 + 1/(mu*phi)) )
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0,1,2,...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\mu$$

Variance

$$\mu/\phi$$

Relationships

- Relationship pair: $DoublePoisson1(\mu,\phi) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\phi = 1$
- Relationship reference(s): [Cameron and Trivedi, 2013]

References and sources

`http://support.sas.com/resources/papers/proceedings09/250-2009.pdf`
[Cameron and Trivedi, 2013]

UncertML mapping:

no UncertML mapping found.

Epanechnikov1

name	Epanechnikov 1 (ID: 0001309)
type	continuous
variate	x, scalar
support	$x \in [-1,1]$

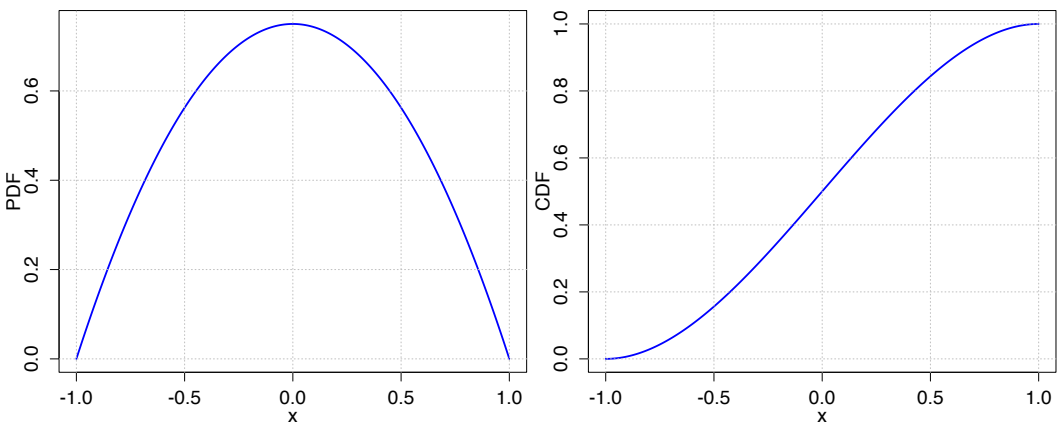

Figure A.24: Epanechnikov1 distribution plotted using the provided R code.

Model

Functions

PDF

$$\frac{3}{4}(1-x^2)$$

PDF in R

```
3/4 * (1-x^2)
```

CDF

$$-\frac{1}{4}x^3+\frac{3}{4}x+\frac{1}{2}$$

CDF in R

```
-1/4*x^3 + 3/4*x + 1/2
```

Characteristics

Mean

$$0$$

Variance

$$\frac{1}{5}$$

Relationships

References and sources

http://doc.openturns.org/openturns-1.6/sphinx/user_manual/_generated/openturns.Epanechnikov.html#openturns.Epanechnikov

UncertML mapping:

no UncertML mapping found.

Erlang1

name	Erlang 1 (ID: 0000392)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$ (Forbes), $x \in (0, +\infty)$ (Leemis)

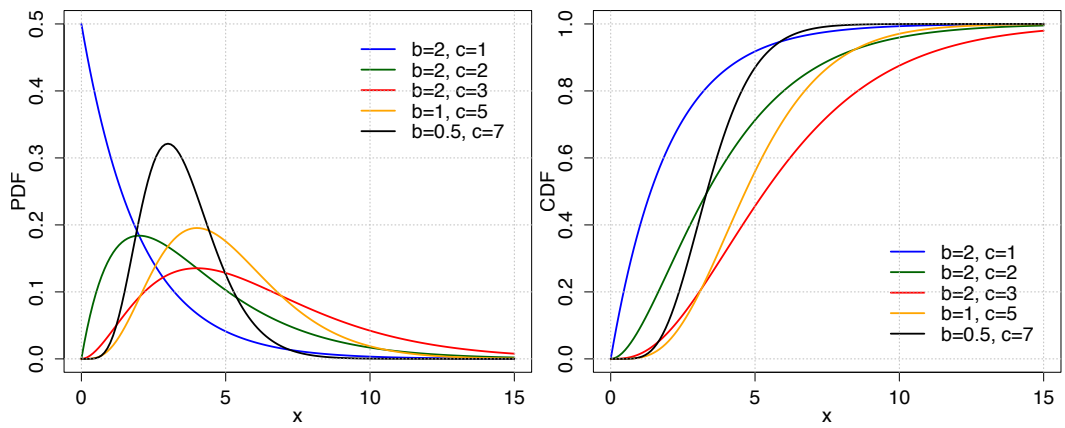

Figure A.25: Erlang1 distribution plotted using the provided R code.

Model

The Erlang variate is the sum of a number of exponential variates. It was developed as the distribution of waiting time and message length in telephone traffic. If the durations of individual calls are exponentially distributed, the duration of a succession of calls has an Erlang distribution.

Parameter: scale

name	scale
type	scalar
symbol	b
definition	$b > 0$

Parameter: shape

name	shape
type	scalar
symbol	c
definition	$c > 0$

Functions

PDF

$$\frac{(x/b)^{c-1}\exp(-x/b)}{b(c-1)!}$$

PDF in R

```
((x/b)^(c-1) *exp(-x/b)) / (b * factorial(c-1))
```

CDF

$$1-\Big[\exp\Big(-\frac{x}{b}\Big)\Big]\Big(\sum_{i=0}^{c-1}\frac{(x/b)^i}{i!}\Big)$$

CDF in R

```
R function
```

HF

$$\frac{(x/b)^{c-1}}{b(c-1)!\sum_{i=0}^{c-1}\frac{(x/b)^i}{i!}}$$

SF

$$\Big[\exp\Big(-\frac{x}{b}\Big)\Big]\Big(\sum_{i=0}^{c-1}\frac{(x/b)^i}{i!}\Big)$$

Characteristics

Mean

$$bc$$

Mode

$$b(c-1), c \geq 1$$

Variance

$$b^2c$$

Relationships

- Relationship pair: $Erlang1(b,c) \rightarrow Exponential2(\beta)$
- Relationship type: Special case
- Relationship definition: $c = 1, b = \beta$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Gamma1(k,\theta) \rightarrow Erlang1(b,c)$
- Relationship type: Special case
- Relationship definition: $k \in N, k = c, \theta = b$
- Relationship reference(s): [Forbes et al., 2011]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaErlang.pdf`

References and sources

[Forbes et al., 2011]
[Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Erlang.pdf`

UncertML mapping:

no UncertML mapping found.

Exponential1

name	Exponential 1 (ID: 0000418)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

Model

This is a distribution of the time to an event when the probability of the event occurring in the next small time interval does not vary through time. It is also the distribution of the time between events when the number of events in any time interval has a Poisson distribution.

Parameter: rate

name	rate or inverse scale
type	scalar
symbol	λ
definition	$\lambda > 0$

Functions

PDF

$$\lambda e^{-\lambda x}$$

PDF in R

```
lambda*exp(-lambda*x)
```

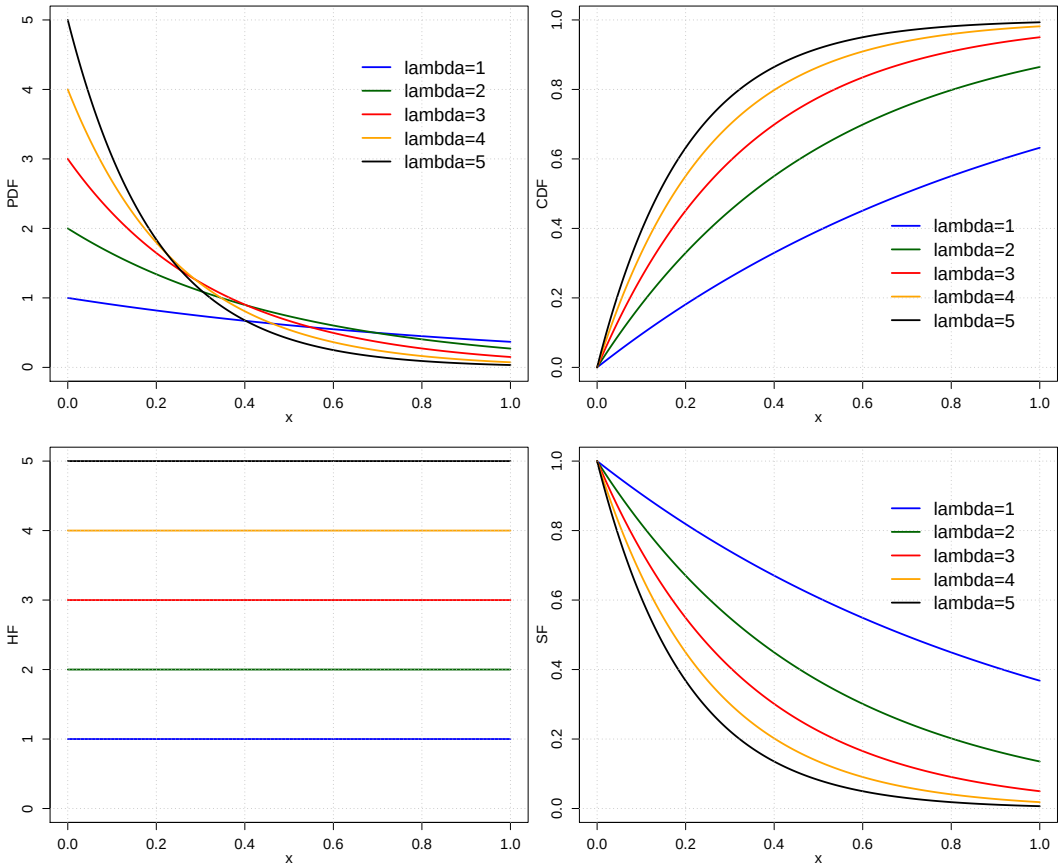

Figure A.26: Exponential1 distribution plotted using the provided R code.

CDF

$$1 - \exp(-\lambda x)$$

CDF in R

```
1 - exp(-lambda*x)
```

HF

$$\lambda$$

HF in R

```
lambda
```

SF

$$\exp(-\lambda x)$$

SF in R

```
exp(-lambda*x)
```

Characteristics

Mean

$$\lambda^{-1}$$

Median

$$\lambda^{-1} \ln(2)$$

Mode

$$0$$

Variance

$$\lambda^{-2}$$

Relationships

- Relationship pair: $Exponential1(\lambda) \rightarrow Exponential2(\beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 1/\lambda$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Rice1(\nu, \sigma) \rightarrow Exponential1(\lambda)$
- Relationship type: Transformation
- Relationship definition: If $R \sim Rice1(0, \sigma)$ then $R^2 \sim Exponential1(\lambda)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Rice_distribution`

- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow Exponential1(\lambda)$
- Relationship type: Reparameterisation
- Relationship definition: $k = c = 1, a = 0, b = 1/\lambda$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Kumaraswamy1(a, b) \rightarrow Exponential1(a)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Kumaraswamy1(a, 1)$ then $-\log(X) \sim Exponential1(a)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Kumaraswamy_distribution`

- Relationship pair: $Gamma1(k, \theta) \rightarrow Exponential1(\lambda)$
- Relationship type: Reparameterisation
- Relationship definition: $k = 1, \theta = 1/\lambda$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Kumaraswamy1(a, b) \rightarrow Exponential1(b)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Kumaraswamy1(1, b)$ then $-(1 - \log(X)) \sim Exponential1(b)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Kumaraswamy_distribution`

- Relationship pair: $Weibull1(\lambda, k) \rightarrow Exponential1(\lambda_{Exponetial})$
- Relationship type: Reparameterisation
- Relationship definition: $k = 1, \lambda_{Exponetial} = 1/\lambda$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $ChiSquared1(k) \rightarrow Exponential1(\lambda)$
- Relationship type: Special case
- Relationship definition: $k = 2$ and $\lambda = 1/2$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ChisquareExponential.pdf`

- Relationship pair: $StandardUniform1(0, 1) \rightarrow Exponential1(\lambda)$
- Relationship type: Transformation
- Relationship definition: $-\frac{1}{\lambda}\log(X)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformExponentialB.pdf`

- Relationship pair: $Exponential2(\beta) \rightarrow Exponential1(\lambda)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda = 1/\beta$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $ParetoTypeI1(x_m, \alpha) \rightarrow Exponential1(\lambda)$
- Relationship type: Transformation
- Relationship definition: $X \sim ParetoTypeI1, Y = \log(X/\lambda) \Rightarrow Y \sim Exponential1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ParetoExponential.pdf`

References and sources

`http://en.wikipedia.org/wiki/Exponential_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/exponential`

Exponential2

name	Exponential 2 (ID: 0000443)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

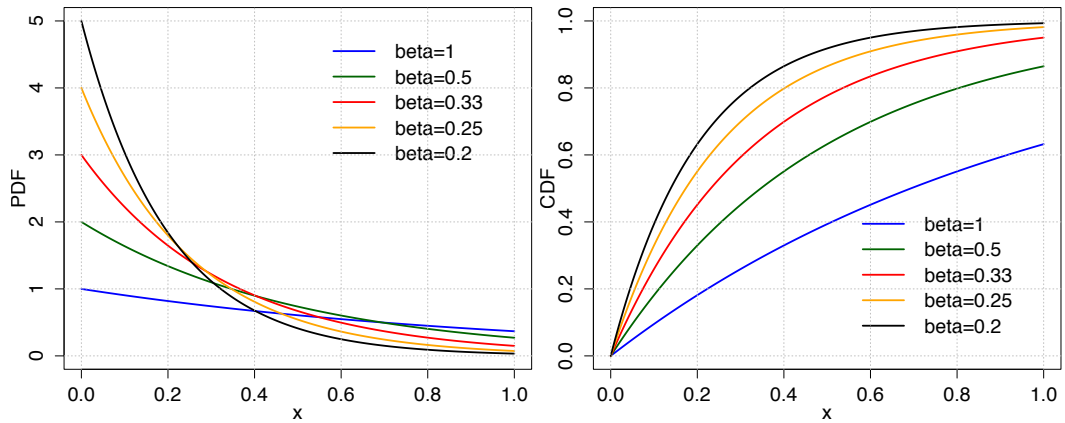

Figure A.27: Exponential2 distribution plotted using the provided R code.

Model

This is a distribution of the time to an event when the probability of the event occurring in the next small time interval does not vary through time. It is also the distribution of the time between events when the number of events in any time interval has a Poisson distribution.

Parameter: mean

name	scale
type	scalar
symbol	β
definition	$\beta > 0$

Functions

PDF

$$1/\beta e^{-x/\beta}$$

PDF in R

```
1/beta*exp(-1/beta*x)
```

CDF

$$1 - \exp(-x/\beta)$$

CDF in R

```
1 - exp(-x/beta)
```

HF

$$1/\beta$$

HF in R

```
1/beta
```

SF

$$e^{-x/\beta}$$

SF in R

```
exp(-x/beta)
```

Characteristics

Mean

$$\beta$$

Median

$$\lambda^{-1}\ln(2)$$

Mode

$$0$$

Variance

$$\beta^2$$

Relationships

- Relationship pair: $Exponential2(\beta) \rightarrow Exponential1(\lambda)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda = 1/\beta$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Exponential2(1) \rightarrow F1(n_1, n_2)$
- Relationship type: Transformation
- Relationship definition: If $X_1, X_2 \sim Exponential2(1)$ mutually independent and identically distributed random variables $\Rightarrow X_1/X_2$ has the $F1$ distribution
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ExponentialF.pdf`

- Relationship pair: $Exponential1(\lambda) \rightarrow Exponential2(\beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 1/\lambda$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Erlang1(b, c) \rightarrow Exponential2(\beta)$
- Relationship type: Special case
- Relationship definition: $c = 1, b = \beta$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Muth1(\kappa) \rightarrow Exponential2(\beta)$
- Relationship type: Limiting
- Relationship definition: As $\kappa \rightarrow 0$ for a Muth1(κ) , the limiting distribution is Exponential2 with mean 1.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/MuthExponential.pdf`

References and sources

`http://www.itl.nist.gov/div898/handbook/eda/section3/eda3667.htm`
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Exponential.pdf`

UncertML mapping:

no UncertML mapping found.

ExponentiallyModifiedGaussian1

name	Exponentially modified Gaussian 1 (ID: 0000470)
type	continuous
variate	x, scalar
support	$x \in R$

Model

The exponentially modified Gaussian distribution (EMG) describes the sum of independent normal and exponential random variables.

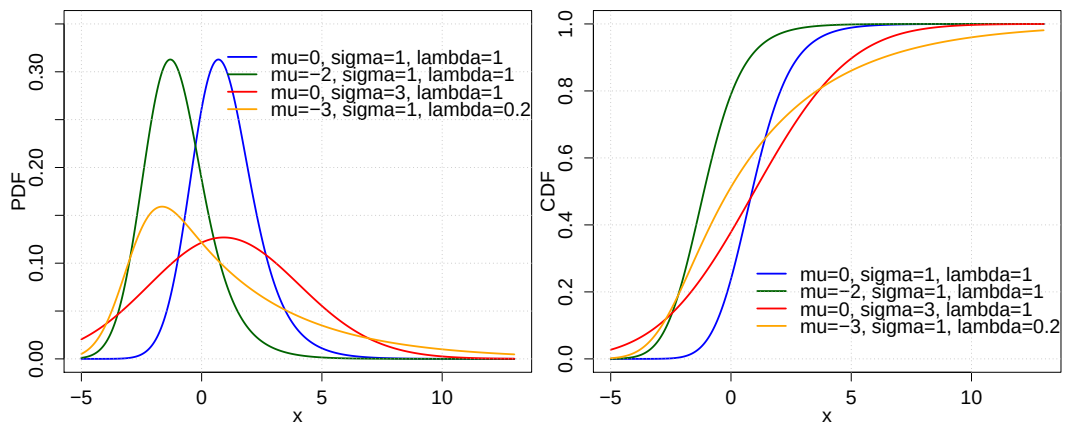

Figure A.28: ExponentiallyModifiedGaussian1 distribution plotted using the provided R code.

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: stdev

name	standard deviation
type	scalar
symbol	σ
definition	$\sigma > 0$

Parameter: rate

name	rate or inverse scale
type	scalar
symbol	λ
definition	$\lambda > 0$

Functions

PDF

$$\frac{\lambda}{2} e^{\frac{\lambda}{2}(2\mu+\lambda\sigma^2-2x)} \mathrm{erfc}(\frac{\mu+\lambda\sigma^2-x}{\sqrt{2}\sigma})$$

PDF in R

```
lambda/2*exp(lambda/2*(2*mu+lambda*sigma^2-2*x))*erfc((mu+lambda*sigma^2-x)/(sqrt(2)*sigma)) \\
erfc <- function(x) 2 * pnorm(x * sqrt(2), lower = FALSE)
```

CDF

$$\Phi(u,0,v) - e^{-u+v^2/2+\log(\Phi(u,v^2,v))} \text{ with } u = \lambda(x-\mu) \text{ and } v = \lambda\sigma$$

CDF in R

```
CDF = function(x,mu,sigma,lambda) {
  u = lambda * (x - mu); v = lambda * sigma;
  CDF_Normal1(u, 0, v) - exp(-u + v^2/2 + log(CDF_Normal1(u, v^2, v)))};
CDF_Normal1 = function(x,mu,sigma) {  1/2 * (1 + erf((x-mu)/(sigma*sqrt(2)))) }
```

Characteristics

Mean

$$\mu + 1/\lambda$$

Variance

$$\sigma^2 + 1/\lambda^2$$

Relationships

References and sources

`https://en.wikipedia.org/wiki/Exponentially_modified_Gaussian_distribution`
[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

F1

name	F 1 (ID: 0000492)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

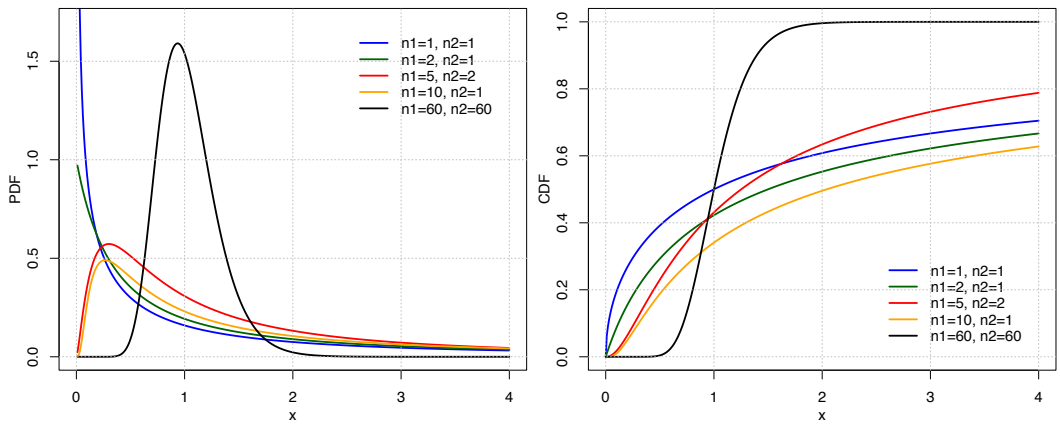

Figure A.29: F1 distribution plotted using the provided R code.

Model

The F variate is the ratio of two chi-squared variates. Chi-squared is the distribution of the variance between data and a theoretical model. The F distribution provides a basis for comparing the ratios of subsets of these variances associated with different factors.

Parameter: numerator

name	degree of freedom
type	scalar
symbol	n_1
definition	$n_1 > 0$

Parameter: denominator

name	degree of freedom
type	scalar
symbol	n_2
definition	$n_2 > 0$

Functions

PDF

$$\frac{\Gamma(\frac{n_1+n_2}{2})(\frac{n_1}{n_2})^{n_1/2}x^{n_1/2-1}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})\left[\frac{n_1}{n_2}x+1\right]^{(n_1+n_2)/2}}$$

PDF in R

```
gamma((n1 + n2)/2)*(n1/n2)^(n1/2)*x^(n1/2-1)/(gamma(n1/2)*gamma(n2/2)*(n1/n2*x+1)^((n1+n2)/2))
```

CDF

$$I_{\frac{n_1 x}{n_1 x + n_2}}\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$$

CDF in R

```
Rbeta(n1*x / (n1*x + n2), n1/2, n2/2)
```

Characteristics

Mean

$$n_2/(n_2 - 2), n_2 > 2$$

Mode

$$\frac{n_2(n_1 - 2)}{n_1(n_2 + 2)}, n_1 > 2$$

Variance

$$\frac{2n_2^2(n_1 + n_2 - 2)}{n_1(n_2 - 2)^2(n_2 - 4)}, n_2 > 4$$

Relationships

- Relationship pair: $F1(n_1, n_2) \rightarrow ChiSquared1(n)$
- Relationship type: Limiting
- Relationship definition: If $X \sim F1(n_1, n_2)$, the limiting distribution of $n_1 X$ as $n_2 \rightarrow \infty$ is the chi-square distribution with n_1 degrees of freedom
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/FChisquare.pdf

- Relationship pair: $ChiSquared1(n) \rightarrow F1(n_1, n_2)$
- Relationship type: Transformation
- Relationship definition: If $X_1 \sim ChiSquared1(n_1), X_2 \sim ChiSquared1(n_2)$ are independent random variables $\Rightarrow \frac{X_1/n_1}{X_2/n_2} \sim F1(n_1, n_2)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ChisquareF.pdf

- Relationship pair: $Exponential2(1) \rightarrow F1(n_1, n_2)$
- Relationship type: Transformation
- Relationship definition: If $X_1, X_2 \sim Exponential2(1)$ mutually independent and identically distributed random variables $\Rightarrow X_1/X_2$ has the $F1$ distribution
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ExponentialF.pdf

- Relationship pair: $StudentT1(\nu) \rightarrow F1(n_1, n_2)$
- Relationship type: Transformation
- Relationship definition: If $X \sim StudentT1(\nu) \Rightarrow Y = X^2 \sim F(1, \nu)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TF.pdf

References and sources

[Forbes et al., 2011]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/F.pdf
http://en.wikipedia.org/wiki/F-distribution

UncertML mapping:

http://www.uncertml.org/distributions/f

FoldedNormal1

name	Folded Normal 1 (ID: 0001386)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

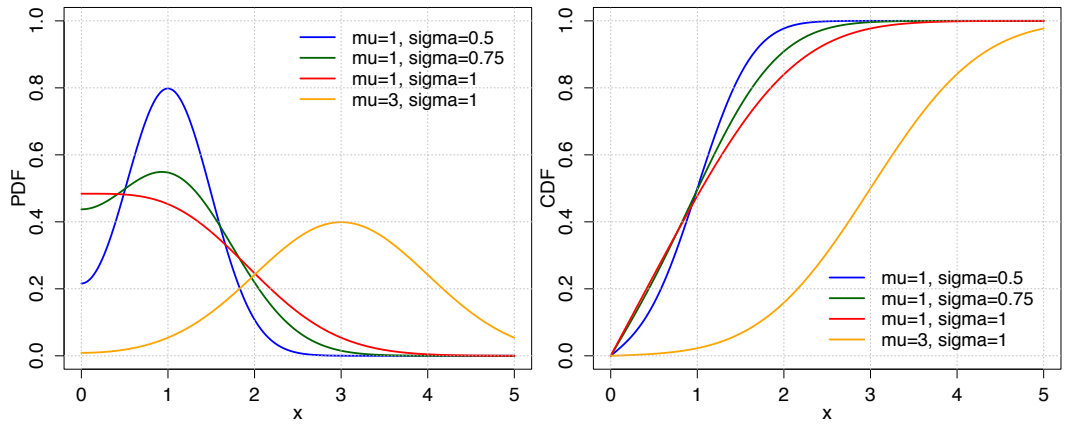

Figure A.30: FoldedNormal1 distribution plotted using the provided R code.

Model

The folded normal distribution is a probability distribution related to the normal distribution. Given a normally distributed random variable X with mean μ and variance σ^2, the random variable $Y = |X|$ has a folded normal distribution. The distribution is called Folded because probability mass to the left of the $x = 0$ is folded over by taking the absolute value.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x+\mu)^2}{2\sigma^2}}$$

PDF in R

```
1/(sigma*sqrt(2*pi))*exp(-(x-mu)^2/(2*sigma^2)) + 1/(sigma*sqrt(2*pi))*exp(-(x+mu)^2/(2*sigma^2))
```

CDF

$$\frac{1}{2}\left[\operatorname{erf}\left(\frac{x+\mu}{\sigma\sqrt{2}}\right) + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

CDF in R

```
1/2 * (erf((x-mu)/(sigma*sqrt(2))) + erf((x+mu)/(sigma*sqrt(2))))
```

Characteristics

Mean

$$\sigma\sqrt{\tfrac{2}{\pi}}\,e^{(-\mu^2/2\sigma^2)} + \mu\left(1 - 2\,\Phi(\tfrac{-\mu}{\sigma})\right)$$

Variance

$$\mu^2 + \sigma^2 - \left(\sigma\sqrt{\tfrac{2}{\pi}}\,e^{(-\mu^2/2\sigma^2)} + \mu\left(1 - 2\,\Phi(\tfrac{-\mu}{\sigma})\right)\right)^2$$

Relationships

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow FoldedNormal1(\mu,\sigma)$
- Relationship type: Transformation
- Relationship definition: Given a normally distributed random variable X with mean μ and variance σ^2, the random variable $Y = |X|$ has a folded normal distribution
- Relationship reference(s): `https://en.wikipedia.org/wiki/Folded_normal_distribution`

References and sources

`https://en.wikipedia.org/wiki/Folded_normal_distribution`

UncertML mapping:

no UncertML mapping found.

Frechet1

name	Frechet 1 (ID: 0000517)
type	continuous
variate	x, scalar
support	$x \in R^+$

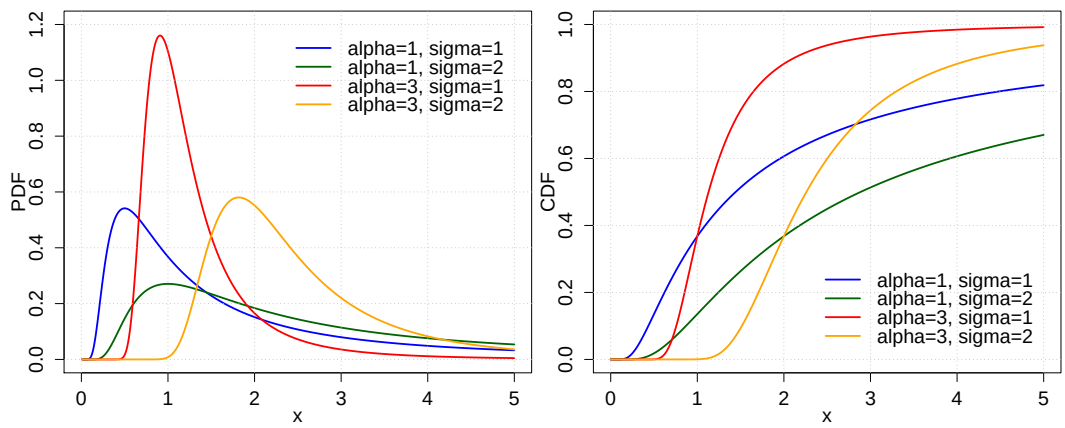

Figure A.31: Frechet1 distribution plotted using the provided R code.

Model

The Frechet distribution, also known as inverse Weibull distribution, is a special case of the generalized extreme value distribution.

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha \in R^+$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma \in R^+$

Functions

PDF

$$\frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-\alpha-1}\exp\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)$$

PDF in R

```
alpha/sigma * (x/sigma)^(-alpha-1) * exp(-(x/sigma)^(-alpha))
```

CDF

$$\exp\left(-\left(\frac{\sigma}{x}\right)^{\alpha}\right)$$

CDF in R

```
exp(-(sigma/x)^alpha)
```

Characteristics

Mean

$$\begin{cases} \sigma\Gamma\left(1-\frac{1}{\alpha}\right) & \text{for } \alpha > 1 \\ \infty & \text{otherwise} \end{cases}$$

Median

$$\frac{\sigma}{\sqrt[\alpha]{\log_e(2)}}$$

Mode

$$\sigma\left(\frac{\alpha}{1+\alpha}\right)^{1/\alpha}$$

Variance

$$\begin{cases} \sigma^2\left(\Gamma\left(1-\frac{2}{\alpha}\right)-\left(\Gamma\left(1-\frac{1}{\alpha}\right)\right)^2\right) & \text{for } \alpha > 2 \\ \infty & \text{otherwise} \end{cases}$$

Relationships

- Relationship pair: $Frechet1(\alpha,\sigma) \rightarrow Weibull1(\lambda,k)$
- Relationship type: Transformation
- Relationship definition: $X \sim Frechet1(\alpha,\sigma) \rightarrow X^{-1} \sim Weibull(\lambda = 1/\sigma, k = \alpha)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution` [STAN Development Team, 2015b]
- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \rightarrow Frechet1(\alpha,\sigma)$
- Relationship type: Special case
- Relationship definition: Frechet1(α,σ) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a = 0, \theta = \sigma, \alpha = 1, \beta rightarrow - \beta$
- Relationship reference(s): [Crooks, 2010]
- Relationship pair: $Frechet2(\alpha,\sigma,m) \rightarrow Frechet1(\alpha,\sigma)$
- Relationship type: Special case
- Relationship definition: $m = 0$
- Relationship reference(s): `http://www.mathwave.com/help/easyfit/html/analyses/distributions/frechet.html`

References and sources

http://www.mathwave.com/help/easyfit/html/analyses/distributions/frechet.html
[STAN Development Team, 2015b]
https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution

UncertML mapping:

no UncertML mapping found.

Frechet2

name	Frechet 2 (ID: 0000543)
type	continuous
variate	x, scalar
support	$x \in (m, \infty)$

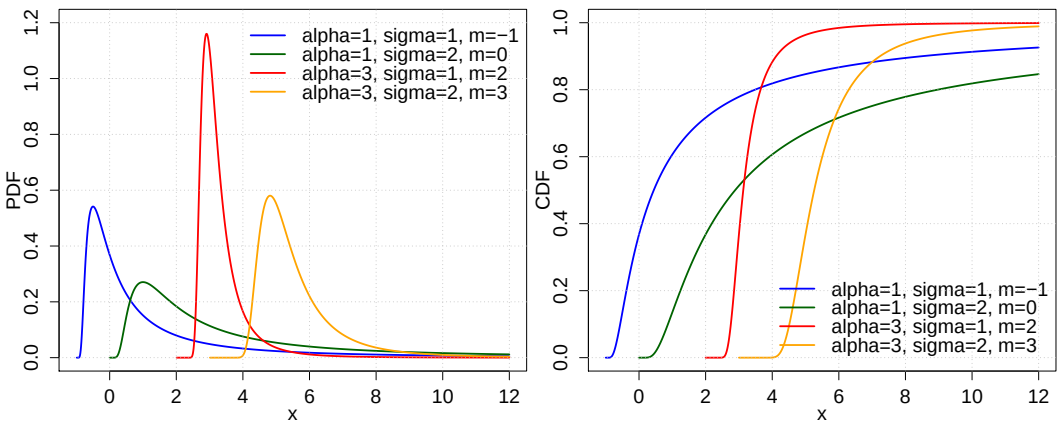

Figure A.32: Frechet2 distribution plotted using the provided R code.

Model

The Frechet distribution, also known as inverse Weibull distribution, is a special case of the generalized extreme value distribution.

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha \in (0, \infty)$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma \in (0, \infty)$

Parameter: locationOfMinimum

name	location of minimum
type	scalar
symbol	m
definition	$m \in (-\infty, \infty)$

Functions

PDF

$$\frac{\alpha}{\sigma}\left(\frac{x-m}{\sigma}\right)^{-\alpha-1}\exp\left(-\left(\frac{x-m}{\sigma}\right)^{-\alpha}\right)$$

PDF in R

```
alpha/sigma * ((x-m)/sigma)^(-alpha-1) * exp(-((x-m)/sigma)^(-alpha))
```

CDF

$$\exp\left(-\left(\frac{\sigma}{x-m}\right)^{\alpha}\right)$$

CDF in R

```
exp(-(sigma/(x-m))^alpha)
```

Characteristics

Mean

$$\begin{cases} m+\sigma\Gamma\left(1-\frac{1}{\alpha}\right) & \text{for } \alpha>1 \\ \infty & \text{otherwise} \end{cases}$$

Median

$$m+\frac{\sigma}{\sqrt[\alpha]{\log_e(2)}}$$

Mode

$$m+\sigma\left(\frac{\alpha}{1+\alpha}\right)^{1/\alpha}$$

Variance

$$\begin{cases} \sigma^2\left(\Gamma\left(1-\frac{2}{\alpha}\right)-\left(\Gamma\left(1-\frac{1}{\alpha}\right)\right)^2\right) & \text{for } \alpha>2 \\ \infty & \text{otherwise} \end{cases}$$

Relationships

- Relationship pair: $Frechet2(\alpha,\sigma,m) \rightarrow Frechet1(\alpha,\sigma)$
- Relationship type: Special case
- Relationship definition: $m=0$
- Relationship reference(s): `http://www.mathwave.com/help/easyfit/html/analyses/distributions/frechet.html`

- Relationship pair: $Frechet2(\alpha,\sigma,m) \rightarrow Weibull1(\lambda,k)$
- Relationship type: Transformation
- Relationship definition: $X \sim Frechet2(\alpha,\sigma,m)$ with $m=0 \rightarrow X^{-1} \sim Weibull(\lambda=1/\sigma, k=\alpha)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution` [STAN Development Team, 2015b]

- Relationship pair: $StandardUniform1(0,1) \rightarrow Frechet2(\alpha,\sigma,m)$
- Relationship type: Transformation
- Relationship definition: If $X \sim StandardUniform1(0,1)$ then $m+\sigma(-\log(X))^{-1/\alpha} \sim Frechet2(\alpha,\sigma,m)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution`

References and sources

`http://www.mathwave.com/help/easyfit/html/analyses/distributions/frechet.html`
`https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution`

UncertML mapping:

no UncertML mapping found.

Gamma1

name	Gamma 1 (ID: 0000571)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

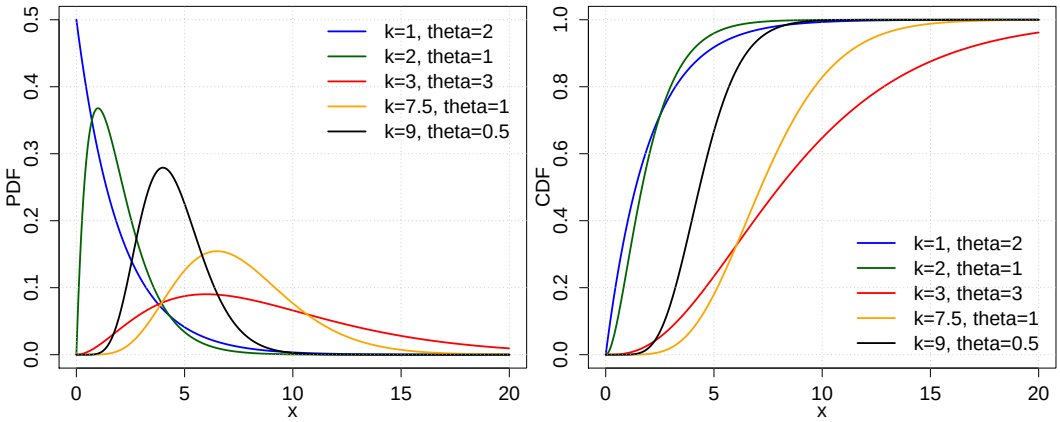

Figure A.33: Gamma1 distribution plotted using the provided R code.

Model

The gamma distribution includes the chi-squared, Erlang, and exponential distributions as special cases, but the shape parameter of the gamma is not confined to integer values. The gamma distribution starts at the origin and has a flexible shape. The parameters are easy to estimate by matching moments.

Parameter: shape

name	shape
type	scalar
symbol	k
definition	$k > 0$

Parameter: scale

name	scale
type	scalar
symbol	θ
definition	$\theta > 0$

Functions

PDF

$$\frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$$

PDF in R

```
1 / (gamma(k) * theta^k) * x^(k-1) * exp(-x/theta)
```

CDF

$$\frac{1}{\Gamma(k)} \gamma\left(k, \frac{x}{\theta}\right)$$

CDF in R

```
1/gamma(k) * Igamma(k,x/theta)
```

SF

$$1 - \frac{\Gamma(\beta, x/\alpha)}{\Gamma(\beta)}$$

Characteristics

Mean

$$k\theta$$

Median

No simple closed form

Mode

$$(k - 1)\theta \text{ for } k \geq 1$$

Variance

$$k\theta^2$$

Relationships

- Relationship pair: $Gamma1(k, \theta) \rightarrow Exponential1(\lambda)$
- Relationship type: Reparameterisation
- Relationship definition: $k = 1, \theta = 1/\lambda$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Gamma1(\alpha, \beta) \rightarrow InverseGamma1(\alpha, \beta)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Gamma1(\alpha, \beta)$ then $X^{-1} \sim InverseGamma1(\alpha, \beta^{-1})$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Inverse-gamma_distribution`

- Relationship pair: $Gamma1(k, \theta) \rightarrow Gamma2(r, \mu)$
- Relationship type: Reparameterisation
- Relationship definition: $r = k, \mu = 1/\theta$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Gamma1(k, \theta) \rightarrow Beta1(\alpha, \beta)$
- Relationship type: Transformation
- Relationship definition: $X1, X2 \sim Gamma1(k, \theta)$ and $Y = X1/(X1 + X2) \Rightarrow Y \sim Beta1(\alpha, \beta)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaBeta.pdf`

- Relationship pair: $Gamma1(k, theta) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $\mu = k\theta, \sigma^2 = k^2\theta, \theta \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaNormal1.pdf`

- Relationship pair: $Gamma1(k, \theta) \rightarrow Erlang1(b, c)$
- Relationship type: Special case
- Relationship definition: $k \in N, k = c, \theta = b$
- Relationship reference(s): [Forbes et al., 2011]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaErlang.pdf`

- Relationship pair: $Gamma1(k, \theta) \rightarrow ChiSquared1(n)$
- Relationship type: Special case
- Relationship definition: $k_{ChiSquared1} = 2k, \theta = 2$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaChisquareT.pdf`

- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow Gamma1(k, \theta)$
- Relationship type: Transformation
- Relationship definition: $k = 1, a = 0$ and renaming parameters: $c = k, b = \theta$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $GeneralizedGamma1(a, d, p) \rightarrow Gamma1(k, \theta)$
- Relationship type: Special case
- Relationship definition: $p = 1, k = d, \theta = a$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Gamma2(r, mu) \rightarrow Gamma1(k, \theta)$
- Relationship type: Reparameterisation
- Relationship definition: $k = r, \theta = 1/\mu$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Amoroso1(a, \theta, \alpha, \beta) \rightarrow Gamma1(k, \theta)$

- Relationship type: Special case
- Relationship definition: Gamma1 distribution is a special case of the Amoroso1$(a, \theta, \alpha, \beta)$ distribution when$a = 0, \beta = 1$
- Relationship reference(s): [Crooks, 2010]

References and sources

`http://en.wikipedia.org/wiki/Gamma_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/gamma`

Gamma2

name	Gamma 2 (ID: 0000597)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

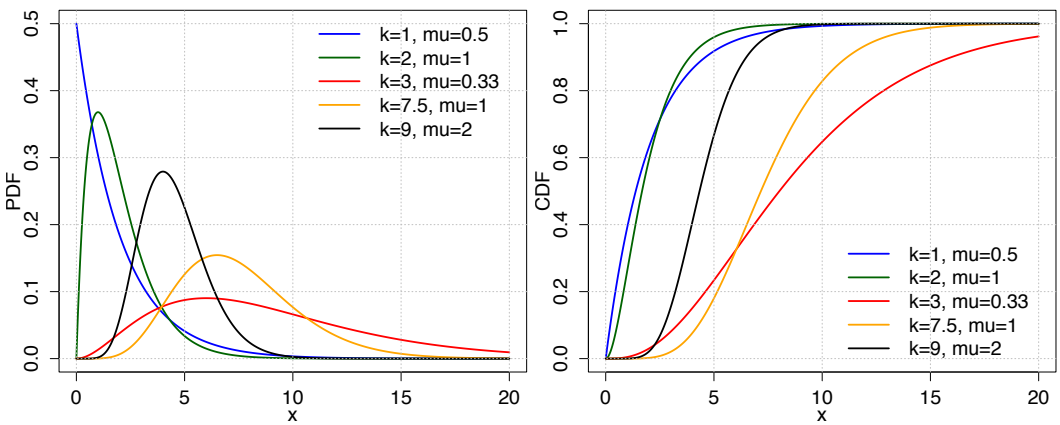

Figure A.34: Gamma2 distribution plotted using the provided R code.

Model

The gamma distribution includes the chi-squared, Erlang, and exponential distributions as special cases, but the shape parameter of the gamma is not confined to integer values. The gamma distribution starts at the origin and has a flexible shape. The parameters are easy to estimate by matching moments.

Parameter: shape

name	shape
type	scalar
symbol	r
definition	$r > 0$

Parameter: rate

name	rate
type	scalar
symbol	μ
definition	$\mu > 0$

Functions

PDF

$$\frac{\mu^r x^{r-1} e^{-\mu x}}{\Gamma(r)}$$

PDF in R

```
(mu^r * x^(r-1) * exp(-mu*x)) / gamma(r)
```

CDF

$$\frac{1}{\Gamma(r)} \gamma\left(r, \mu x\right)$$

CDF in R

```
1/gamma(r) * Igamma(r,mu*x,lower=T)
```

Characteristics

Mean

$$r/\mu$$

Variance

$$r/\mu^2$$

Relationships

- Relationship pair: $Gamma2(r, mu) \rightarrow Gamma1(k, \theta)$
- Relationship type: Reparameterisation
- Relationship definition: $k = r, \theta = 1/\mu$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Gamma1(k, \theta) \rightarrow Gamma2(r, \mu)$
- Relationship type: Reparameterisation
- Relationship definition: $r = k, \mu = 1/\theta$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

GeneralizedGamma1

name	Generalized Gamma 1 (ID: 0000621)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

Model

The generalised gamma distribution is not often used to model life data by itself, but it is sometimes used to determine which of those life distributions should be used to model a particular set of data.

Parameter: scale

name	scale
type	scalar
symbol	a
definition	$a > 0$

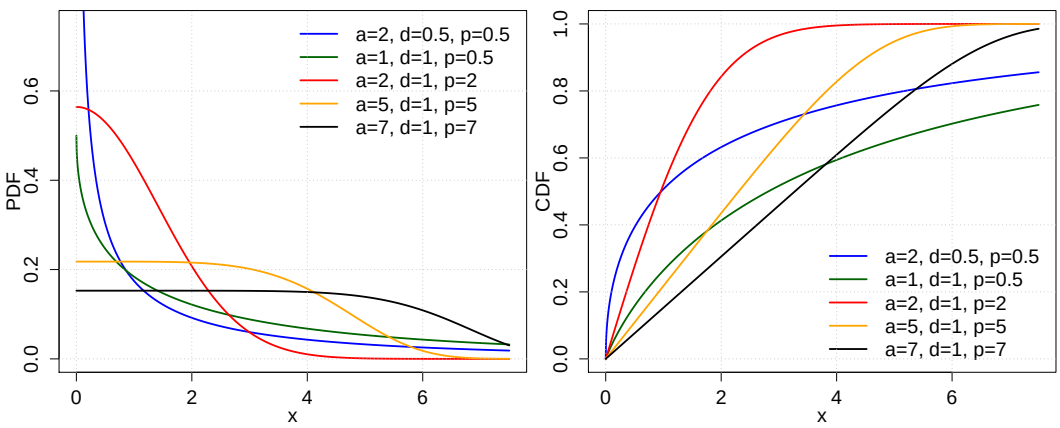

Figure A.35: GeneralizedGamma1 distribution plotted using the provided R code.

Parameter: shape1

name	shape
type	scalar
symbol	d
definition	$d > 0$

Parameter: shape2

name	shape
type	scalar
symbol	p
definition	$p > 0$

Functions

PDF

$$\frac{p/a^d}{\Gamma(d/p)} x^{d-1} e^{-(x/a)^p}$$

PDF in R

```
p/a^d/gamma(d/p) * x^(d-1) * exp(-(x/a)^p)
```

CDF

$$\frac{\gamma(d/p, (x/a)^p)}{\Gamma(d/p)}$$

CDF in R

```
Igamma(d/p, (x/a)^p, lower=T) / gamma(d/p)
```

Characteristics

Mean

$$a \frac{\Gamma((d+1)/p)}{\Gamma(d/p)}$$

Mode

$$a \left(\frac{d-1}{p} \right)^{\frac{1}{p}}, \text{ for } d > 1$$

Variance

$$a^2 \left(\frac{\Gamma((d+2)/p)}{\Gamma(d/p)} - \left(\frac{\Gamma((d+1)/p)}{\Gamma(d/p)} \right)^2 \right)$$

Relationships

- Relationship pair: $GeneralizedGamma1(a, d, p) \rightarrow Gamma1(k, \theta)$
- Relationship type: Special case
- Relationship definition: $p = 1, k = d, \theta = a$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $GeneralizedGamma1(a, d, p) \rightarrow GeneralizedGamma3(r, \mu, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $r = d/p, \beta = p, \mu = 1/a$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $GeneralizedGamma3(r, \mu, \beta) \rightarrow GeneralizedGamma1(a, d, p)$
- Relationship type: Reparameterisation
- Relationship definition: $a = 1/\mu, d = \beta r, p = \beta$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow GeneralizedGamma1(a, d, p)$
- Relationship type: Reparameterisation
- Relationship definition: $a = 0, kc = d,$ and rename $k = p, b = a$
- Relationship reference(s): ProbOnto spec

References and sources

`http://en.wikipedia.org/wiki/Generalized_gamma_distribution`
[Stacy, 1962]

UncertML mapping:

no UncertML mapping found.

GeneralizedGamma2

name	Generalized Gamma 2 (ID: 0000644)
type	continuous
variate	x, scalar
support	$0 < a < x$

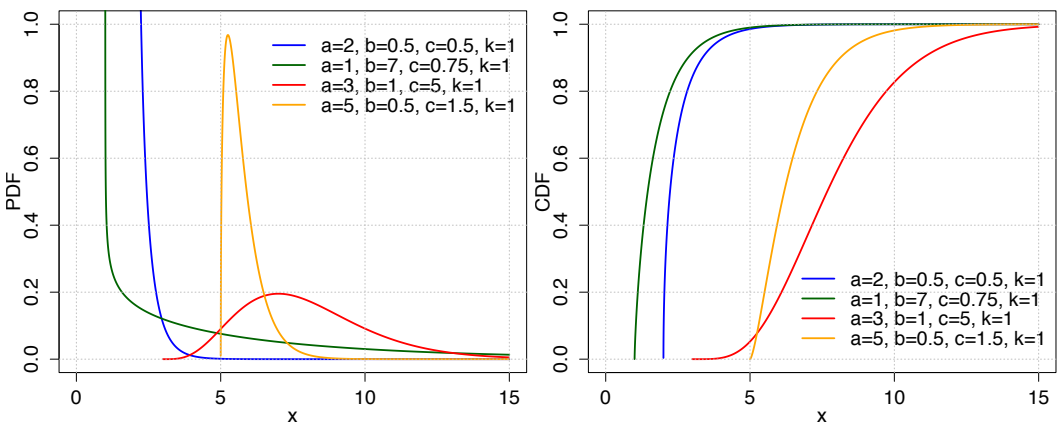

Figure A.36: GeneralizedGamma2 distribution plotted using the provided R code.

Model

The generalised gamma distribution is not often used to model life data by itself, but it is sometimes used to determine which of those life distributions should be used to model a particular set of data.

Parameter: location

name	location
type	scalar
symbol	a
definition	$a > 0$

Parameter: scale

name	scale
type	scalar
symbol	b
definition	$b > 0$

Parameter: shape1

name	shape1
type	scalar
symbol	c
definition	$c > 0$

Parameter: shape2

name	shape2
type	scalar
symbol	k
definition	$k > 0$

Functions

PDF

$$\frac{k(x-a)^{kc-1}}{b^{kc}\Gamma(c)} \exp\Big[-\Big(\frac{x-a}{b}\Big)^k\Big]$$

PDF in R

```
(k*(x-a)^(k*c-1)) / (b^(k*c)*gamma(c)) * exp( -((x-a)/b)^k )
```

CDF

$$\frac{\gamma(c, (\frac{x-a}{b})^k)}{\Gamma(c)}$$

CDF in R

```
Igamma(c, ((x-a)/b)^k, lower=T) / gamma(c)
```

Characteristics

Mean

$$a + b\Gamma(c + 1/k)/\Gamma(c)$$

Mode

$$a + b(c - 1/k)^{1/k}, c > 1/k$$

Variance

$$b^2\{\Gamma(c + 2/k)/\Gamma(c) - [\Gamma(c + 1/k)/\Gamma(c)]^2\}$$

Relationships

- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow Gamma1(k, \theta)$
- Relationship type: Transformation
- Relationship definition: $k = 1, a = 0$ and renaming parameters: $c = k, b = \theta$
- Relationship reference(s): [Forbes et al., 2011]
- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow Exponential1(\lambda)$
- Relationship type: Reparameterisation
- Relationship definition: $k = c = 1, a = 0, b = 1/\lambda$
- Relationship reference(s): [Forbes et al., 2011]
- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow Weibull1(\lambda, k)$
- Relationship type: Reparameterisation
- Relationship definition: $c = 1, a = 0, b = \lambda$
- Relationship reference(s): [Forbes et al., 2011]
- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow ChiSquared1(k)$
- Relationship type: Reparameterisation
- Relationship definition: $a = 0, b = 2, c = k_{ChiSquare1}/2, k = 1$
- Relationship reference(s): [Forbes et al., 2011]
- Relationship pair: $GeneralizedGamma2(a, b, c, k) \rightarrow GeneralizedGamma1(a, d, p)$
- Relationship type: Reparameterisation
- Relationship definition: $a = 0, kc = d,$ and rename $k = p, b = a$
- Relationship reference(s): ProbOnto spec

References and sources

[Forbes et al., 2011]
`http://www.mathwave.com/help/easyfit/html/analyses/distributions/gen_gamma.html`

UncertML mapping:

no UncertML mapping found.

GeneralizedGamma3

name	Generalized Gamma 3 (ID: 0000670)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

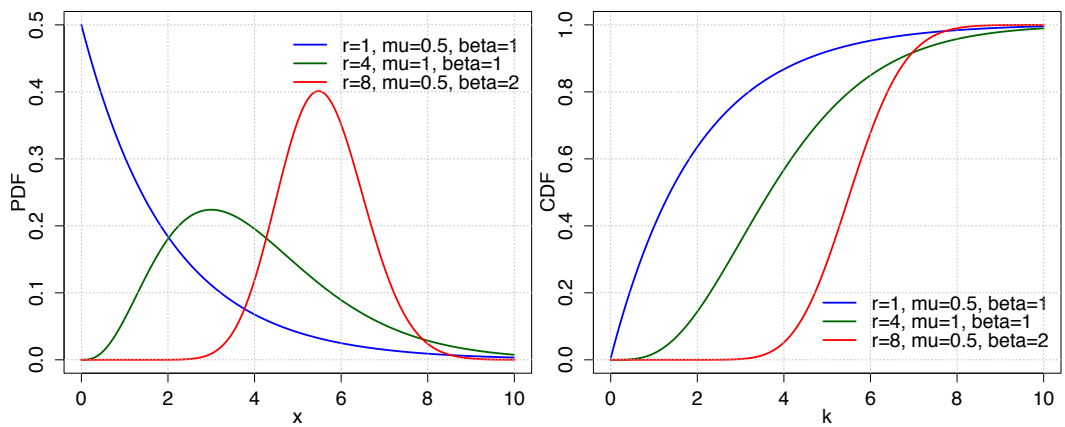

Figure A.37: GeneralizedGamma3 distribution plotted using the provided R code.

Model

The generalised gamma distribution is not often used to model life data by itself, but it is sometimes used to determine which of those life distributions should be used to model a particular set of data.

Parameter: scale

name	scale
type	scalar
symbol	r
definition	$r > 0$

Parameter: shape1

name	shape
type	scalar
symbol	μ
definition	$\mu > 0$

Parameter: shape2

name	shape
type	scalar
symbol	β
definition	$\beta > 0$

Functions

PDF

$$\frac{\beta}{\Gamma(r)}\mu^{\beta r}x^{\beta r-1}\exp[-(\mu x)^{\beta}]$$

PDF in R

```
beta / gamma(r) * mu^(beta*r) * x^(beta*r -1) * exp(-(mu*x)^beta)
```

Characteristics

Relationships

- Relationship pair: $GeneralizedGamma3(r, \mu, \beta) \rightarrow GeneralizedGamma1(a, d, p)$
- Relationship type: Reparameterisation
- Relationship definition: $a = 1/\mu, d = \beta r, p = \beta$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $GeneralizedGamma1(a, d, p) \rightarrow GeneralizedGamma3(r, \mu, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $r = d/p, \beta = p, \mu = 1/a$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

GeneralizedNegativeBinomial1

name	Generalized Negative Binomial 1 (ID: 0000690)
type	discrete
variate	x, scalar
support	$x \in \{0, 1, 2, 3, \dots\}$

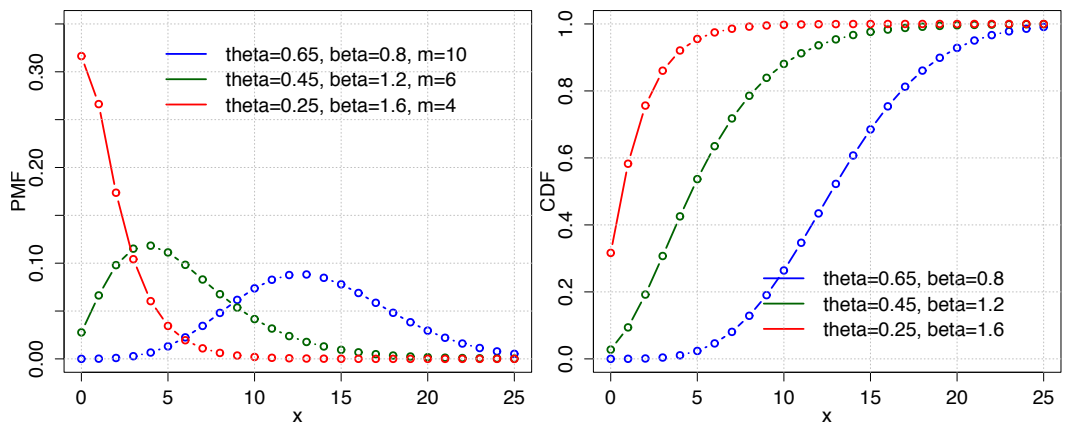

Figure A.38: GeneralizedNegativeBinomial1 distribution plotted using the provided R code.

Model

Parameter: theta

name	scale
type	scalar
symbol	θ
definition	$0 < \theta < 1$

Parameter: beta

name	shape1
type	scalar
symbol	β
definition	$\beta = 0$ or $1 \leq \beta \leq \theta^{-1}$

Parameter: m

name	shape2
type	scalar
symbol	m
definition	$m > 0$

Functions

PMF

$$\frac{m}{m+\beta x}\binom{m+\beta x}{x}\theta^x(1-\theta)^{m+\beta x-x}$$

PMF in R

```
m/(m+beta*x) * choose(m+beta*x,x)*theta^x * (1-theta)^(m+beta*x-x)
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$m\theta(1-\theta\beta)^{-1}$$

Variance

$$m\theta(1-\theta)(1-\theta\beta)^{-3}$$

Relationships

- Relationship pair: $GeneralizedNegativeBinomial1(\theta, \beta, m) \rightarrow Binomial1(n, p)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 0$ and set $m = n, \theta = p$
- Relationship reference(s): [Consul and Famoye, 2006]
- Relationship pair: $GeneralizedNegativeBinomial1(\theta, \beta, m) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 1$ and set $m = r, \theta = p$
- Relationship reference(s): [Consul and Famoye, 2006]
- Relationship pair: $GeneralizedNegativeBinomial1(\theta, \beta, m) \rightarrow InverseBinomial1(k, p)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 2, \theta = 1 - p$
- Relationship reference(s): [Yanagimoto, 1989]

References and sources

[Consul and Famoye, 2006]

UncertML mapping:

no UncertML mapping found.

GeneralizedPoisson1

name	Generalized Poisson 1 (ID: 0000712)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

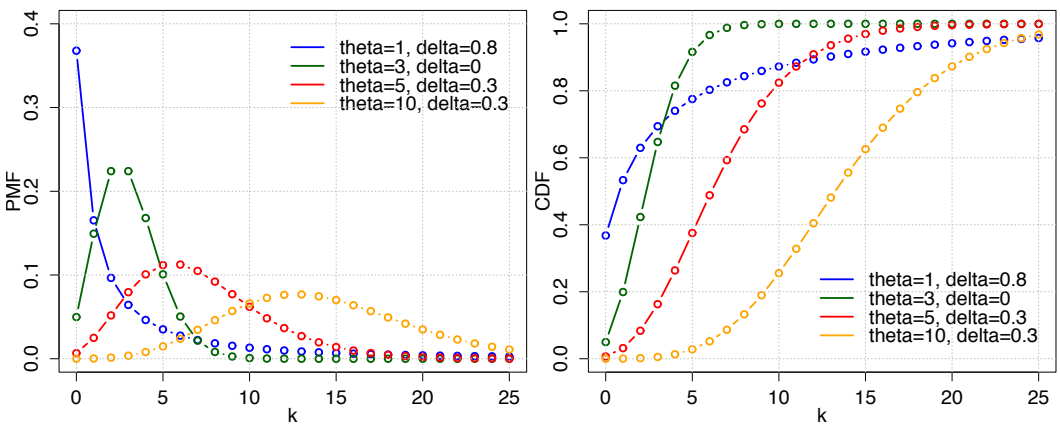

Figure A.39: GeneralizedPoisson1 distribution plotted using the provided R code.

Model

The generalized Poisson model is a model used to account for deviation from equidispersion in data to be analyzed. The additional parameter governs the dispersion of the pmf.

Parameter: rate

name	Poisson intensity
type	scalar
symbol	θ
definition	$\theta > 0$

Parameter: dispersion

name	dispersion
type	scalar
symbol	δ
definition	$max(-1, -\theta/m) < \delta < 1$ with $m(\geq 4)$ the largest positive integer for which $\theta + m\delta > 0$ when $\delta < 0$

Functions

PMF

$$\frac{\theta(\theta + \delta k)^{k-1} \times e^{-\theta-\delta k}}{k!}$$

PMF in R

```
(theta*(theta+k*delta)^(k-1) * exp(-theta-k*delta)) / factorial(k)
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\frac{\theta}{1-\delta}$$

Variance

$$\frac{\theta}{(1-\delta)^3}$$

Relationships

- Relationship pair: $GeneralizedPoisson1(\theta, \delta) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\delta = 0, \theta = \mu$
- Relationship reference(s): [Yang et al., 2007]

- Relationship pair: $GeneralizedPoisson1(\theta, \delta) \rightarrow GeneralizedPoisson2(\mu, \delta)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \theta/(1-\delta)$
- Relationship reference(s): [Yang et al., 2007]
ProbOnto spec

- Relationship pair: $GeneralizedPoisson2(\mu, \delta) \rightarrow GeneralizedPoisson1(\theta, \delta)$
- Relationship type: Reparameterisation
- Relationship definition: $\theta = \mu(1-\delta)$
- Relationship reference(s): [Yang et al., 2007]
ProbOnto spec

References and sources

[Yang et al., 2007]
[Consul and Famoye, 2006]
http://finzi.psych.upenn.edu/library/VGAM/html/genpoisson.html

UncertML mapping:

no UncertML mapping found.

GeneralizedPoisson3

name	Generalized Poisson 3 (ID: 0000758)
type	discrete
variate	y, scalar
support	$y \in \{0, 1, 2, 3, \dots\}$

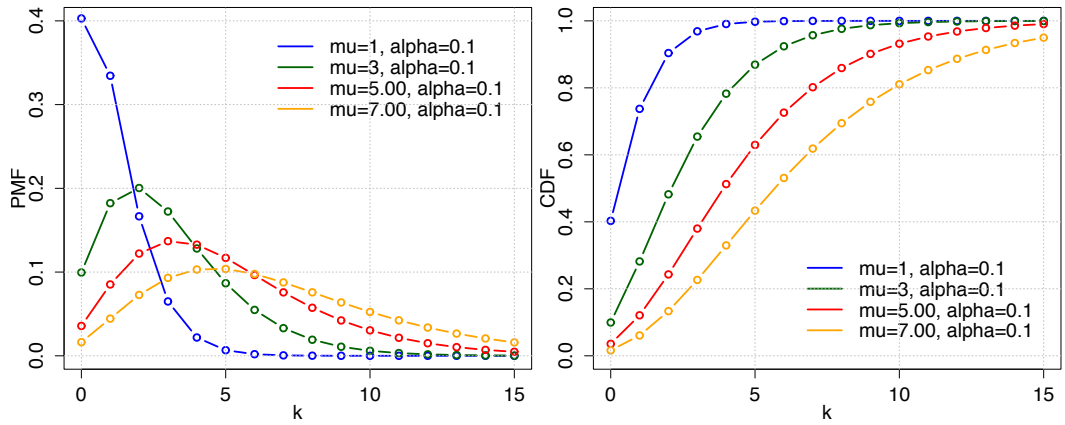

Figure A.40: GeneralizedPoisson3 distribution plotted using the provided R code.

Model

The generalized Poisson model is a model used to account for deviation from equidispersion in data to be analyzed. The additional parameter governs the dispersion of the pmf.

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu > 0$

Parameter: dispersion

name	dispersion
type	scalar
symbol	α
definition	$\alpha > -1, \alpha \in R$

Functions

PMF

$$\left(\frac{\mu}{1+\alpha\mu}\right)^y \frac{(1+\alpha y)^{y-1}}{y!} \exp\left[\frac{-\mu(1+\alpha y)}{1+\alpha\mu}\right]$$

PMF in R

```
(mu/(1+alpha*mu))^y *(1+alpha*y)^(y-1)/factorial(y)*exp(-mu*(1+alpha*y)/(1+alpha*mu))
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\mu$$

Variance

$$\mu(1+\alpha\mu)^2$$

Relationships

- Relationship pair: $GeneralizedPoisson3(\mu,\alpha) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\alpha = 0, \lambda = \mu$
- Relationship reference(s): [Hilbe, 2011]
[Famoye and Singh, 2006]

- Relationship pair: $ZeroInflatedGeneralizedPoisson1(\mu,\alpha,p0) \rightarrow GeneralizedPoisson3(\mu,\alpha)$
- Relationship type: Special case
- Relationship definition: $p0 = 0$
- Relationship reference(s): [Famoye and Singh, 2006]

References and sources

[Famoye and Singh, 2006]
[Hilbe, 2011]
[Ismail and Zamani, 2013]

UncertML mapping:

no UncertML mapping found.

GeneralizedPoisson2

name	GeneralizedPoisson2 (ID: 0000735)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

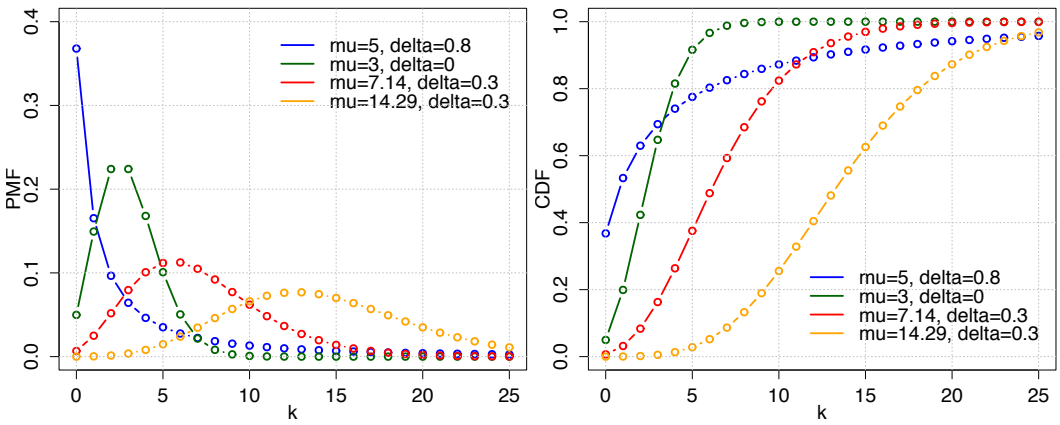

Figure A.41: GeneralizedPoisson2 distribution plotted using the provided R code.

Model

The generalized Poisson model is a model used to account for deviation from equidispersion in data to be analyzed. The additional parameter governs the dispersion of the pmf.

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu > 0$

Parameter: dispersion

name	dispersion
type	scalar
symbol	δ
definition	$max(-1, -\mu/m) < \delta < 1$ with $m(\geq 4)$ the largest positive integer for which $\mu + m\delta > 0$.

Functions

PMF

$$\frac{\mu(1-\delta)[\mu(1-\delta)+\delta k]^{k-1}}{k!} e^{-[\mu(1-\delta)+\delta k]}$$

PMF in R

```
-(mu*(1-delta)*(mu*(1-delta)+delta*k)^(k-1)) / factorial(k) * exp(-(mu*(1-delta)+delta*k))
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\mu$$

Variance

$$\frac{\mu}{(1-\delta)^2}$$

Relationships

- Relationship pair: $GeneralizedPoisson2(\mu, \delta) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\delta = 0, \lambda = \mu$
- Relationship reference(s): [Plan, 2014]

- Relationship pair: $GeneralizedPoisson2(\mu, \delta) \rightarrow GeneralizedPoisson1(\theta, \delta)$
- Relationship type: Reparameterisation
- Relationship definition: $\theta = \mu(1-\delta)$
- Relationship reference(s): [Yang et al., 2007]

ProbOnto spec

- Relationship pair: $GeneralizedPoisson1(\theta, \delta) \rightarrow GeneralizedPoisson2(\mu, \delta)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \theta/(1-\delta)$
- Relationship reference(s): [Yang et al., 2007]

ProbOnto spec

References and sources

[Yang et al., 2007]
[Plan, 2014]

UncertML mapping:

no UncertML mapping found.

Geometric1

name	Geometric 1 (ID: 0000782)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$, number of failures

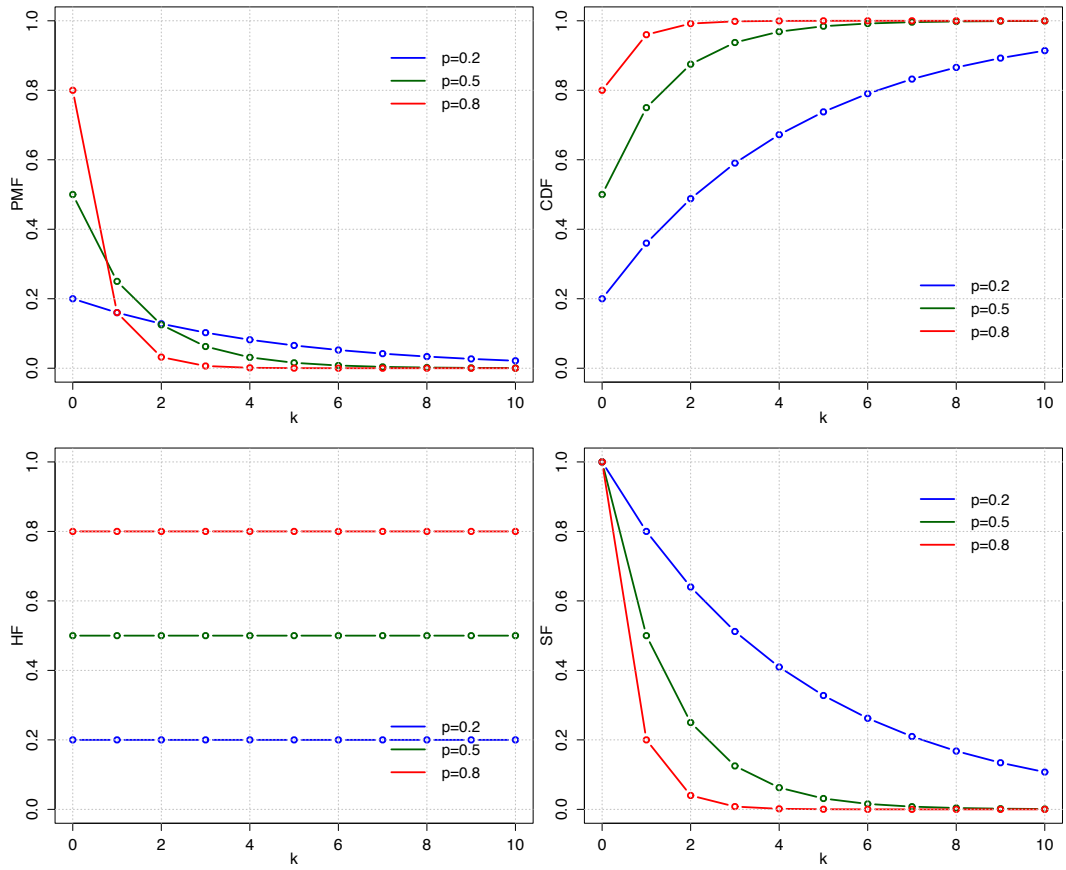

Figure A.42: Geometric1 distribution plotted using the provided R code.

Model

The probability distribution of the number $Y = X - 1$ of failures before the first success, supported on the set $\{0, 1, 2, 3, ...\}$.

Parameter: probability

name	success probability
type	scalar
symbol	p
definition	$0 < p < 1$

Functions

PMF

$$(1-p)^k\, p$$

PMF in R

```
p*(1-p)^k
```

CDF

$$1-(1-p)^{k+1}$$

CDF in R

```
1-(1 - p)^(k+1)
```

HF

$$p$$

HF in R

```
p
```

SF

$$(1-p)^k$$

SF in R

```
(1 - p)^k
```

Characteristics

Mean

$$\frac{1-p}{p}$$

Median

$$\left\lceil \frac{-1}{\log_2(1-p)} - 1 \right\rceil \text{ (not unique if } -1/\log_2(1-p)-1 \text{ is an integer)}$$

Mode

$$0$$

Variance

$$\frac{1-p}{p^2}$$

Relationships

- Relationship pair: $Geometric1(p) \rightarrow NegativeBinomial1(r,p)$
- Relationship type: Transformation
- Relationship definition: $\Sigma X(iid)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GeometricPascal.pdf
- Relationship pair: $NegativeBinomial1(r,p) \rightarrow Geometric1(p)$
- Relationship type: Special case
- Relationship definition: $n=1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalGeometric.pdf
- Relationship pair: $WeibullDiscrete1(p\beta) \rightarrow Geometric1(p)$
- Relationship type: Special case
- Relationship definition: The Geometric1(p) distribution is a special case of the WeibullDiscrete1(p,β) distribution when $\beta=1$.
- Relationship reference(s): http://www.math.wm.edu/~leemis/chart/UDR/PDFs/DiscreteweibullGeometric.pdf

References and sources

[Forbes et al., 2011]
http://en.wikipedia.org/wiki/Geometric_distribution

UncertML mapping:

`http://www.uncertml.org/distributions/geometric`

Gompertz1

name	Gompertz 1 (ID: 0000008)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

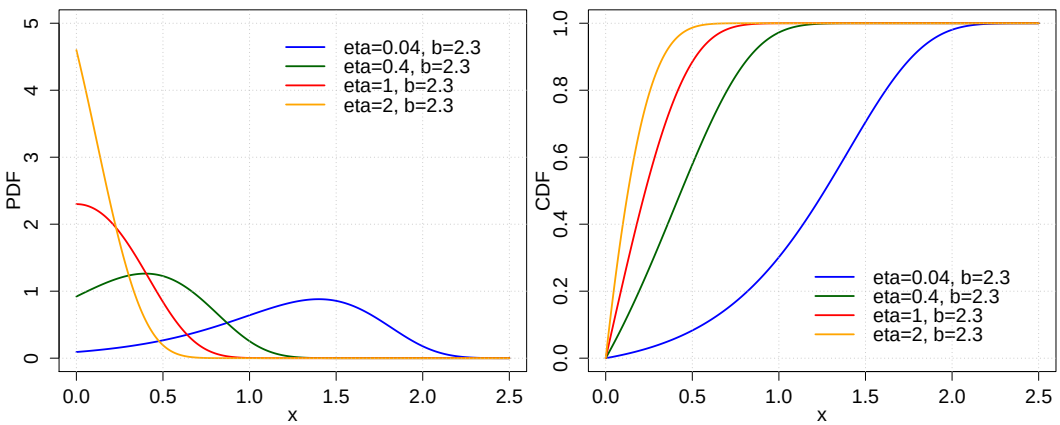

Figure A.43: Gompertz1 distribution plotted using the provided R code.

Model

Gompertz distribution is a continuous probability distribution, named after Benjamin Gompertz (1779 - 1865). The Gompertz distribution is often applied to describe the distribution of adult lifespans by demographers and actuaries.

Parameter: shape

name	shape
type	scalar
symbol	η
definition	$\eta > 0$

Parameter: scale

name	scale
type	scalar
symbol	b
definition	$b > 0$

Functions

PDF

$$b\eta e^{bx} e^{\eta} \exp\left(-\eta e^{bx}\right)$$

PDF in R

```
b*eta*exp(b*x)*exp(eta)*exp(-eta*exp(b*x))
```

CDF

$$1 - \exp\left(-\eta\left(e^{bx} - 1\right)\right)$$

CDF in R

```
1-exp(-eta*(exp(b*x)-1))
```

HF

$$b\eta \exp(-bx)$$

HF in R

```
b *eta * exp(-b*x)
```

SF

$$\exp\left(-\eta\left(e^{bx}-1\right)\right)$$

SF in R

```
exp(-eta*(exp(b*x)-1))
```

Characteristics

Median

$$(1/b)\log\left[(-1/\eta)\log(1/2)+1\right]$$

Relationships

References and sources

https://en.wikipedia.org/wiki/Gompertz_distribution

UncertML mapping:

no UncertML mapping found.

Gumbel1

name	Gumbel 1 (ID: 0000032)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

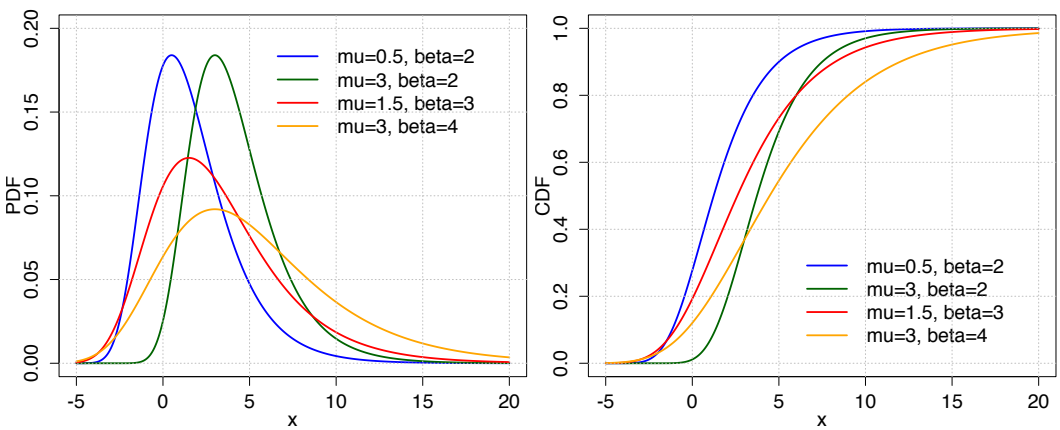

Figure A.44: Gumbel1 distribution plotted using the provided R code.

Model

The extreme value distribution was developed as the distribution of the largest of a number of values and was originally applied to the estimation of flood levels. It has since been applied to the estimation of the magnitude of earthquakes. The distribution may also be applied to the study of athletic and other records.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	β
definition	$\beta > 0, \beta \in R$

Functions

PDF

$$\frac{e^{-e^{-\frac{x-\mu}{\beta}}} e^{-\frac{x-\mu}{\beta}}}{\beta}$$

PDF in R

```
(exp(-exp(-(x-mu)/beta)) * exp(-(x-mu)/beta))/beta
```

CDF

$$e^{-e^{-(x-\mu)/\beta}}$$

CDF in R

```
exp(-exp(-(x-mu)/beta))
```

Characteristics

Mean

$$\mu + \beta\,\gamma_E; \text{ with is Euler constant } \gamma_E$$

Median

$$\mu - \beta\,\ln(\ln(2))$$

Mode

$$\mu$$

Variance

$$\frac{\pi^2}{6}\beta^2$$

Relationships

References and sources

`https://en.wikipedia.org/wiki/Gumbel_distribution`
[Forbes et al., 2011]

UncertML mapping:

no UncertML mapping found.

HalfCauchy1

name	Half Cauchy 1 (ID: 0001187)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

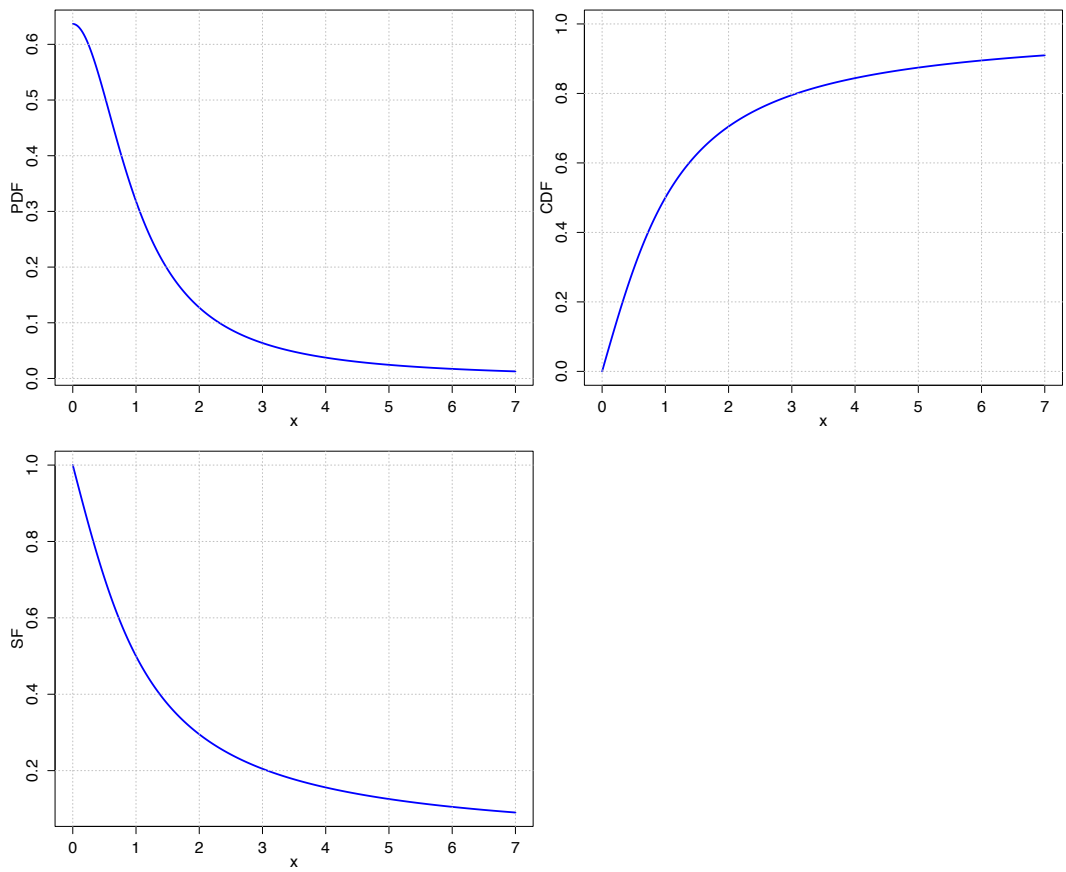

Figure A.45: HalfCauchy1 distribution plotted using the provided R code.

Model

As a heavy tailed distribution, the HalfCauchy1 distribution has been used as an alternative to exponential distribution to model dispersal distances as the former predicts more frequent long distance dispersed events than the latter.

Functions

PDF

$$\frac{2}{\pi}\frac{1}{1+x^2}$$

PDF in R

```
2/pi * 1 / (1 + x^2)
```

CDF

$$\frac{2}{\pi}\arctan(x)$$

CDF in R

```
2/pi * atan(x)
```

SF

$$1-\frac{2}{\pi}\arctan(x)$$

SF in R

```
1 - 2/pi * atan(x)
```

Characteristics

Mean

$$undefined$$

Median

$$\tan(\pi/4)$$

Variance

$$undefined$$

Relationships

References and sources

`http://statperson.com/Journal/StatisticsAndMathematics/Article/Volume3Issue2/3_2_8.pdf`

UncertML mapping:

no UncertML mapping found.

HalfNormal1

name	Half-normal 1 (ID: 0000068)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

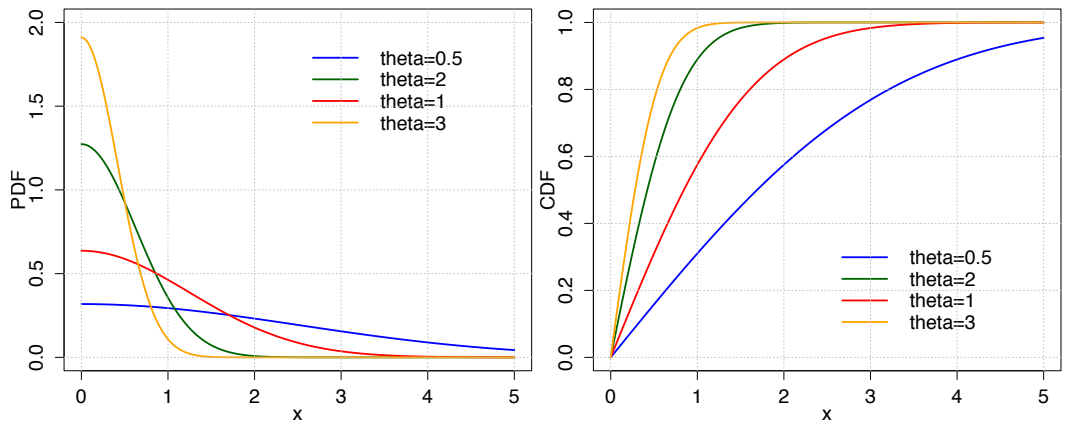

Figure A.46: HalfNormal1 distribution plotted using the provided R code.

Model

Let X follow an ordinary normal distribution, $N(0, \sigma^2)$ then $Y = |X|$ follows a half-normal distribution. Thus, the half-normal distribution is a fold at the mean of an ordinary normal distribution with mean zero.

Parameter: inverseScale

name	inverse scale
type	scalar
symbol	θ
definition	$\theta > 0$

Functions

PDF

$$\frac{2\theta}{\pi}e^{-\theta^2 x^2/\pi}$$

PDF in R

```
2*theta/pi * exp(-theta^2 * x^2 / pi)
```

CDF

$$\mathrm{erf}(\theta x/\sqrt{\pi})$$

CDF in R

```
erf(theta * x / sqrt(pi))
```

Characteristics

Mean

$$1/\theta$$

Variance

$$\frac{\pi - 2}{2\theta^2}$$

Relationships

- Relationship pair: $HalfNormal2(\mu, \sigma) \rightarrow HalfNormal1(\theta)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = 0, \sigma = \sqrt{\pi}/(\theta\sqrt{2})$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $TruncatedNormal1(\mu, \sigma, a, b) \rightarrow HalfNormal1(\theta)$
- Relationship type: Special case
- Relationship definition: $\mu = 0, a = 0, b = \infty$
- Relationship reference(s): `http://reference.wolfram.com/language/ref/HalfNormalDistribution.html`
[Forbes et al., 2011]

References and sources

`http://mathworld.wolfram.com/Half-NormalDistribution.html`
`http://svitsrv25.epfl.ch/R-doc/library/fdrtool/html/halfnorm.html`

UncertML mapping:

no UncertML mapping found.

HalfNormal2

name	Half-normal 2 (ID: 0000982)
type	continuous
variate	x, scalar
support	$x \in [\mu, +\infty)$

Model

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

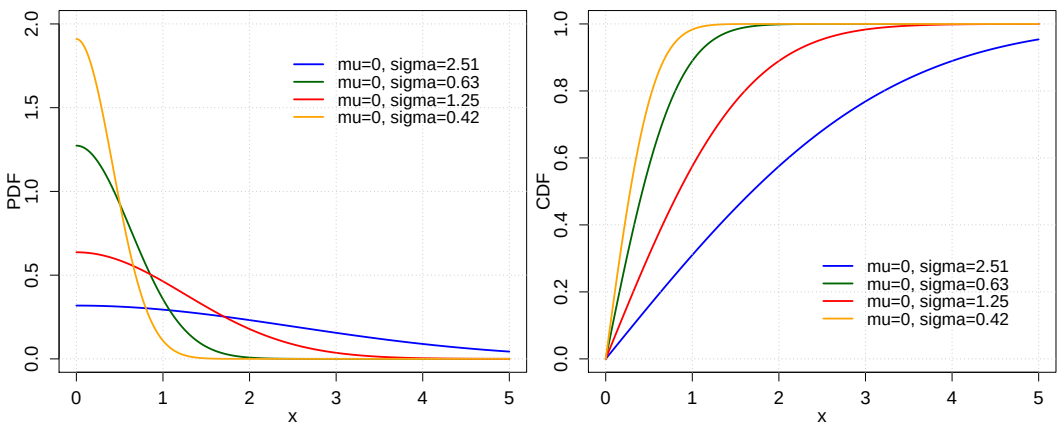

Figure A.47: HalfNormal2 distribution plotted using the provided R code.

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big]$$

PDF in R

```
sqrt(2/pi)*1/sigma * exp(-1/2 * ((x-mu)/sigma)^2)
```

CDF

$$erf\Big(\frac{x-\mu}{\sqrt{2}\sigma}\Big)$$

CDF in R

```
erf((x-mu) / (sqrt(2)*sigma))
```

Characteristics

Mean

$$\mu + \sigma\sqrt{\frac{2}{\pi}}$$

Variance

$$\sigma^2(1-\frac{2}{\pi})$$

Relationships

- Relationship pair: $HalfNormal2(\mu,\sigma) \rightarrow HalfNormal1(\theta)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = 0, \sigma = \sqrt{\pi}/(\theta\sqrt{2})$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \rightarrow HalfNormal2(\mu,\sigma)$
- Relationship type: Special case
- Relationship definition: HalfNormal2(μ,σ) distribution is a special case of the Amoroso1(a,θ,α,β) distribution for $\mu = 0$ and when $a = 0, \theta = \sqrt{2\sigma^2}, \alpha = 1/2, \beta = 2$
- Relationship reference(s): [Crooks, 2010]

References and sources

`http://nl.mathworks.com/help/stats/half-normal-distribution.html`

UncertML mapping:

no UncertML mapping found.

HyperbolicSecant1

name	Hyperbolic secant 1 (ID: 0000098)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

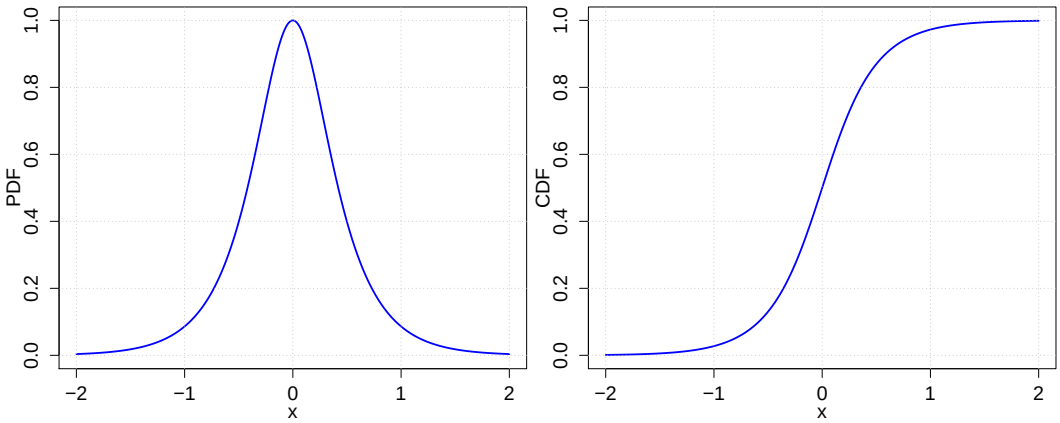

Figure A.48: HyperbolicSecant1 distribution plotted using the provided R code.

Model

The hyperbolic secant distribution is a continuous probability distribution whose probability density function and characteristic function are proportional to the hyperbolic secant function. The hyperbolic secant function is equivalent to the inverse hyperbolic cosine, and thus this distribution is also called the inverse-cosh distribution.

Functions

PDF

$$\operatorname{sech}\left(\pi x\right)$$

PDF in R

```
sech(pi*x)
```

CDF

$$\frac{\pi + 2\,\arctan(\sinh(\pi x))}{2\pi}$$

CDF in R

```
(pi + 2*arctan(sinh(pi*x)))/(2*pi)
```

Characteristics

Mean

$$0$$

Median

$$0$$

Mode

$$0$$

Variance

$$1$$

Relationships

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Hyperbolicsecant.pdf`

UncertML mapping:

no UncertML mapping found.

Hypergeometric1

name	Hypergeometric 1 (ID: 0000126)
type	discrete
variate	k, scalar
support	$k \in \{\max(0,\, n+K-N),\, \ldots,\, \min(n,\, K)\}$

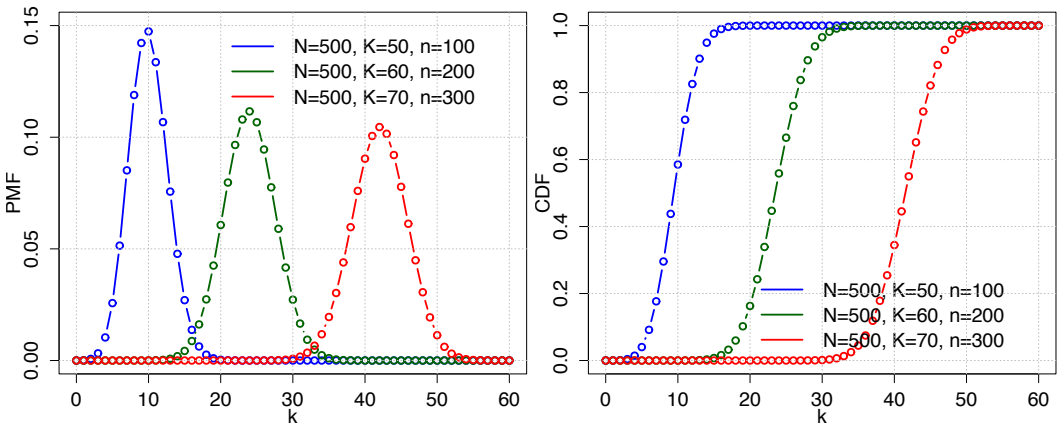

Figure A.49: Hypergeometric1 distribution plotted using the provided R code.

Model

Hypergeometric distribution is a discrete probability distribution that describes the probability of k successes in n draws, without replacement, from a finite population of size N that contains exactly K successes, each draw is either a success or a failure. In contrast, the binomial distribution describes the wherein probability of k successes in n draws with replacement.

Parameter: populationSize

name	population size
type	scalar
symbol	N
definition	$N \in \{0, 1, 2, \ldots\}$

Parameter: numberOfSuccesses

name	number of successes
type	scalar
symbol	K
definition	$K \in \{0, 1, 2, \ldots, N\}$

Parameter: numberOfTrials

name	number of trials
type	scalar
symbol	n
definition	$n \in \{0, 1, 2, \ldots, N\}$

Functions

PMF

$$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

PMF in R

```
choose(K,k)*choose(M-K,n-k)/choose(M,n)
```

CDF

$$1 - \frac{\binom{n}{k+1}\binom{N-n}{K-k-1}}{\binom{N}{K}} {}_3F_2\left[\begin{array}{c} 1,\ k+1-K,\ k+1-n \\ k+2,\ N+k+2-K-n \end{array}; 1\right]$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$n\frac{K}{N}$$

Mode

$$\left\lfloor \frac{(n+1)(K+1)}{N+2} \right\rfloor$$

Variance

$$n\frac{K}{N}\frac{(N-K)}{N}\frac{N-n}{N-1}$$

Relationships

- Relationship pair: $Hypergeometric1(N, K, n) \rightarrow Binomial1(n, p)$
- Relationship type: Limiting
- Relationship definition: $p = K/N, n = n, N \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/HypergeometricBinomial.pdf`

- Relationship pair: $Hypergeometric1(N, K, n) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $X \sim Hypergeometric1(N, K, n) \Rightarrow Y \sim Poisson1(\lambda)$ as K, N and n tend to infinity for K/N small and $nK/N \rightarrow \lambda$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Hypergeometric1(N, K, n) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $X \sim Hypergeometric1(N, K, n) \Rightarrow Y \sim Normal1(\mu, \sigma)$ for large n, but K/N not too small
- Relationship reference(s): [Forbes et al., 2011]

References and sources

http://en.wikipedia.org/wiki/Hypergeometric_distribution

UncertML mapping:

http://www.uncertml.org/distributions/hypergeometric

InverseBinomial1

name	Inverse Binomial 1 (ID: 0000154)
type	discrete
variate	x, scalar
support	$x \in \{0, 1, 2, 3, \dots\}$

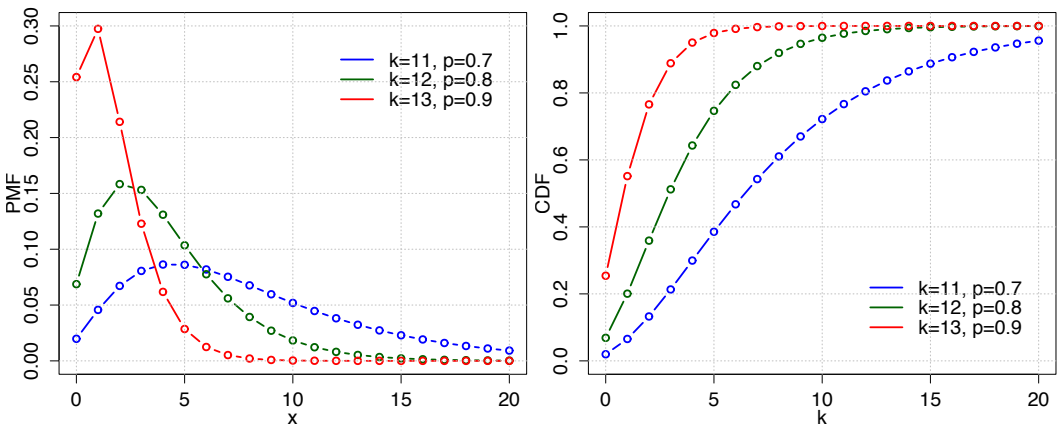

Figure A.50: InverseBinomial1 distribution plotted using the provided R code.

Model

Parameter: index

name	index parameter
type	scalar
symbol	k
definition	$k \in \{0, 1, 2, \dots\}$

Parameter: probability

name	probability
type	scalar
symbol	p
definition	$1/2 < p < 1$

Functions

PMF

$$\frac{k\,\Gamma(2x+k)}{\Gamma(x+1)\,\Gamma(x+k+1)}\,p^{k+x}(1-p)^x$$

PMF in R

```
(k * gamma(2*x+k)) / (gamma(x+1) * gamma(x+k+1)) * p^(x+k) * (1-p)^x
```

Characteristics

Mean

$$k(1-p)/(2p-1)$$

Variance

$$kp(1-p)/(2p-1)^3$$

Relationships

- Relationship pair: $GeneralizedNegativeBinomial1(\theta, \beta, m) \rightarrow InverseBinomial1(k, p)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 2, \theta = 1 - p$
- Relationship reference(s): [Yanagimoto, 1989]

References and sources

[Yanagimoto, 1989]
`https://cran.r-project.org/web/packages/VGAM/VGAM.pdf`

UncertML mapping:

no UncertML mapping found.

InverseChiSquare1

name	Inverse Chi-Square (ID: 0001250)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

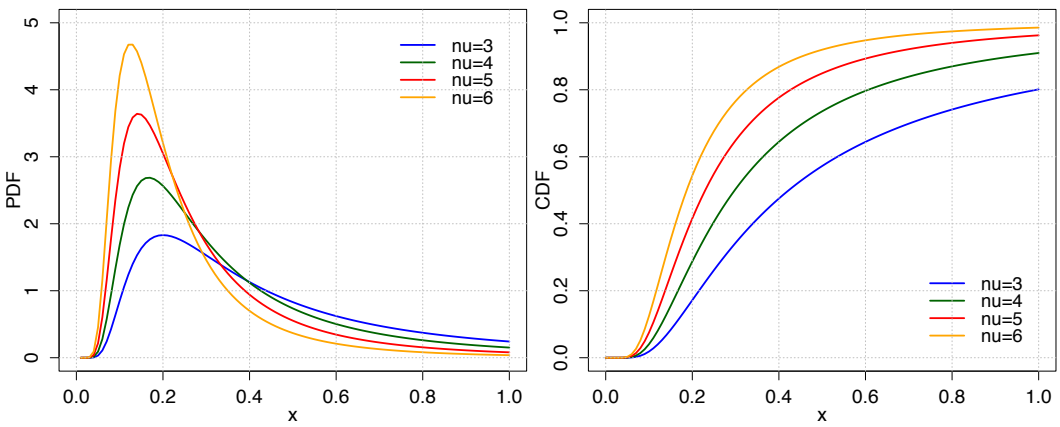

Figure A.51: InverseChiSquare1 distribution plotted using the provided R code.

Model

Inverse-chi-squared distribution is closely related to the chi-squared distribution and its specific importance is that it arises in the application of Bayesian inference to the normal distribution, where it can be used as the prior and posterior distribution for an unknown variance.

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	ν
definition	$\nu \in R^+$

Functions

PDF

$$\frac{2^{-\nu/2}}{\Gamma(\nu/2)} x^{-(\nu/2+1)} \exp\Big(-\frac{1}{2x}\Big)$$

PDF in R

```
2^(-nu/2)/gamma(nu/2)*x^(-(nu/2+1))*exp(-1/(2*x))
```

CDF

$$\Gamma\Big(\frac{\nu}{2}, \frac{1}{2x}\Big) \Big/ \Gamma\Big(\frac{\nu}{2}\Big)$$

CDF in R

```
Igamma(nu/2,1/2/x, lower=F) / gamma(nu/2)
```

Characteristics

Mean

$$\frac{1}{\nu - 2} \text{ for } \nu > 2$$

Mode

$$\frac{1}{\nu + 2}$$

Variance

$$\frac{2}{(\nu-2)^2(\nu-4)} \text{ for } \nu > 4$$

Relationships

- Relationship pair: $Amoroso1(a, \theta, \alpha, \beta) \rightarrow InverseChiSquare1(k)$
- Relationship type: Special case
- Relationship definition: InverseChiSquare1(k) distribution is a special case of the Amoroso1(a, θ, α, β) distribution when $a = 0, \theta = 1/2, \alpha = k/2, \beta = -1$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $ScaledInverseChiSquare1 \rightarrow InverseChiSquare1$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ Scale-inv-$\chi^2(\nu, \tau^2)$ then $\frac{X}{\tau^2 \nu} \sim$ inv-$\chi^2(\nu)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution`

References and sources

[STAN Development Team, 2015b]
`https://en.wikipedia.org/wiki/Inverse-chi-squared_distribution`

UncertML mapping:

no UncertML mapping found.

InverseGamma1

name	Inverse-Gamma 1 (ID: 0000182)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

Model

The inverse gamma distribution is a two-parameter family of continuous probability distributions on the positive real line, which is the distribution of the reciprocal of a variable distributed according to the gamma distribution

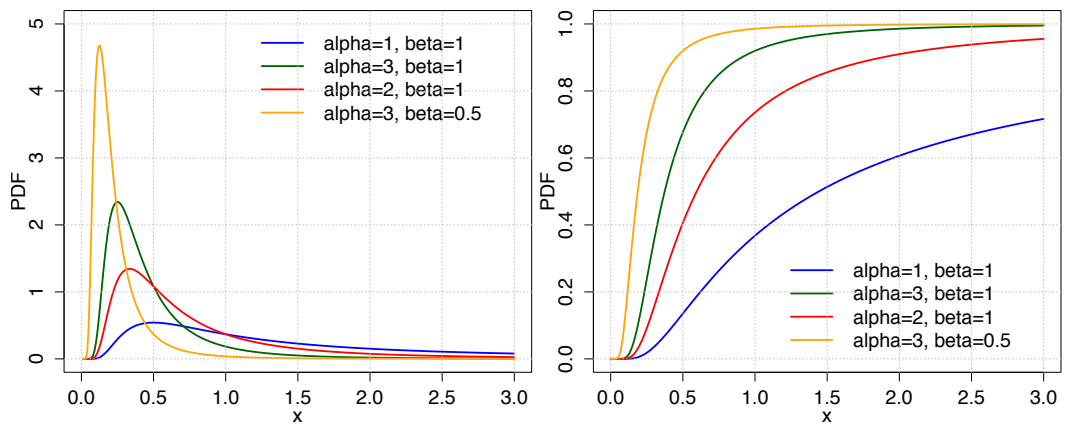

Figure A.52: InverseGamma1 distribution plotted using the provided R code.

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha > 0, \alpha \in R$

Parameter: scale

name	scale
type	scalar
symbol	β
definition	$\beta > 0, \beta \in R$

Functions

PDF

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(\frac{-\beta}{x}\right)$$

PDF in R

```
beta^alpha/gamma(alpha) * x^(-alpha-1) * exp(-beta/x)
```

CDF

$$\frac{\Gamma(\alpha, \beta/x)}{\Gamma(\alpha)}$$

CDF in R

```
Igamma(alpha, beta/x, lower=F) / gamma(alpha)
```

Characteristics

Mean

$$\frac{\beta}{\alpha - 1} \text{ for } \alpha > 1$$

Mode

$$\frac{\beta}{\alpha + 1}$$

Variance

$$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \text{ for } \alpha > 2$$

Relationships

- Relationship pair: $Gamma1(\alpha, \beta) \rightarrow InverseGamma1(\alpha, \beta)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Gamma1(\alpha, \beta)$ then $X^{-1} \sim InverseGamma1(\alpha, \beta^{-1})$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Inverse-gamma_distribution`
- Relationship pair: $ScaledInverseChiSquare1(\nu, \sigma) \rightarrow InverseGamma1(\alpha, \beta)$
- Relationship type: Transformation
- Relationship definition: If $X \sim ScaledInverseChiSquare1(\nu, \sigma)$ then $X \sim InverseGamma1(\nu/2, \nu\sigma^2/2)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution`

References and sources

`http://en.wikipedia.org/wiki/Inverse-gamma_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/inverse-gamma`

InverseGaussian1

name	Inverse Gaussian 1 (ID: 0000212)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

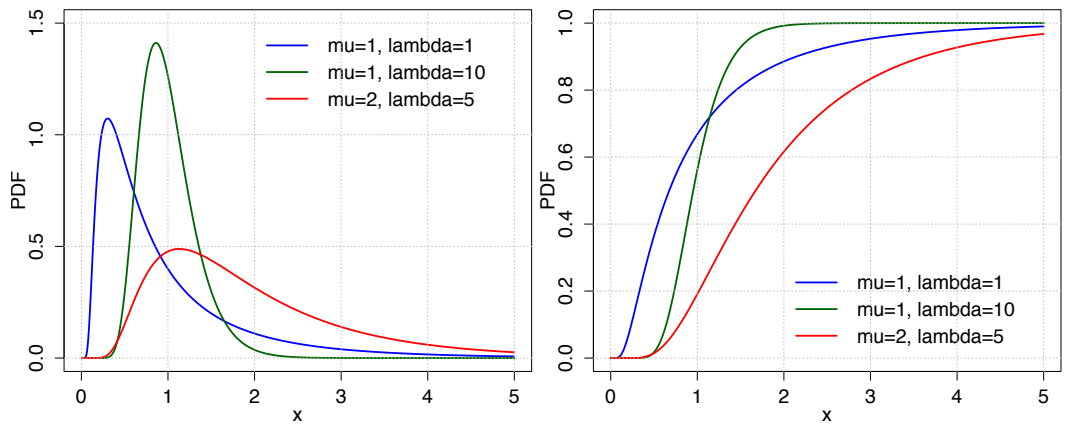

Figure A.53: InverseGaussian1 distribution plotted using the provided R code.

Model

The inverse Gaussian distribution has applications in the study of diffusion processes and as a lifetime distribution model.

Parameter: shape

name	shape
type	scalar
symbol	λ
definition	$\lambda > 0$

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu > 0$

Functions

PDF

$$\sqrt{\frac{\lambda}{2\pi x^3}} \exp\Big(-\frac{\lambda}{2\mu^2 x}(x-\mu)^2 \Big)$$

PDF in R

```
sqrt(lambda/(2*pi*x^3) ) * exp(-lambda/(2*mu^2 x) * (x-mu)^2)
```

CDF

$$\Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\right)$$

CDF in R

```
pnorm(sqrt(lambda/x) * (x/mu-1)) + exp(2*lambda/mu) * pnorm(-sqrt(lambda/x) * (x/mu+1))
```

Characteristics

Relationships

- Relationship pair: $InverseGaussian1(\lambda, \mu) \rightarrow StandardNormal1(0, 1)$
- Relationship type: Limiting
- Relationship definition: $\lambda \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/InversegaussianStandardnormal.pdf

- Relationship pair: $InverseGaussian1(\lambda, \mu) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: $X \sim InverseGaussian1(\lambda, \mu)$ and $Y = \lambda(X-\mu)^2/(\mu^2 X) \Rightarrow Y \sim ChiSquared1(k)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/InversegaussianChisquare.pdf

References and sources

https://en.wikipedia.org/wiki/Inverse_Gaussian_distribution

UncertML mapping:

no UncertML mapping found.

InverseWishart1

name	Inverse-Wishart 1 (ID: 0000232)
type	continuous
variate	X, matrix
support	$X(p \times p)$ – positive-definite matrix

Model

The inverse Wishart distribution is based on the Wishart distribution. In Bayesian statistics it is used as the conjugate prior for the covariance matrix of a multivariate normal distribution.

Parameter: scaleMatrix

name	scale matrix
type	matrix
symbol	Ψ
definition	$\Psi > 0$, positive-definite matrix

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	ν
definition	$\nu > p - 1, \nu \in R$

Functions

PDF

$$\frac{|\Psi|^{\frac{\nu}{2}}}{2^{\frac{\nu p}{2}} \Gamma_p(\frac{\nu}{2})} |X|^{-\frac{\nu+p+1}{2}} e^{-\frac{1}{2}\mathrm{tr}(\Psi X^{-1})}$$

Characteristics

Mean

$$\frac{\Psi}{\nu - p - 1} \text{ for } \nu > p + 1$$

Mode

$$\frac{\Psi}{\nu + p + 1}$$

Relationships

References and sources

`https://en.wikipedia.org/wiki/Inverse-Wishart_distribution`

UncertML mapping:

no UncertML mapping found.

JohnsonSB1

name	Johnson SB 1 (ID: 0001293)
type	continuous
variate	x, scalar
support	$x \in (\mu, \mu + \sigma$

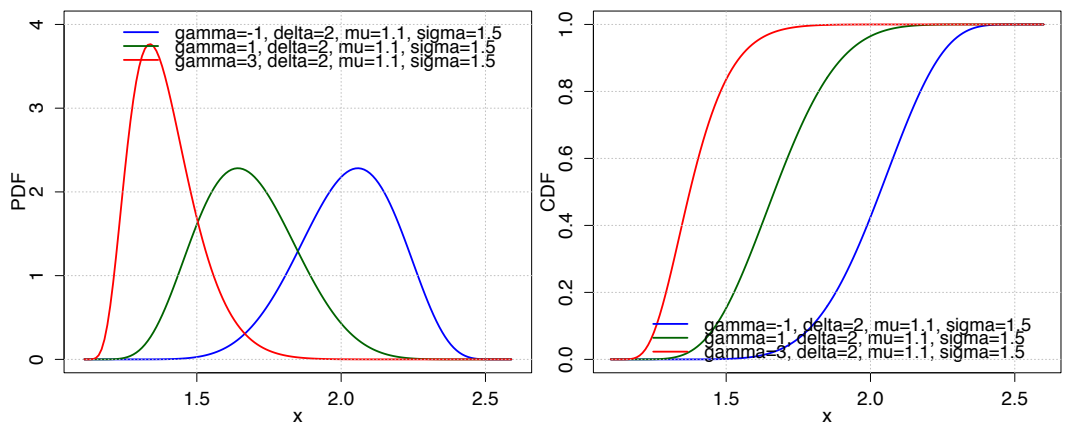

Figure A.54: JohnsonSB1 distribution plotted using the provided R code.

Model

Parameter: shape1

name	shape parameter
type	scalar
symbol	γ
definition	$\gamma \in R$

Parameter: shape2

name	shape parameter
type	scalar
symbol	δ
definition	$\delta > 0$

Parameter: location

name	location parameter
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale parameter
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{e^{-\frac{1}{2}\left(\gamma+\delta\log\left[\frac{x-\mu}{-x+\mu+\sigma}\right]\right)^2}\delta\sigma}{\sqrt{2\pi}(x-\mu)(-x+\mu+\sigma)}$$

PDF in R

```
exp(-1/2*(gamma+delta*log((x-mu)/(-x+mu+sigma)))^2)*delta*sigma/(sqrt(2*pi)*(x-mu)*(-x+mu+sigma))
```

CDF

$$\begin{cases} \frac{1}{2}\text{erfc}\left[-\frac{\gamma+\delta\ \log\left[\frac{x-\mu}{-x+\mu+\sigma}\right]}{\sqrt{2}}\right] & \text{for } \mu < x < \mu + \frac{\sigma}{2} \\ \frac{1}{2}\left(1+\text{erf}\left[\frac{\gamma+\delta\ \log\left[\frac{x-\mu}{-x+\mu+\sigma}\right]}{\sqrt{2}}\right]\right) & \text{for } \mu + \frac{\sigma}{2} \leq x < \mu + \sigma \end{cases}$$

CDF in R

```
CDF1=1/2*erfc(-(gamma+delta*log((x-mu)/(-x+mu+sigma)))/sqrt(2))
CDF2=1/2*(1+erf((gamma+delta*log((x-mu)/(-x+mu+sigma)))/sqrt(2)))
```

Characteristics

Relationships

References and sources

https://reference.wolfram.com/language/ref/JohnsonDistribution.html

UncertML mapping:

no UncertML mapping found.

JohnsonSL1

name	Johnson SL 1 (ID: 0001314)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

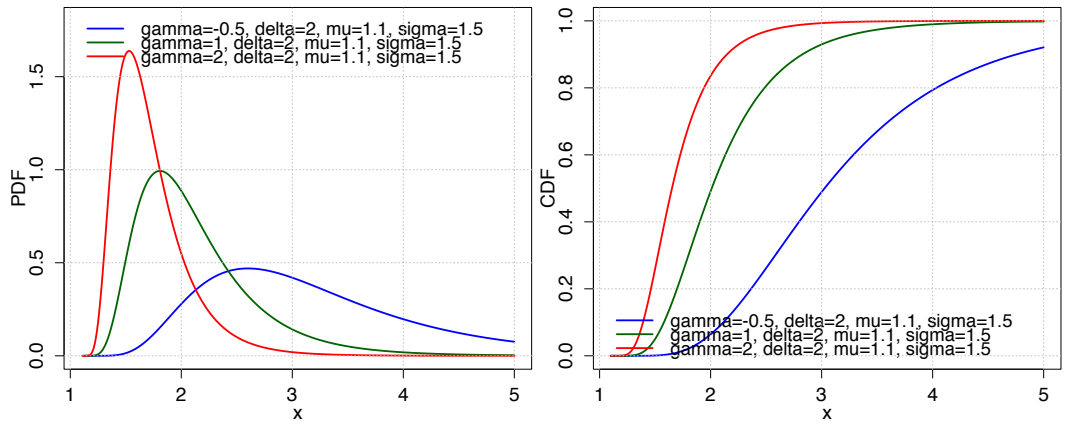

Figure A.55: JohnsonSL1 distribution plotted using the provided R code.

Model

Parameter: shape1

name	shape parameter
type	scalar
symbol	γ
definition	$\gamma \in R$

Parameter: shape2

name	shape parameter
type	scalar
symbol	δ
definition	$\delta > 0$

Parameter: location

name	location parameter
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale parameter
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{e^{-\frac{1}{2}\left(\gamma+\delta\log\left[\frac{x-\mu}{\sigma}\right]\right)^2}\delta}{\sqrt{2\pi}(x-\mu)}$$

PDF in R

```
exp(-1/2*(gamma+delta*log((x-mu)/sigma))^2)*delta/(sqrt(2*pi)*(x-mu))
```

CDF

$$\begin{cases} \frac{1}{2}\text{erfc}\left[-\frac{\gamma+\delta\,\log\left[\frac{x-\mu}{\sigma}\right]}{\sqrt{2}}\right] & \text{for } \mu < x \leq \mu+\sigma \\ \frac{1}{2}\left(1+\text{erf}\left[\frac{\gamma+\delta\,\log\left[\frac{x-\mu}{\sigma}\right]}{\sqrt{2}}\right]\right) & \text{for } x > \mu+\sigma \end{cases}$$

CDF in R

```
CDF1=1/2*erfc(-(gamma+delta*log((x-mu)/sigma))/sqrt(2))
CDF2=1/2*(1+ erf((gamma+delta*log((x-mu)/sigma))/sqrt(2)))
```

Characteristics

Relationships

References and sources

`https://reference.wolfram.com/language/ref/JohnsonDistribution.html`

UncertML mapping:

no UncertML mapping found.

JohnsonSN1

name	Johnson SN 1 (ID: 0001337)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

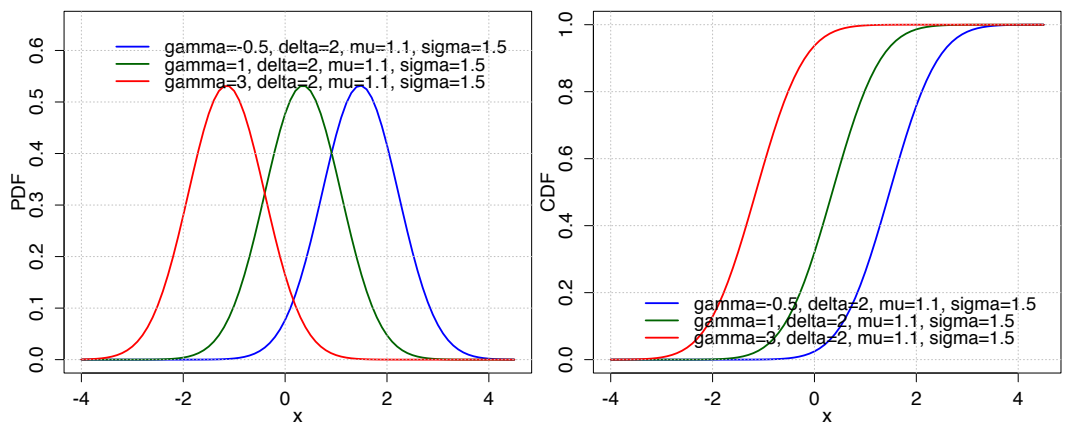

Figure A.56: JohnsonSN1 distribution plotted using the provided R code.

Model

Parameter: shape1

name	shape parameter
type	scalar
symbol	γ
definition	$\gamma \in R$

Parameter: shape2

name	shape parameter
type	scalar
symbol	δ
definition	$\delta > 0$

Parameter: location

name	location parameter
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale parameter
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{e^{-\frac{1}{2}\left(\gamma+\frac{\delta(x-\mu)}{\sigma}\right)^2}\delta}{\sqrt{2\pi}\sigma}$$

PDF in R

```
exp(-1/2*(gamma+(delta*(x-mu))/sigma)^2)*delta/(sqrt(2*pi)*sigma)
```

CDF

$$\frac{1}{2}\mathrm{erfc}\left[-\frac{\delta+\frac{\delta(x-\mu)}{\sigma}}{\sqrt{2}}\right]$$

CDF in R

```
1/2 *erfc(-(gamma+ (delta*(x-mu)/sigma))/sqrt(2))
```

Characteristics

Relationships

References and sources

https://reference.wolfram.com/language/ref/JohnsonDistribution.html

UncertML mapping:

no UncertML mapping found.

JohnsonSU1

name	Johnson SU 1 (ID: 0001328)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

Model

Parameter: shape1

name	shape parameter
type	scalar
symbol	γ
definition	$\gamma \in R$

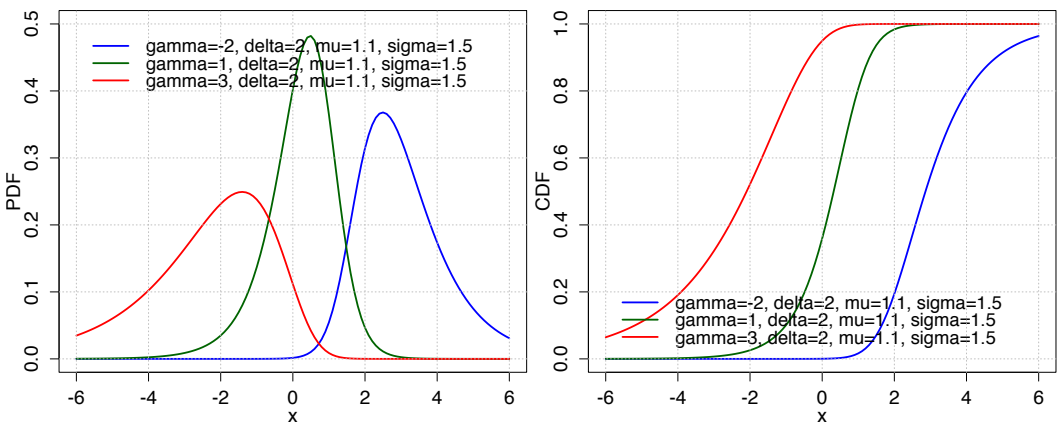

Figure A.57: JohnsonSU1 distribution plotted using the provided R code.

Parameter: shape2

name	shape parameter
type	scalar
symbol	δ
definition	$\delta > 0$

Parameter: location

name	location parameter
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale parameter
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{e^{-\frac{1}{2}\left(\gamma+\delta\,\mathrm{arcsinh}\left[\frac{x-\mu}{\sigma}\right]\right)^2}\delta}{\sqrt{2\pi}\sqrt{(x-\mu)^2+\sigma^2}}$$

PDF in R

```
exp(-1/2 * (gamma+delta *asinh((x-mu)/sigma))^2)* delta / (sqrt(2*pi)*sqrt((x-mu)^2 + sigma^2))
```

CDF

$$\frac{1}{2}\left(1+\mathrm{erf}\left[\frac{\gamma+\delta\,\mathrm{arcsin}\left[\frac{x-\mu}{\sigma}\right]}{\sqrt{2}}\right]\right)$$

CDF in R

```
1/2 *(1+  erf((gamma+delta*asinh((x-mu)/sigma))/sqrt(2)))
```

Characteristics

Relationships

References and sources

https://reference.wolfram.com/language/ref/JohnsonDistribution.html

UncertML mapping:

no UncertML mapping found.

Kumaraswamy1

name	Kumaraswamy 1 (ID: 0001360)
type	continuous
variate	x, scalar
support	$x \in [0, 1]$

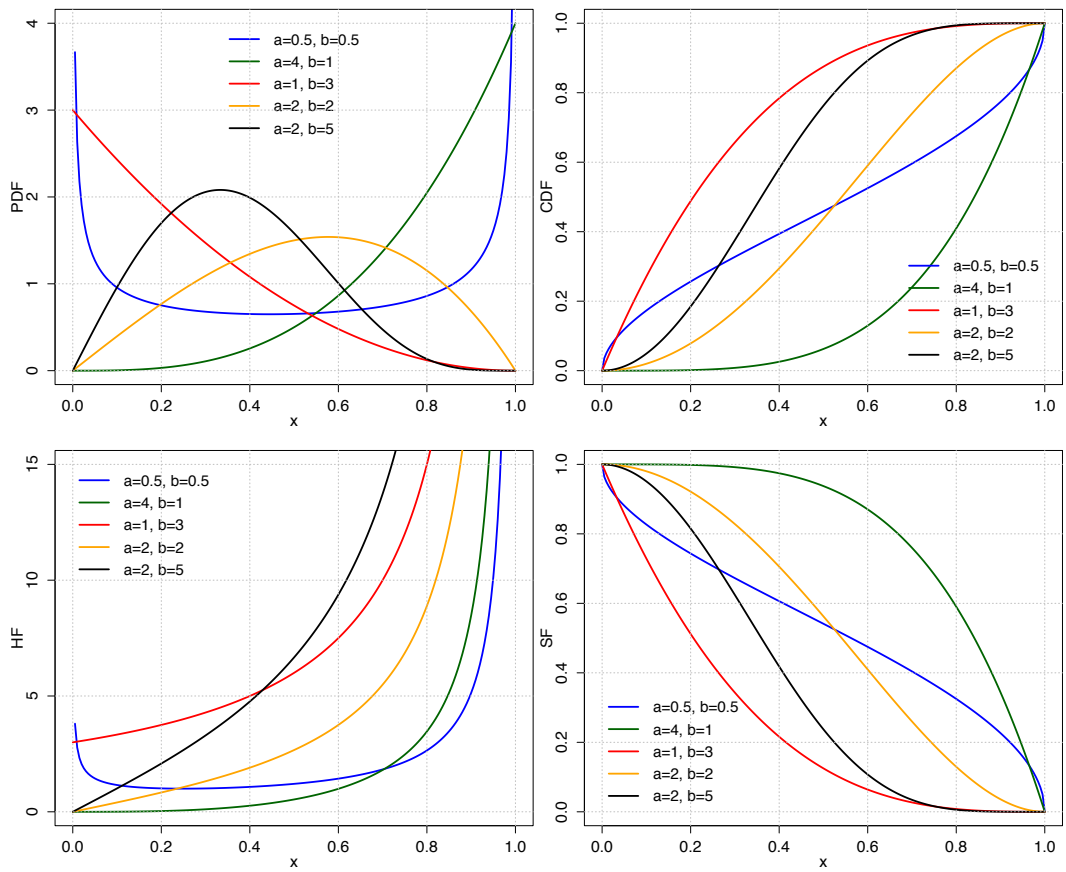

Figure A.58: Kumaraswamy1 distribution plotted using the provided R code.

Model

Kumaraswamy's double bounded distribution is a family of continuous probability distributions defined on the interval $[0, 1]$. It is similar to the Beta distribution, but much simpler to use especially in simulation studies due to the simple closed form of both its probability density function and cumulative distribution function.

Parameter: shape1

name	shape parameter
type	scalar
symbol	a
definition	$a > 0$

Parameter: shape2

name	shape parameter
type	scalar
symbol	b
definition	$b > 0$

Functions

PDF

$$abx^{a-1}(1-x^a)^{b-1}$$

PDF in R

```
a*b*x^(a-1) * (1-x^a)^(b-1)
```

CDF

$$1-(1-x^a)^b$$

CDF in R

```
1-(1-x^a)^b
```

HF

$$\frac{a\,b\,x^{a-1}}{(1-x^a)}$$

HF in R

```
a*b*x^(a-1) / (1-x^a)
```

SF

$$(1-x^a)^b$$

SF in R

```
-(1-x^a)^b
```

Characteristics

Mean

$$\frac{b\Gamma(1+\frac{1}{a})\Gamma(b)}{\Gamma(1+\frac{1}{a}+b)}$$

Median

$$\left(1-2^{-1/b}\right)^{1/a}$$

Mode

$$\left(\frac{a-1}{ab-1}\right)^{1/a} \quad \text{for } a \geq 1, b \geq 1, (a,b) \neq (1,1)$$

Relationships

- Relationship pair: $Kumaraswamy1(a,b) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: If $X \sim Kumaraswamy1(1,1)$ then $X \sim StandardUniform1$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Kumaraswamy_distribution`
- Relationship pair: $Kumaraswamy1(a,b) \rightarrow Exponential1(a)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Kumaraswamy1(a,1)$ then $-\log(X) \sim Exponential1(a)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Kumaraswamy_distribution`
- Relationship pair: $Kumaraswamy1(a,b) \rightarrow Exponential1(b)$
- Relationship type: Transformation
- Relationship definition: If $X \sim Kumaraswamy1(1,b)$ then $-(1-\log(X)) \sim Exponential1(b)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Kumaraswamy_distribution`

References and sources

http://www.itl.nist.gov/div898/software/dataplot/refman2/auxillar/kumcdf.htm
https://en.wikipedia.org/wiki/Kumaraswamy_distribution
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Minimax.pdf

UncertML mapping:

no UncertML mapping found.

Laplace1

name	Laplace 1 (ID: 0000256)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

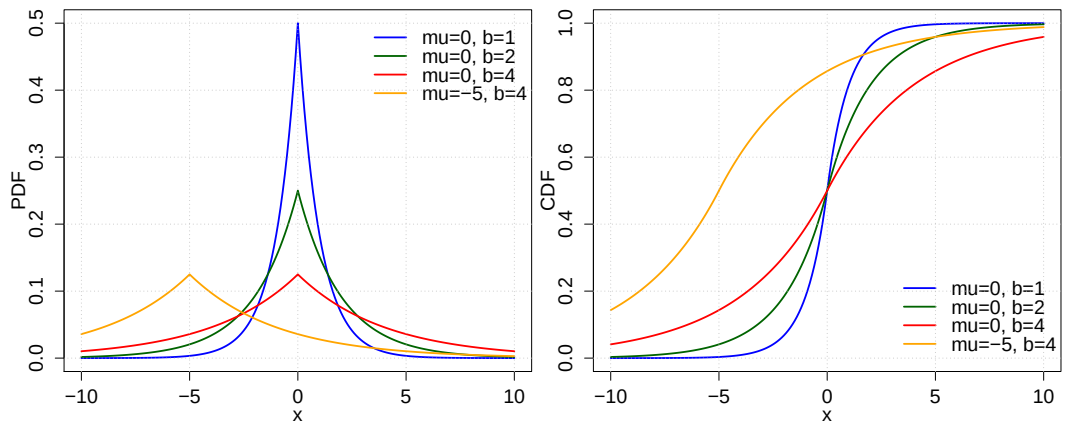

Figure A.59: Laplace1 distribution plotted using the provided R code.

Model

The Laplace distribution, also called the double exponential distribution, is the distribution of differences between two independent variates with identical exponential distributions.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	b
definition	$b > 0, b \in R$

Functions

PDF

$$\frac{1}{2\,b} \exp\left(-\frac{|x-\mu|}{b}\right)$$

PDF in R

```
1/(2*b) * exp(- abs(x-mu)/b )
```

CDF

$$\begin{cases} \frac{1}{2}\exp\left(\frac{x-\mu}{b}\right) & \text{if } x < \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{b}\right) & \text{if } x \geq \mu \end{cases}$$

CDF in R

```
1/2 * exp( (x-mu)/b ) # for x < mu
1- 1/2 * exp( -(x-mu)/b ) # x >= mu
```

Characteristics

Mean

$$\mu$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$2b^2$$

Relationships

- Relationship pair: $Laplace1(\mu, b) \rightarrow Laplace2(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/b$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Laplace2(\mu, \tau) \rightarrow Laplace1(\mu, b)$
- Relationship type: Reparameterisation
- Relationship definition: $b = 1/\tau$
- Relationship reference(s): ProbOnto spec

References and sources

`https://en.wikipedia.org/wiki/Laplace_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/laplace`

Laplace2

name	Laplace 2 (ID: 0000283)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

Model

The Laplace distribution, also called the double exponential distribution, is the distribution of differences between two independent variates with identical exponential distributions.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

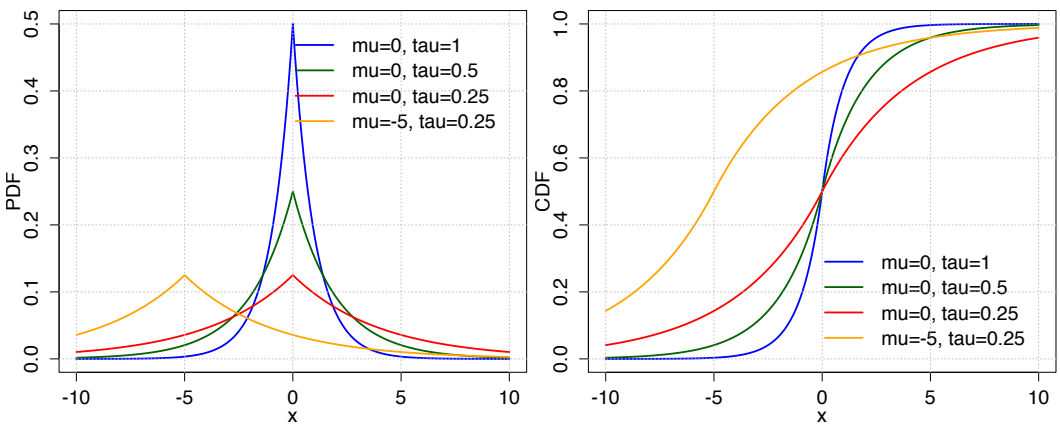

Figure A.60: Laplace2 distribution plotted using the provided R code.

Parameter: tau

name	precision
type	scalar
symbol	τ
definition	$\tau > 0, \tau \in R$

Functions

PDF

$$\frac{\tau}{2} \exp\left(-\tau|x-\mu|\right)$$

PDF in R

```
tau/2 * exp(-tau * abs(x-mu))
```

CDF

$$\begin{cases} \frac{1}{2}\exp\left(\tau(x-\mu)\right) & \text{if } x < \mu \\ 1 - \frac{1}{2}\exp\left(-\tau(x-\mu)\right) & \text{if } x \geq \mu \end{cases}$$

CDF in R

```
1/2 * exp( tau*(x-mu) ) # for x < mu
1- 1/2 * exp( -tau*(x-mu) ) # x >= mu
```

Characteristics

Mean

$$\mu$$

Variance

$$2/\tau^2$$

Relationships

- Relationship pair: $Laplace2(\mu, \tau) \rightarrow Laplace1(\mu, b)$
- Relationship type: Reparameterisation
- Relationship definition: $b = 1/\tau$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Laplace1(\mu, b) \rightarrow Laplace2(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/b$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]
[Lunn, 2012]

UncertML mapping:

no UncertML mapping found.

Levy1

name	Levy 1 (ID: 0001393)
type	continuous
variate	x, scalar
support	$x \in [\mu, +\infty)$

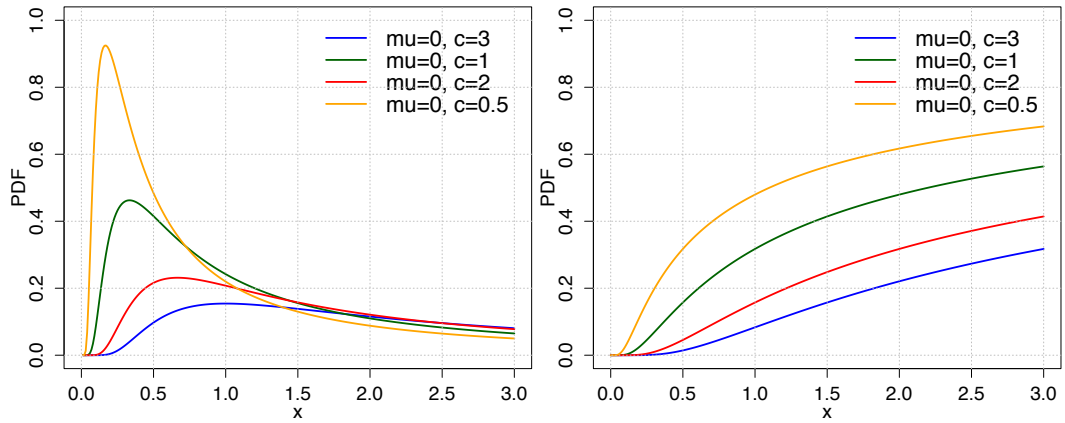

Figure A.61: Levy1 distribution plotted using the provided R code.

Model

Lévy distribution, named after Paul Lévy, is a continuous probability distribution for a non-negative random variable. In spectroscopy, this distribution, with frequency as the dependent variable, is known as a van der Waals profile.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	c
definition	$c > 0, c \in R$

Functions

PDF

$$\sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(x-\mu)}}}{(x-\mu)^{3/2}}$$

PDF in R

```
sqrt(c/2/pi) * exp(-c/2/(x-mu)) / (x-mu)^(3/2)
```

CDF

$$\operatorname{erfc}\left(\sqrt{\frac{c}{2(x-\mu)}}\right)$$

CDF in R

```
erfc(sqrt(c/2/(x-mu)))
```

Characteristics

Mean

$$\infty$$

Median

$$c/2(\operatorname{erfc}^{-1}(1/2))^2 \text{ for } \mu = 0$$

Mode

$$\frac{c}{3} \text{ for } \mu = 0$$

Variance

$$\infty$$

Relationships

- Relationship pair: $Normal1(\mu, \sigma) \rightarrow Levy1(\mu, c)$
- Relationship type: Transformation
- Relationship definition: $Y \sim \text{Normal}(\mu, \sigma) \Rightarrow (Y - \mu)^{-2} \sim \text{Levy}(0, 1/\sigma)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/L%C3%A9vy_distribution#Related_distributions`
- Relationship pair: $Amoroso1(a, \theta, \alpha, \beta) \rightarrow Levy1(\mu, c)$
- Relationship type: Special case
- Relationship definition: Levy1(μ, c) distribution is a special case of the Amoroso1$(a, \theta, \alpha, \beta)$ distribution when $\theta = c/2, \alpha = 1/2, \beta = -1$
- Relationship reference(s): [Crooks, 2010]

References and sources

`https://en.wikipedia.org/wiki/L%C3%A9vy_distribution`
(27 July 2016)

UncertML mapping:

no UncertML mapping found.

LKJCorrelation1

name	LKJ Correlation 1 (ID: 0001402)
type	continuous
variate	Σ, matrix
support	positive-definite, symmetric matrix with unit diagonal (i.e., a correlation matrix)

Model

LKJ1 distribution is a distribution over positive-definite symmetric matrices with unit diagonal, that is correlation matrices. Implicit parameterisation of the LKJ1 correlation matrix density in terms of its Cholesky factor.

Parameter: shape

name	shape
type	scalar
symbol	η
definition	$\eta > 0$

Functions

PDF

$$\propto \det(\Sigma)^{(\eta-1)}$$

Characteristics

Relationships

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

LKJCorrelation2

name	LKJ Correlation 2 (ID: 0001088)
type	continuous
variate	L, matrix
support	Cholesky factor of a symmetric positive-definite matrix with unit diagonal (i.e., a correlation matrix)

Model

Implicit parameterization of the LKJ1 correlation matrix density in terms of its Cholesky factor.

Parameter: shape

name	shape
type	scalar
symbol	η
definition	$\eta > 0$

Functions

PDF

$$\propto |J| \det(LL^T)^{(\eta-1)} = \prod_{k=2}^{K} L_{kk}^{K-k+2\eta-2}$$

Characteristics

Relationships

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

LogLogistic1

name	Log-Logistic 1 (ID: 0000377)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

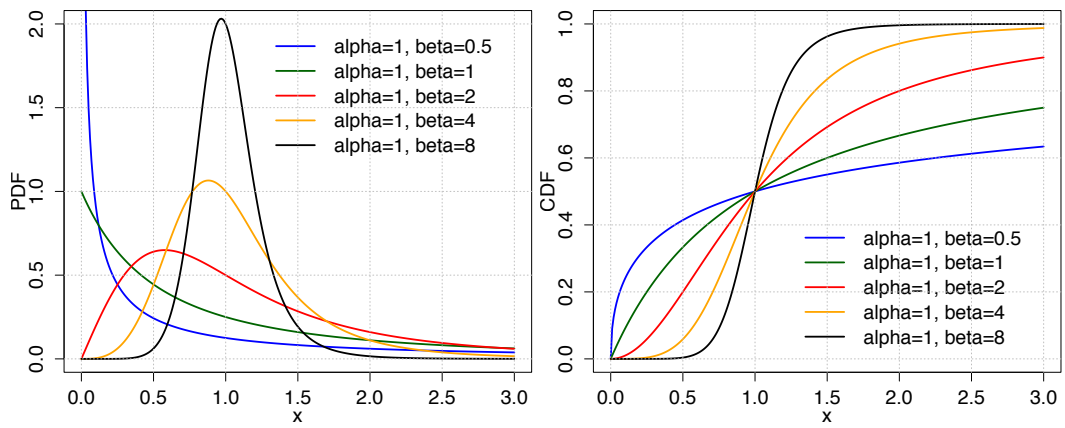

Figure A.62: LogLogistic1 distribution plotted using the provided R code.

Model

Parameter: scale

name	scale
type	scalar
symbol	α
definition	$\alpha > 0$

Parameter: shape

name	shape
type	scalar
symbol	β
definition	$\beta > 0$

Functions

PDF

$$\frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{(1+(x/\alpha)^{\beta})^2}$$

PDF in R

```
(beta/alpha)*(x/alpha)^(beta-1) / (1+(x/alpha)^beta)^2
```

CDF

$$\frac{1}{1+(x/\alpha)^{-\beta}}$$

CDF in R

```
1 / (1+(x/alpha)^(-beta))
```

Characteristics

Mean

$$\frac{\alpha\pi/\beta}{\sin(\pi/\beta)} \text{ if } \beta > 1, \text{ else undefined}$$

Median

$$\alpha$$

Mode

$$\alpha\Big(\frac{\beta-1}{\beta+1}\Big)^{1/\beta} \text{ if } \beta > 1, 0 \text{ otherwise}$$

Variance

$$\alpha^2\Big(\frac{2\pi/\beta}{\sin(2\pi/\beta)} - \frac{(\pi/\beta)^2}{\sin^2(\pi/\beta)}\Big), \text{ for } \beta > 2$$

Relationships

- Relationship pair: $LogLogistic1(\alpha, \beta) \rightarrow Logistic1(\mu, s)$
- Relationship type: Transformation
- Relationship definition: If $X \sim LogLogistic1(\alpha, \beta) \Rightarrow Y = log(X) \sim Logistic1(\mu, s)$ with $\mu = \log(\alpha), s = 1/\beta)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Log-logistic_distribution#Related_distributions`
- Relationship pair: $LogLogistic1(\alpha, \beta) \rightarrow LogLogistic2(\lambda, \kappa)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda = 1/\alpha, \kappa = \beta$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogLogistic2(\lambda, \kappa) \rightarrow LogLogistic1(\alpha, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = 1/\lambda, \beta = \kappa$
- Relationship reference(s): ProbOnto spec

References and sources

`https://en.wikipedia.org/wiki/Log-logistic_distribution`

UncertML mapping:

no UncertML mapping found.

LogLogistic2

name	Log-Logistic 2 (ID: 0000402)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

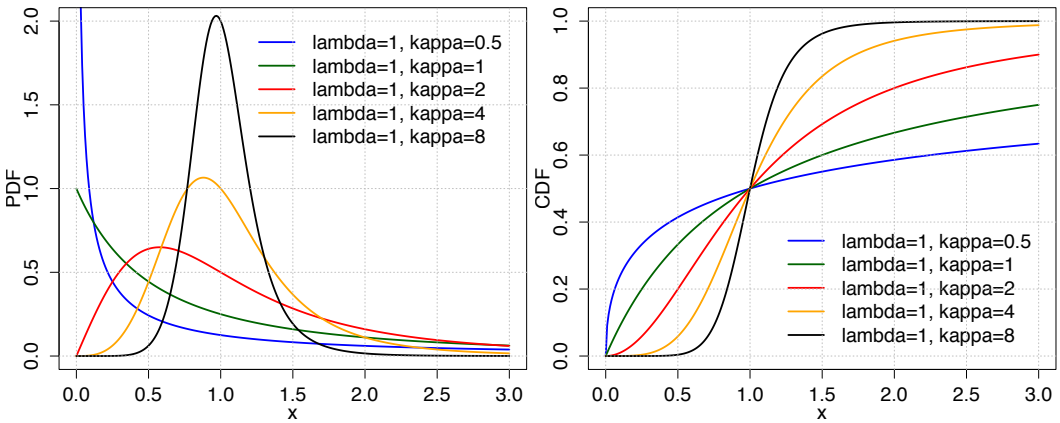

Figure A.63: LogLogistic2 distribution plotted using the provided R code.

Model

Parameter: scale

name	scale
type	scalar
symbol	λ
definition	$\lambda > 0$

Parameter: shape

name	shape
type	scalar
symbol	κ
definition	$\kappa > 0$

Functions

PDF

$$\frac{\lambda\kappa(\lambda x)^{\kappa-1}}{(1+(\lambda x)^{\kappa})^2}$$

PDF in R

```
(lambda*kappa*(lambda*x)^(kappa-1)) / (1+(lambda*x)^kappa)^2
```

CDF

$$\frac{(\lambda x)^{\kappa}}{1+(\lambda x)^{\kappa}}$$

CDF in R

```
(lambda*x)^kappa / (1+(lambda*x)^kappa)
```

HF

$$\frac{\lambda\kappa(\lambda x)^{\kappa-1}}{1+(\lambda x)^{\kappa}}$$

HF in R

```
lambda * kappa * (lambda * x)^(kappa-1) / (1+(lambda * x)^kappa)
```

SF

$$\frac{1}{1+(\lambda x)^{\kappa}}$$

SF in R

```
1/(1+(lambda*x)^kappa)
```

Characteristics

Mean

$$\frac{\pi}{\kappa\lambda\sin(\pi/\kappa)}$$

Median

$$1/\lambda$$

Variance

$$\frac{\pi\big(2\kappa[1-\cos(\frac{\pi}{\kappa})^2]+\pi\sin\big(\frac{\pi(\kappa+2)}{\kappa}\big)\big)}{\sin\big(\frac{\pi(\kappa+2)}{\kappa}\big)\big(\cos^2(\frac{\pi}{\kappa})-1\big)(\lambda\kappa)^2}$$

Relationships

- Relationship pair: $LogLogistic2(\lambda,\kappa) \rightarrow LogLogistic1(\alpha,\beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = 1/\lambda, \beta = \kappa$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $StandardUniform1 \rightarrow LogLogistic2(\lambda,\kappa)$
- Relationship type: Transformation
- Relationship definition: If $X \sim StandardUniform1$ and $Y = \frac{1}{\lambda}\left(\frac{1-X}{X}\right)^{1/\kappa} \Rightarrow Y \sim LogLogistic2(\lambda,\kappa)$
- Relationship reference(s): NA
- Relationship pair: $LogLogistic1(\alpha,\beta) \rightarrow LogLogistic2(\lambda,\kappa)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda = 1/\alpha, \kappa = \beta$
- Relationship reference(s): ProbOnto spec

References and sources

[Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Loglogistic.pdf`

UncertML mapping:

no UncertML mapping found.

LogNormal1

name	Log-Normal 1 (ID: 0000428)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

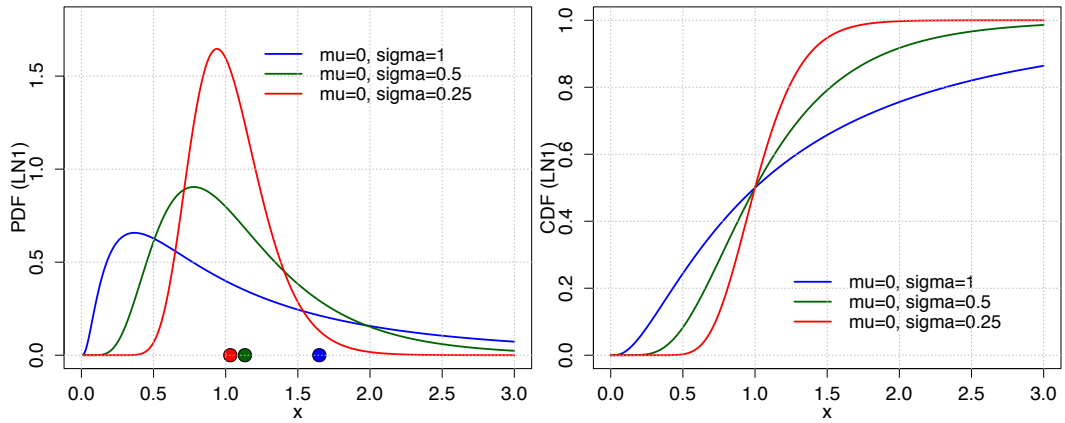

Figure A.64: LogNormal1 distribution plotted using the provided R code.

Model

The lognormal distribution is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. LogNormal1 is parameterized with mean, μ, and standard deviation, σ, both on the log-scale.

Parameter: meanLog

name	mean of log(x)
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: stdevLog

name	shape
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{1}{x\sigma\sqrt{2\pi}}\, e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}$$

PDF in R

```
1/(x*sigma*sqrt(2*pi)) * exp((-(log(x)-mu)^2)/(2*sigma^2))
```

CDF

$$\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\Big[\frac{\log x - \mu}{\sqrt{2}\sigma}\Big]$$

CDF in R

```
1/2 + 1/2 *erf( (log(x)-mu)/(sqrt(2)*sigma) )
```

Characteristics

Mean

$$e^{\mu+\sigma^2/2}$$

Median

$$e^{\mu}$$

Mode

$$e^{\mu-\sigma^2}$$

Variance

$$(e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$$

Relationships

- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: $\log(X)$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalLognormal.pdf`
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = \exp\left(\mu + \frac{1}{2}\sigma^2\right), \sigma_N = \exp\left(\mu + \frac{1}{2}\sigma^2\right)\sqrt{\exp(\sigma^2) - 1}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, v = \sigma^2$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma_g = \exp(\sigma)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma = \sigma$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), cv = \sqrt{\exp(\sigma^2) - 1}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \tau = 1/\sigma^2$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Normal1(\mu, \sigma) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: $\exp(X)$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalLognormal.pdf` [Forbes et al., 2011]
- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal1(\mu, \sigma)$

- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right), \sigma = \sqrt{\log(1+\sigma_N^2/\mu_N^2)}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \sigma = \sqrt{v}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \sigma = \log(\sigma_g)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \sigma = \sigma$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $PowerNormal1(\lambda, \mu, \sigma) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Special case
- Relationship definition: $PowerNormal1$ distribution converges to LogNormal1 one when $\lambda \rightarrow 0$
- Relationship reference(s): [Lavielle, 2014]
- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \sigma = \sqrt{\log(cv^2+1)}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal5(\mu, \tau) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \sigma = 1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

References and sources

`http://en.wikipedia.org/wiki/Log-normal_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/log-normal`

LogNormal2

name	Log-Normal 2 (ID: 0000453)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

Model

The lognormal distribution is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. LogNormal2 is parameterized with mean, μ, and variance, v, both on the log-scale.

Parameter: meanLog

name	mean of log(x)
type	scalar
symbol	μ
definition	$\mu \in R$

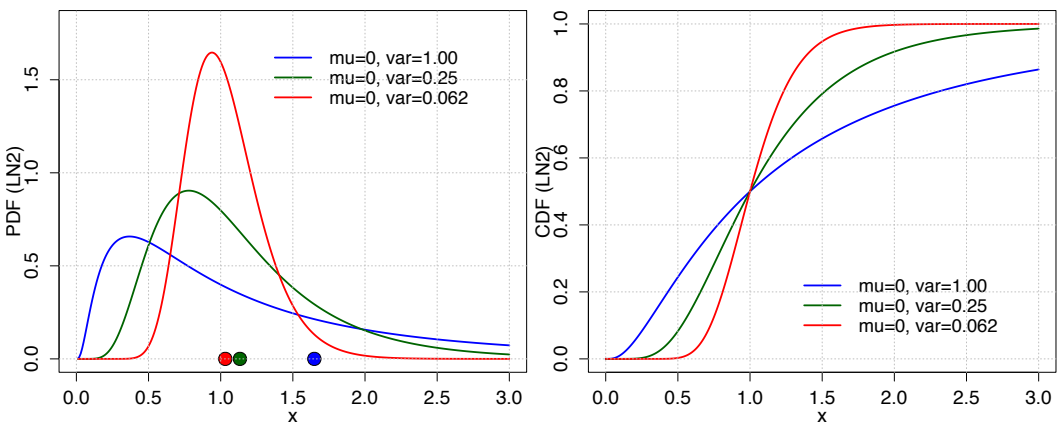

Figure A.65: LogNormal2 distribution plotted using the provided R code.

Parameter: varLog

name	shape
type	scalar
symbol	v
definition	$v > 0$

Functions

PDF

$$\frac{1}{x\sqrt{v}\sqrt{2\pi}}\, e^{-\frac{(\log x-\mu)^2}{2v}}$$

PDF in R

```
1/(x*sqrt(v)*sqrt(2*pi)) * exp(-(ln(x)-mu)^2/(2*v))
```

CDF

$$\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\Big[\frac{\log x-\mu}{\sqrt{2}\sqrt{var}}\Big]$$

CDF in R

```
1/2 + 1/2 * erf( (log(x)-mu) / (sqrt(2)*sqrt(var)) )
```

Characteristics

Mean

$$e^{\mu+v/2}$$

Median

$$e^{\mu}$$

Mode

$$e^{\mu-v}$$

Variance

$$(e^{v}-1)e^{2\mu+v}$$

Relationships

- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = \exp(\mu + v/2), \sigma_N = \exp(\mu + v/2)\sqrt{\exp(v) - 1}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation

- Relationship definition: $\mu = \mu, \sigma = \sqrt{v}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma_g = \exp(\sqrt{v})$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma = \sqrt{v}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), cv = \sqrt{\exp(v) - 1}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \tau = 1/v$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal5(\mu, \tau) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, v = 1/\tau$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log\left(\mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}\right), v = \log(1 + \sigma_N^2/\mu_N^2)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, v = \sigma^2$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), v = \log(\sigma_g^2)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), v = \sigma^2$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), v = \log(cv^2 + 1)$
- Relationship reference(s): ProbOnto spec

References and sources

`http://en.wikipedia.org/wiki/Log-normal_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/log-normal`

LogNormal3

name	Log-Normal 3 (ID: 0000478)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

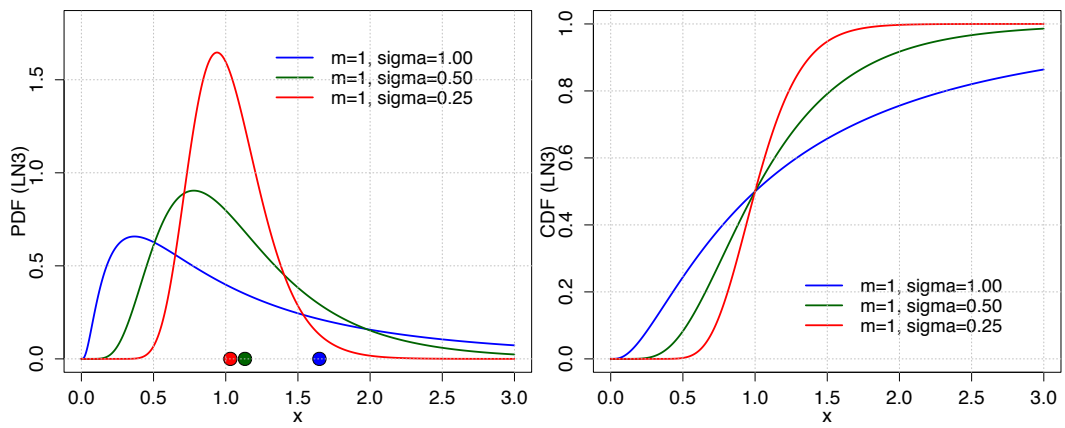

Figure A.66: LogNormal3 distribution plotted using the provided R code.

Model

The lognormal distribution is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. LogNormal3 is parameterized with median, m, on the natural scale and standard deviation, σ, on the log-scale.

Parameter: median

name	median / geometric mean
type	scalar
symbol	m
definition	$m > 0$

Parameter: stdevLog

name	shape
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{1}{x\sigma\sqrt{2\pi}}\, e^{-\frac{[\log(x/m)]^2}{2\sigma^2}}$$

PDF in R

```
1/(x*sigma*sqrt(2*pi)) * exp(-(log(x/m))^2 / (2*sigma^2))
```

CDF

$$\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\Big[\frac{\log x - \log m}{\sqrt{2}\sigma}\Big]$$

CDF in R

```
1/2 + 1/2 * erf( (log(x)-log(m)) / (sqrt(2)*sigma) )
```

Characteristics

Mean

$$m e^{\frac{1}{2}\sigma^2}$$

Median

$$m$$

Mode

$$m/e^{\sigma^2}$$

Variance

$$m^2 e^{\sigma^2}[e^{\sigma^2} - 1]$$

Relationships

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, cv = \sqrt{\exp(\sigma^2) - 1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \tau = 1/\sigma^2$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = m\ \exp(\sigma^2/2), \sigma_N = m\ \exp(\sigma^2/2)\sqrt{\exp(\sigma^2) - 1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \sigma = \sigma$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma_g = \exp(\sigma)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), v = \sigma^2$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma = \sqrt{\log(cv^2 + 1)}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu, \tau) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma = 1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}, \sigma = \sqrt{\log(1 + \sigma_N^2/\mu_N^2)}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma = \sigma$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma = \log(\sigma_g)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma = \sqrt{v}$
- Relationship reference(s): ProbOnto spec

References and sources

[Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Lognormal.pdf`
[Forbes et al., 2011]
[Pharsight, 2011]
`http://www.itl.nist.gov/div898/handbook/eda/section3/eda3669.htm`

UncertML mapping:

no UncertML mapping found.

LogNormal4

name	Log-Normal 4 (ID: 0000500)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

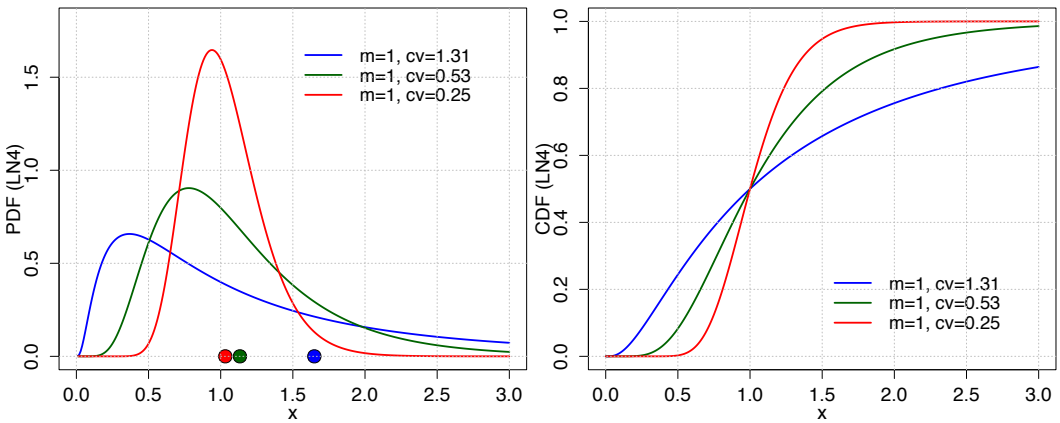

Figure A.67: LogNormal4 distribution plotted using the provided R code.

Model

The lognormal distribution is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. LogNormal4 is parameterized with median, m, and coefficient of variation, cv, both on the natural scale.

Parameter: median

name	median / geometric mean
type	scalar
symbol	m
definition	$m > 0$

Parameter: coefVar

name	coefficient of variation
type	scalar
symbol	cv
definition	$cv > 0$

Functions

PDF

$$\frac{1}{x\sqrt{\log(cv^2+1)}\sqrt{2\pi}}\; e^{-\frac{[\log(x/m)]^2}{2\ln(cv^2+1)}}$$

PDF in R

```
1/(x*sqrt(log(cv^2+1))*sqrt(2*pi)) * exp( -(log(x/m))^2 / (2*log(cv^2+1)) )
```

CDF

$$\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\Big[\frac{\log x - \log m}{\sqrt{2}\sqrt{\log(cv^2+1)}}\Big]$$

CDF in R

```
1/2 + 1/2 * erf( (log(x)-log(m)) / (sqrt(2*log(cv^2+1))) )
```

Characteristics

Mean

$$m\sqrt{cv^2+1}$$

Median

$$m$$

Mode

$$m/(cv^2+1)$$

Variance

$$m^2(cv^2+1)cv^2$$

Relationships

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma = \sqrt{\log(cv^2+1)}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \tau = 1/\log(cv^2+1)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = m\sqrt{cv^2+1}, \sigma_N = m\ cv\ \sqrt{cv^2+1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \sigma = \sqrt{\log(cv^2+1)}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma_g = \exp\big(\sqrt{\log(cv^2+1)}\big)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), v = \log(cv^2+1)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, cv = \sqrt{\exp(\sigma^2)-1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu, \tau) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), cv = \sqrt{\exp(1/\tau)-1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}, cv = \sigma_N/\mu_N$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), cv = \sqrt{\exp(\sigma^2)-1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal4(m, cv)$

- Relationship type: Reparameterisation
- Relationship definition: $m = m, cv = \sqrt{\exp\big(\log^2(\sigma_g)\big) - 1}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), cv = \sqrt{\exp(v) - 1}$
- Relationship reference(s): ProbOnto spec

References and sources

no source recorded.

UncertML mapping:

no UncertML mapping found.

LogNormal5

name	Log-Normal 5 (ID: 0000526)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

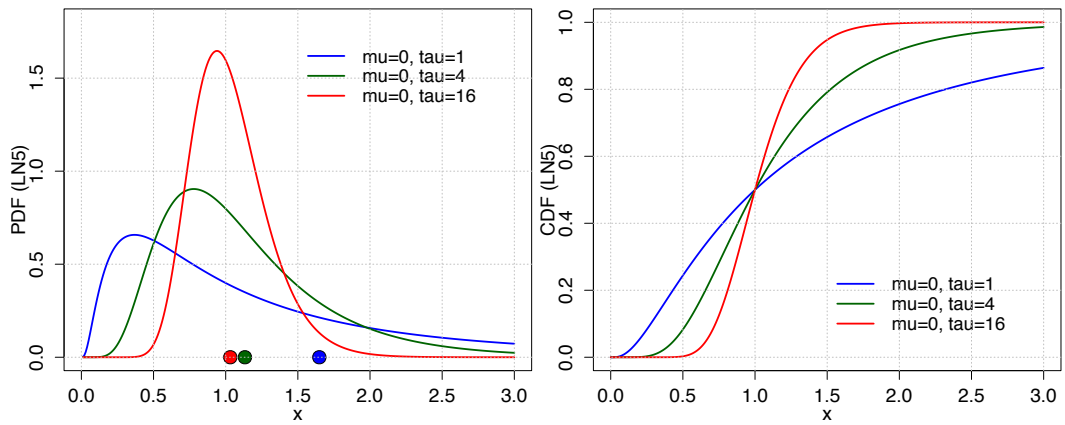

Figure A.68: LogNormal5 distribution plotted using the provided R code.

Model

The lognormal distribution is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. LogNormal4 is parameterized with mean, μ, and standard deviation, σ, both on the log-scale.

Parameter: meanLog

name	mean of log(x)
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: precision

name	precision
type	scalar
symbol	τ
definition	$\tau > 0$

Functions

PDF

$$\sqrt{\frac{\tau}{2\pi}}\frac{1}{x}e^{-\frac{\tau}{2}(\log x-\mu)^2}$$

PDF in R

```
sqrt(tau / (2*pi)) * (1/x) * exp(- (tau/2)*(log(x)-mu)^2 )
```

CDF

$$\frac{1}{2}+\frac{1}{2}\operatorname{erf}\Big[\frac{\log x-\mu}{\sqrt{2/\tau}}\Big]$$

CDF in R

```
1/2 + 1/2 * erf( (log(x)-mu) / sqrt(2/tau) )
```

Characteristics

Mean

$$e^{\mu+\frac{1}{2\tau}}$$

Median

$$e^{\mu}$$

Mode

$$e^{\mu-\frac{1}{\tau}}$$

Variance

$$e^{2\mu+\frac{1}{\tau}}[e^{\frac{1}{\tau}}-1]$$

Relationships

- Relationship pair: $LogNormal5(\mu,\tau)\rightarrow LogNormal2(\mu,v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\mu, v=1/\tau$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu,\tau)\rightarrow LogNormal3(m,\sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m=\exp(\mu), \sigma=1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu,\tau)\rightarrow LogNormal4(m,cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m=\exp(\mu), cv=\sqrt{\exp(1/\tau)-1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu,\tau)\rightarrow LogNormal7(\mu_N,\sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N=\exp\big(\mu+1/(2\tau)\big), \sigma_N=\exp\big(\mu+1/(2\tau)\big)\sqrt{\exp\big(1/\tau\big)-1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu,\tau)\rightarrow LogNormal1(\mu,\sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\mu, \sigma=1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu,\tau)\rightarrow LogNormal6(m,\sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m=\exp(\mu), \sigma_g=\exp(1/\sqrt{\tau})$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m,\sigma)\rightarrow LogNormal5(\mu,\tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\log(m), \tau=1/\sigma^2$

- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \tau = 1/\log(cv^2 + 1)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right), \tau = 1/\log\left(1+\sigma_N^2/\mu_N^2\right)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \tau = 1/\sigma^2$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \tau = 1/\log^2(\sigma_g)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \tau = 1/v$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]
[Lunn, 2012]
[Plummer, 2003]
`http://www2.uaem.mx/r-mirror/web/packages/LaplacesDemon/LaplacesDemon.pdf`

UncertML mapping:

no UncertML mapping found.

LogNormal6

name	Log-Normal 6 (ID: 0000553)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

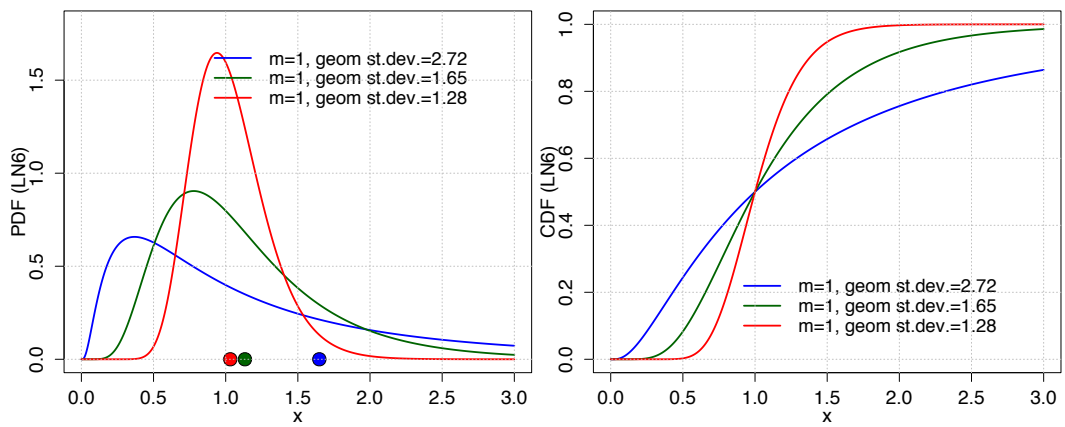

Figure A.69: LogNormal6 distribution plotted using the provided R code.

Model

The lognormal distribution is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. LogNormal6 is parameterized with mean, μ, and precision, τ, both on the log-scale.

Parameter: median

name	median / geometric mean
type	scalar
symbol	m
definition	$m > 0$

Parameter: geomStdev

name	shape
type	scalar
symbol	σ_g
definition	$\sigma_g > 0$

Functions

PDF

$$\frac{1}{x\log(\sigma_g)\sqrt{2\pi}}\exp\left[\frac{-[\log(x/m)]^2}{2\log^2(\sigma_g)}\right]$$

PDF in R

```
1/(x*log(sigma_g)*sqrt(2*pi))*exp(-(log(x/m))^2/(2*log(sigma_g)^2))
```

CDF

$$\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left[\frac{\log x-\log m}{\sqrt{2}\log(\sigma_g)}\right]$$

CDF in R

```
1/2 + 1/2 * erf( (log(x)-log(m)) / (sqrt(2)*log(sigma_g)) )
```

Characteristics

Mean

$$m\,e^{\frac{1}{2}\log^2(\sigma_g)}$$

Median

$$m$$

Mode

$$m/e^{\log^2(\sigma_g)}$$

Variance

$$m^2e^{\log^2(\sigma_g)}[e^{\log^2(\sigma_g)}-1]$$

Relationships

- Relationship pair: $LogNormal6(m,\sigma_g) \rightarrow LogNormal7(\mu_N,\sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = m\exp\left(\frac{1}{2}\log^2(\sigma_g)\right), \sigma_N = m\exp\left(\frac{1}{2}\log^2(\sigma_g)\right)\sqrt{\exp\left[\log^2(\sigma_g)\right]-1}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal6(m,\sigma_g) \rightarrow LogNormal1(\mu,\sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \sigma = \log(\sigma_g)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal6(m,\sigma_g) \rightarrow LogNormal2(\mu,v)$

- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), v = \log(\sigma_g^2)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma = \log(\sigma_g)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, cv = \sqrt{\exp\big(\log^2(\sigma_g)\big) - 1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log(m), \tau = 1/\log^2(\sigma_g)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \mu_N/\sqrt{1 + \sigma_N^2/\mu_N^2}, \sigma_g = \exp\left(\sqrt{\log(1 + \sigma_N^2/\mu_N^2)}\right)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma_g = \exp(\sigma)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma_g = \exp(\sqrt{v})$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma_g = \exp(\sigma)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = m, \sigma_g = \exp\big(\sqrt{\log(cv^2 + 1)}\big)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu, \tau) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \exp(\mu), \sigma_g = \exp(1/\sqrt{\tau})$
- Relationship reference(s): ProbOnto spec

References and sources

[Limpert et al., 2001]

UncertML mapping:

no UncertML mapping found.

LogNormal7

name	Log-Normal 7 (ID: 0001028)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

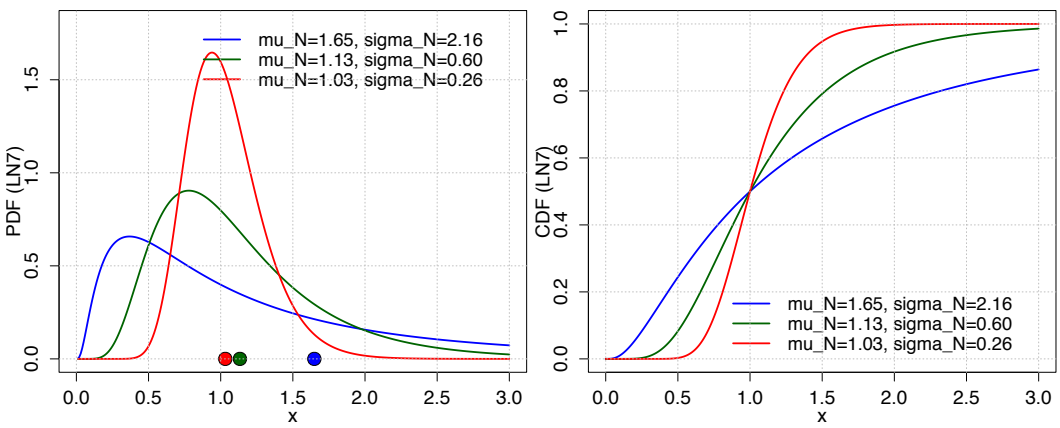

Figure A.70: LogNormal7 distribution plotted using the provided R code.

Model

The lognormal distribution is applicable to random variables that are constrained by zero but have a few very large values. The resulting distribution is asymmetrical and positively skewed. LogNormal7 is parameterized with mean, μ_N, and standard deviation, σ_N, both on the natural scale.

Parameter: mean

name	mean of x
type	scalar
symbol	μ_N
definition	$\mu_N \in R$

Parameter: stdev

name	standard deviation
type	scalar
symbol	σ_N
definition	$\sigma_N > 0$

Functions

PDF

$$\frac{1}{x\sqrt{2\pi \log\left(1+\sigma_N^2/\mu_N^2\right)}} \exp\left(\frac{-\left[\log(x) - \log\left(\frac{\mu_N}{\sqrt{1+\sigma_N^2/\mu_N^2}}\right)\right]^2}{2\log\left(1+\sigma_N^2/\mu_N^2\right)}\right)$$

PDF in R

```
1 / (x * sqrt(2 * pi * log(1+sigma_N^2/mu_N^2 ) ))
* exp( -( log(x) - log(mu_N/sqrt(1+sigma_N^2/mu_N^2)) )^2 / (2*log(1+sigma_N^2/mu_N^2)) )
```

CDF

$$\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\Big[\frac{\log x - \log(\frac{\mu_N}{\sqrt{\sigma_N^2/\mu_N^2+1}})}{\sqrt{2\log(1+\sigma_N^2/\mu_N^2)}}\Big]$$

CDF in R

```
1/2 + 1/2 * erf( (log(x)-log(mu_N/sqrt(sigma_N^2/mu_N^2 + 1))) / sqrt(2 * log(1+sigma_N^2/mu_N^2)) )
```

Characteristics

Mean

$$\mu_N$$

Median

$$\frac{\mu_N}{\sqrt{1+\sigma_N^2/\mu_N^2}}$$

Mode

$$\frac{\mu_N}{(1+\sigma_N^2/\mu_N^2)^{3/2}}$$

Variance

$$\sigma_N^2$$

Relationships

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right), \sigma = \sqrt{\log(1+\sigma_N^2/\mu_N^2)}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right), v = \log(1+\sigma_N^2/\mu_N^2)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal3(m, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}, \sigma = \sqrt{\log(1+\sigma_N^2/\mu_N^2)}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal4(m, cv)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}, cv = \sigma_N/\mu_N$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal5(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \log\left(\mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}\right), \tau = 1/\log\left(1+\sigma_N^2/\mu_N^2\right)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal7(\mu_N, \sigma_N) \rightarrow LogNormal6(m, \sigma_g)$
- Relationship type: Reparameterisation
- Relationship definition: $m = \mu_N/\sqrt{1+\sigma_N^2/\mu_N^2}, \sigma_g = \exp\left(\sqrt{\log(1+\sigma_N^2/\mu_N^2)}\right)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = \exp\left(\mu + \frac{1}{2}\sigma^2\right), \sigma_N = \exp\left(\mu + \frac{1}{2}\sigma^2\right)\sqrt{\exp(\sigma^2) - 1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal2(\mu, v) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = \exp(\mu + v/2), \sigma_N = \exp(\mu + v/2)\sqrt{\exp(v) - 1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal3(m, \sigma) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = m\ \exp(\sigma^2/2), \sigma_N = m\ \exp(\sigma^2/2)\sqrt{\exp(\sigma^2) - 1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal4(m, cv) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = m\sqrt{cv^2 + 1}, \sigma_N = m\ cv\ \sqrt{cv^2 + 1}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $LogNormal5(\mu, \tau) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = \exp\left(\mu + 1/(2\tau)\right), \sigma_N = \exp\left(\mu + 1/(2\tau)\right)\sqrt{\exp\left(1/\tau\right) - 1}$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogNormal6(m, \sigma_g) \rightarrow LogNormal7(\mu_N, \sigma_N)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu_N = m \exp\left(\frac{1}{2}\log^2(\sigma_g)\right), \sigma_N = m \exp\left(\frac{1}{2}\log^2(\sigma_g)\right)\sqrt{\exp\left[\log^2(\sigma_g)\right] - 1}$
- Relationship reference(s): ProbOnto spec

References and sources

[The PopED Development Team, 2016]

UncertML mapping:

no UncertML mapping found.

LogUniform1

name	Log-Uniform 1 (ID: 0000580)
type	continuous
variate	x, scalar
support	$x \in (min, max)$

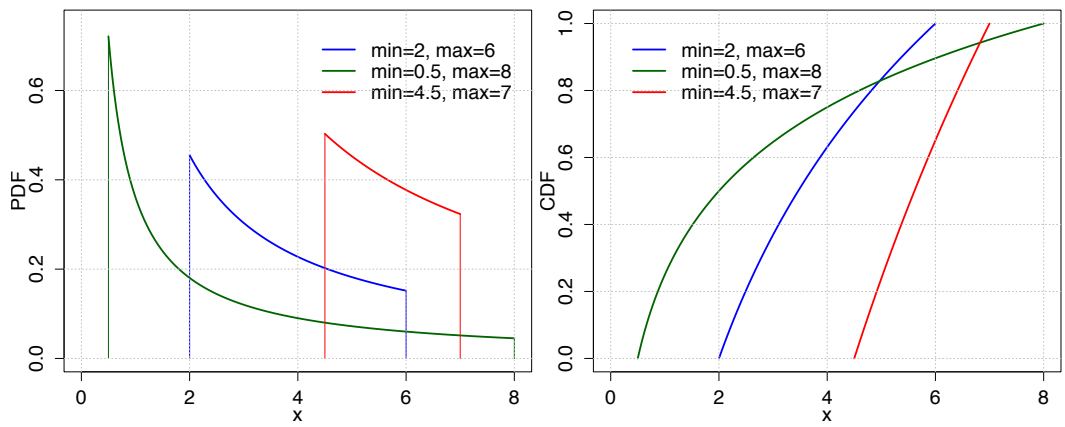

Figure A.71: LogUniform1 distribution plotted using the provided R code.

Model

The log-uniform distribution is useful in cases where inputs cover large ranges of values, but little is known about their underlying distribution. In a log-uniform distribution, the log-transformed random variable is assumed to be uniformly distributed.

Parameter: minimum

name	minimum
type	scalar
symbol	min
definition	$min > 0$

Parameter: maximum

name	maximum
type	scalar
symbol	max
definition	$max \geq min$

Functions

PDF

$$\frac{1}{x(\log(max) - \log(min))}$$

PDF in R

```
1/(x*(log(max) - log(min)))
```

CDF

$$\frac{\log(x) - \log(min)}{\log(max) - \log(min)}$$

CDF in R

```
(log(x) - log(min)) / (log(max) - log(min))
```

Characteristics

Mean

$$\frac{max - min}{\log(max) - \log(min)}$$

Variance

$$\frac{max^2 - min^2}{2[\log(max) - \log(min)]} - \left(\frac{max - min}{\log(max) - \log(min)}\right)^2$$

Relationships

References and sources

http://www.vosesoftware.com/ModelRiskHelp/index.htm#Distributions/Continuous_distributions/LogUniform_distribution.htm

UncertML mapping:

no UncertML mapping found.

Logistic1

name	Logistic 1 (ID: 0000307)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

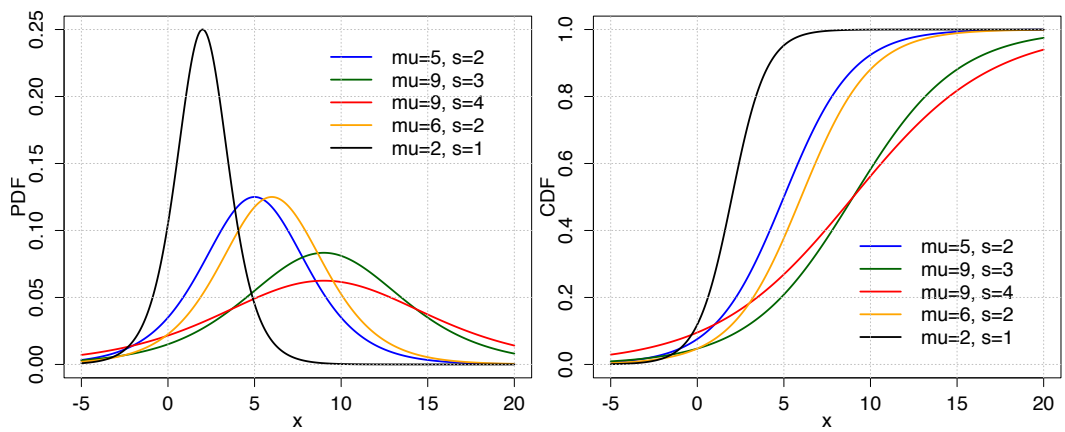

Figure A.72: Logistic1 distribution plotted using the provided R code.

Model

The distribution function of the logistic is used as a model for growth. For example, with a new product we often find that growth is initially slow, then gains momentum, and finally slows down when the market is saturated or some form of equilibrium is reached.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	s
definition	$s > 0, s \in R$

Functions

PDF

$$\frac{e^{-\frac{x-\mu}{s}}}{s\left(1+e^{-\frac{x-\mu}{s}}\right)^2}$$

PDF in R

```
exp(-(x-mu)/s) / (s*(1+exp(-(x-mu)/s))^2)
```

CDF

$$\frac{1}{1+e^{-\frac{x-\mu}{s}}}$$

CDF in R

```
1/(1+exp(-(x-mu)/s))
```

Characteristics

Mean

$$\mu$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$\frac{s^2\pi^2}{3}$$

Relationships

- Relationship pair: $Logistic1(\mu, s) \rightarrow StandardLogistic1$
- Relationship type: Special case
- Relationship definition: $\mu = 0, s = 1$
- Relationship reference(s): -
- Relationship pair: $Logistic1(\mu, s) \rightarrow Logistic2(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/s$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $LogLogistic1(\alpha, \beta) \rightarrow Logistic1(\mu, s)$

- Relationship type: Transformation
- Relationship definition: If $X \sim LogLogistic1(\alpha, \beta) \Rightarrow Y = log(X) \sim Logistic1(\mu, s)$ with $\mu = \log(\alpha), s = 1/\beta)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Log-logistic_distribution#Related_distributions`
- Relationship pair: $Logistic2(\mu, \tau) \rightarrow Logistic1(\mu, s)$
- Relationship type: Reparameterisation
- Relationship definition: $s = 1/\tau$
- Relationship reference(s): ProbOnto spec

References and sources

`http://en.wikipedia.org/wiki/Logistic_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/logistic`

Logistic2

name	Logistic 2 (ID: 0000331)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

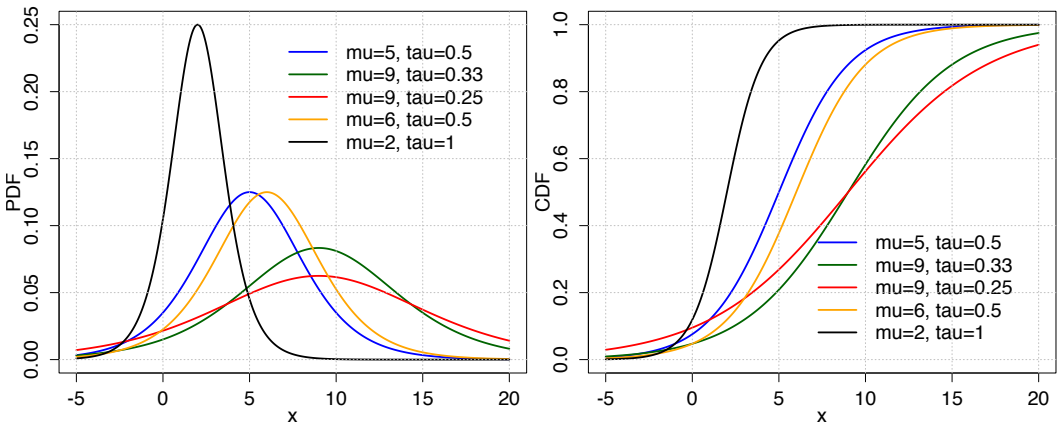

Figure A.73: Logistic2 distribution plotted using the provided R code.

Model

The distribution function of the logistic is used as a model for growth. For example, with a new product we often find that growth is initially slow, then gains momentum, and finally slows down when the market is saturated or some form of equilibrium is reached.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: inverseScale

name	inverse scale
type	scalar
symbol	τ
definition	$\tau > 0, \tau \in R$

Functions

PDF

$$\frac{\tau e^{-\tau(x-\mu)}}{\left(1+e^{-\tau(x-\mu)}\right)^2}$$

PDF in R

```
(tau * exp(-tau*(x-mu))) / (1+exp(-tau*(x-mu)))^2
```

CDF

$$\frac{1}{1+e^{-\tau(x-\mu)}}$$

CDF in R

```
1/(1+exp(-tau*(x-mu)))
```

Characteristics

Mean

$$\mu$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$\frac{\pi^2}{3\tau^2}$$

Relationships

- Relationship pair: $Logistic2(\mu, \tau) \rightarrow StandardLogistic1$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \tau = 1$
- Relationship reference(s): -
- Relationship pair: $Logistic2(\mu, \tau) \rightarrow Logistic1(\mu, s)$
- Relationship type: Reparameterisation
- Relationship definition: $s = 1/\tau$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Logistic1(\mu, s) \rightarrow Logistic2(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/s$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

LogitNormal1

name	Logit Normal 1 (ID: 0000352)
type	continuous
variate	x, scalar
support	$x \in (0, 1)$

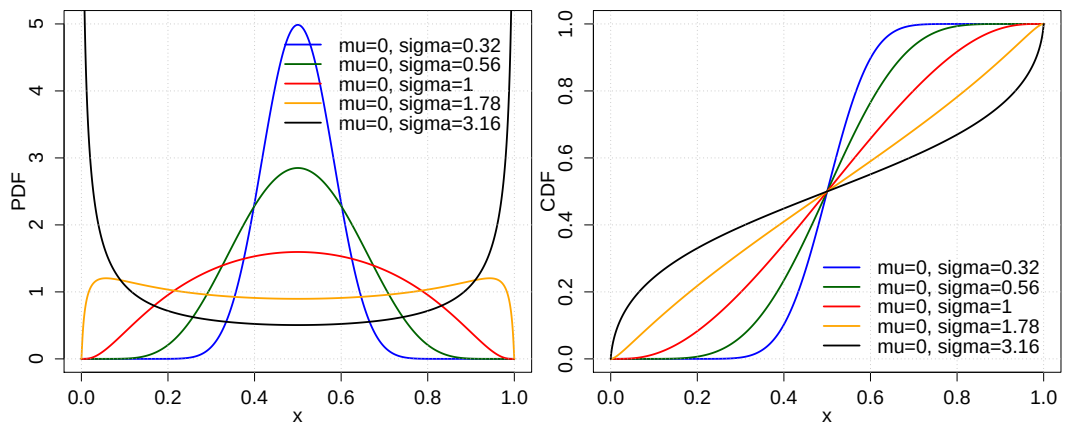

Figure A.74: LogitNormal1 distribution plotted using the provided R code.

Model

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma > 0, \sigma \in R$

Functions

PDF

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(\mathrm{logit}(x)-\mu)^2}{2\sigma^2}}\, \frac{1}{x(1-x)}$$

PDF in R

```
1/(sigma * sqrt(2*pi)) * exp(-(logit(x) - mu)^2)/(2*sigma^2)) * 1 / (x*(1-x))
```

CDF

$$\frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\mathrm{logit}(x) - \mu}{\sqrt{2\sigma^2}}\right)\right]$$

CDF in R

```
1/2*(1 + erf((logit(x)-mu)/sqrt(2*sigma^2))))
```

Characteristics

Mean

no analytical solution

Median

$$\frac{\exp(\mu)}{1 + \exp(\mu)}$$

Mode

no analytical solution

Variance

no analytical solution

Relationships

- Relationship pair: $LogitNormal1(\mu, \sigma) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: X is logit-normally distributed, then $Y = logit(X) = \log(X/(1-X))$ is normally distributed
- Relationship reference(s): `https://en.wikipedia.org/wiki/Logit-normal_distribution`

- Relationship pair: $Normal1(\mu, \sigma) \rightarrow LogitNormal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: If Y is a random variable with a normal distribution, and P is the logistic function, then $X = P(Y)$ has a logit-normal distribution
- Relationship reference(s): `https://en.wikipedia.org/wiki/Logit-normal_distribution`

References and sources

`https://en.wikipedia.org/wiki/Logit-normal_distribution`

UncertML mapping:

no UncertML mapping found.

Lomax1

name	Lomax 1 (ID: 0000604)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

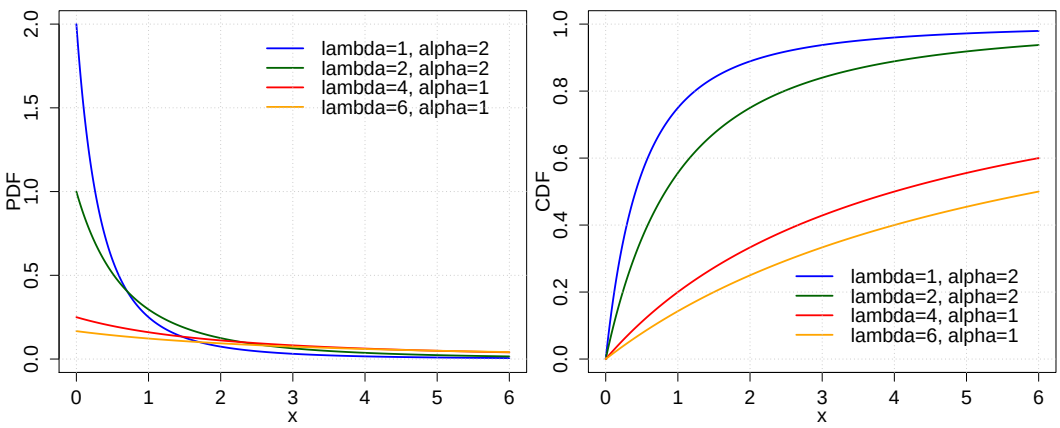

Figure A.75: Lomax1 distribution plotted using the provided R code.

Model

The Lomax distribution is a heavy-tail probability distribution used in business, economics, actuarial science, queueing theory and Internet traffic modeling. It is named after K. S. Lomax. It is essentially a Pareto distribution that has been shifted so that its support begins at zero.

Parameter: scale

name	scale
type	scalar
symbol	λ
definition	$\lambda > 0$

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha > 0$

Functions

PDF

$$\frac{\alpha}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-(\alpha+1)}$$

PDF in R

```
alpha/lambda* (1+ x/lambda)^(-(alpha+1))
```

CDF

$$1-\left[1+\frac{x}{\lambda}\right]^{-\alpha}$$

CDF in R

```
1- (1+ x/lambda)^(-alpha)
```

Characteristics

Mean

$$\frac{\lambda}{\alpha-1} \text{ for } \alpha > 1 \text{ otherwise undefined}$$

Median

$$\lambda(\sqrt[\alpha]{2}-1)$$

Mode

$$0$$

Variance

$$\frac{\lambda^2\alpha}{(\alpha-1)^2(\alpha-2)} \text{ for } \alpha > 2; \infty \text{ for } 1 < \alpha \leq 2; \text{Otherwise undefined}$$

Relationships

- Relationship pair: $ParetoTypeII1(\mu, \lambda, \alpha) \rightarrow Lomax1(\lambda, \alpha)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = 0$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Pareto_distribution#Pareto_types_I.E2.80.93IV`

References and sources

`https://en.wikipedia.org/wiki/Lomax_distribution`

UncertML mapping:

no UncertML mapping found.

Makeham1

name	Makeham 1 (ID: 0001202)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

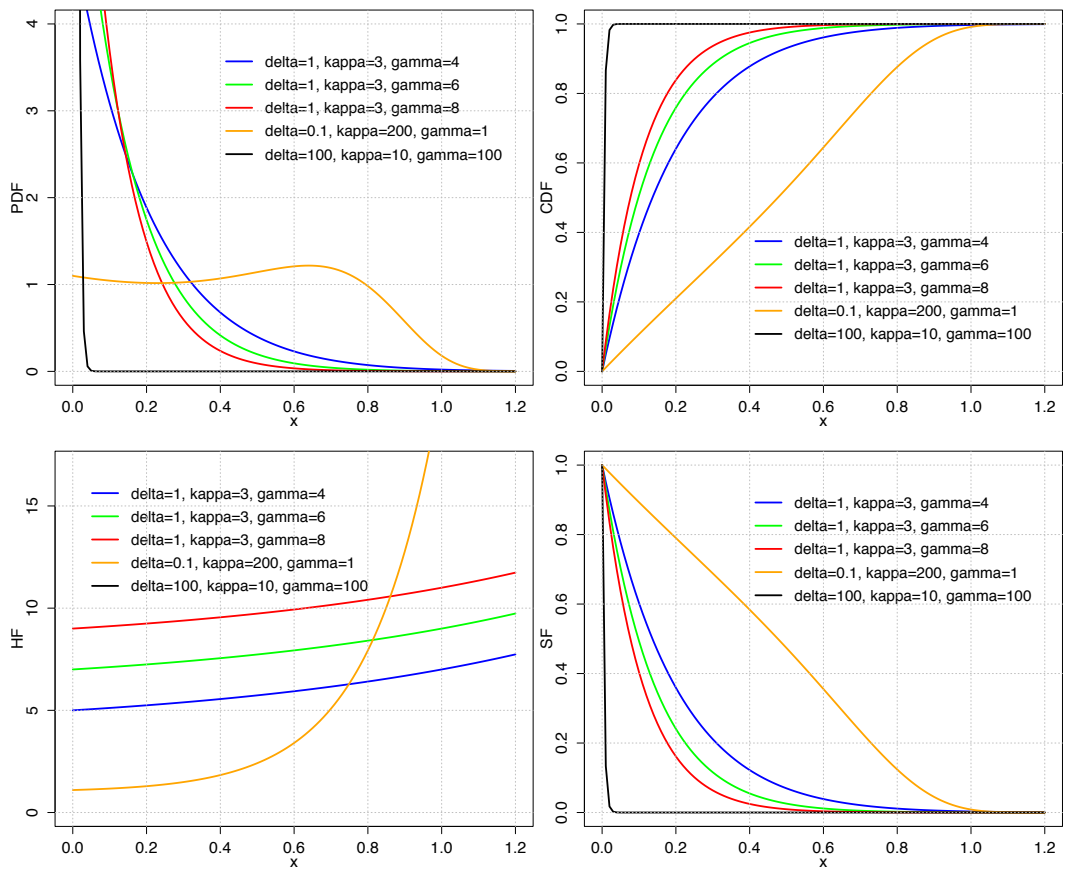

Figure A.76: Makeham1 distribution plotted using the provided R code.

Model

Parameter: delta

name	NA
type	scalar
symbol	δ
definition	$\delta > 0$

Parameter: kappa

name	NA
type	scalar
symbol	κ
definition	$\kappa > 0$

Parameter: gamma

name	NA
type	scalar
symbol	γ
definition	$\gamma > 0$

Functions

PDF

$$(\gamma + \delta\kappa^x)e^{-\gamma x - \delta(\kappa^x - 1)/\log(\kappa)}$$

PDF in R

```
(gamma+delta*kappa^x)*exp(-gamma*x -delta*(kappa^x - 1)/log(kappa) )
```

CDF

$$1 - e^{-(\gamma x \log(\kappa) + \delta \kappa^x - \delta)/\log(\kappa)}$$

CDF in R

```
1 - exp( -(gamma*x*log(kappa)+delta*kappa^x -delta)/log(kappa) )
```

HF

$$\gamma + \delta \kappa^x$$

HF in R

```
gamma + delta*kappa^x
```

SF

$$e^{-(\gamma x \log(\kappa) + \delta \kappa^x - \delta)/\log(\kappa)}$$

SF in R

```
exp( -(gamma*x*log(kappa)+delta*kappa^x -delta)/log(kappa) )
```

Characteristics

Mean

mathematically intractable

Median

mathematically intractable

Mode

mathematically intractable

Variance

mathematically intractable

Relationships

References and sources

http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Makeham.pdf

UncertML mapping:

no UncertML mapping found.

MaxwellBoltzmann1

name	Maxwell Boltzmann 1 (ID: 0001223)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

Model

Maxwell-Boltzmann distribution was first defined and used in physics for describing particle speeds in idealised gases where the particles move freely inside a stationary container without interacting with one another, except for very brief collisions in which they exchange energy and momentum with each other or with their thermal environment.

Parameter: scale

name	scale parameter
type	scalar
symbol	a
definition	$a > 0$

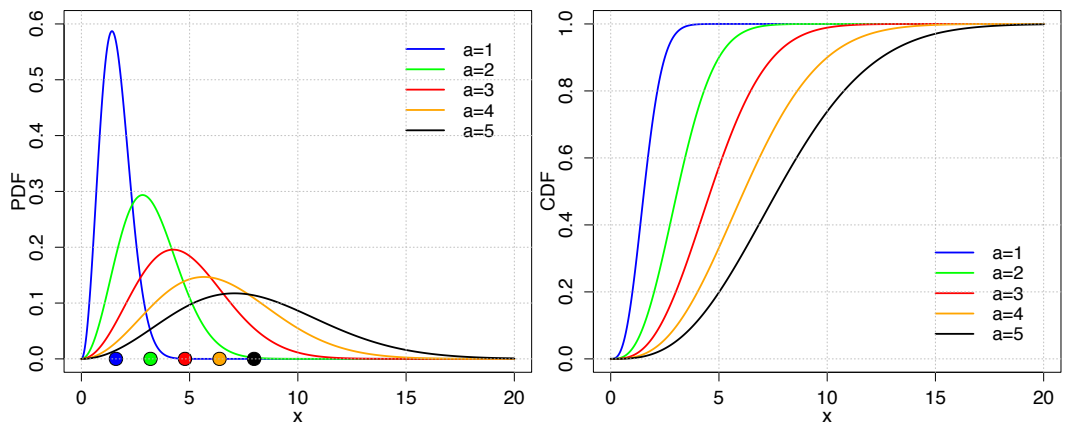

Figure A.77: MaxwellBoltzmann1 distribution plotted using the provided R code.

Functions

PDF

$$\sqrt{\frac{2}{\pi}}\frac{x^2 e^{-x^2/\left(2a^2\right)}}{a^3}$$

PDF in R

```
sqrt(2/pi) * (x^2 * exp(-x^2/(2*a^2)) )/a^3
```

CDF

$$\operatorname{erf}\left(\frac{x}{\sqrt{2}a}\right) - \sqrt{\frac{2}{\pi}}\frac{x e^{-x^2/\left(2a^2\right)}}{a}$$

CDF in R

```
erf(x/(sqrt(2)*a)) -sqrt(2/pi) * (x * exp(-x^2/(2*a^2)) )/a
```

Characteristics

Mean

$$2a\sqrt{\frac{2}{\pi}}$$

Median

$$\sqrt{2}a$$

Variance

$$\frac{a^2(3\pi - 8)}{\pi}$$

Relationships

- Relationship pair: $Normal1(\mu, \sigma) \rightarrow MaxwellBoltzmann1(\sigma)$
- Relationship type: Transformation
- Relationship definition: If U_1, U_2, U_3 are independent normal variables with mean0 and standard deviation $\sigma > 0$ then $X = \sqrt{U_1^2 + U_3^2 + U_3^2}$ has the Maxwell distribution with scale parameter σ
- Relationship reference(s): `http://www.math.uah.edu/stat/special/Maxwell.html`

References and sources

`http://mathworld.wolfram.com/MaxwellDistribution.html`
`https://en.wikipedia.org/wiki/Maxwell%E2%80%93Boltzmann_distribution`

UncertML mapping:

no UncertML mapping found.

MixtureDistribution1

name	Mixture Distribution 1 (ID: 0000630)
type	continuous
variate	-, -
support	-

Model

A mixture distribution has a distribution function with a representation as a convex combination of other specific probability distribution functions.

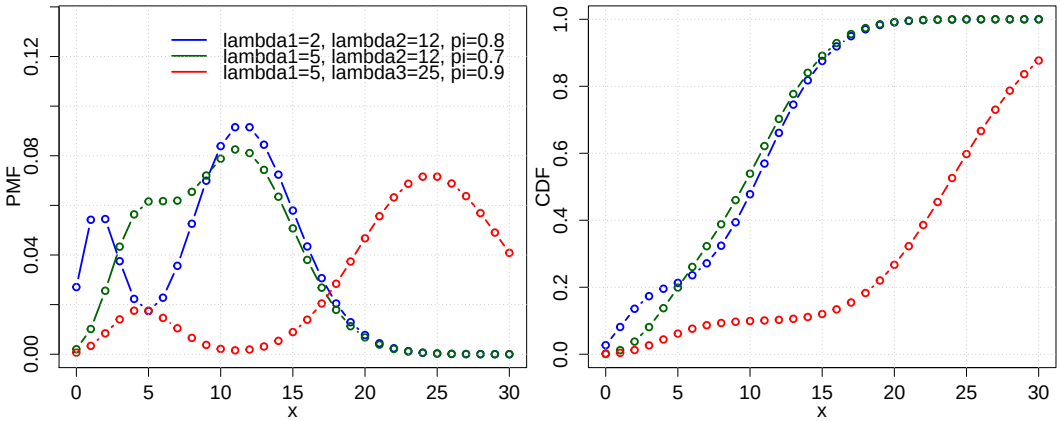

Figure A.78: Example 1: PMF and CDF of the Mixture Poisson distribution plotted using the formula for for various values as shown in the legend of the left plot. The PMF reads: $(1-\pi_1)\,\lambda_1^k/k!\,\exp(-\lambda_1) + \pi_1\,\lambda_2^k/k!\,\exp(-\lambda_2)$. The CDF reads: $(1-\pi_1)\,\Gamma(\lfloor k+1, \lambda_1\rfloor)/\lfloor k\rfloor! + \pi_1\,\Gamma(\lfloor k+1, \lambda_2\rfloor)/\lfloor k\rfloor!$.

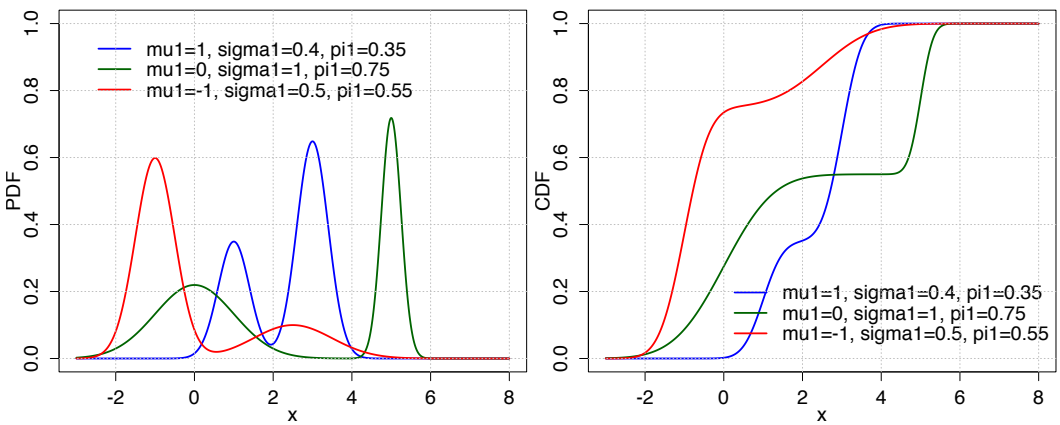

Figure A.79: Example 2: PDF and CDF of the Mixture Normal distribution plotted using the formula for for various values as shown in the legend of the left plot. The PDF reads: $(1-\pi_1) \times 1/(\sigma_1\sqrt{2\pi})\exp(-(x-\mu_1)^2/(2\sigma_1^2)) + \pi_1 \times 1/(\sigma_2\sqrt{2\pi})\exp(-(x-\mu_2)^2/(2\sigma_2^2))$. The CDF reads: $(1-\pi_1) \times 1/2(1 + erf((x-\mu_1)/(\sigma_1\sqrt{2}))) + \pi_1 \times 1/2(1 + erf((x-\mu_2)/(\sigma_2\sqrt{2})))$.

Parameter: weight

name	mixing coefficients
type	vector
symbol	$\pi_1, \ldots, \pi_k$
definition	$\Sigma_{i=1}^K \pi_i = 1; 0 \leq \pi_i \leq 1$

Functions

PDF

$$f(x;\pi,\theta) = \sum_{i=1}^{K} \pi_i \; p_i(x;\theta_i) \text{ where } p_i(x;\theta_i) \text{ the PDF of the } i^{th} \text{ component with parameters } \theta_i$$

Characteristics

Relationships

References and sources

`https://en.wikipedia.org/wiki/Mixture_distribution`
[Forbes et al., 2011]

UncertML mapping:

`http://www.uncertml.org/distributions/mixture-model`

Multinomial1

name	Multinomial 1 (ID: 0000654)
type	discrete
variate	X, vector
support	$X_i \in \{0,\ldots,n\}, \Sigma X_i = n$

Model

For n independent trials each of which leads to a success for exactly one of k categories, with each category having a given fixed success probability, the multinomial distribution gives the probability of any particular combination of numbers of successes for the various categories.

Parameter: numberOfTrials

name	number of trials
type	scalar
symbol	n
definition	$n > 0, n \in N$

Parameter: probabilityOfSuccess

name	event probabilities
type	vector
symbol	$p_1,\ldots,p_k$
definition	$p_1,\ldots,p_k, \Sigma p_i = 1$

Functions

PMF

$$\frac{n!}{x_1!\cdots x_k!} p_1^{x_1}\cdots p_k^{x_k}$$

Characteristics

Mean

$$E\{X_i\} = np_i$$

Variance

$$Var(X_i) = np_i(1-p_i); \quad Cov(X_i,X_j) = -np_ip_j \quad (i \neq j)$$

Relationships

References and sources

`http://en.wikipedia.org/wiki/Multinomial_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/multinomial`

MultivariateGaussianProcess1

name	Multivariate Gaussian Process Distribution 1 (ID: 0000676)
type	continuous
variate	x, matrix
support	$x \in R^{K \times N}$

Model

Parameter: kernelMatrix

name	symmetric, positive definite kernel matrix
type	matrix
symbol	Σ
definition	$\Sigma \in R^{N \times N}, N \in N$

Parameter: inverseScale

name	vector of positive inverse scales
type	vector
symbol	w
definition	$w \in R^{K}, K \in N$

Functions

PDF

$$\prod_{i=1}^{K} \text{MultivariateNormal1}(x_i|0, w_i^{-1}\Sigma)$$

Characteristics

Relationships

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

MultivariateGaussianProcess2

name	Multivariate Gaussian Process Distribution 2 (ID: 0000698)
type	continuous
variate	x, matrix
support	$x \in R^{K \times N}$

Model

Parameter: choleskyFactor

name	lower triangular matrix
type	matrix
symbol	L
definition	$L \in R^{N \times N}$, lower triangular and such that LL^T is positive kernel definite, with $N \in N$

Parameter: inverseScale

name	vector of positive inverse scales
type	vector
symbol	w
definition	$w \in R^K, K \in N$

Functions

PDF

$$\prod_{i=1}^{K} \text{MultivariateNormal1}(x_i|0, w_i^{-1} LL^T)$$

Characteristics

Relationships

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

MultivariateNormal1

name	Multivariate Normal 1 (ID: 0000719)
type	continuous
variate	x, vector
support	$-\infty < x_i < \infty, i = 1, ..., k$

Model

A multivariate normal distribution is a multivariate extension of the normal distribution.

Parameter: mean

name	location
type	vector
symbol	μ
definition	$\mu \in R^k$

Parameter: covarianceMatrix

name	covariance matrix
type	matrix
symbol	Σ
definition	$\Sigma \in R^{k \times k}$

Functions

PDF

$$(2\pi)^{-\frac{k}{2}}|\Sigma|^{-\frac{1}{2}}\, e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

CDF

no analytic expression

Characteristics

Mean

$$\mu$$

Mode

$$\mu$$

Variance

$$\Sigma$$

Relationships

References and sources

`http://en.wikipedia.org/wiki/Multivariate_normal_distribution`
[Forbes et al., 2011]

UncertML mapping:

`http://www.uncertml.org/distributions/multivariate-normal`

MultivariateNormal2

name	Multivariate Normal 2 (ID: 0000742)
type	continuous
variate	x, vector
support	$x \in R^k$

Model

A multivariate normal distribution is a multivariate extension of the normal distribution.

Parameter: mean

name	location
type	vector
symbol	μ
definition	$\mu \in R^k$

Parameter: precisionMatrix

name	precision matrix
type	matrix
symbol	T
definition	inverse of the covariance matrix

Functions

PDF

$$(2\pi)^{-d/2}|T|^{\frac{1}{2}}\ \exp\big(-\frac{1}{2}(x-\mu)'T(x-\mu)\big)$$

CDF

no analytic expression

Characteristics

Mean

$$\mu$$

Mode

$$\mu$$

Variance

$$T^{-1}$$

Relationships

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

MultivariateNormal3

name	Multivariate Normal 3 (ID: 0000765)
type	continuous
variate	x, vector
support	$x \in R^K$

Model

A multivariate normal distribution is a multivariate extension of the normal distribution.

Parameter: mean

name	location
type	vector
symbol	μ
definition	$\mu \in R^K$

Parameter: choleskyFactor

name	lower triangular matrix
type	matrix
symbol	L
definition	$L \in R^{K\times K}$, lower triangular and such that LL^T is positive definite, with $K \in N$

Functions

PDF

$$(2\pi)^{-\frac{k}{2}}(LL^T)^{-\frac{1}{2}}\ e^{-\frac{1}{2}(x-\mu)^T(LL^T)^{-1}(x-\mu)}$$

Characteristics

Mean

$$\mu$$

Mode

$$\mu$$

Variance

$$LL^T$$

Relationships

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

MultivariateStudentT1

name	Multivariate (Student) T 1 (ID: 0000790)
type	continuous
variate	x, vector
support	$x \in R^p$

Model

The multivariate Student's t distribution is a generalisation of the univariate Student's t to two or more variables. It is a distribution for random vectors of correlated variables, each element of which has a univariate Student's t distribution.

Parameter: mean

name	location
type	vector
symbol	μ
definition	$\mu = [\mu_1, \ldots, \mu_p]^T, \mu_i \in R$

Parameter: covarianceMatrix

name	covariance matrix
type	matrix
symbol	Σ
definition	Σ, positive-definite real $p \times p$ matrix

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	ν
definition	$\nu \geq 2$

Functions

PDF

$$\frac{\Gamma\left[(\nu+p)/2\right]}{\Gamma(\nu/2)\nu^{p/2}\pi^{p/2}\left|\Sigma\right|^{1/2}\left[1+\frac{1}{\nu}(x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu)\right]^{(\nu+p)/2}}$$

CDF

no analytic expression

Characteristics

Mean

$$\begin{cases} \mu & \text{for } \nu > 1 \\ undefined & \text{else} \end{cases}$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$\begin{cases} \frac{\nu}{\nu-2}\Sigma & \text{for } \nu > 2 \\ undefined & \text{else} \end{cases}$$

Relationships

References and sources

`http://en.wikipedia.org/wiki/Multivariate_t-distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/multivariate-student-t`

MultivariateStudentT2

name	Multivariate (Student) T 2 (ID: 0000016)
type	continuous
variate	x, vector
support	$x \in R^d, k \geq 2$

Model

The multivariate Student's t distribution is a generalisation of the univariate Student's t to two or more variables. It is a distribution for random vectors of correlated variables, each element of which has a univariate Student's t distribution.

Parameter: mean

name	location
type	vector
symbol	μ
definition	$\mu = [\mu_1, \ldots, \mu_d]^T, \mu_i \in R$

Parameter: precisionMatrix

name	precision matrix
type	matrix
symbol	T
definition	Inverse of the covariance matrix

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	k
definition	$k \geq 2$

Functions

PDF

$$\frac{\Gamma((k+d)/2)}{\Gamma(k/2)k^{d/2}\pi^{d/2}}|T|^{1/2}\Big[1+\frac{1}{k}(x-\mu)'T(x-\mu)\Big]^{-(k+d)/2}$$

Characteristics

Relationships

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

Muth1

name	Muth 1 (ID: 0001344)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

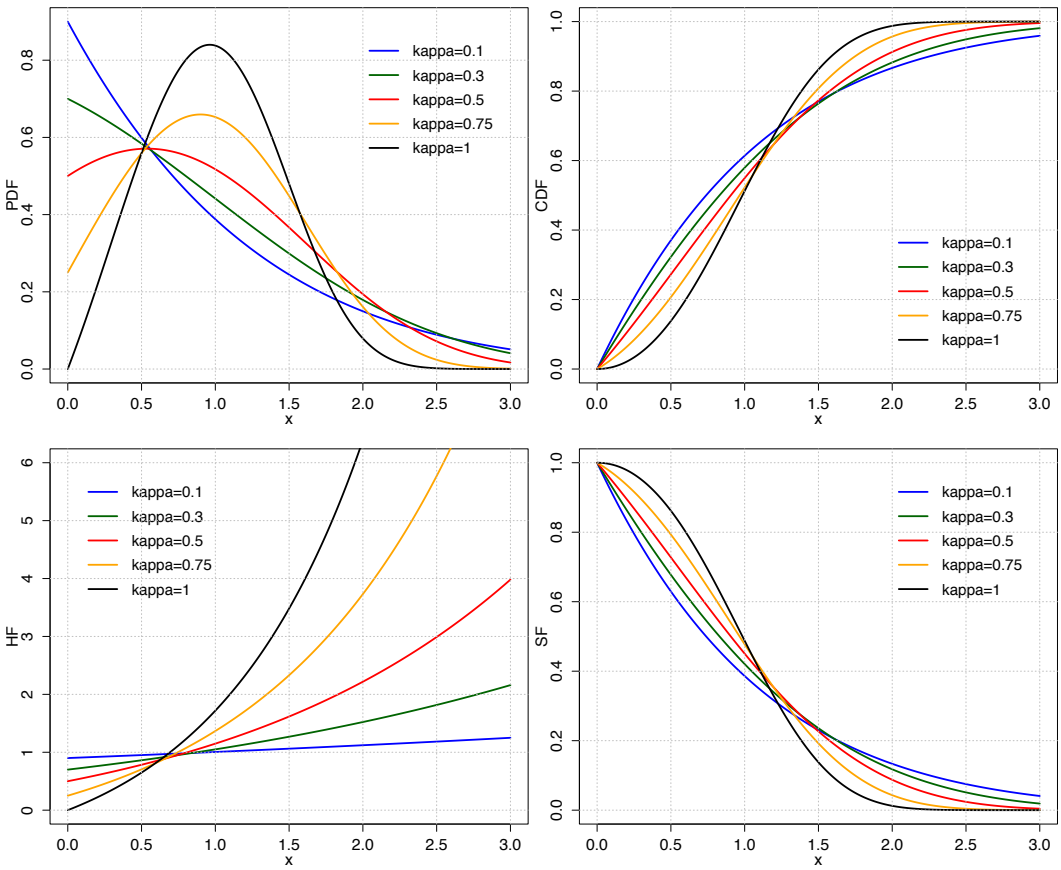

Figure A.80: Muth1 distribution plotted using the provided R code.

Model

The Muth distribution is a continuous random variable introduced in the context of reliability theory.

Parameter: shape

name	shape parameter
type	scalar
symbol	κ
definition	$0 \leq \kappa \leq 1$

Functions

PDF

$$(e^{\kappa x} - \kappa)e^{\left[-\frac{e^{\kappa x}}{\kappa}+\kappa x+\frac{1}{\kappa}\right]}$$

PDF in R

```
(exp(kappa*x)-kappa)*exp( -exp(kappa*x)/x + kappa*x + 1/kappa )
```

CDF

$$1 - e^{\left[-\frac{e^{\kappa x}}{\kappa}+\kappa x+\frac{1}{\kappa}\right]}$$

CDF in R

```
1-exp( -exp(kappa*x)/x + kappa*x + 1/kappa )
```

HF

$$e^{\kappa x} - \kappa$$

HF in R

```
exp(kappa*x) - kappa
```

SF

$$e^{\left[-\frac{e^{\kappa x}}{\kappa}+\kappa x+\frac{1}{\kappa}\right]}$$

SF in R

```
exp( -exp(kappa*x)/x + kappa*x + 1/kappa )
```

Characteristics

Mean

$$1$$

Relationships

- Relationship pair: $Muth1(\kappa) \rightarrow Exponential2(\beta)$
- Relationship type: Limiting
- Relationship definition: As $\kappa \rightarrow 0$ for a Muth1(κ) , the limiting distribution is Exponential2 with mean 1.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/MuthExponential.pdf`

References and sources

`http://www.itl.nist.gov/div898/software/dataplot/refman2/auxillar/mutpdf.htm`
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Muth.pdf`

UncertML mapping:

no UncertML mapping found.

Nakagami1

name	Nakagami 1 (ID: 0000041)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

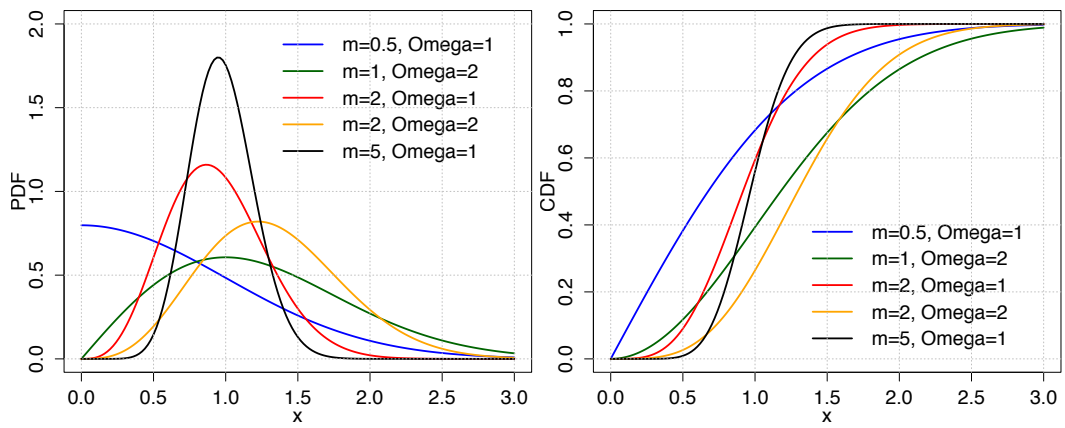

Figure A.81: Nakagami1 distribution plotted using the provided R code.

Model

The Nakagami distribution is commonly used in communication theory to model scattered signals that reach a receiver using multiple paths.

Parameter: shape

name	shape
type	scalar
symbol	m
definition	$m > 0$

Parameter: spread

name	spread
type	scalar
symbol	Ω
definition	$\Omega > 0$

Functions

PDF

$$\frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} \exp(-\frac{m}{\omega} x^2)$$

PDF in R

```
2*m^m / (gamma(m)*Omega^m)*x^(2*m-1)*exp(-m/Omega*x^2)
```

CDF

$$\frac{\gamma(m, \frac{m}{\Omega} x^2)}{\Gamma(m)}$$

CDF in R

```
Igamma(m,m/Omega*x^2,lower=T)/gamma(m)
```

Characteristics

Mean

$$\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{\frac{1}{2}}$$

Median

$$\sqrt{\Omega}$$

Mode

$$\frac{\sqrt{2}}{2} \left(\frac{(2m-1)\Omega}{m}\right)^{1/2}$$

Variance

$$\Omega\Big(1 - \frac{1}{m}\Big(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m)}\Big)^2\Big)$$

Relationships

References and sources

`https://en.wikipedia.org/wiki/Nakagami_distribution`

UncertML mapping:

no UncertML mapping found.

NegativeBinomial1

name	Negative Binomial 1 (ID: 0000074)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$ – number of failures

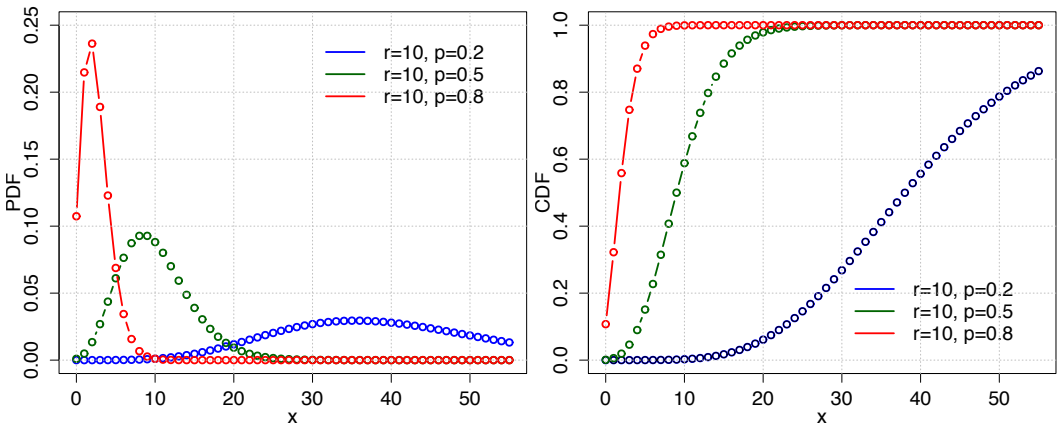

Figure A.82: NegativeBinomial1 distribution plotted using the provided R code.

Model

NegativeBinomial1 distribution is a discrete probability distribution of the number of failures before the rth success in repeated mutually independent Bernoulli trials, each with probability of success p. Compare NegativeBinomial4.

Parameter: numberOfSuccesses

name	number of successes
type	scalar
symbol	r
definition	$r > 0, r \in N$

Parameter: probability

name	success probability
type	scalar
symbol	p
definition	$p \in (0, 1)$

Functions

PMF

$$\binom{k+r-1}{k} p^r (1-p)^k$$

PMF in R

```
choose(k+r-1,k) * p^r * (1-p)^k
```

CDF

$$1 - I_{1-p}(k+1, r)$$

CDF in R

```
1 - Rbeta(1-p, k+1, r, lower = T)
```

Characteristics

Mean

$$\frac{r(1-p)}{p}$$

Mode

$$\lfloor \frac{(1-p)(r-1)}{p} \rfloor$$

Variance

$$\frac{r(1-p)}{p^2}$$

Relationships

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial5(\alpha, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = r, \beta = p/(1-p)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = 1 - p$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow Geometric1(p)$
- Relationship type: Special case
- Relationship definition: $n = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalGeometric.pdf

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $\mu = n(1-p), n \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalNormal.pdf

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $\mu = np, n \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalPoisson.pdf

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial3(\mu, \phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\phi = r, \mu = r(1-p)/p$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Reparameterisation

- Relationship definition: $\tau = 1/r, \lambda = r(1-p)/p$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial5(\alpha, \beta) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = \alpha, p = \beta/(1+\beta)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = 1 - p$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Geometric1(p) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Transformation
- Relationship definition: $\Sigma X(iid)$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GeometricPascal.pdf`
- Relationship pair: $NegativeBinomial3(\mu, \phi) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = \phi, p = r/(\mu + r)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial2(\lambda, \tau) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = 1/\tau, p = 1/(1 + \tau\lambda)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $BetaNegativeBinomial1(\alpha, \beta, n) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ NegativeBinomial1(n,p) and $p \sim$ Beta1(α, β) then $X \sim$ NegativeBinomial1(α, β, n)
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalBetapascal.pdf`
- Relationship pair: $ConwayMaxwellPoisson1(\lambda, \nu) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Transformation
- Relationship definition: For $\nu = 0$ and $\lambda < 1$ the sum of Conway-Maxwell-Poisson distributed variables reduces to the sum of geometric variables, which follows a Negative Binomial distribution with parameters n and $1 - \lambda$
- Relationship reference(s): [Shmueli et al., 2005]

References and sources

[Leemis and Mcqueston, 2008]

UncertML mapping:

no UncertML mapping found.

NegativeBinomial2

name	Negative Binomial 2 (ID: 0000105)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

Parameter: rate

name	Poisson intensity
type	scalar
symbol	λ
definition	$\lambda \in R, \lambda > 0$

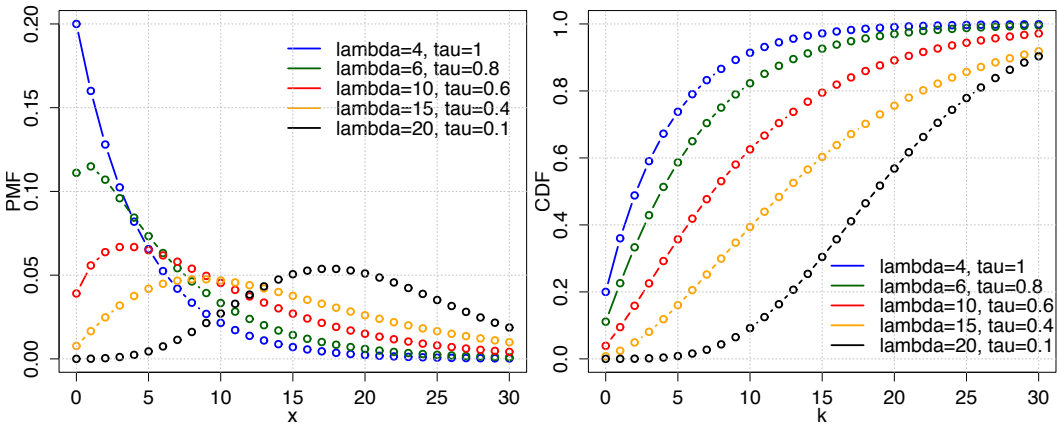

Figure A.83: NegativeBinomial2 distribution plotted using the provided R code.

Parameter: overdispersion

name	overdispersion
type	scalar
symbol	τ
definition	$\tau \in R$

Functions

PMF

$$\frac{\Gamma(k+\frac{1}{\tau})}{k!\,\Gamma(\frac{1}{\tau})}\left(\frac{1}{1+\tau\lambda}\right)^{\frac{1}{\tau}}\left(\frac{\lambda}{\frac{1}{\tau}+\lambda}\right)^{k}$$

PMF in R

```
gamma(k + 1/tau)/(factorial(k) * gamma(1/tau)) * 1/(1+tau*lambda)^(1/tau)
                               * (lambda/(1/tau + lambda))^k
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0,1,2,...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\lambda$$

Variance

$$\lambda(1+\tau\lambda)$$

Relationships

- Relationship pair: $NegativeBinomial2(\lambda,\tau) \rightarrow NegativeBinomial5(\alpha,\beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = 1/\tau, \beta = 1/(\tau\lambda)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial2(\lambda,\tau) \rightarrow NegativeBinomial4(r,p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = 1/\tau, p = \frac{\tau\lambda}{1+\tau\lambda}$
- Relationship reference(s): ProbOnto spec
[Cameron and Trivedi, 2013]
- Relationship pair: $NegativeBinomial2(\lambda,\tau) \rightarrow NegativeBinomial3(\mu,\phi)$

- Relationship type: Reparameterisation
- Relationship definition: $\mu = \lambda, \phi = 1/\tau$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial2(\lambda, \tau) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = 1/\tau, p = 1/(1 + \tau\lambda)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial5(\alpha, \beta) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda = \alpha/\beta, \tau = 1/\alpha$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $ZeroInflatedNegativeBinomial1(\lambda, \tau, p0) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Special case
- Relationship definition: $p0 = 0$
- Relationship reference(s): [Trocóniz et al., 2009]

- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/r, \lambda = \frac{rp}{1-p}$
- Relationship reference(s): ProbOnto spec
[Cameron and Trivedi, 2013]

- Relationship pair: $NegativeBinomial3(\mu, \phi) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda = \mu, \tau = 1/\phi$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/r, \lambda = r(1 - p)/p$
- Relationship reference(s): ProbOnto spec

References and sources

[Hilbe, 2011]
[Cameron and Trivedi, 2013]

UncertML mapping:

no UncertML mapping found.

NegativeBinomial3

name	Negative Binomial 3 (ID: 0000135)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu \in R, \mu > 0$

Parameter: dispersion

name	index parameter
type	scalar
symbol	ϕ
definition	$\phi \in R, \phi > 0$

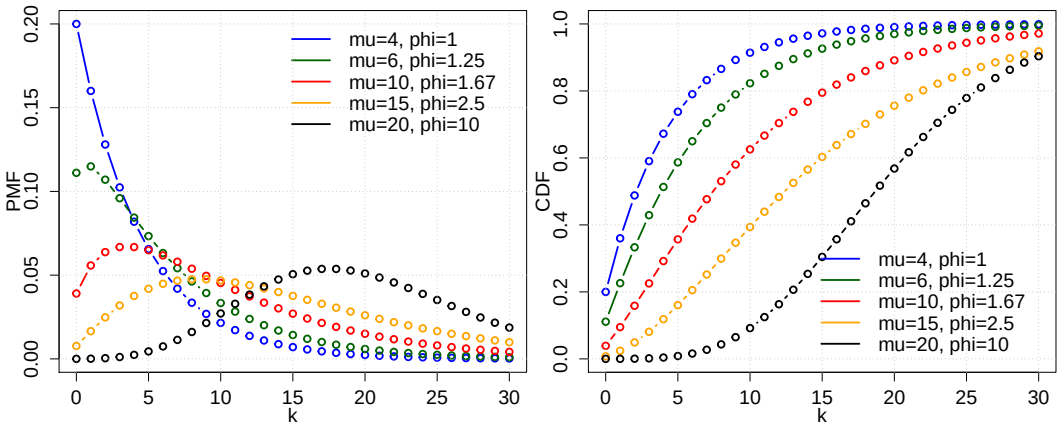

Figure A.84: NegativeBinomial3 distribution plotted using the provided R code.

Functions

PMF

$$\binom{k+\phi-1}{k}\left(\frac{\phi}{\mu+\phi}\right)^{\phi}\left(\frac{\mu}{\mu+\phi}\right)^{k}$$

PMF in R

```
choose(k+phi-1,k) * (phi / (mu + phi))^phi * (mu / (mu + phi))^k
```

Characteristics

Mean

$$\mu$$

Variance

$$\mu+\mu^2/\phi$$

Relationships

- Relationship pair: $NegativeBinomial3(\mu,\phi) \rightarrow NegativeBinomial5(\alpha,\beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha=\phi, \beta=\phi/\mu$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial3(\mu,\phi) \rightarrow NegativeBinomial4(r,p)$
- Relationship type: Reparameterisation
- Relationship definition: $r=\phi, p=\mu/(\phi+\mu)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial3(\mu,\phi) \rightarrow NegativeBinomial1(r,p)$
- Relationship type: Reparameterisation
- Relationship definition: $r=\phi, p=r/(\mu+r)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial3(\mu,\phi) \rightarrow NegativeBinomial2(\lambda,\tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda=\mu, \tau=1/\phi$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial3(\mu,\phi) \rightarrow NegativeBinomial6(\eta,\phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\eta=\log(\mu)$
- Relationship reference(s): [STAN Development Team, 2015b]

- Relationship pair: $NegativeBinomial5(\alpha,\beta) \rightarrow NegativeBinomial3(\mu,\phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\alpha/\beta, \phi=\alpha$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial3(\mu, \phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\phi = r, \mu = rp/(1-p)$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial3(\mu, \phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\phi = r, \mu = r(1-p)/p$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial2(\lambda, \tau) \rightarrow NegativeBinomial3(\mu, \phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \lambda, \phi = 1/\tau$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $NegativeBinomial6(\eta, \phi) \rightarrow NegativeBinomial3(\mu, \phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \exp(\eta)$
- Relationship reference(s): [STAN Development Team, 2015b]

References and sources

`cran.r-project.org/web/packages/VGAM/VGAM.pdf`
[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

NegativeBinomial4

name	Negative Binomial 4 (ID: 0000161)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$ – number of successes

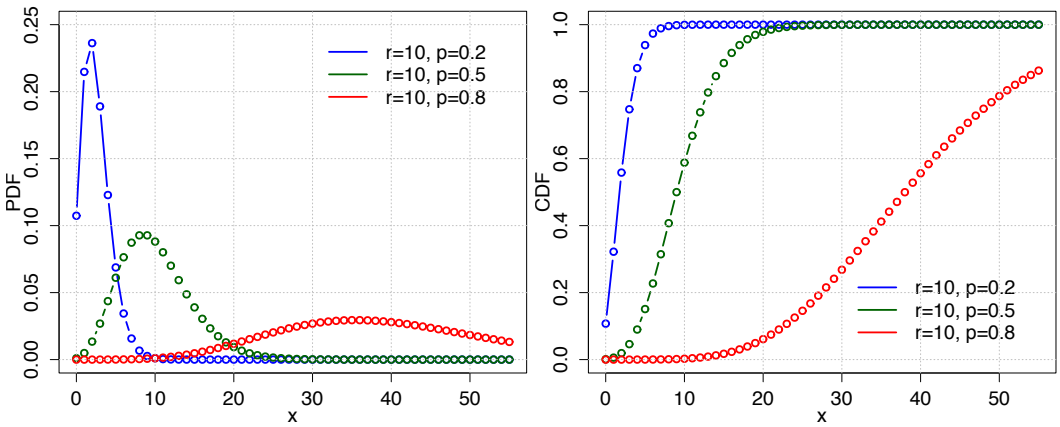

Figure A.85: NegativeBinomial4 distribution plotted using the provided R code.

Model

NegativeBinomial4 distribution is a discrete probability distribution of the number of successes in a sequence of independent and identically distributed Bernoulli trials before a specified (non-random) number of failures, r, occurs. Compare NegativeBinomial1.

Parameter: numberOfFailures

name	number of failures
type	scalar
symbol	r
definition	$r > 0, r \in N$

Parameter: probability

name	success probability
type	scalar
symbol	p
definition	$p \in (0, 1)$

Functions

PMF

$$\binom{k+r-1}{k}(1-p)^r p^k$$

PMF in R

```
choose(k+r-1,k) * (1-p)^r * p^k
```

CDF

$$1 - I_p(k+1, r)$$

CDF in R

```
1 - Rbeta(p, k+1, r,lower = T)
```

Characteristics

Mean

$$\frac{pr}{1-p}$$

Mode

$$\begin{cases} \lfloor \frac{p(r-1)}{1-p} \rfloor & \text{for } r > 1 \\ 0 & \text{for } r \leq 1 \end{cases}$$

Variance

$$\frac{pr}{(1-p)^2}$$

Relationships

- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/r, \lambda = \frac{rp}{1-p}$
- Relationship reference(s): ProbOnto spec
[Cameron and Trivedi, 2013]

- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = 1 - p$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial5(\alpha, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = r, \beta = (1-p)/p$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial3(\mu, \phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\phi = r, \mu = rp/(1-p)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial2(\lambda, \tau) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = 1/\tau, p = \frac{\tau\lambda}{1+\tau\lambda}$
- Relationship reference(s): ProbOnto spec [Cameron and Trivedi, 2013]

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $p = 1 - p$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $GeneralizedNegativeBinomial1(\theta, \beta, m) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $\beta = 1$ and set $m = r, \theta = p$
- Relationship reference(s): [Consul and Famoye, 2006]

- Relationship pair: $NegativeBinomial5(\alpha, \beta) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = \alpha, p = 1/(1 + \beta)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial3(\mu, \phi) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = \phi, p = \mu/(\phi + \mu)$
- Relationship reference(s): ProbOnto spec

References and sources

`https://en.wikipedia.org/wiki/Binomial_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/negative-binomial`

NegativeBinomial5

name	Negative Binomial 5 (ID: 0000190)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

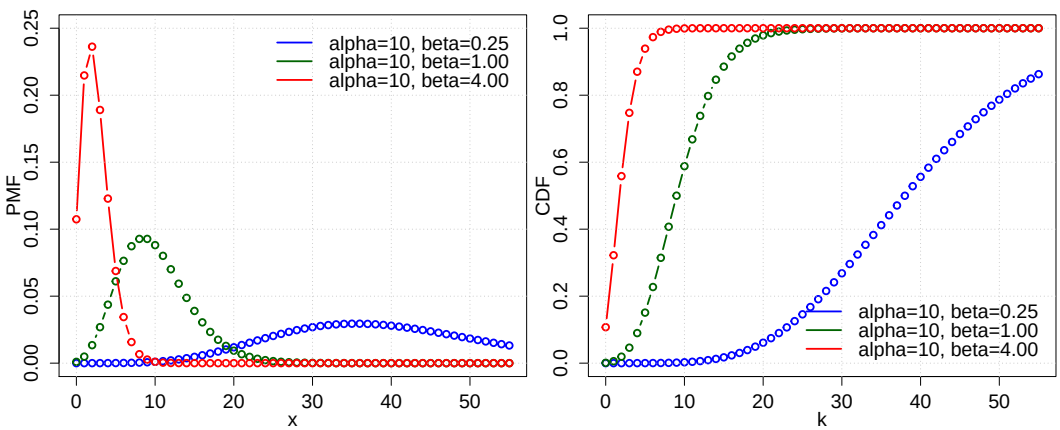

Figure A.86: NegativeBinomial5 distribution plotted using the provided R code.

Model

NegativeBinomial5 is the marginal distribution for a Poisson random variable when the rate parameter has a Gamma(α, β) prior distribution.

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha \in R^+$

Parameter: inverseScale

name	inverse scale
type	scalar
symbol	β
definition	$\beta \in R^+$

Functions

PMF

$$\binom{k+\alpha-1}{\alpha-1}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^{k}$$

PMF in R

```
choose(k+alpha-1,alpha-1) * (beta / (beta + 1))^alpha * (1 / (beta + 1))^k
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
cumsum(PMF)
```

Characteristics

Mean

$$\alpha/\beta$$

Variance

$$\alpha/\beta^2(\beta+1)$$

Relationships

- Relationship pair: $NegativeBinomial5(\alpha, \beta) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\lambda = \alpha/\beta, \tau = 1/\alpha$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial5(\alpha, \beta) \rightarrow NegativeBinomial3(\mu, \phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \alpha/\beta, \phi = \alpha$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial5(\alpha, \beta) \rightarrow NegativeBinomial1(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = \alpha, p = \beta/(1+\beta)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial5(\alpha, \beta) \rightarrow NegativeBinomial4(r, p)$
- Relationship type: Reparameterisation
- Relationship definition: $r = \alpha, p = 1/(1+\beta)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial2(\lambda, \tau) \rightarrow NegativeBinomial5(\alpha, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = 1/\tau, \beta = 1/(\tau\lambda)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial3(\mu, \phi) \rightarrow NegativeBinomial5(\alpha, \beta)$

- Relationship type: Reparameterisation
- Relationship definition: $\alpha = \phi, \beta = \phi/\mu$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow NegativeBinomial5(\alpha, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = r, \beta = p/(1-p)$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $NegativeBinomial4(r, p) \rightarrow NegativeBinomial5(\alpha, \beta)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = r, \beta = (1-p)/p$
- Relationship reference(s): ProbOnto spec

References and sources

[Gelman et al., 2014]

UncertML mapping:

no UncertML mapping found.

NegativeBinomial6

name	Negative Binomial 6 (ID: 0000217)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

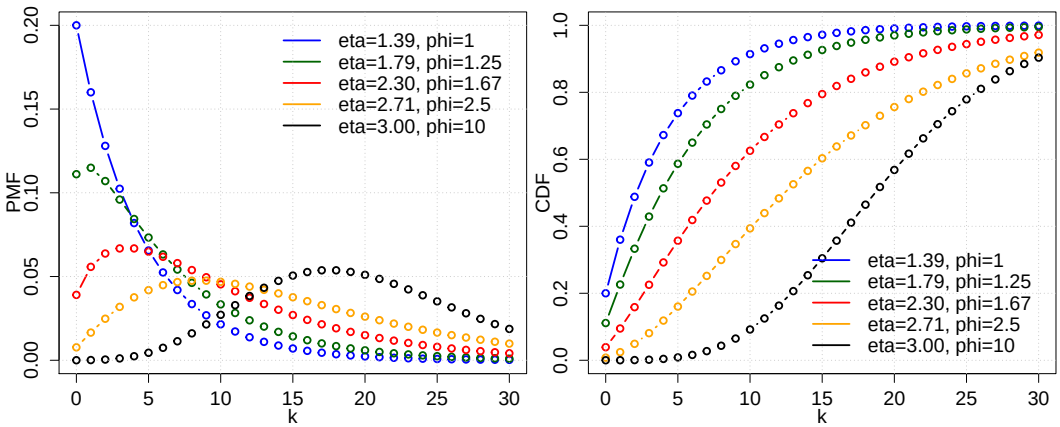

Figure A.87: NegativeBinomial6 distribution plotted using the provided R code.

Model

Parameter: logMean

name	log mean
type	scalar
symbol	η
definition	$\eta \in R$

Parameter: dispersion

name	index parameter
type	scalar
symbol	ϕ
definition	$\phi \in R, \phi > 0$

Functions

PMF

$$\binom{k+\phi-1}{k}\left(\frac{\exp(\eta)}{\exp(\eta)+\phi}\right)^k\left(\frac{\phi}{\phi+\exp(\eta)}\right)^\phi$$

PMF in R

```
choose(k+phi-1,k) * (exp(eta) / (exp(eta) + phi))^k * (phi / (phi + exp(eta)))^phi
```

Characteristics

Mean

$$\exp(\eta)$$

Variance

$$\exp(\eta)+\exp(\eta)^2/\phi$$

Relationships

- Relationship pair: $NegativeBinomial6(\eta,\phi) \rightarrow NegativeBinomial3(\mu,\phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \exp(\eta)$
- Relationship reference(s): [STAN Development Team, 2015b]
- Relationship pair: $NegativeBinomial3(\mu,\phi) \rightarrow NegativeBinomial6(\eta,\phi)$
- Relationship type: Reparameterisation
- Relationship definition: $\eta = \log(\mu)$
- Relationship reference(s): [STAN Development Team, 2015b]

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

NoncentralBeta1

name	Noncentral Beta 1 (ID: 0001114)
type	continuous
variate	x, scalar
support	$x \in [0,1]$

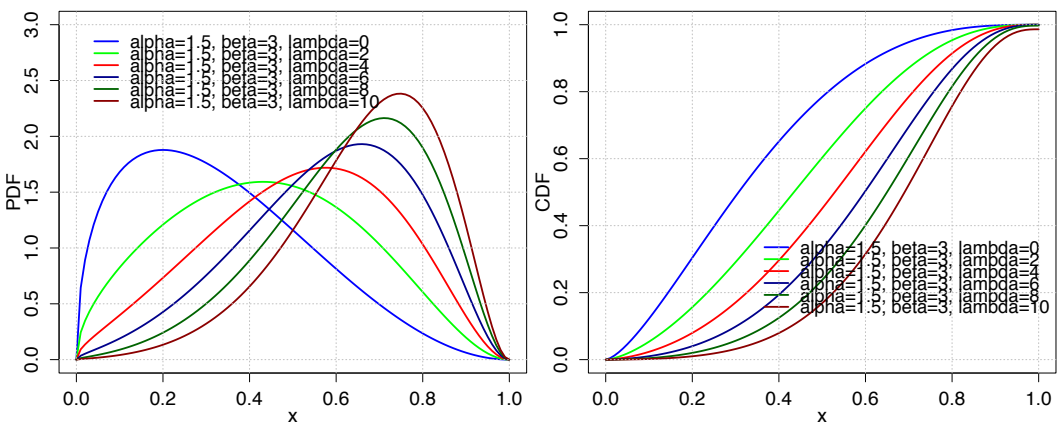

Figure A.88: NoncentralBeta1 distribution plotted using the provided R code.

Model

NoncentralBeta1 is a generalisation of the (central) Beta1 distribution.

Parameter: noncentrality

name	noncentrality parameter
type	scalar
symbol	λ
definition	$\lambda \geq 0$

Parameter: shape1

name	shape parameter
type	scalar
symbol	α
definition	$\alpha > 0$

Parameter: shape2

name	shape parameter
type	scalar
symbol	β
definition	$\beta > 0$

Functions

PDF

$$\sum_{j=0}^{\infty} e^{-\lambda/2} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \frac{x^{\alpha+j-1}\left(1-x\right)^{\beta-1}}{\mathrm{B}\left(\alpha+j,\beta\right)}$$

PDF in R

```
PDFsum = function(x,alpha,beta,lambda,fromValue,toValue) {
  PDF=array(0,length(x));
  for(i in 1:length(x)) {
    for(j in fromValue:toValue) {
      PDF[i] = PDF[i] + exp(-lambda/2) * (lambda/2)^j / factorial(j)
                                * (x[i]^(alpha+j-1)*(1-x[i])^(beta-1)) / beta(alpha+j,beta)
    }
  }
  return(PDF)
}
```

CDF

$$\sum_{j=0}^{\infty} e^{-\lambda/2} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} I_x\left(\alpha+j,\beta\right)$$

CDF in R

```
CDFsum = function(x,alpha,beta,lambda,fromValue,toValue) {
  CDF=array(0,length(x));
  for(j in 1:length(x)) {
    for(k in fromValue:toValue) {
      CDF[j] = CDF[j] + exp(-lambda/2) * (lambda/2)^k / factorial(k) * Rbeta(x[j],alpha+k,beta,lower=T)
    }
  }
  return(CDF)
}
```

Characteristics

Relationships

- Relationship pair: $NoncentralBeta1(\lambda, \alpha, \beta) \rightarrow Beta1(\alpha, \beta)$
- Relationship type: Special case
- Relationship definition: $\lambda = 0$
- Relationship reference(s): [Walck, 2007]

References and sources

[Walck, 2007]

UncertML mapping:

no UncertML mapping found.

NoncentralChiSquared1

name	Noncentral chi-squared 1 (ID: 0001127)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

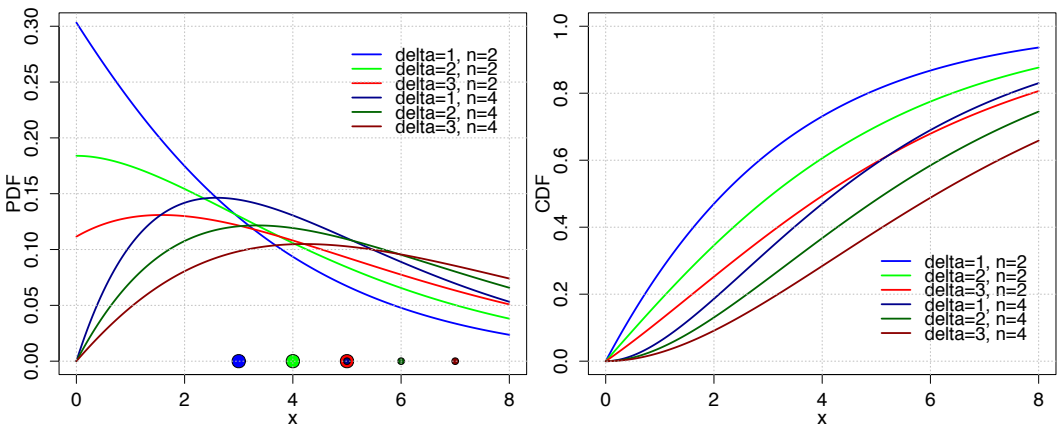

Figure A.89: NoncentralChiSquared1 distribution plotted using the provided R code.

Model

The NoncentralChiSquared1 distribution is a generalisation of the ChiSquared1 distribution.

Parameter: noncentrality

name	noncentrality parameter
type	scalar
symbol	δ
definition	$\delta > 0$

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	n
definition	$n \in N$

Functions

PDF

$$\sum_{k=0}^{\infty} \frac{e^{-\delta/2}(\delta/2)^k}{k!} \frac{e^{-x/2}x^{(n+2k)/2-1}}{2^{(n+2k)/2}\Gamma[(n+2k)/2]}$$

PDF in R

```
sumFromTo = function(x,delta,n,fromValue,toValue) {
  PDF=array(0,length(x));
  for(j in 1:length(x)) {
    for(i in fromValue:toValue) {
      PDF[j] = PDF[j] + exp(-delta/2)*(delta/2)^i/factorial(i)
                              * (exp(-x[j]/2)*x[j]^((n+2*i)/2-1)) / (2^((n+2*i)/2)*gamma((n+2*i)/2))
    }
  }
  return(PDF)
}
```

CDF

$$e^{-\delta/2}\sum_{k=0}^{\infty} \frac{(\delta/2)^k}{k!} Q(x; n+2k) \text{ with } Q(x;p) = \frac{\gamma(p/2, x/2)}{\Gamma(p/2)}$$

CDF in R

```
CDF = function(x,delta,n,lowerLimit,upperLimit) { exp(-delta/2) * CDFsum(x,n,delta,0,Ninfty) }
CDFsum = function(x,n,delta,fromValue,toValue) {
  CDF=array(0,length(x));
  for(j in 1:length(x)) {
    for(k in fromValue:toValue) {
      CDF[j] = CDF[j] + (delta/2)^k/factorial(k) * Igamma((n+2*k)/2,x[j]/2,lower=T)/gamma((n+2*k)/2)
    }
  }
  return(CDF)
}
```

Characteristics

Mean

$$\delta + n$$

Variance

$$2(n + 2\delta)$$

Relationships

- Relationship pair: $NoncentralF1(\lambda, m, n) \rightarrow NoncentralChiSquared1(m, \lambda)$
- Relationship type: Transformation
- Relationship definition: $F \sim NoncentralF1(\lambda, m, n)$, as $n \rightarrow \infty$ then
nF approaches a non-central chi-square distribution with m degrees of freedom and non-central parameter λ
- Relationship reference(s): [Walck, 2007]

References and sources

http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Noncentralchisquare.pdf

UncertML mapping:

no UncertML mapping found.

NoncentralF1

name	Noncentral F 1 (ID: 0001143)
type	continuous
variate	$F' = \frac{x_1/m}{x_2/n}$ with $x_1 \sim ChiSquared1(m), x_2 \sim ChiSquared1(n)$, scalar
support	$F' \in (0, +\infty)$

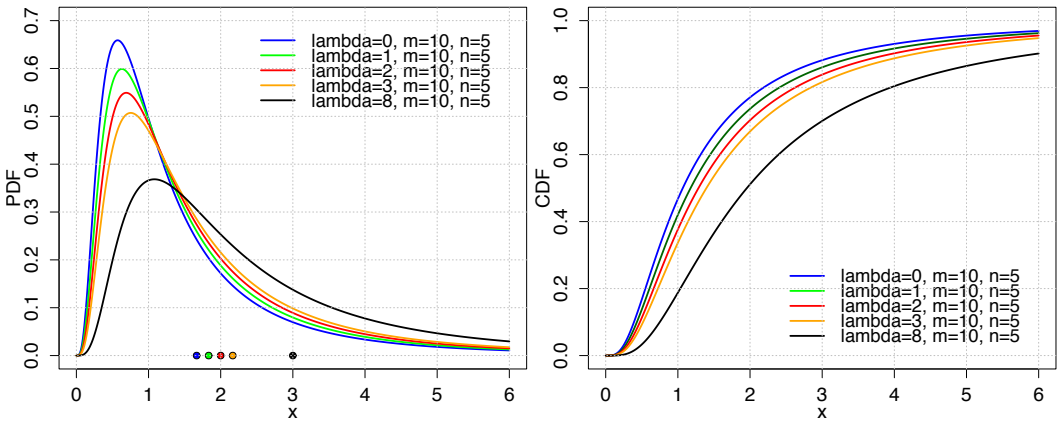

Figure A.90: NoncentralF1 distribution plotted using the provided R code.

Model

The NoncentralF1 is a generalisation of the F1 distribution. It describes the distribution of the quotient $\frac{x_1/m}{x_2/n}$, where the numerator has a non-central chi-squared distribution with m degrees of freedom and the denominator has a central chi-squared distribution n degrees of freedom.

Parameter: noncentrality

name	noncentrality parameter
type	scalar
symbol	λ
definition	$\lambda \geq 0$

Parameter: degreesOfFreedom1

name	degrees of freedom
type	scalar
symbol	m
definition	$m \in N$

Parameter: degreesOfFreedom2

name	degrees of freedom
type	scalar
symbol	n
definition	$n \in N$

Functions

PDF

$$e^{-\frac{\lambda}{2}} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{2}\right)^r \frac{\Gamma\left(\frac{m+n}{2}+r\right)}{\Gamma\left(\frac{m}{2}+r\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{\frac{m}{2}+r} \frac{(F')^{\frac{m}{2}-1+r}}{\left(1+\frac{mF'}{n}\right)^{\frac{1}{2}(m+n)+r}}$$

PDF in R

```
PDF = function(x,lambda,m,n,lowerLimit,upperLimit) { exp(-lambda/2)
                          * PDFsum(x,lambda,m,n,lowerLimit,upperLimit) }
PDFsum = function(x,lambda,m,n,fromValue,toValue) {
  PDF=array(0,length(x));
  for(j in 1:length(x)) {
    for(r in fromValue:toValue) {
      PDF[j] = PDF[j] + 1/factorial(r)*(lambda/2)^r*gamma((m+n)/2+r)/gamma(m/2+r)/gamma(n/2)
                          * (m/n)^(m/2+r) * x[j]^(m/2-1+r) / (1+m*x[j]/n)^(1/2*(m+n)+r)
    }
  }
  return(PDF)
}
```

CDF

$$\int_0^x f(x), \text{ with f the PDF}$$

CDF in R

```
cumsum(PDF*rep(stepSize,length(PDF)))
```

Characteristics

Mean

$$\frac{n}{m}\frac{m+lambda}{n-2}$$

Variance

$$\left(\frac{n}{m}\right)^2 \frac{2}{(n-2)(n-4)}\left[\frac{(\lambda+m)^2}{n-2}+2\lambda+m\right]$$

Relationships

- Relationship pair: $NoncentralF1(\lambda, m, n) \rightarrow NoncentralChiSquared1(m, \lambda)$
- Relationship type: Transformation
- Relationship definition: $F \sim NoncentralF1(\lambda, m, n)$, as $n \rightarrow \infty$ then nF approaches a non-central chi-square distribution with m degrees of freedom and non-central parameter λ
- Relationship reference(s): [Walck, 2007]

References and sources

[Walck, 2007]

UncertML mapping:

no UncertML mapping found.

NoncentralT1

name	NoncentralT (ID: 0001162)
type	continuous
variate	$t' = \frac{x}{\sqrt{y/n}}$ with $x \sim Normal2(\delta, 1), y \sim ChiSquared1(n)$., scalar
support	$t' \in (0, +\infty)$

Model

NoncentralT1 distribution is of importance in hypotheses testing if we are interested in the probability of committing a Type II error implying that we would accept an hypothesis although it was wrong.

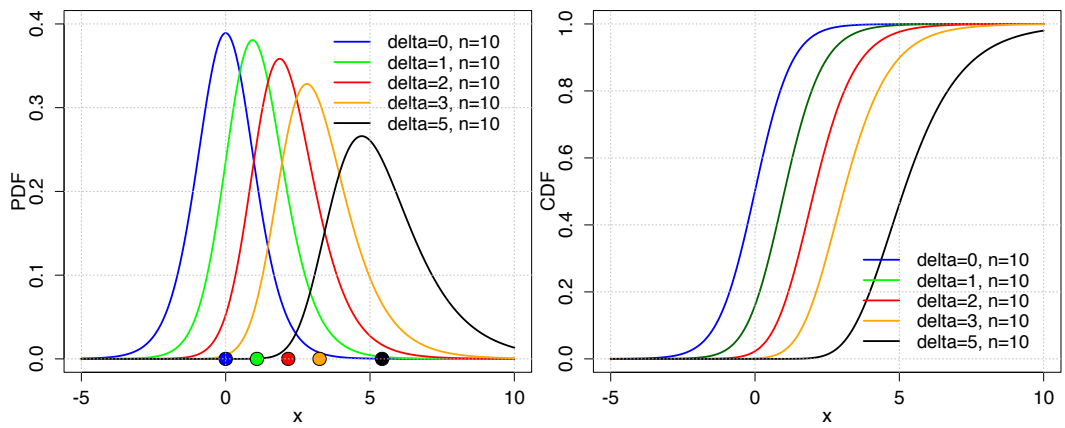

Figure A.91: NoncentralT1 distribution plotted using the provided R code.

Parameter: noncentrality

name	noncentrality parameter
type	scalar
symbol	λ
definition	$\lambda \geq 0$

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	n
definition	$n \in N$

Functions

PDF

$$\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \sum_{r=0}^{\infty} \frac{(t'\delta)^r}{r!n^{\frac{r}{2}}} \left(1+\frac{t'^2}{n}\right)^{-\frac{n+r+1}{2}} 2^{\frac{r}{2}}\,\Gamma\left(\frac{n+r+1}{2}\right)$$

PDF in R

```
exp(-delta^2/2)/(sqrt(n*pi) * gamma(n/2)) * PDFsum(x,delta,n,lowerLimit,upperLimit)
PDFsum = function(x,delta,n,fromValue,toValue) {
  PDF=array(0,length(x));
  for(j in 1:length(x)) {
    for(r in fromValue:toValue) {
      PDF[j] = PDF[j] + (x[j]*delta)^r / (factorial(r)*n^(r/2)) * (1+x[j]^2/n)^(-(n+r+1)/2)
                        * 2^(r/2) *gamma((n+r+1)/2)
    }
  }
  return(PDF)
}
```

Characteristics

Mean

$$sqrt(n/2) * gamma((n-1)/2)/gamma(n/2) * delta$$

Relationships

- Relationship pair: $NoncentralT1(\delta, n) \rightarrow StudentT1(n)$
- Relationship type: Special case
- Relationship definition: $\delta = 0$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NoncentraltT.pdf`

References and sources

[Walck, 2007]
https://en.wikipedia.org/wiki/Noncentral_t-distribution

UncertML mapping:

no UncertML mapping found.

Normal1

name	Normal 1 (ID: 0000239)
type	continuous
variate	x, scalar
support	$x \in R$

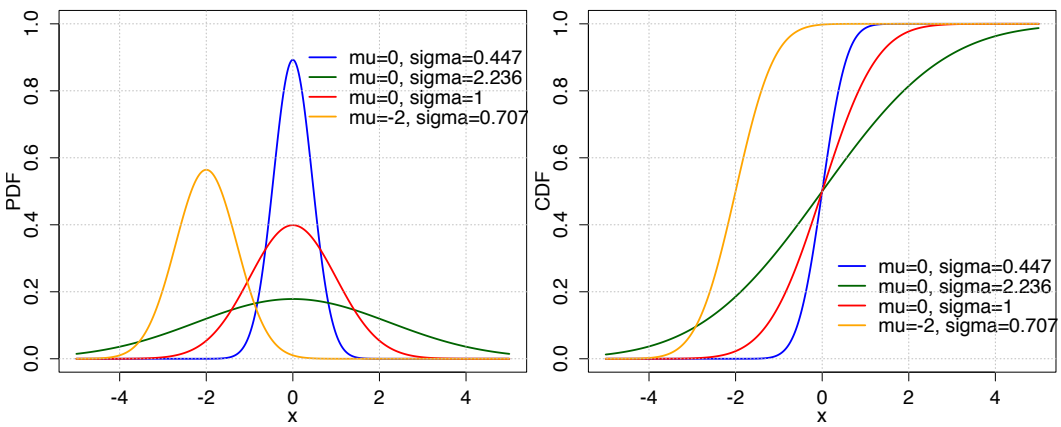

Figure A.92: Normal1 distribution plotted using the provided R code.

Model

The normal distribution is applicable to a very wide range of phenomena and is the most widely used distribution in statistics. It was originally developed as an approximation to the binomial distribution when the number of trials is large and the Bernoulli probability p is not close to 0 or 1. It is also the asymptotic form of the sum of random variables under a wide range of conditions. Normal1 is parameterized with mean, μ, and standard deviation, σ.

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: stdev

name	standard deviation
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

PDF in R

```
1/(sigma*sqrt(2*pi))*exp(-(x-mu)^2/(2*sigma^2))
```

CDF

$$\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

CDF in R

```
1/2 * (1 + erf((x-mu)/(sigma*sqrt(2))))
```

Characteristics

Mean

$$\mu$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$\sigma^2$$

Relationships

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow LogNormal1(\mu,\sigma)$
- Relationship type: Transformation
- Relationship definition: $\exp(X)$
- Relationship reference(s): [Leemis and Mcqueston, 2008] http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalLognormal.pdf [Forbes et al., 2011]

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow StandardNormal1(0,1)$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \sigma = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008] http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalStandardnormalT.pdf

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow Normal2(\mu,v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, v = \sigma^2$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow Normal3(\mu,\tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \tau = 1/\sigma^2$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow StandardNormal1(0,1)$
- Relationship type: Transformation
- Relationship definition: $X \sim Normal1(\mu,\sigma); Y = (X-\mu)/\sigma; Y \sim StandardNormal1$
- Relationship reference(s): [Leemis and Mcqueston, 2008] http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalStandardnormalT.pdf

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow Levy1(\mu,c)$
- Relationship type: Transformation
- Relationship definition: $Y \sim \text{Normal}(\mu,\sigma) \Rightarrow (Y-\mu)^{-2} \sim \text{Levy}(0,1/\sigma)$
- Relationship reference(s): https://en.wikipedia.org/wiki/L%C3%A9vy_distribution#Related_distributions

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow FoldedNormal1(\mu,\sigma)$
- Relationship type: Transformation
- Relationship definition: Given a normally distributed random variable X with mean μ and variance σ^2, the random variable $Y = |X|$ has a folded normal distribution
- Relationship reference(s): https://en.wikipedia.org/wiki/Folded_normal_distribution

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow ChiSquared1(n)$

- Relationship type: Transformation
- Relationship definition: If $X_i \sim N(\mu, \sigma), i = 1, 2, ..., n$ are mutually independent and identically distributed random variables and $Y = \sum_{i=1}^{n}((X_i - \mu)/\sigma)^2 \Rightarrow Y \sim ChiSquared1(n)$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalChisquare.pdf`

- Relationship pair: $Normal1(\mu, \sigma) \rightarrow MaxwellBoltzmann1(\sigma)$
- Relationship type: Transformation
- Relationship definition: If U_1, U_2, U_3 are independent normal variables with mean0 and standard deviation $\sigma > 0$ then $X = \sqrt{U_1^2 + U_3^2 + U_3^2}$ has the Maxwell distribution with scale parameter σ
- Relationship reference(s): `http://www.math.uah.edu/stat/special/Maxwell.html`

- Relationship pair: $Normal1(\mu, \sigma) \rightarrow LogitNormal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: If Y is a random variable with a normal distribution, and P is the logistic function, then $X = P(Y)$ has a logit-normal distribution
- Relationship reference(s): `https://en.wikipedia.org/wiki/Logit-normal_distribution`

- Relationship pair: $SinhArcsinh2(\mu, \sigma, \epsilon, \delta) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Special case
- Relationship definition: $\epsilon = 0, \delta = 1$
- Relationship reference(s): [Rubio et al., 2013]

- Relationship pair: $Poisson1(\lambda) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $\sigma^2 = \lambda, \mu = \lambda, \lambda \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PoissonNormal.pdf`

- Relationship pair: $LogNormal1(\mu, \sigma) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: $\log(X)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalLognormal.pdf`

- Relationship pair: $Normal2(\mu, v) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \sigma = \sqrt{v}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $TruncatedNormal1(\mu, \sigma, a, b) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Special case
- Relationship definition: $a = -\infty, b = \infty$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Normal3(\mu, \tau) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \sigma = 1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Binomial1(n, p) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: For $X \sim Binomial1(n, p)$ as $n \rightarrow \infty, X$ is approximately normally distributed $Normal1(\mu, \sigma)$ with $\mu = np, \sigma = np(1 - p)$.
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BinomialNormal.pdf`

- Relationship pair: $StandardNormal1(0, 1) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: $X \sim StandardNormal1$ and $Y = \mu + \sigma X \Rightarrow Y \sim Normal1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandardnormalNormal.pdf`

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $\mu = n(1 - p), n \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalNormal.pdf`

- Relationship pair: $Gamma1(k, theta) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting

- Relationship definition: $\mu = k\theta, \sigma^2 = k^2\theta, \theta \to \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaNormal1.pdf`

- Relationship pair: $Beta1(alpha, beta) \to Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $\alpha = \beta, \beta \to \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BetaNormal.pdf`

- Relationship pair: $LogitNormal1(\mu, \sigma) \to Normal1(\mu, \sigma)$
- Relationship type: Transformation
- Relationship definition: X is logit-normally distributed, then $Y = logit(X) = \log(X/(1-X))$ is normally distributed
- Relationship reference(s): `https://en.wikipedia.org/wiki/Logit-normal_distribution`

- Relationship pair: $Amoroso1(a, \theta, \alpha, \beta) \to Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: Normal1(μ, σ) distribution is a special case of the Amoroso1$(a, \theta, \alpha, \beta)$ distribution when $a = \mu - \sigma\sqrt{\alpha}, \theta = \sigma/\sqrt{\alpha}, \beta = 1$ and $\alpha \to \infty$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Hypergeometric1(N, K, n) \to Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $X \sim Hypergeometric1(N, K, n) \Rightarrow Y \sim Normal1(\mu, \sigma)$ for large n, but K/N not too small
- Relationship reference(s): [Forbes et al., 2011]

References and sources

`http://en.wikipedia.org/wiki/Normal_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/normal`

Normal2

name	Normal 2 (ID: 0000265)
type	continuous
variate	x, scalar
support	$x \in R$

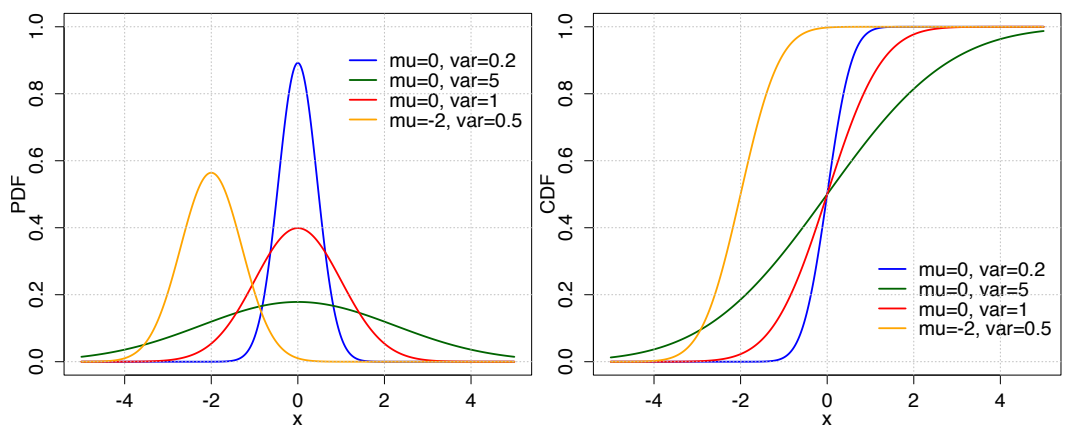

Figure A.93: Normal2 distribution plotted using the provided R code.

Model

The normal distribution is applicable to a very wide range of phenomena and is the most widely used distribution in statistics. It was originally developed as an approximation to the binomial distribution when the number of trials is large and the Bernoulli probability p is not close to 0 or 1. It is also the asymptotic form of the sum of random variables under a wide range of conditions. Normal2 is parameterized with mean, μ, and variance, v.

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: var

name	variance
type	scalar
symbol	v
definition	$v > 0$

Functions

PDF

$$\frac{1}{\sqrt{v}\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2v}}$$

PDF in R

```
1/(sqrt(var)*sqrt(2*pi))*exp(-(x-mu)^2/(2*var))
```

CDF

$$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{v}\sqrt{2}}\right)\right]$$

CDF in R

```
1/2 * (1 + erf((x-mu)/(sqrt(var)*sqrt(2))))
```

Characteristics

Mean

$$\mu$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$v$$

Relationships

- Relationship pair: $Normal2(\mu, v) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \sigma = \sqrt{v}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Normal2(\mu, v) \rightarrow Normal3(\mu, \tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, \tau = 1/v$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Normal1(\mu, \sigma) \rightarrow Normal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, v = \sigma^2$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Normal3(\mu, \tau) \rightarrow Normal2(\mu, v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = \mu, v = 1/\tau$
- Relationship reference(s): ProbOnto spec

References and sources

`http://en.wikipedia.org/wiki/Normal_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/normal`

Normal3

name	Normal 3 (ID: 0000290)
type	continuous
variate	x, scalar
support	$x \in R$

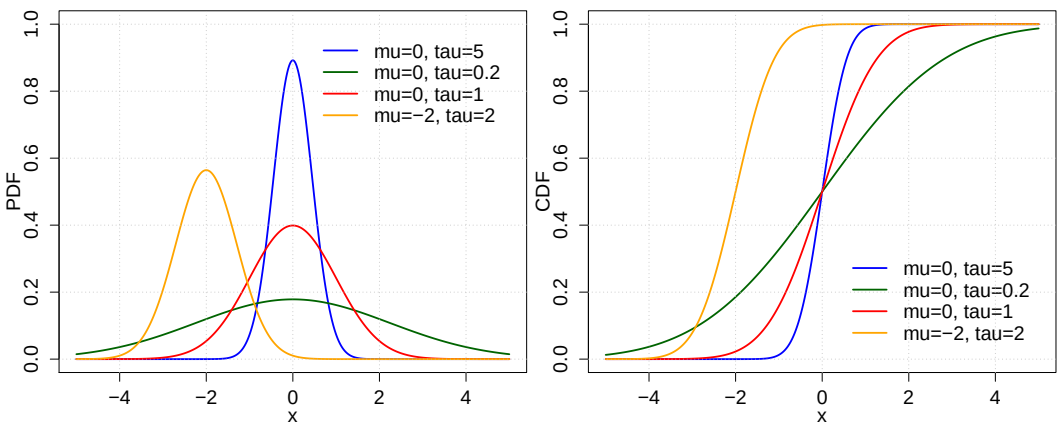

Figure A.94: Normal3 distribution plotted using the provided R code.

Model

The normal distribution is applicable to a very wide range of phenomena and is the most widely used distribution in statistics. It was originally developed as an approximation to the binomial distribution when the number of trials is large and the Bernoulli probability p is not close to 0 or 1. It is also the asymptotic form of the sum of random variables under a wide range of conditions. Normal3 is parameterized with mean, μ, and precision, τ.

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: precision

name	precision
type	scalar
symbol	τ
definition	$\tau > 0$

Functions

PDF

$$\sqrt{\frac{\tau}{2\pi}}e^{-\frac{\tau}{2}(x-\mu)^2}$$

PDF in R

```
sqrt(tau/(2*pi))*exp(-tau/2*(x-mu)^2)
```

CDF

$$\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{x-\mu}{\sqrt{1/\tau}\sqrt{2}}\right)\right]$$

CDF in R

```
1/2*(1+erf((x-mu)/(sqrt(1/tau)*sqrt(2))))
```

Characteristics

Mean

$$\mu$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$1/\tau$$

Relationships

- Relationship pair: $Normal3(\mu,\tau) \rightarrow Normal1(\mu,\sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\mu, \sigma=1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Normal3(\mu,\tau) \rightarrow Normal2(\mu,v)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\mu, v=1/\tau$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow Normal3(\mu,\tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\mu, \tau=1/\sigma^2$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Normal2(\mu,v) \rightarrow Normal3(\mu,\tau)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu=\mu, \tau=1/v$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]
http://artax.karlin.mff.cuni.cz/r-help/library/LaplacesDemon/html/dist.Normal.Precision.html

UncertML mapping:

no UncertML mapping found.

NormalInverseGamma1

name	Normal-inverse-gamma 1 (ID: 0000316)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty), \sigma^2 \in (0, +\infty)$

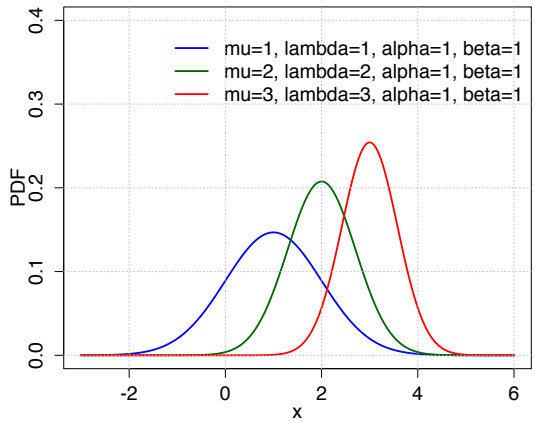

Figure A.95: NormalInverseGamma1 distribution plotted using the provided R code.

Model

"The normal-inverse-gamma distribution (or Gaussian-inverse-gamma distribution) is a four-parameter family of multivariate continuous probability distributions. It is the conjugate prior of a normal distribution with unknown mean and variance."

Parameter: mean

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: lambda

name	lambda
type	scalar
symbol	λ
definition	$\lambda > 0, \lambda \in R$

Parameter: alpha

name	shape
type	scalar
symbol	α
definition	$\alpha > 0, \alpha \in R$

Parameter: beta

name	scale
type	scalar
symbol	β
definition	$\beta > 0, \beta \in R$

Functions

PDF

$$\frac{\sqrt{\lambda}}{\sigma\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-\frac{2\beta+\lambda(x-\mu)^2}{2\sigma^2}}$$

PDF in R

```
sqrt(lambda)/(sigma*sqrt(2*pi)) * beta^alpha/gamma(alpha) * (1/sigma^2)^(alpha + 1)
                                                    * exp(- (2*beta+lambda*(x-mu)^2)/(2*sigma^2))
```

Characteristics

Relationships

References and sources

http://en.wikipedia.org/wiki/Normal-inverse-gamma_distribution

UncertML mapping:

http://www.uncertml.org/distributions/normal-inverse-gamma

OrderedLogistic1

name	Ordered Logistic 1 (ID: 0000340)
type	discrete
variate	k, scalar
support	$k \in \{1, \ldots, K\}$

Model

Parameter: cutpoint

name	ordered cutpoints
type	vector
symbol	c
definition	$c \in R^{K-1}$ such that $c_k < c_{k-1}$ for $k \in \{1, ..., K-2\}, K \in N, K > 2$

Parameter: predictor

name	predictor
type	scalar
symbol	η
definition	$\eta \in R$

Functions

PMF

$$\begin{cases} 1 - logit^{-1}(\eta - c_1) & \text{for } k = 1 \\ logit^{-1}(\eta - c_{k-1}) - logit^{-1}(\eta - c_k) & \text{for } 1 < k < K \\ logit^{-1}(\eta - c_{k-1}) & \text{for } k = K \end{cases}$$

Characteristics

Relationships

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

ParetoTypeI1

name	Pareto Type I (ID: 0000361)
type	continuous
variate	x, scalar
support	$x \in [x_m, +\infty)$

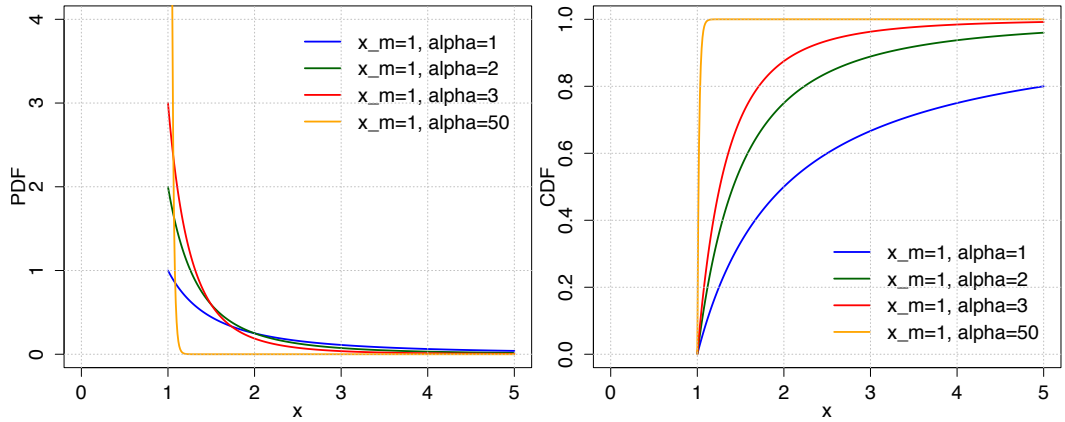

Figure A.96: ParetoTypeI1 distribution plotted using the provided R code.

Model

The Pareto distribution is often described as the basis of the 80/20 rule. For example, 80% of customer complaints regarding a make of vehicle typically arise from 20% of components. Other applications include the distribution of income and the classification of stock in a warehouse on the basis of frequency of movement.

Parameter: scale

name	scale
type	scalar
symbol	x_m
definition	$x_m > 0, x_m \in R$

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha > 0, \alpha \in R$

Functions

PDF

$$\frac{\alpha\, x_m^{\alpha}}{x^{\alpha+1}} \text{ for } x \geq x_m$$

PDF in R

```
(alpha * x_m^alpha) / x^(alpha+1)
```

CDF

$$1 - \left(\frac{x_m}{x}\right)^{\alpha} \text{ for } x \geq x_m$$

CDF in R

```
1-(x_m/x)^alpha
```

Characteristics

Mean

$$\begin{cases} \infty & \text{for } \alpha \leq 1 \\ \frac{\alpha\, x_m}{\alpha - 1} & \text{for } \alpha > 1 \end{cases}$$

Median

$$x_m \sqrt[\alpha]{2}$$

Mode

$$x_m$$

Variance

$$\begin{cases} \infty & \text{for } \alpha \in (1, 2] \\ \frac{x_m^2\, \alpha}{(\alpha-1)^2(\alpha-2)} & \text{for } \alpha > 2 \end{cases}$$

Relationships

- Relationship pair: $ParetoTypeI1(x_m, \alpha) \rightarrow Exponential1(\lambda)$
- Relationship type: Transformation
- Relationship definition: $X \sim ParetoTypeI1, Y = \log(X/\lambda) \Rightarrow Y \sim Exponential1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ParetoExponential.pdf`

- Relationship pair: $StandardUniform1(0, 1) \rightarrow ParetoTypeI1(x_m, \alpha)$
- Relationship type: Transformation
- Relationship definition: $x_m X^{-1/\alpha}$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformPareto.pdf`

References and sources

`http://en.wikipedia.org/wiki/Pareto_distribution`
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Pareto.pdf`

UncertML mapping:

`http://www.uncertml.org/distributions/pareto`

ParetoTypeII1

name	Pareto Type II (ID: 0000386)
type	continuous
variate	x, scalar
support	$x \in [\mu, +\infty)$

Model

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	λ
definition	$\lambda \in R^+$

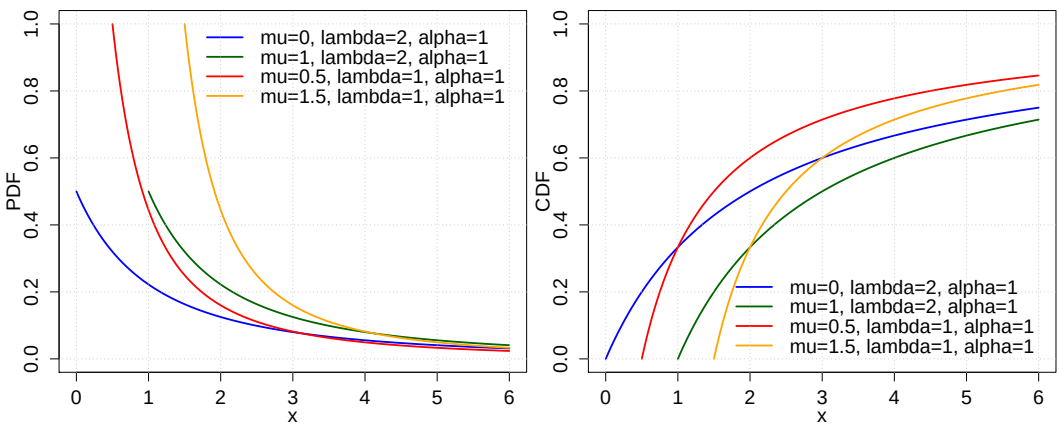

Figure A.97: ParetoTypeII1 distribution plotted using the provided R code.

Parameter: tailIndex

name	tail index
type	scalar
symbol	α
definition	$\alpha > 0, \alpha \in R$

Functions

PDF

$$\frac{\alpha}{\lambda}\left[1+\frac{x-\mu}{\lambda}\right]^{-(\alpha+1)}$$

PDF in R

```
alpha/lambda* (1+ (x-mu)/lambda)^(-(alpha+1))
```

CDF

$$1-\left[1+\frac{x-\mu}{\lambda}\right]^{-\alpha}$$

CDF in R

```
1- (1+ (x-mu)/lambda)^(-alpha)
```

Characteristics

Relationships

- Relationship pair: $ParetoTypeII1(\mu,\lambda,\alpha) \rightarrow Lomax1(\lambda,\alpha)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = 0$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Pareto_distribution#Pareto_types_I.E2.80.93IV`

References and sources

`https://en.wikipedia.org/wiki/Pareto_distribution`
[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

Poisson1

name	Poisson 1 (ID: 0000410)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

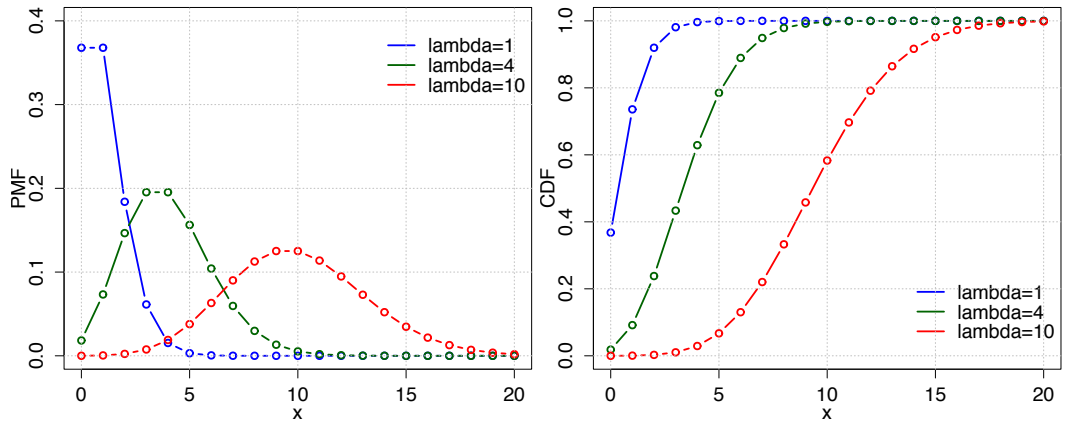

Figure A.98: Poisson1 distribution plotted using the provided R code.

Model

The Poisson distribution is applied in counting the number of rare but open-ended events. A classic example is the number of people per year who become invalids due to being kicked by horses. Another application is the number of faults in a batch of materials.

Parameter: rate

name	Poisson intensity
type	scalar
symbol	λ
definition	$\lambda \in R, \lambda > 0$

Functions

PMF

$$\frac{\lambda^k}{k!} e^{-\lambda}$$

PMF in R

```
lambda^k/factorial(k) * exp(-lambda)
```

CDF

$$\frac{\gamma(\lfloor k+1 \rfloor, \lambda)}{\lfloor k \rfloor !}$$

CDF in R

```
Igamma(floor(k+1), lambda, lower=F) / factorial(floor(k))
```

Characteristics

Mean

$$\lambda$$

Median

$$\approx \lfloor \lambda + 1/3 - 0.02/\lambda \rfloor$$

Mode

$$\lceil \lambda \rceil - 1, \lfloor \lambda \rfloor$$

Variance

$$\lambda$$

Relationships

- Relationship pair: $Poisson1(\lambda) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Limiting
- Relationship definition: $\sigma^2 = \lambda, \mu = \lambda, \lambda \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PoissonNormal.pdf`

- Relationship pair: $Poisson1(\lambda) \rightarrow Poisson2(\alpha)$
- Relationship type: Transformation
- Relationship definition: $\alpha = \log(\lambda)$
- Relationship reference(s): [STAN Development Team, 2015b]

- Relationship pair: $GeneralizedPoisson1(\theta, \delta) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\delta = 0, \theta = \mu$
- Relationship reference(s): [Yang et al., 2007]

- Relationship pair: $ConwayMaxwellPoisson1(\lambda, \nu) \rightarrow Poisson1(\lambda)$
- Relationship type: Transformation
- Relationship definition: For $\nu = 1$ the sum has a Poisson distribution with parameter $n\lambda$
- Relationship reference(s): [Shmueli et al., 2005]

- Relationship pair: $Binomial1(n, p) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $\lambda = np, n \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BinomialPoisson.pdf`

- Relationship pair: $GeneralizedPoisson2(\mu, \delta) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\delta = 0, \lambda = \mu$
- Relationship reference(s): [Plan, 2014]

- Relationship pair: $GeneralizedPoisson3(\mu, \alpha) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\alpha = 0, \lambda = \mu$
- Relationship reference(s): [Hilbe, 2011]
[Famoye and Singh, 2006]

- Relationship pair: $DoublePoisson1(\mu, \phi) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\phi = 1$
- Relationship reference(s): [Cameron and Trivedi, 2013]

- Relationship pair: $ZeroInflatedPoisson1(\lambda, \pi) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\pi = 0$
- Relationship reference(s): [Trocóniz et al., 2009]

- Relationship pair: $ZeroInflatedNegativeBinomial1(\lambda, \tau, p0) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $p0 = 0, \tau \rightarrow 0$
- Relationship reference(s): [Trocóniz et al., 2009]

- Relationship pair: $NegativeBinomial1(r, p) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $\mu = np, n \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PascalPoisson.pdf`

- Relationship pair: $ZeroInflatedGeneralizedPoisson1(\mu, \alpha, p0) \rightarrow Poisson1(\lambda)$
- Relationship type: Reparameterisation

- Relationship definition: $p0 = 0, \alpha = 0, \lambda = \mu$
- Relationship reference(s): ProbOnto spec
- Relationship pair: $Hypergeometric1(N, K, n) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $X \sim Hypergeometric1(N, K, n) \Rightarrow Y \sim Poisson1(\lambda)$ as K, N and n tend to infinity for K/N small and $nK/N \rightarrow \lambda$
- Relationship reference(s): [Forbes et al., 2011]
- Relationship pair: $Poisson2(\alpha) \rightarrow Poisson1(\lambda)$
- Relationship type: Transformation
- Relationship definition: $\lambda = \exp(\alpha)$
- Relationship reference(s): [STAN Development Team, 2015b]

References and sources

`http://en.wikipedia.org/wiki/Poisson_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/poisson`

Poisson2

name	Poisson 2 (ID: 0000437)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

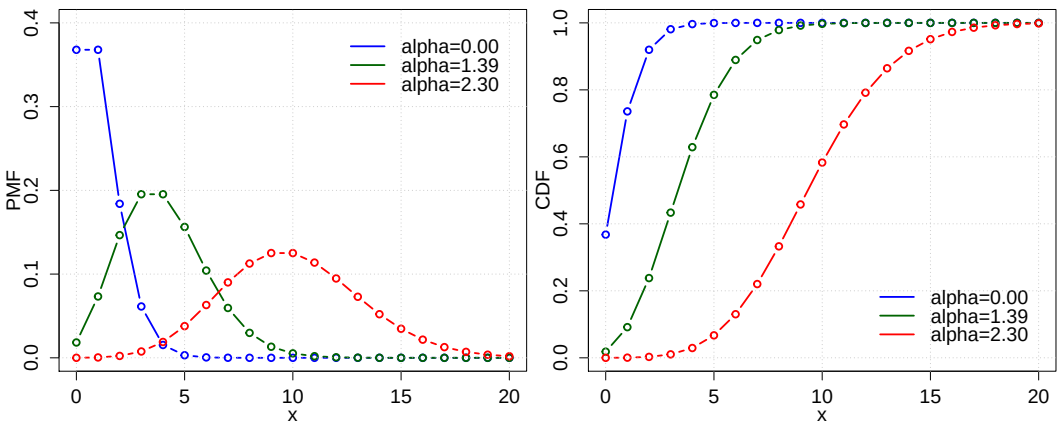

Figure A.99: Poisson2 distribution plotted using the provided R code.

Model

Parameter: logRate

name	log Poisson intensity
type	scalar
symbol	α
definition	$\alpha \in R$

Functions

PMF

$$\frac{1}{k!} \exp\left(k\alpha - \exp(\alpha)\right)$$

PMF in R

```
1/factorial(k) * exp(k*alpha - exp(alpha))
```

CDF

$$\frac{\gamma(\lfloor k+1 \rfloor, \exp(\alpha))}{\lfloor k \rfloor !}$$

CDF in R

```
Igamma(floor(k+1), exp(alpha), lower=F) / factorial(floor(k))
```

Characteristics

Mean

$$\exp(\alpha)$$

Variance

$$\exp(\alpha)$$

Relationships

- Relationship pair: $Poisson2(\alpha) \rightarrow Poisson1(\lambda)$
- Relationship type: Transformation
- Relationship definition: $\lambda = \exp(\alpha)$
- Relationship reference(s): [STAN Development Team, 2015b]
- Relationship pair: $Poisson1(\lambda) \rightarrow Poisson2(\alpha)$
- Relationship type: Transformation
- Relationship definition: $\alpha = \log(\lambda)$
- Relationship reference(s): [STAN Development Team, 2015b]

References and sources

[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

Power1

name	Power 1 (ID: 0001214)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

Parameter: scale

name	scale parameter
type	scalar
symbol	α
definition	$\alpha > 0$

Parameter: shape

name	shape parameter
type	scalar
symbol	β
definition	$\beta > 0$

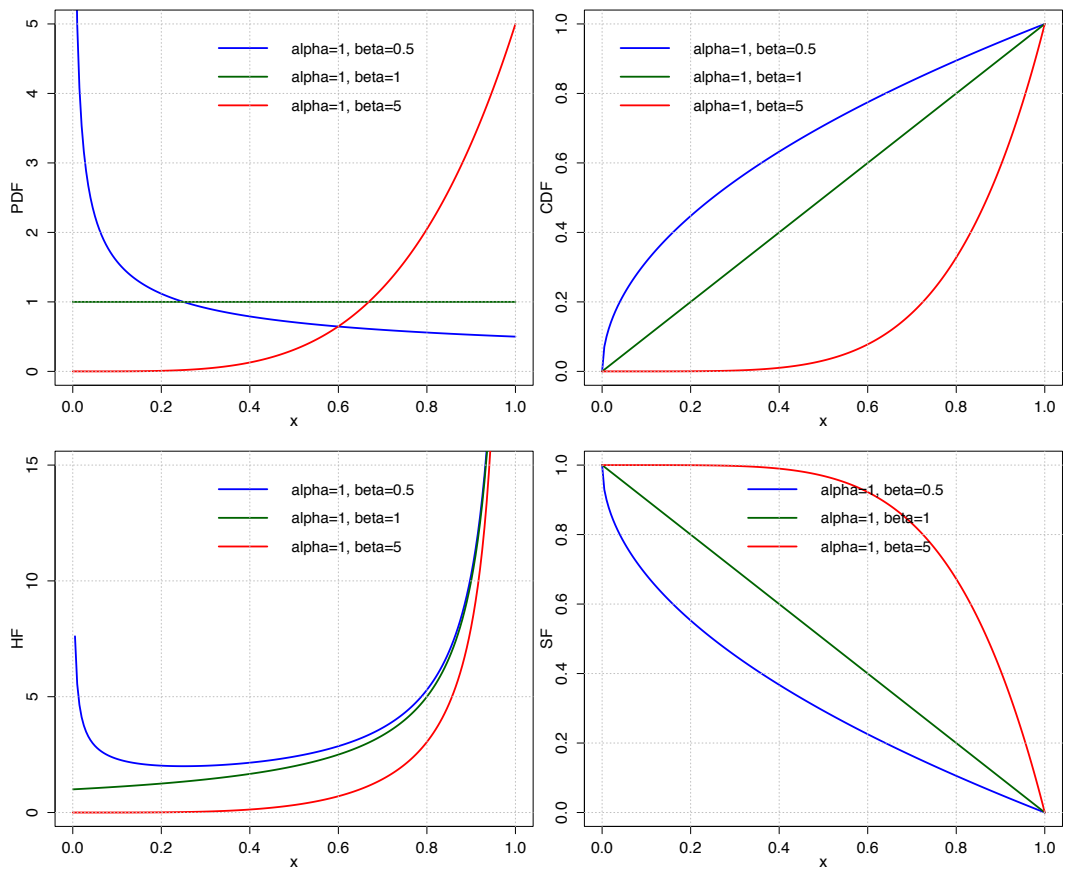

Figure A.100: Power1 distribution plotted using the provided R code.

Functions

PDF

$$\frac{\beta x^{\beta-1}}{\alpha^{\beta}}$$

PDF in R

```
(beta * x^(beta-1))/alpha^beta
```

CDF

$$\frac{x^{\beta}}{\alpha^{\beta}}$$

CDF in R

```
x^beta / alpha^beta
```

HF

$$\frac{\beta x^{\beta-1}\alpha^{-\beta}}{\alpha^{\beta}-x^{\beta}}$$

HF in R

```
(beta * x^(beta-1) * alpha^(-beta)) / (alpha^beta - x^beta)
```

SF

$$1-\frac{x^{\beta}}{\alpha^{\beta}}$$

SF in R

```
1- x^beta / alpha^beta
```

Characteristics

Mean

$$\frac{\alpha\beta}{1+\beta}$$

Variance

$$\frac{\alpha^2\beta}{(2+\beta)(1+\beta)^2}$$

Relationships

- Relationship pair: $Power1(\alpha,\beta) \rightarrow StandardPower1(\beta)$
- Relationship type: Special case
- Relationship definition: The StandardPower1(β) distribution is a special case of the Power1(α,β) distribution when $\alpha = 1$.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PowerStandardpower.pdf`

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Power.pdf`

UncertML mapping:

no UncertML mapping found.

PowerNormal1

name	Power-Normal 1 (ID: 0001230)
type	continuous
variate	x, scalar
support	$x \in R$

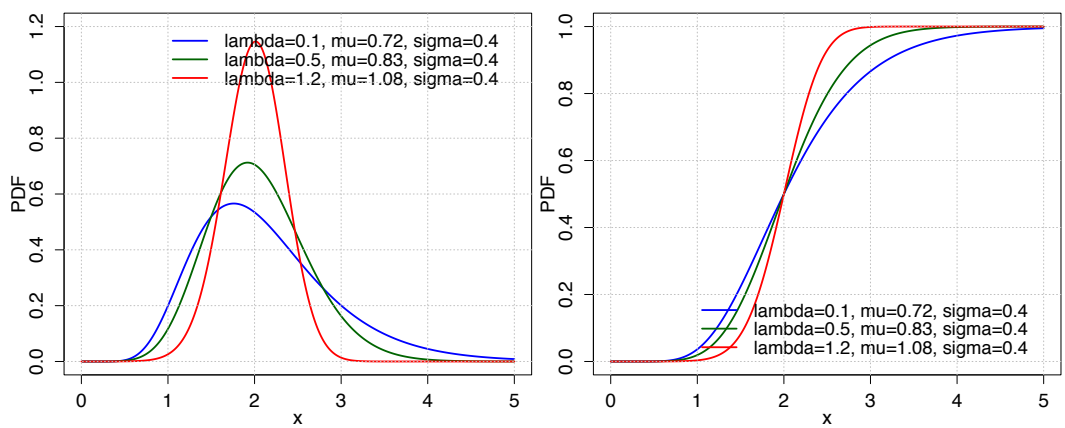

Figure A.101: PowerNormal1 distribution plotted using the provided R code.

Parameter: shape

name	transforming parameter
type	scalar
symbol	λ
definition	$\lambda \in R$

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma \in R, \sigma > 0$

Functions

PDF

$$\frac{1}{K}\frac{1}{\sqrt{2\pi}\sigma}x^{\lambda-1}\exp\left[-\frac{1}{2}\left(\frac{x^{(\lambda)}-\mu}{\sigma}\right)^2\right]$$

with

$$x^{(\lambda)} = \begin{cases}(x^\lambda - 1)/\lambda & \text{for } \lambda \neq 0\\ \log(x) & \text{for } \lambda = 0\end{cases}; \quad K = \begin{cases}\Phi(T) & \text{for } \lambda > 0\\ 1 & \text{for } \lambda = 0\\ \Phi(-T) & \text{for } \lambda < 0\end{cases}; \quad T = 1/(\lambda\sigma) + \mu/\sigma$$

PDF in R

```
1/(K(lambda,mu,sigma) * sqrt(2*pi) * sigma) * x^(lambda-1)
                          * exp(- 1/2 * ((BCtrafo(x,lambda)-mu) / sigma)^2 )
BCtrafo = function(x,lambda) {
  (x^lambda - 1) / lambda
}
erf <- function(x) { 2 * pnorm(x * sqrt(2)) - 1 }
PHI <- function(x) { 1/2 * (1 + erf(x/(sqrt(2)))) }
Tfunc <- function(lambda,mu,sigma) { 1/(lambda*sigma) + mu/sigma }
K = function(lambda,mu,sigma) {
  if (lambda > 0) {
    K = PHI(Tfunc(lambda,mu,sigma))
  } else if (lambda==0) {
    K = 1
  } else {
    K = PHI(-Tfunc(lambda,mu,sigma))
  }
}
```

CDF

$$\begin{cases}\frac{1}{K}(\Phi(Z) - \Phi(-T)) & \text{for } \lambda > 0\\ \Phi(Z) & \text{for } \lambda = 0\\ \frac{1}{K}\Phi & \text{for } \lambda < 0\end{cases}$$

with

$$Z = \frac{(x^\lambda - 1)/\lambda - \mu}{\sigma} \quad \text{and} \quad T = 1/(\lambda\sigma) + \mu/\sigma$$

CDF in R

```
CDF = function(x,lambda,mu,sigma) {
  if (lambda > 0) {
    CDF = 1/K(lambda,mu,sigma) * ( PHI(Zfunc(x,lambda,mu,sigma)) - PHI(-Tfunc(lambda,mu,sigma)) )
  } else if (lambda==0) {
    CDF = PHI(Zfunc(x,lambda,mu,sigma))
  } else if (lambda < 0) {
    CDF = 1/K(lambda,mu,sigma) * PHI(Zfunc(x,lambda,mu,sigma))
  }
```

```
}
Zfunc <- function(x,lambda,mu,sigma) {
  Zfunc = (BCtrafo(x,lambda)-mu) / sigma
  }
```

Characteristics

Relationships

- Relationship pair: $PowerNormal1(\lambda, \mu, \sigma) \rightarrow LogNormal1(\mu, \sigma)$
- Relationship type: Special case
- Relationship definition: $PowerNormal1$ distribution converges to LogNormal1 one when $\lambda \rightarrow 0$
- Relationship reference(s): [Lavielle, 2014]

- Relationship pair: $PowerNormal1(\lambda, \mu, \sigma) \rightarrow TruncatedNormal1(\mu, \sigma, a, b)$
- Relationship type: Special case
- Relationship definition: $PowerNormal1$ converges to $TruncatedNormal1$ when $\lambda \rightarrow 1$
- Relationship reference(s): [Lavielle, 2014]

References and sources

[Freeman and Modarres, 2002]

UncertML mapping:

no UncertML mapping found.

Rayleigh1

name	Rayleigh 1 (ID: 0000462)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

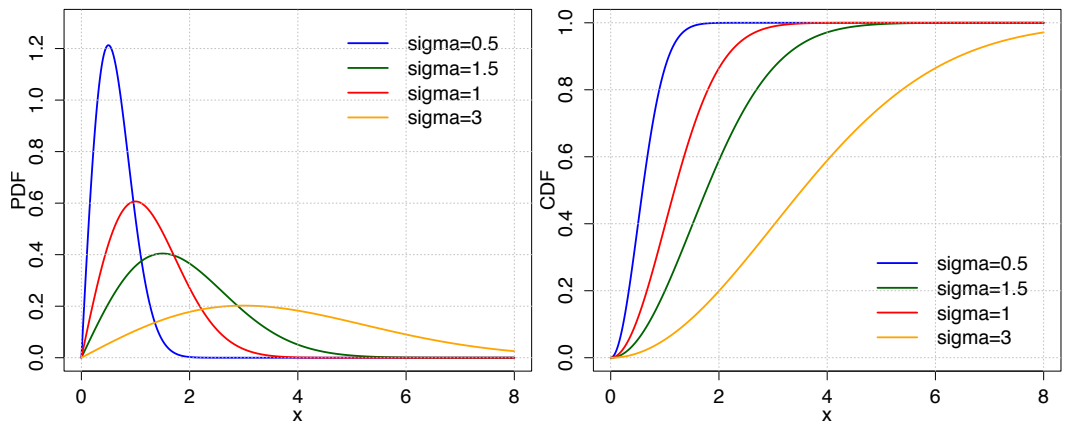

Figure A.102: Rayleigh1 distribution plotted using the provided R code.

Model

If the component velocities of a particle in the x and y directions are two independent normal random variables with zero means and equal variances, then the distance the particle travels per unit time is distributed Rayleigh.

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}$$

PDF in R

```
x/sigma^2 * exp(-x^2/(2*sigma^2))
```

CDF

$$1 - e^{-x^2/(2\sigma^2)}$$

CDF in R

```
1 - exp(-x^2/(2*sigma^2))
```

Characteristics

Mean

$$\sigma\sqrt{\frac{\pi}{2}}$$

Median

$$\sigma\sqrt{\log(4)}$$

Mode

$$\sigma$$

Variance

$$\frac{4-\pi}{2}\sigma^2$$

Relationships

- Relationship pair: $Amoroso1(a, \theta, \alpha, \beta) \rightarrow Rayleigh1(\sigma)$
- Relationship type: Special case
- Relationship definition: Rayleigh1(σ) distribution is a special case of the Amoroso1(a, θ, α, β) distribution when $a = 0, \theta = \sqrt{2\sigma^2}, \alpha = 1, \beta = 2$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $Rice1(\nu, \sigma) \rightarrow Rayleigh1(\sigma)$
- Relationship type: Special case
- Relationship definition: If $R \sim Rice1(0, \sigma)$ then $R \sim Rayleigh1(\sigma)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Rice_distribution`

- Relationship pair: $Weibull1(\lambda, k) \rightarrow Rayleigh1(\sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $k = 2, \lambda = \sqrt{2}\sigma$
- Relationship reference(s): ProbOnto spec

References and sources

`https://en.wikipedia.org/wiki/Rayleigh_distribution`

UncertML mapping:

no UncertML mapping found.

Rice1

name	Rice 1 (ID: 0000487)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

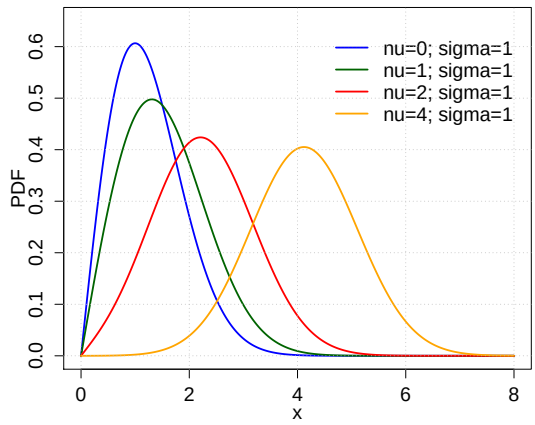

Figure A.103: Rice1 distribution plotted using the provided R code.

Model

In probability theory, the Rice distribution, Rician distribution or Ricean distribution is the probability distribution of the magnitude of a circular bivariate normal random variable with potentially non-zero mean. It was named after Stephen O. Rice.

Parameter: noncentrality

name	noncentrality parameter
type	scalar
symbol	ν
definition	$\nu \geq 0$

Parameter: scale

name	scale parameter
type	scalar
symbol	σ
definition	$\sigma > 0$

Functions

PDF

$$\frac{x}{\sigma^2} \exp\left(\frac{-(x^2+\nu^2)}{2\sigma^2}\right) I_0\left(\frac{x\nu}{\sigma^2}\right)$$

PDF in R

```
x/sigma^2*exp(-(x^2+nu^2)/2/sigma^2) * besselI(x*nu/sigma^2, 0, expon.scaled = FALSE)
```

Characteristics

Relationships

- Relationship pair: $Rice1(\nu, \sigma) \rightarrow Exponential1(\lambda)$
- Relationship type: Transformation
- Relationship definition: If $R \sim Rice1(0, \sigma)$ then $R^2 \sim Exponential1(\lambda)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Rice_distribution`

- Relationship pair: $Rice1(\nu, \sigma) \rightarrow Rayleigh1(\sigma)$
- Relationship type: Special case
- Relationship definition: If $R \sim Rice1(0, \sigma)$ then $R \sim Rayleigh1(\sigma)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Rice_distribution`

References and sources

`http://nl.mathworks.com/help/stats/rician-distribution.html`
`https://en.wikipedia.org/wiki/Rice_distribution`

UncertML mapping:

no UncertML mapping found.

ScaledInverseChiSquare1

name	Scaled Inverse Chi-Square (ID: 0000509)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

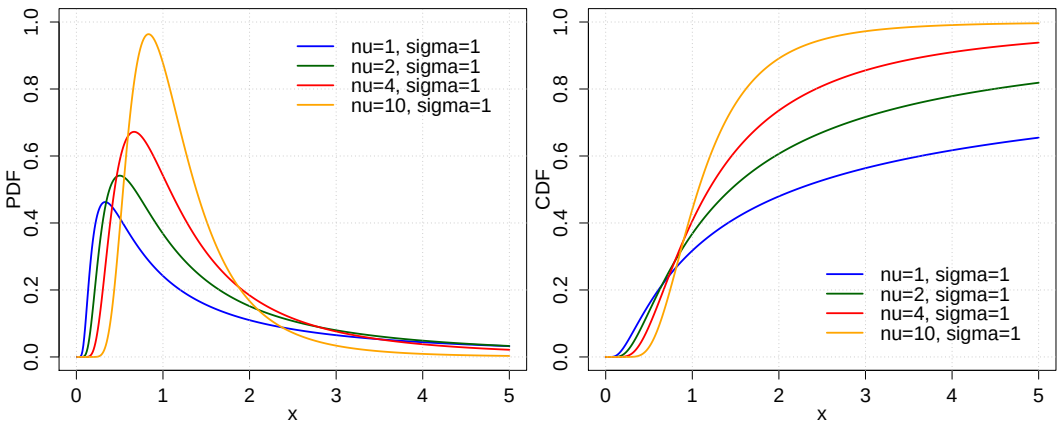

Figure A.104: ScaledInverseChiSquare1 distribution plotted using the provided R code.

Model

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	ν
definition	$\nu \in R^+$

Parameter: scale

name	scaling parameter
type	scalar
symbol	σ
definition	$\sigma \in R^+$

Functions

PDF

$$\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \sigma^{\nu} x^{-(\nu/2+1)} \exp\left(-\frac{1}{2}\nu\sigma^2\frac{1}{x}\right)$$

PDF in R

```
(nu/2)^(nu/2)/gamma(nu/2)*sigma^nu*x^(-(nu/2+1))*exp(-1/2*nu*sigma^2*1/x)
```

CDF

$$\Gamma\left(\frac{\nu}{2}, \frac{\sigma^2\nu}{2x}\right) \Big/ \Gamma\left(\frac{\nu}{2}\right)$$

CDF in R

```
Igamma(nu/2,sigma^2*nu/2/x, lower=F) / gamma(nu/2) \text{ with } Igamma(a, x, lower=F)
```

Characteristics

Mean

$$\frac{\nu\sigma^2}{\nu - 2} \text{ for } \nu > 2$$

Mode

$$\frac{\nu\sigma^2}{\nu + 2}$$

Variance

$$\frac{2\nu^2\sigma^4}{(\nu-2)^2(\nu-4)} \text{ for } \nu > 4$$

Relationships

- Relationship pair: $ScaledInverseChiSquare1(\nu, \sigma) \rightarrow ScaledInverseChiSquare1(\nu, \sigma)$
- Relationship type: Transformation
- Relationship definition: If $X \sim ScaledInverseChiSquare1(\nu, \sigma)$ then $kX \sim ScaledInverseChiSquare1(\nu, k\sigma)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution`
- Relationship pair: $ScaledInverseChiSquare1 \rightarrow InverseChiSquare1$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ Scale-inv-$\chi^2(\nu, \tau^2)$ then $\frac{X}{\tau^2\nu} \sim$ inv-$\chi^2(\nu)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution`
- Relationship pair: $ScaledInverseChiSquare1(\nu, \sigma) \rightarrow InverseGamma1(\alpha, \beta)$
- Relationship type: Transformation
- Relationship definition: If $X \sim ScaledInverseChiSquare1(\nu, \sigma)$ then $X \sim InverseGamma1(\nu/2, \nu\sigma^2/2)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution`
- Relationship pair: $ScaledInverseChiSquare1(\nu, \sigma) \rightarrow ScaledInverseChiSquare1(\nu, \sigma)$
- Relationship type: Transformation
- Relationship definition: If $X \sim ScaledInverseChiSquare1(\nu, \sigma)$ then $kX \sim ScaledInverseChiSquare1(\nu, k\sigma)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution`

References and sources

[STAN Development Team, 2015b]
`https://en.wikipedia.org/wiki/Scaled_inverse_chi-squared_distribution`

UncertML mapping:

no UncertML mapping found.

SinhArcsinh1

name	Sinh-Arcsinh 1 (ID: 0001257)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

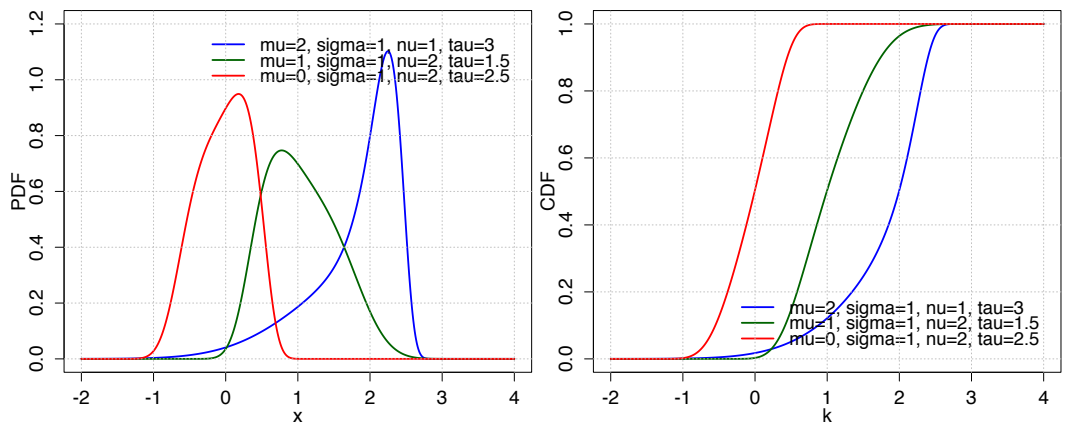

Figure A.105: SinhArcsinh1 distribution plotted using the provided R code.

Model

This four-parameter family has symmetric and skewed members and allows for tailweights that are both heavier and lighter than those of the generating distribution.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma > 0$

Parameter: skewness

name	left hand tail
type	scalar
symbol	ν
definition	$\nu > 0$

Parameter: kurtosis

name	right hand tail
type	scalar
symbol	τ
definition	$\tau > 0$

Functions

PDF

$$\frac{c}{\sqrt{2\pi}\sigma(1+z^2)^{1/2}} e^{-r^2/2} \text{ with}$$

$$r = \frac{1}{2}\left(\exp[\tau \sinh^{-1}(z)] - \exp[-\nu \sinh^{-1}(z)]\right)$$

$$c = \frac{1}{2}\left(\tau \exp[\tau \sinh^{-1}(z)] + \nu \exp[-\nu \sinh^{-1}(z)]\right)$$

$$z = (x-\mu)/\sigma$$

PDF in R

```
cfunc(x,mu,sigma,nu,tau) / (sqrt(2*pi) * sigma * (1+zfunc(x,mu,sigma)^2)^(1/2))
                                    * exp(-rfunc(x,mu,sigma,nu,tau)^2/2)
rfunc = function(x,mu,sigma,nu,tau) { 1/2 * ( exp(tau * asinh(zfunc(x,mu,sigma)))
                                    - exp(-nu * asinh(zfunc(x,mu,sigma))) ) }
cfunc = function(x,mu,sigma,nu,tau) { 1/2 * ( tau * exp(tau * asinh(zfunc(x,mu,sigma)))
                                    + nu * exp(-nu * asinh(zfunc(x,mu,sigma))) ) }
zfunc = function(x,mu,sigma) { (x-mu)/sigma }
```

Characteristics

Relationships

References and sources

https://cran.r-project.org/web/packages/gamlss.dist/gamlss.dist.pdf

UncertML mapping:

no UncertML mapping found.

SinhArcsinh2

name	Sinh-Arcsinh 2 (ID: 0001279)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

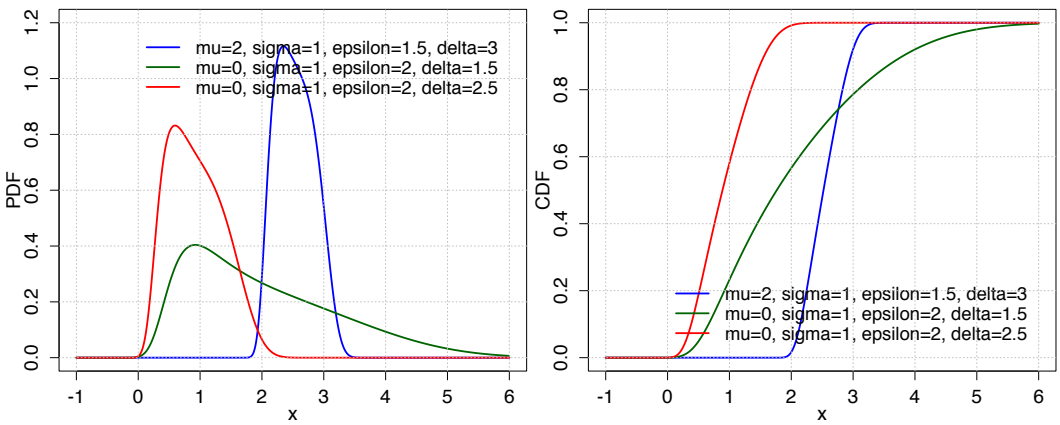

Figure A.106: SinhArcsinh2 distribution plotted using the provided R code.

Model

The sinh-arcsinh distribution, [Jones and Pewsey, 2009], represents an interesting model obtained as a parametric change of variable. This distribution, which is described in the next section, contains two shape parameters that can be interpreted as skewness and kurtosis parameters, and has tractable expressions for the density and distribution functions. Another appealing property is that it contains models with both heavier or lighter tails than those of the normal distribution.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma > 0$

Parameter: skewness

name	skewness
type	scalar
symbol	ϵ
definition	$\epsilon \in R$

Parameter: kurtosis

name	kurtosis
type	scalar
symbol	δ
definition	$\delta > 0$

Functions

PDF

$$PDF(x;\mu,\sigma,\epsilon,\delta) = \phi[H(x;\mu,\sigma,\epsilon,\delta)]\, h(x;\mu,\sigma,\epsilon,\delta)$$

$$h(x;\mu,\sigma,\epsilon,\delta) = \frac{\delta \cosh\left(\delta \operatorname{arcsinh}\left(\frac{x-\mu}{\sigma}\right) - \epsilon\right)}{\sigma\sqrt{1+\left(\frac{x-\mu}{\sigma}\right)^2}}$$

$$H(x;\mu,\sigma,\epsilon,\delta) = \sinh\left(\delta \operatorname{arcsinh}\left(\frac{x-\mu}{\sigma}\right) - \epsilon\right)$$

$$\phi = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

PDF in R

```
PDF = function(x,mu,sigma,epsilon,delta) { phi(Hfunc(x,mu,sigma,epsilon,delta))
                                          * hfunc(x,mu,sigma,epsilon,delta) }
hfunc = function(x,mu,sigma,epsilon,delta) { (delta*cosh(delta*asinh((x-mu)/sigma)-epsilon))
                                           /(sigma*sqrt(1+((x-mu)/sigma)^2)) }
Hfunc = function(x,mu,sigma,epsilon,delta) { sinh(delta*asinh((x-mu)/sigma)-epsilon) }
phi = function(x) { 1/(sqrt(2*pi))*exp(-x^2/2) }
```

CDF

$$S_0(x;\mu,\sigma,\epsilon,\delta) = \Phi[H(x;\mu,\sigma,\epsilon,\delta)]$$

$$\text{with } \Phi(x) = \frac{1}{2}(1 + erf(x/\sqrt{2}))$$

$$\text{and } H(x;\mu,\sigma,\epsilon,\delta) = \sinh\left(\delta \operatorname{arcsinh}\left(\frac{x-\mu}{\sigma}\right) - \epsilon\right)$$

CDF in R

```
S0func = function(x,mu,sigma,epsilon,delta) { PHI(Hfunc(x,mu,sigma,epsilon,delta)) }
Hfunc = function(x,mu,sigma,epsilon,delta) { sinh(delta*asinh((x-mu)/sigma)-epsilon) }
PHI = function(x) { 1/2 * (1 + erf(x/(sqrt(2)))) }
```

Characteristics

Relationships

- Relationship pair: $SinhArcsinh2(\mu, \sigma, \epsilon, \delta) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Special case
- Relationship definition: $\epsilon = 0, \delta = 1$
- Relationship reference(s): [Rubio et al., 2013]
- Relationship pair: $SinhArcsinh2(\mu, \sigma, \epsilon, \delta) \rightarrow StandardNormal1$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \sigma = 1, \epsilon = 0, \delta = 1$
- Relationship reference(s): [Rubio et al., 2013]

References and sources

[Rubio et al., 2013]

UncertML mapping:

no UncertML mapping found.

SkewNormal1

name	Skew Normal (ID: 0000535)
type	continuous
variate	x, scalar
support	$x \in R$

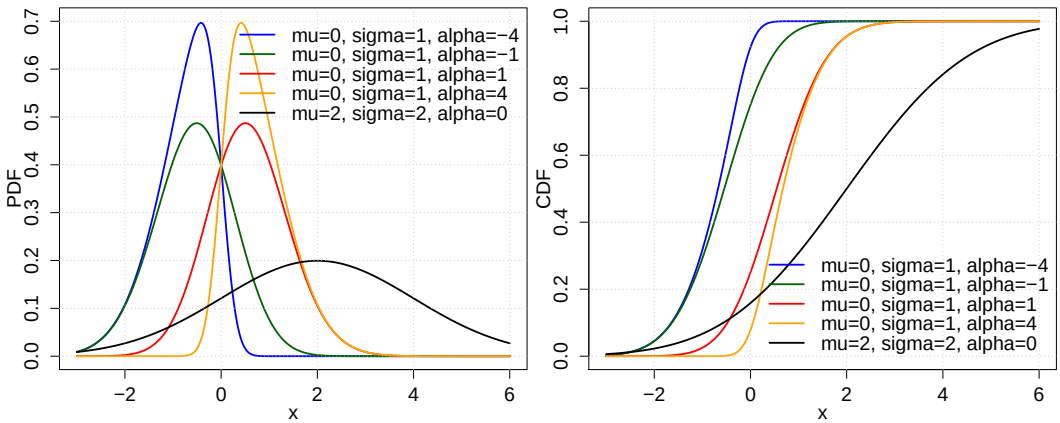

Figure A.107: SkewNormal1 distribution plotted using the provided R code.

Model

In probability theory and statistics, the skew normal distribution is a continuous probability distribution that generalises the normal distribution to allow for non-zero skewness.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\sigma > 0$

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha \in R$

Functions

PDF

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{1}{2}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big]\Big[1+\mathrm{erf}\Big(\alpha\Big(\frac{x-\mu}{\sigma\sqrt{2}}\Big)\Big)\Big]$$

PDF in R

```
1/(sigma*sqrt(2*pi)) * exp(-1/2*((x-mu)/sigma)^2) * (1+erf(alpha*((x-mu)/(sigma*sqrt(2)))))
```

CDF

$$\Phi\Big(\frac{x-\mu}{\sigma}\Big) - 2T\Big(\frac{x-\mu}{\sigma}, \alpha\Big)$$

CDF in R

```
Phi((x-mu)/sigma) - 2*T.Owen((x-mu)/sigma,alpha)
```

Characteristics

Mean

$$\mu + \sigma\delta\sqrt{\frac{2}{\pi}} \text{ where } \delta = \frac{\alpha}{\sqrt{1+\alpha^2}}$$

Variance

$$\sigma^2\left(1 - \frac{2\delta^2}{\pi}\right) \text{ where } \delta = \frac{\alpha}{\sqrt{1+\alpha^2}}$$

Relationships

- Relationship pair: $SkewNormal1(\mu, \sigma, \alpha) \rightarrow StandardNormal1$
- Relationship type: Special case
- Relationship definition: If $X \sim SkewNormal1(\mu, \sigma, 0)$ then $X \sim StandardNormal1$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Skew_normal_distribution` `http://azzalini.stat.unipd.it/SN/Intro/intro.html`
- Relationship pair: $SkewNormal1(\mu, \sigma, \alpha) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: If $X \sim SkewNormal1(\mu, \sigma, \alpha)$ then $X^2 \sim ChiSquared1(1)$
- Relationship reference(s): `http://azzalini.stat.unipd.it/SN/Intro/intro.html`

References and sources

[STAN Development Team, 2015b]
`https://en.wikipedia.org/wiki/Skew_normal_distribution`
`http://www.boost.org/doc/libs/1_60_0/libs/math/doc/html/math_toolkit/dist_ref/dists/skew_normal_dist.html`

UncertML mapping:

no UncertML mapping found.

StandardCauchy1

name	Standard Cauchy 1 (ID: 0001300)
type	continuous
variate	x, scalar
support	$x \in R$

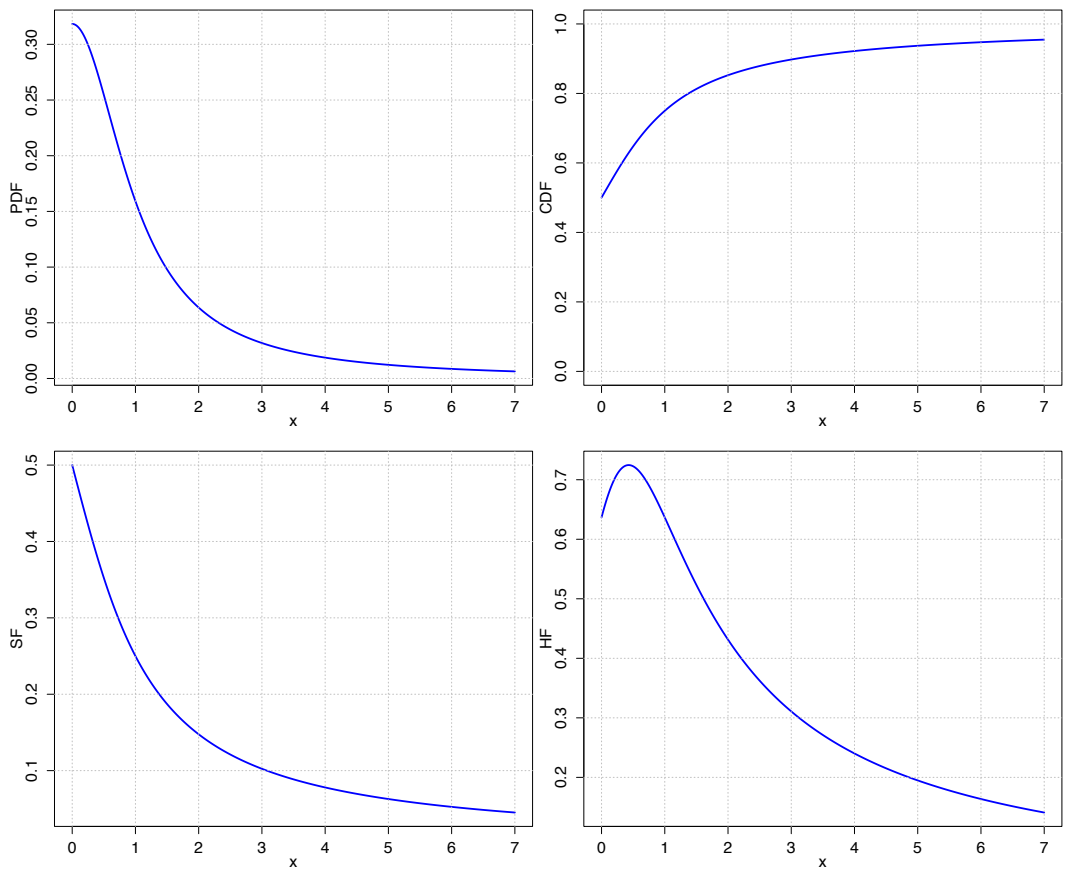

Figure A.108: StandardCauchy1 distribution plotted using the provided R code.

Model

StandardCauchy1 is a special case of Cauchy1.

Functions

PDF

$$\frac{1}{\pi}\frac{1}{1+x^2}$$

PDF in R

```
1/pi/(1+x^2)
```

CDF

$$\frac{\pi + 2\arctan(x)}{2\pi}$$

CDF in R

```
(pi + 2*atan(x))/2/pi
```

HF

$$\frac{2}{(1+x^2)(\pi - 2\arctan(x))}$$

HF in R

```
2/(1+x^2)/(pi - 2*atan(x))
```

SF

$$\frac{\pi - 2\arctan(x)}{2\pi}$$

SF in R

```
(pi - 2*atan(x))/2/pi
```

Characteristics

Mean

$$undefined$$

Median

$$0$$

Mode

$$0$$

Variance

$$undefined$$

Relationships

- Relationship pair: $Cauchy1(x_0, \gamma) \rightarrow StandardCauchy1$
- Relationship type: Special case
- Relationship definition: $x0 = 0, \gamma = 1$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Cauchy_distribution`

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Standardcauchy.pdf`

UncertML mapping:

no UncertML mapping found.

StandardLogistic1

name	Standard Logistic 1 (ID: 0001321)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

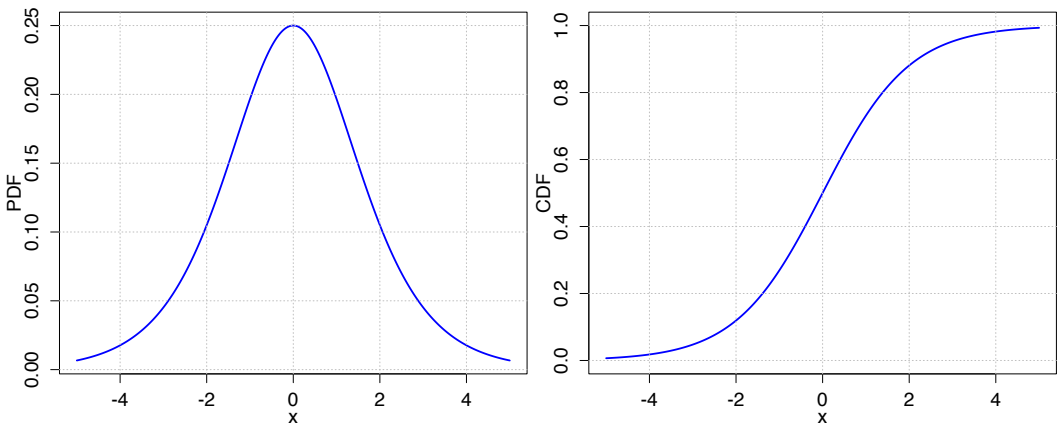

Figure A.109: StandardLogistic1 distribution plotted using the provided R code.

Model

Functions

PDF

$$\frac{e^x}{(1+e^x)^2}$$

PDF in R

```
exp(x) / (1+exp(x))^2
```

CDF

$$\frac{e^x}{1+e^x}$$

CDF in R

```
exp(x)/(1+exp(x))
```

Characteristics

Mean

$$0$$

Median

$$0$$

Mode

$$0$$

Variance

$$\frac{\pi^2}{3}$$

Relationships

- Relationship pair: $Logistic1(\mu, s) \rightarrow StandardLogistic1$
- Relationship type: Special case
- Relationship definition: $\mu = 0, s = 1$
- Relationship reference(s): -
- Relationship pair: $Logistic2(\mu, \tau) \rightarrow StandardLogistic1$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \tau = 1$
- Relationship reference(s): -

References and sources

http://www.math.uah.edu/stat/special/Logistic.html

UncertML mapping:

no UncertML mapping found.

StandardNormal1

name	Standard Normal 1 (ID: 0000562)
type	continuous
variate	x, scalar
support	$x \in R$

Model

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu = 0$

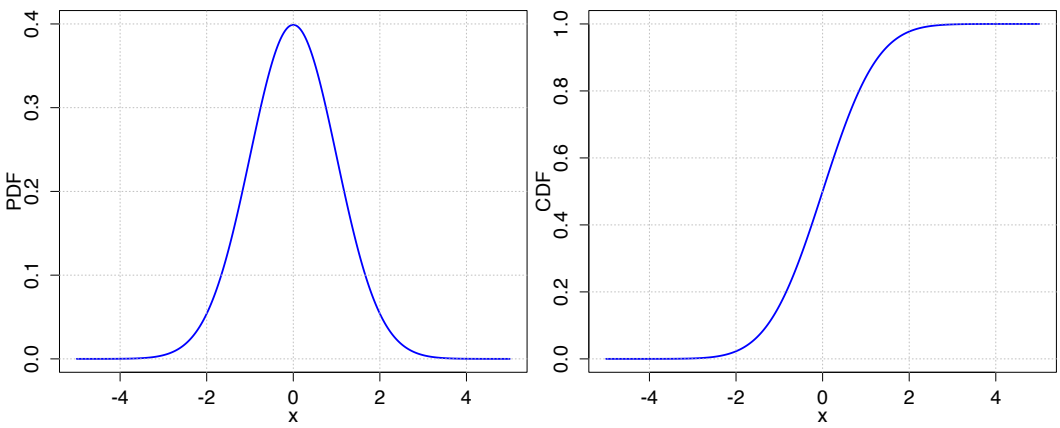

Figure A.110: StandardNormal1 distribution plotted using the provided R code.

Parameter: stdev

name	standard deviation
type	scalar
symbol	σ
definition	$\sigma = 1$

Functions

PDF

$$\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

PDF in R

```
1/(sqrt(2*pi))*exp(-x^2/2)
```

CDF

$$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

CDF in R

```
1/2 * (1 + erf(x/(sqrt(2))))
```

HF

$$\frac{\sqrt{\frac{2}{\pi}}}{e^{\frac{x^2}{2}}\left(2 - \operatorname{erfc}\left(\frac{-x}{\sqrt{2}}\right)\right)}$$

HF in R

```
sqrt(2/pi)/(exp(x^2/2)*(2-erfc(-x/sqrt(2))))
```

SF

$$1 - \frac{1}{2}\operatorname{erfc}\left(-\frac{x}{\sqrt{2}}\right)$$

SF in R

```
1 - 1/2 *erfc(-x/sqrt(2))
```

Characteristics

Mean

$$0$$

Median

$$0$$

Mode

$$0$$

Variance

$$1$$

Relationships

- Relationship pair: $StandardNormal1(0,1) \rightarrow ChiSquared1(k)$
- Relationship type: Transformation
- Relationship definition: ΣX^2
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandardnormalChisquare.pdf`

- Relationship pair: $StandardNormal1(0,1) \rightarrow Normal1(\mu,\sigma)$
- Relationship type: Transformation
- Relationship definition: $X \sim StandardNormal1$ and $Y = \mu + \sigma X \Rightarrow Y \sim Normal1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandardnormalNormal.pdf`

- Relationship pair: $StudentT1(\nu) \rightarrow StandardNormal1(0,1)$
- Relationship type: Limiting
- Relationship definition: $X \sim StudentT1(\nu) \Rightarrow StandardNormal1(0,1)$ as ν tends to infinity. The approximation is reasonable for $\nu \geq 30$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $SkewNormal1(\mu,\sigma,\alpha) \rightarrow StandardNormal1$
- Relationship type: Special case
- Relationship definition: If $X \sim SkewNormal1(\mu,\sigma,0)$ then $X \sim StandardNormal1$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Skew_normal_distribution`
`http://azzalini.stat.unipd.it/SN/Intro/intro.html`

- Relationship pair: $SinhArcsinh2(\mu,\sigma,\epsilon,\delta) \rightarrow StandardNormal1$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \sigma = 1, \epsilon = 0, \delta = 1$
- Relationship reference(s): [Rubio et al., 2013]

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow StandardNormal1(0,1)$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \sigma = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalStandardnormalT.pdf`

- Relationship pair: $InverseGaussian1(\lambda,\mu) \rightarrow StandardNormal1(0,1)$
- Relationship type: Limiting
- Relationship definition: $\lambda \rightarrow \infty$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/InversegaussianStandardnormal.pdf`

- Relationship pair: $Normal1(\mu,\sigma) \rightarrow StandardNormal1(0,1)$
- Relationship type: Transformation
- Relationship definition: $X \sim Normal1(\mu,\sigma); Y = (X-\mu)/\sigma; Y \sim StandardNormal1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NormalStandardnormalT.pdf`

References and sources

`https://en.wikipedia.org/wiki/Normal_distribution#Standard_normal_distribution`
[Marichev and Trott, 2013]

UncertML mapping:

no UncertML mapping found.

StandardPower1

name	Standard Power 1 (ID: 0001351)
type	continuous
variate	x, scalar
support	$x \in (0, +\infty)$

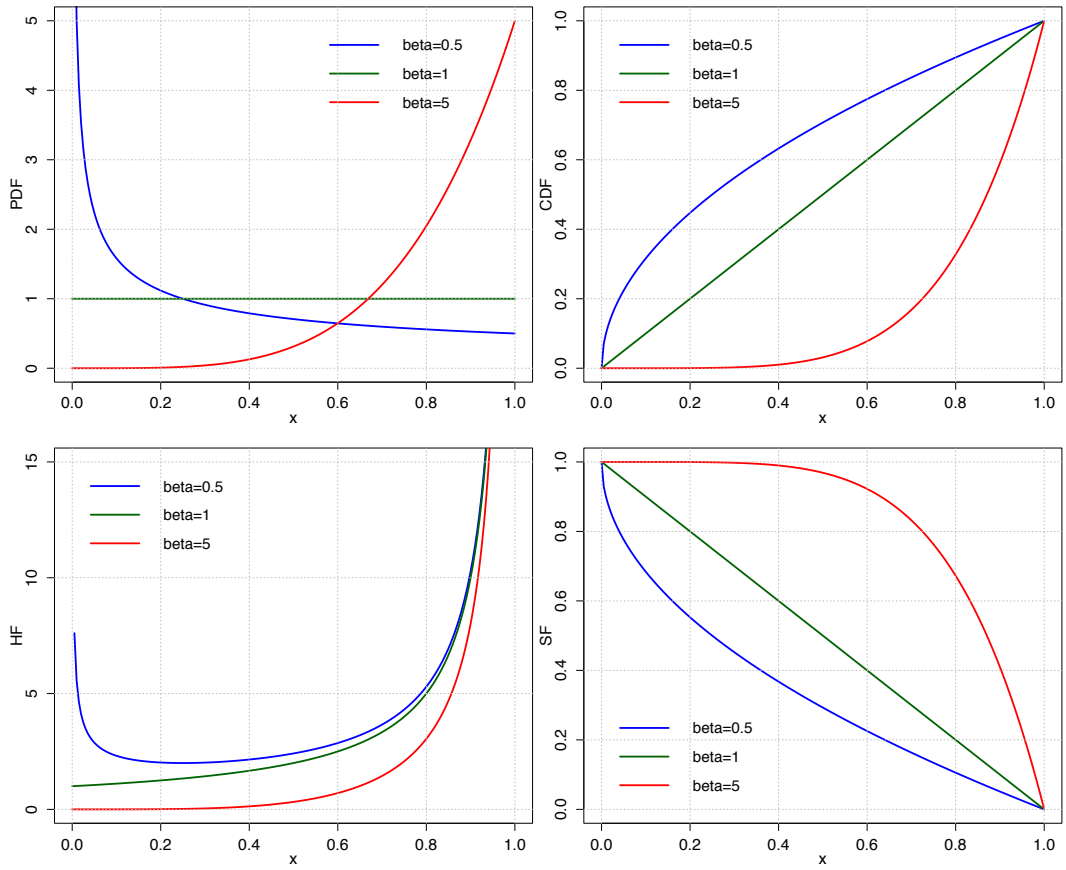

Figure A.111: StandardPower1 distribution plotted using the provided R code.

Model

StandardPower1 is a special case of power distribution, Power1.

Parameter: shape

name	shape parameter
type	scalar
symbol	β
definition	$\beta > 0$

Functions

PDF

$$\beta x^{\beta-1}$$

PDF in R

```
(beta * x^(beta-1))
```

CDF

$$x^{\beta}$$

CDF in R

```
x^beta
```

HF

$$\frac{\beta x^{\beta-1}}{1-x^{\beta}}$$

HF in R

```
beta * x^(beta-1) / (1 - x^beta)
```

SF

$$1-x^{\beta}$$

SF in R

```
1- x^beta
```

Characteristics

Mean

$$\beta/(\beta+1)$$

Median

$$(1/2)^{1/\beta}$$

Variance

$$\frac{\beta}{(2+\beta)(1+\beta)^2}$$

Relationships

- Relationship pair: $StandardPower1(\beta) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: StandardUniform1 distribution is a special case of the StandardPower1(β) distribution when $\beta = 1$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandardpowerStandarduniform.pdf`
- Relationship pair: $Power1(\alpha, \beta) \rightarrow StandardPower1(\beta)$
- Relationship type: Special case
- Relationship definition: The StandardPower1(β) distribution is a special case of the Power1(α, β) distribution when $\alpha = 1$.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/PowerStandardpower.pdf`
- Relationship pair: $StandardUniform1 \rightarrow StandardPower1(\beta)$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ StandardUniform1, then $Y = X^{1/\beta}$ has the StandardPower(β) distribution, where $\beta > 1$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformStandardpowerB.pdf`

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Power.pdf`

UncertML mapping:

no UncertML mapping found.

StandardTriangular1

name	Standard Triangular 1 (ID: 0001370)
type	continuous
variate	x, scalar
support	$0 \leq x \leq 1$

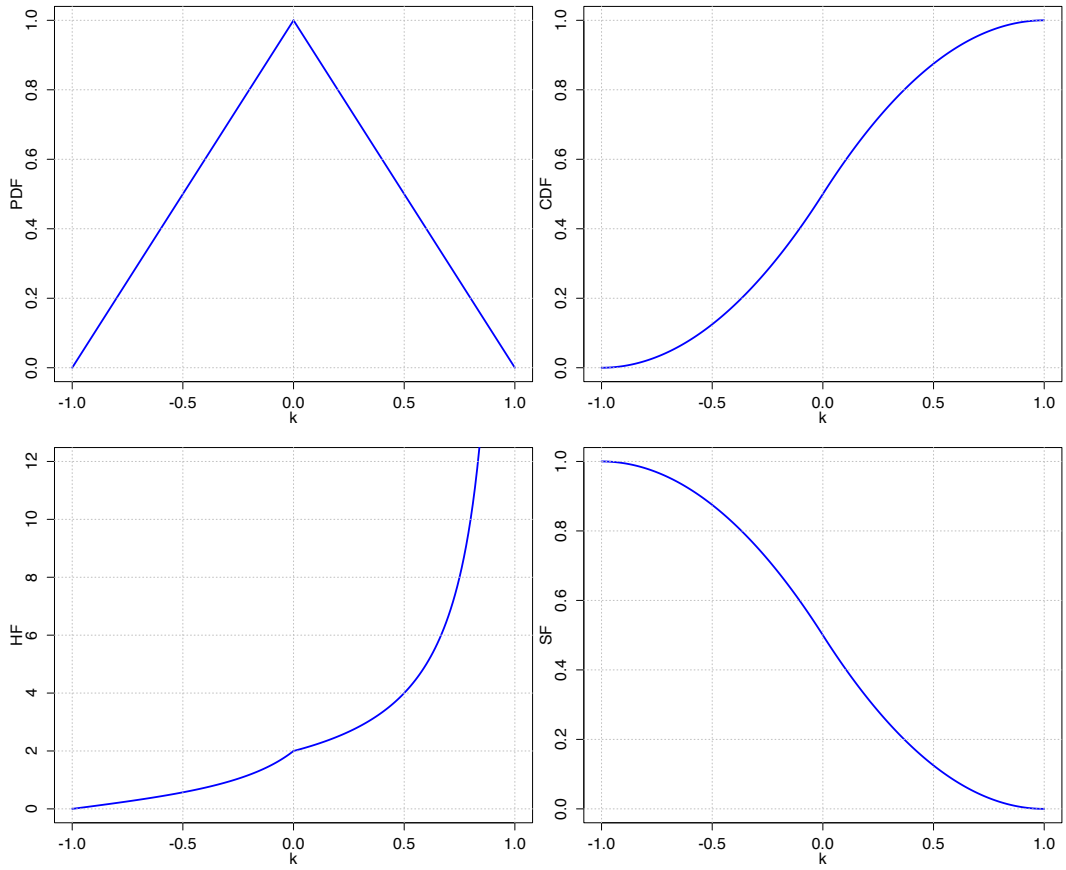

Figure A.112: StandardTriangular1 distribution plotted using the provided R code.

Model

Functions

PDF

$$\begin{cases} x+1 & \text{for } -1 < x < 0 \\ 1-x & \text{for } 0 \leq x < 1 \end{cases}$$

PDF in R

```
x+1 for -1 < x < 0 \\
1-x for 0 <= x < 1
```

CDF

$$\begin{cases} \frac{1}{2}x^2 + x + \frac{1}{2} & \text{for } -1 < x < 0 \\ -\frac{1}{2}x^2 + x + \frac{1}{2} & \text{for } 0 \leq x < 1 \end{cases}$$

CDF in R

```
CDF1=1/2*x^2+x+1/2
CDF2=-1/2*x^2+x+1/2
```

HF

$$\begin{cases} \frac{2(x+1)}{-x^2-2x+1} & \text{for } -1 < x < 0 \\ \frac{2}{1-x} & \text{for } 0 \leq x < 1 \end{cases}$$

HF in R

```
HF1=(2*(x+1))/(-x^2-2*x+1) for -1 < x < 0
HF2=2/(1-x) for 0 \leq x < 1
```

SF

$$\begin{cases} -\frac{1}{2}x^2 - x + \frac{1}{2} & \text{for } -1 < x < 0 \\ \frac{1}{2}x^2 - x + \frac{1}{2} & \text{for } 0 \leq x < 1 \end{cases}$$

SF in R

```
SF1=-1/2*x^2-x+1/2
SF2=1/2*x^2-x+1/2
```

Characteristics

Mean

$$0$$

Median

$$0$$

Mode

$$0$$

Variance

$$42375$$

Relationships

- Relationship pair: $Triangular1(a, b, c) \rightarrow StandardTriangular1$
- Relationship type: Special case
- Relationship definition: $StandardTriangular1$ distribution is a special case of the $Triangular1(a, b, c)$ distribution when $a = -1, b = 1, c = 0$.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TriangularStandardtriangular.pdf`

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Standardtriangular.pdf`

UncertML mapping:

no UncertML mapping found.

StandardTwoSidedPower1

name	Standard Two-Sided Power 1 (ID: 0001379)
type	continuous
variate	x, scalar
support	$x \in (0, 1)$

Model

Parameter: location

name	location
type	scalar
symbol	m
definition	$a < m < b$

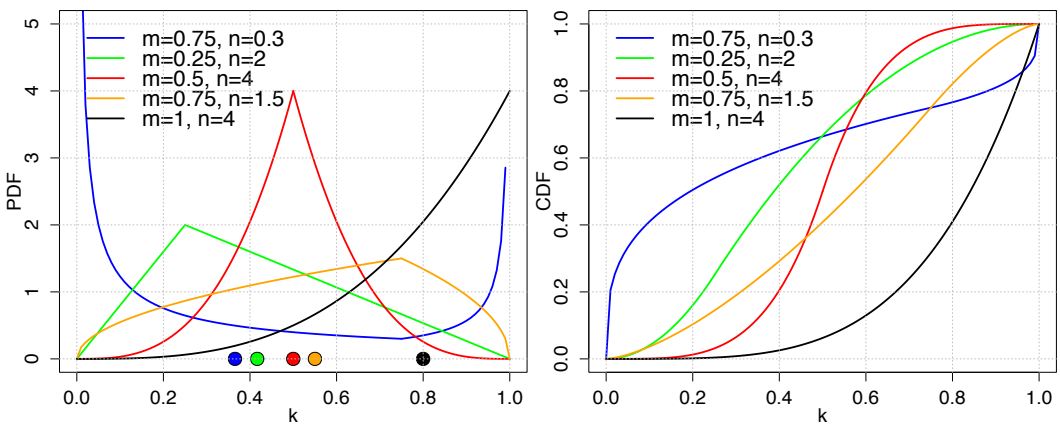

Figure A.113: StandardTwoSidedPower1 distribution plotted using the provided R code.

Parameter: shape

name	shape
type	scalar
symbol	n
definition	$n > 0$

Functions

PDF

$$\begin{cases} n\left(\frac{x}{m}\right)^{n-1} & \text{for } 0 < x < m \\ n\left(\frac{1-x}{1-m}\right)^{n-1} & \text{for } m \leq x < 1 \end{cases}$$

PDF in R

```
PDF1 = function(x,m,n) { n*(x/m)^(n-1) }
PDF2 = function(x,m,n) { n*((1-x)/(1-m))^(n-1) }
```

CDF

$$\begin{cases} x^n m^{1-n} & \text{for } 0 < x < m \\ 1-(1-m)((1-x)/(1-m))^n & \text{for } m \leq x < 1 \end{cases}$$

CDF in R

```
CDF1 = function(x,m,n) { x^n * m^(1-n) }
CDF2 = function(x,m,n) { 1-(1-m)*((1-x)/(1-m))^n }
```

Characteristics

Mean

$$\frac{(n-1)m+1}{n+1}$$

Variance

$$\frac{n-2(n-1)m(1-m)}{(n+2)(n+1)^2}$$

Relationships

- Relationship pair: $StandardTwoSidedPower1(m,n) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: $n = 1$
- Relationship reference(s): -
- Relationship pair: $TwoSidedPower1(a,b,m,n) \rightarrow StandardTwoSidedPower1(m,n)$
- Relationship type: Special case
- Relationship definition: $a = 0, b = 1$
- Relationship reference(s): [Van Dorp and Kotz, 2002]

References and sources

[Van Dorp and Kotz, 2002]

UncertML mapping:

no UncertML mapping found.

StandardUniform1

name	Standard Uniform 1 (ID: 0000587)
type	continuous
variate	x, scalar
support	$x \in [0, 1]$

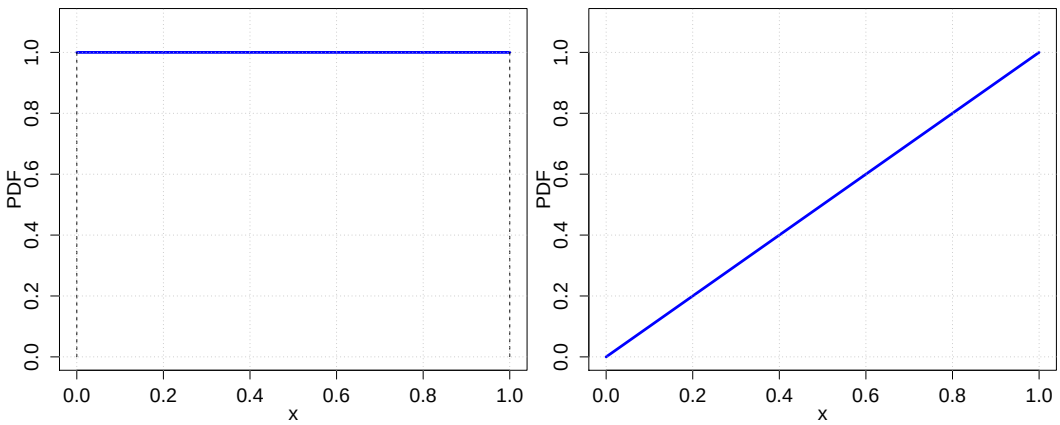

Figure A.114: StandardUniform1 distribution plotted using the provided R code.

Model

Parameter: minimum

name	minimum
type	scalar
symbol	a
definition	$a = 0$

Parameter: maximum

name	maximum
type	scalar
symbol	b
definition	$b = 1$

Functions

PDF

$$1$$

PDF in R

```
1
```

CDF

$$x$$

CDF in R

```
x
```

SF

$$1 - x$$

SF in R

```
1-x
```

Characteristics

Mean

$$0.5$$

Median

$$0.5$$

Mode

$$\text{any value in } [0, 1]$$

Relationships

- Relationship pair: $StandardUniform1(0,1) \rightarrow Exponential1(\lambda)$
- Relationship type: Transformation
- Relationship definition: $-\frac{1}{\lambda}\log(X)$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformExponentialB.pdf`

- Relationship pair: $StandardUniform1(0,1) \rightarrow ParetoTypeI1(x_m, \alpha)$
- Relationship type: Transformation
- Relationship definition: $x_m X^{-1/\alpha}$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformPareto.pdf`

- Relationship pair: $StandardUniform1 \rightarrow StandardPower1(\beta)$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ StandardUniform1, then $Y = X^{1/\beta}$ has the StandardPower(β) distribution, where $\beta > 1$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformStandardpowerB.pdf`

- Relationship pair: $StandardUniform1 \rightarrow Benford1$
- Relationship type: Transformation
- Relationship definition: If $X \sim$ StandardUniform1, then $Y = \lfloor 10^X \rfloor$ has the Benford1 distribution
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformBenford.pdf`

- Relationship pair: $StandardUniform1(0,1) \rightarrow Beta1(alpha, beta)$
- Relationship type: Transformation
- Relationship definition: $aX_1, X_2, ..., X_n(iid) \sim StandardNormal1 \Rightarrow$ with $\alpha = r$ and $\beta = n-r+1, X_{(r)} \sim Beta(\alpha, \beta)$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformBeta.pdf`

- Relationship pair: $StandardUniform1(0,1) \rightarrow Uniform1(a, b)$
- Relationship type: Transformation
- Relationship definition: $X \sim StandardNormal1$ and $Y = a + (b-a)X \Rightarrow Y \sim Uniform1$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformUniform.pdf`

- Relationship pair: $StandardUniform1(0,1) \rightarrow Frechet2(\alpha, \sigma, m)$
- Relationship type: Transformation
- Relationship definition: If $X \sim StandardUniform1(0,1)$ then $m + \sigma(-\log(X))^{-1/\alpha} \sim Frechet2(\alpha, \sigma, m)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution`

- Relationship pair: $StandardUniform1 \rightarrow LogLogistic2(\lambda, \kappa)$
- Relationship type: Transformation
- Relationship definition: If $X \sim StandardUniform1$ and $Y = \frac{1}{\lambda}\left(\frac{1-X}{X}\right)^{1/\kappa} \Rightarrow Y \sim LogLogistic2(\lambda, \kappa)$

- Relationship reference(s): NA

- Relationship pair: $Kumaraswamy1(a, b) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: If $X \sim Kumaraswamy1(1, 1)$ then $X \sim StandardUniform1$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Kumaraswamy_distribution`

- Relationship pair: $StandardPower1(\beta) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: StandardUniform1 distribution is a special case of the StandardPower1(β) distribution when $\beta = 1$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandardpowerStandarduniform.pdf`

- Relationship pair: $Uniform1(a, b) \rightarrow StandardUniform1(0, 1)$
- Relationship type: Special case
- Relationship definition: $a = 0, b = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008] `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/UniformStandarduniform.pdf`

- Relationship pair: $StandardTwoSidedPower1(m, n) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: $n = 1$
- Relationship reference(s): -

- Relationship pair: $Beta1(\alpha, \beta) \rightarrow StandardUniform1$
- Relationship type: Special case
- Relationship definition: $\alpha = 1, \beta = 1$
- Relationship reference(s): [Forbes et al., 2011]

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Standarduniform.pdf`

UncertML mapping:

no UncertML mapping found.

StudentT1

name	Student's t-distribution 1 (ID: 0000613)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

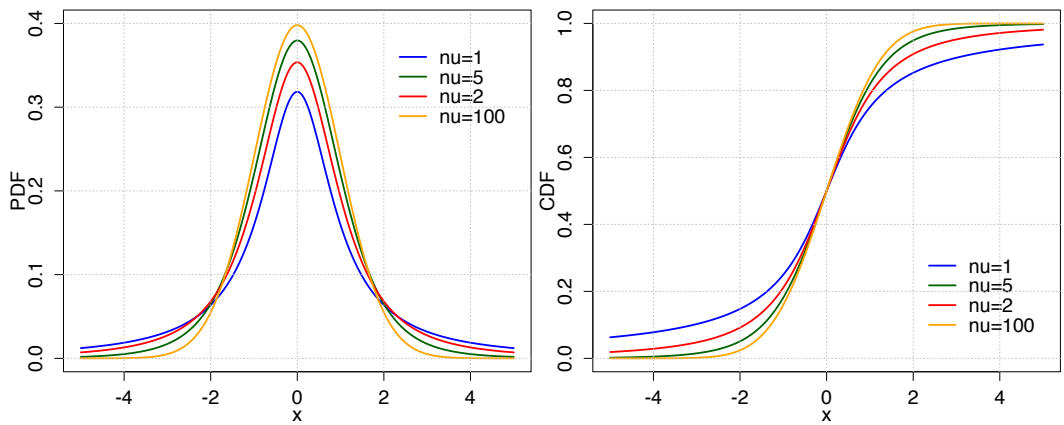

Figure A.115: StudentT1 distribution plotted using the provided R code.

Model

The Student's t distribution is used to test whether the difference between the means of two samples of observations is statistically significant. For example, the heights of a random sample of basketball players could be compared with the heights from a random sample of football players.

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	ν
definition	$\nu > 0, \nu \in R$

Functions

PDF

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

PDF in R

```
gamma((nu+1)/2)/(sqrt(nu*pi)*gamma(nu/2))*(1+x^2/nu)^(-(nu+1)/2)
```

CDF

$$\frac{1}{2}+x\Gamma\left(\frac{\nu+1}{2}\right)\times\frac{{}_2F_1\left(\frac{1}{2},\frac{\nu+1}{2};\frac{3}{2};-\frac{x^2}{\nu}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}$$

CDF in R

```
1/2+x*gamma((nu+1)/2)*hypergeo(1/2,(nu+1)/2,3/2,-x^2/nu)/( sqrt(pi*nu) *gamma(nu/2))
```

Characteristics

Mean

$$\begin{cases} 0 & \text{for } \nu > 1 \\ undefined & \text{else} \end{cases}$$

Median

$$0$$

Mode

$$0$$

Variance

$$\begin{cases} \frac{\nu}{\nu-2} & \text{for } \nu > 2 \\ \infty & \text{for } 1 < \nu \leq 2 \\ undefined & \text{else} \end{cases}$$

Relationships

- Relationship pair: $StudentT1(\nu) \rightarrow StandardNormal1(0,1)$
- Relationship type: Limiting
- Relationship definition: $X \sim StudentT1(\nu) \Rightarrow StandardNormal1(0,1)$ as ν tends to infinity.
The approximation is reasonable for $\nu \geq 30$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $StudentT1(\nu) \rightarrow F1(n_1, n_2)$
- Relationship type: Transformation
- Relationship definition: If $X \sim StudentT1(\nu) \Rightarrow Y = X^2 \sim F(1,\nu)$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TF.pdf`

- Relationship pair: $StudentT2(\mu,\tau,k) \rightarrow StudentT1(\nu)$
- Relationship type: Reparameterisation

- Relationship definition: $\mu = 0, \tau = 1$
- Relationship reference(s): NA
- Relationship pair: $NoncentralT1(\delta, n) \rightarrow StudentT1(n)$
- Relationship type: Special case
- Relationship definition: $\delta = 0$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/NoncentraltT.pdf`
- Relationship pair: $StudentT3(\nu, \mu, \sigma) \rightarrow StudentT1(\nu)$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \sigma = 1$
- Relationship reference(s): ProbOnto spec

References and sources

`http://en.wikipedia.org/wiki/Student's_t-distribution`
`http://nl.mathworks.com/help/stats/students-t-distribution.html`

UncertML mapping:

`http://www.uncertml.org/distributions/student-t`

StudentT2

name	Student's t-distribution 2 (ID: 0000635)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

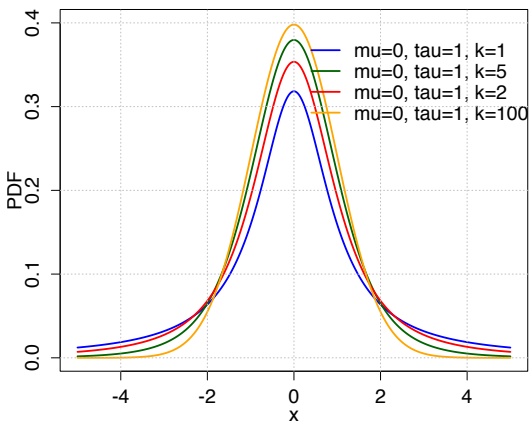

Figure A.116: StudentT2 distribution plotted using the provided R code.

Model

The Student's t distribution is used to test whether the difference between the means of two samples of observations is statistically significant. For example, the heights of a random sample of basketball players could be compared with the heights from a random sample of football players.

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	τ
definition	$\tau > 0$

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	k
definition	$k \geq 2$

Functions

PDF

$$\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\sqrt{\frac{\tau}{k\pi}}\left[1+\frac{\tau}{k}(x-\mu)^2\right]^{-\frac{k+1}{2}}$$

PDF in R

```
gamma((k+1)/2)/gamma(k/2)*sqrt(tau/(k*pi))*(1+tau/k*(x-mu)^2)^(-(k+1)/2)
```

Characteristics

Mean

$$\mu$$

Mode

$$\mu \text{ for } k > 1$$

Variance

$$\frac{1}{\tau}\frac{k}{k-2} \text{ for } k > 2$$

Relationships

- Relationship pair: $StudentT2(\mu, \tau, k) \rightarrow StudentT1(\nu)$
- Relationship type: Reparameterisation
- Relationship definition: $\mu = 0, \tau = 1$
- Relationship reference(s): NA

- Relationship pair: $StudentT2(\mu, \tau, k) \rightarrow StudentT3(\nu, \mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\sigma = 1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $StudentT3(\nu, \mu, \sigma) \rightarrow StudentT2(\mu, \tau, k)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/\sigma^2$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

StudentT3

name	Student's t-distribution 3 (ID: 0001091)
type	continuous
variate	x, scalar
support	$x \in (-\infty, +\infty)$

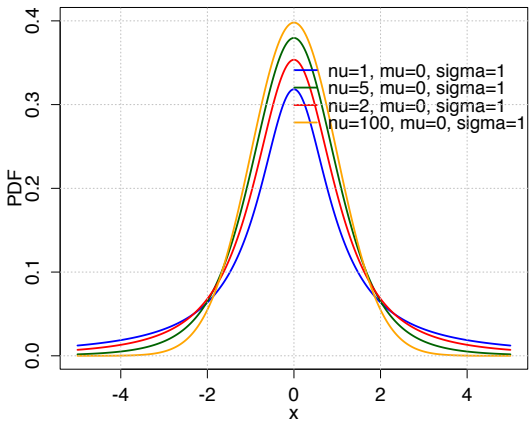

Figure A.117: StudentT3 distribution plotted using the provided R code.

Model

The Student's t distribution is used to test whether the difference between the means of two samples of observations is statistically significant. For example, the heights of a random sample of basketball players could be compared with the heights from a random sample of football players.

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	ν
definition	$\nu > 0, \nu \in R$

Parameter: location

name	mean
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: scale

name	scale
type	scalar
symbol	σ
definition	$\mu \in R^+$

Functions

PDF

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}\sigma} \left(1 + \frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-\frac{\nu+1}{2}}$$

PDF in R

```
gamma((nu+1)/2)/gamma(nu/2)*1/sqrt(nu*pi)/sigma*(1+1/nu*((x-mu)/sigma)^2)^(-(nu+1)/2)
```

Characteristics

Relationships

- Relationship pair: $StudentT3(\nu, \mu, \sigma) \rightarrow StudentT1(\nu)$
- Relationship type: Special case
- Relationship definition: $\mu = 0, \sigma = 1$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $StudentT3(\nu, \mu, \sigma) \rightarrow StudentT2(\mu, \tau, k)$
- Relationship type: Reparameterisation
- Relationship definition: $\tau = 1/\sigma^2$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $StudentT2(\mu, \tau, k) \rightarrow StudentT3(\nu, \mu, \sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $\sigma = 1/\sqrt{\tau}$
- Relationship reference(s): ProbOnto spec

References and sources

[STAN Development Team, 2015b]
`https://en.wikipedia.org/wiki/Student%27s_t-distribution#Non-standardized_Student.27s_t-distribution`
`http://nl.mathworks.com/help/stats/t-location-scale-distribution.html`

UncertML mapping:

no UncertML mapping found.

Trapezoidal1

name	Trapezoidal 1 (ID: 0001044)
type	continuous
variate	x, scalar
support	$x \in [a, d]$

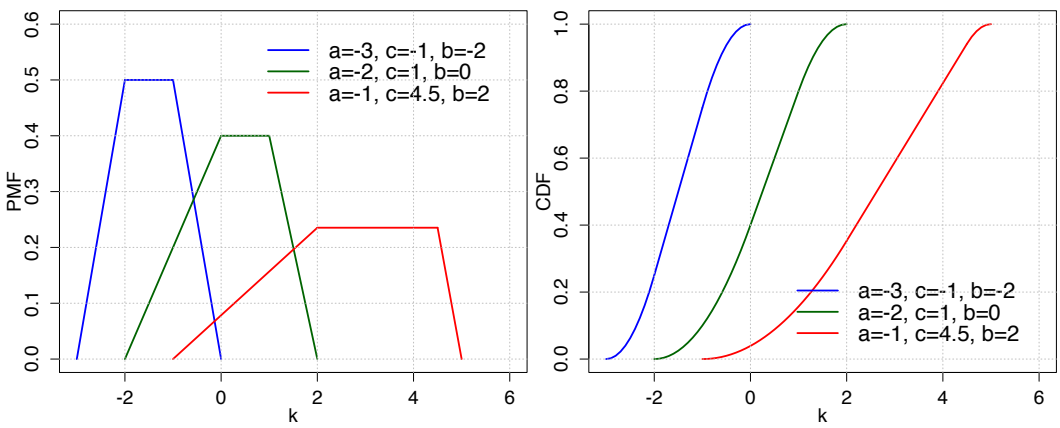

Figure A.118: Trapezoidal1 distribution plotted using the provided R code.

Model

Trapezoidal distributions may be appropriate for modeling processes that can be represented by the following three stages: 1 - growth stage, 2 - period of relative stability stage, 3 - decay stage. The trapezoidal model is probably the simplest model of this type (i.e., the growth and decline stages are linear and the stability stage is constant). The generalized trapezoidal distribution allows more flexibility in modeling these stages.

Parameter: lowerBound

name	Lower bound
type	scalar
symbol	a
definition	$a < d$

Parameter: levelStart

name	Level start
type	scalar
symbol	b
definition	$a \leq b < c$

Parameter: levelEnd

name	Level end
type	scalar
symbol	c
definition	$b < c \leq d$

Parameter: upperBound

name	Upper bound
type	scalar
symbol	d
definition	$c \leq d$

Functions

PDF

$$\begin{cases} \dfrac{2}{d+c-a-b}\dfrac{x-a}{b-a} & a \leq x < b \\ \dfrac{2}{d+c-a-b} & b \leq x < c \\ \dfrac{2}{d+c-a-b}\dfrac{d-x}{d-c} & c \leq x \leq d \end{cases}$$

PDF in R

```
PMF1 = 2/(d+c-a-b)*(x-a)/(b-a)
PMF2 = 2/(d+c-a-b)
PMF3 = 2/(d+c-a-b)*(d-x)/(d-c)
```

CDF

$$\begin{cases} \dfrac{1}{d+c-a-b}\dfrac{1}{b-a}(x-a)^2 & a \leq x < b \\ \dfrac{1}{d+c-a-b}(2x-a-b) & b \leq x < c \\ 1-\dfrac{1}{d+c-a-b}\dfrac{1}{d-c}(d-x)^2 & c \leq x \leq d \end{cases}$$

CDF in R

```
CDF1 = 1/(d+c-a-b)*1/(b-a)*(x-a)^2
CDF2 = 1/(d+c-a-b)*(2*x-a-b)
CDF3 = 1-1/(d+c-a-b)*1/(d-c)*(d-x)^2
```

Characteristics

Mean

$$\frac{2}{6(d+c-a-b)}\left(d^2+cd+c^2-b^2-ab-a^2\right)$$

Variance

$$\frac{2^2}{72(d+c-a-b)^2}\big(d^4+2cd^3-3bd^3-3ad^3-3bcd^2-3acd^2+4b^2d^2+4abd^2 \\ +4a^2d^2+2c^3d-3bc^2d-3ac^2d+4b^2cd+4abcd+4a^2cd-3b^3d-3ab^2d \\ -3a^2bd-3a^3d+c^4-3bc^3-3ac^3+4b^2c^2+4abc^2+4a^2c^2-3b^3c-3ab^2c-3a^2bc \\ -3a^3c+b^4+2ab^3+2a^3b+a4\big)$$

Relationships

- Relationship pair: $Trapezoidal1(a,b,c,d) \rightarrow Triangular1(a,b,c)$
- Relationship type: Special case
- Relationship definition: $b=c$
- Relationship reference(s): [Garvey, 2000]

References and sources

`http://doc.openturns.org/openturns-1.6/sphinx/user_manual/_generated/openturns.Trapezoidal.html#openturns.Trapezoidal`
[Garvey, 2000]
`http://www.itl.nist.gov/div898/software/dataplot/refman2/auxillar/trapdf.htm`

UncertML mapping:

no UncertML mapping found.

Triangular1

name	Triangular 1 (ID: 0000661)
type	continuous
variate	x, scalar
support	$a \leq x \leq b$

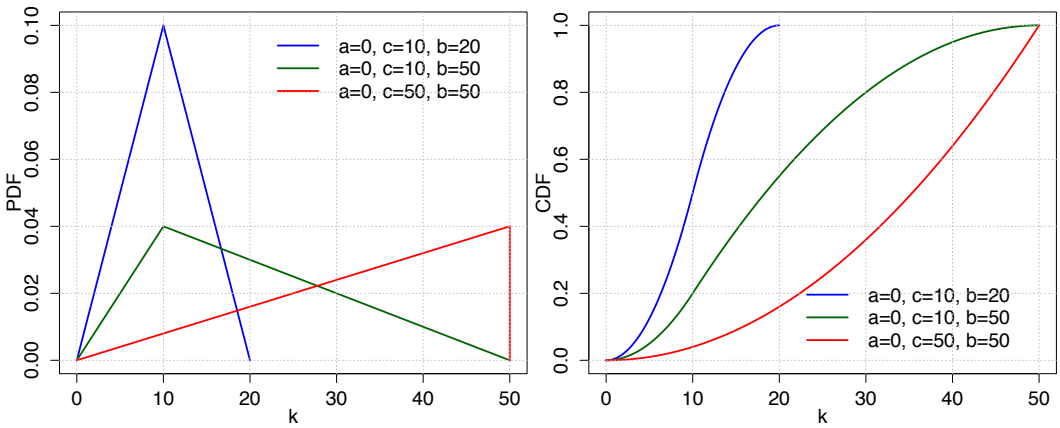

Figure A.119: Triangular1 distribution plotted using the provided R code.

Model

The triangular distribution is typically used as a subjective description of a population for which there is only limited sample data, and especially in cases where the relationship between variables is known but data is scarce (possibly because of the high cost of collection). It is based on a knowledge of the minimum and maximum and an inspired guess [2] as to the modal value. For these reasons, the triangle distribution has been called a lack of knowledge distribution.

Parameter: lowerLimit

name	lower limit
type	scalar
symbol	a
definition	$a \in R$

Parameter: upperLimit

name	upper limit
type	scalar
symbol	b
definition	$b \in R, a < b$

Parameter: shape

name	shape (mode)
type	scalar
symbol	c
definition	$c \in R$

Functions

PDF

$$\begin{cases} 2(x-a)/[(b-a)(c-a)] & \text{for } a \leq x \leq c \\ 2(b-x)/[(b-a)(b-c)] & \text{for } c \leq x \leq b \end{cases}$$

PDF in R

```
2*(x-a) / ((b-a)*(c-a)) for a <= x <= c \\
2*(b-x) / ((b-a)*(b-c)) for c <= x <= b
```

CDF

$$\begin{cases} (x-a)^2/[(b-a)(c-a)] & \text{for } a \leq x \leq c \\ 1-(b-x)^2/[(b-a)(b-c)] & \text{for } c \leq x \leq b \end{cases}$$

CDF in R

```
(x-a)^2 / ((b-a)*(c-a)) for a <= x <= c \\
1 - (b-x)^2 / ((b-a)*(b-c)) for c <= x <= b
```

Characteristics

Mean

$$(a+b+c)/3$$

Mode

$$c$$

Variance

$$(a^2+b^2+c^2-ab-ac-bc)/18$$

Relationships

- Relationship pair: $Triangular1(a,b,c) \rightarrow StandardTriangular1$
- Relationship type: Special case
- Relationship definition: $StandardTriangular1$ distribution is a special case of the $Triangular1(a,b,c)$ distribution when $a=-1, b=1, c=0$.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TriangularStandardtriangular.pdf`
- Relationship pair: $Trapezoidal1(a,b,c,d) \rightarrow Triangular1(a,b,c)$
- Relationship type: Special case
- Relationship definition: $b=c$

- Relationship reference(s): [Garvey, 2000]
- Relationship pair: $TwoSidedPower1(a, b, m, n) \rightarrow Triangular1(a, b, m)$
- Relationship type: Special case
- Relationship definition: $n = 2$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TSPTriangular.pdf`

References and sources

[Forbes et al., 2011]

UncertML mapping:

no UncertML mapping found.

TruncatedNormal1

name	Truncated Normal 1 (ID: 0000681)
type	continuous
variate	x, scalar
support	$x \in [a, b]$

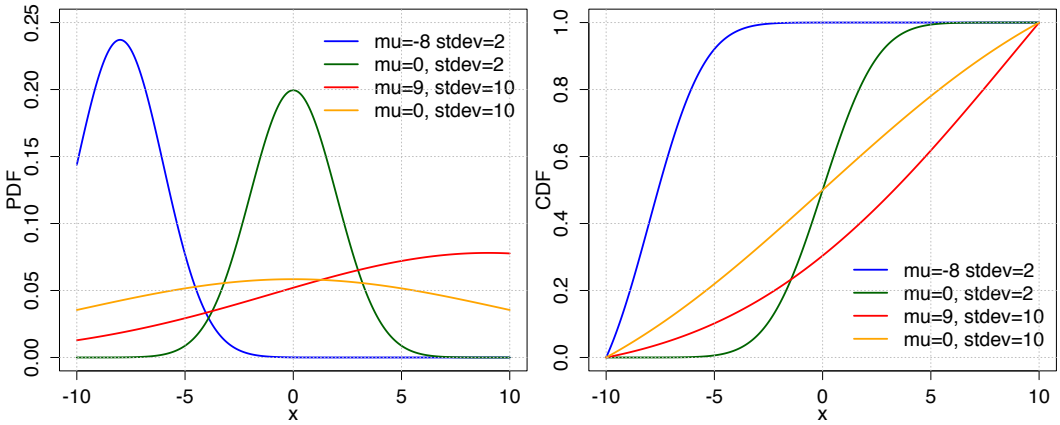

Figure A.120: TruncatedNormal1 distribution plotted using the provided R code.

Model

In probability and statistics, the truncated normal distribution is the probability distribution of a normally distributed random variable whose value is either bounded below or above (or both).

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: stdev

name	standard deviation
type	scalar
symbol	σ
definition	$\sigma > 0$

Parameter: lowerBound

name	lower bound
type	scalar
symbol	a
definition	$a \in R$

Parameter: upperBound

name	upper bound
type	scalar
symbol	b
definition	$b \in R, b > a$

Functions

PDF

$$\frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}$$

PDF in R

```
( 1/sigma * phi((x-mu)/sigma) ) / ( Phi((b-mu)/sigma)-Phi((a-mu)/sigma) )

phi = function(x) {  1/(sqrt(2*pi))*exp(-x^2/2) }
Phi = function(x) {  1/2 * (1 + erf(x/(sqrt(2)))) }
erf = function(x) {  2 * pnorm(x * sqrt(2)) - 1 }
```

CDF

$$\frac{\Phi(\frac{x-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}$$

CDF in R

```
( Phi((x-mu)/sigma)-Phi((a-mu)/sigma) ) / ( Phi((b-mu)/sigma)-Phi((a-mu)/sigma) )

Phi = function(x) {  1/2 * (1 + erf(x/(sqrt(2)))) }
erf = function(x) {  2 * pnorm(x * sqrt(2)) - 1 }
```

Characteristics

Mean

$$\mu + \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\sigma$$

Variance

$$\sigma^2\Big[1 + \frac{\frac{a-\mu}{\sigma}\phi(\frac{a-\mu}{\sigma}) - \frac{b-\mu}{\sigma}\phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \Big(\frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\Big)^2\Big]$$

Relationships

- Relationship pair: $TruncatedNormal1(\mu, \sigma, a, b) \rightarrow Normal1(\mu, \sigma)$
- Relationship type: Special case
- Relationship definition: $a = -\infty, b = \infty$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $TruncatedNormal1(\mu, \sigma, a, b) \rightarrow HalfNormal1(\theta)$
- Relationship type: Special case
- Relationship definition: $\mu = 0, a = 0, b = \infty$
- Relationship reference(s): `http://reference.wolfram.com/language/ref/HalfNormalDistribution.html`
[Forbes et al., 2011]

- Relationship pair: $PowerNormal1(\lambda, \mu, \sigma) \rightarrow TruncatedNormal1(\mu, \sigma, a, b)$

- Relationship type: Special case
- Relationship definition: $PowerNormal1$ converges to $TruncatedNormal1$ when $\lambda \to 1$
- Relationship reference(s): [Lavielle, 2014]

References and sources

`https://en.wikipedia.org/wiki/Truncated_normal_distribution`
[Forbes et al., 2011]

UncertML mapping:

no UncertML mapping found.

TwoSidedPower1

name	Two-Sided Power 1 (ID: 0001151)
type	continuous
variate	x, scalar
support	$x \in (a, b)$

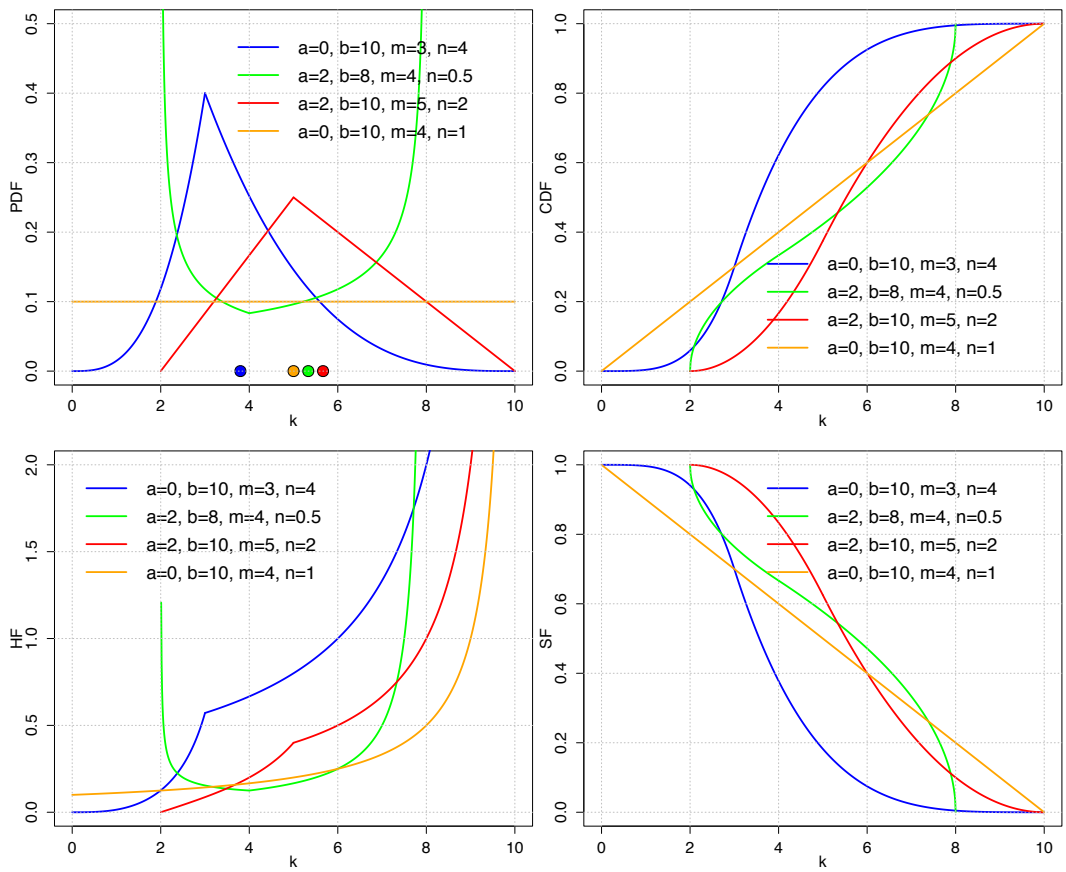

Figure A.121: TwoSidedPower1 distribution plotted using the provided R code.

Model

Parameter: lowerBound

name	Lower bound
type	scalar
symbol	a
definition	$a \in R$

Parameter: upperBound

name	upper bound
type	scalar
symbol	b
definition	$b \in R, b > m$

Parameter: location

name	location
type	scalar
symbol	m
definition	$a < m < b$

Parameter: shape

name	shape
type	scalar
symbol	n
definition	$n > 0$

Functions

PDF

$$\begin{cases} \frac{n(x-a)^{n-1}}{(b-a)(m-a)^{n-1}} & \text{for } a < x < m \\ \frac{n(b-x)^{n-1}}{(b-a)(b-m)^{n-1}} & \text{for } m \leq x < b \end{cases}$$

PDF in R

```
PDF1 = function(x,a,b,m,n) { (n*(x-a)^(n-1))/((b-a)*(m-a)^(n-1)) }
PDF2 = function(x,a,b,m,n) { (n*(b-x)^(n-1))/((b-a)*(b-m)^(n-1)) }
```

CDF

$$\begin{cases} \frac{(x-a)^n(m-a)^{1-n}}{b-a} & \text{for } a < x < m \\ -\frac{a+b(b-x)^n(b-m)^{-n}-b-m(b-x)^n(b-m)^{-n}}{b-a} & \text{for } m \leq x < b \end{cases}$$

CDF in R

```
CDF1 = function(x,a,b,m,n) { ((x-a)^n*(m-a)^(1-n))/(b-a) }
CDF2 = function(x,a,b,m,n) { -(a+b*(b-x)^n*(b-m)^(-n)-b-m*(b-x)^n*(b-m)^(-n))/(b-a) }
```

HF

$$\begin{cases} -\frac{n(x-a)^{n-1}(m-a)^{1-n}}{a-b+(m-a)(x-a)^n(m-a)^{-n}} & \text{for } a < x < m \\ \frac{n}{b-x} & \text{for } m \leq x < b \end{cases}$$

HF in R

```
HF1 = function(x,a,b,m,n) { -(n*(x-a)^(n-1)*(m-a)^(1-n))/(a-b+(m-a)*(x-a)^n*(m-a)^(-n)) }
HF2 = function(x,a,b,m,n) { n/(b-x) }
```

SF

$$\begin{cases} -\frac{-b+a+(x-a)^n(m-a)^{1-n}}{b-a} & \text{for } a < x < m \\ \frac{(b-x)^n(b-m)^{1-n}}{b-a} & \text{for } m \leq x < b \end{cases}$$

SF in R

```
SF1 = function(x,a,b,m,n) { -(-b+a+(x-a)^n*(m-a)^(1-n))/(b-a) }
SF2 = function(x,a,b,m,n) { ((b-x)^n*(b-m)^(1-n))/(b-a) }
```

Characteristics

Mean

$$\frac{b - m + mn + a}{n + 1}$$

Variance

$$\frac{-2mnb + 2m^2n + a^2n + b^2n - 2mna + 2bm - 2ba + 2am - 2m^2}{(n+1)^2(n+2)}$$

Relationships

- Relationship pair: $TwoSidedPower1(a, b, m, n) \rightarrow Uniform1(a, b)$
- Relationship type: Special case
- Relationship definition: $n = 1$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TSPUniform.pdf`
- Relationship pair: $TwoSidedPower1(a, b, m, n) \rightarrow Triangular1(a, b, m)$
- Relationship type: Special case
- Relationship definition: $n = 2$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TSPTriangular.pdf`
- Relationship pair: $TwoSidedPower1(a, b, m, n) \rightarrow StandardTwoSidedPower1(m, n)$
- Relationship type: Special case
- Relationship definition: $a = 0, b = 1$
- Relationship reference(s): [Van Dorp and Kotz, 2002]

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TSP.pdf`

UncertML mapping:

no UncertML mapping found.

Uniform1

name	Uniform 1 (ID: 0000703)
type	continuous
variate	x, scalar
support	$x \in [a, b]$

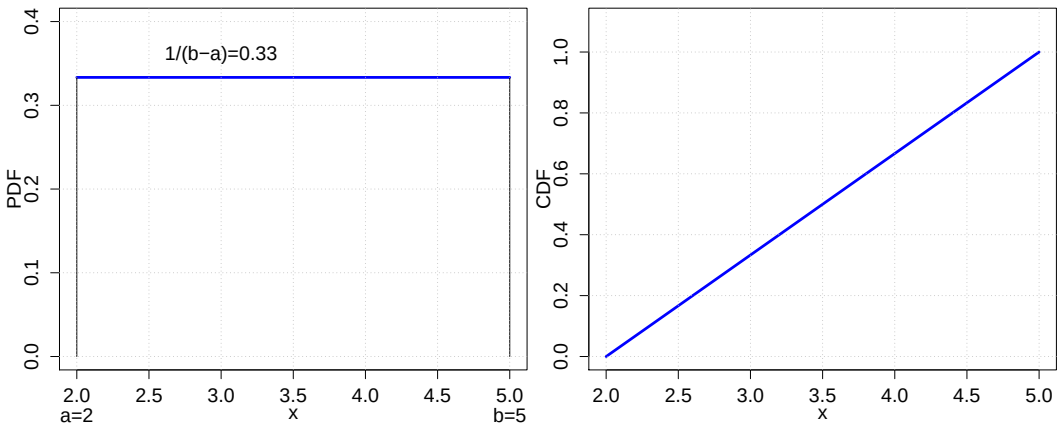

Figure A.122: Uniform1 distribution plotted using the provided R code.

Model

Every value in the range of the distribution is equally likely to occur. This is the distribution taken by uniform random numbers. It is widely used as the basis for the generation of random numbers for other statistical distributions.

Parameter: minimum

name	minimum
type	scalar
symbol	a
definition	$a \in R$

Parameter: maximum

name	maximum
type	scalar
symbol	b
definition	$b \in R, a < b$

Functions

PDF

$$\begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

PDF in R

```
1/(b-a)
```

CDF

$$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b) \\ 1 & \text{for } x \geq b \end{cases}$$

CDF in R

```
(x-a)/(b-a)
```

Characteristics

Mean

$$\frac{1}{2}(a+b)$$

Median

$$\frac{1}{2}(a+b)$$

Mode

$$\text{any value in } [a,b]$$

Variance

$$\frac{1}{12}(b-a)^2$$

Relationships

- Relationship pair: $Uniform1(a,b) \rightarrow StandardUniform1(0,1)$
- Relationship type: Special case
- Relationship definition: $a = 0, b = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/UniformStandarduniform.pdf`
- Relationship pair: $TwoSidedPower1(a,b,m,n) \rightarrow Uniform1(a,b)$
- Relationship type: Special case
- Relationship definition: $n = 1$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/TSPUniform.pdf`
- Relationship pair: $StandardUniform1(0,1) \rightarrow Uniform1(a,b)$
- Relationship type: Transformation
- Relationship definition: $X \sim StandardNormal1$ and $Y = a + (b-a)X \Rightarrow Y \sim Uniform1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/StandarduniformUniform.pdf`

References and sources

http://en.wikipedia.org/wiki/Uniform_distribution_(continuous)

UncertML mapping:

http://www.uncertml.org/distributions/uniform

UniformDiscrete1

name	Uniform Discrete 1 (ID: 0000727)
type	discrete
variate	k, scalar
support	$k \in \{a, a+1, ..., b-1, b\}$

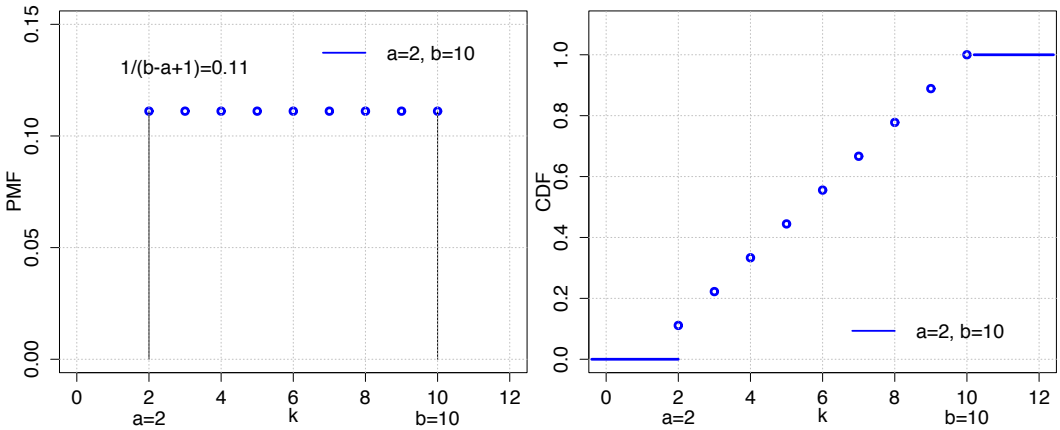

Figure A.123: UniformDiscrete1 distribution plotted using the provided R code.

Model

Parameter: minimum

name	minimum
type	scalar
symbol	a
definition	$a \in \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$

Parameter: maximum

name	maximum
type	scalar
symbol	b
definition	$b \in \{\dots, -2, -1, 0, 1, 2, 3, \dots\}, b \geq a$

Functions

PMF

$$1/(b-a+1)$$

PMF in R

```
1/(b-a+1)
```

CDF

$$(k-a+1)/(b-a+1)$$

CDF in R

```
(k-a+1)/(b-a+1)
```

Characteristics

Mean

$$(a+b)/2$$

Median

$$(a+b)/2$$

Variance

$$\frac{(b-a+1)^2-1}{12}$$

Relationships

- Relationship pair: $UniformDiscrete1(a,b) \rightarrow UniformDiscrete2(n)$
- Relationship type: Special case
- Relationship definition: $a=0, b=n$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/DiscreteuniformRectangular.pdf`
- Relationship pair: $Zipf1(\alpha,n) \rightarrow UniformDiscrete1(a,b)$
- Relationship type: Special case
- Relationship definition: $\alpha=0, a=1, b=n$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ZipfDiscreteuniform.pdf`

References and sources

[Leemis and Mcqueston, 2008]
[Marichev and Trott, 2013]

UncertML mapping:

no UncertML mapping found.

UniformDiscrete2

name	Uniform Discrete 2 (ID: 0000750)
type	discrete
variate	k, scalar
support	$k \in \{0,1,2,\ldots,n\}$

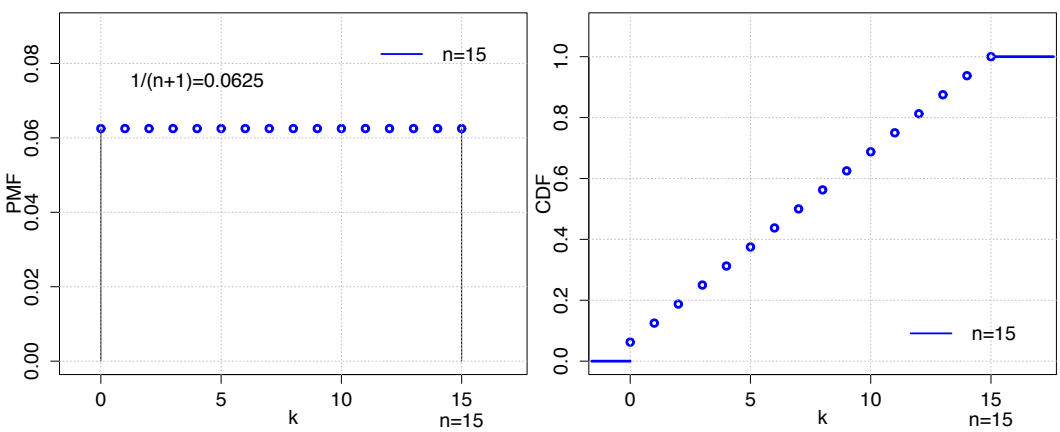

Figure A.124: UniformDiscrete2 distribution plotted using the provided R code.

Model

Parameter: numberOfValues

name	number of values
type	scalar
symbol	n
definition	$n \in N$

Functions

PMF

$$1/(n+1)$$

PMF in R

```
1/(n+1)
```

CDF

$$\frac{k+1}{n+1}$$

CDF in R

```
(k+1)/(n+1)
```

HF

$$\frac{1}{n-k}$$

HF in R

```
1/(n-k)
```

SF

$$\frac{n-k}{n+1}$$

SF in R

```
(n-k)/(n+1)
```

Characteristics

Mean

$$\frac{n}{2}$$

Variance

$$\frac{n(n+2)}{12}$$

Relationships

- Relationship pair: $UniformDiscrete1(a,b) \rightarrow UniformDiscrete2(n)$
- Relationship type: Special case
- Relationship definition: $a = 0, b = n$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/DiscreteuniformRectangular.pdf

- Relationship pair: $BetaBinomial1(n,\alpha,\beta) \rightarrow UniformDiscrete2(n)$
- Relationship type: Special case
- Relationship definition: $\alpha = 1, \beta = 1$
- Relationship reference(s): [Leemis and Mcqueston, 2008]
http://www.math.wm.edu/~leemis/chart/UDR/PDFs/BetabinomialRectangular.pdf

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Rectangular.pdf`
[Forbes et al., 2011]

UncertML mapping:

no UncertML mapping found.

VonMises1

name	Von Mises 1 (ID: 0000773)
type	continuous
variate	x, scalar
support	$x \in$ any interval of length 2π

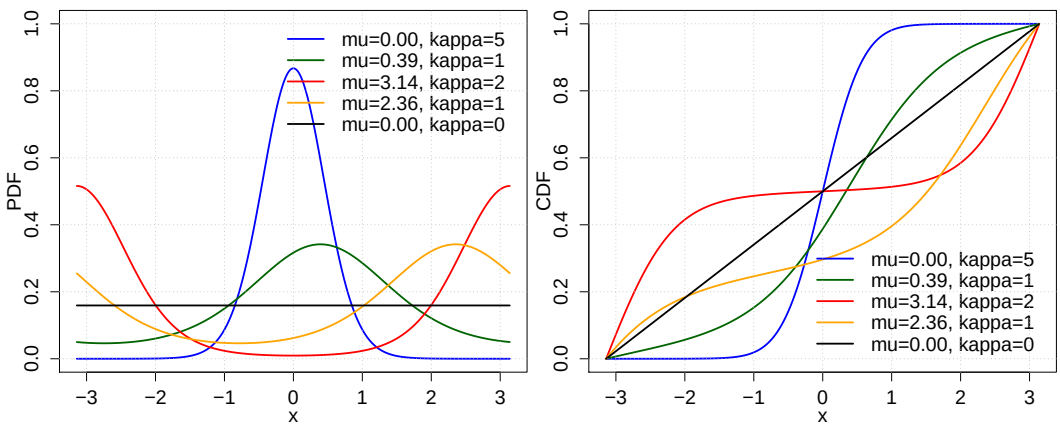

Figure A.125: VonMises1 distribution plotted using the provided R code.

Model

The von Mises distribution can be regarded as the circular analogue of the normal distribution on the line. The distribution is unimodal, symmetric about μ, and infinitely divisible.

Parameter: location

name	location
type	scalar
symbol	μ
definition	$\mu \in R$

Parameter: concentration

name	concentration
type	scalar
symbol	κ
definition	$\kappa \in R^+$

Functions

PDF

$$\frac{\exp(\kappa \cos(x-\mu))}{2\pi I_0(\kappa)}$$

PDF in R

```
exp(kappa * cos(x-mu)) / (2*pi * besselI(kappa, 0, expon.scaled = FALSE))
```

CDF

$$\frac{1}{2\pi I_0(\kappa)}\Big[xI_0(\kappa) + 2\sum_{j=0}^{\infty} I_j(\kappa)\frac{sin(j(x-mu))}{j}\Big]$$

CDF in R

```
1/(2*pi*besselI(kappa,0,expon.scaled=FALSE))
* ( x*besselI(kappa, 0, expon.scaled = FALSE) + 2*sumFromTo(x,mu,kappa,0,infty))
# with
sumFromTo = function(x,mu,kappa,fromValue,toValue) {
    sigma=0;
    for(i in fromValue:toValue) {
      sigma = sigma + besselI(kappa,i,expon.scaled=FALSE) * sin(i*(x-mu))/i
    }
  return(sigma)
}
```

Characteristics

Mean

$$\mu$$

Median

$$\mu$$

Mode

$$\mu$$

Variance

$$1 - \frac{I_1(\kappa)}{I_1(0)} \text{ (circular variance)}$$

Relationships

References and sources

https://en.wikipedia.org/wiki/Von_Mises_distribution
[STAN Development Team, 2015b]

UncertML mapping:

no UncertML mapping found.

Weibull1

name	Weibull 1 (ID: 0000800)
type	continuous
variate	x, scalar
support	$x \in [0, +\infty)$

Model

The Weibull distribution is used in reliability and survival analysis to model the lifetime of an object, the lifetime of a organism, or a service time. The accelerated life and Cox proportional hazards model are identical when the baseline distribution is Weibull.

Parameter: scale

name	scale
type	scalar
symbol	λ
definition	$\lambda \in (0, +\infty)$

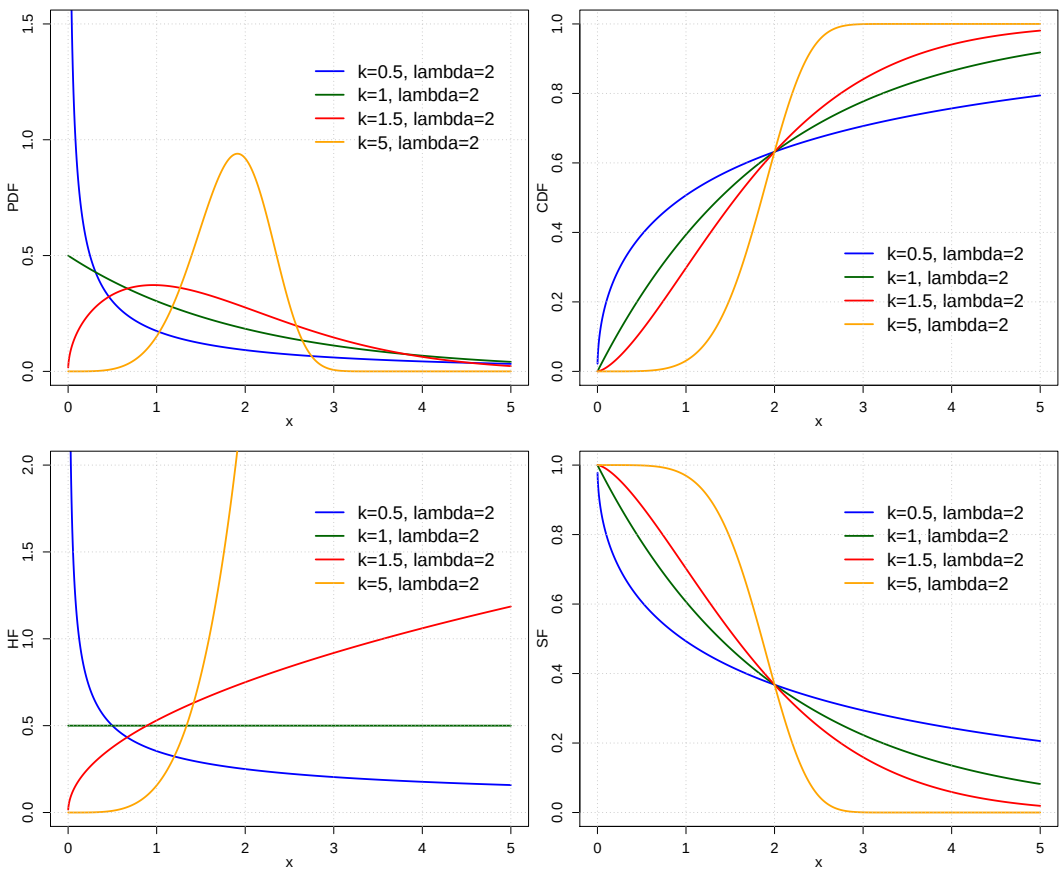

Figure A.126: Weibull1 distribution plotted using the provided R code.

Parameter: shape

name	shape
type	scalar
symbol	k
definition	$k \in (0, +\infty)$

Functions

PDF

$$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

PDF in R

```
k/lambda * (x/lambda)^(k-1) * exp(-(x/lambda)^k)
```

CDF

$$1 - \exp(-(x/\lambda)^k)$$

CDF in R

```
1- exp(-(x/lambda)^k)
```

HF

$$\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}$$

HF in R

```
k/lambda *(x/lambda)^(k-1)
```

SF

$$\exp(-(x/\lambda)^k)$$

SF in R

```
exp(-(x/lambda)^k)
```

Characteristics

Mean

$$\lambda\,\Gamma(1+1/k)$$

Median

$$\lambda(\log(2))^{1/k}$$

Mode

$$\begin{cases} \lambda\left(\frac{k-1}{k}\right)^{\frac{1}{k}} & k>1 \\ 0 & k=1 \end{cases}$$

Variance

$$\lambda^2\left[\Gamma\left(1+\frac{2}{k}\right)-\left(\Gamma\left(1+\frac{1}{k}\right)\right)^2\right]$$

Relationships

- Relationship pair: $Weibull1(\lambda,k) \to Weibull2(\lambda,v)$
- Relationship type: Reparameterisation
- Relationship definition: $v=k, \lambda=1/\lambda^k$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Weibull1(\lambda,k) \to Exponential1(\lambda_{Exponetial})$
- Relationship type: Reparameterisation
- Relationship definition: $k=1, \lambda_{Exponetial}=1/\lambda$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Weibull1(\lambda,k) \to Rayleigh1(\sigma)$
- Relationship type: Reparameterisation
- Relationship definition: $k=2, \lambda=\sqrt{2}\sigma$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Amoroso1(a,\theta,\alpha,\beta) \to Weibull1(\lambda,k)$
- Relationship type: Special case
- Relationship definition: Weibull1(λ,k) distribution is a special case of the Amoroso1(a,θ,α,β) distribution when $a=0, \theta=\lambda, \alpha=1$
- Relationship reference(s): [Crooks, 2010]

- Relationship pair: $GeneralizedGamma2(a,b,c,k) \to Weibull1(\lambda,k)$
- Relationship type: Reparameterisation
- Relationship definition: $c=1, a=0, b=\lambda$
- Relationship reference(s): [Forbes et al., 2011]

- Relationship pair: $Weibull2(\lambda,v) \to Weibull1(\lambda,k)$
- Relationship type: Reparameterisation
- Relationship definition: $k=v, \lambda=\lambda^{-1/v}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Frechet1(\alpha,\sigma) \to Weibull1(\lambda,k)$
- Relationship type: Transformation
- Relationship definition: $X \sim Frechet1(\alpha,\sigma) \to X^{-1} \sim Weibull(\lambda=1/\sigma, k=\alpha)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution` [STAN Development Team, 2015b]

- Relationship pair: $Frechet2(\alpha,\sigma,m) \to Weibull1(\lambda,k)$
- Relationship type: Transformation
- Relationship definition: $X \sim Frechet2(\alpha,\sigma,m)$ with $m=0 \to X^{-1} \sim Weibull(\lambda=1/\sigma, k=\alpha)$
- Relationship reference(s): `https://en.wikipedia.org/wiki/Fr%C3%A9chet_distribution` [STAN Development Team, 2015b]

References and sources

`http://en.wikipedia.org/wiki/Weibull_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/weibull`

Weibull2

name	Weibull 2 (ID: 0000022)
type	continuous
variate	x, scalar
support	$x > 0$

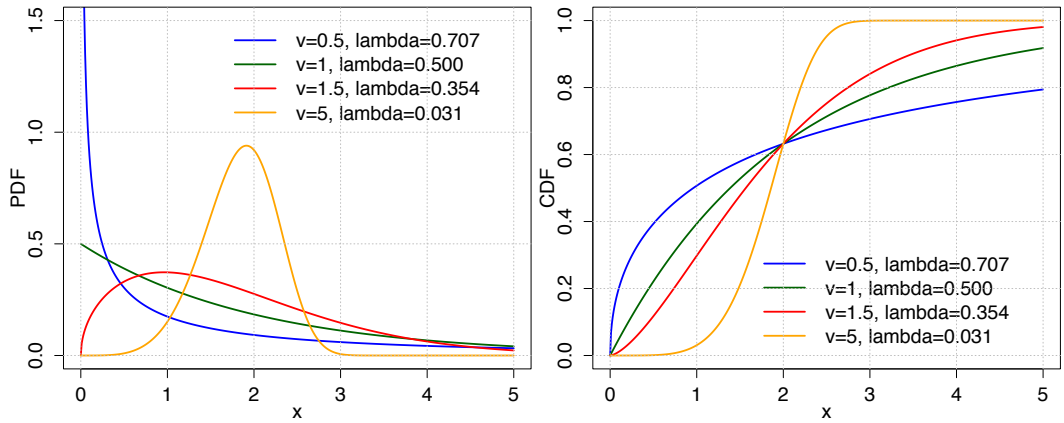

Figure A.127: Weibull2 distribution plotted using the provided R code.

Model

The Weibull distribution is used in reliability and survival analysis to model the lifetime of an object, the lifetime of a organism, or a service time. The accelerated life and Cox proportional hazards model are identical when the baseline distribution is Weibull.

Parameter: lambda

name	lambda
type	scalar
symbol	λ
definition	–

Parameter: shape

name	shape
type	scalar
symbol	v
definition	–

Functions

PDF

$$v\lambda\, x^{v-1} e^{-\lambda x^v}$$

PDF in R

```
v*lambda * x^(v-1) * exp(-lambda * x^v)
```

CDF

$$1 - \exp(-x^v \lambda)$$

CDF in R

```
1- exp(-x^v * lambda)
```

HF

$$v\lambda\, x^{v-1}$$

HF in R

```
v*lambda * x^(v-1)
```

SF

$$\exp(-x^v \lambda)$$

SF in R

```
exp(-x^v * lambda)
```

Characteristics

Relationships

- Relationship pair: $Weibull2(\lambda, v) \rightarrow Weibull1(\lambda, k)$
- Relationship type: Reparameterisation
- Relationship definition: $k = v, \lambda = \lambda^{-1/v}$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $Weibull1(\lambda, k) \rightarrow Weibull2(\lambda, v)$
- Relationship type: Reparameterisation
- Relationship definition: $v = k, \lambda = 1/\lambda^k$
- Relationship reference(s): ProbOnto spec

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

WeibullDiscrete1

name	Weibull Discrete 1 (ID: 0001176)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, \dots\}$

Model

This distribution has application in reliability when the response of interest is a discrete variable.

Parameter: shape1

name	shape parameter
type	scalar
symbol	p
definition	$0 < p < 1$

Parameter: shape2

name	shape parameter
type	scalar
symbol	β
definition	$\beta > 0$

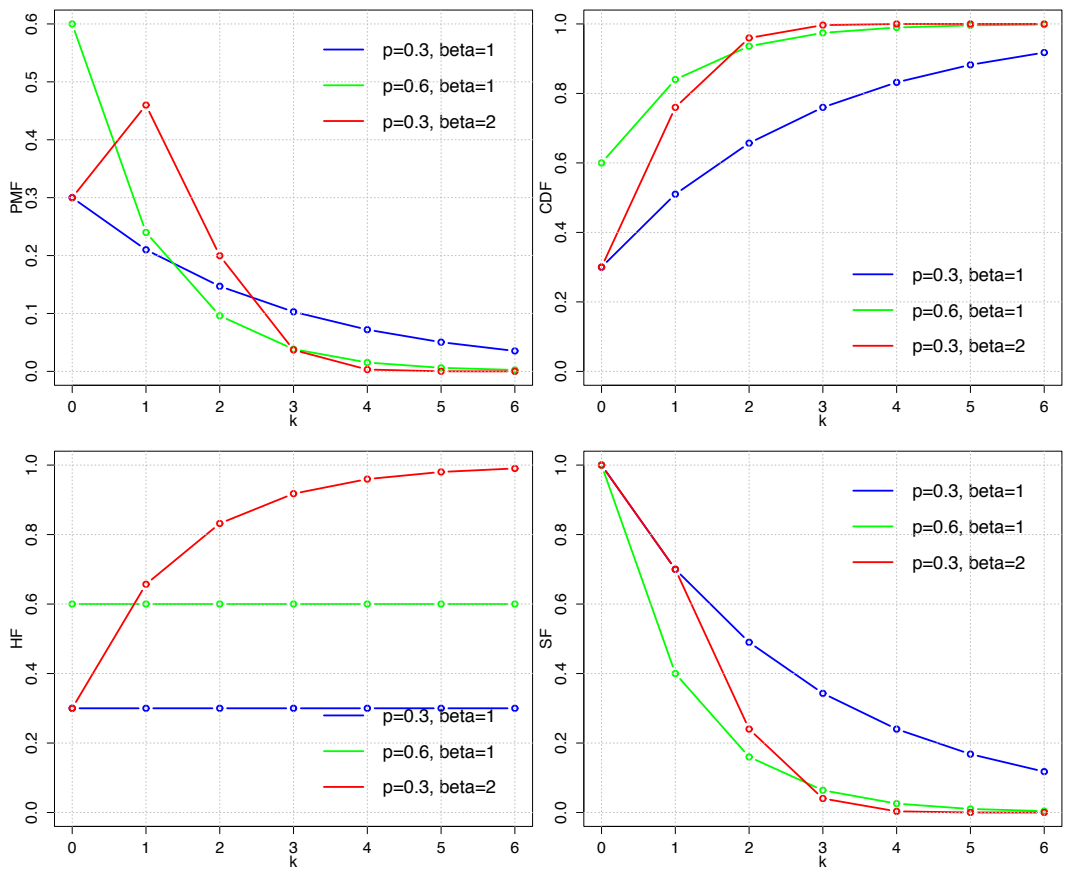

Figure A.128: WeibullDiscrete1 distribution plotted using the provided R code.

Functions

PMF

$$(1-p)^{x^\beta} - (1-p)^{(x+1)^\beta}$$

PMF in R

```
(1-p)^(x^beta) - (1-p)^((x+1)^beta)
```

CDF

$$1-(1-p)^{(x+1)^\beta}$$

CDF in R

```
1-(1-p)^((x+1)^beta)
```

HF

$$1-(1-p)^{(x+1)^\beta - x^\beta}$$

HF in R

```
1-(1-p)^((x+1)^beta-x^beta)
```

SF

$$(1-p)^{x^\beta}$$

SF in R

```
(1-p)^(x^beta)
```

Characteristics

Relationships

- Relationship pair: $WeibullDiscrete1(p\beta) \rightarrow Geometric1(p)$
- Relationship type: Special case
- Relationship definition: The Geometric1(p) distribution is a special case of the WeibullDiscrete1(p, β) distribution when β 1.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/DiscreteweibullGeometric.pdf`

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Discreteweibull.pdf`
`http://www.itl.nist.gov/div898/software/dataplot/refman2/auxillar/diwpdf.htm`

UncertML mapping:

no UncertML mapping found.

WienerDiffusionModel1

name	Wiener Diffusion Model 1 (ID: 0000050)
type	continuous
variate	x, scalar
support	NA

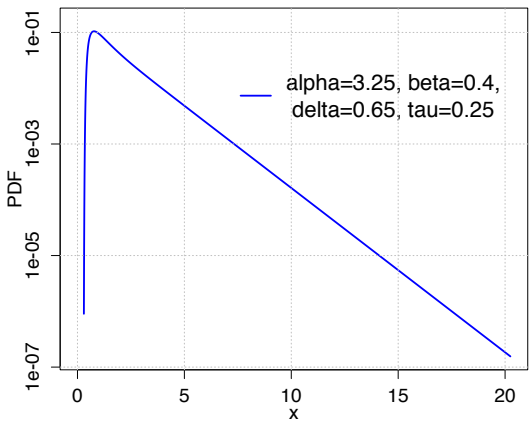

Figure A.129: WienerDiffusionModel1 distribution plotted using the provided R code.

Model

Parameter: boundSeparation

name	boundary separation
type	scalar
symbol	α
definition	$\alpha \in R^+$

Parameter: initialBias

name	initial bias
type	scalar
symbol	β
definition	$\beta \in [0, 1]$

Parameter: driftRate

name	drift rate
type	scalar
symbol	δ
definition	$\delta \in R$

Parameter: nondecisionTime

name	nondecision time
type	scalar
symbol	τ
definition	$\tau \in R^+$

Functions

PDF

$$\frac{\alpha}{(x-\tau)^{3/2}} \exp\Big(-\delta\alpha\beta - \frac{\delta^2(x-\tau)}{2}\Big) \sum_{k=-\infty}^{\infty} (2k+\beta)\phi\Big(\alpha\frac{2k+\beta}{\sqrt{x-\tau}}\Big)$$

Characteristics

Relationships

References and sources

[STAN Development Team, 2015b]
[Blurton et al., 2012]

UncertML mapping:

no UncertML mapping found.

WignerSemicircle1

name	Wigner Semicircle 1 (ID: 0001194)
type	continuous
variate	x, scalar
support	$x \in (-R, R)$

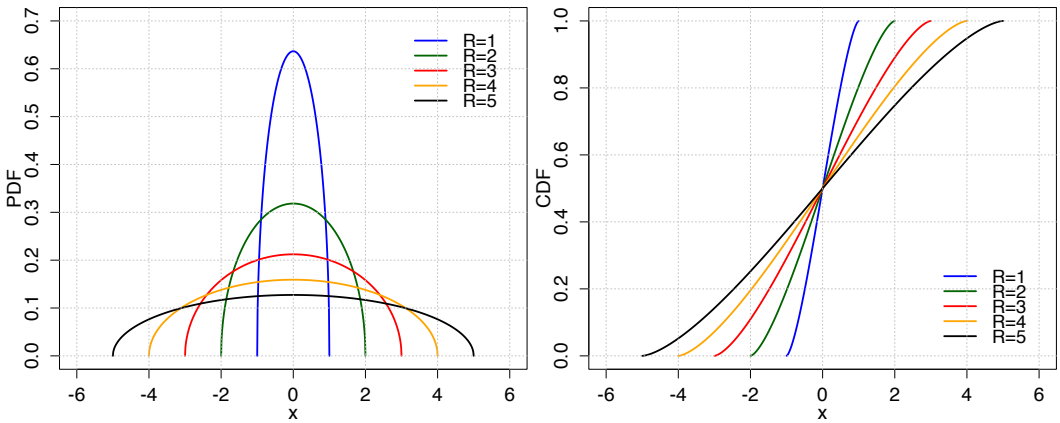

Figure A.130: WignerSemicircle1 distribution plotted using the provided R code.

Model

WignerSemicircle1 distribution arises as the limiting distribution of eigenvalues of many random symmetric matrices as the size of the matrix approaches infinity.

Parameter: radius

name	radius
type	scalar
symbol	R
definition	$R > 0$

Functions

PDF

$$\frac{2}{\pi R^2}\sqrt{R^2 - x^2}$$

PDF in R

```
2/(pi * R^2) * sqrt(R^2-x^2)
```

CDF

$$\frac{1}{2}\frac{x\sqrt{R^2 - x^2}}{\pi R^2} + \frac{\arcsin(\frac{x}{R})}{\pi}$$

CDF in R

```
1/2 * (x*sqrt(R^2-x^2))/(pi * R^2) + asin(x/R)/pi
```

Characteristics

Mean

$$0$$

Median

$$0$$

Mode

$$0$$

Variance

$$R^2/4$$

Relationships

References and sources

https://en.wikipedia.org/wiki/Wigner_semicircle_distribution

UncertML mapping:

no UncertML mapping found.

Wishart1

name	Wishart 1 (ID: 0000082)
type	continuous
variate	X, matrix
support	$X(p \times p)$ – positive definite matrix

Model

The Wishart distribution is a generalization to multiple dimensions of the chi-squared distribution, or, in the case of non-integer degrees of freedom, of the gamma distribution. It is a family of probability distributions defined over symmetric, nonnegative-definite matrix-valued random variables ("random matrices"). These distributions are of importance in the estimation of covariance matrices in multivariate statistics.

Parameter: scaleMatrix

name	scale matrix
type	matrix
symbol	V
definition	$V > 0, p \times p -$ positive definite matrix

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	n
definition	$n > p - 1$

Functions

PDF

$$\frac{|X|^{\frac{n-p-1}{2}} e^{-\frac{\mathrm{tr}(V^{-1}X)}{2}}}{2^{\frac{np}{2}} |V|^{\frac{n}{2}} \Gamma_p(\frac{n}{2})}$$

Characteristics

Mean

$$nV$$

Mode

$$(n-p-1)V \text{ for } n \leq p+1$$

Variance

$$Var(X_{ij}) = n\left(v_{ij}^2 + v_{ii}v_{jj}\right)$$

Relationships

References and sources

`http://en.wikipedia.org/wiki/Wishart_distribution`

UncertML mapping:

`http://www.uncertml.org/distributions/wishart`

Wishart2

name	Wishart 2 (ID: 0000112)
type	continuous
variate	X, matrix
support	$X(p \times p) -$ symmetric, positive definite matrix

Model

Parameter: inverseScaleMatrix

name	inverse scale matrix
type	matrix
symbol	R
definition	$p \times p -$ symmetric, positive definite matrix

Parameter: degreesOfFreedom

name	degrees of freedom
type	scalar
symbol	k
definition	$-$

Functions

PDF

$$|R|^{k/2}|x|^{(k-p-1)/2}e^{-\frac{1}{2}\,tr(Rx)}$$

Characteristics

Relationships

References and sources

[Spiegelhalter et al., 2003]

UncertML mapping:

no UncertML mapping found.

YuleSimon1

name	YuleSimon1 (ID: 0001236)
type	discrete
variate	k, scalar
support	$k \in \{1, 2, \dots\}$

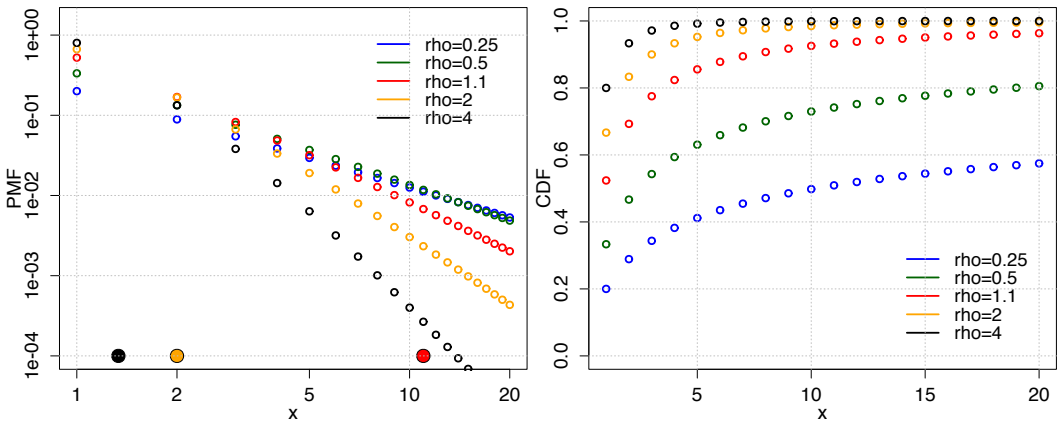

Figure A.131: YuleSimon1 distribution plotted using the provided R code.

Model

The Yule-Simon distribution arose originally as the limiting distribution of a particular stochastic process studied by Yule as a model for the distribution of biological taxa and subtaxa.

Parameter: shape

name	shape parameter
type	scalar
symbol	ρ
definition	$\rho > 0$

Functions

PMF

$$\rho B(k, \rho + 1)$$

PMF in R

```
rho*beta(k,rho+1)
```

CDF

$$1 - kB(k, \rho + 1)$$

CDF in R

```
1 - k*beta(k,rho+1)
```

Characteristics

Mean

$$\frac{\rho}{\rho - 1} \text{ for } \rho > 1$$

Mode

$$1$$

Variance

$$\frac{\rho^2}{(\rho - 1)^2(\rho - 2)} \text{ for } \rho > 2$$

Relationships

References and sources

https://artax.karlin.mff.cuni.cz/r-help/library/VGAM/html/yulesimon.html
https://en.wikipedia.org/wiki/Yule%E2%80%93Simon_distribution

UncertML mapping:

no UncertML mapping found.

ZeroInflatedGeneralizedPoisson1

name	Zero-Inflated Generalized Poisson 1 (ID: 0000169)
type	discrete
variate	y, scalar
support	$y \in \{0, 1, 2, 3, \dots\}$

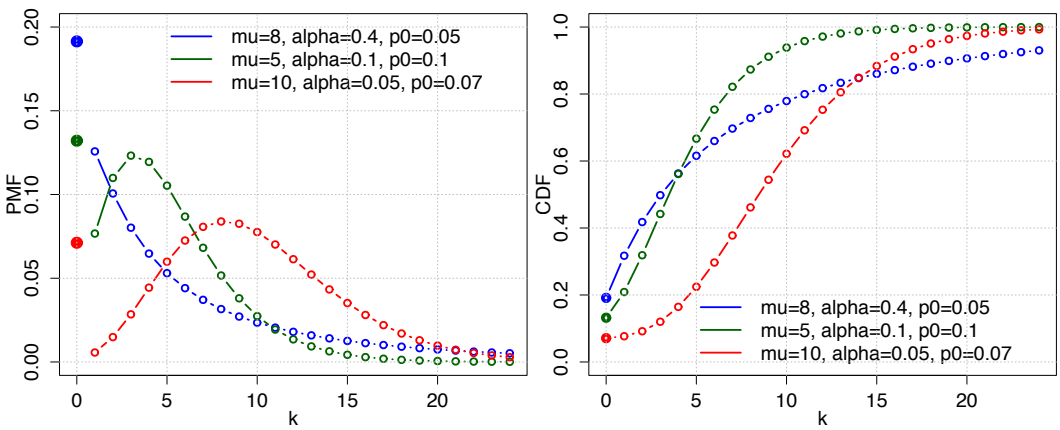

Figure A.132: ZeroInflatedGeneralizedPoisson1 distribution plotted using the provided R code.

Model

Parameter: mean

name	mean
type	scalar
symbol	μ
definition	$\mu > 0$

Parameter: dispersion

name	dispersion
type	scalar
symbol	α
definition	$\alpha > -1, \alpha \in R$

Parameter: probabilityOfZero

name	probability of zero
type	scalar
symbol	$p0$
definition	$0 < p0 < 1, p0 \in R$

Functions

PMF

$$\begin{cases} p0 + (1-p0)\exp\left[\frac{-\mu}{1+\alpha\mu}\right] & \text{for } y = 0 \\ (1-p0)\left(\frac{\mu}{1+\alpha\mu}\right)^y \frac{(1+\alpha y)^{y-1}}{y!} \exp\left[\frac{-\mu(1+\alpha y)}{1+\alpha\mu}\right] & \text{for } y > 0 \end{cases}$$

PMF in R

```
PMF1=p0 + (1-p0) * exp(-mu/(1+ alpha*mu)) # for y = 0
PMF2=(1-p0)*( mu/(1+alpha*mu))^y*((1+alpha*y)^(y-1))/factorial(y)
                        * exp((-mu*(1+alpha*y))/(1+alpha*mu)) # for y > 0
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
c(PMF1,cumsum(PMF2)+PMF1)
```

Characteristics

Mean

$$(1-p0)\mu$$

Variance

$$(1-p0)[\mu^2 + \mu(1+\alpha\mu)^2] - (1-p0)^2\mu^2$$

Relationships

- Relationship pair: $ZeroInflatedGeneralizedPoisson1(\mu, \alpha, p0) \rightarrow GeneralizedPoisson3(\mu, \alpha)$
- Relationship type: Special case
- Relationship definition: $p0 = 0$
- Relationship reference(s): [Famoye and Singh, 2006]

- Relationship pair: $ZeroInflatedGeneralizedPoisson1(\mu, \alpha, p0) \rightarrow Poisson1(\lambda)$
- Relationship type: Reparameterisation
- Relationship definition: $p0 = 0, \alpha = 0, \lambda = \mu$
- Relationship reference(s): ProbOnto spec

- Relationship pair: $ZeroInflatedGeneralizedPoisson1(\mu, \alpha, p0) \rightarrow ZeroInflatedPoisson1(\lambda, \pi)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = 0, \lambda = \mu$
- Relationship reference(s): [Famoye and Singh, 2006]

References and sources

[Famoye and Singh, 2006]

UncertML mapping:

no UncertML mapping found.

ZeroInflatedNegativeBinomial1

name	Zero-Inflated Negative Binomial 1 (ID: 0000142)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

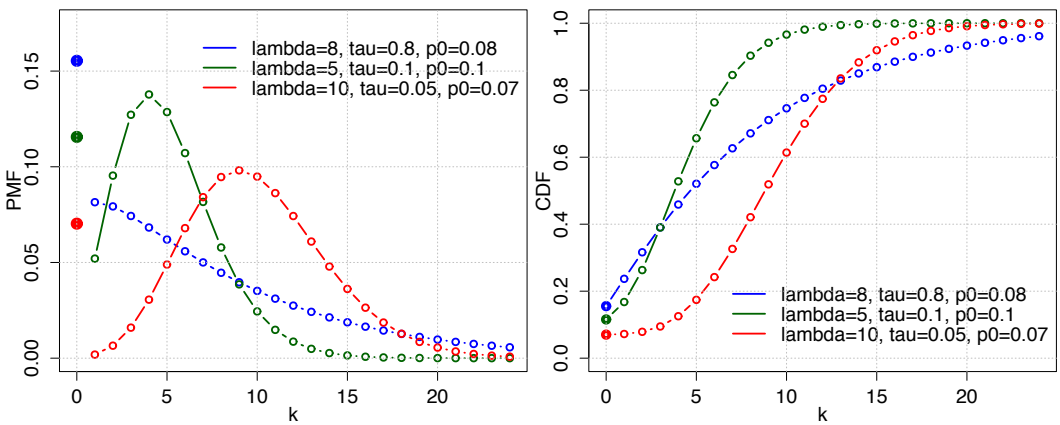

Figure A.133: ZeroInflatedNegativeBinomial1 distribution plotted using the provided R code.

Model

Parameter: rate

name	mean
type	scalar
symbol	λ
definition	$\lambda > 0$

Parameter: overdispersion

name	overdispersion
type	scalar
symbol	τ
definition	$-$

Parameter: probabilityOfZero

name	probability of zero
type	scalar
symbol	$p0$
definition	$0 < p0 < 1, p0 \in R$

Functions

PMF

$$\begin{cases} p0 + (1-p0)\left(\frac{1}{1+\tau\lambda}\right)^{1/\tau} & \text{for } y = 0 \\ (1-p0)\frac{\Gamma(y+1/\tau)}{y!\Gamma(1/\tau)}\left(\frac{1}{1+\tau\lambda}\right)^{1/\tau}\left(\frac{\lambda}{1/\tau+\lambda}\right)^{y} & \text{for } y > 0 \end{cases}$$

PMF in R

```
PMF1=p0 + (1-p0) * (1/ (1 + tau * lambda))^(1/tau) # for y=0
PMF2=(1-p0) * gamma(y+1/tau) / (factorial(y) *gamma(1/tau))
* (1/(1 + tau*lambda))^(1/tau) * (lambda/(1/tau + lambda))^y # for y>0
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
c(PMF1,cumsum(PMF2)+PMF1)
```

Characteristics

Mean

$$\mu$$

Relationships

- Relationship pair: $ZeroInflatedNegativeBinomial1(\lambda, \tau, p0) \rightarrow NegativeBinomial2(\lambda, \tau)$
- Relationship type: Special case
- Relationship definition: $p0 = 0$
- Relationship reference(s): [Trocóniz et al., 2009]
- Relationship pair: $ZeroInflatedNegativeBinomial1(\lambda, \tau, p0) \rightarrow ZeroInflatedPoisson1(\lambda, \pi)$
- Relationship type: Limiting
- Relationship definition: $\tau \rightarrow 0$
- Relationship reference(s): [Trocóniz et al., 2009]
- Relationship pair: $ZeroInflatedNegativeBinomial1(\lambda, \tau, p0) \rightarrow Poisson1(\lambda)$
- Relationship type: Limiting
- Relationship definition: $p0 = 0, \tau \rightarrow 0$
- Relationship reference(s): [Trocóniz et al., 2009]

References and sources

[Famoye and Singh, 2006]
[Trocóniz et al., 2009]

UncertML mapping:

no UncertML mapping found.

ZeroInflatedPoisson1

name	Zero-inflated Poisson 1 (ID: 0000197)
type	discrete
variate	k, scalar
support	$k \in \{0, 1, 2, 3, \dots\}$

Model

Parameter: rate

name	Poisson intensity
type	scalar
symbol	λ
definition	$\lambda \in R, \lambda > 0$

Parameter: probabilityOfZero

name	probability of extra zeros
type	scalar
symbol	π
definition	$0 < \pi < 1, \pi \in R$

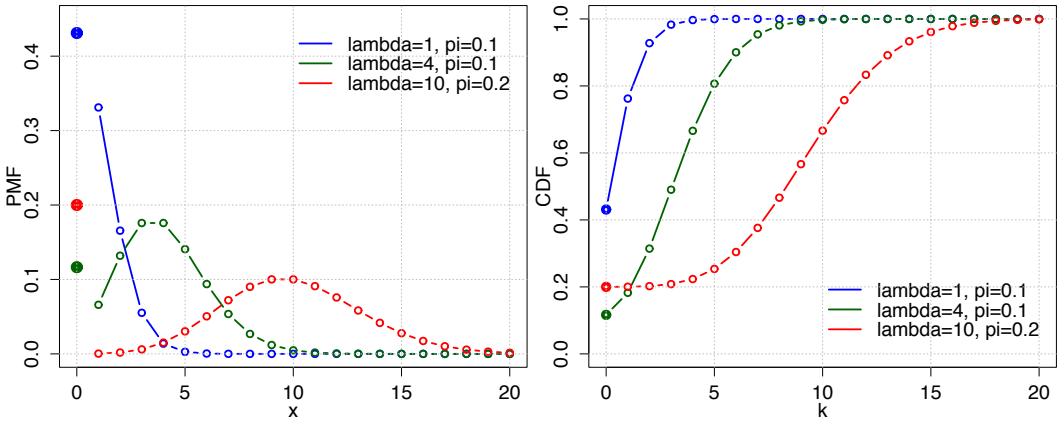

Figure A.134: ZeroInflatedPoisson1 distribution plotted using the provided R code.

Functions

PMF

$$\begin{cases} \pi + (1-\pi)e^{-\lambda} & \text{for } k = 0 \\ (1-\pi)e^{-\lambda}\frac{\lambda^k}{k!} & \text{for } k > 0 \end{cases}$$

PMF in R

```
PMF1=pi + (1-pi)*exp(-lambda) # if k=0
PMF2=(1-pi)*exp(-lambda) * lambda^k/factorial(k) # if k>0
```

CDF

$$\Sigma_{i=1}^{x} f(i), x \in \{0, 1, 2, ...\} \text{ with } f \text{ the PMF}$$

CDF in R

```
c(PMF1,cumsum(PMF2)+PMF1)
```

Characteristics

Mean

$$(1-\pi)\lambda$$

Variance

$$\lambda(1-\pi)(1+\lambda\pi)$$

Relationships

- Relationship pair: $ZeroInflatedPoisson1(\lambda, \pi) \rightarrow Poisson1(\lambda)$
- Relationship type: Special case
- Relationship definition: $\pi = 0$
- Relationship reference(s): [Trocóniz et al., 2009]
- Relationship pair: $ZeroInflatedNegativeBinomial1(\lambda, \tau, p0) \rightarrow ZeroInflatedPoisson1(\lambda, \pi)$
- Relationship type: Limiting
- Relationship definition: $\tau \rightarrow 0$
- Relationship reference(s): [Trocóniz et al., 2009]
- Relationship pair: $ZeroInflatedGeneralizedPoisson1(\mu, \alpha, p0) \rightarrow ZeroInflatedPoisson1(\lambda, \pi)$
- Relationship type: Reparameterisation
- Relationship definition: $\alpha = 0, \lambda = \mu$
- Relationship reference(s): [Famoye and Singh, 2006]

References and sources

`http://en.wikipedia.org/wiki/Zero-inflated_model#Zero-inflated_Poisson`
[Plan, 2014]

UncertML mapping:

no UncertML mapping found.

Zeta1

name	Zeta 1 (ID: 0001263)
type	discrete
variate	x, scalar
support	$x \in \{1, 2, 3, \dots\}$

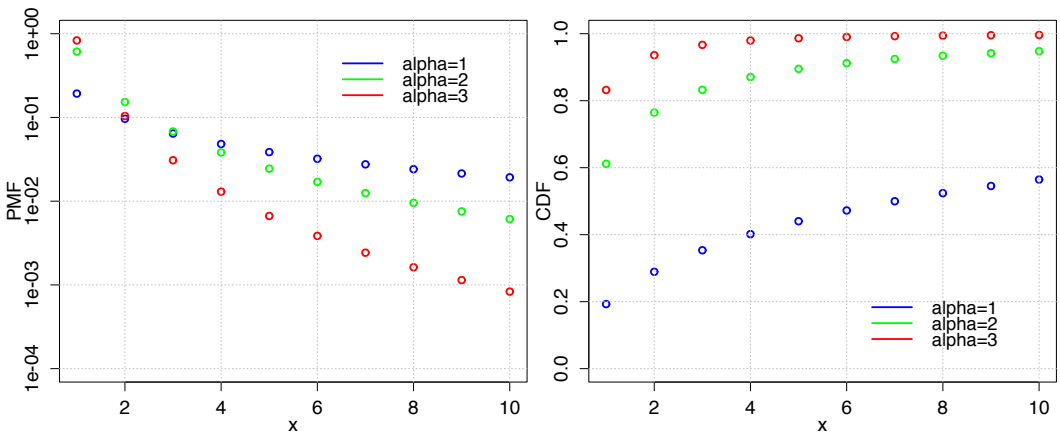

Figure A.135: Zeta1 distribution plotted using the provided R code.

Model

The zeta distribution is a discrete probability distribution. If X is a zeta-distributed random variable with parameter α, then the probability that X takes the integer value x is given by the zeta PMF.

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha \in R, \alpha > 0$

Functions

PMF

$$\frac{1}{x^{\alpha}\zeta(\alpha)}$$

PMF in R

```
1 / (x^alpha * zeta(alpha))
```

CDF

$$\frac{\sum_{i=1}^{x}(1/i)^{\alpha}}{\zeta(\alpha)}$$

CDF in R

```
Hfunc(x,alpha)/zeta(\alpha)
Hfunc = function(x,alpha) {
  Hsum=array(0,length(x));
  for(i in 1:length(x)) {
    Hsum[i] = (1/i)^alpha
  }
  return(cumsum(Hsum))
}
```

Characteristics

Mean

$$\zeta(\alpha-1)/\zeta(\alpha)$$

Variance

$$\frac{\zeta(\alpha)\zeta(\alpha-2)-\zeta(\alpha-1)^2}{\zeta(\alpha)^2}$$

Relationships

- Relationship pair: $Zipf1(\alpha, n) \rightarrow Zeta1(\alpha)$
- Relationship type: Limiting
- Relationship definition: The limiting distribution of a $Zipf1(\alpha, n)$ random variable as $n \rightarrow \infty$ is the $Zeta1(\alpha)$ distribution.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ZipfZeta.pdf`

References and sources

`http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Zipf.pdf`

UncertML mapping:

no UncertML mapping found.

Zipf1

name	Zipf 1 (ID: 0001286)
type	discrete
variate	x, scalar
support	$x \in \{1, 2, 3, \dots\}$

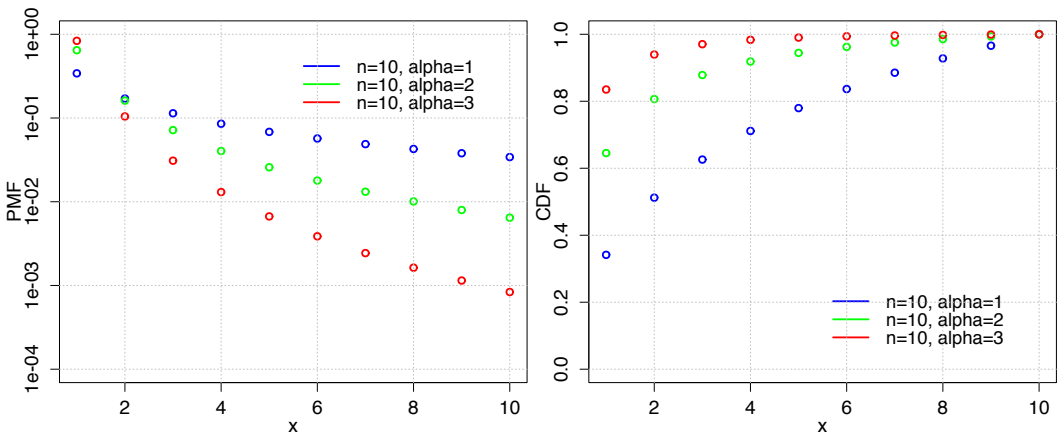

Figure A.136: Zipf1 distribution plotted using the provided R code.

Model

Zipf's law is an empirical law formulated using mathematical statistics that refers to the fact that many types of data studied in the physical and social sciences can be approximated with a Zipfian distribution, one of a family of related discrete power law probability distributions.

Parameter: shape

name	shape
type	scalar
symbol	α
definition	$\alpha \in R, \alpha > 0$

Parameter: numberOfElements

name	number of elements
type	scalar
symbol	n
definition	$n \in N^+$

Functions

PMF

$$\frac{1}{x^\alpha H_{n,\alpha}} H_{n,\alpha} = \sum_{i=1}^{n} (1/i)^\alpha$$

PMF in R

```
1 / (x^alpha * Hfunc(n,alpha))\\
Hfunc = function(n,alpha) {
  Hsum=0;
  for(i in 1:n) {
    Hsum = Hsum + (1/i)^alpha
  }
  return(Hsum)
}
```

CDF

$H_{x,\alpha}/H_{n,\alpha}$, with $x = 1, 2, ..., n$

$H = function(n, alpha)\{Hsum = 0; for(i\ in\ 1:n)\{Hsum = Hsum + (1/i)^{alpha}\}\ return(Hsum)\}$

CDF in R

```
CDF = function(x,n,alpha) {
  H=array(0,length(x));
  for(j in 1:length(x)) {
    H[j] = Hfunc(x[j],alpha) / Hfunc(n,alpha)
  }
  cat(H)
  return(H)
}
```

Characteristics

Mean

$$H_{n,\alpha-1}/H_{n,\alpha}$$

Variance

$$(H_{n,\alpha-2}H_{n,\alpha} - H^2_{n,\alpha-1})/H^2_{n,\alpha}$$

Relationships

- Relationship pair: $Zipf1(\alpha, n) \rightarrow UniformDiscrete1(a, b)$
- Relationship type: Special case
- Relationship definition: $\alpha = 0, a = 1, b = n$
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ZipfDiscreteuniform.pdf`
- Relationship pair: $Zipf1(\alpha, n) \rightarrow Zeta1(\alpha)$
- Relationship type: Limiting
- Relationship definition: The limiting distribution of a $Zipf1(\alpha, n)$ random variable as $n \rightarrow \infty$ is the $Zeta1(\alpha)$ distribution.
- Relationship reference(s): `http://www.math.wm.edu/~leemis/chart/UDR/PDFs/ZipfZeta.pdf`

References and sources

http://www.math.wm.edu/~leemis/chart/UDR/PDFs/Zipf.pdf

UncertML mapping:

no UncertML mapping found.

A.3 Parameterisations of the negative binomial (NB1 vs NB4)

There are at least three ways to formulate and interpret an experiment for the independent Bernoulli trials with a fixed number of outcomes (successes, failures or both) with the goal to estimate its probability distribution. The negative binomial (NB) model which formalises this experiment is formulated using two out of three following parameters dependent on the formulation:

- number of trials (i.e. total number successes or failures)
- number of successes
- number of failures

The possible formulations for such experiment are

Option 1 observing k failures before obtaining the rth success (most common)

Option 2 obtaining r successes until k failures have occurred

Option 3 number of trials (successes or failures), n, required before the rth success occurs.

The definition of the negative binomial in UncertML is based on the English Wikipedia, which features *Option 2*[1], a rather rare form of this distribution. This might lead to mistakes because tools such as Matlab, R and winBUGS, to mention here only few, use the most common formulation, *Option 1*, see Figure A.137 and Table A.1 capturing the differences.

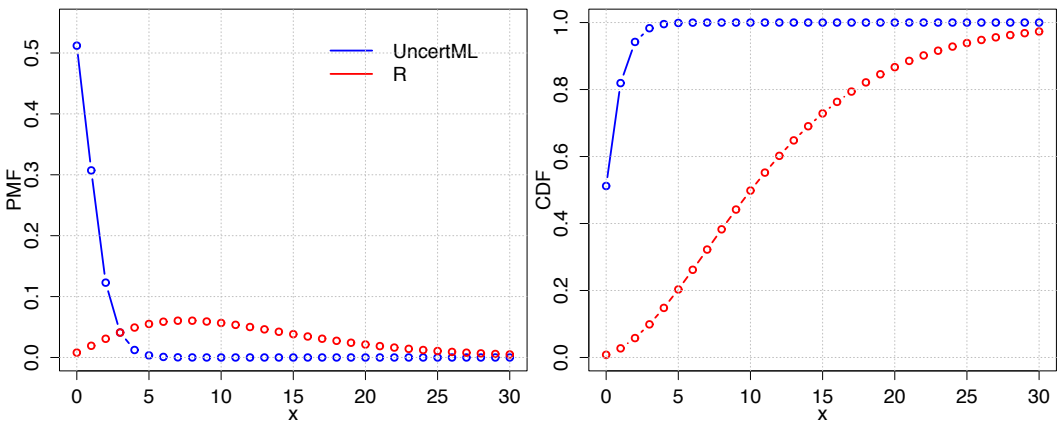

Figure A.137: Comparison of PMF and CDF plots for *Option 1* (supported in UncertML) and *Option 2* (supported in R). In blue the functions for the UncertML formulation, in red as implemented in R for parameters $r = n = 3$ and $p = 0.2$. Using the formulas given in Table A.1 one can calculate the means as 0.75 and 12, for *Option 1* and *Option 2*, respectively.

	UncertML	R
PMF	$f(x;r,p) = \binom{x+r-1}{x} p^x (1-p)^r$	$f(x;n,p) = \binom{x+n-1}{x} p^n (1-p)^x$
Support	$x \in 0, 1, 2, \ldots$ number of **successes**	$x \in 0, 1, 2, \ldots$ number of **failures**
Parameters	$p \in [0,1]$ – probability of success r – number of **failures**	$p \in (0,1]$ – probability of success n – number of **successes**
Mean	$\frac{rp}{1-p}$	$\frac{n(1-p)}{p}$

Table A.1: UncertML versus R: comparison of the model formulation, parameter interpretation and the support variable. The PMF equations of these two models are very similar so that the differences can go unnoticed to most modellers which might in turn result in unwanted results, see figure A.137.

See Table A.2 for the overview of the occurrence of these options in the literature and software tools.

[1]This was not always the case as the change from the standard formulation to the current one happened around 2010. The majority of users who commented on this change are unhappy about it, see the English and French Wikipedia Talk/Discussion pages. Because there is no agreement to reverse these changes, which would require significant effort, the page remains as is.

A.3.1 Differences explained

The above mentioned formulation options, see also table A.1, are better understood if one compares different language version for the articles on negative binomial distribution in [Wikipedia (English), 2015], [Wikipedia (French), 2015] and [Wikipedia (German), 2015].

English Wikipedia version (Option 1) – supported in UncertML

Interpretation: distribution of the number of successes, k, until r failures have occurred.

$$P_{NB}(k;r,p) = \binom{k+r-1}{k} p^k (1-p)^r, \quad \mathrm{E}(X=k) = \frac{rp}{(1-p)}$$

- Support
  - $k \in \{0,1,2,3,\dots\}$ – number of **successes**
- Parameters
  - $r > 0$ – number of **failures** until the experiment is stopped
  - $p \in (0,1)$ – success probability in each experiment

French Wikipedia version (Option 2)

Interpretation: distribution of the number of failures, k, before obtaining n successes

$$P_{NB}(k;n,p) = \binom{k+n-1}{k} p^n (1-p)^k, \quad \mathrm{E}(X=k) = \frac{n(1-p)}{p}$$

- Support
  - $k \in \{0,1,2,3,\dots\}$ – number of **failures**
- Parameters
  - $n > 0$ – number of **successes** until the experiment is stopped (fr: *le nombre de succès attendus*)
  - $p \in (0,1)$ – success probability in each experiment (fr: *la probabilitè d'un succès*)

German Wikipedia version (Option 2)

The German Wiki page describes two alternative representations and interpolations of this distribution. We present here that which is listed as the alternative representation. (The alternative is of type referred above as 'Option 3'.)
Interpretation: distribution of the number of failures, k, before obtaining r successes. (ger.: *NB Distribution beschreibt die Anzahl, k, der Misserfolge bis zum Eintreten des r-ten Erfolgs.*)

$$P_{NB}(k;r,p) = \binom{k+r-1}{k} p^r (1-p)^k, \quad \mathrm{E}(X=k) = \frac{r(1-p)}{p}$$

- Support
  - $k \in \{0,1,2,3,\dots\}$ – number of **failures** (ger: *Anzahl Misserfolge*)
- Parameters
  - $r > 0$ – number of **successes** until the experiment is stopped (ger: *Anzahl Erfolge bis zum Abbruch*)
  - $p \in (0,1)$ – success probability in each experiment, (ger: *Einzel-Erfolgs-Wahrscheinlichkeit*)

$$P_{NB}(k;r,p) = \binom{k+r-1}{k} p^r (1-p)^k.$$

A.3.2 Comparison of formulation support

The following table gives an overview of supported NB formulation options across over 30 literature sources and specialised software tools.

Source	PMF	Probability	Support	Variate
	Option 1			
	probability of observing a fixed number of failures before certain number of success			
[Feller, 1968]	$\binom{r+k-1}{k}p^r(1-p)^{\mathbf{k}}$	$0<p<1$	$k=0,1,2,...$	#failures
[Hilbe, 2011]	$\binom{y+r-1}{y}p^r(1-p)^{\mathbf{y}}$	$0<p<1$	$0\leq y<\infty$	#failures
[Forbes et al., 2011]	$\binom{x+r-1}{x}p^r(1-p)^{\mathbf{x}}$	$0<p<1$	$0\leq x<\infty$	#failures
[Leemis and Mcqueston, 2008]	$\binom{x+r-1}{x}p^r(1-p)^{\mathbf{x}}$	$0<p<1$	$0\leq x<\infty$	–
[Devroye, 1986]	$\binom{x+n-1}{x}p^n(1-p)^{\mathbf{x}}$	$p\in(0,1)$	$x\geq 0$	#failures
[Dobson, 2002]	$\binom{y+r-1}{y}p^r(1-p)^{\mathbf{y}}$	–	y	–
[Zelterman, 2004]	$\binom{c+y-1}{c-1}p^c(1-p)^{\mathbf{y}}$	–	$y=0,1,...$	#failures
[Hardin et al., 2007]	$\binom{y+r-1}{r-1}p^r(1-p)^{\mathbf{y}}$	$0<p<1$	$y=0,1,2,...$	#failures
[Bolker, 2008]	$\frac{(n+x-1)!}{(n-1)!\,x!}p^n(1-p)^{\mathbf{x}}$	$0<p<1$	$x\geq 0$	#failures
[Scheaffer and Young, 2009]	$\binom{x+r-1}{r-1}p^r(1-p)^{\mathbf{x}}$	–	$x=0,1,...$	#failures
[Bonate, 2011]	$\frac{\Gamma(y+k)}{\Gamma(k)y!}p^k(1-p)^{\mathbf{y}}$	–	$y\geq 0$	–
[Yu et al., 2013]	$\binom{r+x-1}{x}p^r(1-p)^{\mathbf{x}}$	$p\in(0,1)$	$x\geq 0$	#failures
[Dodge, 2008]	$C^x_{z+x-1}p^r(1-p)^{\mathbf{k}}$	–	k	#failures
[Lesaffre and Lawson, 2012]	$\binom{\theta+n-1}{\theta}\pi^n(1-\pi)^{\theta}$	$0\leq p\leq 1$	$\theta\in\{0,1,2,...,n\}$	#failures
[Law, 2007]	$\binom{x+s-1}{x}p^s(1-p)^{\mathbf{x}}$	$p\in(0,1)$	$x\in\{0,1,2,...\}$	#failures
[Vidakovic, 2011]	$\binom{r+x-1}{x}p^r(1-p)^{\mathbf{x}}$	–	$x=0,1,2,...$	#failures
[Cook, 2009]	$\binom{r+x-1}{x}p^r(1-p)^{\mathbf{x}}$	$0<p<1$	$x\geq 0$	#failures
Matlab, Stats Toolbox	$\binom{x+r-1}{x}p^r(1-p)^{\mathbf{x}}$	$0<p<1$	$0\leq x<\infty$	#failures
[R Core Team, 2013]	$\frac{\Gamma(x+n)}{\Gamma(n)x!}p^n(1-p)^{\mathbf{x}}$	$0<p\leq 1$	$x\in 0,1,...$	#failures
[Yee, 2008, Yee, 2015]	$\binom{y+k-1}{y}p^k(1-p)^{\mathbf{y}}$	$0<p<1$	$y\in 0,1,2,...$	–
[SAS Institute Inc., 2014]	$\frac{(n+x-1)!}{x!\,(n-1)!}p^n(1-p)^{\mathbf{x}}$	$p\in(0,1)$	$x\in\{0,1,...\}$	–
[Lunn et al., 2002]	$\binom{x+r-1}{x}p^r(1-p)^{\mathbf{x}}$	–	$0\leq x<\infty$	#failures[2]
[Plummer, 2003]	$\binom{x+r-1}{x}p^r(1-p)^{\mathbf{x}}$	$0<p\leq 1$	$x\geq 0$	–
[Marichev and Trott, 2013]	$\binom{m+n-1}{n-1}p^n(1-p)^{\mathbf{m}}$	$0<p\leq 1$	$m\in 0,1,2,...$	–
VoseSoftware.com	$\binom{s+x-1}{x}p^s(1-p)^{\mathbf{x}}$	$0<p\leq 1$	$x\in 0,1,...$	#failures
boost.org	$\binom{x+r-1}{x}p^r(1-p)^{\mathbf{x}}$	$0\leq p\leq 1$	x	#failures
cplusplus.com	$\binom{k+i-1}{i}p^k(1-p)^{\mathbf{i}}$	–	i	#failures
Mathwave.com	$\binom{x+n-1}{x}p^n(1-p)^{\mathbf{x}}$	$0<p<1$	$x\in 0,1,...$	–
Wolfram MathWorld	$\binom{x+r-1}{r-1}p^r(1-p)^{\mathbf{x}}$	–	x	#failures
[Wikipedia (French), 2015]	$\binom{k+r-1}{k}p^r(1-p)^{\mathbf{k}}$	$p\in(0,1)$	$k\in 0,1,2,3,...$	#failures
[Wikipedia (German), 2015]♣	$\binom{k+r-1}{k}p^r(1-p)^{\mathbf{k}}$	$p\in(0,1)$	$k\in 0,1,2,3,...$	#failures
massmatics.de♣	$\binom{k+r-1}{k}p^r(1-p)^{\mathbf{k}}$	$0<p<1$	k	#failures
[Chen and Blitzstein, 2015]	$\binom{r+n-1}{r-1}p^r q^{\mathbf{n}}$	$0<p<1$	n	#failures
[Walck, 2007]♣	$\binom{n+k-1}{n}p^k(1-p)^{\mathbf{n}}$	$0\leq p\leq 1$	$n\geq 0$	#failures
	Option 2			
	probability of observing a fixed number of successes before certain number of failures			

[2]Interpretation based on the description of winBUGS in [Vidakovic, 2011].

[Agresti, 2013]	$\binom{y+k-1}{y}(1-p)^k p^{\mathbf{y}}$	–	$y \in 0, 1, 2, ...$	#successes
[Cameron and Trivedi, 2013]	$\binom{y+r-1}{r-1}(1-p)^r p^{\mathbf{y}}$	$0 < p < 1$	$y \in 0, 1, 2, ...$	–
[UncertML Team, 2014]	$\binom{x+r-1}{x}(1-p)^r p^{\mathbf{x}}$	$p \in [0, 1]$	$x \in 0, 1, 2, 3, ...$	#successes
[Wikipedia (English), 2015]	$\binom{k+r-1}{k}(1-p)^r p^{\mathbf{k}}$	$p \in (0, 1)$	$k \in 0, 1, 2, 3, ...$	#successes
[Consul and Famoye, 2006]	$\binom{x+n-1}{x}(1-p)^{\mathbf{n}} p^x$	$0 < p < 1$	$x = 0, 1, 2, 3, ...$	–
openturns.org	$\frac{\Gamma(k+r)}{\Gamma(r)\Gamma(k+1)}(1-p)^{\mathbf{r}} p^k$	$0 < p < 1$	$k \in \mathbb{N}$	–
nemetrian.com	$\binom{x+r-1}{x}(1-p)^{\mathbf{r}} p^x$	$0 < p < 1$	$x = 0, 1, 2, ...$	#successes
Option 3				
probability of number of trials required to achieve certain number of successes				
[Chen and Peace, 2010]	$\binom{n-1}{k-1} p^k (1-p)^{\mathbf{n}-k}$	–	$n \in k, k+1, ...$	#trials
[Dinov et al., 2015]	$\binom{x-1}{k-1} p^k (1-p)^{\mathbf{x}-k}$	$p \in (0, 1]$	$x \in k, k+1, ...$	#trials
[Song and Chen, 2011]	$\binom{x-1}{k-1} p^k (1-p)^{\mathbf{x}-k}$	$0 \leq p \leq 1$	$x \in k, k+1, ...$	–
[Heckert and Filliben, 2003]	$\binom{x-1}{k-1} p^k (1-p)^{\mathbf{x}-k}$	$0 < p < 1$	–	#trials
massmatics.de ♣	$\binom{n-1}{r-1} p^r (1-p)^{\mathbf{n}-r}$	$0 < p < 1$	$n \in N, n \geq r$	#trials
[Wikipedia (German), 2015]♣	$\binom{n-1}{r-1} p^r (1-p)^{\mathbf{n}-r}$	$0 < p < 1$	$n \in N, n \geq r$	#trials
[Walck, 2007]♣	$\binom{r-1}{k-1} p^k (1-p)^{\mathbf{r\text{-}k}}$	$0 \leq p \leq 1$	$n \geq k, k > 0$	#trials
[Heckert and Filliben, 2003]	$\binom{x-1}{k-1} p^k (1-p)^{\mathbf{x\text{-}k}}$	$0 \leq p \leq 1$	–	#trials

Table A.2: Overview of different formulation types of NB described or supported in literature and specialised software tools. *Option 1* and *Option 2* are parameterised with the number of success or failures, respectively, while *Option 3* with the total number of successes and failures. All options use the probability of success, p, as second parameter. Some of the well known resources, [STAN Development Team, 2015a] or [Gelman et al., 2014], have not been listed as they feature another version(s) of the negative binomial, NB2 or NB5, see section 3.1. ♣ These sources list *Option 1* as an alternative to *Option 3*.

Note, the first term of the PMF, in Option 1 & 2, can be formulated in various ways and the following equations illustrate their equivalence

$$C^x_{z+x-1} = \binom{x+z-1}{x} = \binom{x+z-1}{z-1} = \frac{(x+z-1)!}{x!\,(z-1)!} = \frac{\Gamma(x+z)}{x!\,\Gamma(z)}.$$

The R and UncertML versions are stored in ProbOnto, as NB1 and NB4, respectively.

A.4 Generalised Negative Binomial Distribution

This section addresses an apparent name conflict within the family of discrete distributions. Negative binomial distribution is sometimes called 'Inverse Binomial'. We have investigated this and found out that this might lead to misunderstandings due to the fact that *Inverse binomial* distribution exists and is different, albeit related, from the *Negative binomial*.

A special form of the *negative binomial* distribution, NB, was first discussed by Blaise Pascal almost 350 years ago in 1679, according to [Dodge, 2008] and [Johnson et al., 2005]. Then in 1719 another French mathematician Montmort, formulated the negative binomial as the distribution of the number of tosses of a coin necessary to achieve a fixed number of heads, [Tripathi, 2006]. The generalisation for both *binomial* and NB distributions called the *generalised negative binomial distribution*, GNB, was proposed almost 300 years later by two Canadian mathematicians [Jain and Consul, 1971]. The PMF of the GNB is defined for $0 < \alpha < 1$ and $|\alpha\beta| < 1$ and reads in Jain & Consul paper

$$b_\beta(x, n, \alpha) = \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\,\alpha^x(1-\alpha)^{n+\beta x - x}, n > 0, x = 0, 1, 2, 3, ...$$

such that $b_\beta(x, n, \alpha) = 0$ for $x \leq m$ if $n + \beta m < 0$.

Interestingly, following distributions are special cases of the GNB distribution

- binomial, B(n,p)
- negative binomial, NB(r,p)[3]
- inverse binomial, IB(k,p)

which will be shown in the following sections.

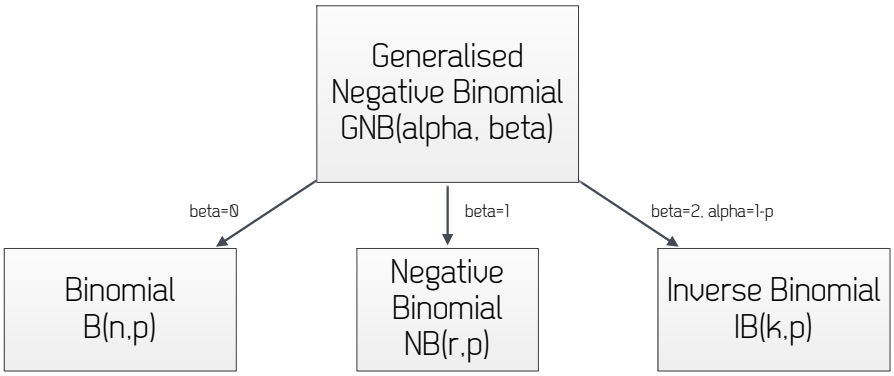

Figure A.138: GNB and its family, see also [Consul and Famoye, 2006].

Relationships

GNB(α,β) $\rightarrow$ B(n,p)

According to [Jain and Consul, 1971], GNB reduces to B for $\beta = 0$ and indeed this can be shown (replacing α with p) as follows

$$b_\beta(x, n, \alpha) \rightarrow P_B(x; n, p) : \frac{n\,\Gamma(n + \beta x)}{x!\,\Gamma(n + \beta x - x + 1)}\,\alpha^x(1-\alpha)^{n+\beta x - x} \rightarrow \frac{n\,\Gamma(n)}{x!\,\Gamma(n - x + 1)}\,p^x(1-p)^{n-x}$$

with the first term in the last expression $\frac{n\,\Gamma(n)}{x!\,\Gamma(n-x+1)} = \frac{n(n-1)!}{x!(n-x)!} = \frac{n!}{x!(n-x)!} = \binom{n}{x}$ we get the expected result

$$P_B(x; n, p) = \binom{n}{x} p^x(1-p)^{n-x}.$$

[3]This corresponds to the NB1 parameterisation of the negative binomial distribution in ProbOnto.

GNB(α,β) $\rightarrow$ NB4(r,p)

According also to [Jain and Consul, 1971], GNB reduces to NB for $\beta = 1$ and indeed this can be shown (replacing α with p and n with r) as follows

$$b_\beta(x,n,\alpha) \rightarrow P_{NB4}(x;r,p) : \frac{n\,\Gamma(n+\beta x)}{x!\,\Gamma(n+\beta x - x + 1)}\,\alpha^x(1-\alpha)^{n+\beta x - x} \rightarrow \frac{r\,\Gamma(r+x)}{x!\,\Gamma(r+1)}\,p^x(1-p)^r$$

with the first term $\frac{r\,\Gamma(r+x)}{x!\,\Gamma(r+1)} = \frac{r\,(r+x-1)!}{x!\,r!} = \frac{(r+x-1)!}{x!\,(r-1)!} = \binom{r+x-1}{x}$ we get the correct PMF

$$P_{NB4}(x;r,p) = \binom{r+x-1}{x} p^x(1-p)^r.$$

It is important to stress that it is the parameterisation NB4 which results as the special case for $\beta = 1$, see also discussion in appendix A.3, and not the common NB1.

GNB(α,β) $\rightarrow$ IB(k,p)

[Yanagimoto, 1989], proposed the *inverse binomial* distribution as another special case of GNB for $\beta = 2, \alpha = 1-p$ and $n = k$, which can be derived as the following shows

$$b_\beta(x,n,\alpha) \rightarrow P_{IB}(x;k,p) : \frac{n\,\Gamma(n+\beta x)}{x!\,\Gamma(n+\beta x - x + 1)}\,\alpha^x(1-\alpha)^{n+\beta x - x} \rightarrow \frac{k\,\Gamma(k+2x)}{x!\,\Gamma(k+x+1)}\,(1-p)^x p^{k+x}$$

and the result follows in agreement with the formulation in [Yanagimoto, 1989], i.e.

$$P_{IB}(x;k,p) = \frac{k\,\Gamma(2x+k)}{\Gamma(x+1)\,\Gamma(x+k+1)}\,p^{k+x}(1-p)^x,$$

and from $|\alpha\beta| < 1$ and $0 < \alpha < 1$ one can derive the required condition for p, $1/2 < p < 1$.

Interestingly, IB has a medical application. Yanagimoto used the distribution to estimate the proportion of discharged patients who can be expected to stay completely free from some disease. In the original paper a dataset for relapse of pulmonary tuberculosis was analysed.

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