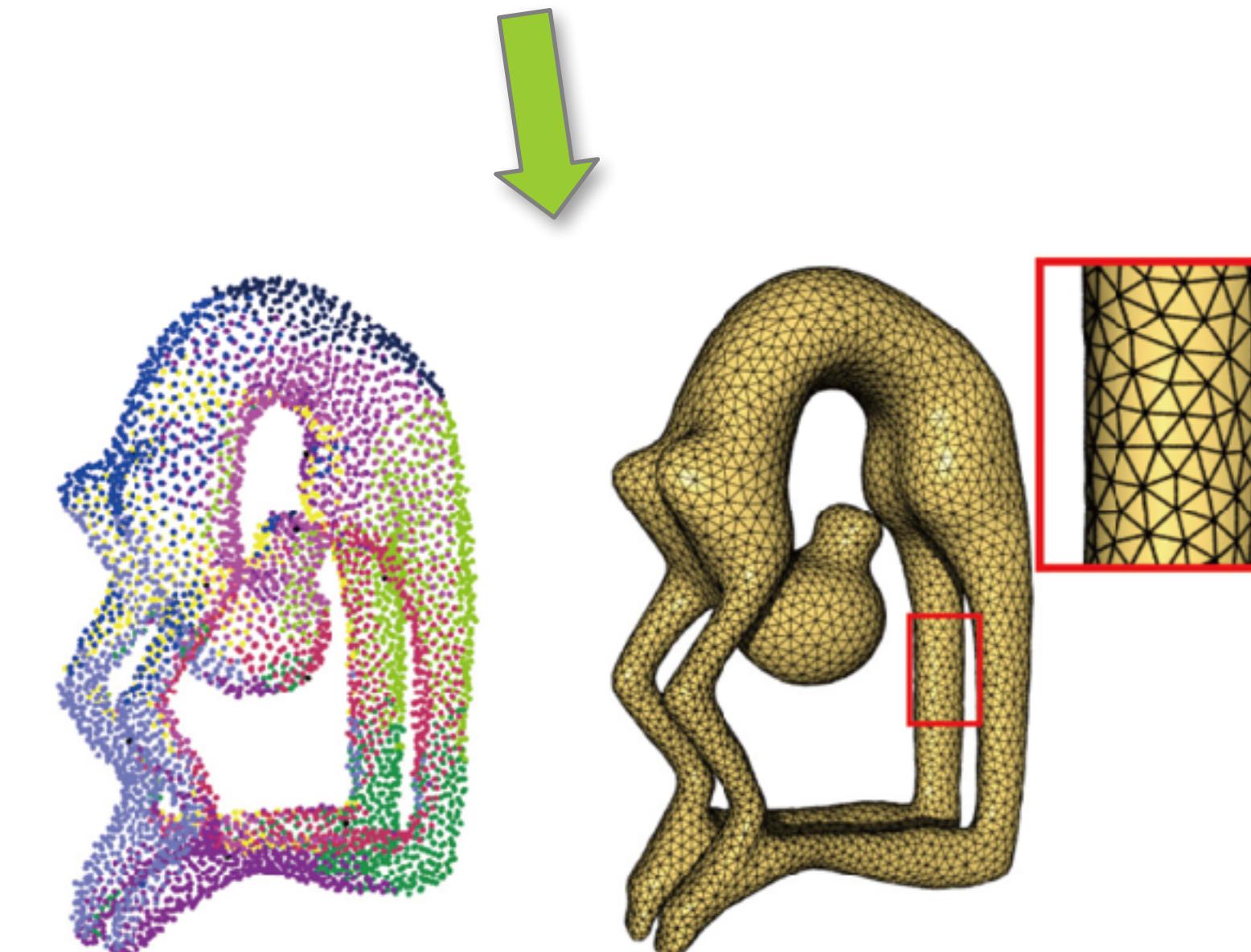
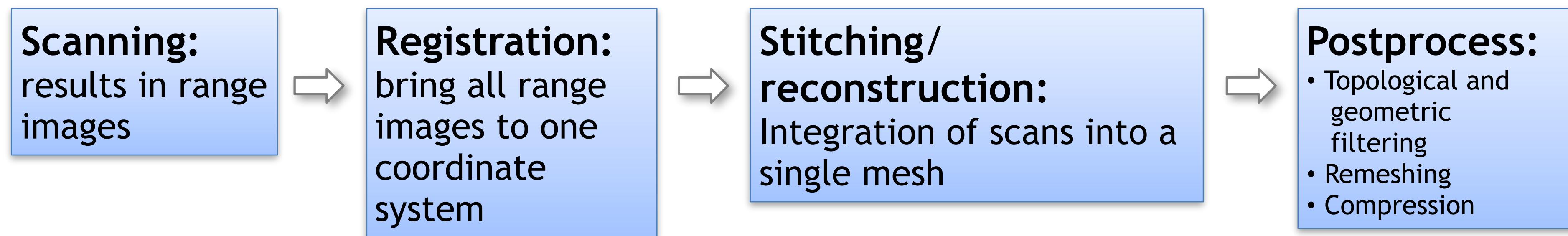


# 03 - Reconstruction

Acknowledgements: Olga Sorkine-Hornung  
CSCI-GA.3033-018 - Geometric Modeling - Daniele Panozzo

# Geometry Acquisition Pipeline



# Digital Michelangelo Project



1G sample points → 8M triangles

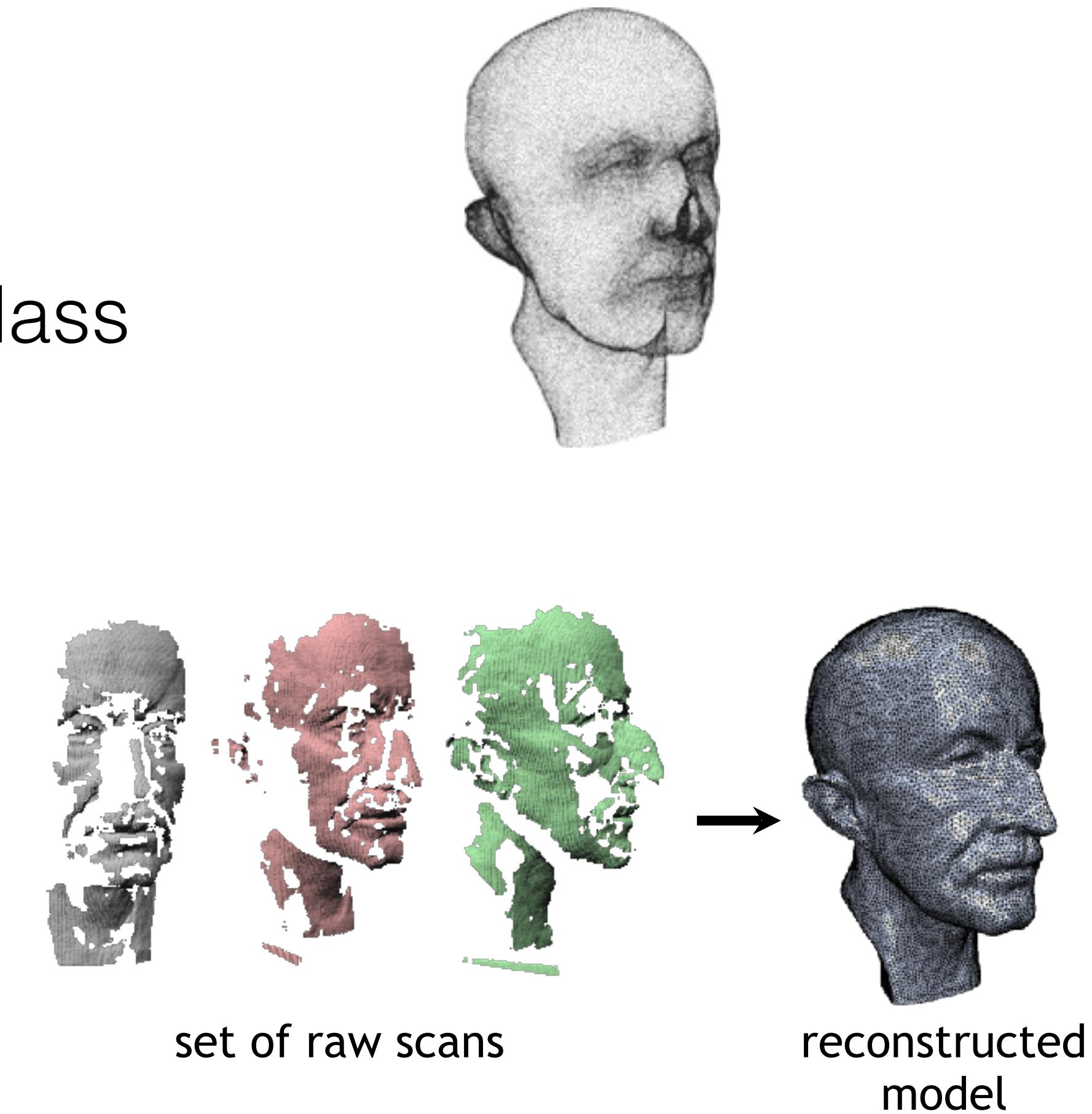


4G sample points → 8M triangles



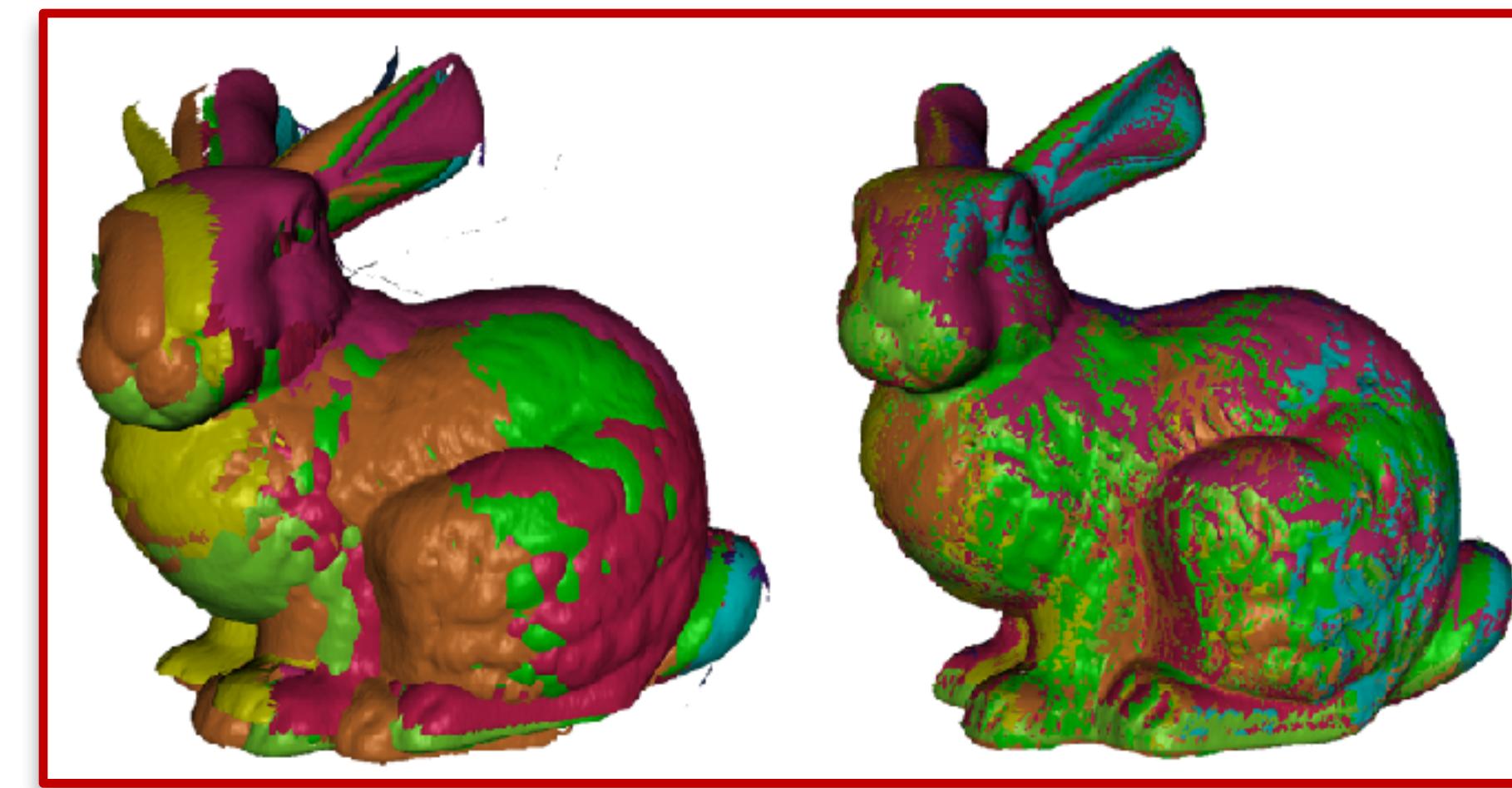
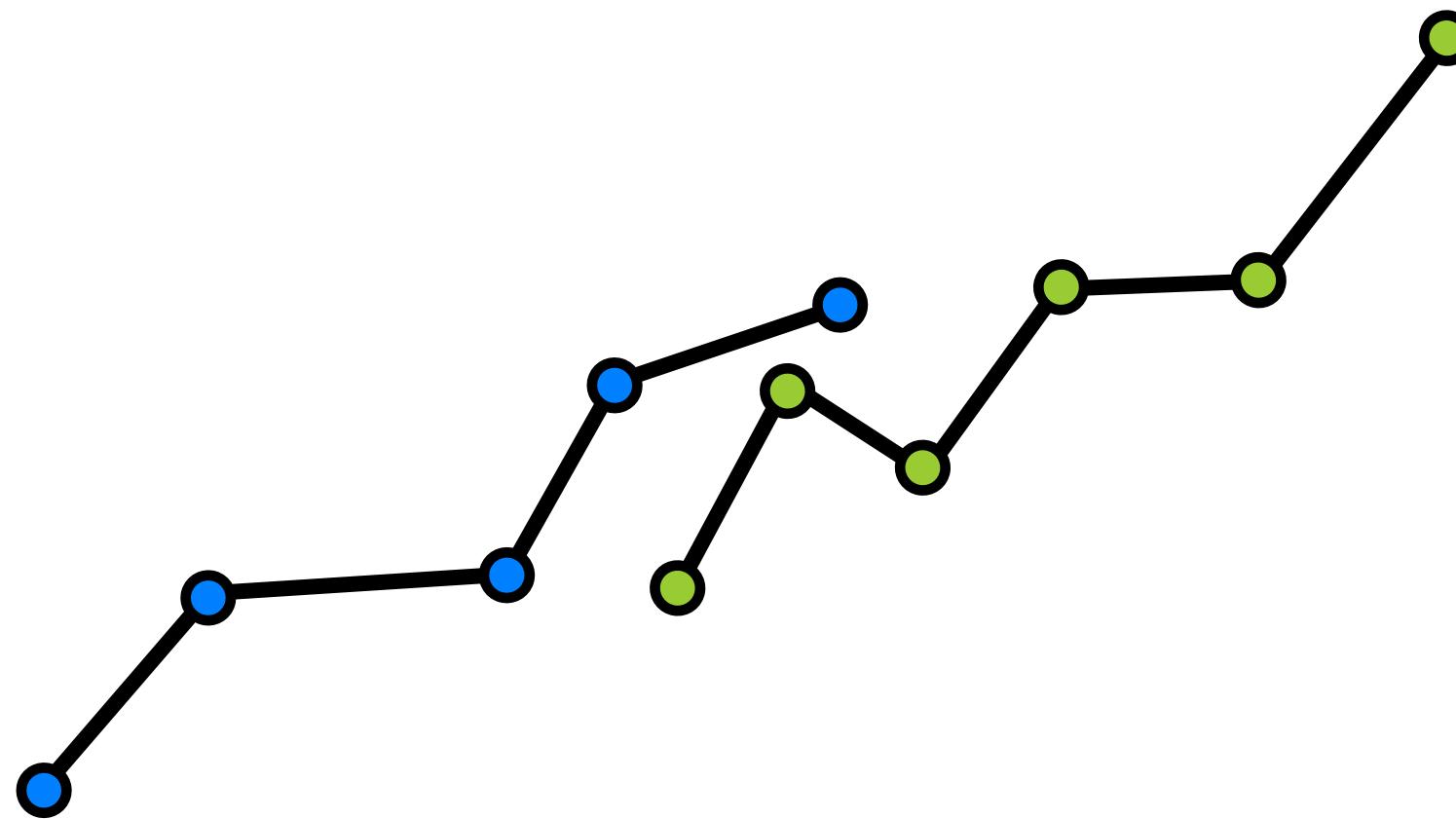
# Input to Reconstruction Process

- Input option 1: just a set of 3D points, irregularly spaced
  - Need to estimate normals → next class
- Input option 2:  
normals come from the range scans



# How to Connect the Dots?

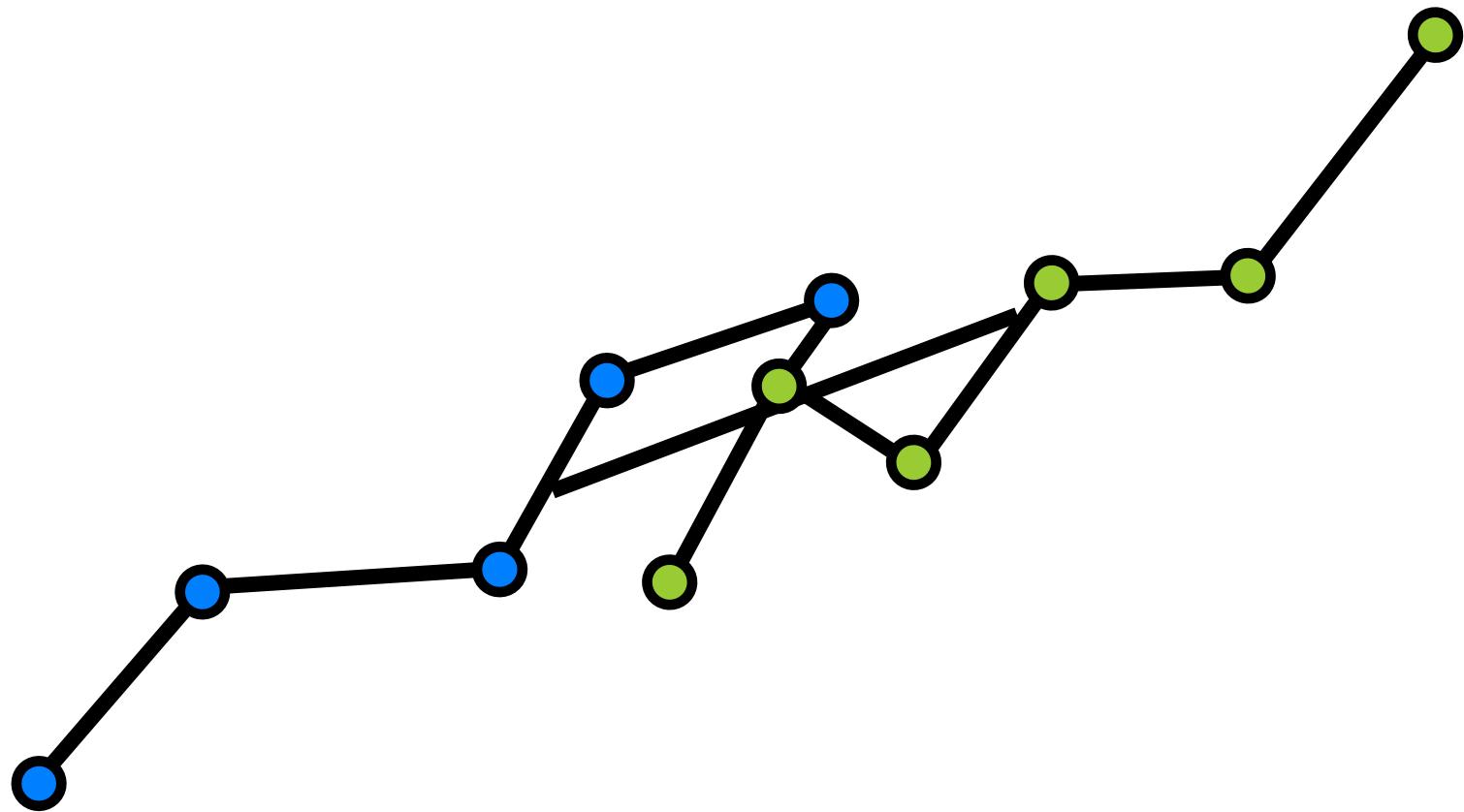
- **Explicit reconstruction:**  
stitch the range scans together



“Zippered Polygon Meshes from Range Images”, Greg Turk and Marc Levoy, ACM SIGGRAPH 1994

# How to Connect the Dots?

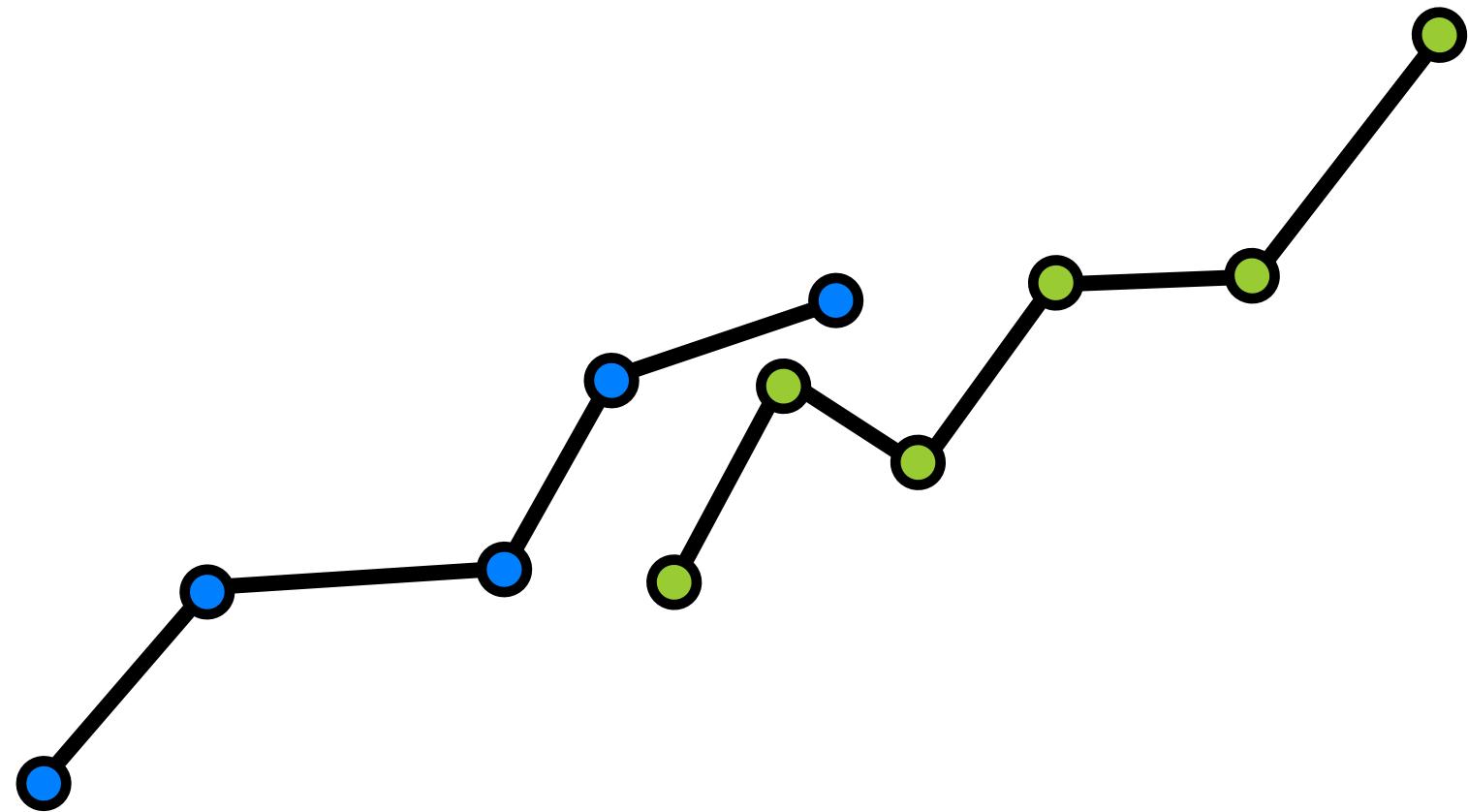
- **Explicit reconstruction:**  
stitch the range scans together



- Connect sample points by triangles
- Exact interpolation of sample points
- Bad for noisy or misaligned data
- Can lead to holes or non-manifold situations

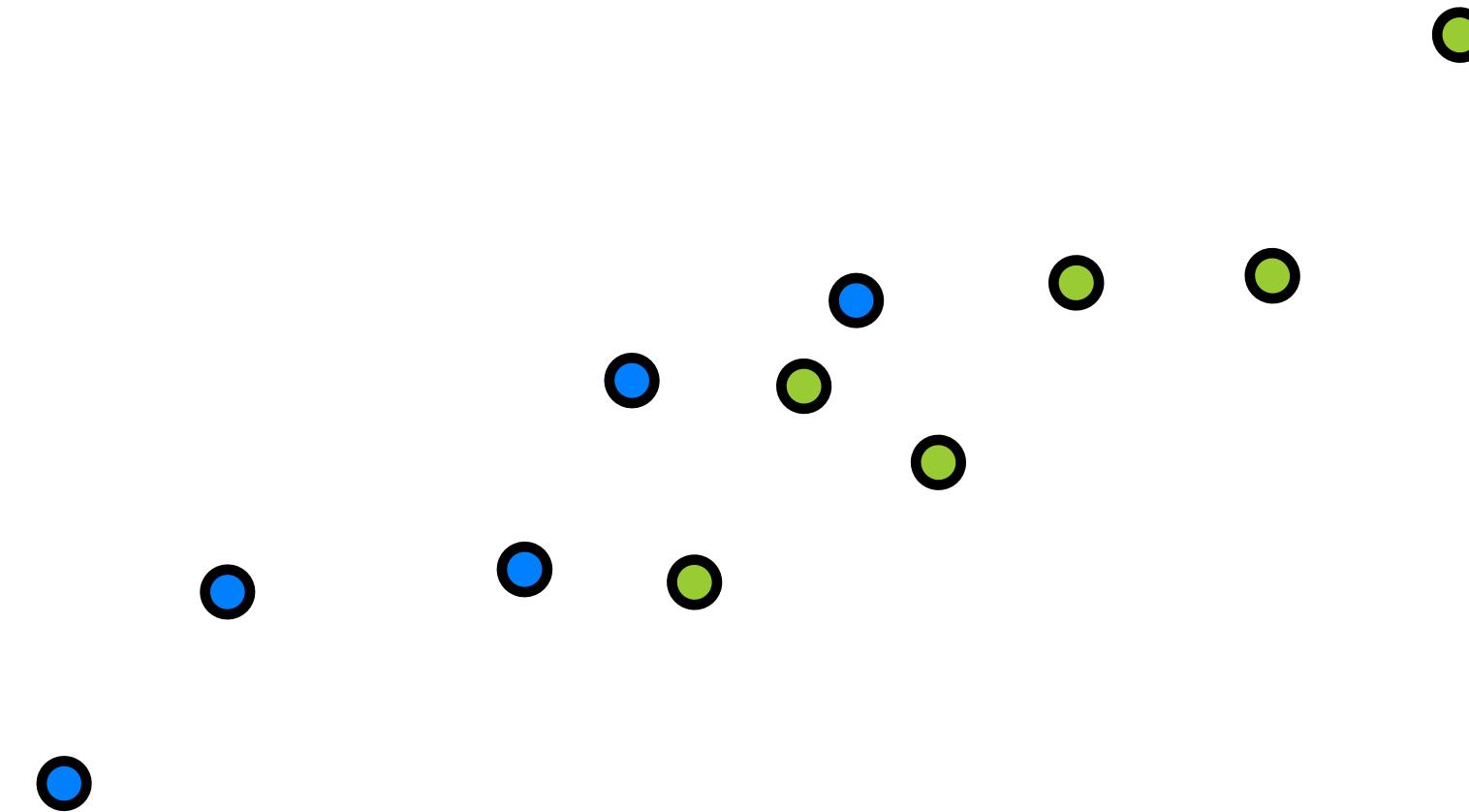
# How to Connect the Dots?

- **Implicit reconstruction:** estimate a signed distance function (SDF); extract 0-level set mesh using Marching Cubes



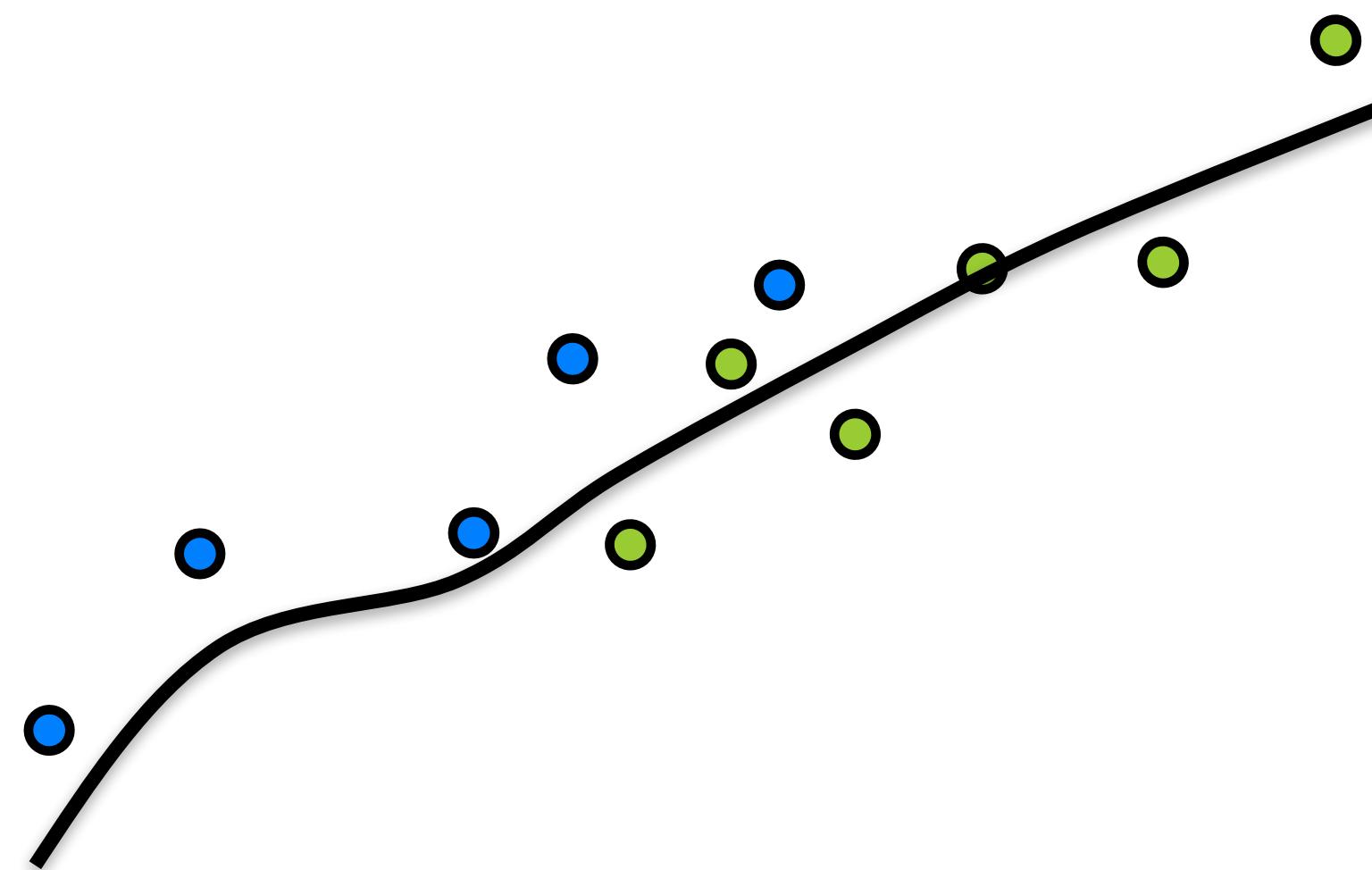
# How to Connect the Dots?

- **Implicit reconstruction:** estimate a signed distance function (SDF); extract 0-level set mesh using Marching Cubes



# How to Connect the Dots?

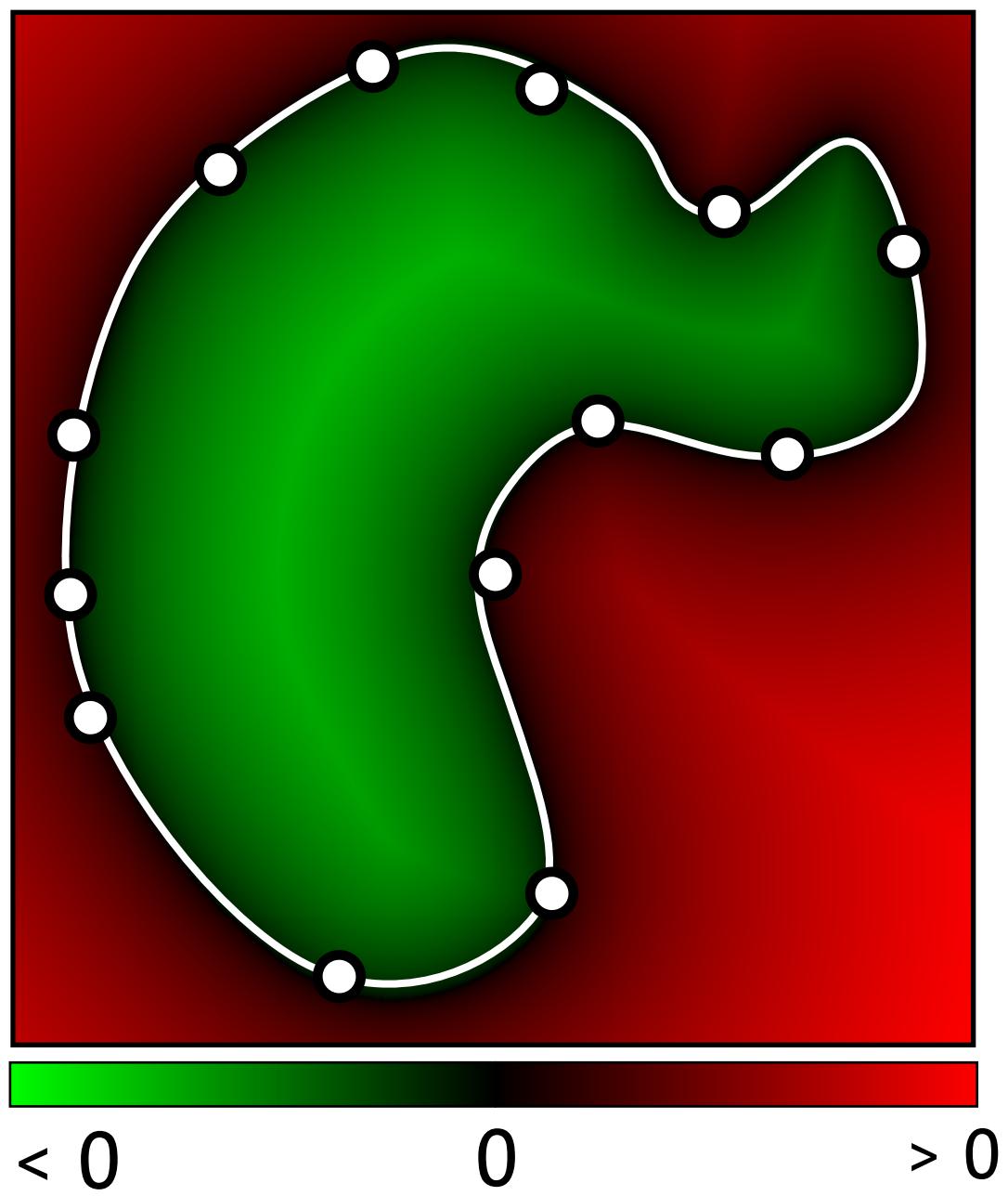
- **Implicit reconstruction:** estimate a signed distance function (SDF); extract 0-level set mesh using Marching Cubes



- Approximation of input points
- Watertight manifold results by construction

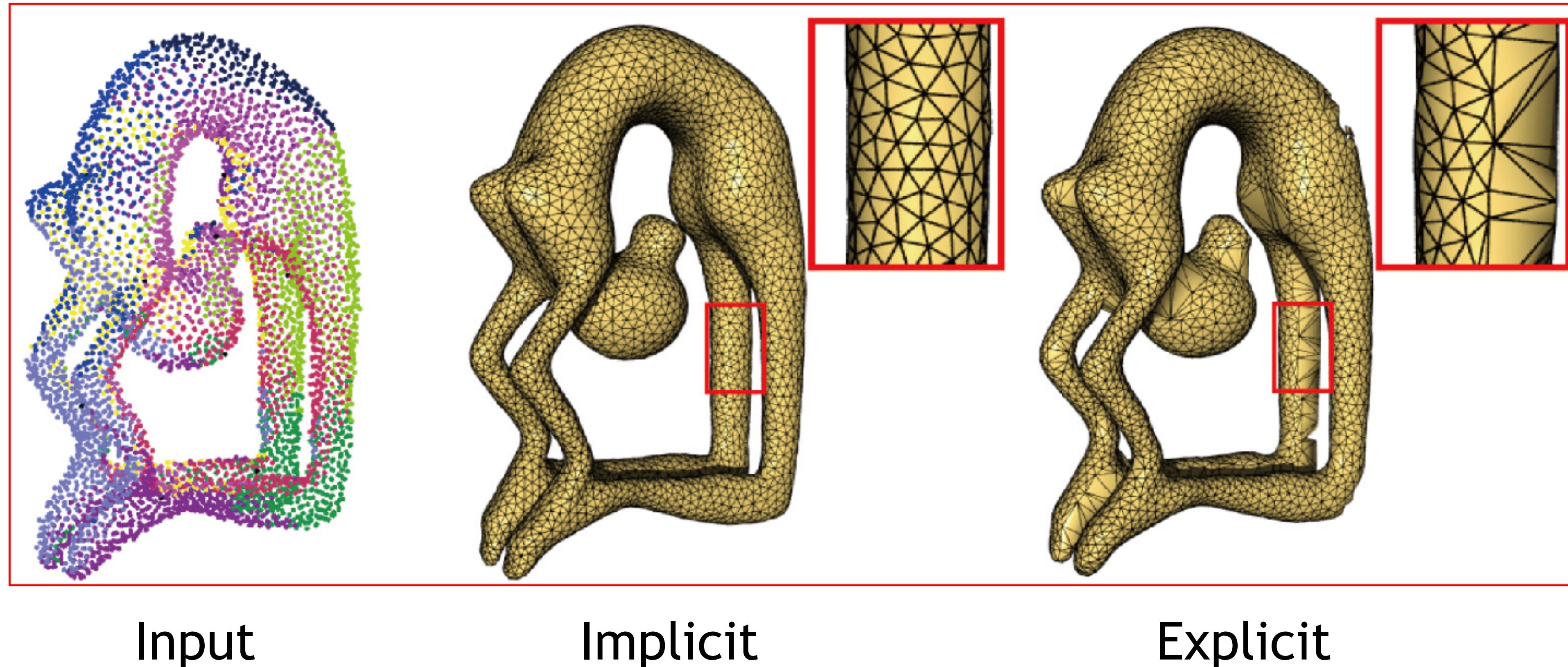
# How to Connect the Dots?

- **Implicit reconstruction:** estimate a signed distance function (SDF); extract 0-level set mesh using Marching Cubes



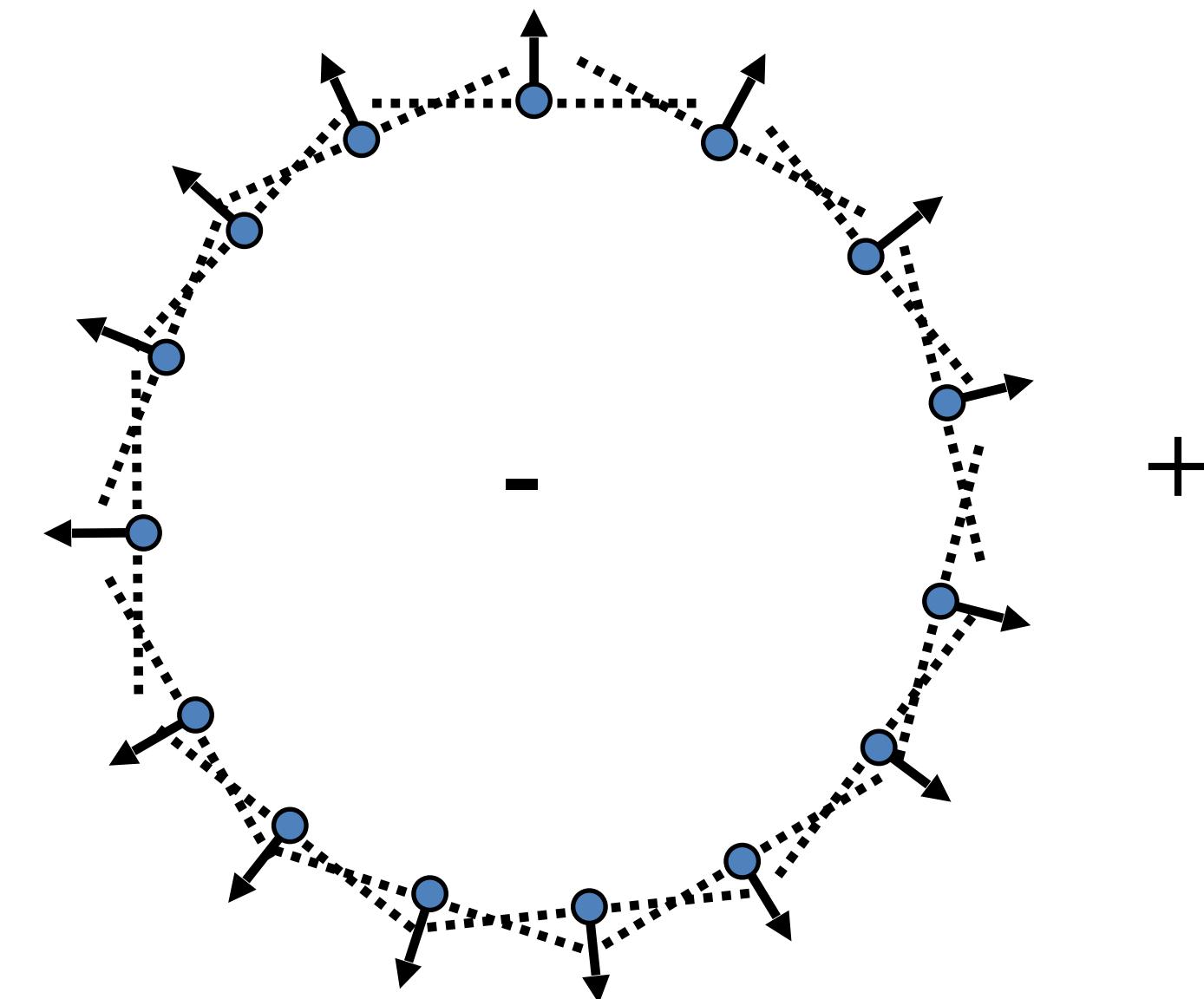
- Approximation of input points
- Watertight manifold results by construction

# Implicit vs. Explicit



# SDF from Points and Normals

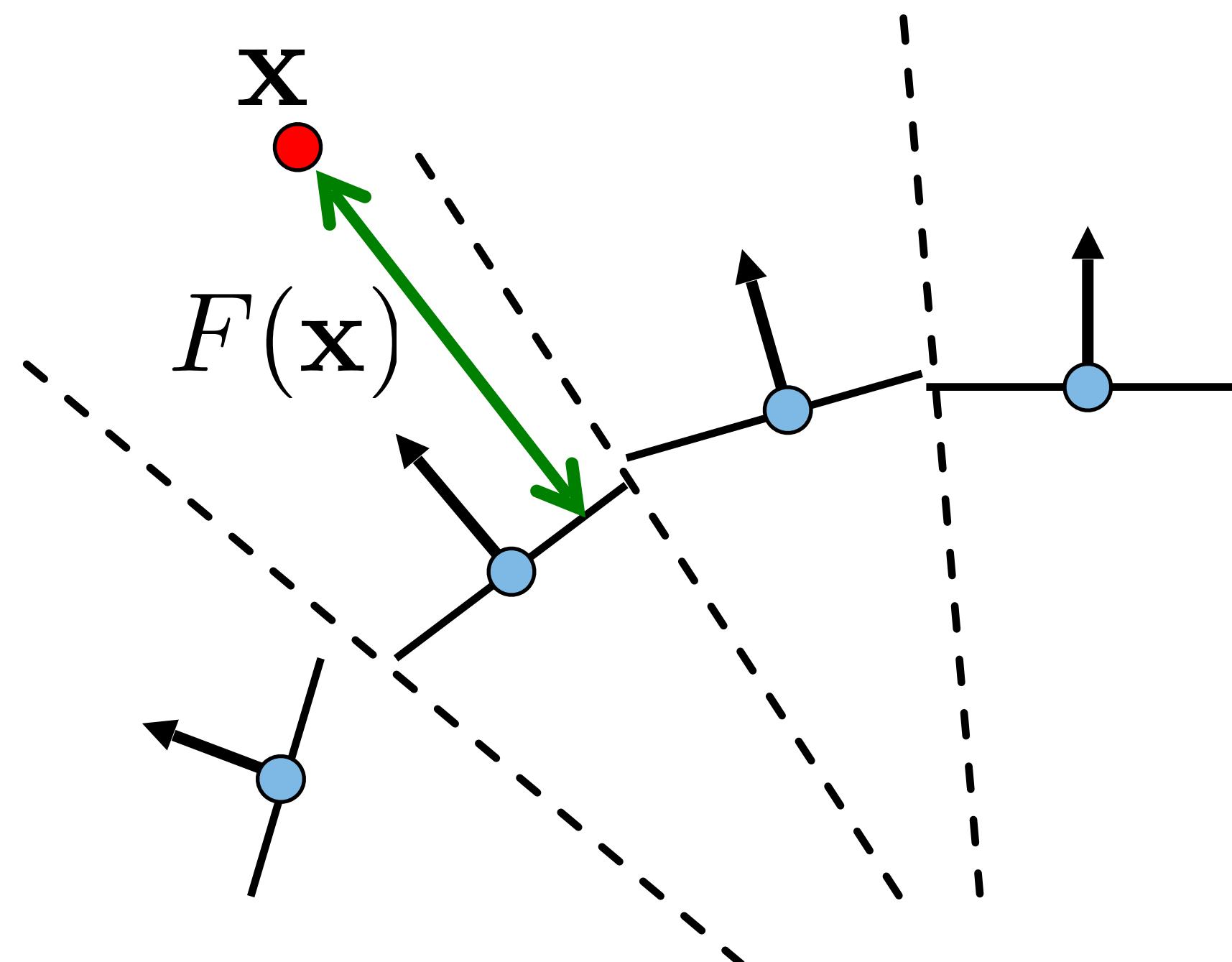
- Compute signed distance to the tangent plane of the closest point
- Normals help to distinguish between inside and outside



“Surface reconstruction from unorganized points”, Hoppe et al., ACM SIGGRAPH 1992  
<http://research.microsoft.com/en-us/um/people/hoppe/proj/recon/>

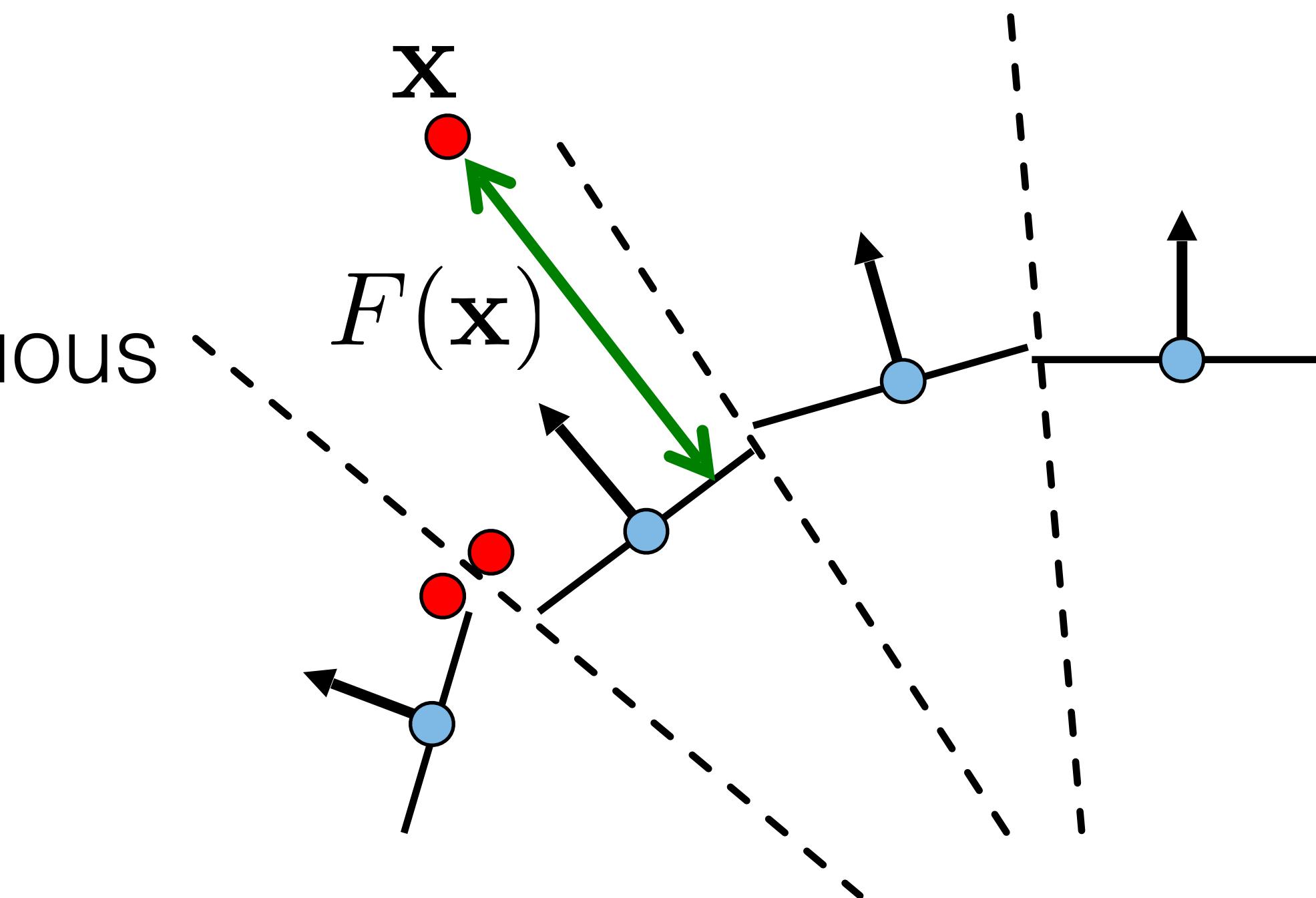
# SDF from Points and Normals

- Compute signed distance to the tangent plane of the closest point
- Problem??



# SDF from Points and Normals

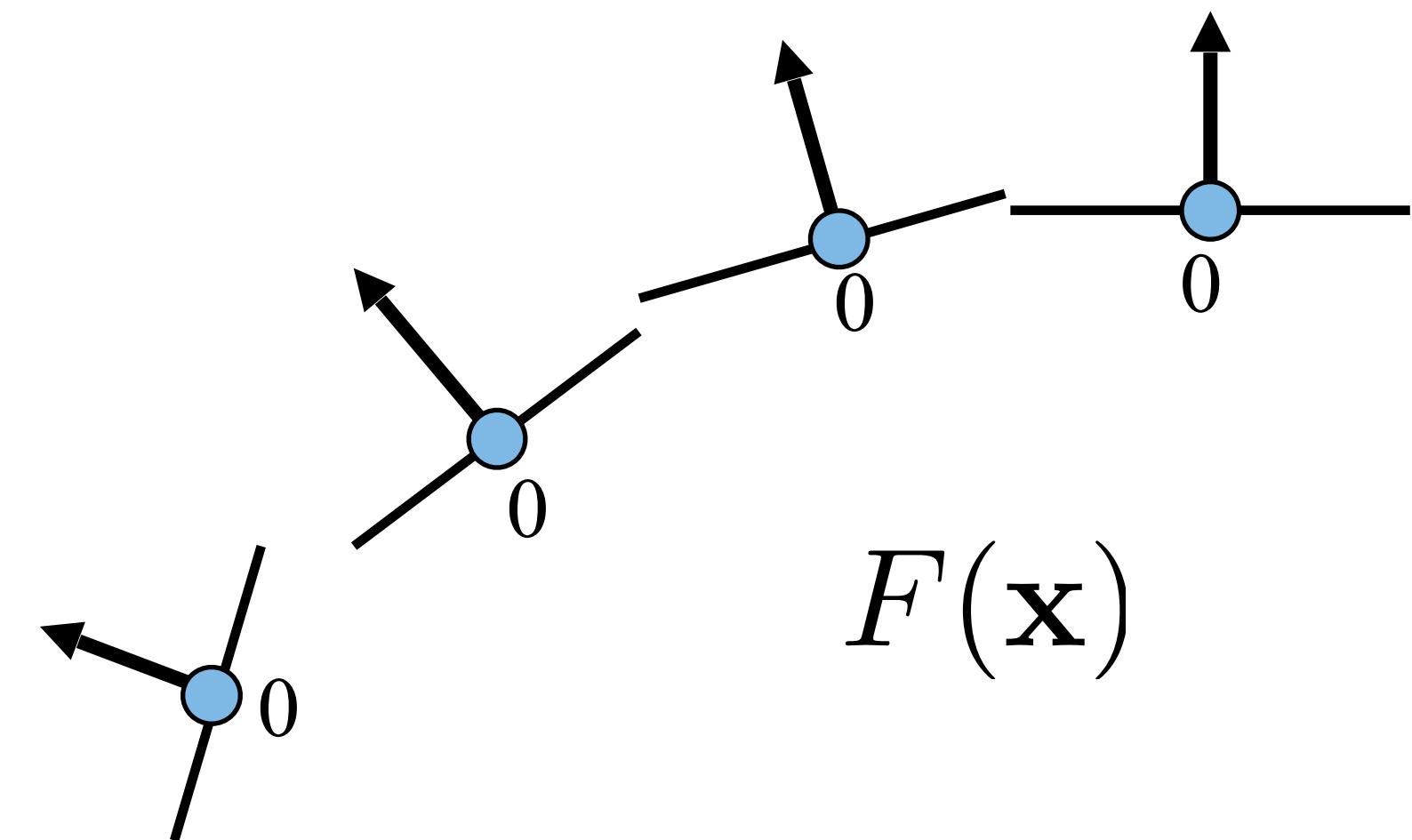
- Compute signed distance to the tangent plane\* of the closest point
- The function will be discontinuous



\* The Hoppe92 paper computes the tangent planes slightly differently (by PCA on k-nearest-neighbors of each data point, see next class), but the consequences are still the same.

# Smooth SDF

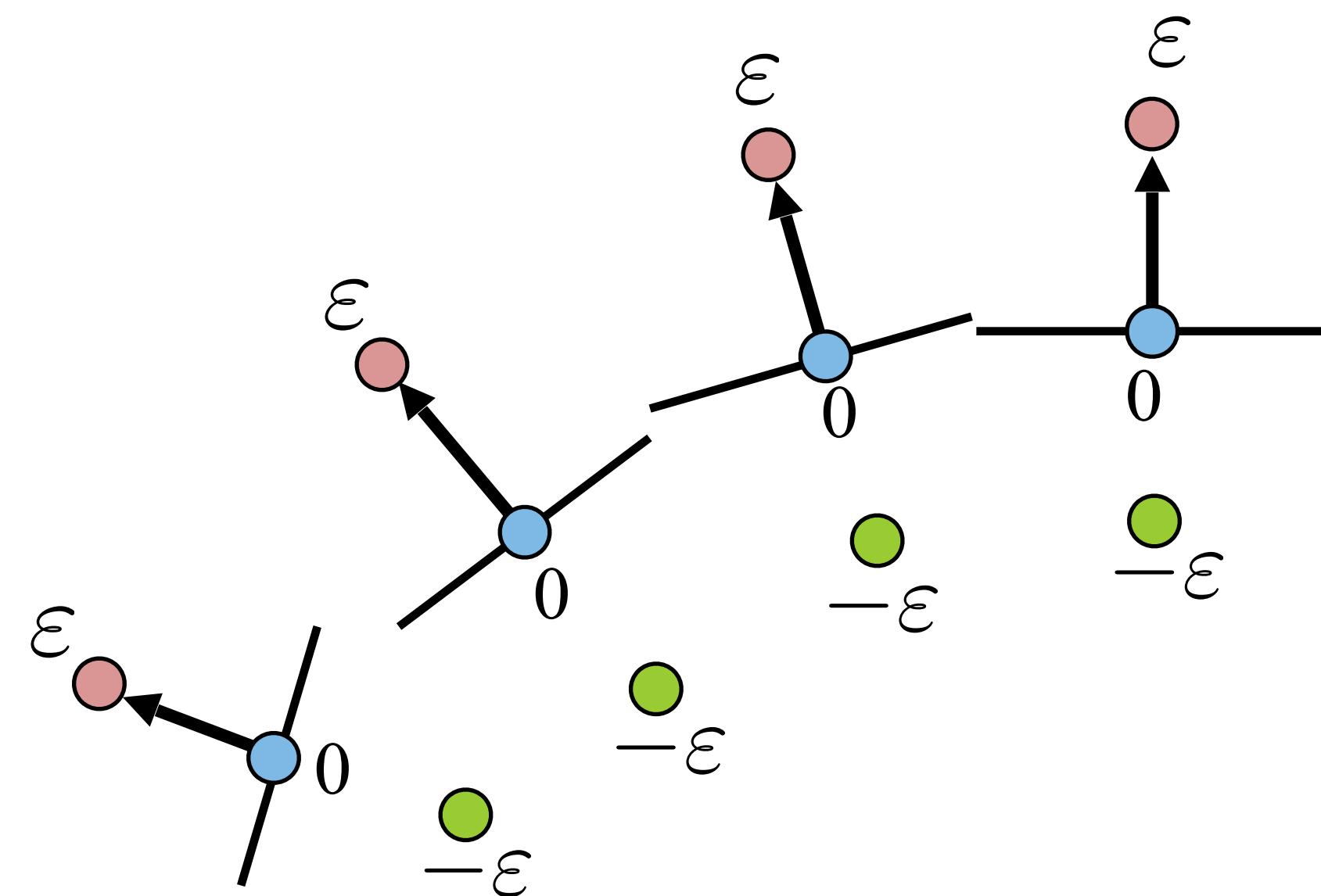
- Instead find a smooth formulation for  $F$ .
- Scattered data interpolation:
  - $F(\mathbf{p}_i) = 0$
  - $F$  is smooth
  - Avoid trivial  $F \equiv 0$



“Reconstruction and representation of 3D objects with radial basis functions”, Carr et al., ACM SIGGRAPH 2001

# Smooth SDF

- Scattered data interpolation:
  - $F(\mathbf{p}_i) = 0$
  - $F$  is smooth
  - Avoid trivial  $F \equiv 0$
- Add off-surface constraints



$$F(\mathbf{p}_i + \varepsilon \mathbf{n}_i) = \varepsilon$$

$$F(\mathbf{p}_i - \varepsilon \mathbf{n}_i) = -\varepsilon$$

“Reconstruction and representation of 3D objects with radial basis functions”, Carr et al., ACM SIGGRAPH 2001

# Radial Basis Function Interpolation

- RBF: Weighted sum of shifted, smooth kernels

$$F(\mathbf{x}) = \sum_{i=0}^{N-1} w_i \varphi(\|\mathbf{x} - \mathbf{c}_i\|)$$

Scalar weights  
Unknowns

Smooth kernels  
(basis functions)  
centered at constrained  
points.  
For example:  
 $\varphi(r) = r^3$

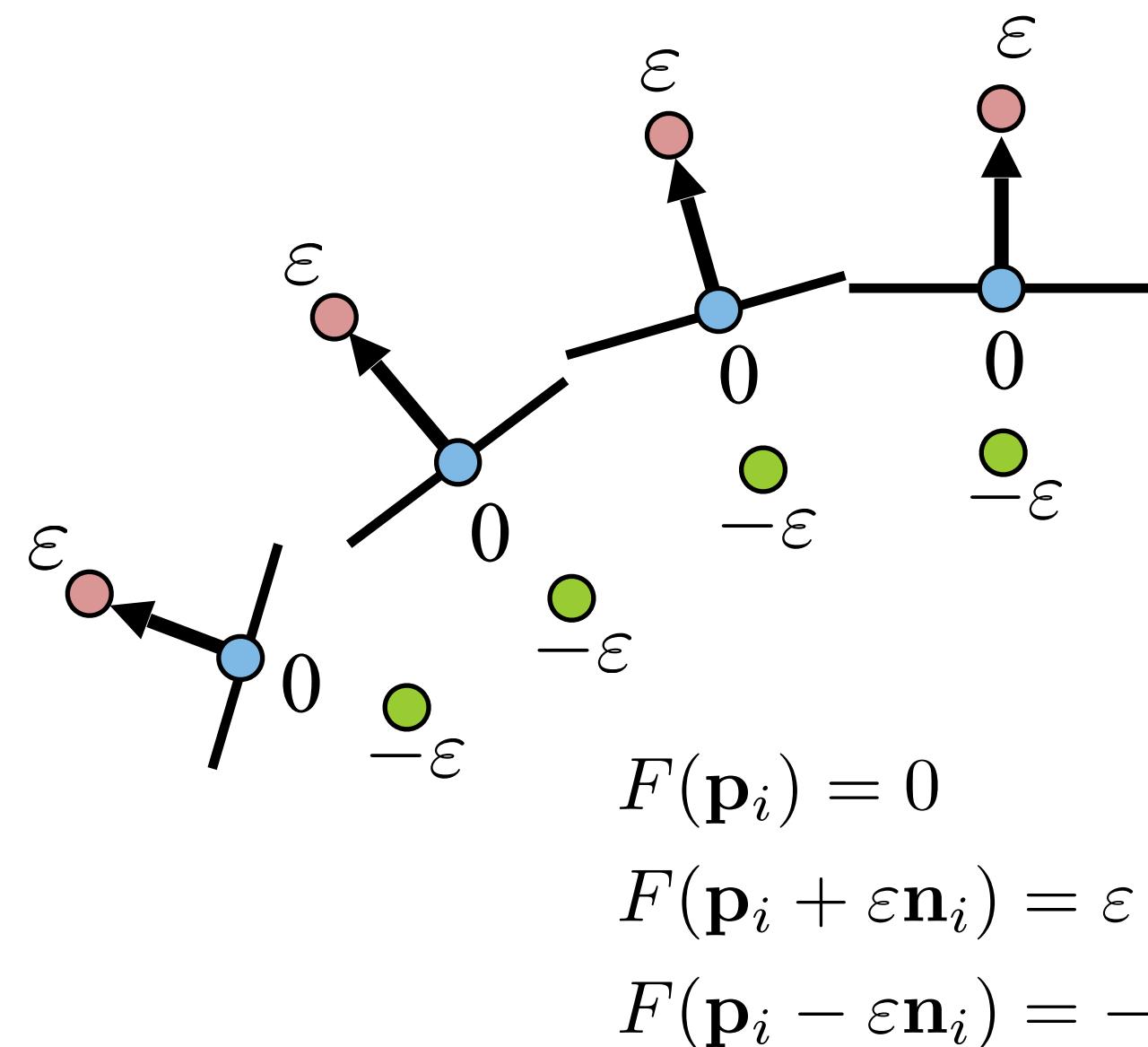
$$N = 3n$$

# Radial Basis Function Interpolation

- Interpolate the constraints:

$$\{\mathbf{c}_{3i}, \mathbf{c}_{3i+1}, \mathbf{c}_{3i+2}\} = \{\mathbf{p}_i, \mathbf{p}_i + \varepsilon \mathbf{n}_i, \mathbf{p}_i - \varepsilon \mathbf{n}_i\}$$

$$\forall j = 0, \dots, N-1, \quad \sum_{i=0}^{N-1} w_i \varphi(\|\mathbf{c}_j - \mathbf{c}_i\|) = d_j$$



# Radial Basis Function Interpolation

- Interpolate the constraints:

$$\{\mathbf{c}_{3i}, \mathbf{c}_{3i+1}, \mathbf{c}_{3i+2}\} = \{\mathbf{p}_i, \mathbf{p}_i + \varepsilon \mathbf{n}_i, \mathbf{p}_i - \varepsilon \mathbf{n}_i\}$$

- Symmetric linear system to get the weights:

$$\begin{pmatrix} \varphi(\|\mathbf{c}_0 - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_0 - \mathbf{c}_{N-1}\|) \\ \vdots & \ddots & \vdots \\ \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_{N-1}\|) \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \end{pmatrix} = \begin{pmatrix} d_0 \\ \vdots \\ d_{N-1} \end{pmatrix}$$

# RBF Kernels

$$\varphi(r) = r^3$$

- Triharmonic:
  - Globally supported
  - Leads to dense symmetric linear system
  - $C^2$  smoothness
  - Works well for highly irregular sampling

# RBF Kernels

- Polyharmonic spline

$$\varphi(r) = r^k \log(r), \quad k = 2, 4, 6 \dots$$
$$\varphi(r) = r^k, \quad k = 1, 3, 5 \dots$$

- Multiquadratic

$$\varphi(r) = \sqrt{r^2 + \beta^2}$$

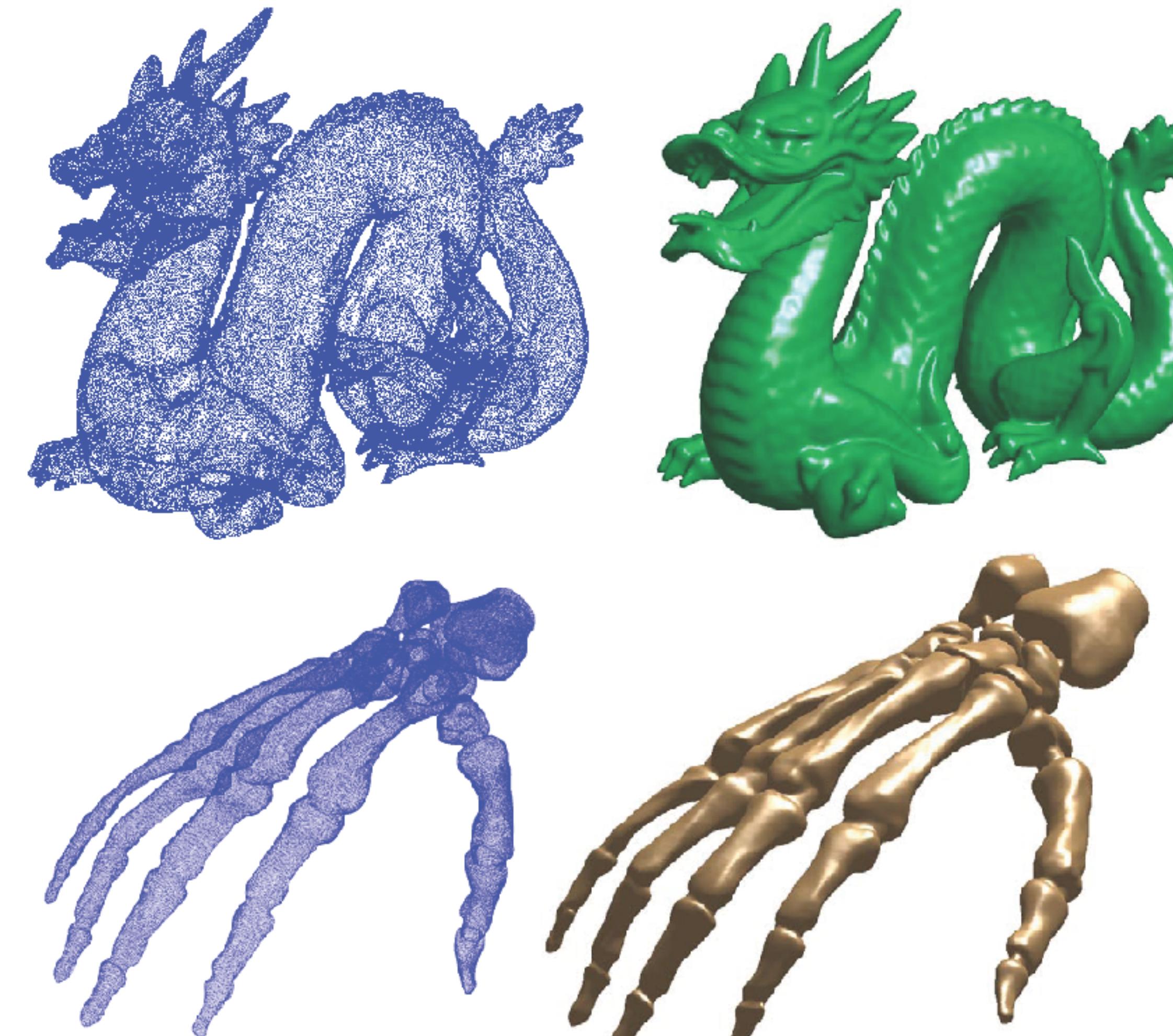
- Gaussian

$$\varphi(r) = e^{-\beta r^2}$$

- B-Spline (compact support)

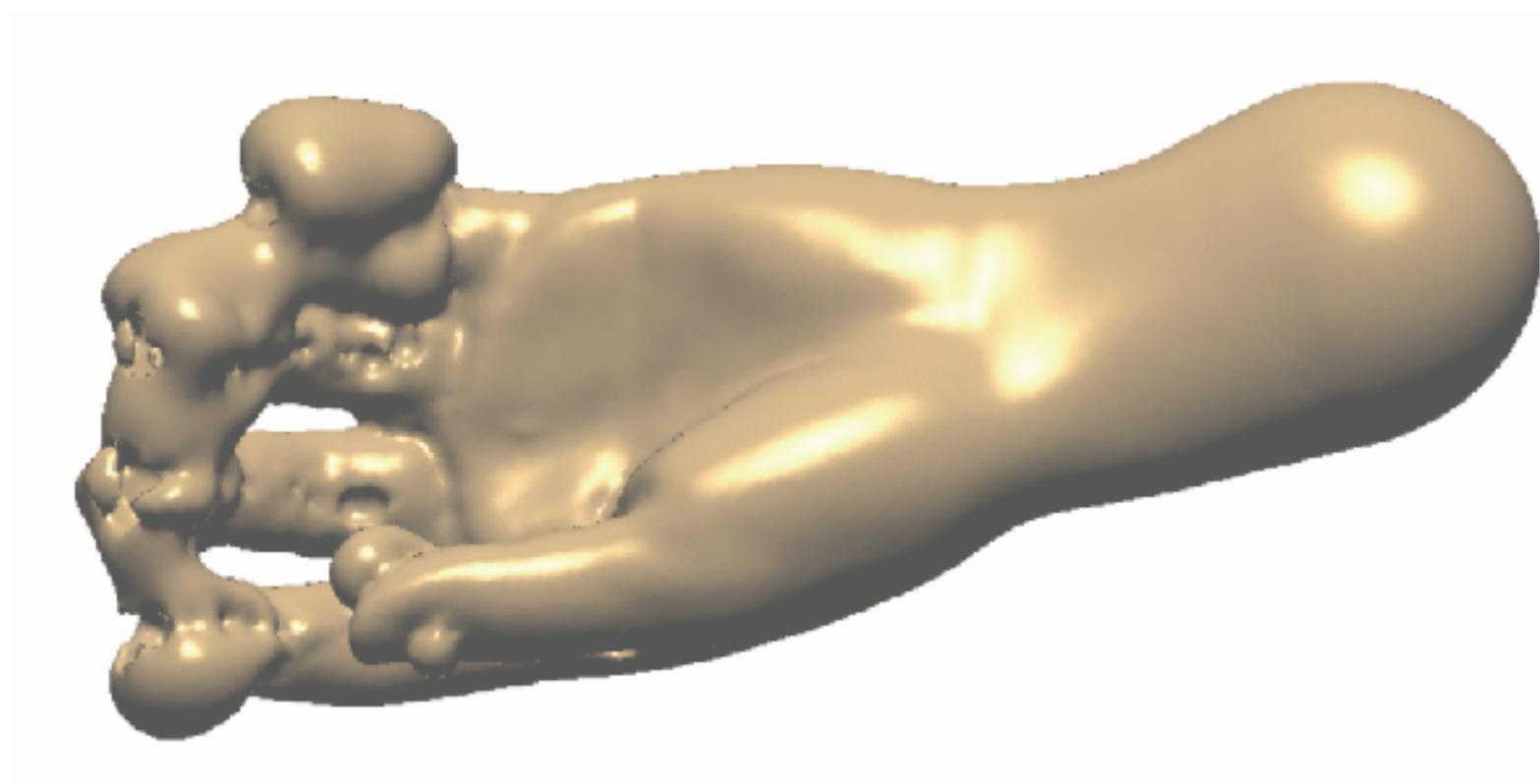
$$\varphi(r) = \text{piecewise-polynomial}(r)$$

# RBF Reconstruction Examples

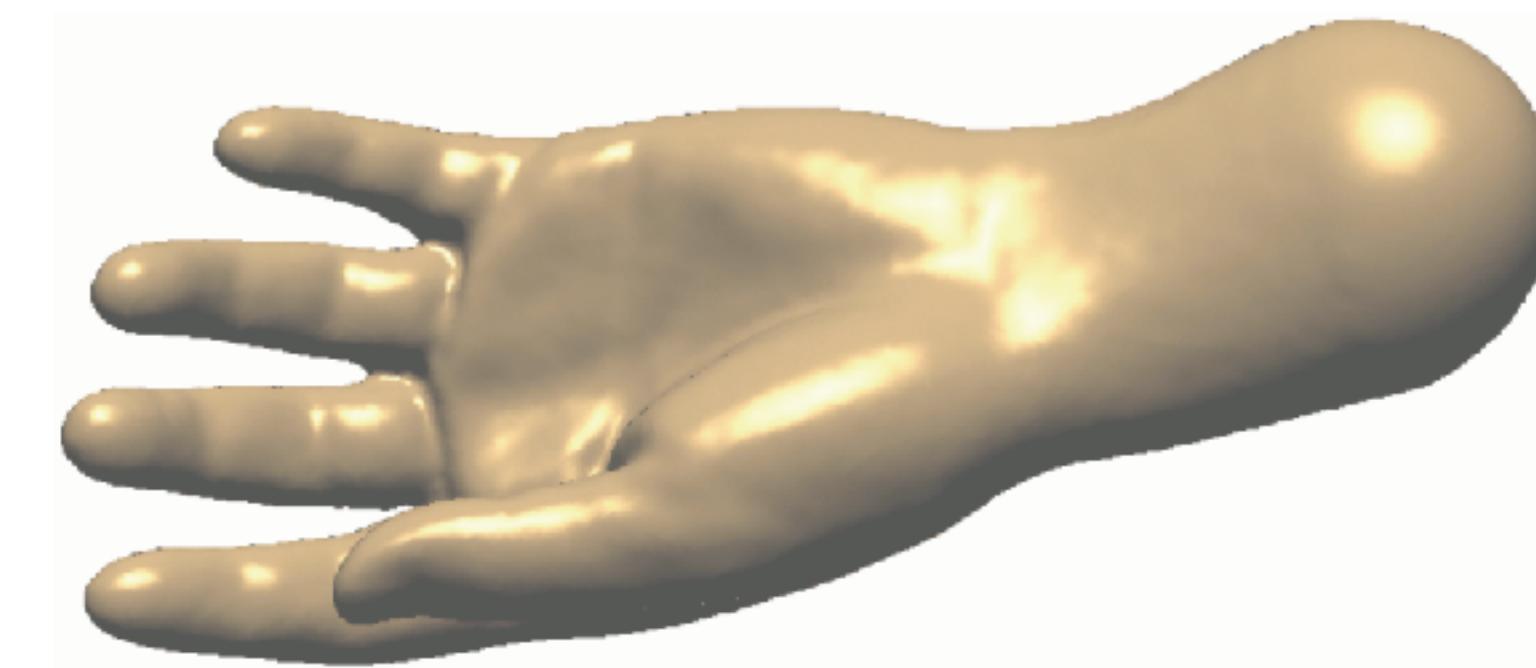


“Reconstruction and representation of 3D objects with radial basis functions”, Carr et al., ACM SIGGRAPH 2001

# Off-Surface Points



Insufficient number/  
badly placed off-surface points



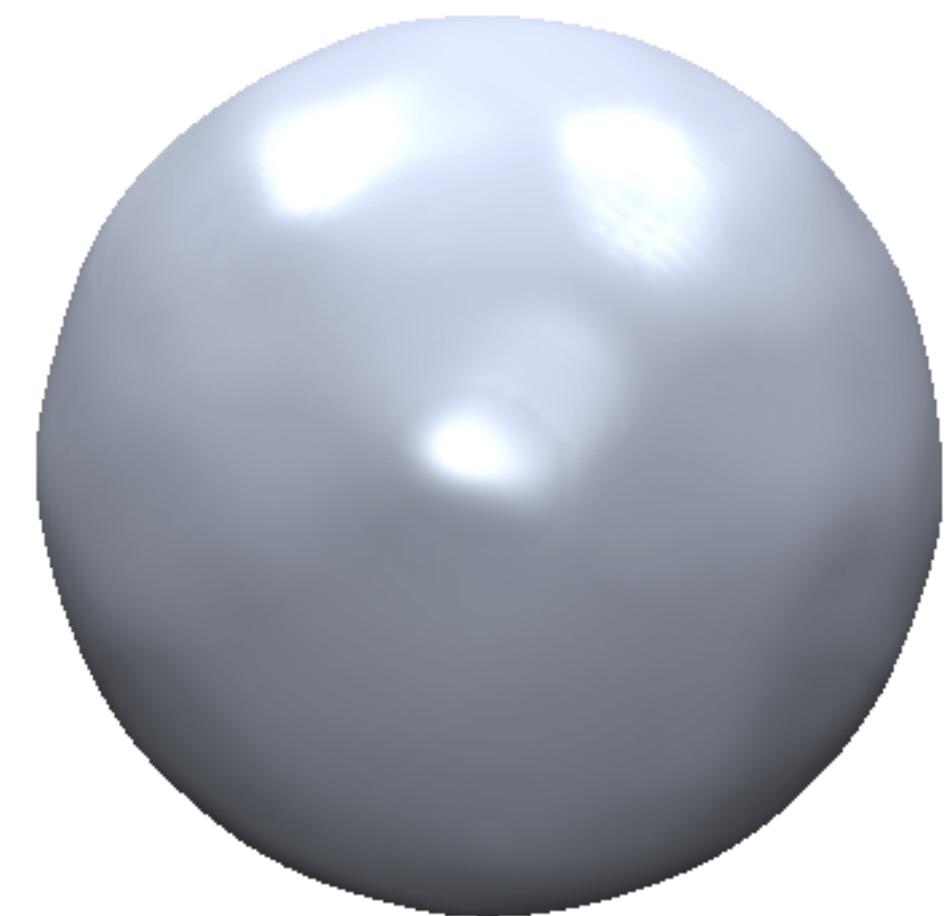
Properly chosen off-surface points

“Reconstruction and representation of 3D objects with radial basis functions”, Carr et al., ACM SIGGRAPH 2001

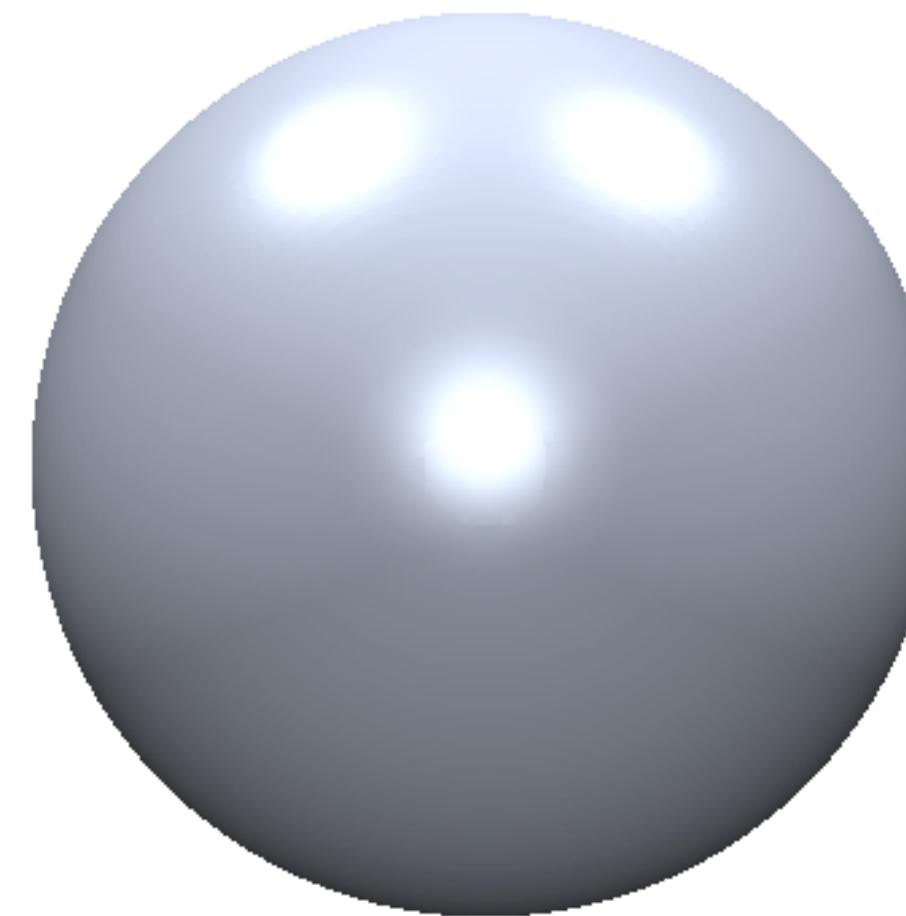
# Comparison of the various SDFs so far



Distance  
to plane



Compact RBF



Global RBF  
Triharmonic

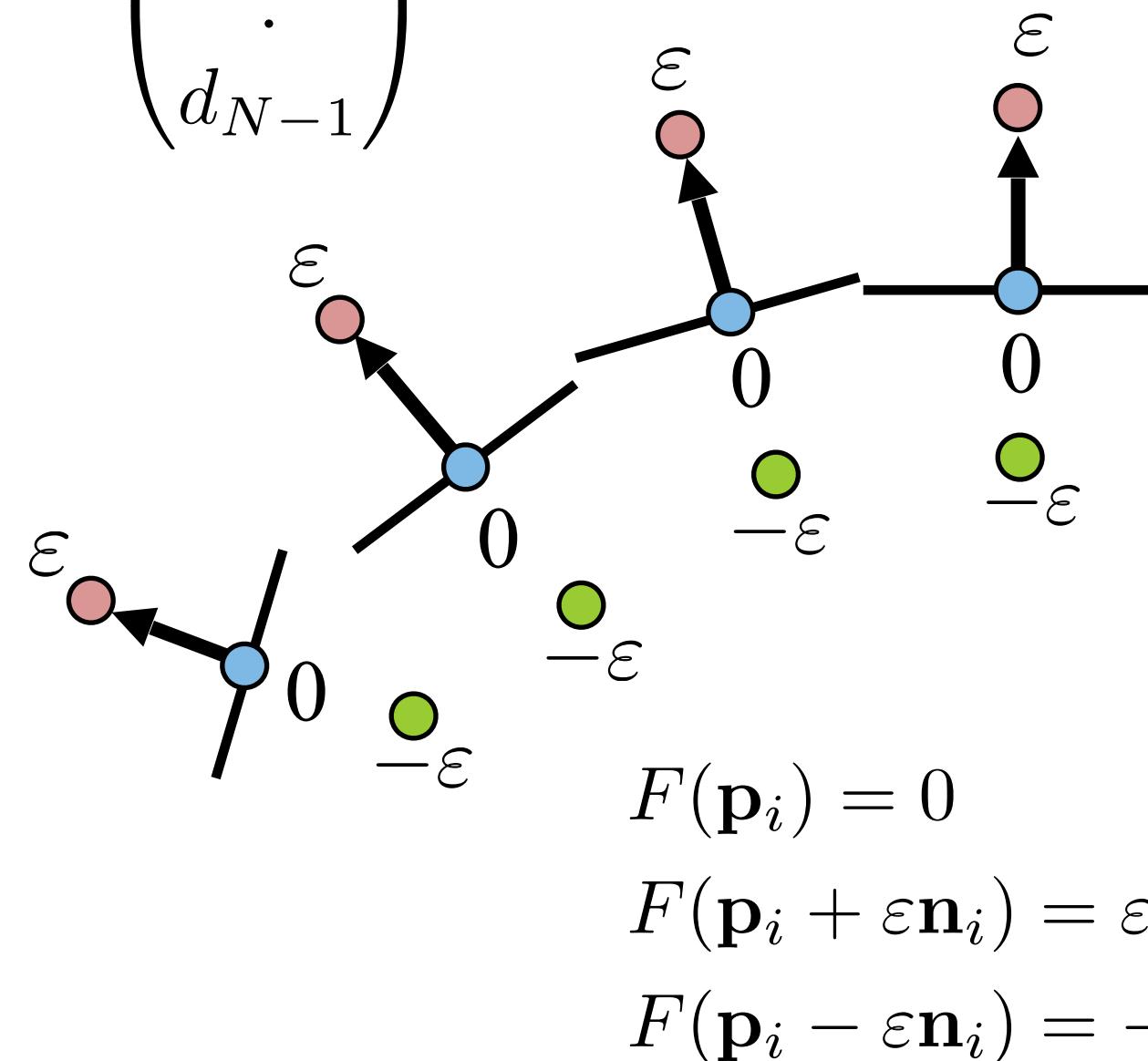
# RBF – Discussion

- Global definition!

$$F(\mathbf{x}) = \sum_{i=0}^{N-1} w_i \varphi(\|\mathbf{x} - \mathbf{c}_i\|)$$

$$\{\mathbf{c}_{3i}, \mathbf{c}_{3i+1}, \mathbf{c}_{3i+2}\} = \{\mathbf{p}_i, \mathbf{p}_i + \varepsilon \mathbf{n}_i, \mathbf{p}_i - \varepsilon \mathbf{n}_i\}$$

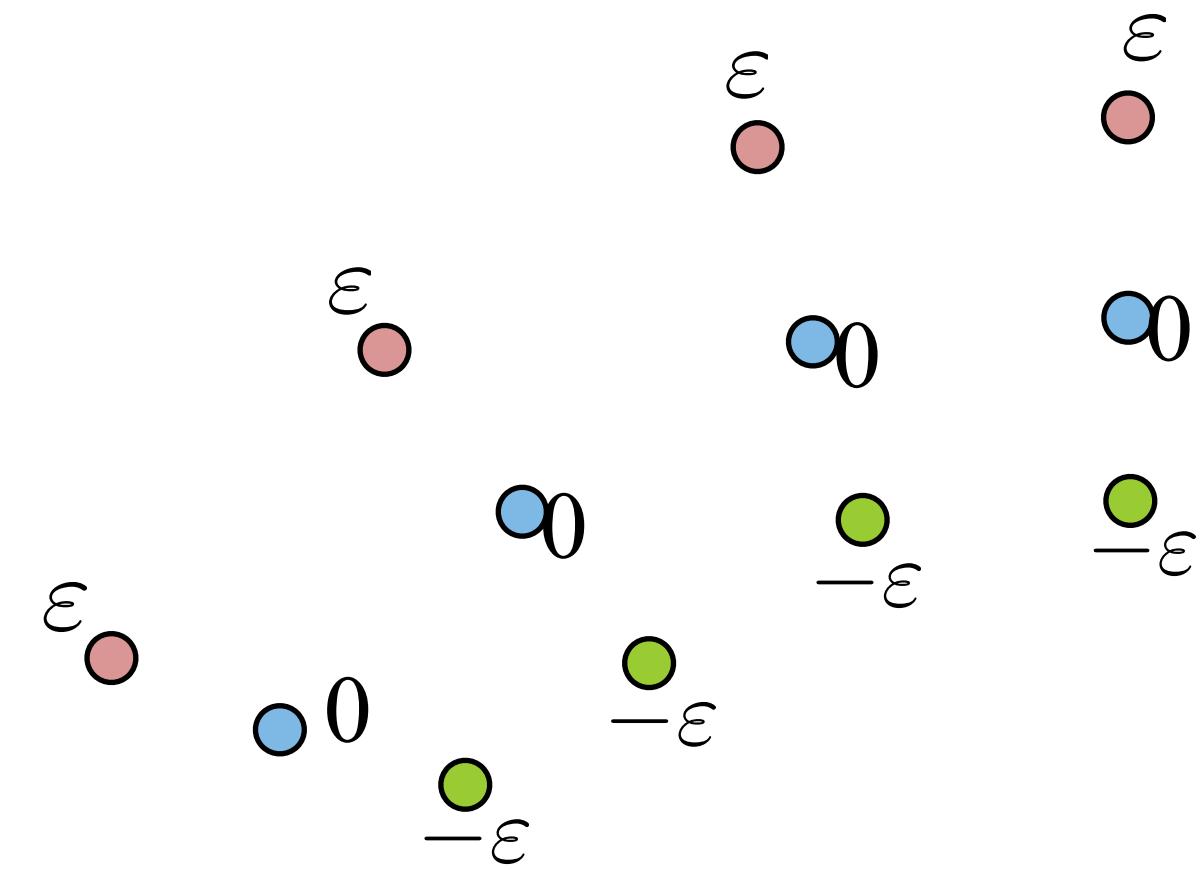
$$\begin{pmatrix} \varphi(\|\mathbf{c}_0 - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_0 - \mathbf{c}_{N-1}\|) \\ \vdots & \ddots & \vdots \\ \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_0\|) & \dots & \varphi(\|\mathbf{c}_{N-1} - \mathbf{c}_{N-1}\|) \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \end{pmatrix} = \begin{pmatrix} d_0 \\ \vdots \\ d_{N-1} \end{pmatrix}$$



- Global optimization of the weights, even if the basis functions are local

# Moving Least Squares (MLS)

- Do purely **local** approximation of the SDF
- Weights change depending on where we are evaluating
- The beauty: the “stitching” of all local approximations, seen as one function  $F(\mathbf{x})$ , is smooth everywhere!
  - We get a **globally** smooth function but only do **local** computation

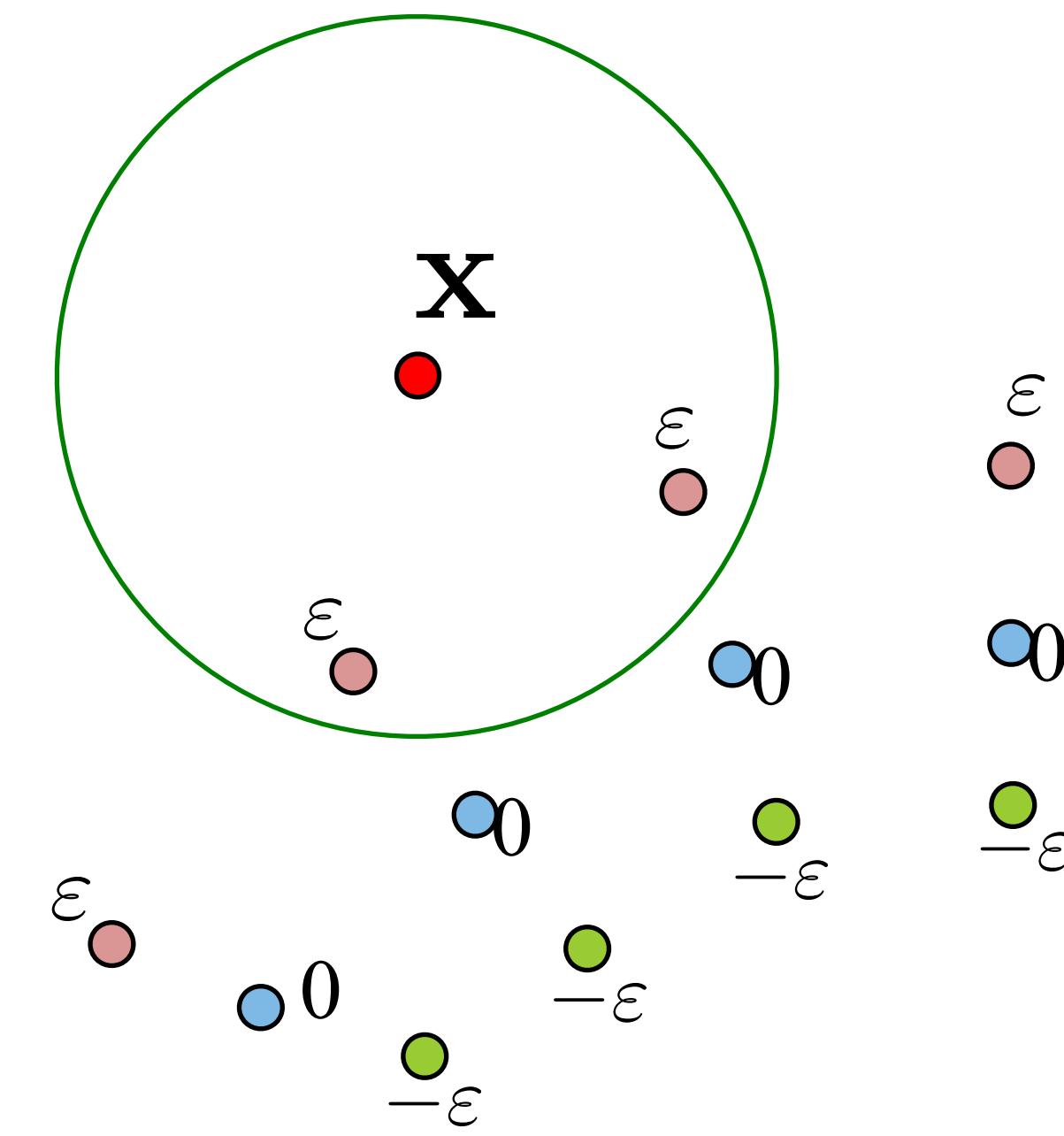


“Interpolating and Approximating Implicit Surfaces from Polygon Soup”, Shen et al.,  
ACM SIGGRAPH 2004

<http://graphics.berkeley.edu/papers/Shen-IAI-2004-08/index.html>

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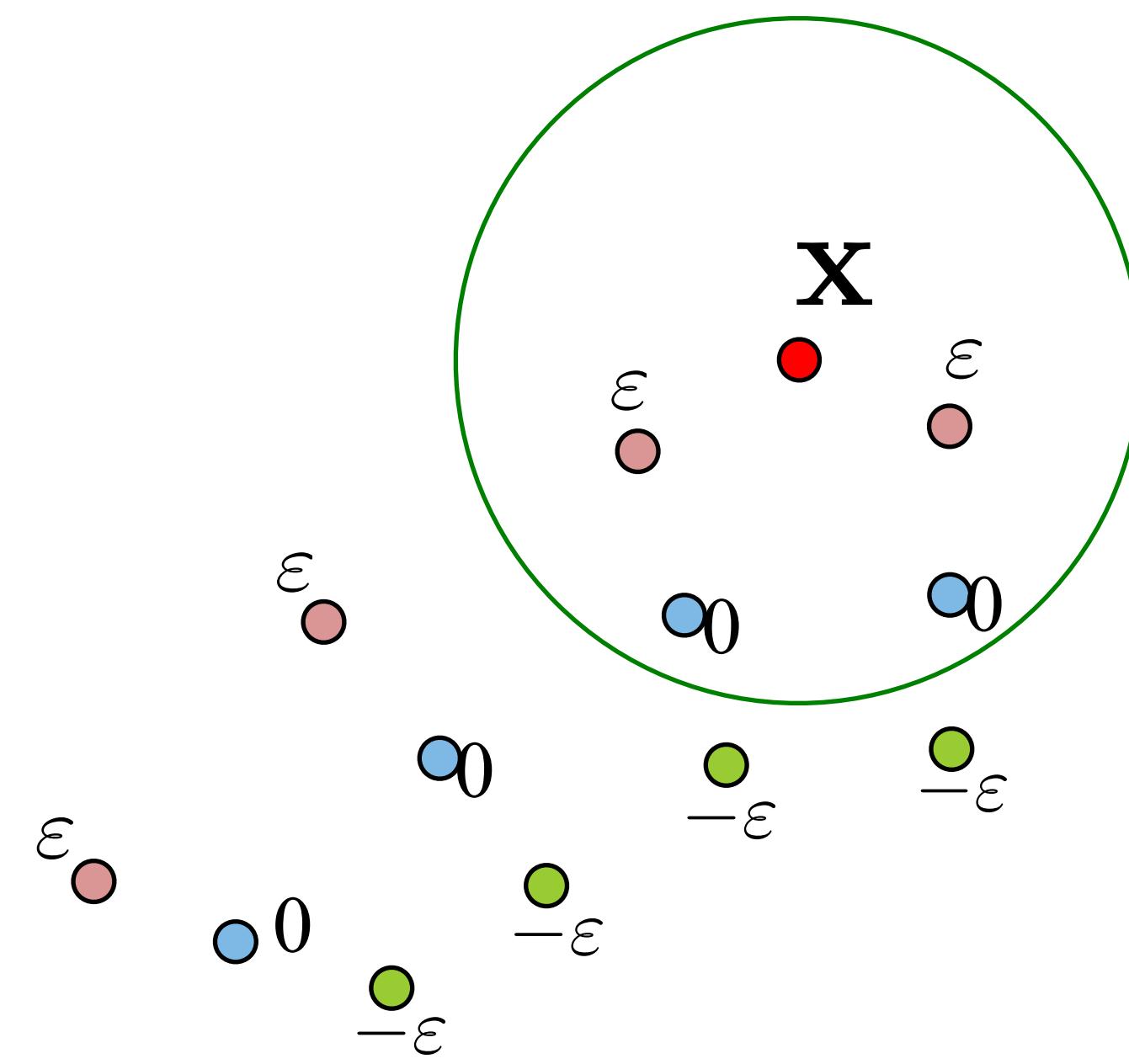


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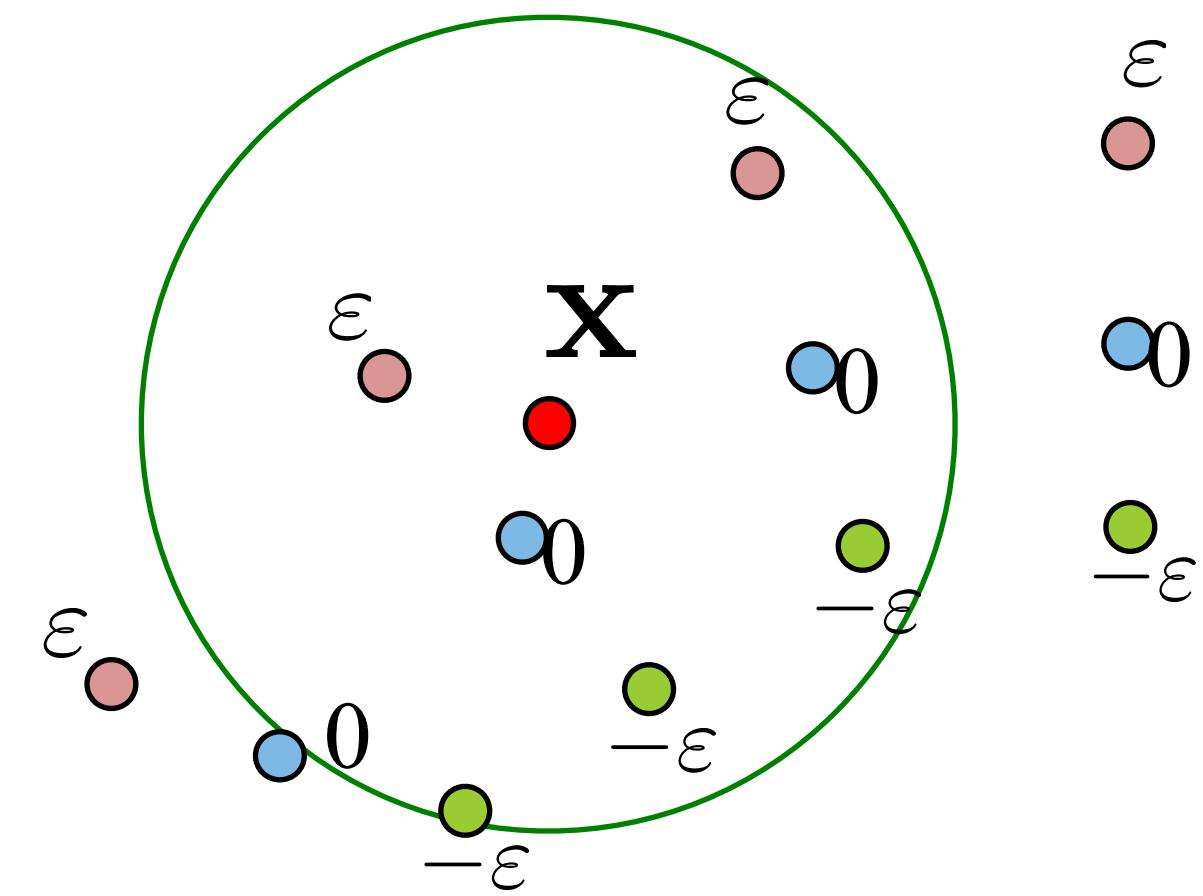


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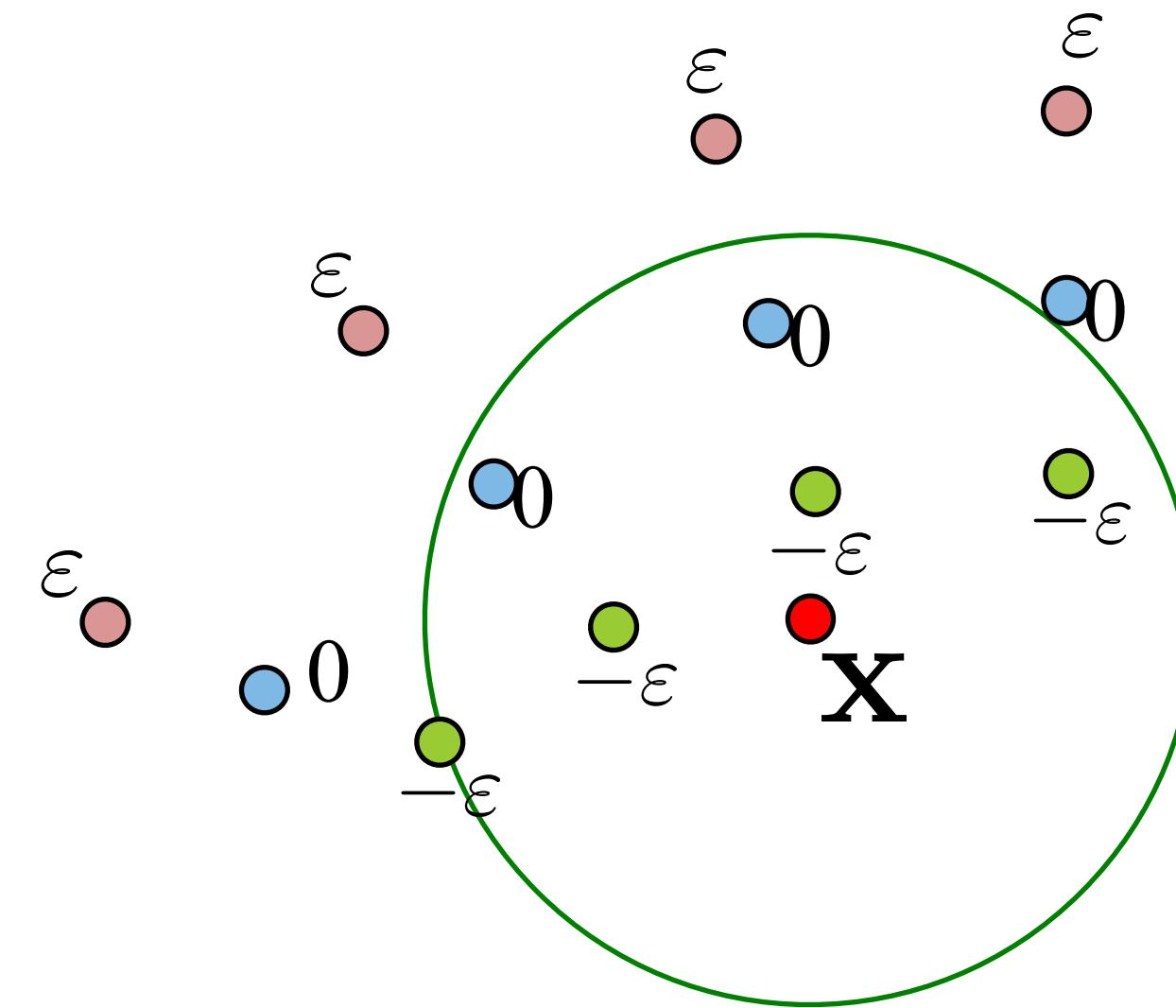
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ACM SIGGRAPH 2004

<http://graphics.berkeley.edu/papers/Shen-IAI-2004-08/index.html>



# Least-Squares Approximation

- Polynomial least-squares approximation
  - Choose degree,  $k$

$$f \in \Pi_k^3 : f(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5xy + \dots + a_*z^k$$

$$f(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T \mathbf{a}$$

$$\mathbf{a} = (a_1, a_2, \dots, a_*)^T, \quad \mathbf{b}(\mathbf{x})^T = (1, x, y, z, x^2, xy, \dots, z^k)$$

- Find  $\mathbf{a}$  that minimizes sum of squared differences

$$\operatorname{argmin}_{f \in \Pi_k^3} \sum_{i=0}^{N-1} (f(\mathbf{c}_i) - d_i)^2 \quad \text{or:} \quad \operatorname{argmin}_{\mathbf{a}} \sum_{i=0}^{N-1} (\mathbf{b}(\mathbf{c}_i)^T \mathbf{a} - d_i)^2$$

# MOVING Least-Squares Approximation

- Polynomial least-squares approximation
  - Choose degree,  $k$

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$$\mathbf{a} = (a_1, a_2, \dots, a_*)^T, \quad \mathbf{b}(\mathbf{x})^T = (1, x, y, z, x^2, xy, \dots, z^k)$$

- Find  $\mathbf{a}_x$  that minimizes WEIGHTED sum of squared differences

$$f_{\mathbf{x}} = \operatorname{argmin}_{f \in \Pi_k^3} \sum_{i=0}^{N-1} \theta(\|\mathbf{x} - \mathbf{c}_i\|) (f(\mathbf{c}_i) - d_i)^2 \text{ or:}$$

$$\mathbf{a}_{\mathbf{x}} = \operatorname{argmin}_{\mathbf{a}} \sum_{i=0}^{N-1} \theta(\|\mathbf{x} - \mathbf{c}_i\|) (\mathbf{b}(\mathbf{c}_i)^T \mathbf{a} - d_i)^2$$



# MOVING Least-Squares Approximation

- Polynomial least-squares approximation
  - Choose degree,  $k$

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$$f(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T \mathbf{a}$$

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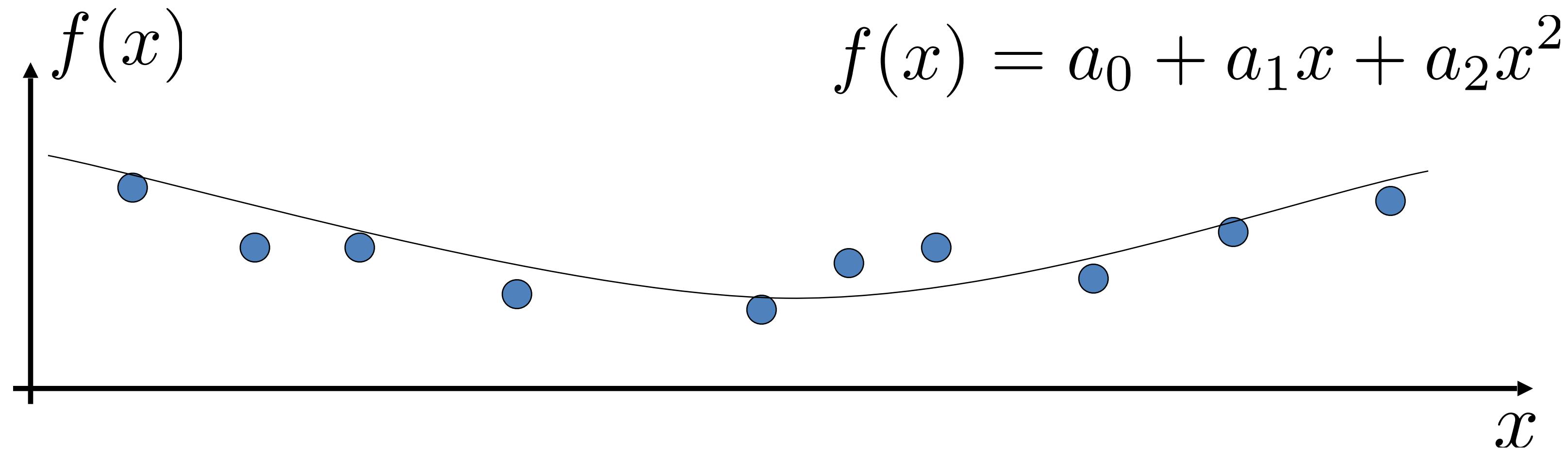
- Find  $\mathbf{a}_x$  that minimizes WEIGHTED sum of squared differences
- The value of the SDF is the obtained approximation evaluated at  $\mathbf{x}$ :

$$F(\mathbf{x}) = f_{\mathbf{x}}(\mathbf{x}) = \mathbf{b}(\mathbf{x})^T \mathbf{a}_{\mathbf{x}}$$



# MLS – 1D Example

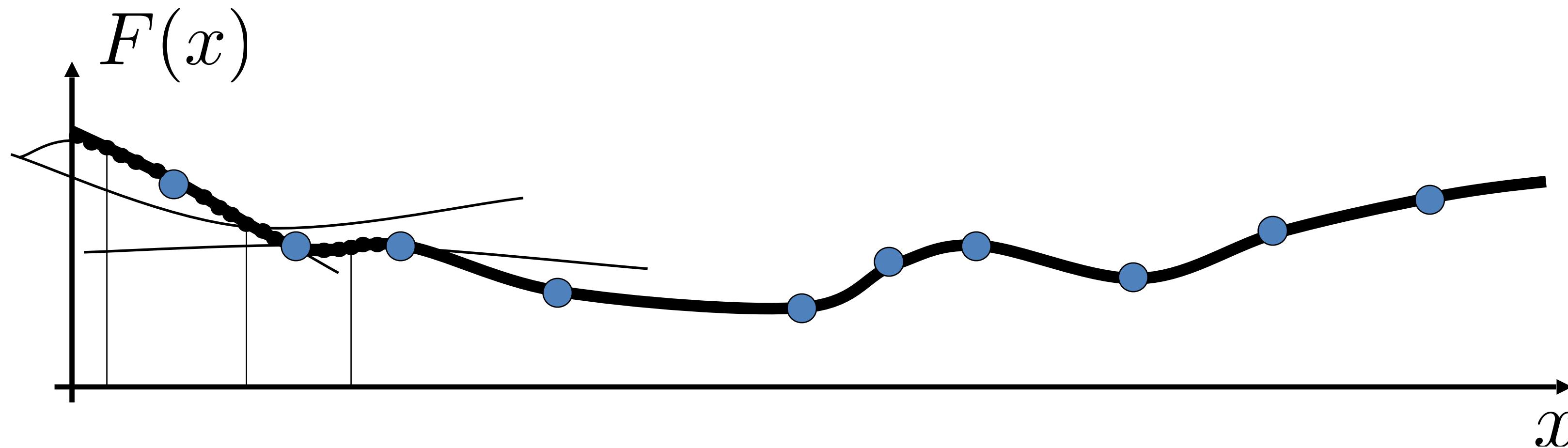
- Global approximation in  $\Pi_2^1$



$$f = \operatorname{argmin}_{f \in \Pi_2^1} \sum_{i=0}^{N-1} (f(c_i) - d_i)^2$$

# MLS – 1D Example

- MLS approximation using functions in  $\Pi_2^1$



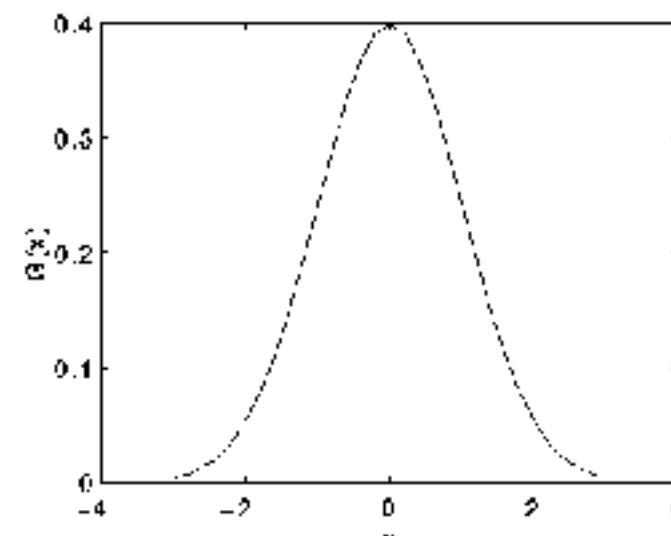
$$F(x) = f_x(x), \quad f_x = \operatorname{argmin}_{f \in \Pi_2^1} \sum_{i=0}^{N-1} \theta(\|c_i - x\|) (f(c_i) - d_i)^2$$

# Weight Functions

- Gaussian

- $h$  is a smoothing parameter

$$\theta(r) = e^{-\frac{r^2}{h^2}}$$



$$\theta(r) = (1 - r/h)^4(4r/h + 1)$$

- Wendland function

- Defined in  $[0, h]$  and  $\theta(0) = 1, \theta(h) = 0, \theta'(h) = 0, \theta''(h) = 0$

- Singular function

$$\theta(r) = \frac{1}{r^2 + \varepsilon^2}$$

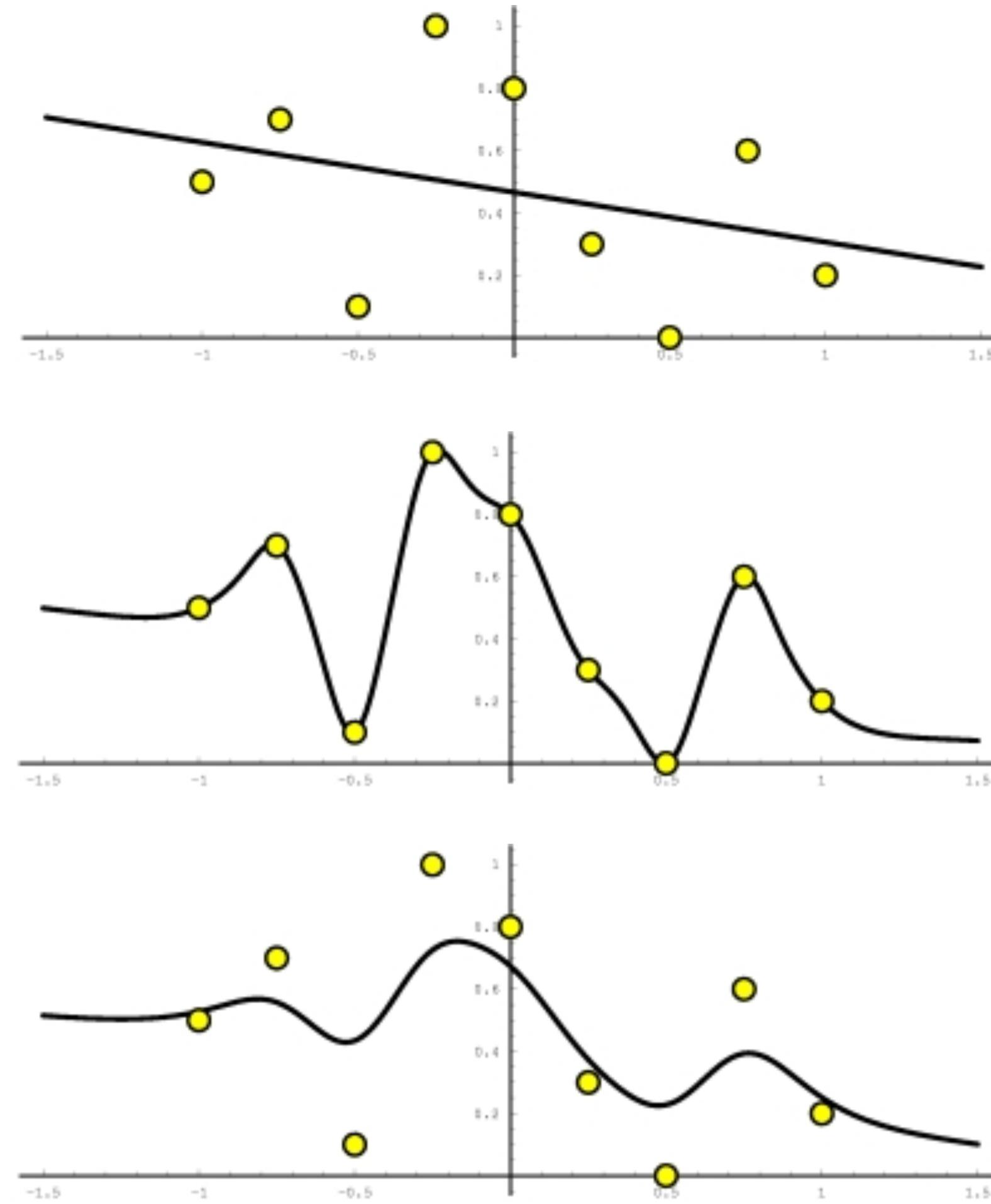
- For small  $\varepsilon$ , weights large near  $r=0$  (interpolation)

# Dependence on Weight Function

- Global least squares with linear basis
- MLS with (nearly) singular weight function
- MLS with approximating weight function

$$\theta(r) = \frac{1}{r^2 + \varepsilon^2}$$

$$\theta(r) = e^{-\frac{r^2}{h^2}}$$

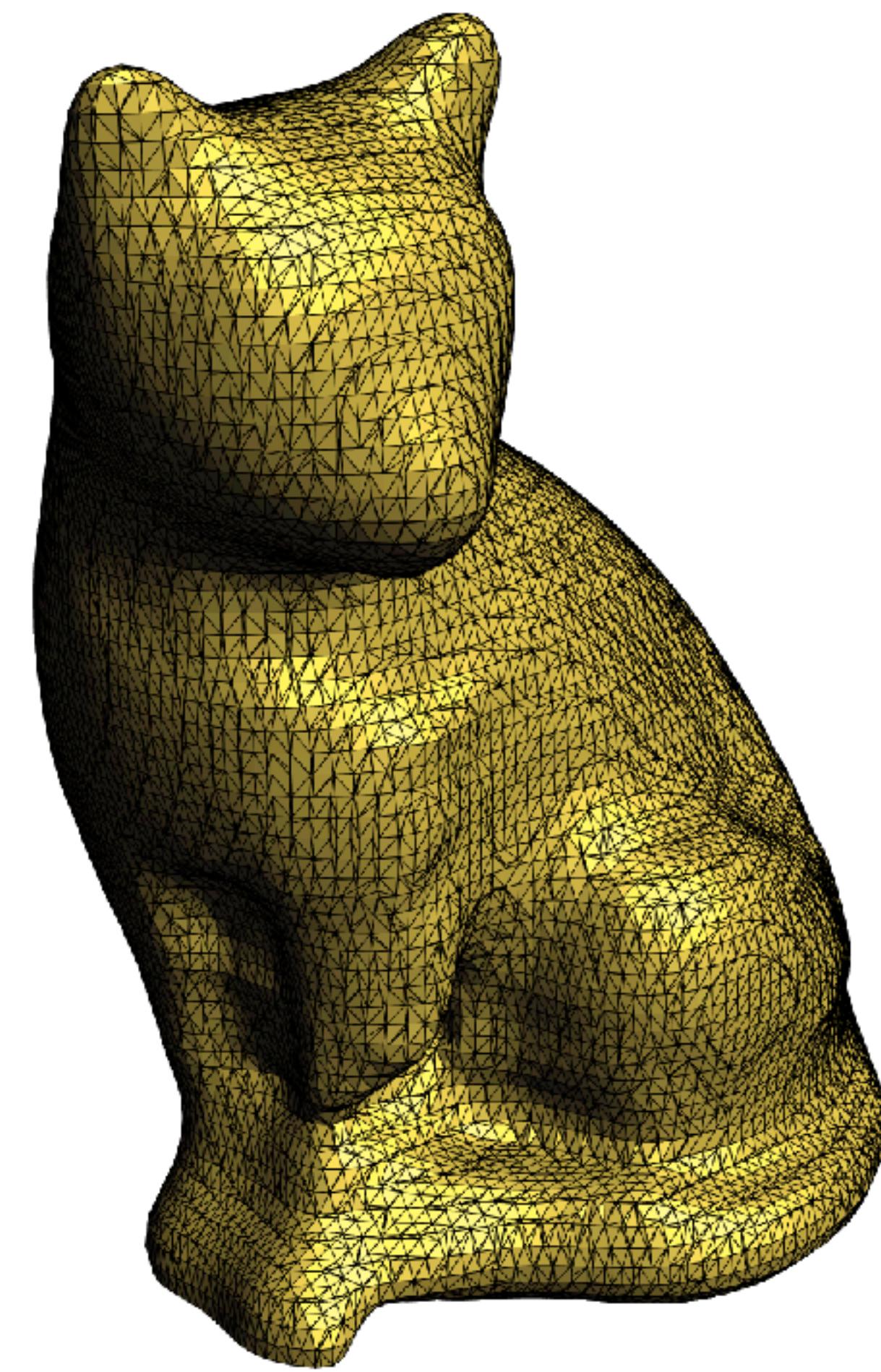
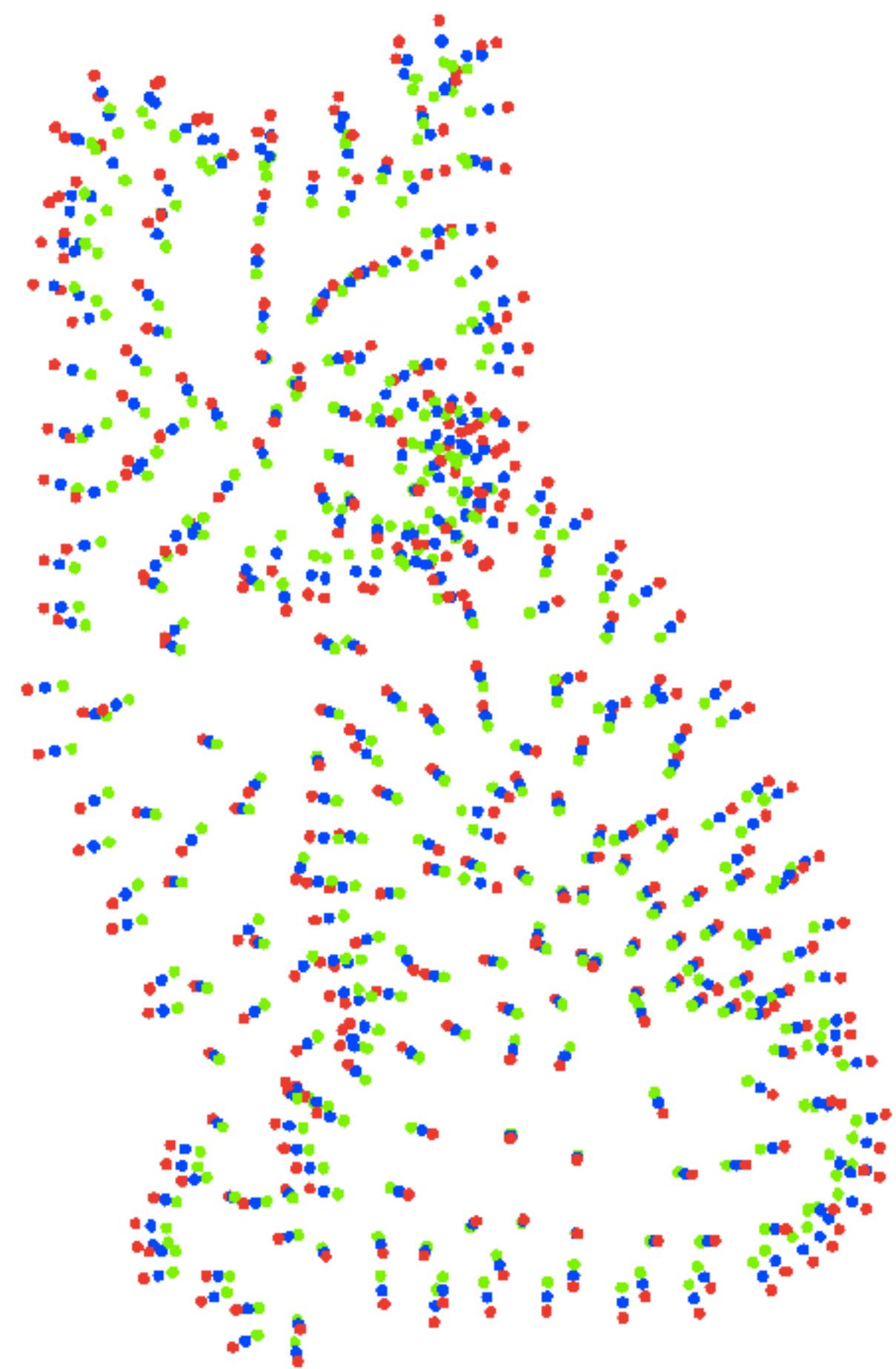


# Dependence on Weight Function

- The MLS function  $F$  is continuously differentiable if and only if the weight function  $\theta$  is continuously differentiable
- In general,  $F$  is as smooth as  $\theta$

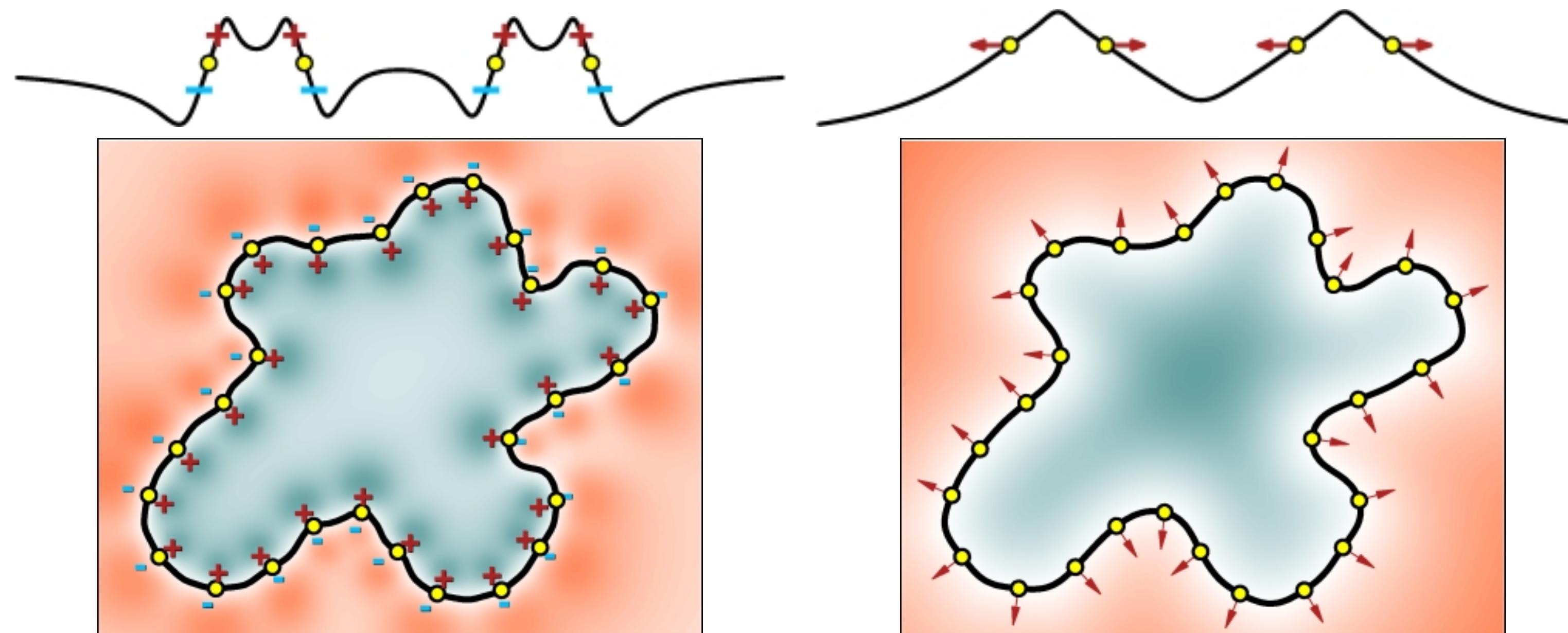
$$F(\mathbf{x}) = f_{\mathbf{x}}(\mathbf{x}), \quad f_{\mathbf{x}} = \operatorname{argmin}_{f \in \Pi_k^d} \sum_{i=0}^{N-1} \theta(\|\mathbf{c}_i - \mathbf{x}\|) (f(\mathbf{c}_i) - d_i)^2$$

# Example: Reconstruction



# MLS SDF – Possible Improvement

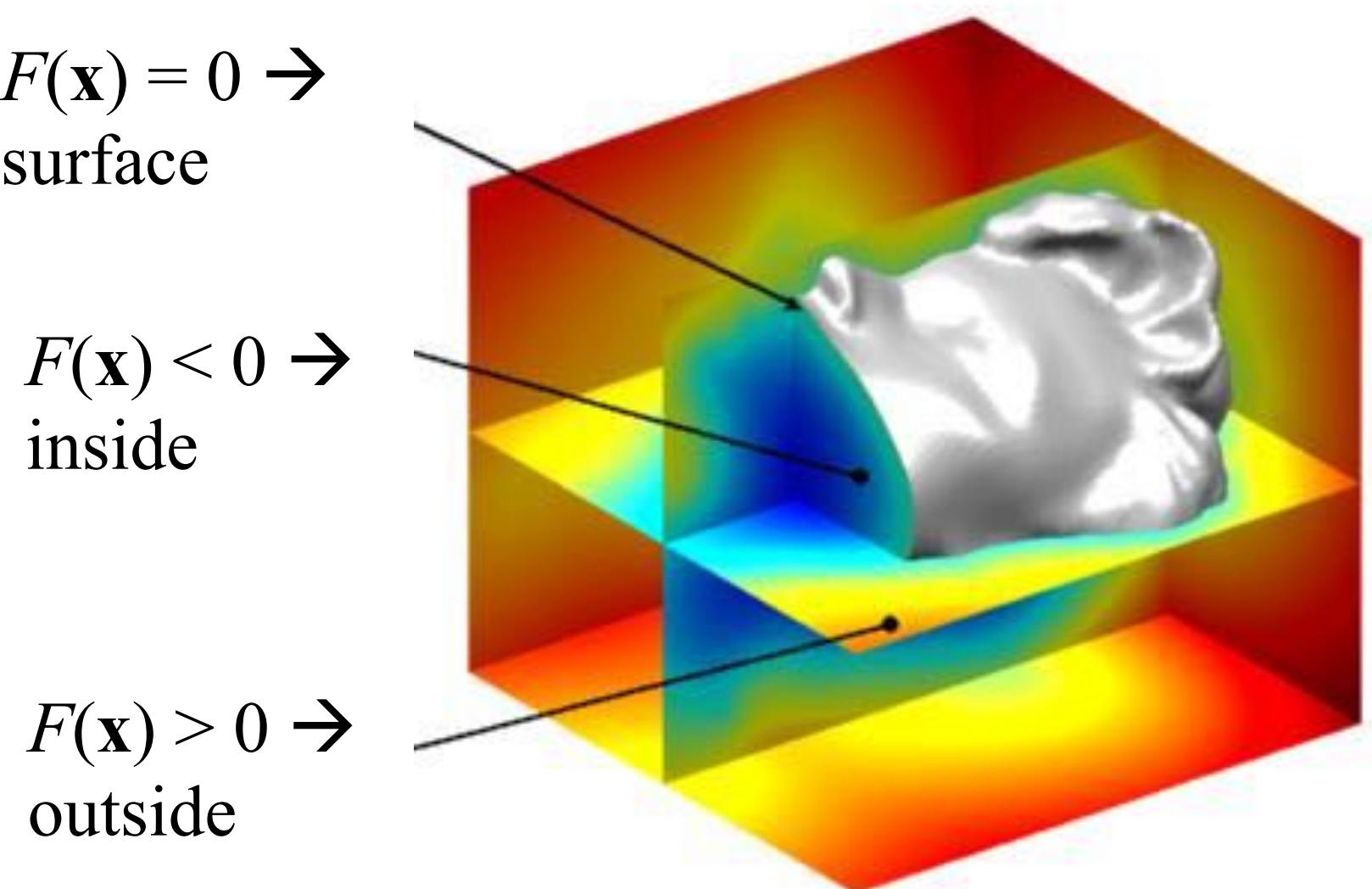
- Point constraints vs. true normal constraints



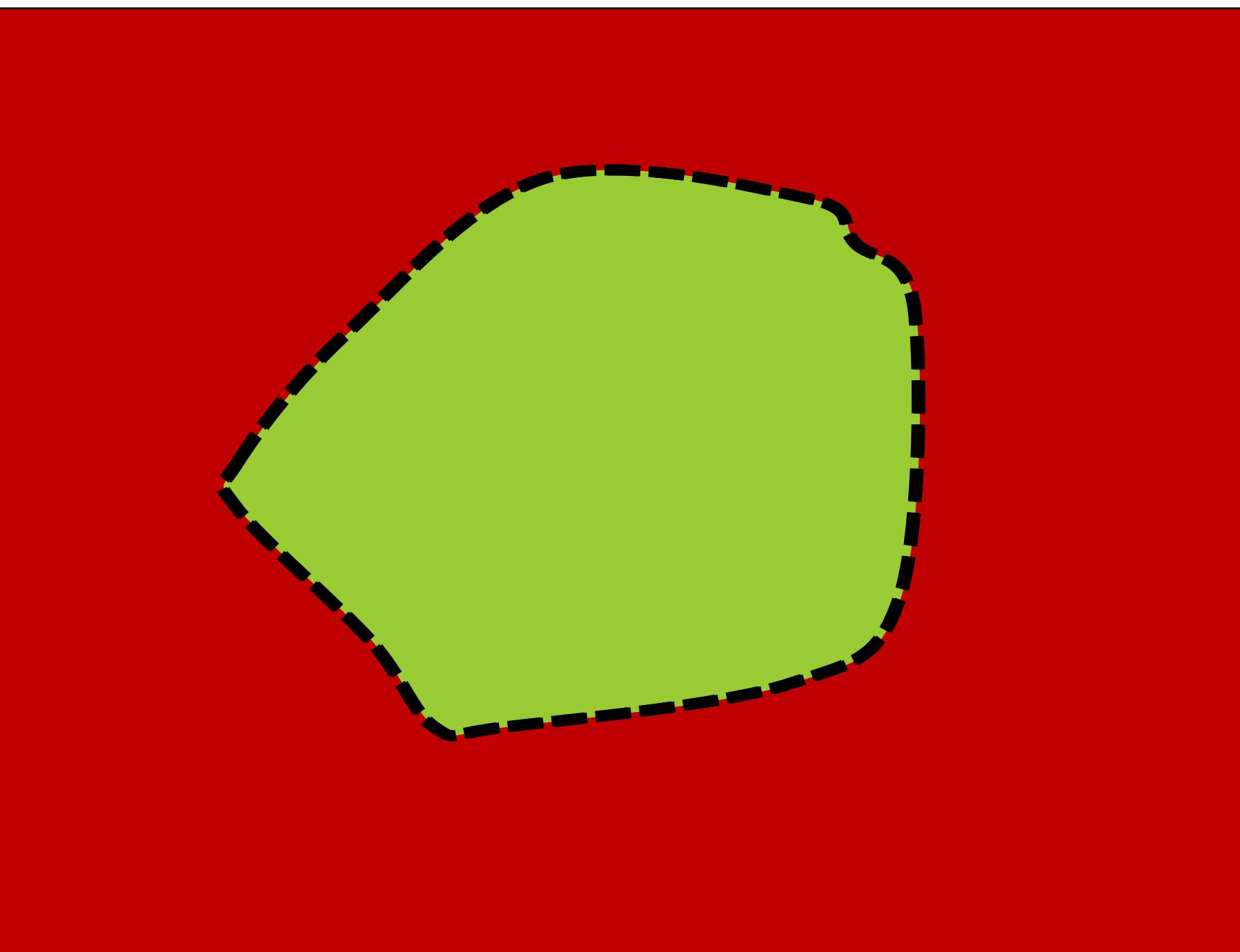
- Details: see [Shen et al. SIGGRAPH 2004] and the bonus assignment in Ex2

# Extracting the Surface

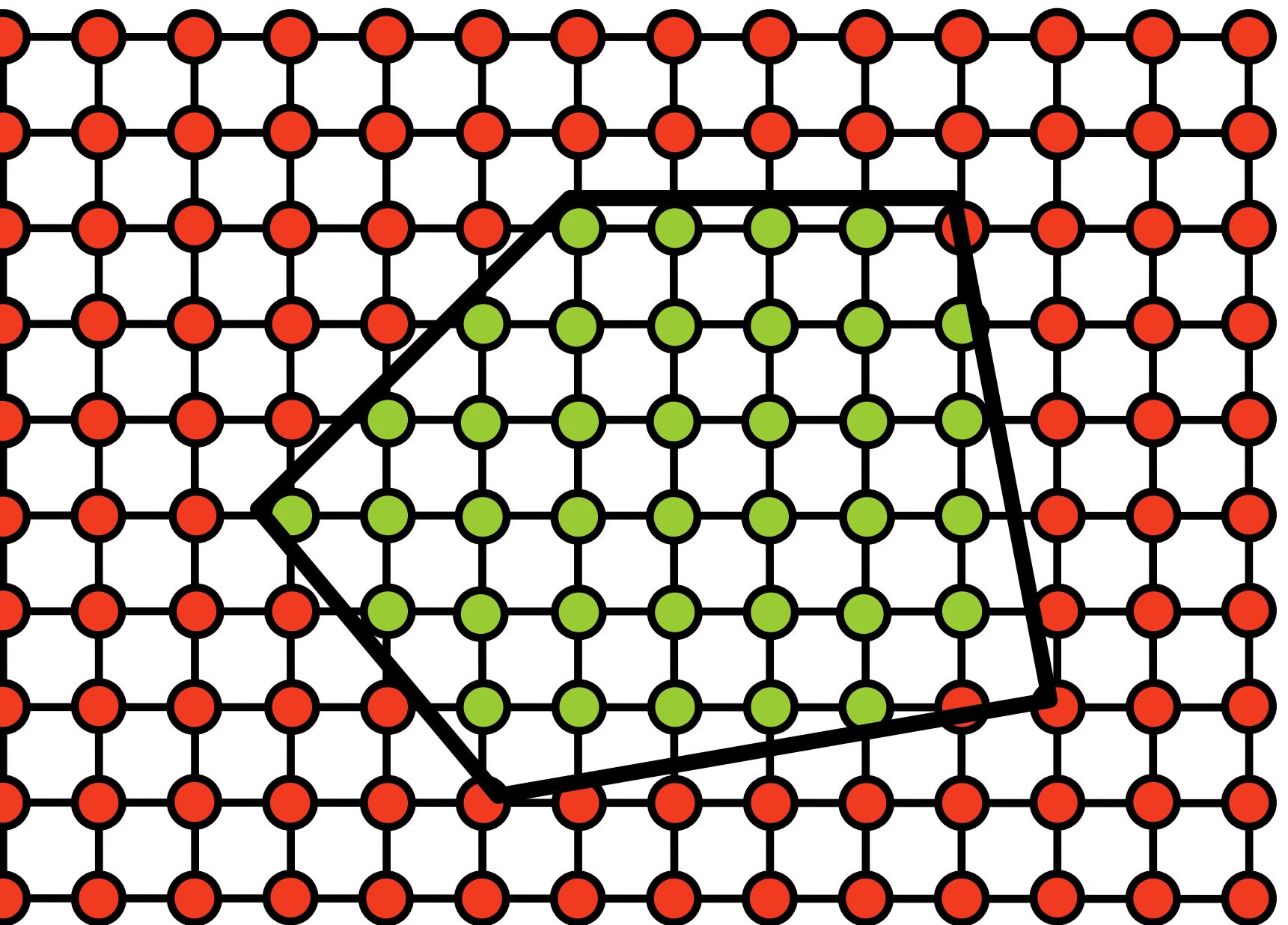
- Wish to compute a manifold mesh of the level set



# Sample the SDF

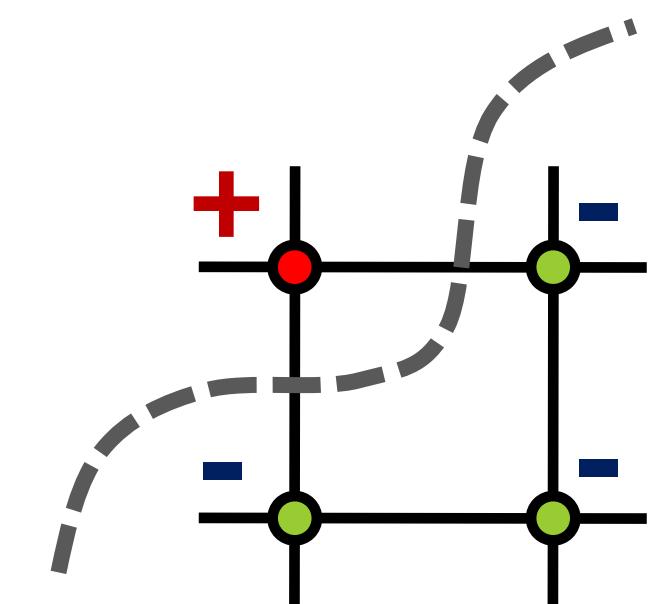


# Sample the SDF

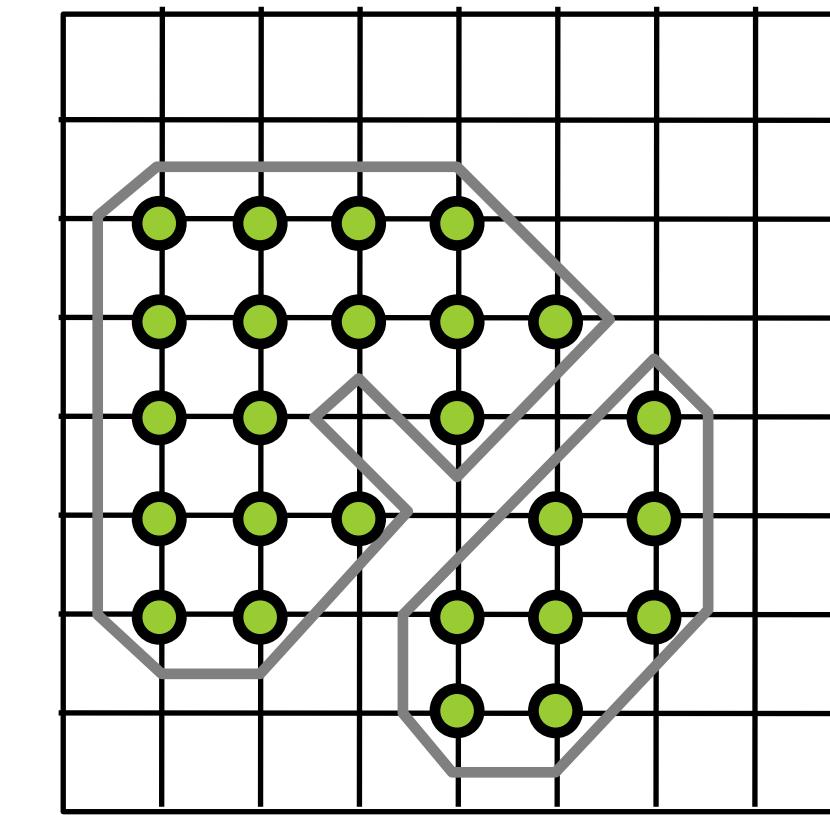
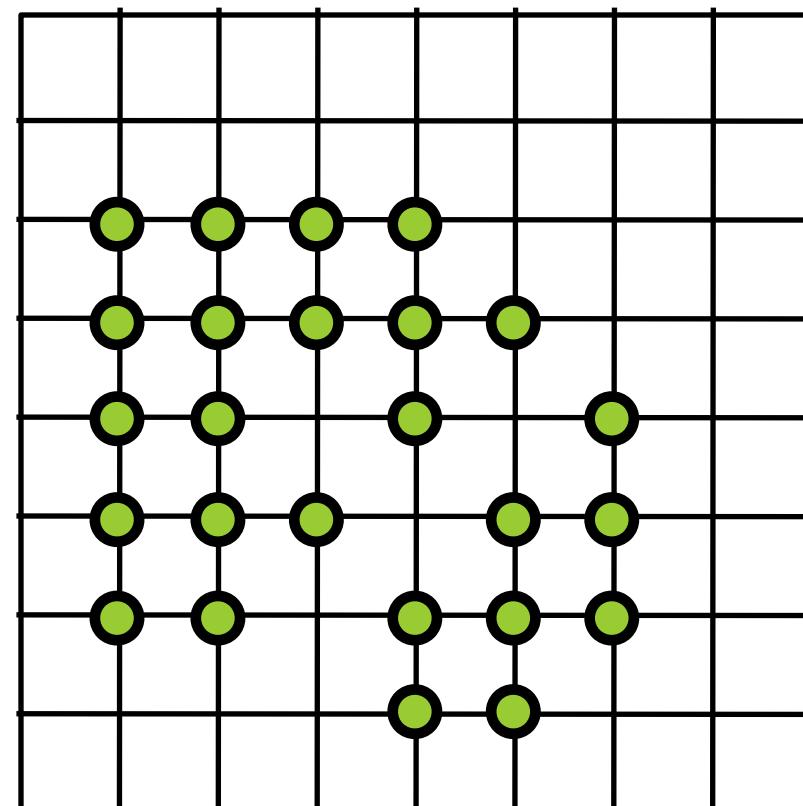


# Tessellation

- Want to approximate an implicit surface with a mesh
- Can't explicitly compute all the roots
  - Sampling the level set is difficult (root finding)
- Solution: find approximate roots by trapping the implicit surface in a grid (lattice)

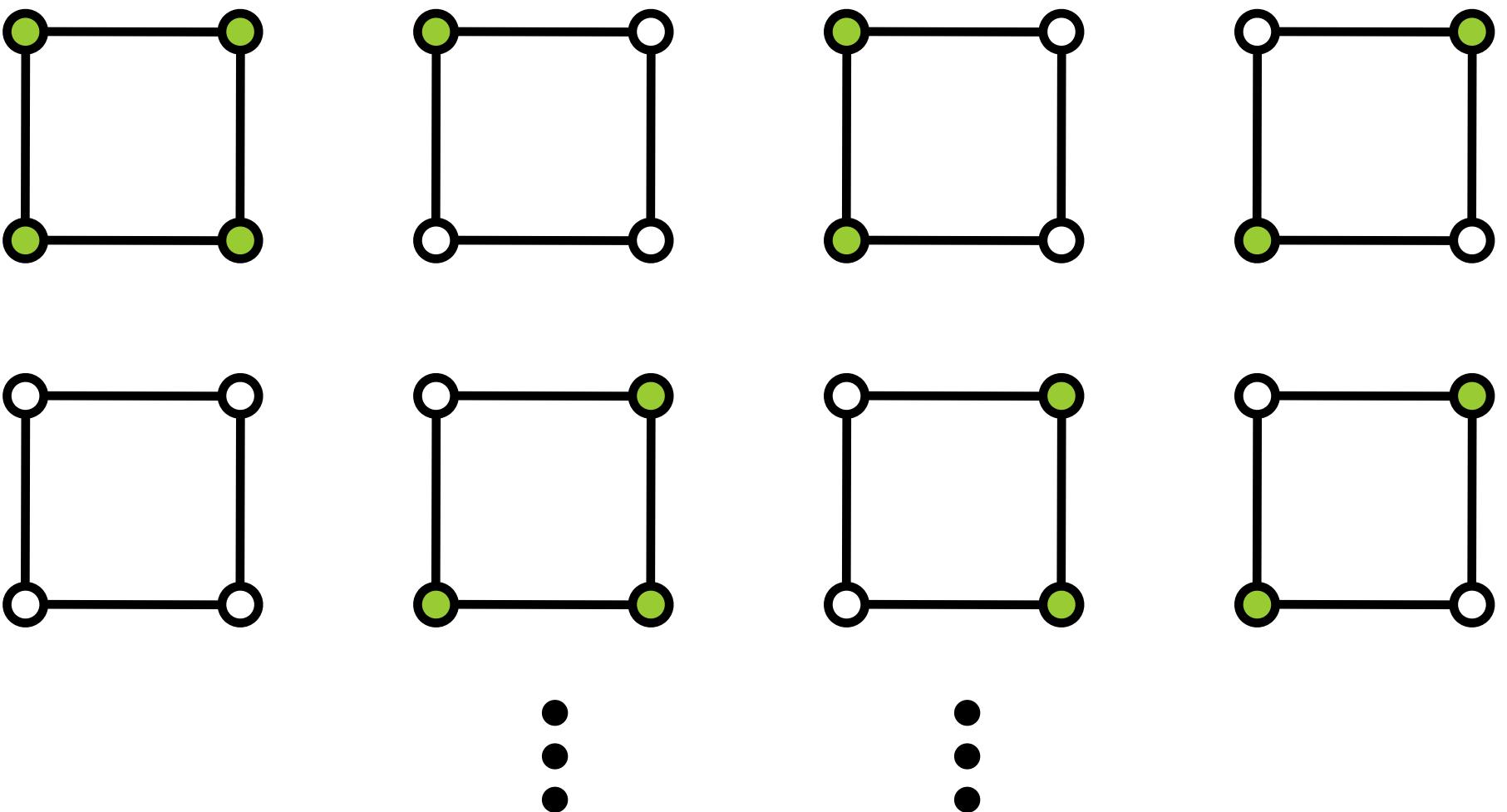


$$\bullet F(\mathbf{x}) < 0$$



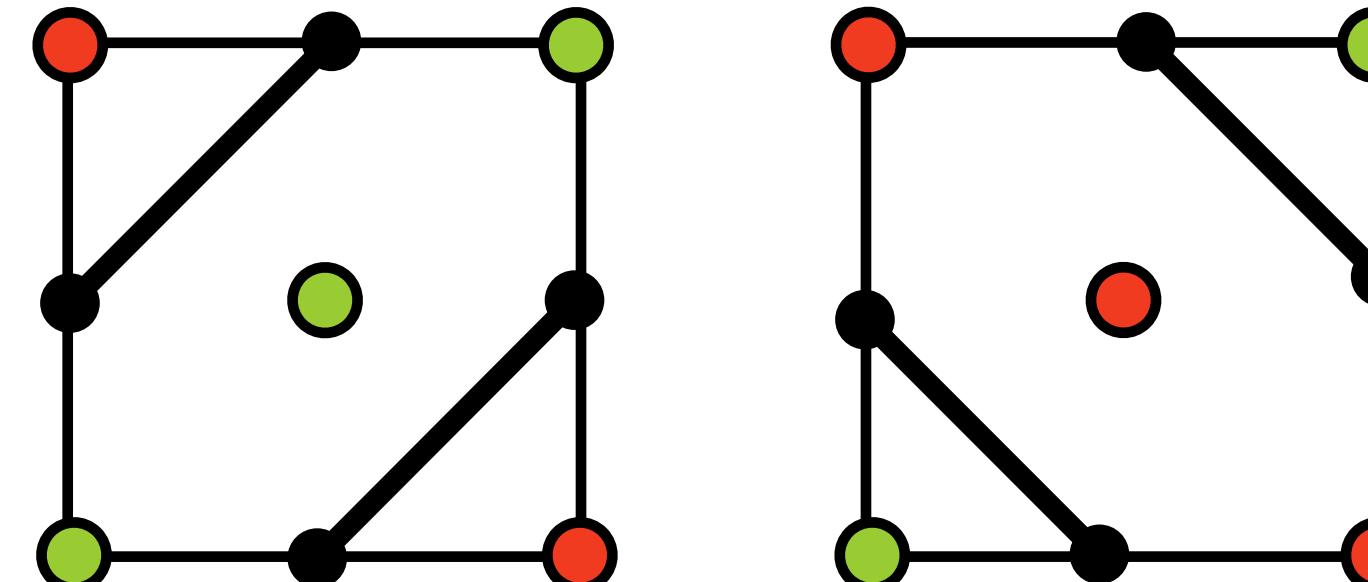
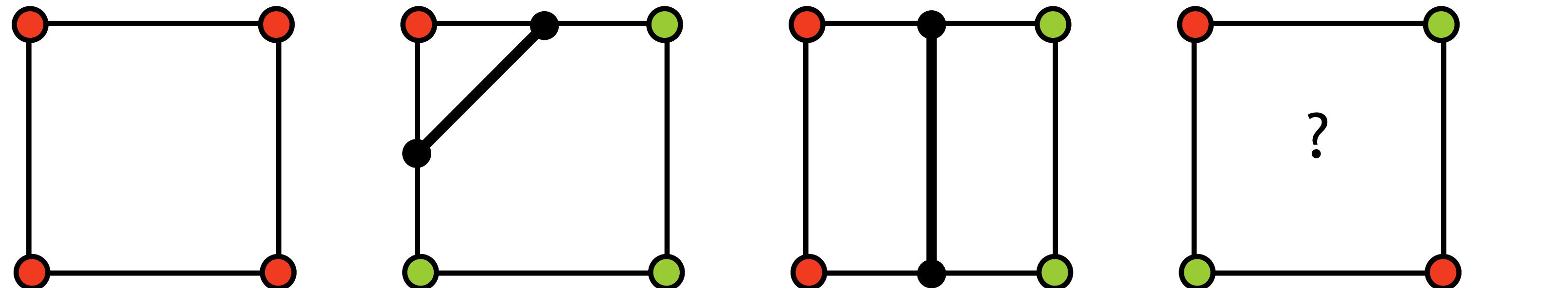
# Marching Squares

- 16 different configurations in 2D
- 4 equivalence classes (up to rotational and reflection symmetry + complement)



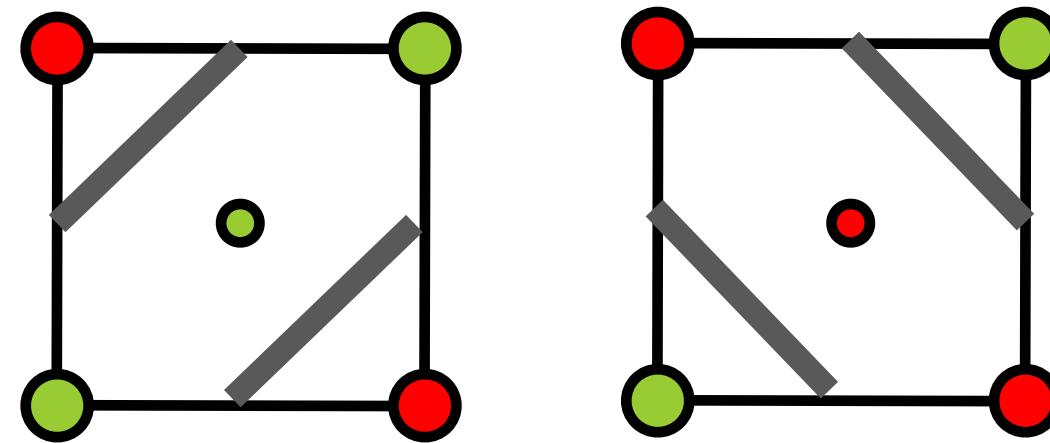
# Tessellation in 2D

- 4 equivalence classes (up to rotational and reflection symmetry + complement)

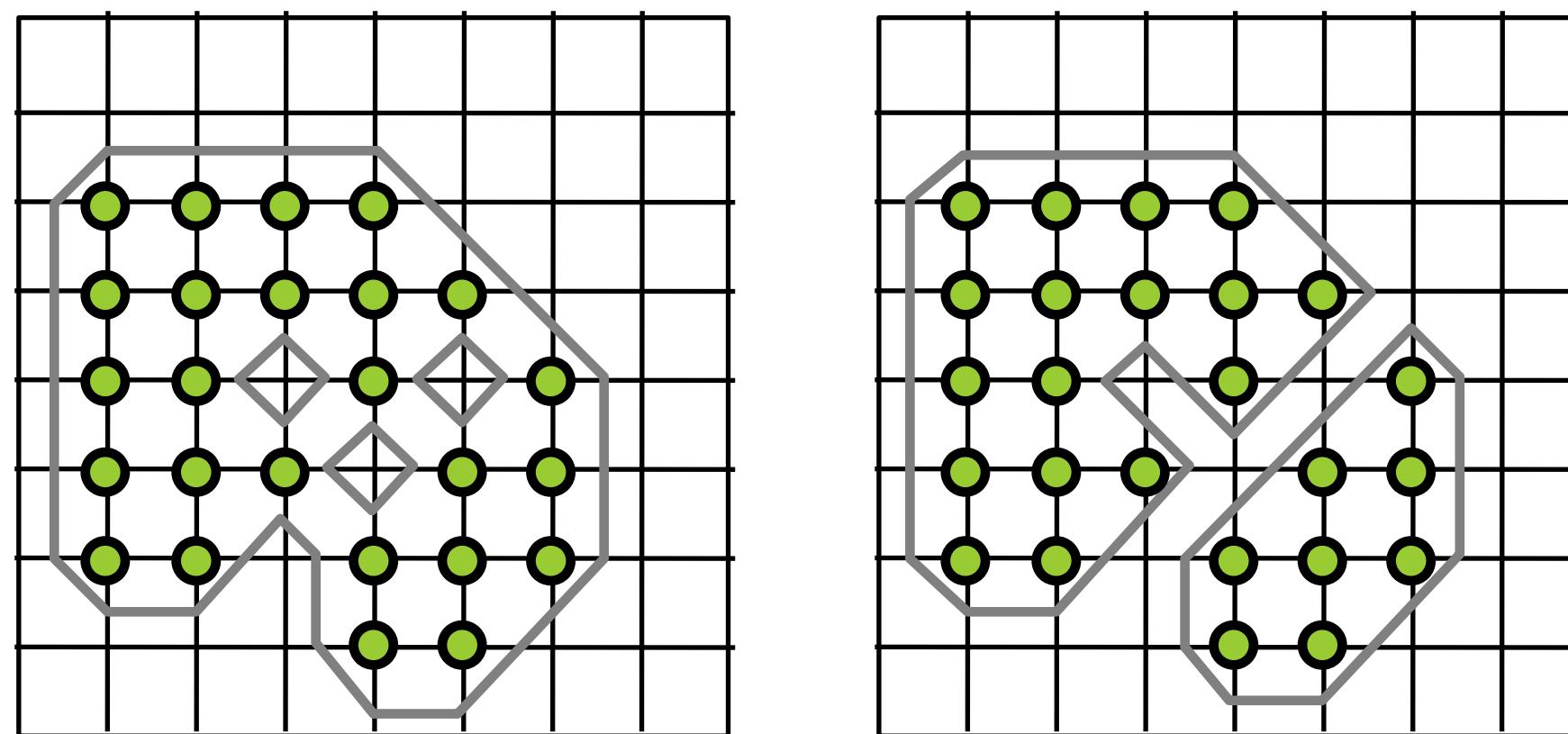


# Tessellation in 2D

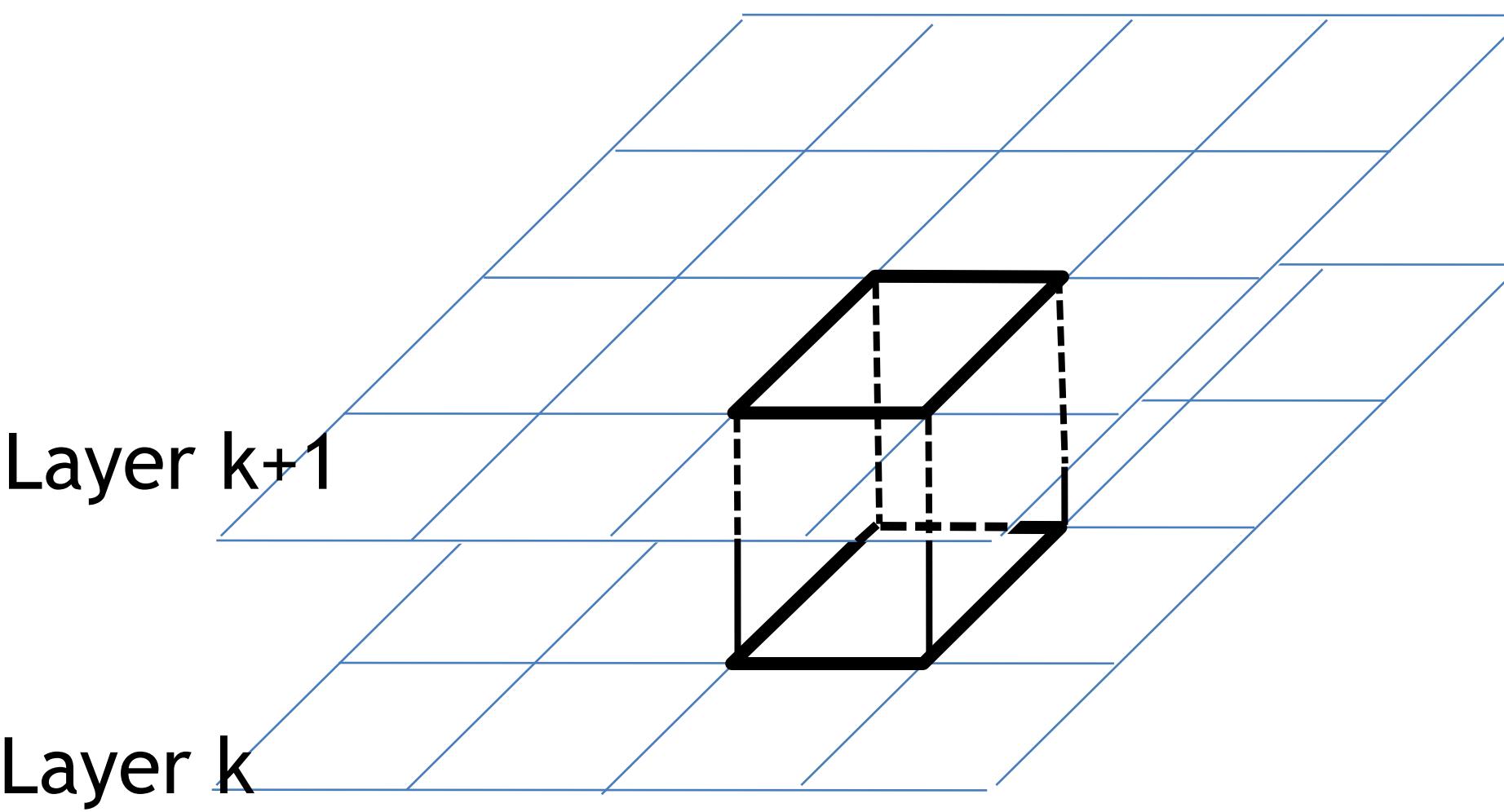
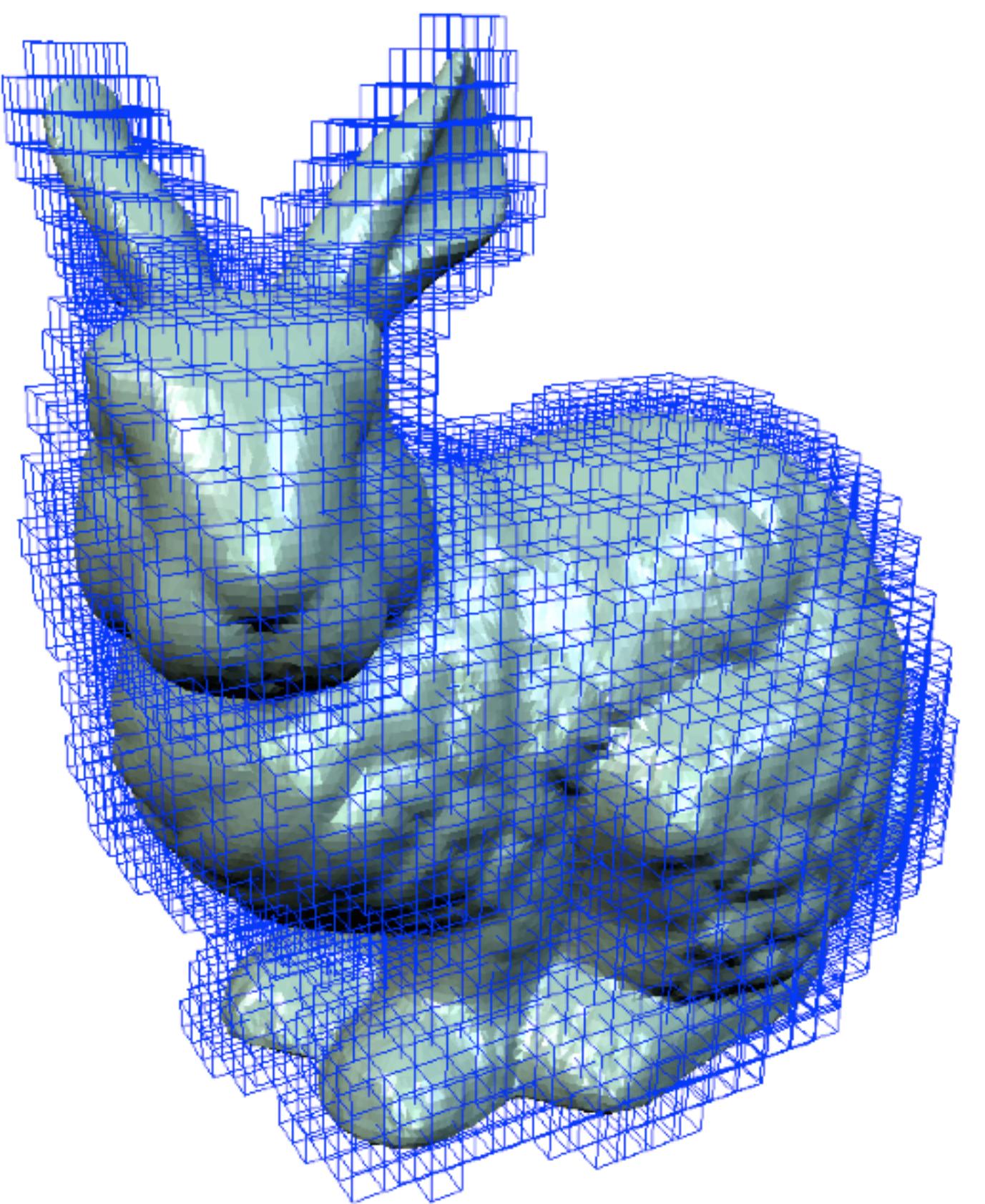
- Case 4 is ambiguous:



- Always pick consistently to avoid problems with the resulting mesh

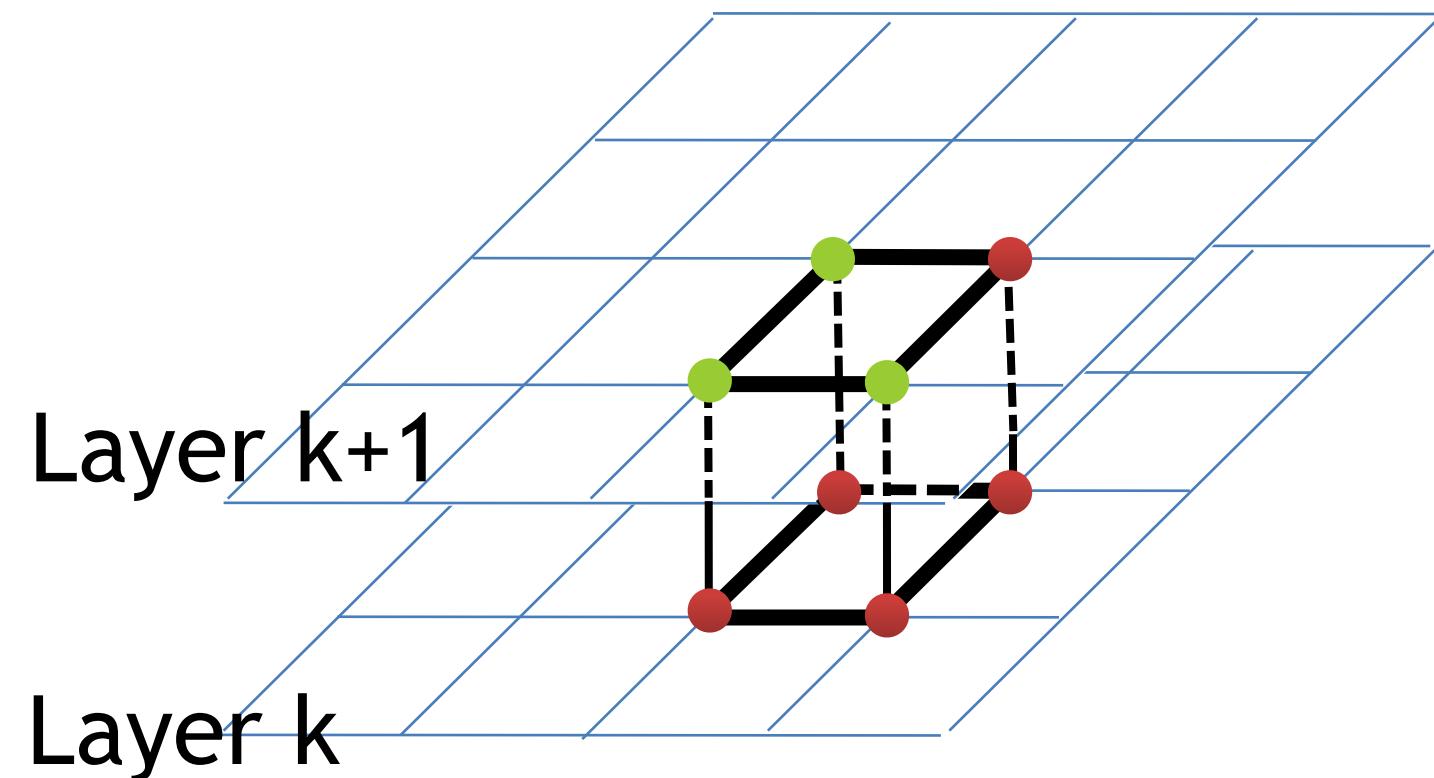


# 3D: Marching Cubes



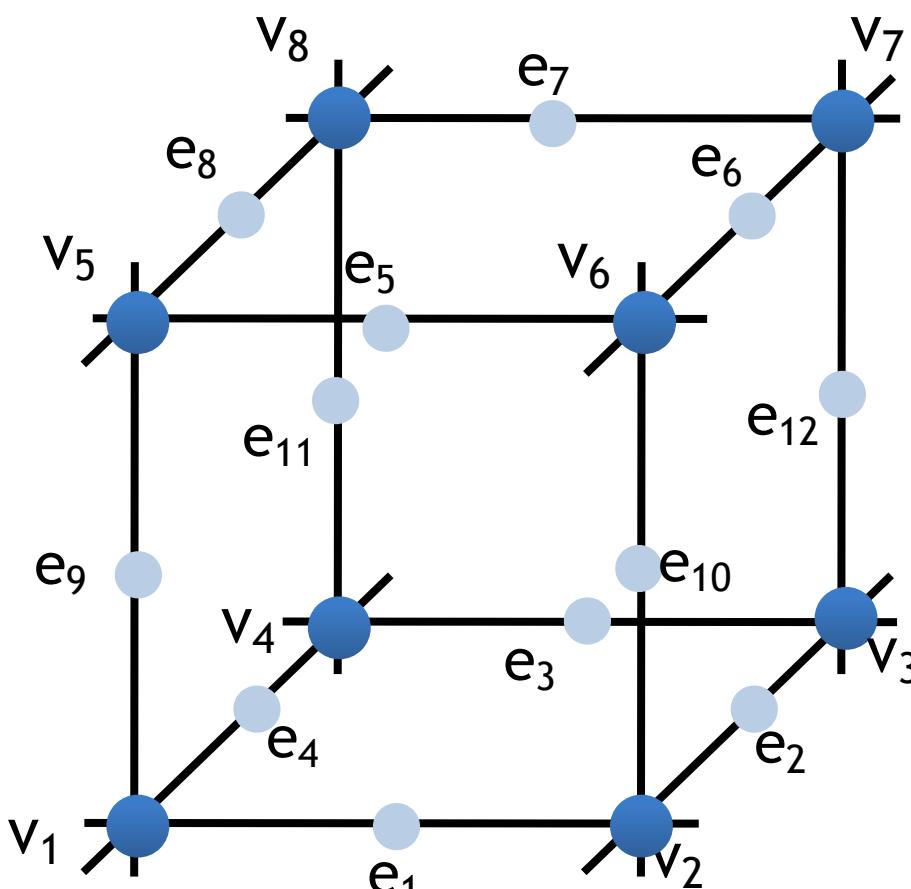
# Marching Cubes

- Marching Cubes (Lorensen and Cline 1987)
  1. Load 4 layers of the grid into memory
  2. Create a cube whose vertices lie on the two middle layers
  3. Classify the vertices of the cube according to the implicit function (inside, outside or on the surface)



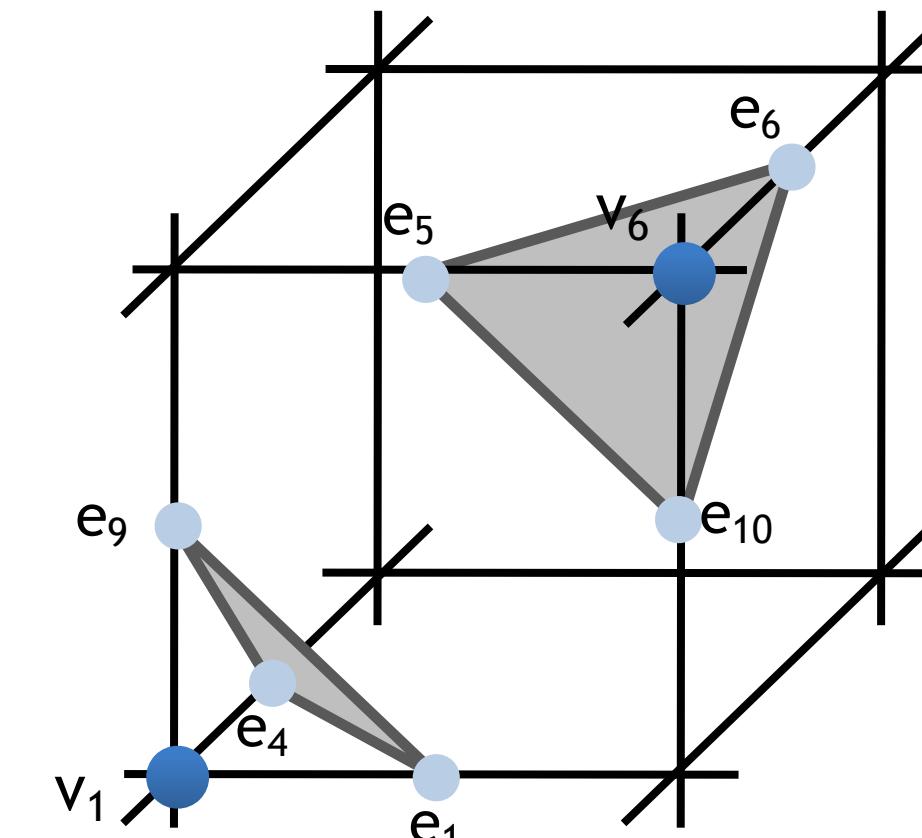
# Marching Cubes

4. Compute case index. We have  $2^8 = 256$  cases (0/1 for each of the eight vertices) – can store as 8 bit (1 byte) index.



index = 

$v_1$	$v_2$	$v_3$	$v_4$	$v_5$	$v_6$	$v_7$	$v_8$
-------	-------	-------	-------	-------	-------	-------	-------



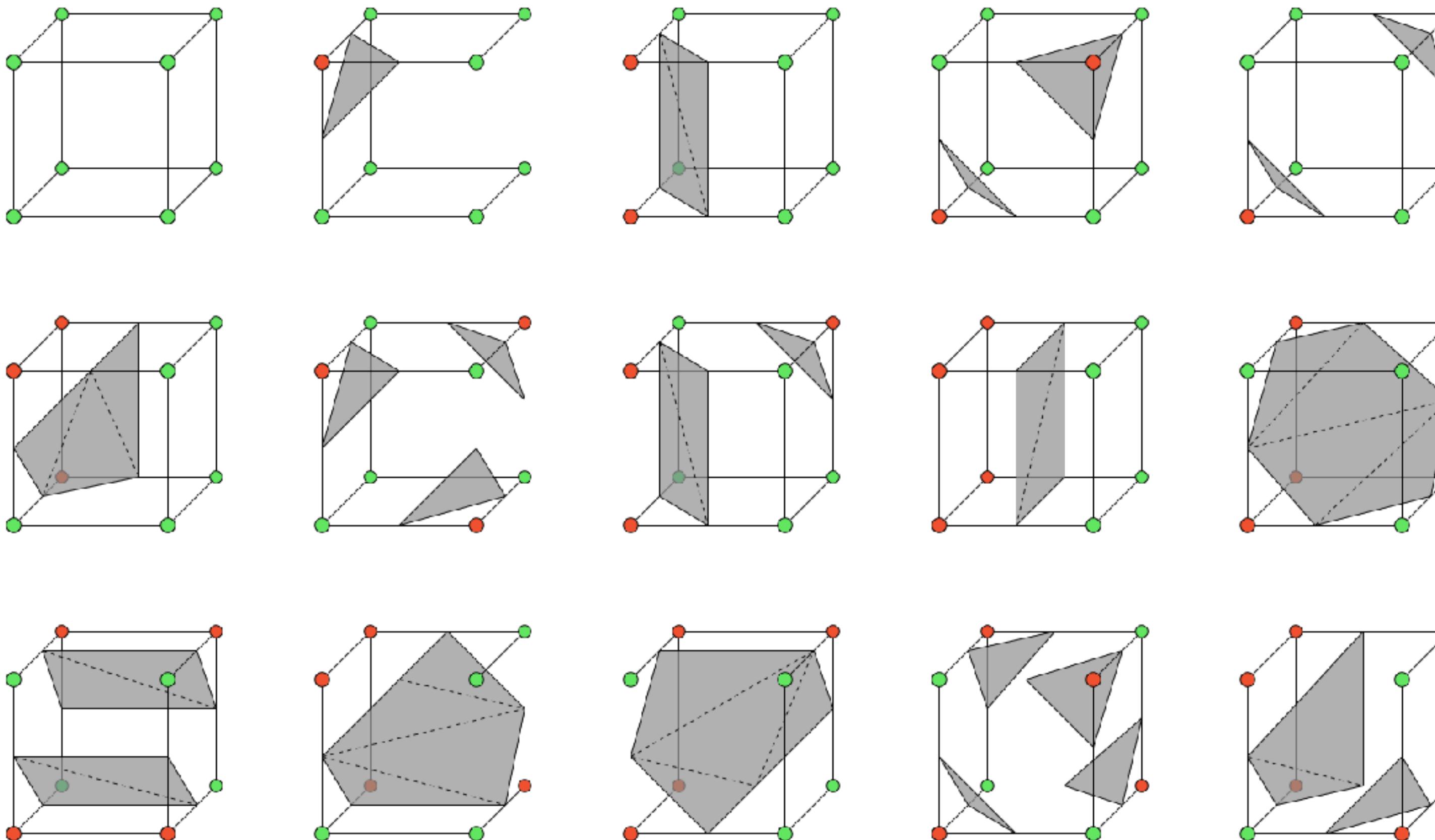
index = 

0	0	1	0	0	0	0	1
---	---	---	---	---	---	---	---

 = 33

# Marching Cubes

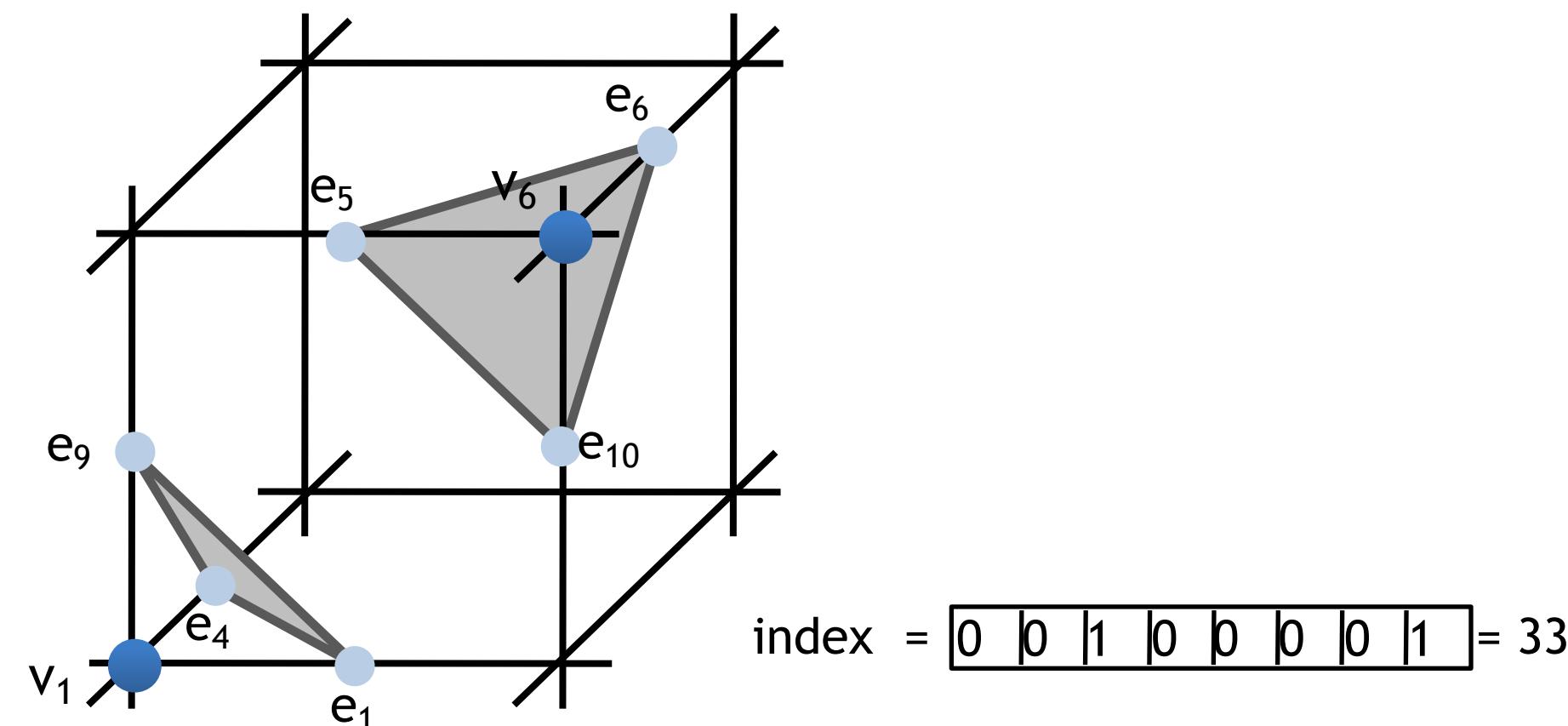
- Unique cases (by rotation, reflection and complement)



# Tessellation

## 3D – Marching Cubes

5. Using the case index, retrieve the connectivity in the look-up table
  - Example: the entry for index 33 in the look-up table indicates that the cut edges are  $e_1; e_4; e_5; e_6; e_9$  and  $e_{10}$ ; the output triangles are  $(e_1; e_9; e_4)$  and  $(e_5; e_{10}; e_6)$ .

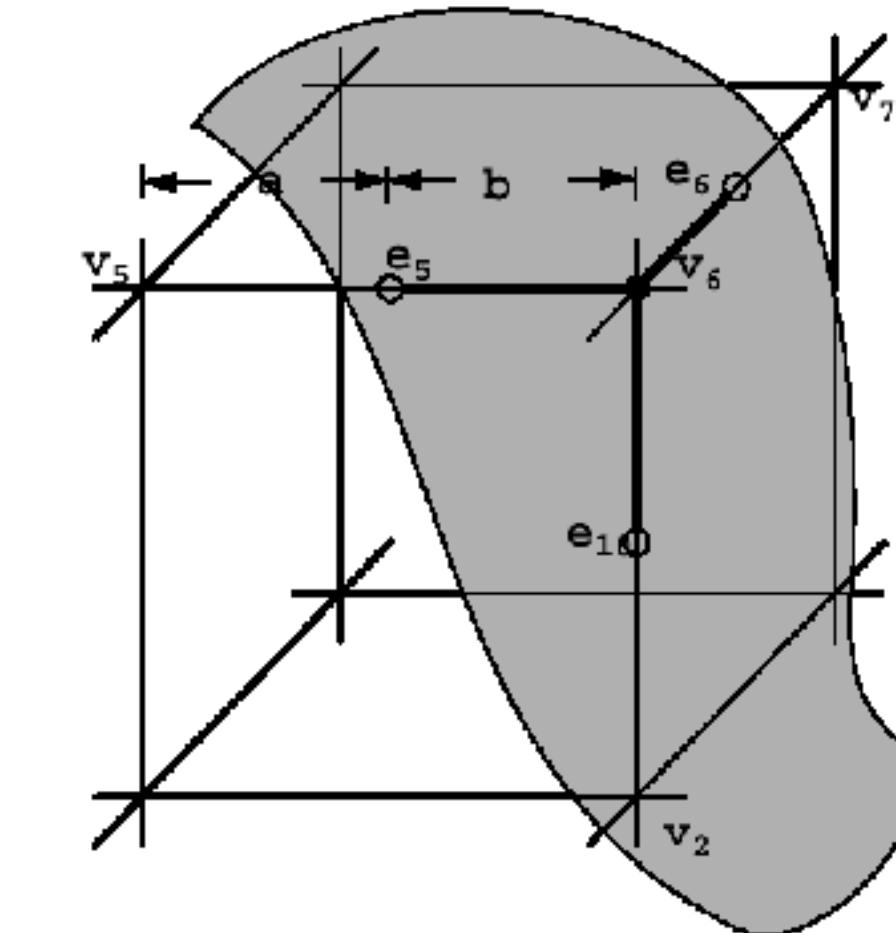


# Marching Cubes

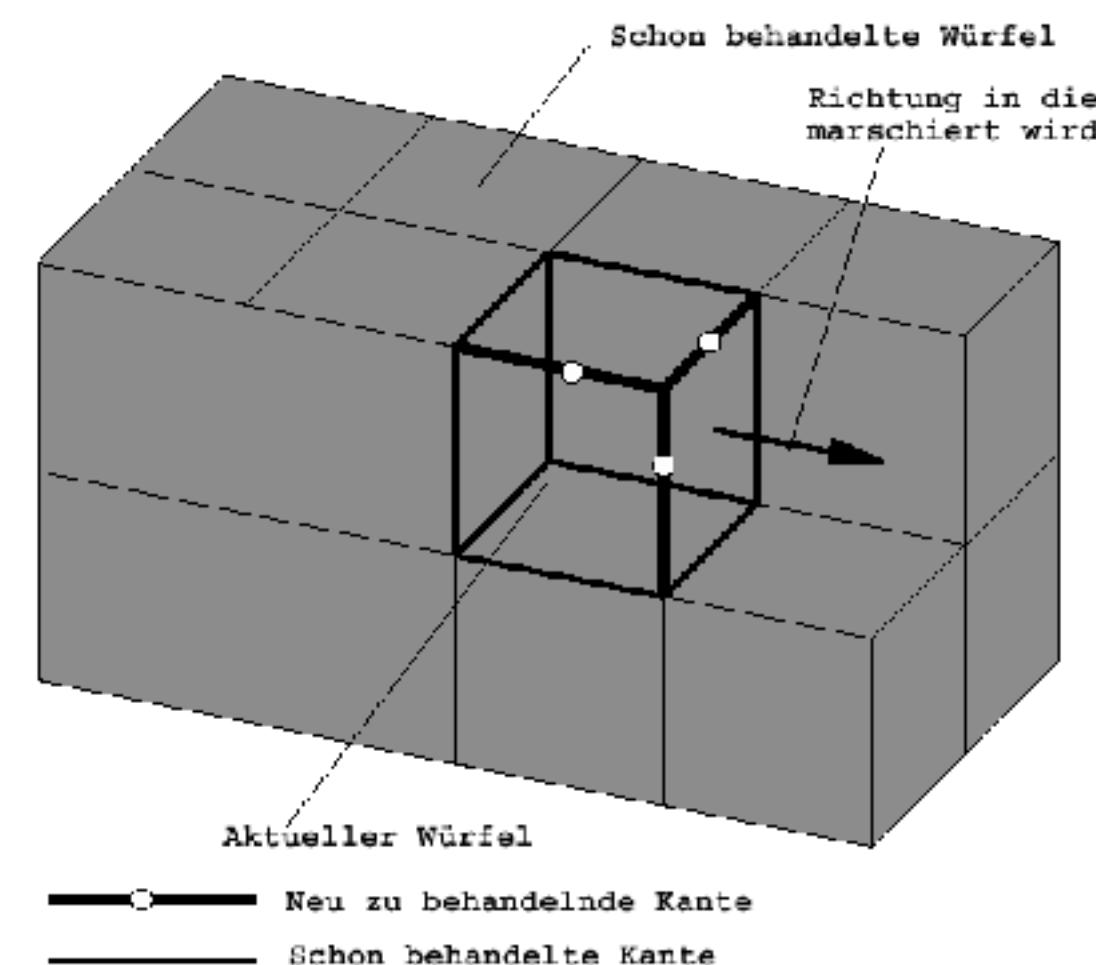
6. Compute the position of the cut vertices by linear interpolation:

$$\mathbf{v}_s = t\mathbf{v}_a + (1 - t)\mathbf{v}_b$$

$$t = \frac{F(\mathbf{v}_b)}{F(\mathbf{v}_b) - F(\mathbf{v}_a)}$$

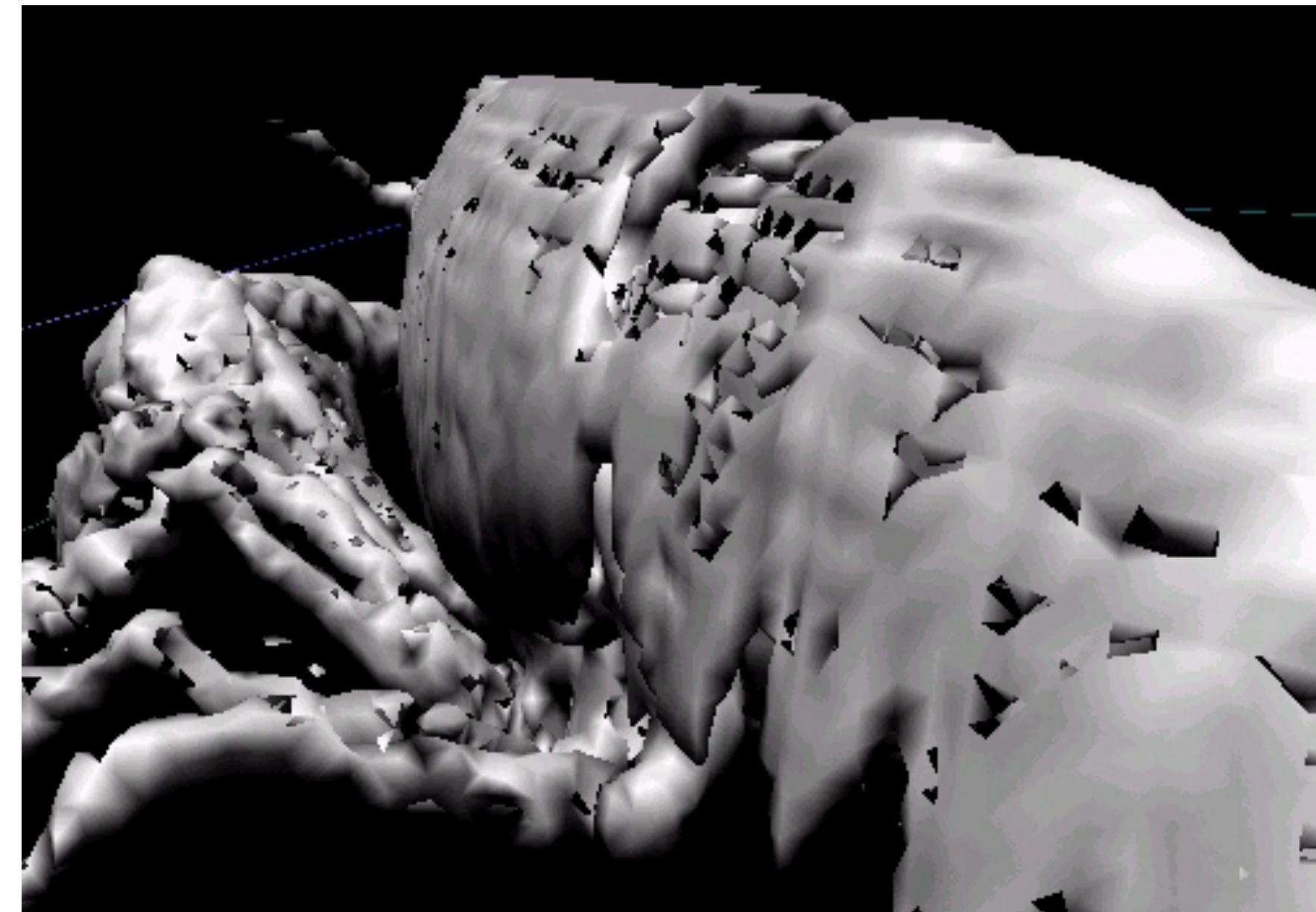
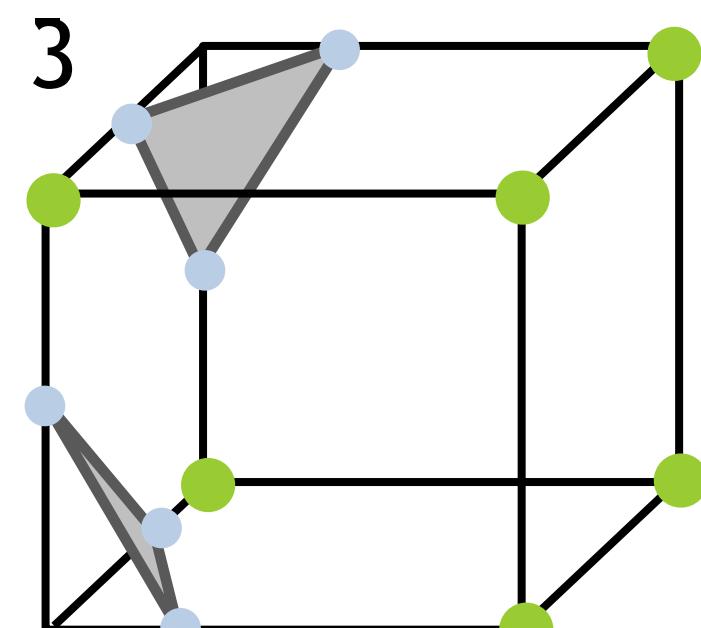
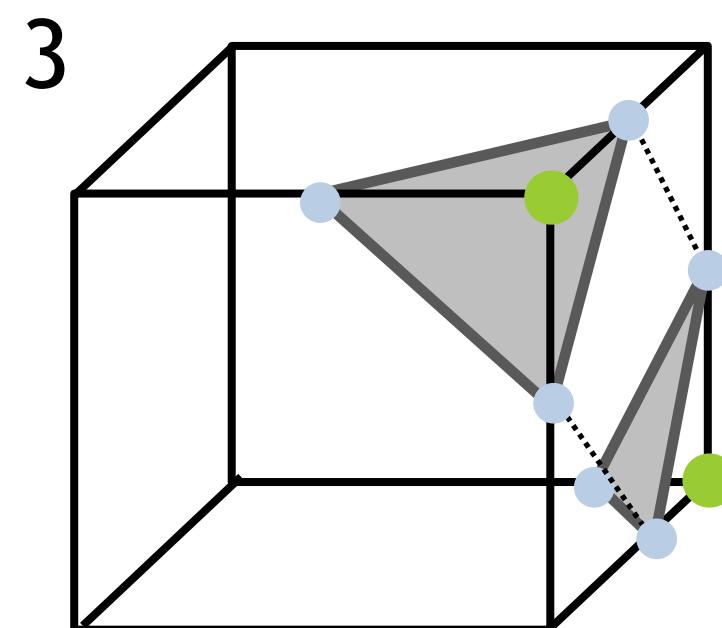


7. Move to the next cube



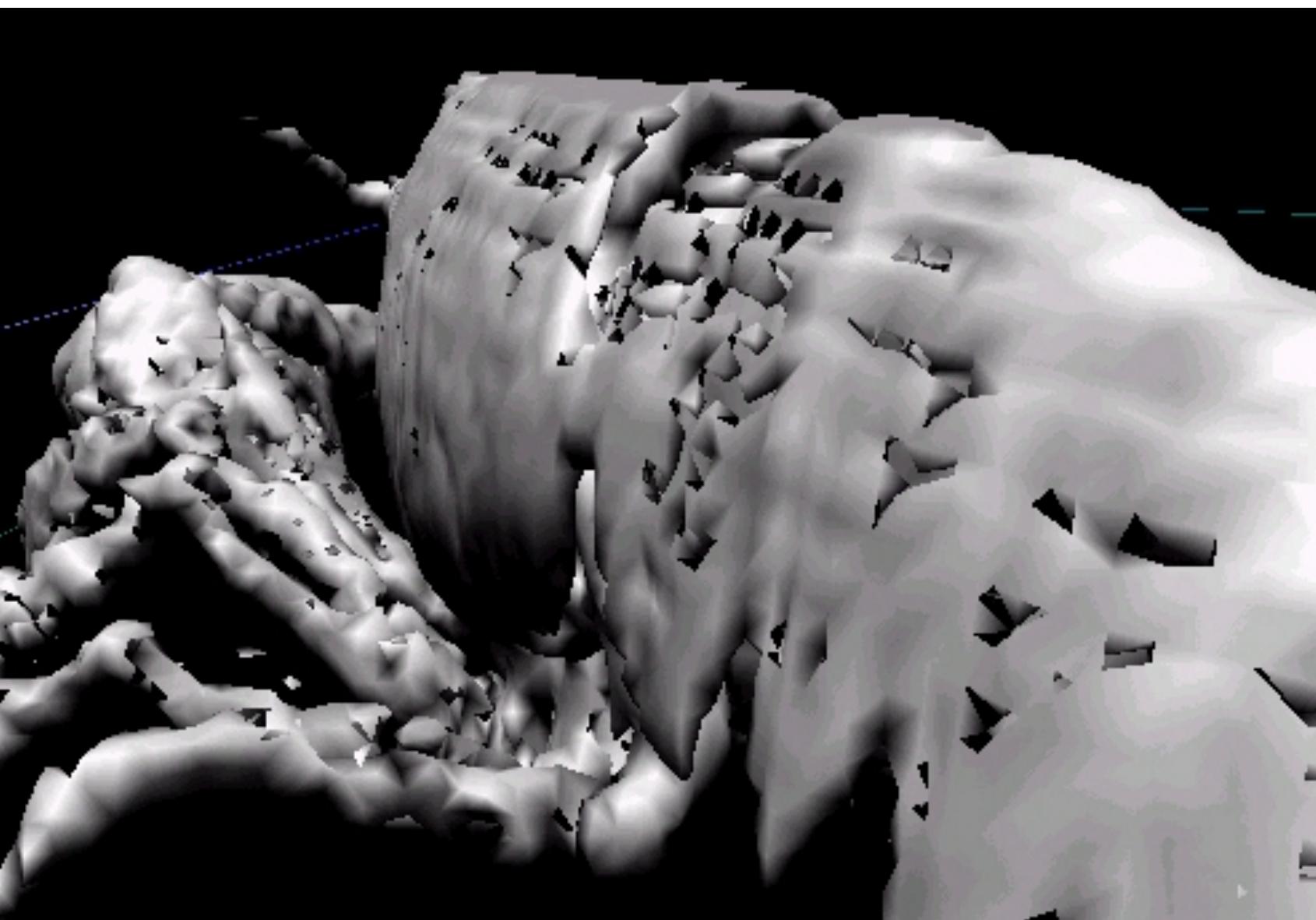
# Marching Cubes – Problems

- Have to make consistent choices for neighboring cubes – otherwise get holes

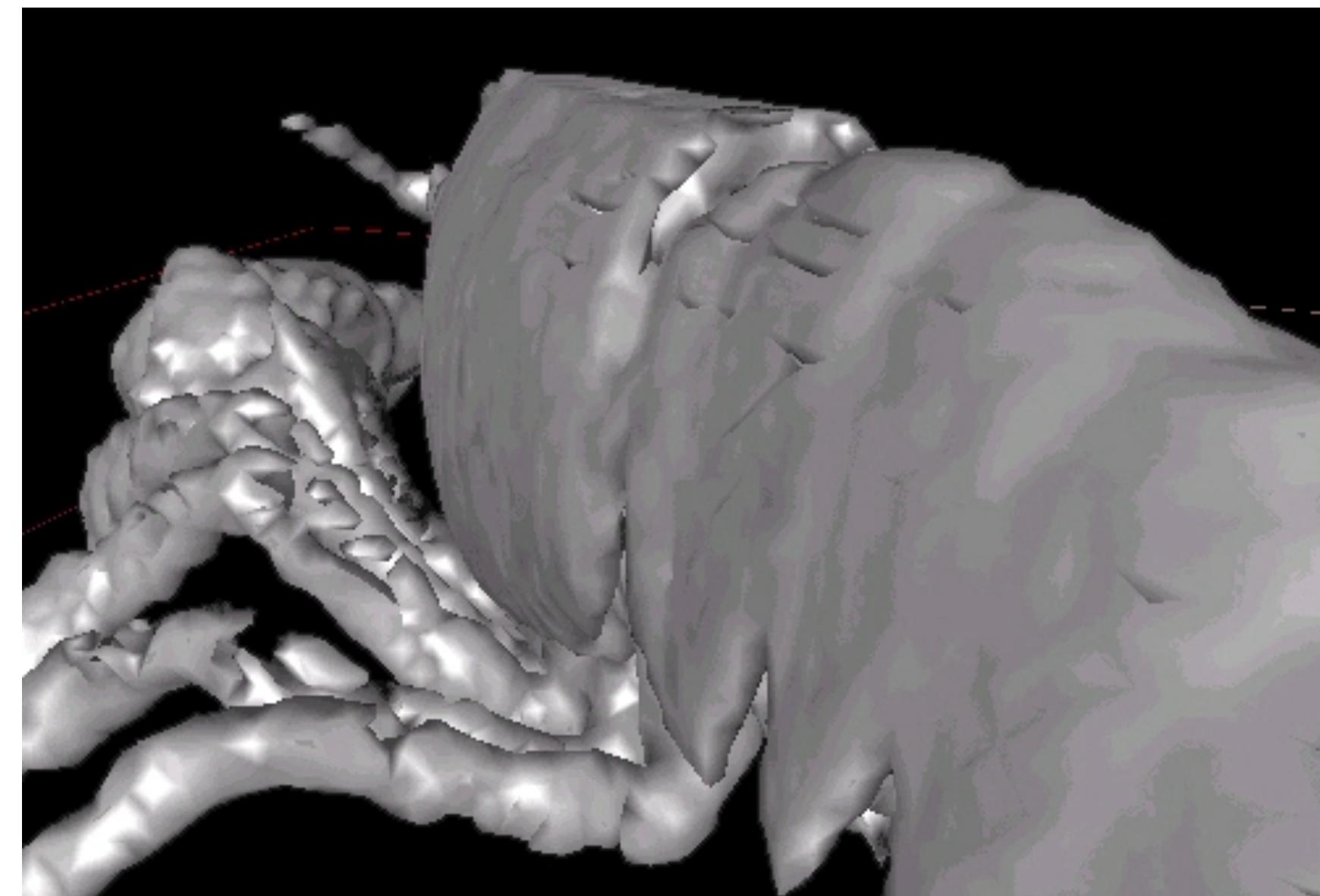


# Marching Cubes – Problems

- Resolving ambiguities



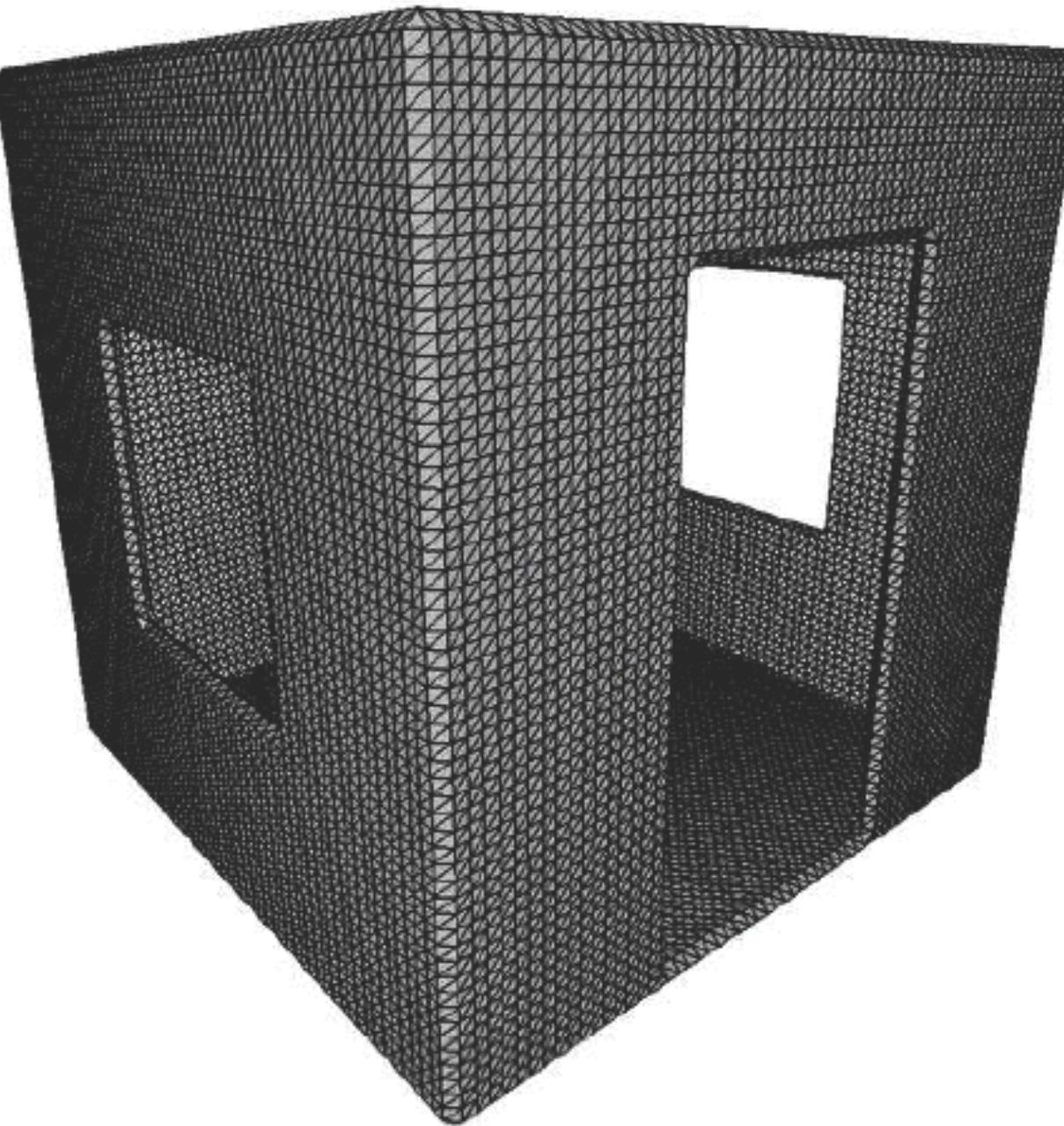
Ambiguity



No Ambiguity

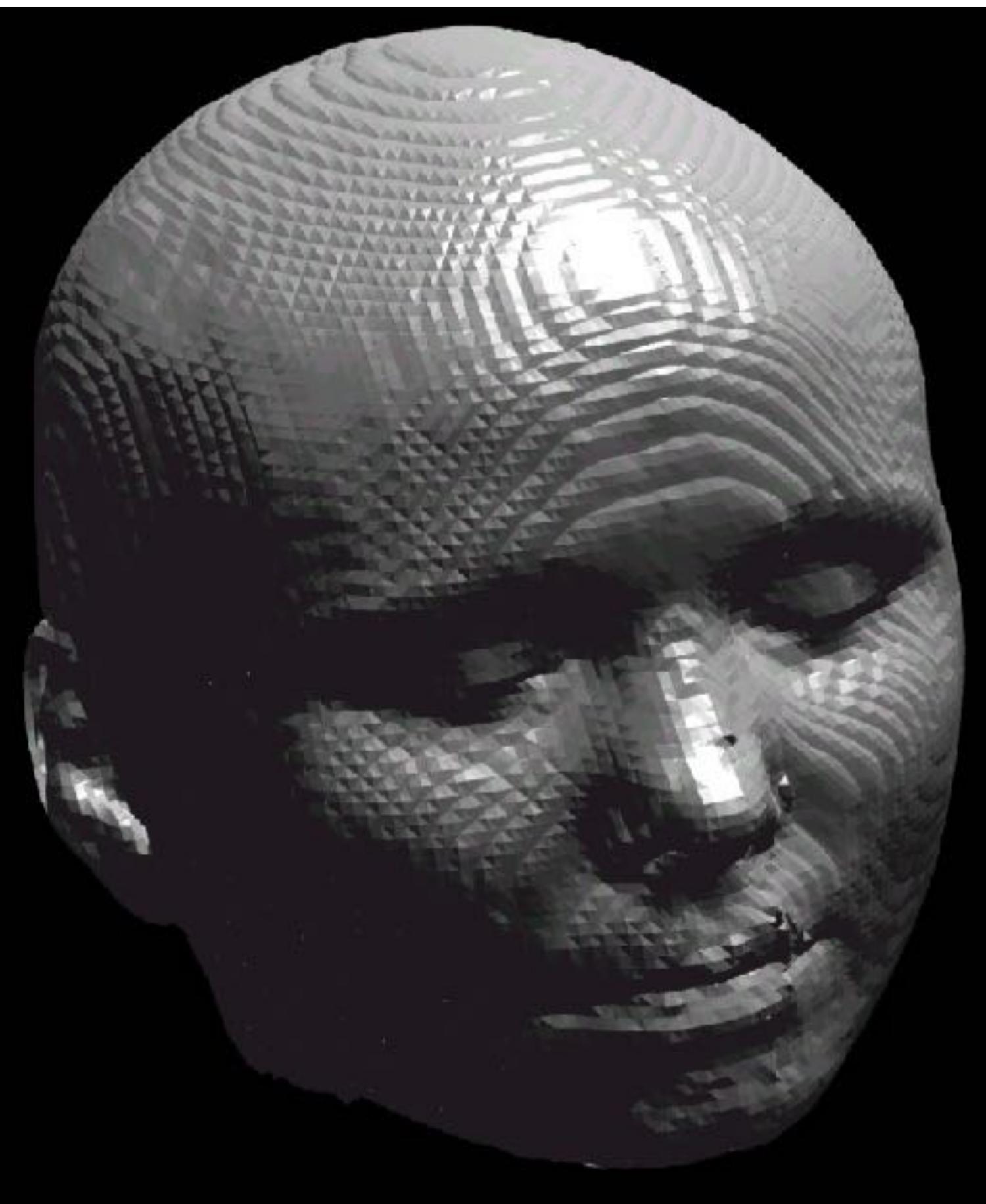
# Marching Cubes – Problems

- Grid not adaptive
- Many polygons required to represent small features



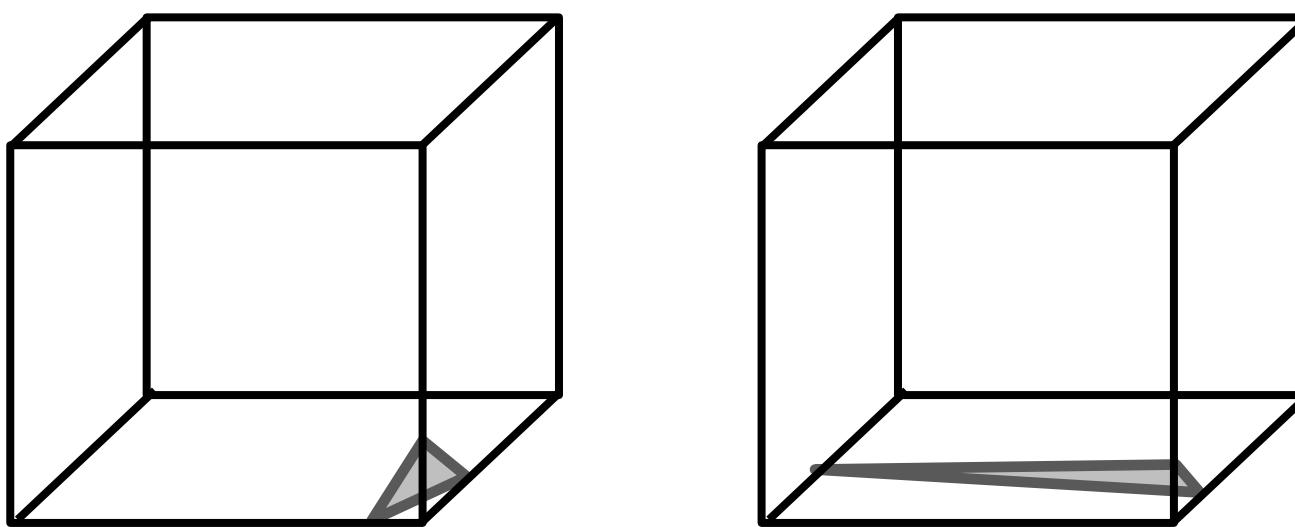
Images from: “Dual Marching Cubes: Primal Contouring of Dual Grids”  
by Schaeffer et al.

# Marching Cubes – Problems



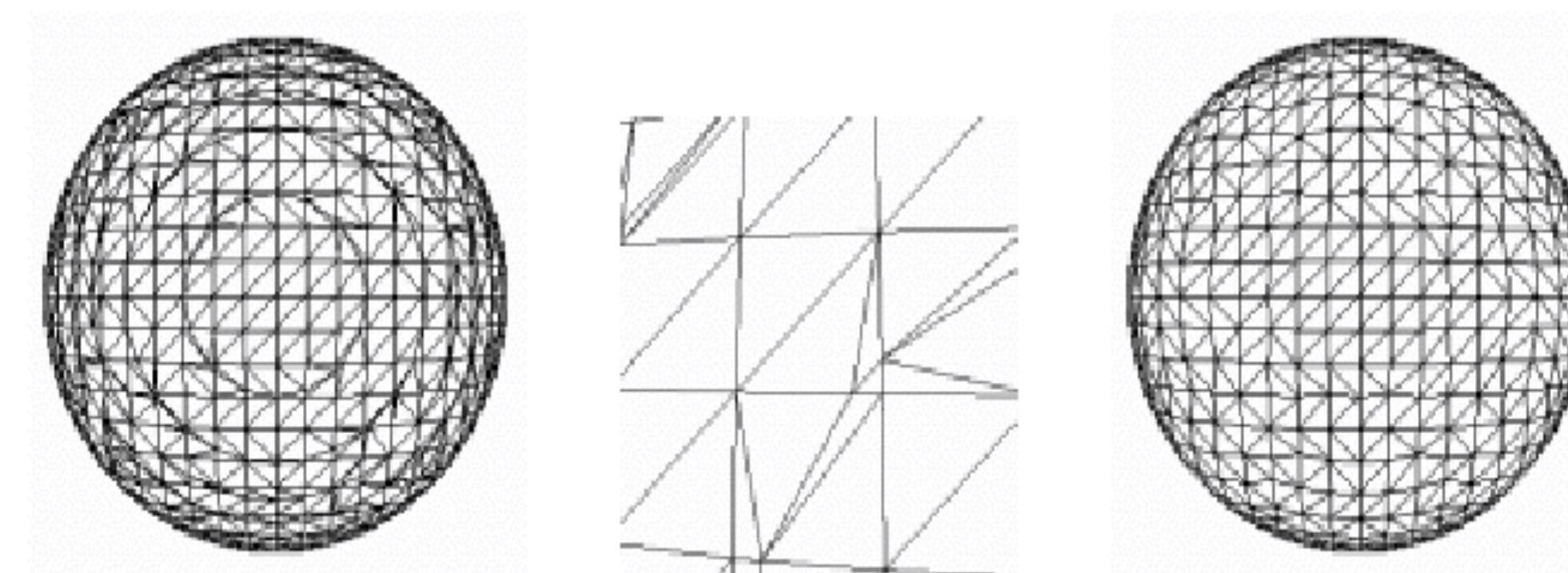
# Marching Cubes – Problems

- Problems with short triangle edges
  - When the surface intersects the cube close to a corner, the resulting tiny triangle doesn't contribute much area to the mesh
  - When the intersection is close to an edge of the cube, we get skinny triangles (bad aspect ratio)
- Triangles with short edges waste resources but don't contribute to the surface mesh representation



# Grid Snapping

- Solution: threshold the distances between the created vertices and the cube corners
- When the distance is smaller than  $d_{\text{snap}}$  we snap the vertex to the cube corner
- If more than one vertex of a triangle is snapped to the same point, we discard that triangle altogether



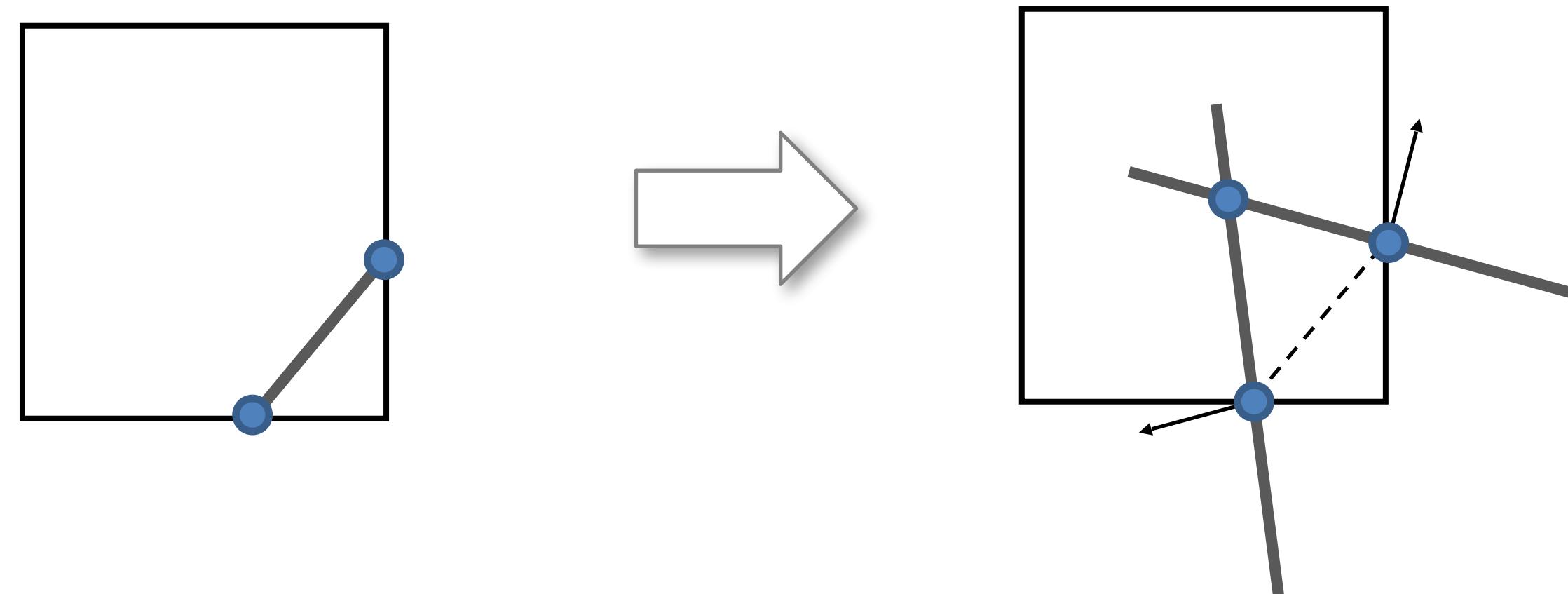
# Grid Snapping

- With Grid-Snapping one can obtain significant reduction of space consumption

Parameter	0	0,1	0,2	0,3	0,4	0,46	0,495
Vertices	1446	1398	1254	1182	1074	830	830
Reduction	0	3,3	13,3	18,3	25,7	42,6	42,6

# Sharp Corners and Features

- (Kobbelt et al. 2001):
  - Evaluate the normals (use gradient of  $F$ )
  - When they significantly differ, create additional vertex



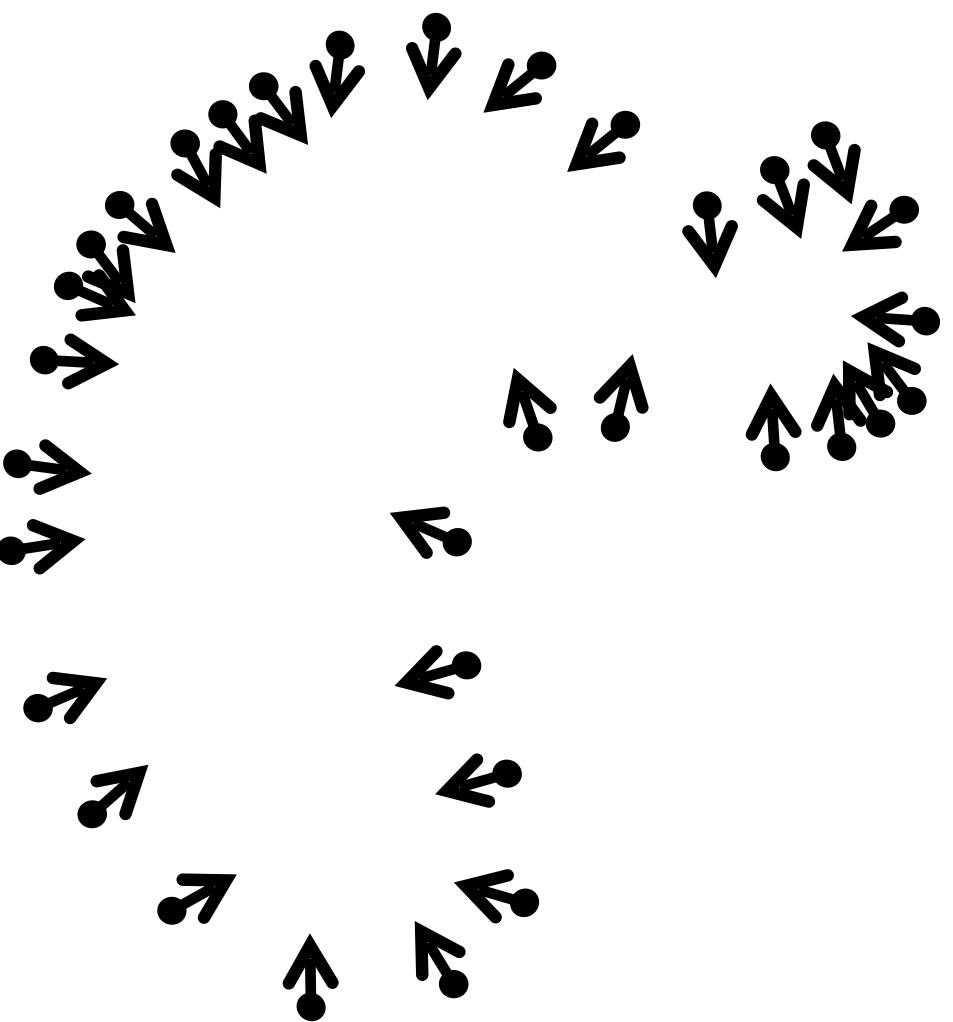
# Global RBF vs. Local MLS

- RBF:
  - sees the whole data set, can make for very smooth surfaces
  - global (dense) system to solve – expensive
- MLS:
  - sees only a small part of the dataset, can get confused by noise
  - local linear solves – cheap

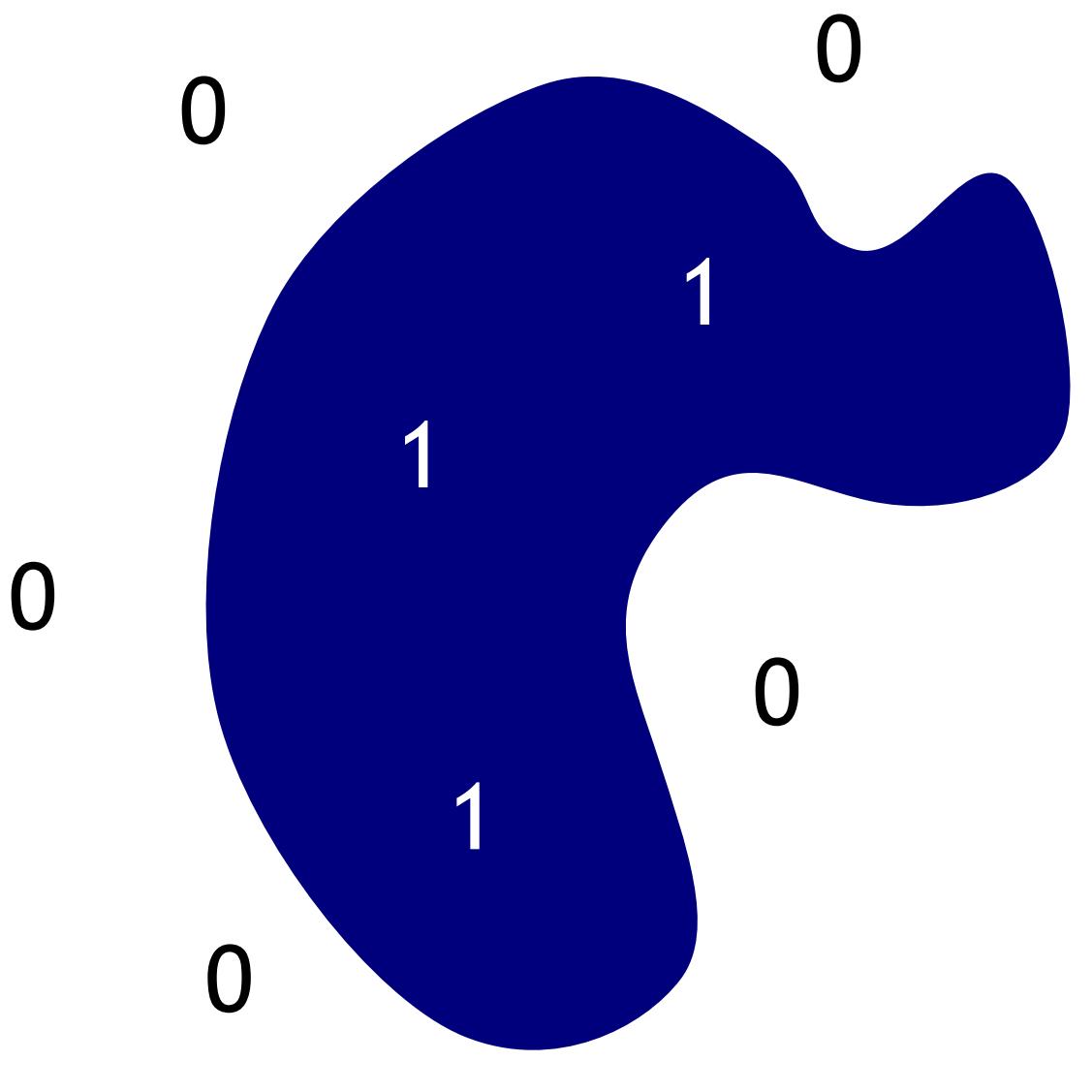
# Poisson Surface Reconstruction

- Very popular modern method, code available: M. Kazhdan, M. Bolitho and H. Hoppe, Symposium on Geometry Processing 2006  
<http://www.cs.jhu.edu/~misha/Code/PoissonRecon/>
- Global fitting of an *indicator function* using PDE
  - Robust to noise, sparse, computationally tractable
- You will try out the code in Ex2 and compare with MLS results

# Poisson Surface Reconstruction



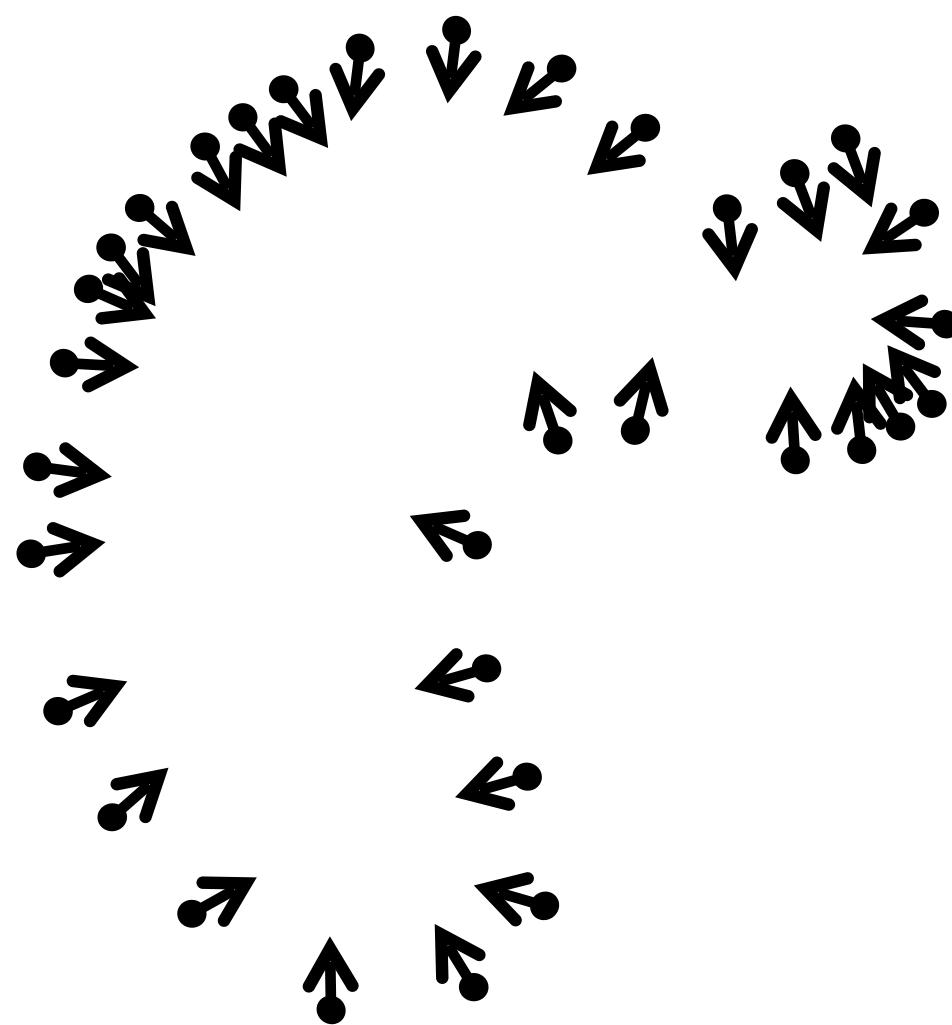
Oriented points



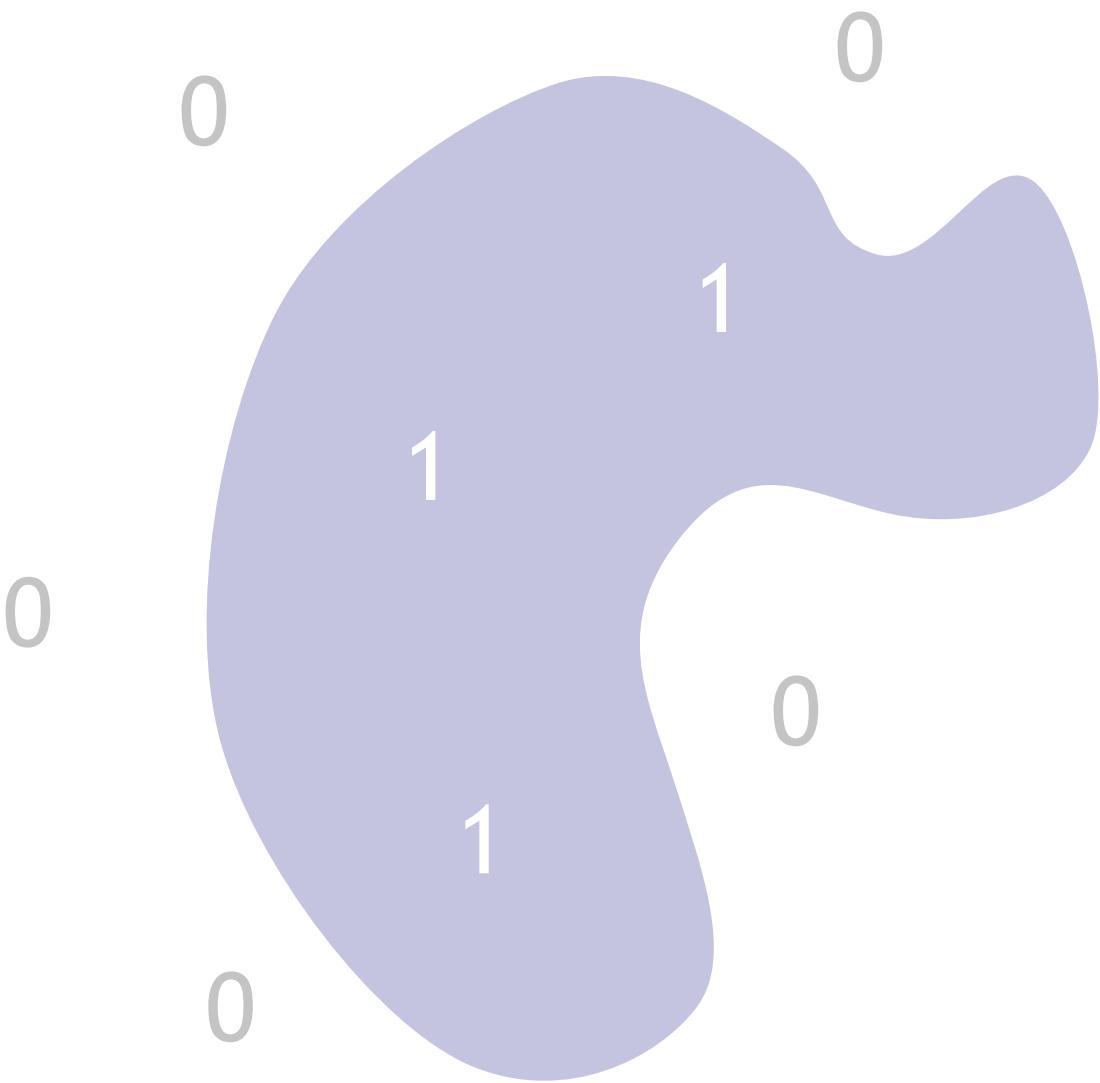
Indicator function

$$\chi_{\mathcal{M}}$$

# Poisson Surface Reconstruction



Oriented points

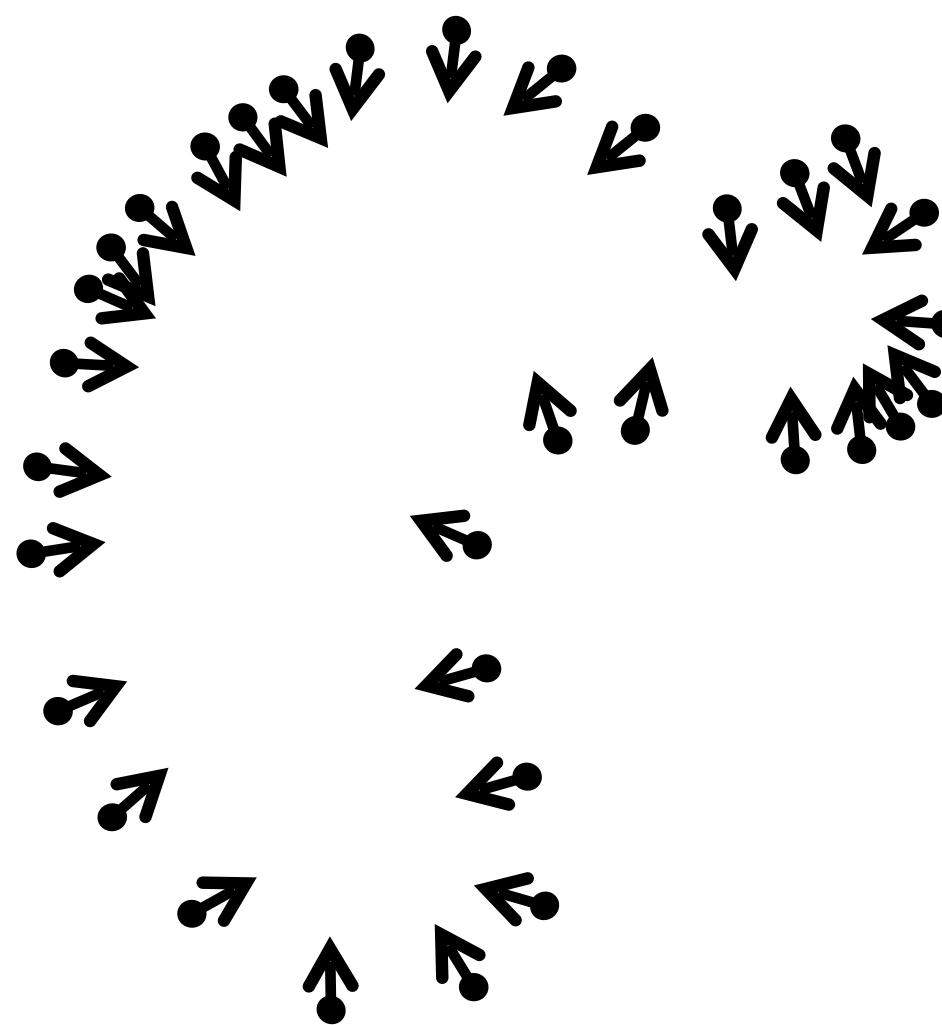


Indicator function

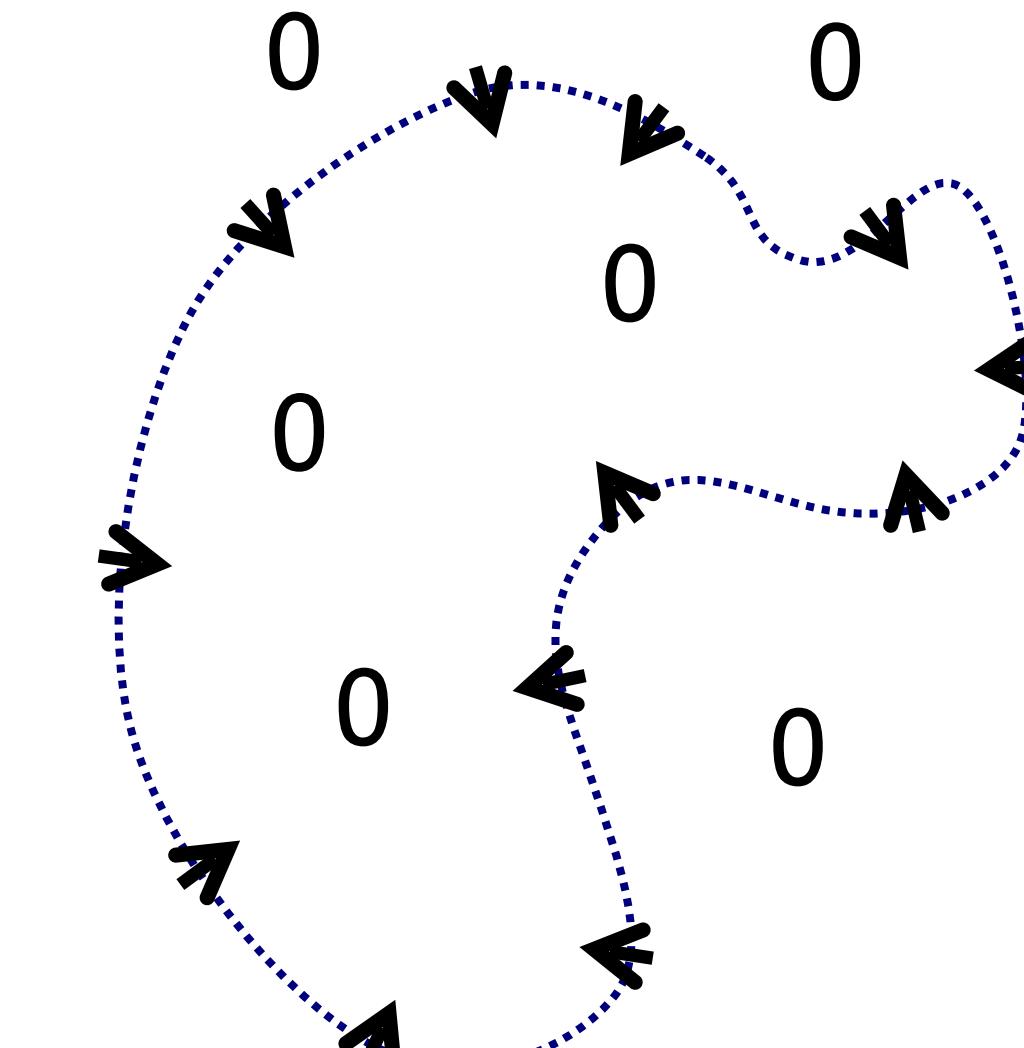
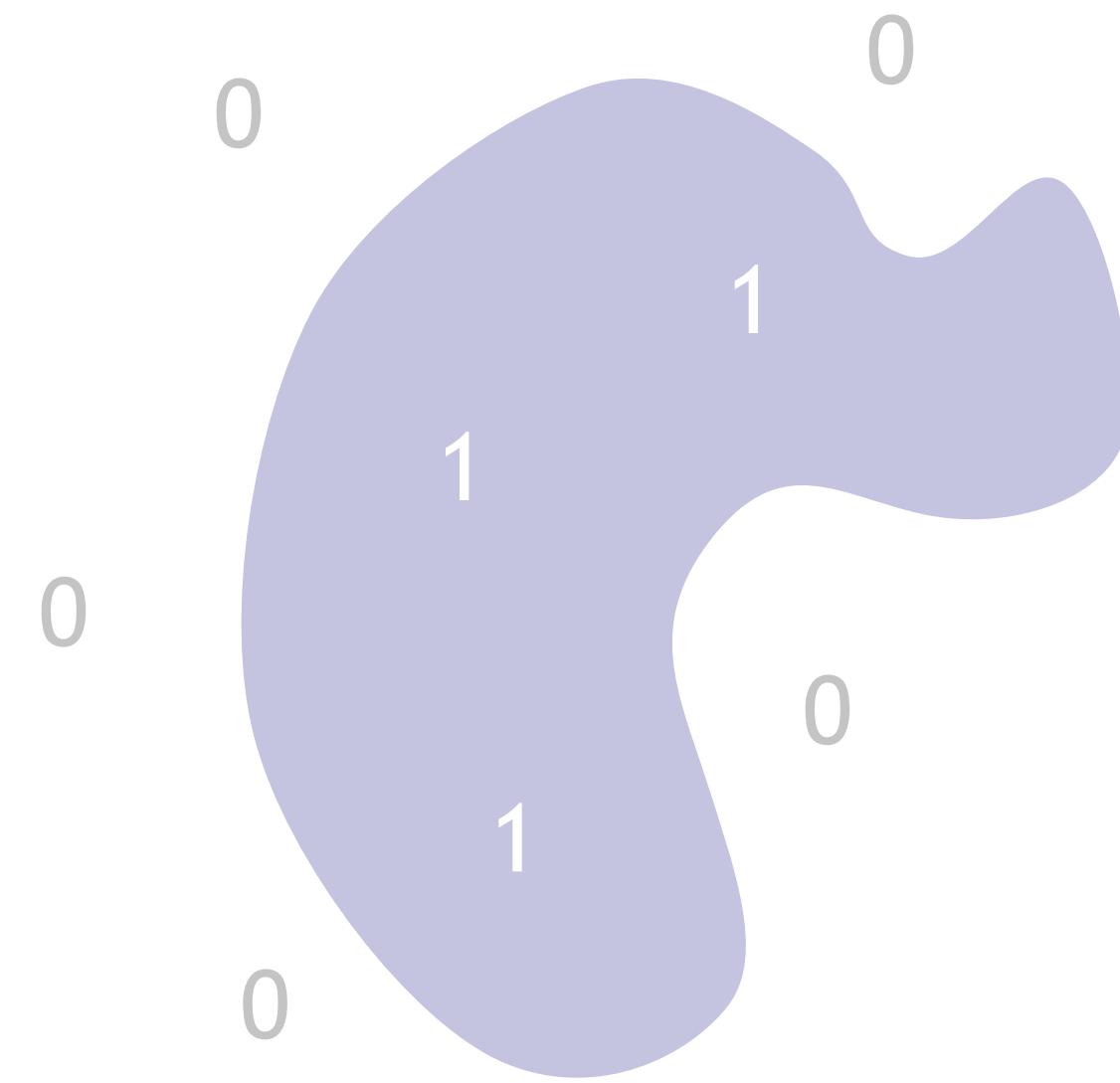
$$\chi_{\mathcal{M}}$$

We don't know the indicator function 😞

# Poisson Surface Reconstruction



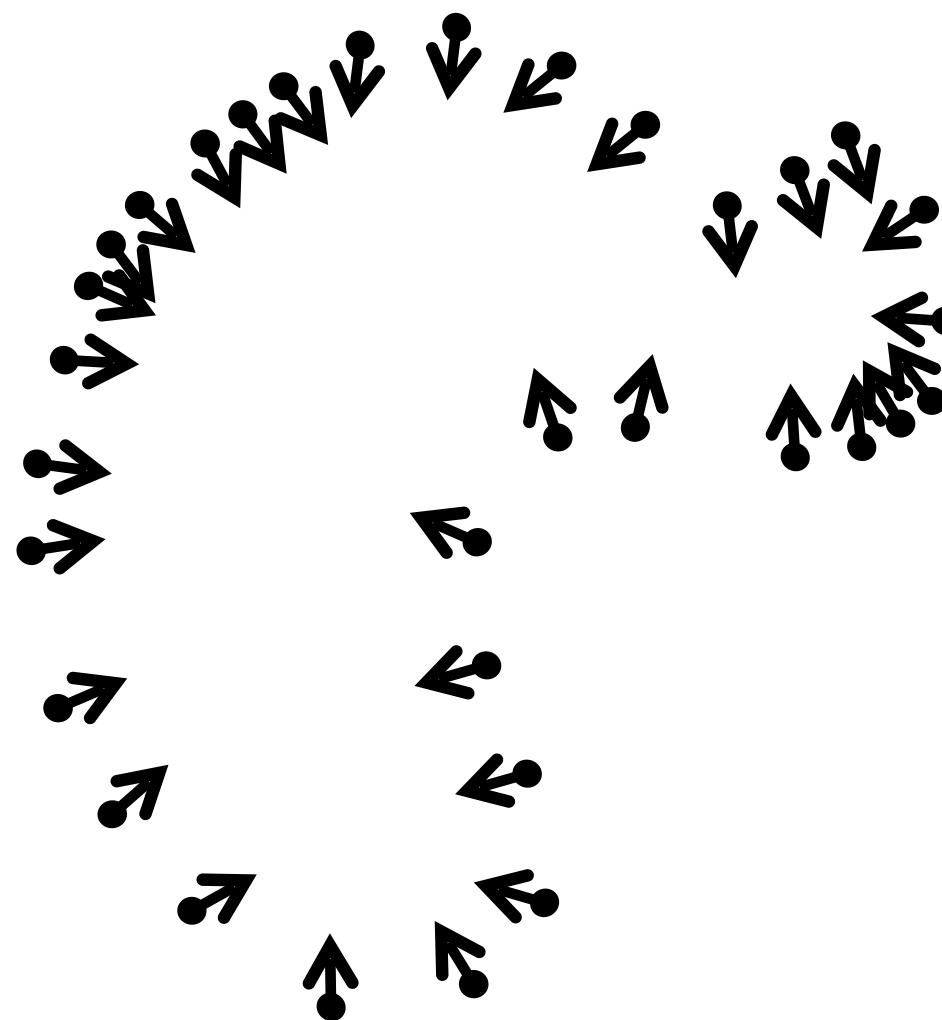
Oriented points



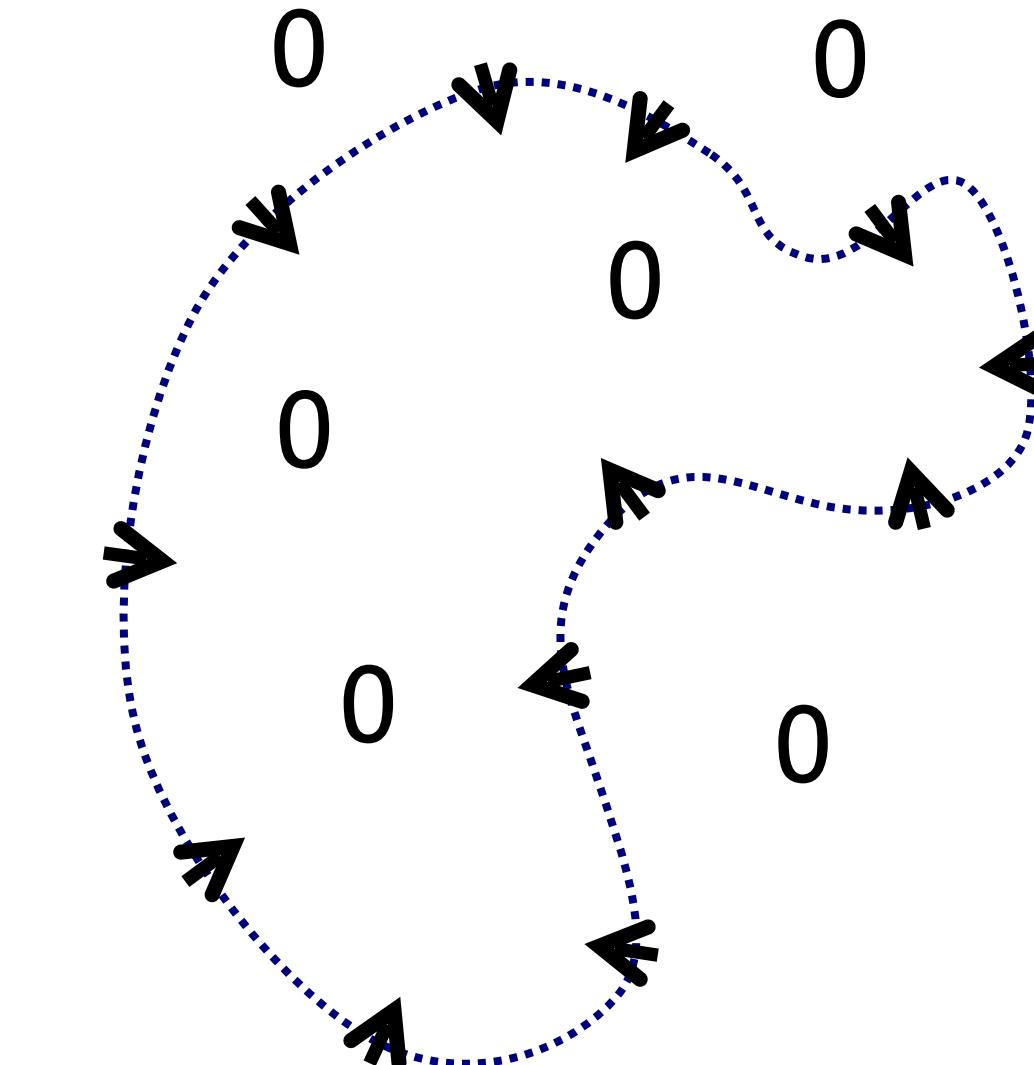
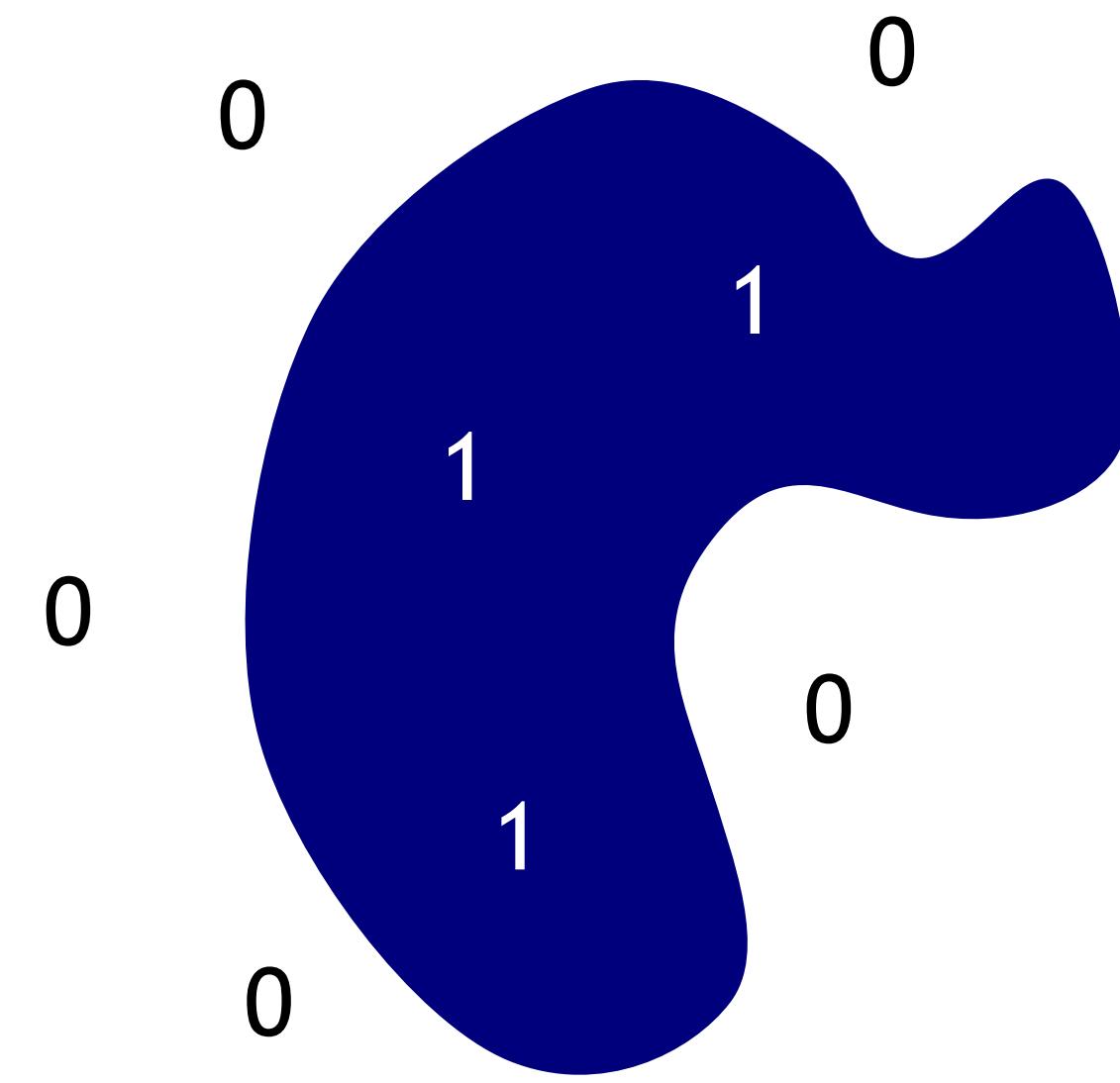
But we can estimate its gradient! 😊



# Poisson Surface Reconstruction



Oriented points



$$\chi_M$$

$$\nabla \chi_M$$

Reconstruct  $\chi$  by solving  
the Poisson equation

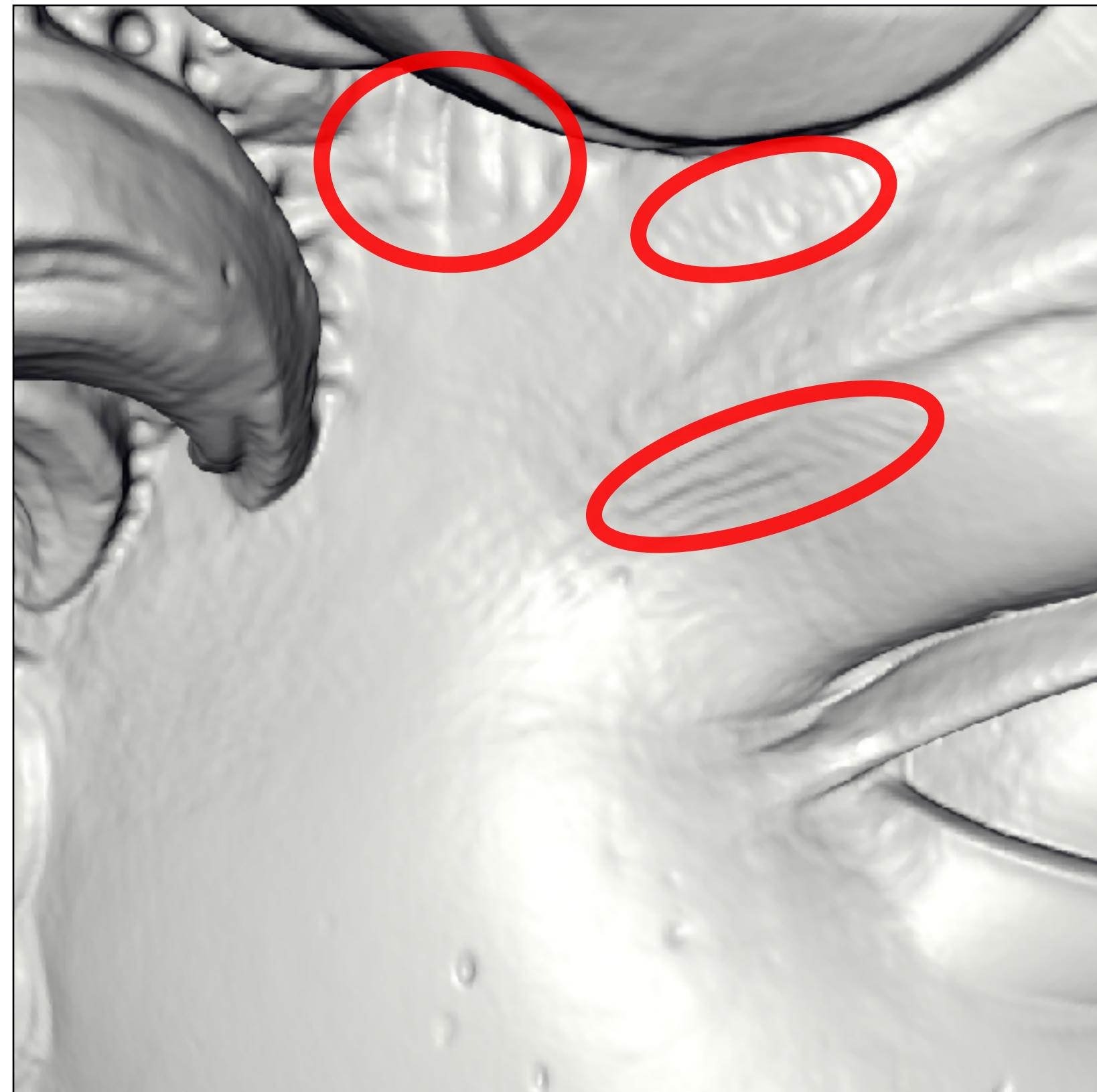
$$\Delta \chi_M = \operatorname{div} \nabla \chi_M$$

# Michelangelo's David

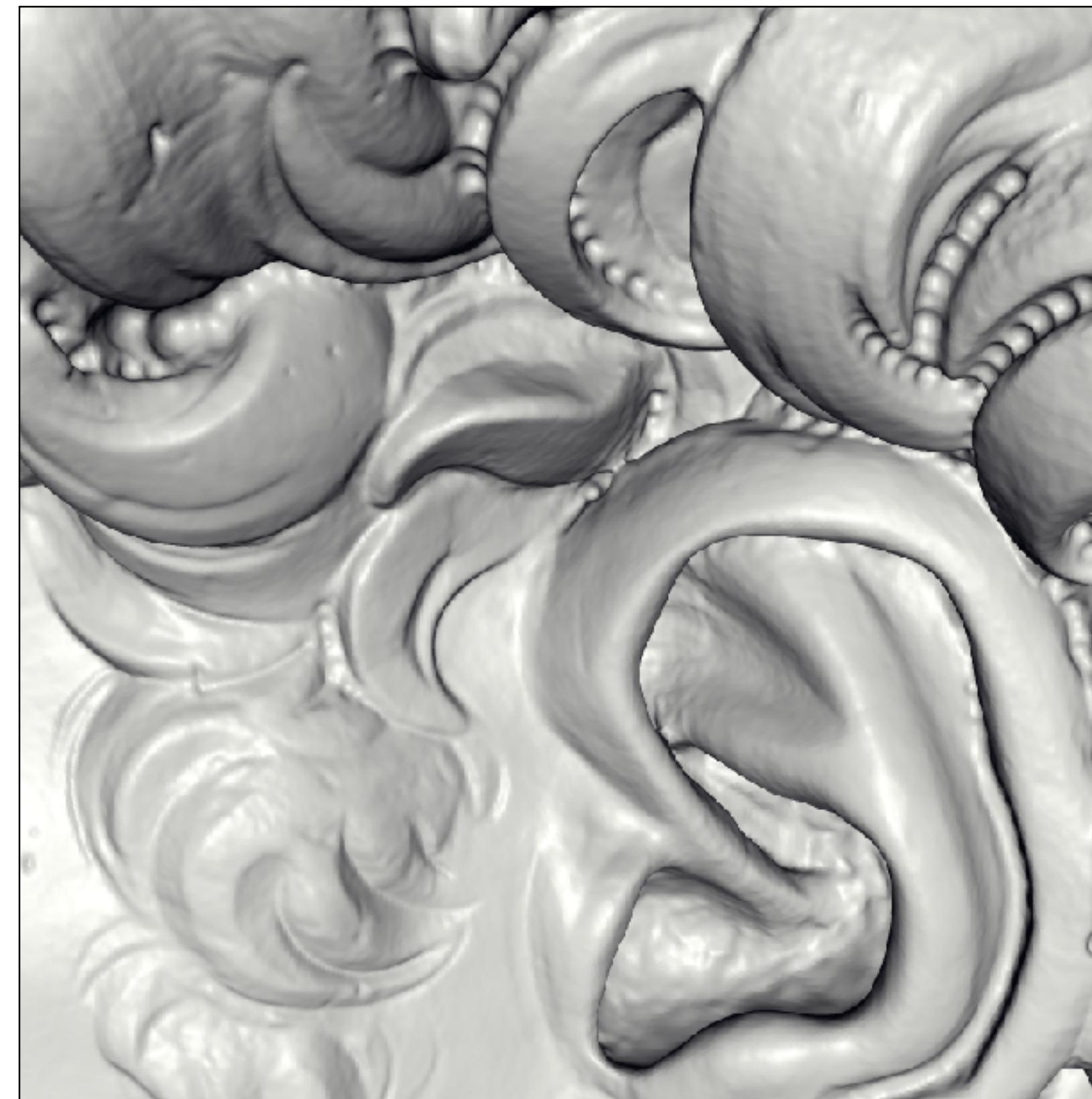


215 million data points from 1000 scans  
22 million triangle reconstruction  
Compute Time: 2.1 hours  
(this was in year 2006)  
Peak Memory: 6600MB

# David – Chisel marks



# David – Drill Marks



# David – Eye



# Thank you