Q1) Give four examples of Abelian Group / Commutative Group with explaplenation.

Mathematical Logic

- Proposition - A statement/ proposition is a declorative sentence which is either true or false but not both

Eg. 1. The sun rises in the east. → This is a Proposition.
2. Do you know Tamil? → Not Proposition
3. $5+7=10$ → Proposition
4. $x-5=0$ → Not a Proposition.
5. Give me the pen. → Not a Proposition
6. The temperature of Darjeeling is now 23°C. → Proposition
7. The sun will set today at 5:45 pm. → Proposition

- A proposition obtained by the combination of two or more elementary proposition (only a single propoposition variable/ constant) by means of the logical connectives/operator is called a compound or composite proposition.

- The words or phrases or symbols used to form a compound proposition are called the logical connectives.

5 logical connectives

i) Negation. ($\sim p$ or $\neg p$) ($\neg(\neg p) = p$)
ii) Conjunction.
iii) Disjunction
iv) Conditional
v) Bicondition

> Conjunction -
Logic: (AND)

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

iii) Disjunction ($\vee$) -

(Logic OR)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

iv) Conditional ($\longrightarrow$ or $\Rightarrow$) - $p \rightarrow q$

antecedent or hypothesis (p); consequent / conclusion (q)

read as 'if p then q'.

$p \rightarrow q$ it is also called implication.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

v) Biconditional ($\longleftrightarrow$) -

(logic XOR).

p	q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

* Converse, Inverse & contrapositive of an Implication

Let, $p \longrightarrow q$ be an implication. Then,

i) The converse of $p \rightarrow q$ is the implication $q \rightarrow p$.

ii) The inverse of $p \rightarrow q$ is the implication $\sim p \rightarrow \sim q$.

iii) The contrapositive of $p \rightarrow q$ is the implication $\sim q \rightarrow \sim p$

Truth Table of converse.

p	q	q → p
T	T	T
T	F	T
F	T	F
F	F	T

Truth Table of Inverse.

p	q	~p	~q	~p → ~q
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Truth Table of contrapositive.

p	q	~p	~q	~q → ~p
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Q) Construct the truth table of the following -

i) $(\sim p \wedge q) \vee q$.

p	q	~p	~p ∧ q	(~p ∧ q) ∨ p.
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

iii) $(p \to q) \to (\sim q \to \sim p)$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \to (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

* Consider the following.

p: "I will study Discrete Mathematics"
q: "I will go to a movie"
r: "I am in a good mood".

- If i am not in a good mood then I will go to movie else i will study discrete mathematics.

A. $(\sim r \to q) \vee p$.

- If i am in a good mood then either i will study discrete mathematics or i will not go to movie else i will go to a movie.

A. $(r \to (p \vee \sim q)) \vee q$

p	q	r	$\sim q$	$p \vee \sim q$	$r \to p \vee \sim q$	$(r \to (p \vee \sim q)) \vee q$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

- I will not go to movie and I will study Discrete math if and only if I am in a good mood.

A. $(\sim q \wedge p) \longleftrightarrow r$

Tautology, Contradiction, Contingency:

- A statement which is true for all possible values of its contituent propositional variables is called a tautology.

- → A statement which is false for all possible values of its constituent propositional variables is called a contradiction.

- → A statement which is neither a tautology nor a contradiction is called a contingency.

p	q	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$	$\sim p \vee q$
T	T	F	T	F	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	T	F	T
			↓ Tautology (All True)	↓ Contradiction (All False)	↓ Contingency (Mixed)

Logical Equivalence

- Two Propositions P and Q are said to be logically equivalent if $P \longleftrightarrow Q$ is a tautology. It is represented by $P \equiv Q$.

Check $p \longleftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \longleftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$(p\leftrightarrow q)\leftrightarrow(p\to q)\wedge(q\to p)$
T
T
T
T

Since, $(p\to q)\wedge(q\to p)$ is a tautology

$\therefore\ P\longleftrightarrow Q \equiv (p\to q)\wedge(q\to p)$

ii) $p\to q \equiv \sim p \vee q$

p	q	$p\to q$	$\sim p$	$\sim p\vee q$	$(p\to q)\longleftrightarrow(\sim p\vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

iii) $\sim(p\wedge q) \equiv \sim p \vee q$

p	q	$\sim p$	$p\wedge q$	$\sim(p\wedge q)$	$\sim p\vee q$	$\sim(p\wedge q)\leftrightarrow \sim p\vee q$
T	T	F	T	F	T	F
T	F	F	F	T	F	F
F	T	T	F	T	T	T
F	F	T	F	T	T	T

Algebra of Proposition

The operations $\sim, \wedge, \vee$ for any proposition p, q, r satisfy the following properties -

1) Commutative Properties

a) $p\vee q \equiv q\vee p$

b) $p\wedge q \equiv q\wedge p$

2) Associative Properties

a) $p\vee(q\vee r) \equiv (p\vee q)\vee r$

b) ~~$p\vee q$~~ $p\wedge(q\wedge r) \equiv (p\wedge q)\wedge r$.

3) **Distributive Properties**

a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

4) **Idempotent Properties**

a) $p \vee p \equiv p$

b) $p \wedge p \equiv p$

5) **Properties of Complement**

a) $p \vee \sim p \equiv T$

b) $p \wedge \sim p \equiv T$

c) $\sim T \equiv F$

d) $\sim F \equiv T$.

* 6) **De Morgan's Law**

e) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

f) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

* 7) **Involution Property**

g) $\sim (\sim p) \equiv p$.

8) **Identity Property**

a) $p \wedge T \equiv p$

b) $p \wedge F \equiv F$

c) $p \vee T \equiv T$

d) $p \vee F \equiv p$.

Well - Formed Formula (WFF) :-

A statement formula is a logical equation consisting of variables, parenthesis and connecting symbols.

A statement formula is not a statement and has no truth value. If we substitute definite statement in place of a variable in a given formula we get then a statement

A statement formula is called a Well formed formula (wff) if it can be generated by the following rules.

* <W1> - A statement variable 'p' standing alone is a well formed formula.

* <W2> - If 'p' is well-formed formula then '$\sim p$' is also a well-formed formula.

* <W3> - If 'p' and 'q' both are well formed formula then $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$, $(p \leftrightarrow q)$ is also a well formed formula.

* <W4> - A string of symbols is a well-formed formula if and only if it is obtained by finitely;
  E.g.- $\sim(p \vee q)$, $p \wedge \sim q$, $p \rightarrow (\sim p \wedge \sim q)$, $(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ etc.

Normal Forms (Cannonical Forms).

* The problem of determining in a finite number of steps whether a given formula is a tautology, contradiction or a contingency atleast is known as a 'decision' problem in propopositional calculus.

We can check using the Truth Table
Construct of Truth Table becomes tedious when the no.

of variables increases. * form of expression say P' and Q' such that a simple comparision b/w

$n \rightarrow 2^n \rightarrow$ elements in Truth Table.

$\downarrow$

variables

A better approach is to transform the expression b/w say P and Q to some standard * ** such that a simple comparision P' and Q' shows that $P \equiv Q$ or not. These standard forms are called the normal forms.

There are 2 types of normal forms -

1) DNF (Disjunctive Normal Form)
2) CNF (Conjunctive Normal Form)

How to

- Find a specific formula truth table?

$P(p, q, r) \longrightarrow$ Truth Table

p	q	r		P(p, q, r)
T	T	T	—	T
T	T	F		F
T	F	T		T
T	F	F	—	T
F	T	T	—	F
F	T	F		F
F	F	T	—	T
F	F	F		F

For True value :- (internal connected by AND (conjuction), expressions are connected by OR (disjunction)).

SOP $P(p, q, r) \equiv (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$ — (i).

For False value -

$\cancel{P(p, q, r) \equiv (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee \sim r)}$

Disjunctive Normal Form (DNF) , Sum of Product (SOP)

A logical expression which is equivalent to a given logical expression which consists of a sum of the given logical expression.

Eg - $p \vee (q \wedge r)$, $(p \wedge q) \vee (q \wedge \sim p)$, $(p \wedge q) \vee (\sim p \wedge r) \vee (r \wedge \sim q)$

Procedure to obtain the DNF of a given logical expression.

- **Step 1** - Remove conditional ($\rightarrow$), biconditional ($\leftrightarrow$) from the given expression and obtain an equivalent expression containing the connectives negation ($\sim$), disjunction ($\vee$), conjunction ($\wedge$).

- ~~Step2~~ Replace $p \rightarrow q$ with $\sim p \vee q$.

  Replace $p \leftrightarrow q$ with $(p \wedge q) \vee (\sim p \wedge \sim q)$ (SOP)

  $\equiv (\sim p \vee q) \wedge (p \vee \sim q)$ (POS)

- **Step 2** - Use De Morgan's Law to eliminate negation before sum and product by using the double negation.

- **Step 3** - Apply Distributive Property until a sum of elementary product form is obtained.

Q) Find the DNF of the expression $\sim(p \vee q) \leftrightarrow (p \wedge q)$.

Ans)- $\sim(p \vee q) \leftrightarrow (p \wedge q)$.

~~Using~~ Removing Biconditional,

$\Rightarrow (\sim(p \vee q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge (\sim(p \wedge q)))$

Using De Morgan's Law,

$\equiv ((\sim p \wedge \sim q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge (\sim p \vee \sim q))$

~~Using Distributive Property,~~

$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee (((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q))$

Using Distributive Property,

$\equiv (\sim p \wedge \sim q \wedge p \wedge q) \vee ((p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q))$

(Ans).

Q) Find the CNF of the expression $\sim(p \vee q) \leftrightarrow (p \wedge q)$.

Ans) Removing Biconditional, (from $\sim(p \vee q) \leftrightarrow (p \wedge q)$)

$\equiv ((p \vee q) \vee (p \wedge q)) \wedge (\sim(p \vee q) \vee \sim(p \wedge q))$.

Using De Morgan's Law,

$\equiv ((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \wedge \sim q) \vee (\sim p \vee \sim q))$

Using Distributive Property,

$\equiv (((p \vee q) \vee p) \wedge ((p \vee q) \vee q)) \wedge \cancel{(((\sim p \wedge \sim q) \vee \sim p) \vee ((\sim p \wedge \sim q)}$

$((\sim p \vee (\sim p \vee \sim q)) \wedge (\sim q \vee (\sim p \vee \sim q))$

$\equiv (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim p \vee \sim q) \wedge (\sim q \vee \sim p \vee \sim q)$

Without forming truth table prove that -

$p \vee (p \wedge q) \equiv p$.

A). $p \vee (p \wedge q) \equiv (p \wedge T) \vee (p \wedge q)$ [as $p \wedge T \equiv p$].

Using Distributive Property -

$p \wedge (T \vee q) \equiv p \wedge T \equiv p$

[as $q \vee T = T$]

$q \wedge T \equiv q$

g) $\sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p.$

$\sim p \vee (q \wedge T)$

LHS,

$\sim(p \vee q) \vee (\sim p \wedge q)$

$\equiv ((\sim p \vee q) \vee \sim p) \wedge ((\sim p \vee q) \vee q))$

$\equiv (\sim p \vee q) \wedge (\sim p \vee q)$

$\equiv (\sim p \vee q)$

LHS,

$\sim(p \vee q) \vee (\sim p \wedge q)$

De Morgan's Law,

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

$\equiv \sim p \wedge (\sim q \vee q)$

$\equiv \sim p \wedge T$ [as $q \vee \sim q \equiv T$]

$\equiv \sim p$ [as $p \wedge T \equiv p$]

18/4/22.

Q) $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q)) \equiv q$, without using Truth Table

LHS, Distributive Property. De Morgan's Law.

$\equiv (p \wedge \sim p) \vee (p \wedge q) \vee (q \wedge (\sim p \vee \sim q))$

$\equiv F \vee (p \wedge q) \vee (q \wedge \sim p) \vee (q \wedge \sim q)$ [Distributive Property]

$\equiv F \vee (p \wedge q) \vee (q \wedge \sim p) \vee F$ [Property of complement].

$\equiv (p \wedge q) \vee (q \wedge \sim p)$ [Property of Identity].

$\equiv q \wedge (p \vee \sim p)$ [Distributive Property].

$\equiv q \wedge T$ [Property of complement].

$\equiv q$ [Property of Identity].

Q) $p \rightarrow (q \rightarrow r) \equiv p \wedge q \rightarrow r$

$\equiv p \rightarrow (\sim q \vee r)$ (Removing Conditionals ~~Reducing to Normal Form~~).

$\equiv \sim p \vee (\sim q \vee r)$

LHS,

$\equiv (\sim p \vee \sim q) \vee r$ [Associative Property].

$\equiv \sim (p \wedge q) \vee r$, [De Morgan's Law].

$\equiv p \wedge q \rightarrow r$ (Proved).

Q) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

LHS,

$\equiv$ ~~$(\sim p \wedge \sim q)$~~ $((\sim p \wedge \sim q) \wedge r) \vee (q \wedge r) \vee (p \wedge r)$ [Associative Prop.]

$\equiv (\sim (p \vee q) \wedge r) \vee (r \wedge (p \vee q))$ [De Morgan's Law] [Associative Prop.]

$\equiv r \wedge (\sim (p \vee q) \vee (p \vee q))$ [Distributive Prop.]

$\equiv r \wedge T$ [Property of complement]

$\equiv r$ [Identity Prop.]

Precedence of Logical Connections

$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

Mathematical Logic in Proof

If an implication $p \rightarrow q$ or $p \leftrightarrow q$ is a tautology where p and q maybe elementary or compound propositions involving any number of propositional variables we say that q logically follows from p.

Let an implication $(p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \rightarrow q$ — (i) be a tautology then this implication is true regardless of the truth values of any of its components. In such a case q logically follows from $p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n$.
Then we can write this expression

$$\left.\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{array}\right\} \rightarrow (2) \text{ (Premises or Hypothesis)}$$

$\therefore q$, (conclusion)

The statements $p_1, p_2, p_3 \dots p_n$ in ~~expression~~ equation (ii) is called the premises or hypothesis and q is called the conclusion of the hypothesis.

* A sequence of statements where is last one is the conclusion and all others are premises is called an argument.

* An argument is said to be a logically valid argument iff the conjunction of all the premises implies the conclusion that is the premises are all true, the conclusion must also be true.

Procedure for Testing the validity of an argument using Truth Table

Step 1 - First identify the premises and conclusion of the given argument

Step 2 - construct the Truth Table showing the Truth Values of all the premises. ~~and~~

Step 3 - Identify the row or rows in which all the premises are true. These special rows are called critical rows.

Step 4 - In each critical row determine whether the conclusion of the argument is also true.

(a) If in each critical row, the conclusion of the argument is true then the given argument is a valid argument.

(b) If there is atleast one critical row in which the conclusion is false then the given argument is said to be an invalid argument.

Q) Check the given argument is valid or invalid

$$\left.\begin{array}{l} p \to q \\ p \end{array}\right\} \text{Premises.}$$

$$\therefore q \quad \} \to \text{conclusion}$$

Ans) =

p	q	$p \to q$	$(p \to q) \wedge p$	
T	T	T	T	→ Critical Row.
T	F	F	F	
F	T	T	F	
F	F	T	F	

~~∵ $(p \to q) \wedge q$ is not a tautology.~~

∴ The argument is ~~invalid~~ valid.

---- x ----

p	q	p → q	~p	~q
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Critical Row

Method of Proof

① **Direct Proof** - This method uses the implication $(h_1 \wedge h_2 \wedge h_3 \wedge \dots \wedge h_n) \rightarrow c$ which is proved to be a tautology where $(h_1, h_2, h_3 \dots h_n)$ is a hypothesis and c is the conclusion. Eg- The product of two odd integers is always odd.

Proof : Let m & n be two odd integers then there exists integers r & s such that $m = 2r+1$ and $n = 2s+1$

Then $mn = (2r+1)(2s+1)$

$= (4rs + 2r + 2s) + 1$

$=$ even number + 1 = odd number.

* Thus (m is a an odd integer) ∧ (n is an odd integer)

→ (mn is an odd integer). is universally valid. Hence, the proof.

② **Indirect Proof** - Indirect Proof uses the tautology $(p \rightarrow q) \leftrightarrow ((\sim q) \rightarrow (\sim p))$. This states that an implication is equivalent to its contrapositive. Thus to prove p implies q indirectly we assume that q is false $(\sim q)$ and show that p is then false $(\sim p)$. Let n be an integer. Prove that if n^2 is odd then n is also odd.

Proof: Let us consider the statement p: "n^2 is odd"

q: "n is odd". Then it is

required to $p \rightarrow q$ is true when both p and q are true. Instead to prove the contrapositive $\sim q \rightarrow \sim p$. Suppose that n is not odd $(\sim q)$ i.e n is even.

Let $n = 2k$, where k is an integer.
$n^2 = 4k^2 = 2 \cdot 2k^2$.
So, n^2 is an even no. $(\sim p)$.

Thus, we show that if n is even then n^2 is also even (which is actually the contrapositive of the given statement). Hence, the implication if "n^2 is odd then n is odd", is universally true. Hence the prove.

Proof By Contradiction:
This method is based on the tautology $((p \rightarrow q) \wedge (\sim q)) \rightarrow (\sim p)$. In other words, this argument is valid -

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

Q) Prove that $\sqrt{2}$ is not a rational number.

Let, q : "$\sqrt{2}$ is a rational no."

We assume that $(\sim q)$ is true.
Then, there exists integers m and n that are mutually prime (co-prime).

$\therefore \sqrt{2} = \frac{m}{n} \Rightarrow 2n^2 = m^2$ — (1)

m^2 is even number.
If m^2 is even then m is also even no.

Let $m = 2k$, k is an integer;

Substituting in equation (i):

$$4k^2 = 2n^2.$$
$$\Rightarrow n^2 = 2k^2,$$

$\therefore n^2$ is an even number;
$\therefore n$ is also even.

Thus, n and m both are even and they are not prime to each other as they have a common factor 2. Hence, ~p statement is false. Thus 'q' is true. (Hence Proved).

Proof by Cases

This method utilizes the fact, that the implication.

$$h_1 \vee h_2 \vee h_3 \vee \dots \vee h_n \rightarrow c.$$
$$(h_1 \rightarrow c) \wedge (h_2 \rightarrow c) \wedge \dots \wedge (h_n \rightarrow c).$$

where $h_1, h_2, h_3, \dots, h_n$ are hypothesis & c is conclusion

Thus if the cases $h_1 \rightarrow c$, $h_2 \rightarrow c$, ..., $h_n \rightarrow c$ are each proved to be true seperately. then $h_1 \vee h_2 \vee \dots \vee h_n \rightarrow c$ will be proved.

Prove that for every +ve integer n, $n^3 + n$ is always even

Proof: Let, h_1 : "n is +ve even integer".
h_2 : "n is +ve odd integer".
c : "$n^3 + n$ is an even integer".

Case 1: To prove $h_1 \rightarrow c$ is true,
Let, n be a +ve even integer.

Then, $n = 2k$ for some +ve integer k.

$\therefore n^3 + n = 8k^3 + 2k = 2(4k^3 + k)$ is even.

Thus $h_1 \rightarrow c$ is true

case 2. To prove $h_2 \rightarrow c$ is true,

let, n be a +ve odd integer.

Then, $n = 2k + 1$ for some +ve integer k.

$\therefore n^3 + n = (2k+1)^3 (2k+1)$

$= (8k^3 + 3 \cdot 4k^2 + 3 \cdot 2k \cdot 1 + 1) + (2k+1)$.

$= 8k^3 + 12k^2 + 8k + 2$.

$= 2(4k^3 + 6k^2 + 4k + 1)$ is even.

Thus $h_2 \rightarrow c$ is true.

Since, $h_1 \rightarrow c$ & $h_2 \rightarrow c$ ~~and~~ are both

Proof by the Principles of Mathematical Induction

This is an important of proof of those theorems that are based on ~~the~~ positive integers or naturalnumbers.

Formal statement - Let P(n) be a proposition involved in positive integer n then P(n) is true for all positive integral values of n provided that :

i) P(1) is true.

ii) P(m+1) is true whenever P(m) is true for some +ve integer m.

Thus, there are 3 steps in the proof by Mathematical Induction.

* Inductive base : verify that P(1) is true.
* Inductive hypothesis : Assume that P(m) is true for some arbitrary +ve integer m
* Inductive step - Verify that P(m+1) is true on the basis of inductive hypothesis.

Problem : Using the Principle of Mathematical Induction prove the following :

i) $4^{n+1} + 5^{2n-1}$ is divisible by 21, $n \in \mathbb{Z}^+$

A) Inductive Base - let n = 1.

$\therefore 4^{1+1} + 5^{(2\times1)-1}$

$= 4^2 + 5^{2-1}$

$= 16 + 5$

$= 21$

21 is divisible by 21.

$\therefore$ For P(1), the statement is true.

- **Inductive Hypothesis:** For P(m) i.e n = m

$\therefore 4^{m+1} + 5^{2m-1} = P(m)$

- Assuming P(m) is true, i.e P(m) is divisible by 21.

Inductive Step.

$\therefore P(m+1) = 4^{m+1+1} + 5^{2m+1-1}$

$= 4^{m+1} \cdot 4 + 5^{2m+1} \cdot 5^{-1}$

Proof - Let us consider the proposition:

P(n): "$4^{n+1} + 5^{2n-1}$" is divisible by 21, $n \in Z^+$ ".

Inductive base - For n = 1, $4^{1+1} + 5^{2\times1-1} = 21$ which is divisible by 21.

Hence, P(1) is true.

Inductive Hypothesis

$4^{m+1} + 5^{2m-1} = 21k$ for some +ve integer k — (1).

Inductive Step $4^{(m+1)+1} + 5^{2(m+1)-1}$

$= 4^{m+1} \cdot 4 + 25 \cdot 5^{2m-1}$

Q) If n is odd then $n(n^2-1)$ is divisible by 24. Proof it by mathematical induction technique.

Ans) = Let us consider the following proposition :

$P(n)$: " $n(n^2-1)$ is divisible by 24".

Inductive Base - For $n=1$, $1(1^2-1)=0$ which is divisible by 24

Hence, $P(1)$ is true.

- **Inductive Hypothesis** - $m(m^2-1) = 24k$ for some +ve integer k - (1).

- **Inductive step** - $(m+2)((m+2)^2-1)$

$= (m+2)(m^2+4m+4-1)$

$= (m+2)(m^2+4m+3)$

$\cancel{= m(m+1).(m+2)}$.

$= (m+2)(m^2+4m+3)$

$= (2p+1+2)((2p+2)^2+4(2p+1)+3)$. [$\because$ m is odd]

$= (2p+3)(4p^2+4p+1+8p+4+3)$.

$= (2p+3)(4p^2+12p+8)$,

m+1-1)

(m+1+1)

m(m+2).

$(m+2)[(m+2)^2-1]$

$= (m+2)[m^2+4m+4-1]$

$= (m+2)(m^2+4m+3)$.

$= (m^3+4m^2+3m+2m^2+8m+6)$

$= (m^3+6m^2+11m+6)$.

$\cancel{= m(m^2}$

$= (m^3-m+6m^2+12m+6)$,

$= m(m^2-1) + 6(m+1)^2$

$= 24k + 6(2p)^2$ [$\because$ m+1 is even as m is odd]

$= 24k + 24p^2$

$= 24(k+p^2) = 24k_1 \Rightarrow$ div. by 24.