Set Theory.

Date : 8/2/2022.

A set is a group of homogeneous elements. which are unique.
The elements should be well defined.

A = {Sunday, Monday, Tuesday, wednesday, Thursday, Friday, Saturday} ⟶ days of a week. (it is well defined rule)

Well defined: A set represented with some rules. That rules guide the elements should be present in the set or not. This rule should be well defined.

A set should have same no. of elements universally.

Notation

- When talking about a set we usually denote the set with a capital letter.
- Roster notation is the method of describing a set by listing each element of the set.
- Example: Let set A = The set of odd numbers greater than zero and less than 10.

  The roster notation of A = {1, 3, 5, 7, 9}.

- Example: Let set T = All prime numbers less than 20.
  The roster notation of A = {2, 3, 5, 7, 11, 13, 17, 19}

More on Notation

- Sometimes we can't list all the elements of a set. For instance $\mathbb{Z}$ = The set of integer numbers. We can't write out all the integers, there infinitely many integers. So we adopt a convention using dots....
- The dots mean continue on in this pattern forever and ever

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

• $W = \{0, 1, 2, 3, \ldots\}$: This is the set of whole numbers.

Set - Builder Notation

• When it is not convenient to list all the elements of a set, we use a notation that employs the rules in which an element is a member of the set. This is called set - builder ~~builder~~ notation

• Example $A = \{x : x$ is +ve odd number and $x < 10\}$ (":" → such that)

$A = \{x : x$ is +ve and $x < 100000\}$

• Example $B = \{y : y$ is -ve integer and divisible by $5\}$.

$B = \{\ldots, -20, -15, -10, -5\}$

Special Set of Numbers.

• N = The set of natural numbers. $= \{1, 2, 3, \ldots\}$

• W = The set of whole numbers $= \{0, 1, 2, 3 \ldots\}$

• Z = The set of integers. $= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

• Q = The set of rational numbers [p/q, $q \neq 0$, $p, q \to$ integers]

$= \{x \mid x = p/q$, where p and q are elements of Z and $q \neq 0\}$

• H = The set of irrational numbers.

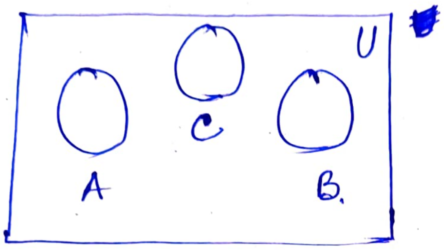
• R = The set of real numbers.

• C = The set of complex numbers.

Date : 9/2/2022

Universal Set (U)

Venn Diagram



Subset : if all the ~~set~~ elements of a set is present in ~~the elem~~ other set then the 1st set is a subset of the other.

A

B.

$B \subseteq A$, $B \subset A$ → proper subset.

atleast one element of B should be less than A

no. of elem of B < no. of elem of A.

$A = \{1, 2, 3, 4, 5\}$

$B = \{1, 4, 5\}$

$B \subset A$

$A \supset B$

A is a superset of B.

Cardinality : The no. of elements of a set is called cardinality of the set.

$A = \{1, 5, 6, 9, 13\}$

$|A| = 5.$

or

$n(A) = 5$

Empty set (Null Set) : It contains no element. it is denoted by ϕ or $\{\}$. It is subset of any set.

$A = \{0\} \rightarrow$ not an empty set.

$n(A) = 1$.

$|\{\ \}| = 0$.

<u>Power Set</u>: ~~The~~ It consists of all the possible subsets of a given set.

$A = \{1, 2, 3\}$

$P(A) =$

$\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

8 subsets.

A, $|(A)| = n$

$|P(A)| = 2^n$.

Singleton set: If a set contains one element.

$A = \{1\}$

$P(A) = \{\phi, \{1\}\}$

$n(P(A)) = 2$.

Doubleton set: If a set contains two elements.

Every set is a proper subset of universal set U

Date : 10/2/2022

S and T such that $S \subset T$ and $T \subset S$ thus

$S = T$ (antisymmetric)

'$\subset$' $\rightarrow$ set of inclusion

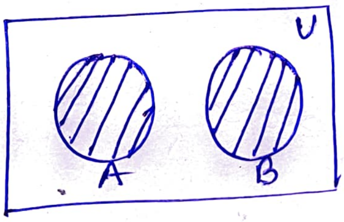
1) $S \subset S$ (reflexive property)

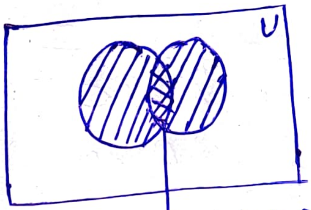
$S \subset T$, $T \subset V \Rightarrow S \subset V$ (transistivity)

Set operations

Union : Let A and B two subset of U (universal set)

$\therefore A \cup B = \{x : x \in A \text{ or } x \in B\}$
$\hookrightarrow$ union.





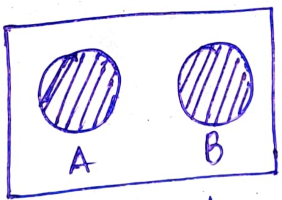
$A \cup B$ $\qquad$ $\hookrightarrow A \cap B$.

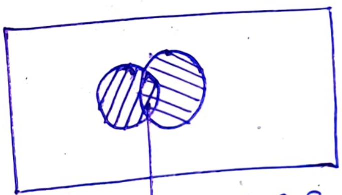
Intersection :

A and B two subsets of universal sets.

$\cap \rightarrow$ symbol of intersection

$A \cap B = \{x : x \in A \text{ and } x \in B\}$





$\hookrightarrow A \cap B =$ common section

$A \cap B = \phi$

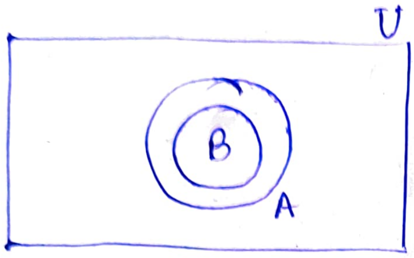
If $A \cap B = \phi$ then A and B are disjoint sets.

eg: $A = \{$set of odd no.s$\}$ $\Rightarrow A \cap B = \phi$
$B = \{$set of even no.s$\}$

Properties of Union and Intersection:

① Consistency property:

$B \subset A$, $A \cup B = A$ and $A \cap B = B$ are mutually equivalents that is one implies other 2.



$\Rightarrow A \cup B = A$

$\Rightarrow A \cap B = B.$

Consistency property   Date: 11/2/2022

$B \subset A$, $A \cup B = A$, $A \cap B = B$.

1.
i) $A \cup \phi = A$, $A \cap \phi = \phi$, $\phi \subset A$
ii) $A \cup U = U$, $A \cap U = A$, $A \subset U$
iii) $A \cup A = A$, $A \cap A = A$ (Idempotent property) $A \subset A$

2. Commutative property

$A \cup B = B \cup A$, $A \cap B = B \cap A$

~~3. A ∪ (B ∪ C) = (A ∪ B) ∪ C~~

3. Associative property

i) $A \cup (B \cup C) = (A \cup B) \cup C$

ii) $A \cap (B \cap C) = (A \cap B) \cap C.$

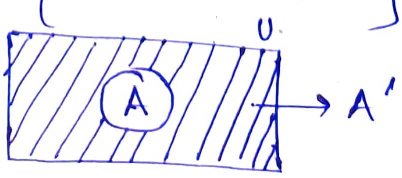
4. Distributive property

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Complementation:

The complementation of a subset A is a subset of U denoted by A' or $A^c = \{x \in U : x \notin A\}$



Properties of complementation :

i) $A \cup A' = U$, $A \cap A' = \phi$

ii) $(A')' = A$

De Morgan's Law :

i) $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

Let, $P = (A \cup B)'$

$Q = A' \cap B'$

Let, x be an element $\in P$.

$x \in P \Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B' \Rightarrow x \in Q$.

$x \in P \Rightarrow x \in Q$

$P \subset Q$. —— (1)

Date: 15/2/2022.

Let, $P = (A \cup B)'$

$Q = A' \cap B'$

$\cancel{x \in P}$

$\cancel{\Rightarrow x \in (A \cup B)'}$

$\cancel{\Rightarrow x \notin A \cup B}$

$\cancel{\Rightarrow x \in A' \text{ and } x \in B'}$

$\cancel{\Rightarrow x \in A'}$

$y \in Q$

$\Rightarrow y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin A \cup B$

$\Rightarrow y \in (A \cup B)'$

$\Rightarrow y \in P$.

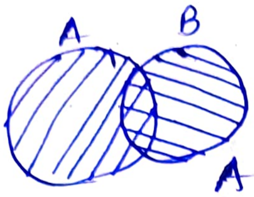
$\therefore Q \subset P$ —(2)

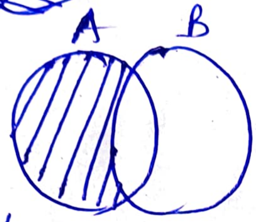
$\therefore P = Q$

Set Difference

The difference of two subsets A and B is a subset of U denoted by A - B and it is defined by ~~A ⊖ B of~~

$A - B = \{ x \in A : x \notin B \}$



A - B = 

$A - B = \phi$

→ A is a subset of B.

$A - B = A$ → A is a disjoint set, or B is a null set.

Properties:

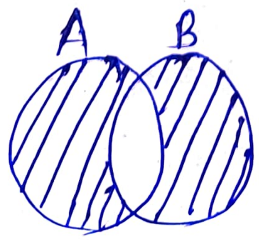
i) $A - (B \cap C) = (A - B) \cup (A - C)$

ii) $A - (B \cup C) = (A - B) \cap (A - C)$

Symmetric Difference

The symmetric difference of two subsets A and B is a subset of U denoted by $A \Delta B = (A - B) \cup (B - A)$

$A \Delta B$ is a set of all those elements which belong either to A or to B but not to both sets.



$A \Delta B$.

Tutorial

Date : 16/2/2022

Discrete mathematics is the study of discrete objects where mathematics theorem will be applied by applying computer science algorithms to solve any problem.

$A = \{1, 3, 7, 9\}$. ✓ → set of odd no.

$A = \{1, 2, 2, 3, 4\}$ ✗ → ambiguous elements

$A = \{1, 2, a, b\}$. ✓ → set of digits and alphabets.

$A = \{1, 2, 3, \{\}\}$. ✓

Equivalence Relation :

$R = \{1, 2, 3\}$

R → reflexive, S → symmetric, T → transitive

R S T → duplicate of each other

$R = \{(1,1), (2,2), (3,3)\}$

$(1,2)$ $(2,3)$ $(1,3)$. → transitive

a : b b : c a : c

$R = \{(1,1), (2,2), (3,3), (2,1), (1,2)\}$.

Binary relation → subset of cartesian product of a given set.

$\rho \subset S \times S$.

If A is a set & R_1, R_2 be 2 equivalence relations on A.

$\therefore R_1 \cup R_2$ may not be equivalence relation.

As it cannot be Transitive.

But $R_1 \cap R_2$ has to be equivalent.

~~R S T~~ R_1 R_2

	Union	Intersection
R	✓	✓
S	✓	✓
T	✗ (may not)	✓

Date: 18/2/2022

Cartesian product of two sets

Let, $A = \{1, 2\}$, $B = \{3, 4, 5\}$

$A \times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

$B \times A = \{(3,1), (3,2), (4,1), (4,2), (5,1), (5,2)\}$

$\therefore A \times B \neq B \times A$

If $A = \phi$ or $B = \phi$ or A and $B = \phi$, then

$A \times B = B \times A = \phi$

Let, $A = \{a, b, c\}$, $B = \{d, e\}$, $C = \{a, d\}$

i) $A \times B = \{(a,d), (a,e), (b,d), (b,e), (c,d), (c,e)\}$

ii) $B \times A = \{(d,a), (d,b), (d,c), (e,a), (e,b), (e,c)\}$

iii) $A \times (B \cup C) = \{a,b,c\} \times \{d,e,a\}$

$= \{(a,d), (a,e), (a,a), (b,d), (b,e), (b,a), (c,d), (c,e), (c,a)\}$

iv) $A \cap B = \phi$, $(A \cap B) \times C = \phi \times \{c\} = \phi$

~~v) $(A \cap B) \times C =$~~

v) $(A \cap C) \times B = \{a\} \times \{d,e\} = \{(a,d), (a,e)\}$

vi) $A \times (B - C) = A \times \{e\} = \{a,b,c\} \times \{e\}$

$= \{(a,e), (b,e), (c,e)\}$

Relations: —

Let, $A = \{$Mohan, Sohan, David, Karim$\}$

$B = \{$Rita, Mary, Fatima$\}$

Rita $\xrightarrow{\text{brother}}$ Mohan. Mary $\xrightarrow{\text{brother}}$ David

Rita $\xrightarrow{\text{brother}}$ Sohan Fatima $\xrightarrow{\text{brother}}$ Karim.

R is a relation "is a brother of"

Mohan R Rita

Sohan R Rita

R = { (Mohan, Rita), ... (Karim, Fatima)}

i) If R is a ϕ, then R is called void relation

ii) If $R = A \times B$, then it is called universal relation.

iii) If R is a relation defined from $A \to A$ set, it is called a relation defined on A.

iv) If R is a relation, $R = \{ (a,a) \ \forall a \in A \}$ is called identity relation.

A is called domain in $R : A \to A$ and B is called range.

- Let S be a non empty set. A binary relation (ρ) on set S is a subset of the cartesian product $S \times S$.

If (a,b) be an element at $S \times S$ and

$$(a,b) \in \rho \Rightarrow a \rho b.$$

Date : 22/2/2022

Equivalence Relation:

Let S be a non empty set and ρ be a binary relation on S. The relation ρ is said to be reflexive if $(a,a) \in \rho$

i.e, $a \rho a$ hold $\forall$ $a \in S$.
for all.

The relation ρ is said to be symmetric if for any two elements

$(a,b) \in \rho \Rightarrow a \rho b$ holds.

$\Rightarrow (b,a) \in \rho$ where $(a,b) \in S$.

$\therefore a \rho b \Rightarrow b \rho a$

The relation is said to <u>transitive</u> if for any 3 elements $(a, b, c) \in S$ such that if $(a,b) \in \rho$ and $(b,c) \in \rho$

$\Rightarrow (a,c) \in \rho$.

The relation ρ on set S is said to be an equivalence relation on set S or an RST relation on S if ρ is reflexive, symmetric as well as transitive.

Q. Let the relation ρ is defined on the set $\mathbb{Z}$ by "$a \rho b$ iff $(a-b)$ divisible by 5". check the given relation is equivalence or not.

~~Reflexive:~~

~~Let $(a,a) \in Z$ such that~~

~~$(a,a) \in \rho$~~

~~$(a-a) \in a \rho a$ holds & which is divisible by 5.~~

~~$\Rightarrow a \rho a$ holds~~

~~Symmetric:~~

~~Let $(a,b) \in Z$ such that $a \rho b$ hold.~~

~~$a \rho b$ holds $\Rightarrow a-b$ is divisible by 5.~~

~~$\Rightarrow b-a$ is divisib~~

Soln:

Let $a \in \mathbb{Z}$ then

$(a-a)$ is divisible by 5

$\therefore a \rho a$ holds $\forall$ a in $\mathbb{Z}$.

So ρ is reflexive relation.

Let $(a,b) \in \mathbb{Z}$ and $a \rho b$ hold then

$(a-b)$ is divisible by 5.

$\therefore (b-a)$ is divisible by 5.

$\Rightarrow b \rho a$ holds. So ρ is symmetric

Let $(a,b,c) \in Z$ and $a\rho b$ and $b\rho c$ both hold.

then $(a-b)$ and $(b-c)$ both are divisible by 5

$\therefore (a-c)$ is divisible by 5. $(a-c)=(a-b)+(b-c)$

$\Rightarrow a\rho c$ hold.

thus $a\rho b$ and $b\rho c$

$\Rightarrow a\rho c$ holds.

Since ρ is RST, so ρ is an equivalence relation on Z.

2. A relation ρ on set N is given by $\rho = \{(a,b) \in N \times N :$ a is a divisor of b$\}$. Examine whether it is equivalence relation.

Let $a \in N$ then

a is divisible by a.

$\therefore a\rho a$ holds $\forall$ a in N.

So ρ is reflexive.

Let $(a,b) \in Z$ and $a\rho b$ hold then.

b is divisible by a.

$\Rightarrow$ a is not divisible by b.

$\Rightarrow b\rho a$ does not hold.

$\therefore$ So ρ is not symmetric.

Let $(a,b,c) \in$ ~~Z~~ N and $a\rho b$ and $b\rho c$ both holds

then a divides b and b divides c.

$$\frac{c}{a} = \frac{b}{a} \times \frac{c}{b}.$$

~~then~~ ~~too~~ thus $a\rho b$ and $b\rho c$.

$\Rightarrow a\rho c$ holds

$\therefore \rho$ is transitive.

$A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$R_2 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

$R_1 \cup R_2 \rightarrow (1,3)$ ✗ not there

may not be equivalent

$R_1 \cap R_2 = \{(1,1), (2,2), (3,3)\}$ → Reflexive → Symmetric } ✓ equivalence.

Partial Order Relation: Date: 23/2/2022

Let S be a non empty set. A relation ρ on set S is said to be antisymmetric if $a \rho b$ and $b \rho a$ hold. It implies $a = b$ for $(a, b) \in S$.

The relation ρ defined on R by "$x \rho y$" iff $x \leq y$. for $(x, y) \in R$.

Let $x_1 \rho y_1 \Rightarrow x_1 \leq y_1$.

$y_1 \rho x_1 \Rightarrow y_1 \leq x_1$

$\Rightarrow x_1 = y_1$ [this is the definition of antisymmetric relationship]

Let X be a non empty set. The relation ρ defined on $P(X)$ by "$A \rho B$" iff A is a subset of B" for $(A, B) \in P(X)$

$A \rho B$, $A \subset B$

$B \rho A$, $B \subset A$

then $A = B$.

Let S be a non empty set. The relation ρ on S is said to be a partial order relation if ρ is reflexive, antisymmetric and transitive. The relation of partial order is often denoted by ($\leq$) → notion of partial order relation.

Eg: $(R, \leq)$, it is partial order relation on set R.

<u>Poset</u>: A non empty set S together with the relation of partial order on S is called poset. (Partially ordered set). It is denoted by $(S, \leq)$.

$x \leq y$ means.

"x is less than or equal y"

for $x, y \in R$

Date : 1/3/2022

Given $(S, \leq)$ poset. Relation $\succcurlyeq$ defined by "$a \succcurlyeq b$ iff $b \leq a$", for $a, b \in S$. Show $(S, \succcurlyeq)$ is a poset.

Let, $a \leq a \ \forall a \in S$
$\Rightarrow$ reflexive
$\Rightarrow a \succcurlyeq a$ for $a \in S$.
$\Rightarrow \succcurlyeq$ is reflexive.

Let, $a \succcurlyeq b$ and $b \succcurlyeq a$ hold for $a, b \in S$.

$a \succcurlyeq b \Rightarrow b \leq a$
$b \succcurlyeq a \Rightarrow a \leq b$
$b \succcurlyeq a$ and $a \leq b$,
$a = b$.

$\Rightarrow \succcurlyeq$ is antisymmetric.

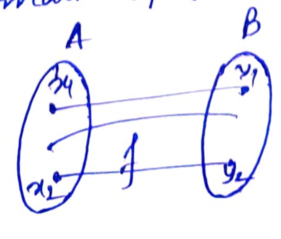
Let, $a \succcurlyeq b$ and $b \succcurlyeq c$ for $a, b, c \in S$.

$a \succcurlyeq b \Rightarrow b \leq a$
$b \succcurlyeq c \Rightarrow c \leq b$.
$\Rightarrow c \leq b$ and $b \leq a$.
$\Rightarrow c \leq a$.
$\Rightarrow a \succcurlyeq c$, $\succcurlyeq$ is transitive.

Mapping :

Let A, B two non empty sets. A mapping f from A to B set is a rule that assigns to each element x of set A. A definite element y in set B, ~~AtoB~~

A $\rightarrow$ domain of f.
B $\rightarrow$ codomain of f.



Let $f : A \rightarrow B$ ~~where x∈A~~ and $x \in A$. Then the unique element of $y \in B$, that corresponds to x by the mapping f is called the f-image of x ($f(x)$)

If $f(x) = y$ we say that f maps x to y. The set of all f-images i.e, $\{f(x) : x \in A\}$ is denoted by $f(A)$ and is said to be the image set of f: mapping ~~to~~ for the range set of f. $D(f) \rightarrow$ domain

$R(f) \rightarrow$ range.

Let, $S = \{1, 2, 3, 4\}$, $T = \{a, b, c, d\}$

i) $f1 = \{(1,a), (1,b), (2,c), (3,c), (4,d)\}$

ii) $f2 = \{(1,a), (2,b), (3,c)\}$

iii) $f3 = \{(1,b), (2,b), (3,c), (4,d)\}$.

iv) $f4 = \{(1,b), (2,c), (3,d), (4,a)\}$

i) As it is not definite, $f1$ is not mapping.

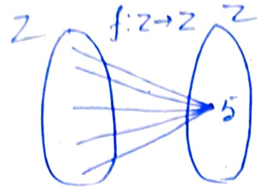
ii) Not Mapping.

iii) Mapping

iv) Mapping

Constant Mapping Date : 8/3/2022

A mapping $f: A \to B$ is said to be a constant mapping if f maps each element of A to one and the same element of B set.



$f(Z) = 5$

co-domain is a singleton set.

A mapping $f: A \to A$ where A is a non-empty set, is a said to be an identity mapping on A if $f(x) = x, x \in A$

Equal mapping

Two mappings $f: A \to B$ and $g: A \to C$ is said to be equal if $f(x) = g(x) \ \forall \ x \in A$.

Let, $S = \{x \in R : x > 0\}$

$f: S \to R$ be defined $f(x) = \frac{|x|}{x}$, $x \in S$ and

$g: S \to R$ be defined $g(x) = 1$, $x \in S$.

Binary Composition (BC)

Let A be a non empty set. A binary composition on A is a mapping $f: A \times A \to A$. This mapping f is generally denoted by $\circ$. For a pair of elements (a, b), in set A, the image of pair (a, b) under the binary composition $\circ$ is denoted by $a \circ b$. Eg: $(b, a) \longrightarrow b \circ a$

On the set Z, let $\circ$ stand for the binary composition addition the $2 \circ 3 = 5$, $6 \circ 4 = 10$

If $\circ$ indicates multiplication then $2 \circ 3 = 6$, $3 \circ 0 = 0$.

If $\circ$ indicates subtraction then $2 \circ 1 = 1$, $1 \circ 2 = -1$.

'Subtraction' is not a binary composition on the set $\mathbb{N}$

Eg: $5 \circ 3 = 2 \in N$

$3 \circ 5 = -2 \notin N$

A binary composition $\circ$ is said to be defined on a non empty set A if $a \circ b \in A$ for all $a, b \in A$.

In this case the set A is said to be closed under the binary composition $\circ$.

Let $\circ$ be a binary ~~combination~~ composition on a nonempty set A i.e, $\circ$ is said to be commutative if $a \circ b = b \circ a$, $a, b \in A$.

It is said to be associative if $a \circ (b \circ c) = (a \circ b) \circ c$, $a, b \in A$.

Date : 9/3/2022

Groupoid

Let G be a non empty set on which a binary composition $\circ$ is defined. Some algebraic structure is imposed on G by the composition $\circ$ and $(G, \circ)$ becomes an algebraic system. This algebraic system $(G, \circ)$ is said to be a groupoid.

$(\mathbb{Z}, +)$ make a groupoid.

A groupoid $(G, \circ)$ is said to be a commutative groupoid if the binary composition is commutative.

An element e in G is said to be an identity element in the groupoid $(G, \circ)$ if $a \circ e = e \circ a = a$, where $a \in G$.

eg: $0 + a = a + 0 = a$.

An element e in G is said to be a right identity element in the groupoid $(G, \circ)$ if $a \circ e = a \ \forall a$ in G.

Left identity element $\Rightarrow e \circ a = a$.

Theorem:

If a groupoid $(G, \circ)$ contains an identity element then that element is unique.

Proof:

If possible let there be two identity element e and f in $(G, \circ)$. Then $a \circ e = e \circ a = a$ and $f \circ a = a \circ f = a \ \forall a \in G$.

Now, $e \circ f = e$ by property of f [f is identity element]

$e \circ f = f$ by property of e $\therefore e = f$

Theorem:

If a groupoid (G, o) contains a left identity as well as right identity element then they are equal and the equal element is the identity element of the groupoid.

Tutorial

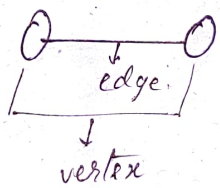
GRAPH THEORY

$G(V, E)$ $V \to$ set of vertex $E \to$ set of edges

$|V| \to$ order of a graph (no. of vertices) $|E| \to$ size of a graph. (no. of edges)

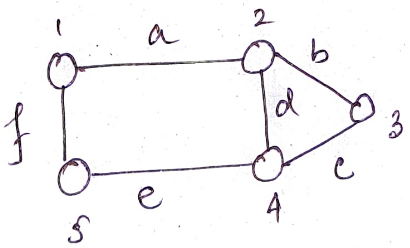
1. **Adjacent vertex:**

There should be a common edge between two vertices



2. **Adjacent edge:**

There should be a common vertex between two edges.



a and b are adjacent to each other

Similarly, f and a are adjacent edges.

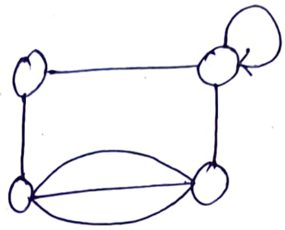
3. **Self loop** :- (SL)  [exception of adjacent edge].

4. **Multiedges** (11) :- 

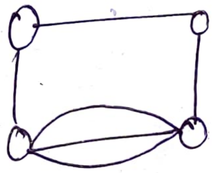
When we have multiple edges joining two vertices.

Classification of graphs based on multiedges :

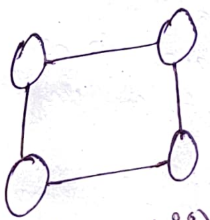
(i) Pseudograph : It contains self loop as well as multiedges.
(SL, ||)



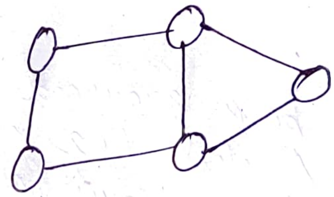
ii) Multigraph : It ~~contal~~ contains multiedges (||)



iii) Simple graph : It contains neither self loops nor multiedges.



OR

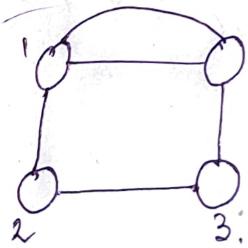


Two types : (i) Regular (ii) complete

Theorems :

Handshaking lemma :—

The sum of the degree of all the vertices should be equal to twice of the edges.



degree of ~~edges~~ vertex.

$$d(V) = 2 * |E|$$

$$d(V) = \{3, 3, 2, 2\} = 10$$

No. of edges = 2'5

$$\therefore d(V) = 2 \times 5 = 10.$$

Degree : The no. of edges associated of a vertex is called degree of the vertex.

Proof :

Let e be the left identity and f be the right identity in the given groupoid. Then $e \circ a = a$ by the property of e.

Also, $a \circ f = a$ by the property of f.

Now $e \circ f = f$ by the property of e.

$e \circ f = e$ by the property of f.

$\Rightarrow e = f$.

This proves that e is an identity element in the groupoid. By the previous theorem e is the unique identity element in the groupoid.

Inverse

Let $(G, \circ)$ be a groupoid containing the identity element e. An element a in G is said to be invertible if there exist an element a' in G $(a' \in G)$ such that $a' \circ a = a \circ a' = e$.

a' is said to be an inverse of a in the groupoid.

An element a in G is said to be left invertible if there exist an element b in G such that $b \circ a = e$.

Right invertible $\longrightarrow$ $a \circ c = e$.

eg: -1.

$(-1) * (-1) = 1$.

$\therefore -1$ is a right & left invertible

$(Q, *) \rightarrow$ except 0 all elements in Q have invertible elements in Q.

If e be a just a left identity element in the groupoid $(G, \circ)$ then an element a in G set is said to be left e-invertible, If there exist an element b in G such that $b \circ a = e$. and a is said to be right e-invertible if there exist an element c in G such that

$a \circ c = e$. Here b is said to be a left e-inverse of a and c is said to be a right e-inverse of a.

$b \circ a = e$ [a is left e-invertible element]

$a \circ c = e$ [a is right e-invertible element]

Date : 14/3/2022

1. $(\mathbb{Z}, *)$ where $*$ is defined by $a * b = a + 2b$, $a, b \in \mathbb{Z}$.

→ Only right identity element is present and ie, 0(zero).

$a * 0 = a + 2 * 0 = a$.

$a * e = a$.

$0 * b = 0 + 2b = 2b \neq b$.

$e * b = b$. (definition of left identity element)

2.(i) $5 \in \mathbb{Z}$. 5 is a left 0 invertible element?

Is 5 is a right 0 invertible element?

(ii) $6 \in \mathbb{Z}$. 6 is left 0 invertible element?

6 is right 0 invertible element?

$-\frac{5}{2}$

(i) $x * 5 = 0$

$\underline{-10} + 2 \times 5 = 0$.

$-10 \in \mathbb{Z}$.

∴ 5 is left 0 invertible element

$5 * x = 0$.

$5 + 2.b = 0$.

$5 + 2(-\frac{5}{2}) = 0$.

$\therefore -\frac{5}{2} \notin \mathbb{Z}$.

∴ 5 is not right 0 invertible element

(ii) $x * 6 = 0$.

$-12 + 2 \times 6 = 0$.

$-12 \in \mathbb{Z}$.

∴ 6 is left 0 invertible element

~~$x * 6 = 0$.~~

$6 * x = 0$

$6 + 2x = 0$

$6 + 2(-3) = 0$.

$-3 \in \mathbb{Z}$.

∴ 6 is right invertible element.

Semigroup:

A groupoid $(G, \circ)$ is said to be a semigroup if the binary composition $\circ$ is associative in nature.

eg: $(Z, +)$, $(Z, *)$.

Let $(G, \circ)$ is a semigroup and $a \in G$ then $a \circ a \in G$.

$a \circ (a \circ a) = (a \circ a) \circ a$, as $\circ$ is associative.

Dropping the parenthesis, each of them is written $a \circ a \circ a$

Thus $a \circ a \circ a \in G$, $a \circ a \circ a \circ a \circ \in G$, ...

The +ve integral powers of $a \in G$ are defined as follows:

$$a^1 = a,\ a^2 = a \circ a,\ a^3 = a \circ a \circ a, \ldots$$

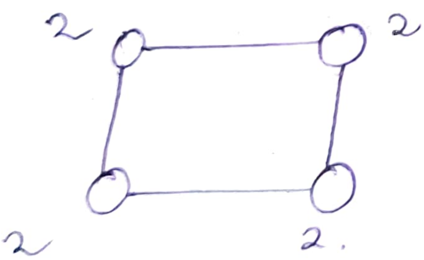
$$a^{n+1} = a^n \circ a \ \forall n,\ n \in N$$

TUTORIAL

Date: 15/3/2022

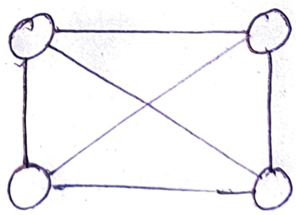
Regular Graph

A regular graph is a simple graph in which every vertex has same degree



Complete Graph

i) A simple graph is a complete graph in which every pair of vertices is adjacent.



K_4.

A complete graph contains maximum number of edges.
It is represented by K_n where $n \rightarrow$ no. of vertices.

$K_1 \longrightarrow 0$ [∵ self loop is not a simple graph]

$K_2 = 1$

$K_3 = 2$.

ii) The degree of every vertex in a complete graph is $n-1$, The no. of edges is $n(n-1)/2 = {}^nC_2$.

If one extra edge would be added, the graph will be no more a simple graph.

How many maximum simple graphs can be formed from a complete graph? $\longrightarrow$ $2^{E} = 2^{\frac{n(n-1)}{2}}$

~~Every regular graph~~

Every complete graph is regular but the reverse may or maynot be true.

Date: 16,

Theorem:

In a monoid $(G, \circ)$ if an element a be left invertible as well as right invertible, then a is invertible and a has the unique inverse in the monoid.

[Semigroup with identity element → Monoid]

Proof:

Let e be the identity element and b be the left inverse and c be a right inverse of a.

$$b \circ a = e \qquad a \circ c = e.$$

$$b \circ (a \circ c) = (b \circ a) \circ c.$$

Now,

$$b \circ (a \circ c) = b \circ e = b$$

$$(b \circ a) \circ c = e \circ c = c.$$

$$\therefore \; b = c.$$

Group

A non empty set G is said to form a group with respect to a binary composition $\circ$ if

(i) G is closed under binary composition $\circ$.

(ii) $\circ$ is associative.

(iii) There exist an element $e \in G$ such that $a \circ e = e \circ a = a \; \forall a \in G$

(iv) For each element $a \in G$ there exist an element a' in G such that $a' \circ a = a \circ a' = e$.

Theorem:

A group $(G, \circ)$ contains only one identity element.

Proof

Let e, f be two identity elements in group.

Then $e \circ a = a \circ e = a$ and $f \circ a = a \circ f = a$ for all a in G.

Now $e \circ f = f$, by property of e

$e \circ f = e$, by property of f.

$$\therefore e = f$$