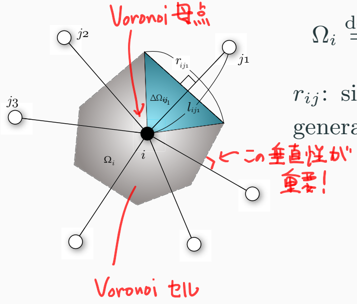
Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 3)



A Voronoi region Ω_i by a Voronoi generator $\boldsymbol{x}_i$ is

$$\Omega_i \overset{\text{def}}{=} \{\boldsymbol{x} |\ \|\boldsymbol{x} - \boldsymbol{x}_i\| \leq \|\boldsymbol{x} - \boldsymbol{x}_j\| \text{ for any } j \neq i\}$$

r_{ij}: size of boundary facet, l_{ij}: distance between generators,

境界facetのサイズ

$$\begin{cases} r_{ij} & \overset{\text{def}}{=} & \|\Omega_i \cap \Omega_j\|, \\ l_{ij} & \overset{\text{def}}{=} & \|\boldsymbol{x}_i - \boldsymbol{x}_j\|. \end{cases}$$

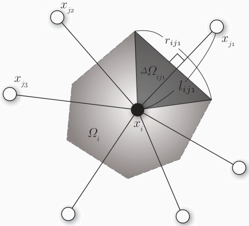
セルの代表点間の距離
(Voronoi generator)

Some properties of Voronoi decomposition:

- The line between generators are orthogonal to boudary, so $(\boldsymbol{x}_j - \boldsymbol{x}_i)/l_{ij} = \boldsymbol{n}_{ij}$ is the outward normal unit unit on the boundary.
- Cost of computation of a 2D region is $\mathrm{O}(n \log n)$ (n: # of generators).
- Mean of # of vertices of Voronoi mesh of a 2D region is 6.

Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 4)

• 通常のメッシュとして Voronoi mesh を用いる



共変

Covariant mesh

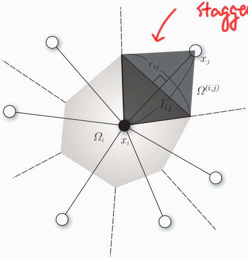
is defined by the Voronoi mesh Ω_i themselves to compute covariant derivatives.

Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 5)

「境界」を意味する領域

・staggered mesh も用意する。

このひし形が staggered mesh



反変

Contravariant mesh

is defined by the region composed by two pyramids those bases are boudary facet of Voronoi meshes to compute contravariant derivatives. (Or, they are a kind of staggered meshes)

共変微分

Covariant derivatives grad_d , div_d, rot_d, Δ_d

これは反変

$$(\mathrm{grad}_\mathrm{d}\ \phi)_i \stackrel{\mathrm{def}}{=} \frac{1}{|\Omega_i|}\sum_{j\in N_i} \phi^{(i,j)} \boldsymbol{n}_{ij} r_{ij},$$

$$(\mathrm{div}_\mathrm{d}\boldsymbol{v})_i \stackrel{\mathrm{def}}{=} \frac{1}{|\Omega_i|}\sum_{j\in N_i} \boldsymbol{n}_{ij}\cdot\boldsymbol{v}^{(i,j)}\, r_{ij},$$

$$(\mathrm{rot}_\mathrm{d}\boldsymbol{v})_i \stackrel{\mathrm{def}}{=} \frac{1}{|\Omega_i|}\sum_{j\in N_i} \boldsymbol{n}_{ij}\times\boldsymbol{v}^{(i,j)}\, r_{ij},$$

$$\text{(gradient of vector field)}\ \left(\boldsymbol{v}\otimes\overleftarrow{\nabla}_\mathrm{d}\right)_i \stackrel{\mathrm{def}}{=} \frac{1}{|\Omega_i|}\sum_{j\in N_i} \boldsymbol{v}^{(i,j)}\otimes\boldsymbol{n}_{ij}\, r_{ij},$$

$$(\Delta_\mathrm{d}\phi)_i \stackrel{\mathrm{def}}{=} \frac{1}{|\Omega_i|}\sum_{j\in N_i}\left(\frac{\phi_j-\phi_i}{l_{ij}}\right) r_{ij},$$

Laplacianは
共変要素を用いて
共変Laplacianをdef.

where the interpolation on the boundary defined by $f^{(i,j)} \stackrel{\mathrm{def}}{=} (f_i+f_j)/2$ for Voronoi mesh.

Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 7)

$$
\begin{aligned}
|\Omega_i| &\overset{\text{def}}{=} \text{volume of Voroni region } \Omega_i, \\
N_i &\overset{\text{def}}{=} \{\, j \mid \boldsymbol{x}_j \text{ is an adjecent generator of } \boldsymbol{x}_i \}, \\
r_{ij} &\overset{\text{def}}{=} \text{size of the boundary facet } \partial\Omega_i \text{ between } \boldsymbol{x}_i \text{ and } \boldsymbol{x}_j, \\
l_{ij} &\overset{\text{def}}{=} \text{distance between Voronoi generator } \boldsymbol{x}_i \text{ and } \boldsymbol{x}_j, \\
\boldsymbol{n}_{ij} &\overset{\text{def}}{=} \text{outward normal unit vector from Voronoi generator } \boldsymbol{x}_i \text{ to } \boldsymbol{x}_j
\end{aligned}
$$

← x_iの近傍点の(添字)集合

Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 8)

反変微分

Contravariant derivatives $\mathrm{grad_d}$, $\mathrm{div_d}$, $\mathrm{rot_d}$, Δ_d

これは共変

$$(\mathrm{grad_d}\ \phi)^{(i,j)} \stackrel{\text{def}}{=} \left(\frac{\phi_j - \phi_i}{l_{ij}}\right)\boldsymbol{n}_{ij},$$

$$(\mathrm{div_d}\boldsymbol{v})^{(i,j)} \stackrel{\text{def}}{=} \left(\frac{\boldsymbol{v}_j - \boldsymbol{v}_i}{l_{ij}}\right)\cdot \boldsymbol{n}_{(i,j)},$$

$$(\mathrm{rot_d}\boldsymbol{v})^{(i,j)} \stackrel{\text{def}}{=} \boldsymbol{n}_{ij}\times\left(\frac{\boldsymbol{v}_j - \boldsymbol{v}_i}{l_{ij}}\right),$$

$$\text{(gradient of vector field)}\quad \left(\boldsymbol{v}\otimes\overleftarrow{\nabla}_\mathrm{d}\right)^{(i,j)} \stackrel{\text{def}}{=} \left(\frac{\boldsymbol{v}_j - \boldsymbol{v}_i}{l_{ij}}\right)\otimes\boldsymbol{n}_{ij}$$

for piecewise constant functions on covariant meshes.

やはり反変 $\phi^{(i,j)}$ から反変 Laplacian を def.

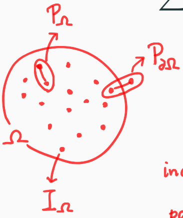
$$(\Delta_\mathrm{d}\phi)^{(i,j)} \stackrel{\text{def}}{=} \left(\mathrm{div_d}((\mathrm{grad_d}\ \phi)_i)\right)^{(i,j)}$$

for piecewise constant functions on contravariant meshes.

Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 9)

Definitions

$\int_\Omega (\underbrace{f_x g}_{a} + \underbrace{f g_x}_{b})\, dx$ の離散版



$$\sum (a,\ b)_\Omega \overset{\text{def}}{=} \overbrace{\sum_{i\in I_\Omega} a_i|\Omega_i|}^{\text{共変積分}} + \overbrace{\sum_{(i,j)\in\overline{P_\Omega}} b^{(i,j)} n \left|\Omega^{(i,j)}\right|}^{\text{反変積分}},$$

$$\sum_{(i,j)\in\overline{P_\Omega}} \overset{\text{def}}{=} \left\{ \sum_{(i,j)\in P_\Omega} + \frac{1}{2} \sum_{(i,j)\in P_{\partial\Omega}} \right\},$$

台形則に相当する

indicies $I_\Omega \overset{\text{def}}{=} \{\, i \mid \boldsymbol{x}_i \in \Omega \},$

pairs $P_\Omega \overset{\text{def}}{=} \{\, (i,j) \mid i,j \in I_\Omega, j \in N_i, i < j \},$

pairs $P_{\partial\Omega} \overset{\text{def}}{=} \{\, (i,j) \mid i \in I_\Omega, j \notin I_\Omega, j \in N_i \},$

Note: Above "n" is # of dimension; i.e., $\Omega \subset \mathbb{R}^n$.

Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 10)

• 先の6つの部分積分の離散版.

Discretized six equations (1/2)

厳密に成り立つことに注意.

$$
\begin{aligned}
\sum \left(f \cdot \mathrm{grad_d}\ g,\ \mathrm{grad_d}\ f \cdot g\right)_\Omega &= \sum_{(i,j)\in P_{\partial\Omega}} \left(\frac{f_i + f_j}{2}\right) g^{(i,j)} \boldsymbol{n}_{ij} r_{ij}, \\
\sum \left(f \cdot \mathrm{div_d} \boldsymbol{v},\ \mathrm{grad_d}\ f \cdot \boldsymbol{v}\right)_\Omega &= \sum_{(i,j)\in P_{\partial\Omega}} \left(\frac{f_i + f_j}{2}\right) \boldsymbol{v}^{(i,j)} \cdot \boldsymbol{n}_{ij} r_{ij}, \\
\sum \left(f \cdot \mathrm{rot_d} \boldsymbol{v},\ \mathrm{grad_d}\ f \times \boldsymbol{v}\right)_\Omega &= \sum_{(i,j)\in P_{\partial\Omega}} \left(\frac{f_i + f_j}{2}\right) \boldsymbol{v}^{(i,j)} \times \boldsymbol{n}_{ij} r_{ij},
\end{aligned}
$$

右辺は boundary term

Discrete integration by parts (multi dimension, Voronoi/Delaunay mesh) # 11)

・先の連続版部分積分ときれいに対応することに注意.

Discretized six equations (2/2)

$$
\begin{aligned}
\sum \left(\boldsymbol{u} \cdot \mathrm{div}_{\mathrm{d}} \boldsymbol{v},\ \left(\boldsymbol{u} \otimes \overleftarrow{\nabla} \right) \boldsymbol{v} \right)_{\Omega} &= \sum_{(i,j) \in P_{\partial\Omega}} \left(\frac{\boldsymbol{u}_i + \boldsymbol{u}_j}{2} \right) \left(\boldsymbol{n}_{ij} \cdot \boldsymbol{v}^{(i,j)} \right) r_{ij}, \\
\sum \left(\boldsymbol{u} \cdot \mathrm{rot}_{\mathrm{d}} \boldsymbol{v},\ -\mathrm{rot}_{\mathrm{d}} \boldsymbol{u} \cdot \boldsymbol{v} \right)_{\Omega} &= - \sum_{(i,j) \in P_{\partial\Omega}} \left(\left(\frac{\boldsymbol{u}_i + \boldsymbol{u}_j}{2} \right) \times \boldsymbol{v}^{(i,j)} \right) \cdot \boldsymbol{n}_{ij} r_{ij}, \\
\sum \left(\boldsymbol{u} \times \mathrm{rot}_{\mathrm{d}} \boldsymbol{v},\ -\mathrm{rot}_{\mathrm{d}} \boldsymbol{u} \times \boldsymbol{v} \right)_{\Omega} &= \sum \left(\boldsymbol{u} \cdot \mathrm{div}_{\mathrm{d}} \boldsymbol{v},\ \mathrm{div}_{\mathrm{d}} \boldsymbol{u} \cdot \boldsymbol{v} \right)_{\Omega} \\
&\quad + \sum_{(i,j) \in P_{\partial\Omega}} \left(\left(\frac{\boldsymbol{u}_i + \boldsymbol{u}_j}{2} \right) \cdot \boldsymbol{n}_{ij} \right) \boldsymbol{v}^{(i,j)} r_{ij} \\
&\quad + \sum_{(i,j) \in P_{\partial\Omega}} \left(\boldsymbol{n}_{ij} \times \left(\frac{\boldsymbol{u}_i + \boldsymbol{u}_j}{2} \right) \right) \times \boldsymbol{v}^{(i,j)} r_{ij}
\end{aligned}
$$