(ADAPTIVE) GROVER FIXED POINT SEARCH FOR QUBO

ABSTRACT. *to be completed later...*

1. INTRODUCTION

to be completed later...

Organization of the paper: In Section 2, ...

2. GROVER FIXED POINTS SEARCH

Input: A symmetric, integer-valued, n-by-n matrix, Q and a constant $c \in \mathbb{Z}$, or, equivalently, a quadratic function on $x \in \{0,1\}^n$ given by

$$f(x) := x^T Q x + c. \tag{2.1}$$

(Note that since $x_i^2 = x_i$, we can move linear terms into the diagonal of Q.)

Output: An estimate for the value

$$M := \max\big(\{\, f(x) \mid x \in \{0,1\}^n \,\}\big).$$

Example 2.1 (Maximal Graph Cuts)**.** *Given a simple, undirected graph, $G = (V,E)$, let Q be its graph Laplacian, defined as*

$$Q_{i,j} = \begin{cases} \deg(v_i), & \textit{if } i = j, \\ -1, & \textit{if } \{v_i, v_j\} \in E, \\ 0, & \textit{otherwise,} \end{cases}$$

$b = 0$ and $c = 0$. Then $V = V^+ \coprod V^-$ is a maximal exactly when $\mathrm{MaxCut}(G) = f(x) = M$*, where $x \in \{0,1\}^n$ is defined as $x_i = 1$ if $v_i \in V^+$ and zero otherwise.*

The Edwards–Erdős bound yields

$$\mathrm{MaxCut}(G) \geqslant B_G := \begin{cases} \frac{2|V|+|E|-1}{4}, & \textit{if (we know that) } G \textit{ is connected,} \\ \frac{|V|}{2} + \sqrt{\frac{|V|}{8} + \frac{1}{64}} - \frac{1}{8}, & \textit{otherwise.} \end{cases}$$

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3. THE ORACLES:

An element $x = (x_1, x_2, \dots, x_n) \in \{0,1\}^n$ is also regarded as a binary number via $x \sim \overline{x_1 x_2 \dots x_n} := \sum_i x_i 2^{n-i}$ and as an element of the computational basis via

$$|x\rangle_n := |x_1\rangle \dots |x_{n-1}\rangle |x_n\rangle,$$

Given a function as in equation (2.1), let us pick $m \gg \log_2(M)$ (in fact, $m = \lceil \log_2(\mathrm{tr}(Q)) \rceil + 1$ works for our purposes). We use the binary 2s complement convention when digitizing integers and we with that in mind, we construct a oracle on $(n+m)$-qubits, U_f, so that

$$U_f |x\rangle_n \big|y\big\rangle_m = |x\rangle_n \big|y - f(x)\big\rangle_m.$$

Note that the $(n+1)^{\text{th}}$ register of $U_f |x\rangle_n \big|y\big\rangle_m$ is $|1\rangle$ exactly when $y < f(x)$.

3.1. **Construction of U_f:**

Let $\mathscr{P}(\theta)$ be the following m-qubit gate

$$\big|y_1\big\rangle \;—\; \boxed{P\big(2^{m-1}\theta\big)} \;—\; e^{i\theta y_1 2^{m-1}} \big|y_1\big\rangle$$

$$\vdots$$

$$\big|y_j\big\rangle \;—\; \boxed{P\big(2^{m-j}\theta\big)} \;—\; e^{i\theta y_j 2^{m-j}} \big|y_j\big\rangle$$

$$\vdots$$

$$\big|y_m\big\rangle \;—\; \boxed{P(\theta)} \;—\; e^{i\theta y_m} \big|y_m\big\rangle$$

Thus $\mathscr{P}(\theta)\big|y\big\rangle_m = e^{i\theta y}\big|y\big\rangle_m$. Note that

$$\big|y\big\rangle_m \;—\; \boxed{\mathrm{QFT}} \;—\; \boxed{\mathscr{P}\big(k\tfrac{2\pi}{2^m}\big)} \;—\; \boxed{\mathrm{QFT}^\dagger} \;—\; |z+k\rangle_m$$

Thus if $f(x) = \sum_{i,j} Q_{i,j} x_i x_j + c$, then we need to add:

(1) $-Q_{i,j}$, exactly when $x_i = x_j = 1$. This amounts to the addition of a $\mathrm{QFT}^\dagger \circ \mathscr{P}\big(-Q_{i,j}\frac{2\pi}{2^m}\big) \circ \mathrm{QFT}$ gate, controlled by the i^{th} and j^{th} register of $|x\rangle_n$,

(2) $-c$, independent of $|x\rangle$. This amounts to the addition of a $\mathrm{QFT}^\dagger \circ \mathscr{P}\big(-c\frac{2\pi}{2^m}\big) \circ \mathrm{QFT}$ gate.

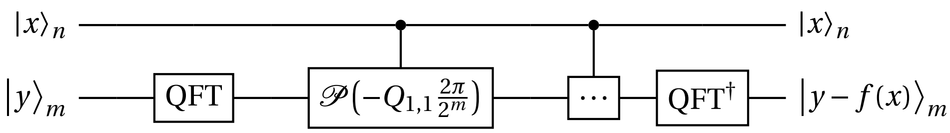
However, the following observation further simplifies the circuit. Let $q_i = \sum_{j=1}^n Q_{i,j}$ and for $i < j$ let $S^{(i,j)}$ be the n-by-n matrix defined via

$$S^{(i,j)}_{k,l} = \begin{cases} 1, & \text{if } k = l \in \{i,j\}, \\ -1, & \text{if } k = i, l = j, \text{ or } k = j, l = j, \\ 0, & \text{otherwise.} \end{cases}$$

Then Q can be written as

$$Q = \operatorname{diag}\big(q_1, q_2, \dots, q_n\big) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Q_{i,j} S^{(i,j)}.$$

Since QFT is unitary, only the first one is needed; similarly, only the last $\mathrm{QFT}^\dagger$ is need. Hence U_f is given by:



Example 3.1. *Let $n = 4$ and $f(x) = 3x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2(x_1x_2 + x_1x_3 + x_1x_4)$. This is equivalent to equation* (2.1)*, with*

$$Q_{1,2} = Q_{1,2} = Q_{1,4} = -1, \quad b_1 = 3, \quad b_2 = b_3 = b_4 = 1,$$

and all other coefficient zero. Furthermore, $m = 3$ works.

Remark 3.2. *When f has symmetries, the above picture can be simplified. For example, in the case of MaxCut, we can assume that, say the last vertex is always in the* 0*-components, thus that qubit register can be eliminated.*

Note that the adjusted cost, $y - f(x)$, is encoded, which can later be used to implements the unitary operators $\exp\big(i\gamma H_f\big)$ without Trotterization as follow: let $\gamma \in \mathbb{R}$ and let us omit the ancilla qubit. Then $\exp\big(i\gamma H_f\big)|x\rangle_n = e^{i\gamma(y-f(x))}|x\rangle$ can be prepared via a $\mathscr{P}\big(\gamma\frac{2\pi}{2^m}\big)$-gate.

4. Application to Grover fixed point search and state preparation

Fix $\delta \in (0,1)$ and y. Let $\lambda := \frac{|C_y|}{2^n}$, where $C_y := \big\{x \in \{0,1\}^n \big| f(x) \geqslant y\big\}$. Finally let $l := \left\lceil \frac{\log_2\left(\frac{2}{\delta}\right)}{2\sqrt{\lambda}} - \frac{1}{2} \right\rceil$. Then, following [?yoder_fixed-point_2014], we can construct a Quantum circuit (using U_f from the previous section), that results in a state $S_l|0\rangle_n\big|y\big\rangle_m$ with the following significance: When the first n qubits are measured in the computational basis, then

$$P\big(x \in C_y\big) = \sum_{x \in C_y} |\langle x|S_l|0\rangle|^2 \geqslant 1 - \delta^2.$$

Let us make the following definitions:

$$
\begin{aligned}
U_S &:= H^{\otimes n} \otimes \mathbb{1}^{\otimes m}, \\
R_0(\alpha) &:= \mathbb{1}^{\otimes(n+m)} + \left(1 - e^{i\alpha}\right) |0\rangle_n \langle 0|_n \otimes \mathbb{1}^{\otimes(1+m)}, \\
R_T(\beta) &:= U_f^\dagger P_{n+1}(\beta) U_f, \\
G(\alpha,\beta) &:= -U_S R_0(\alpha) U_S^\dagger R_T(\beta).
\end{aligned}
$$

Let $(\boldsymbol{\alpha},\boldsymbol{\beta}) = (\alpha_1,\beta_1,\cdots,\alpha_l,\beta_l)$ be given by

$$
\forall j \in \{1,\ldots,l\}: \quad \alpha_j := -\beta_{l-j+1} = 2\cot^{-1}\left(\tan\left(\tfrac{2\pi j}{2l+1}\right)\sqrt{1-\gamma^2}\right),
$$

where $\gamma := \left(T_{1/(2l+1)}(\delta^{-1})\right)^{-1}$ and let

$$
S_l(\boldsymbol{\alpha},\boldsymbol{\beta}) = G(\alpha_l,\beta_l)G(\alpha_{l-1},\beta_{l-1})\cdots G(\alpha_1,\beta_1)U_S.
$$