# COMP 545: Advanced topics in optimization
# From simple to complex ML systems

Lecture 5

# Overview

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    - Talked a little bit about **general smooth optimization** problems
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  - This included some non-convex optimization, but mostly convex
  - The discussion was quite abstract (no particular application)
- For the next 2-3 lectures, we will consider (possibly) the simplest non-convex setting: **sparse model selection**
  - We will provide motivation, background and alternative solutions
  - We will focus on how we can **provably and efficiently solve** such problems

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- We will focus on the cases of (structured) sparsity and low-rankness

(But I'm open to other alternatives as we proceed)

# Sparse linear regression

(Not again man!..)

- Generative model: $y_i = a_i^\top x^\star + w_i$
  - $a_i \in \mathbb{R}^p$ : features
  - $y_i \in \mathbb{R}$ : responses
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- Any suggestions how to solve this?

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**LASSO**

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$$\longrightarrow \quad x_{t+1} = \Pi_{\|\cdot\|_1 \leq \lambda}\left(x_t - \eta \nabla f(x_t)\right)$$

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- Solution #2: convexification + **proximal** gradient descent

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- Solution #3: keep non-convexity + **non-convex projected gradient descent**

**Hard-thresholding**

$$\min_{x\in\mathbb{R}^p} \quad f(x) := \tfrac{1}{2}\|y - Ax\|_2^2 \qquad \text{s.t.} \quad \|x\|_0 \le k$$

$$\longrightarrow \quad x_{t+1} = H_k\left(x_t - \eta \nabla f(x_t)\right)$$

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# But before we proceed..

- Some questions:

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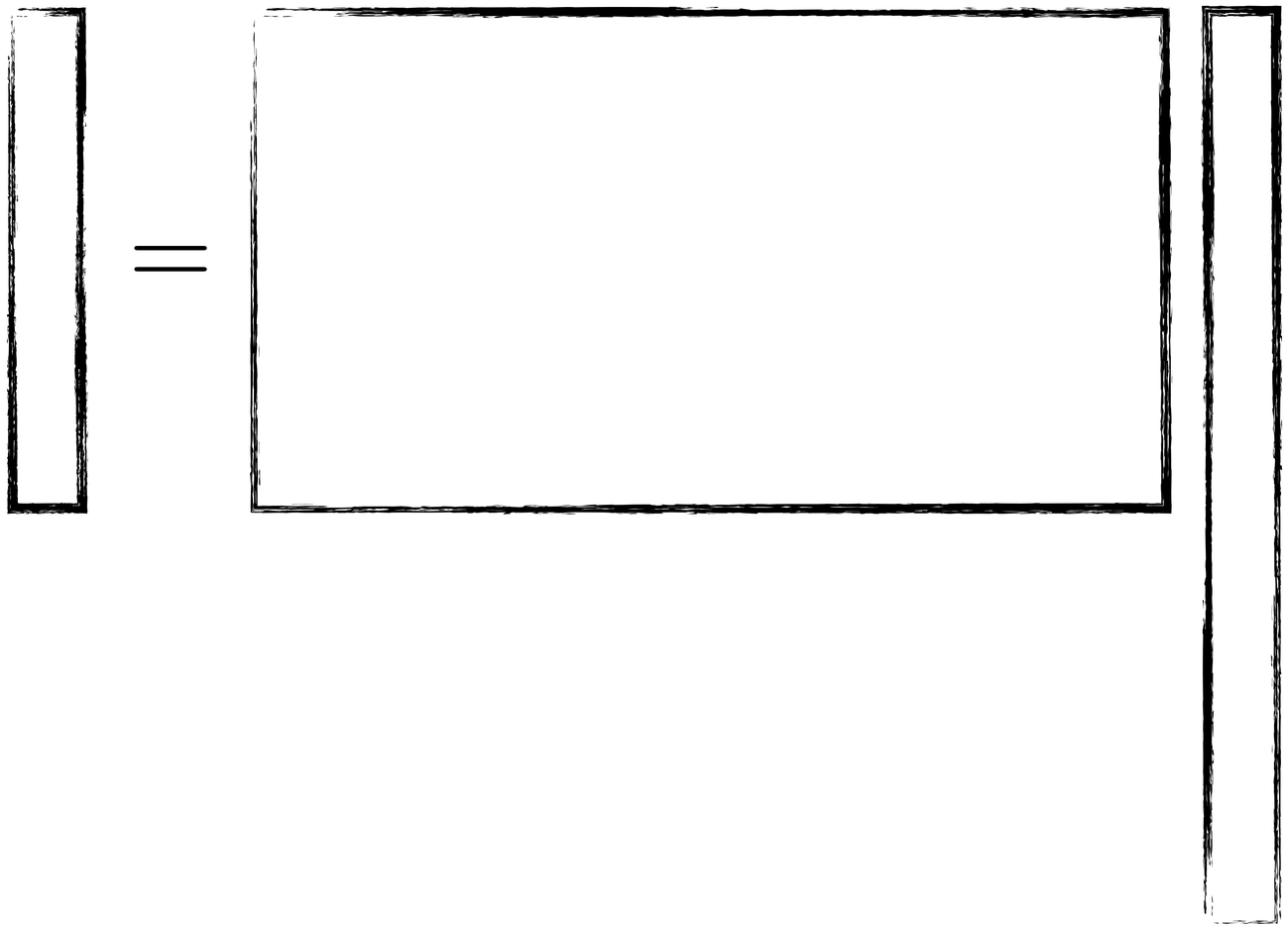
  - Q: "But isn't the problem underdetermined? $(n \ll p)$"
  - A: "Yes, without any constraints, the problem has infinite solutions"
  - Q: "Why then do we have hopes solving this problem?"
  - A: "Under assumptions on $A$, and the relation between $(n, p, k)$, we Will see that on average this problem can be solved in polynomial time"

# Over- vs. under-parameterized



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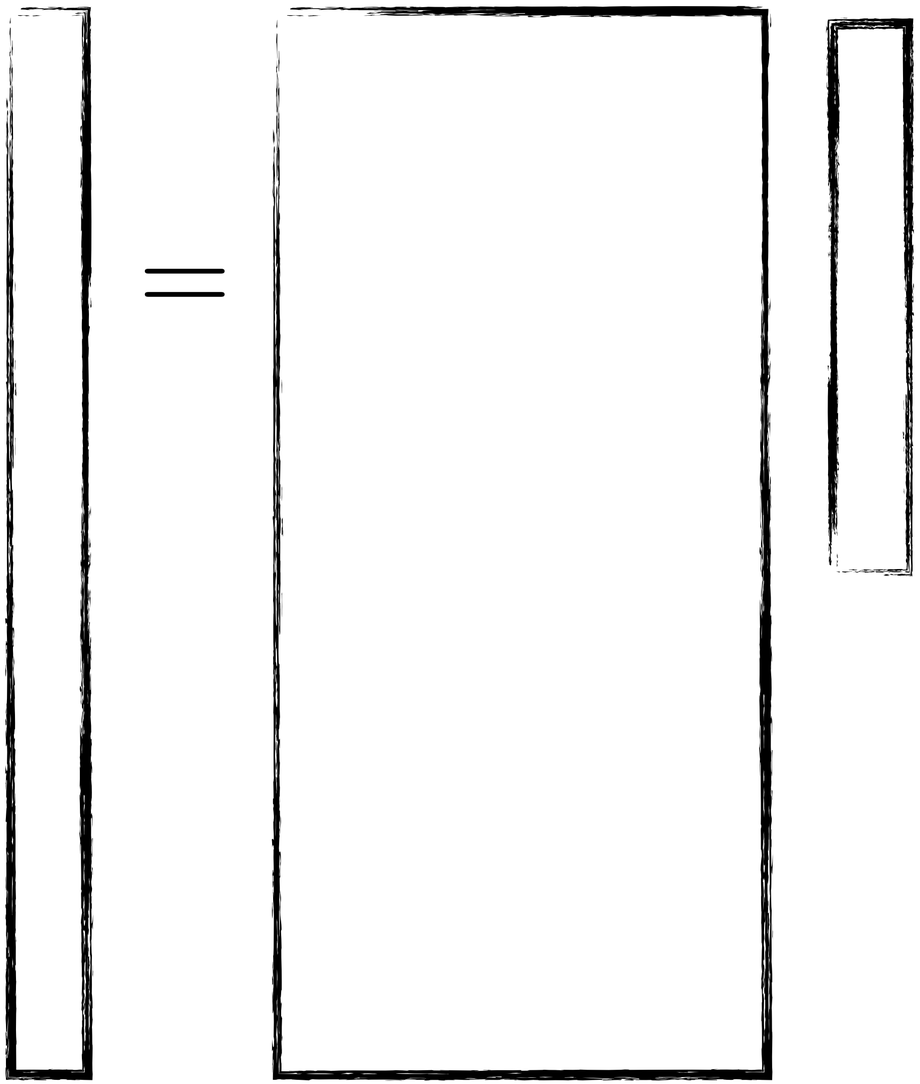

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- Now, imagine yourself implementing this.. What are the hyper-parameters?
  - "How do we set the step size?"
  - "How do we select the initial point? (it is non-convex after all)"
  - "What if we don't know the sparsity level?"
  - "Are there any other tricks we can pull-off?"

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- Property of $I$: isometry

$$(1-\delta)\|x_1 - x_2\|_2^2 \leq \|I(x_1 - x_2)\|_2^2 \leq (1-\delta)\|x_1 - x_2\|_2^2, \quad \text{for some } \delta \in [0,1], \ \forall x_1, x_2 \in \mathbb{R}^p$$

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- We get linear convergence to the global optimum!

# How does it perform in practice?

Demo

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(In our case, we generate it as Gaussian so with high probability we are fine)

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Note: Checking whether a fixed matrix actually satisfies RIP is NP-hard..

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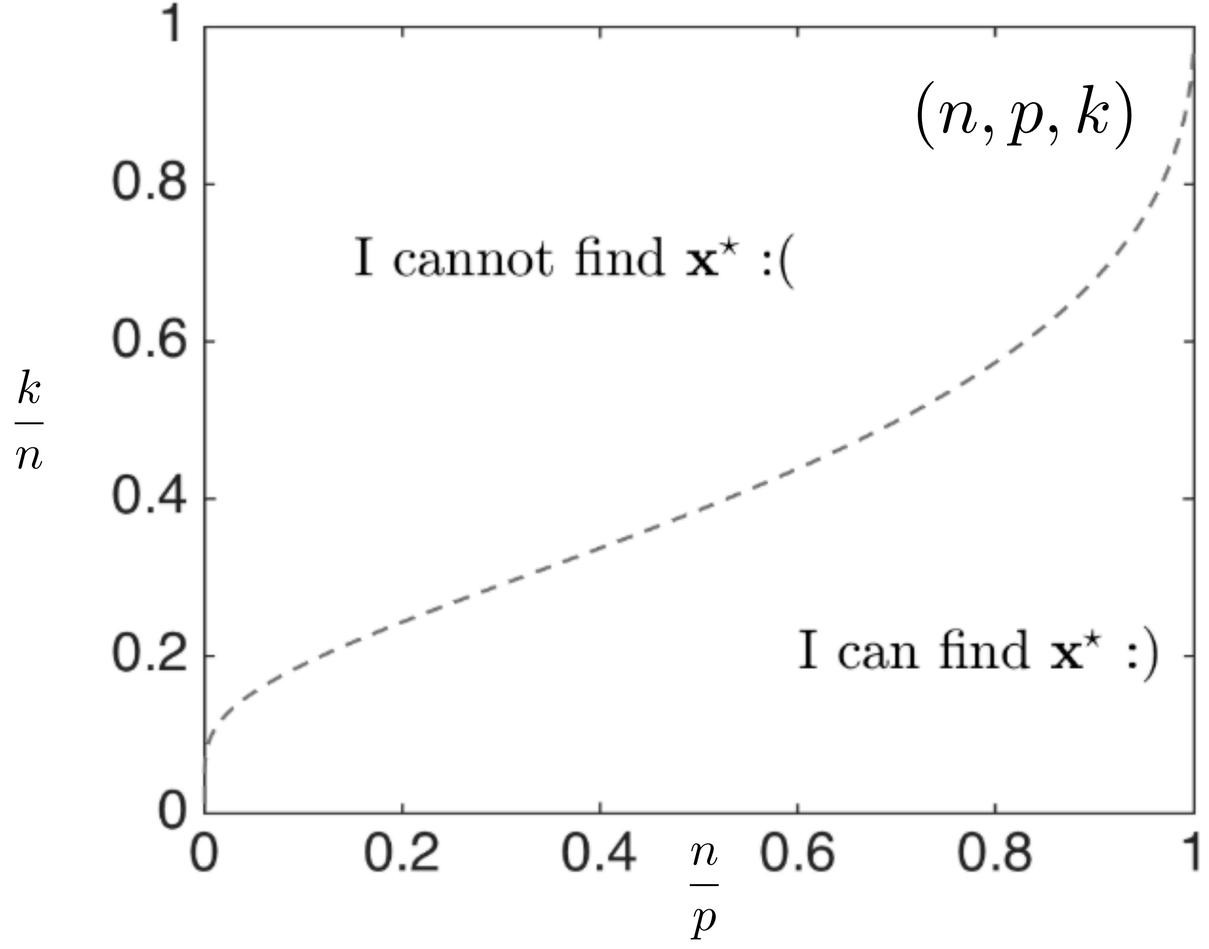


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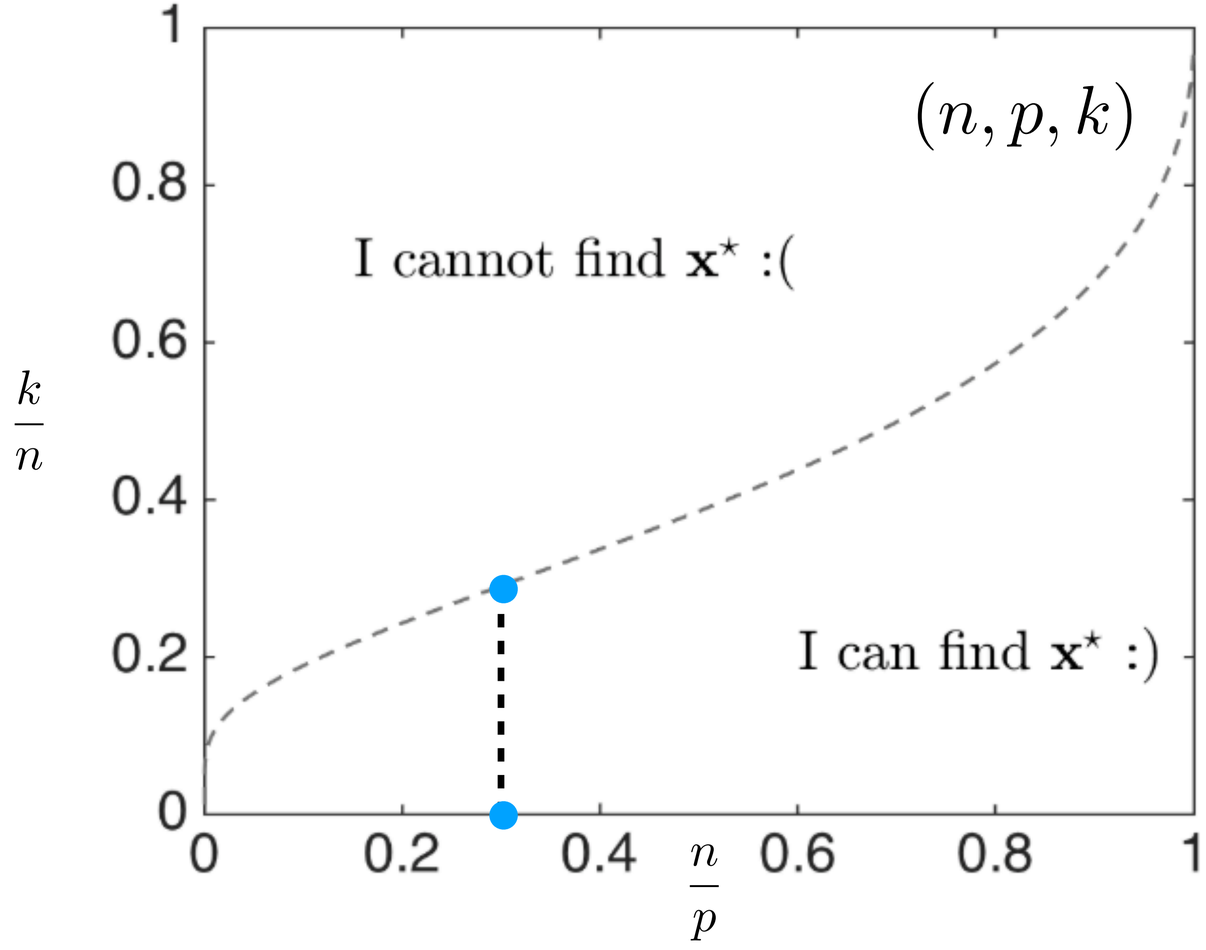


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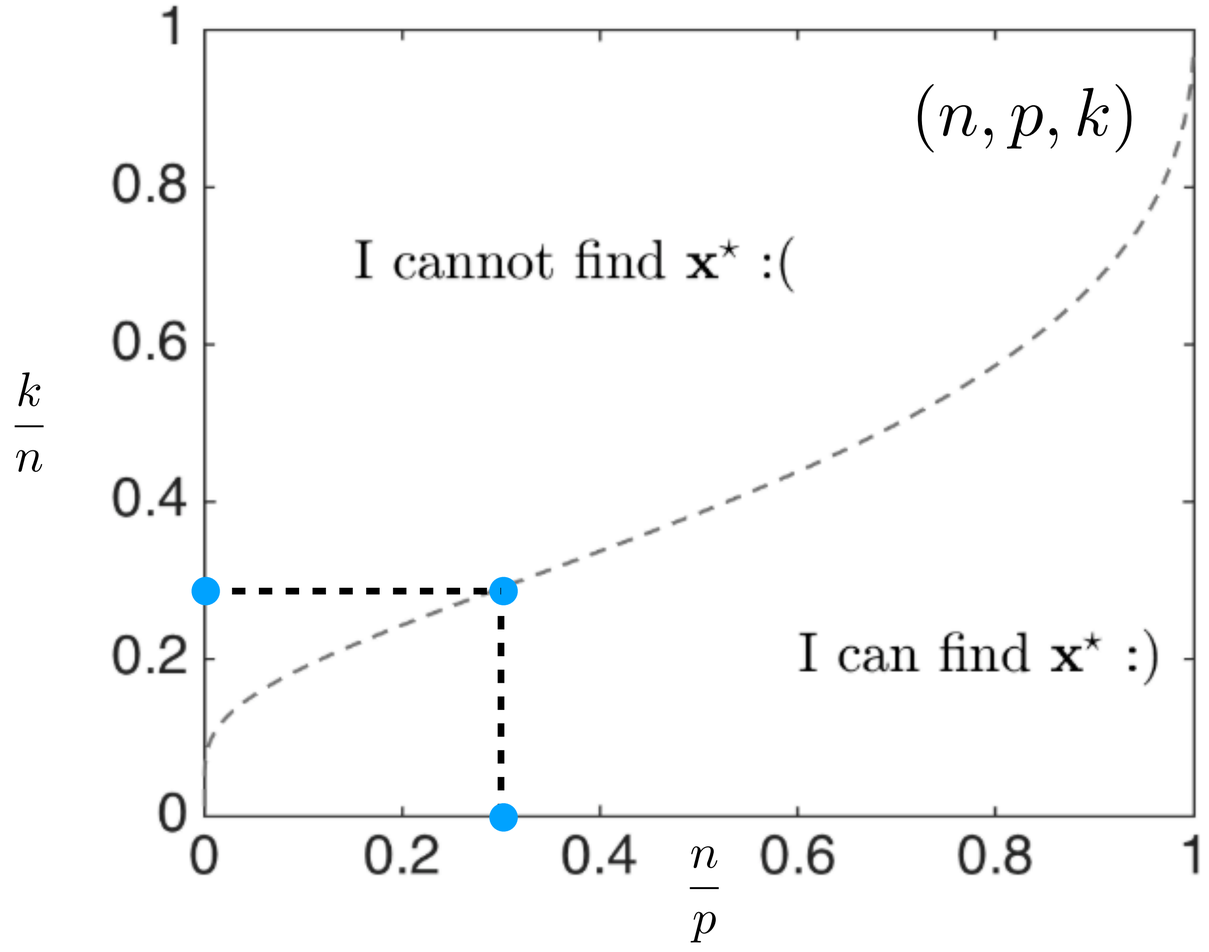


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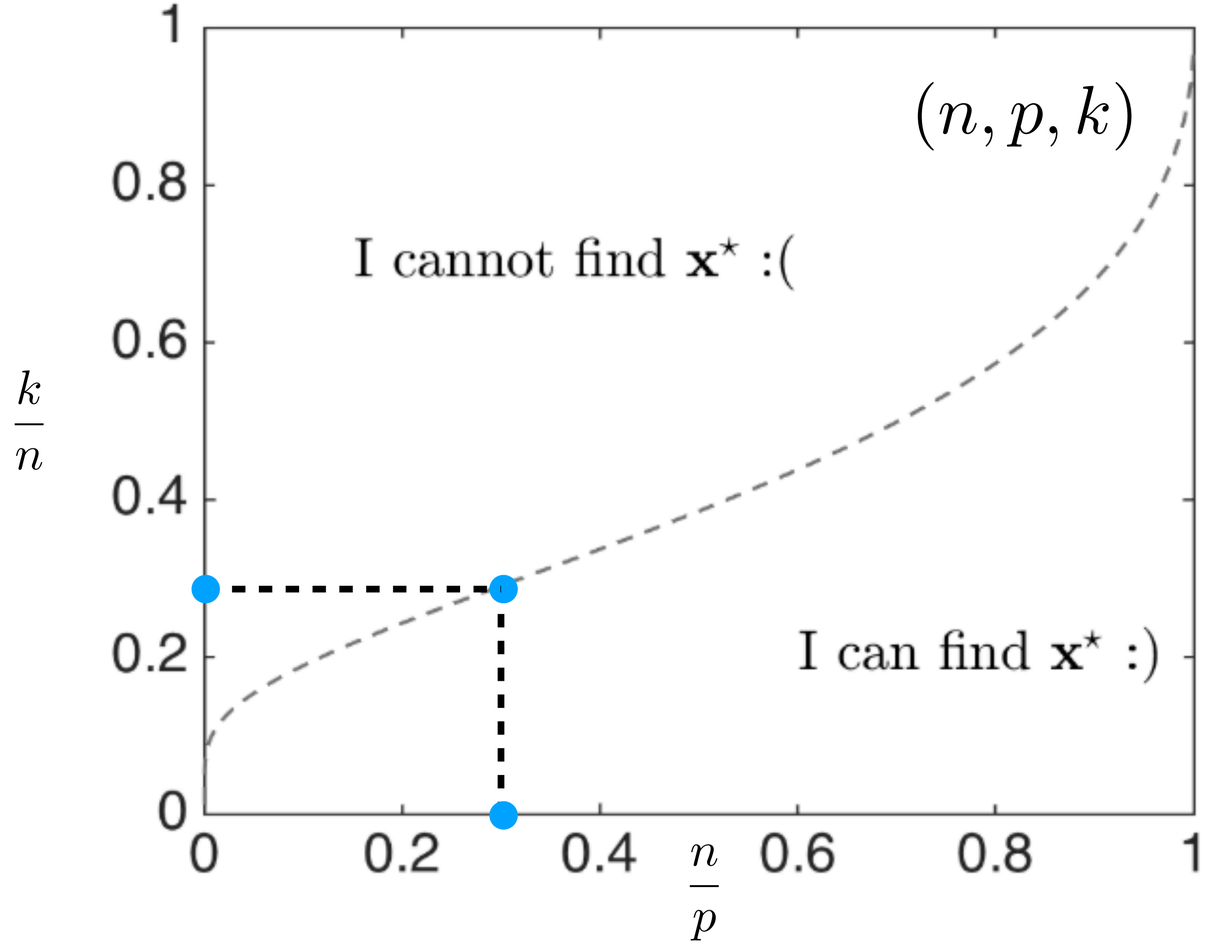


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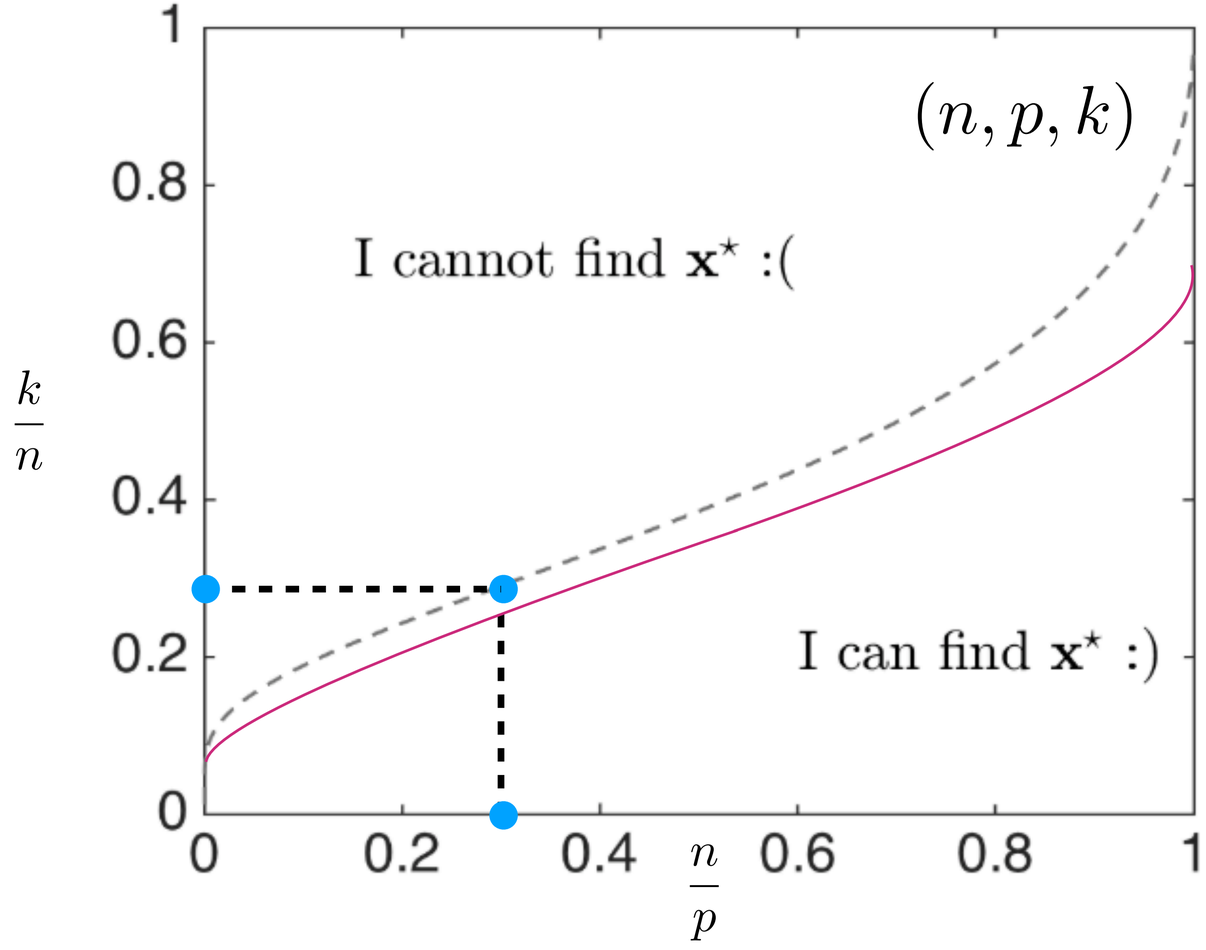


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(In literature section, there is work that validates theoretically this selection)

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(In literature section, there is work that validates theoretically this selection)

- But, is this practical?

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- Q: "How can we then improve its performance?"
- Via hyper-parameter tuning! Can we tune the step size via theory?
- Thus, we propose:

$$\eta = \frac{1}{1+\delta}$$

(In literature section, there is work that validates theoretically this selection)

- But, is this practical? Generally, no!

(Value of $\delta$ is NP-hard to find)

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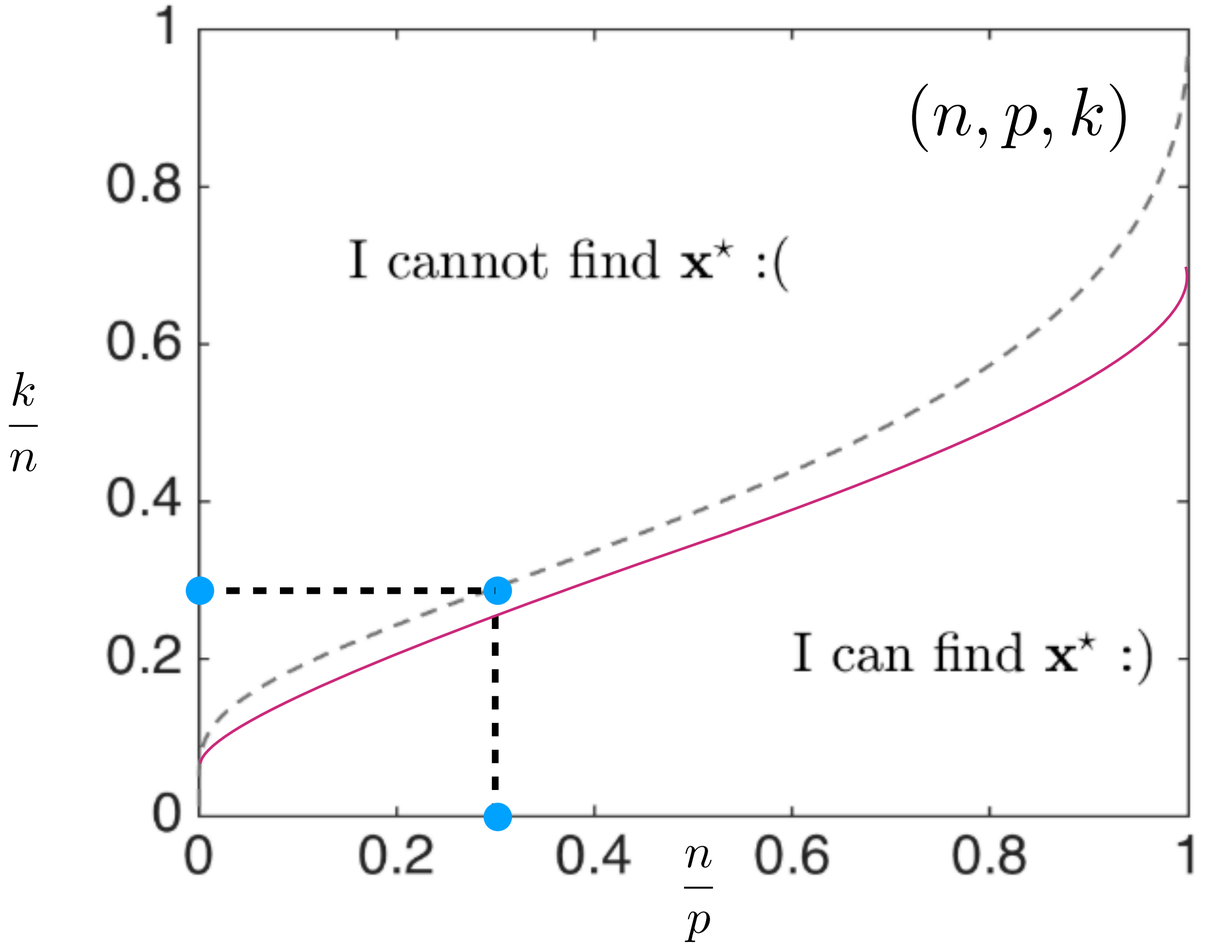
- Q: "Great! Why don't we use that all the time?"
- A: "Because, moving beyond least squares, solving this might be as difficult as the original problem"

# Phase transition update

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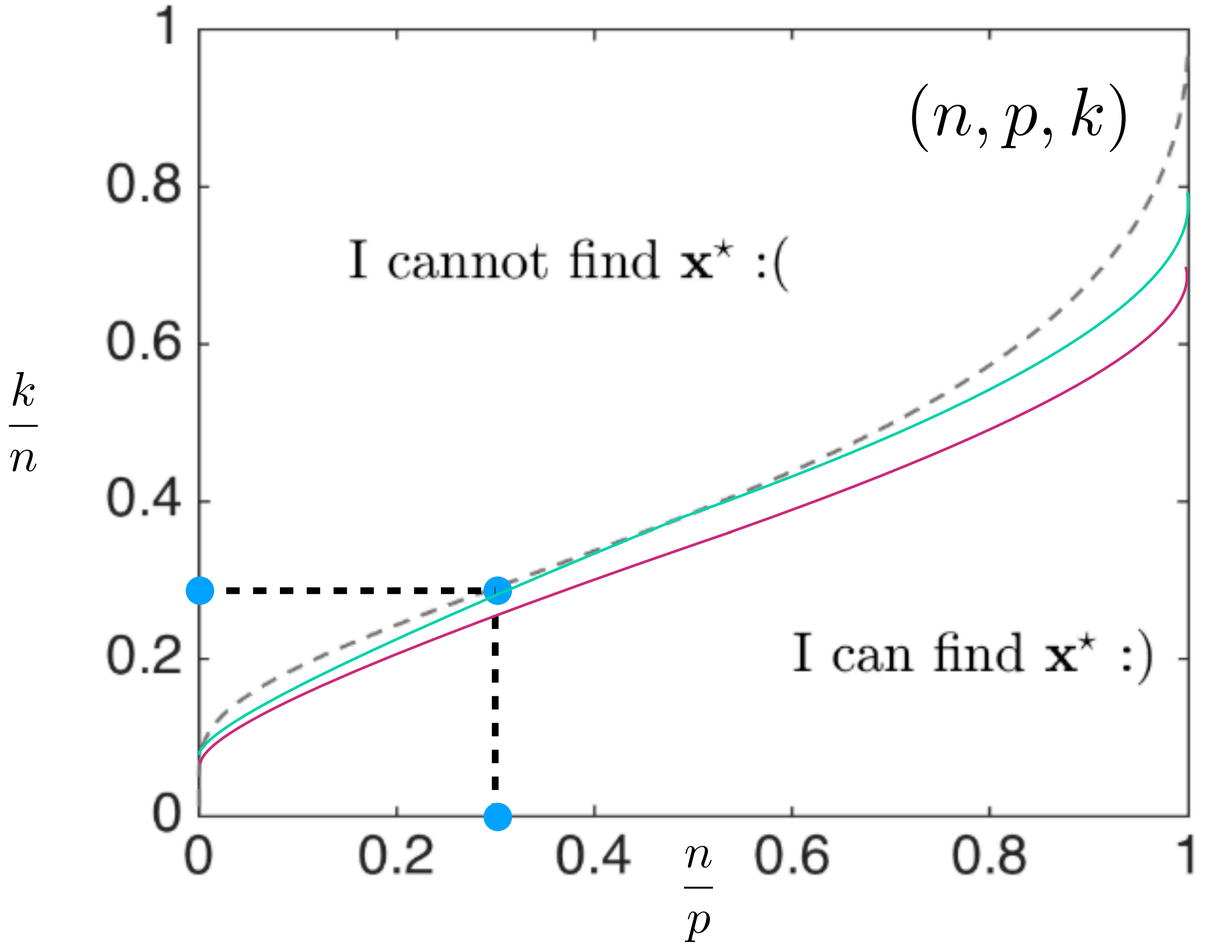


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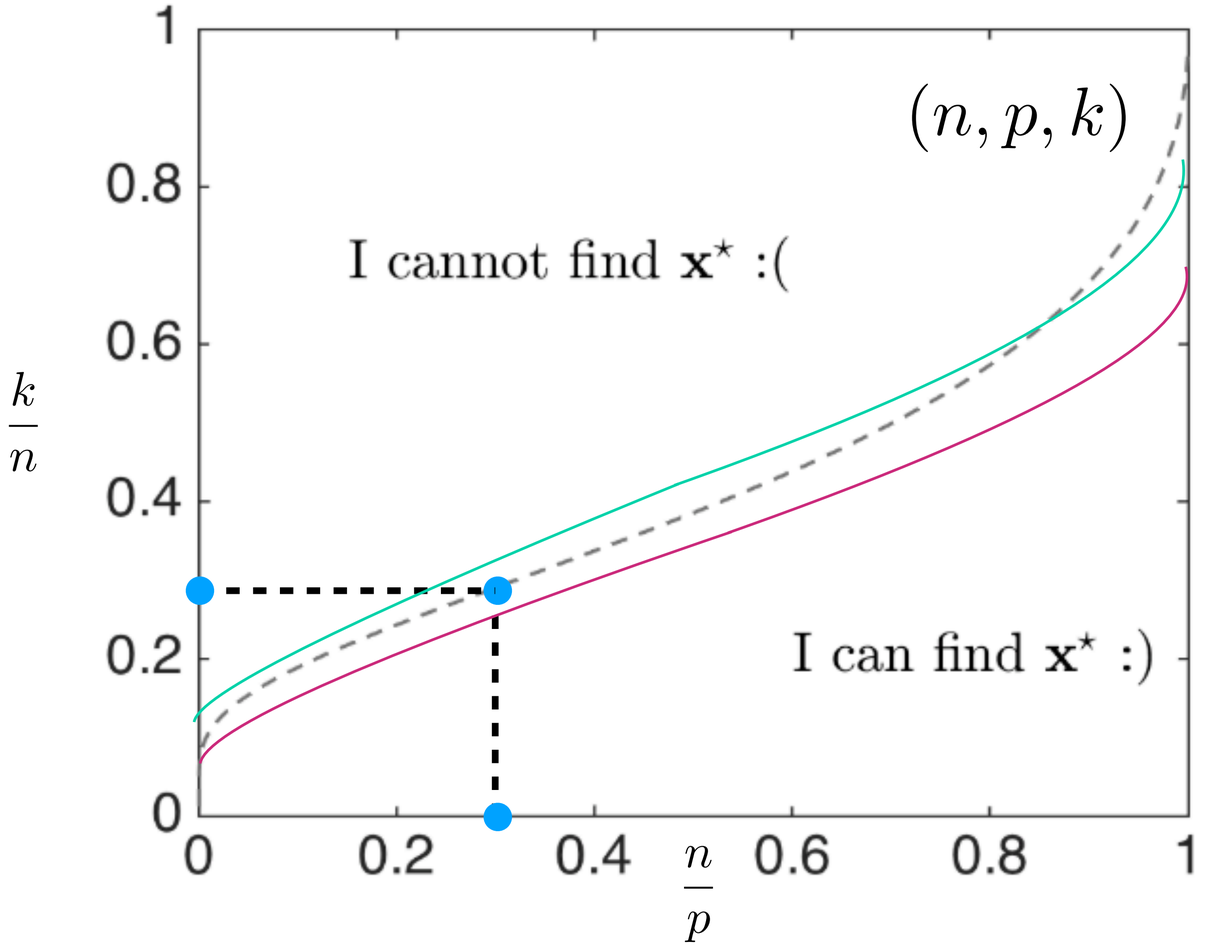


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(Some of these methods can be found in the Review part)

# But does this step size selection work in theory?

Whiteboard

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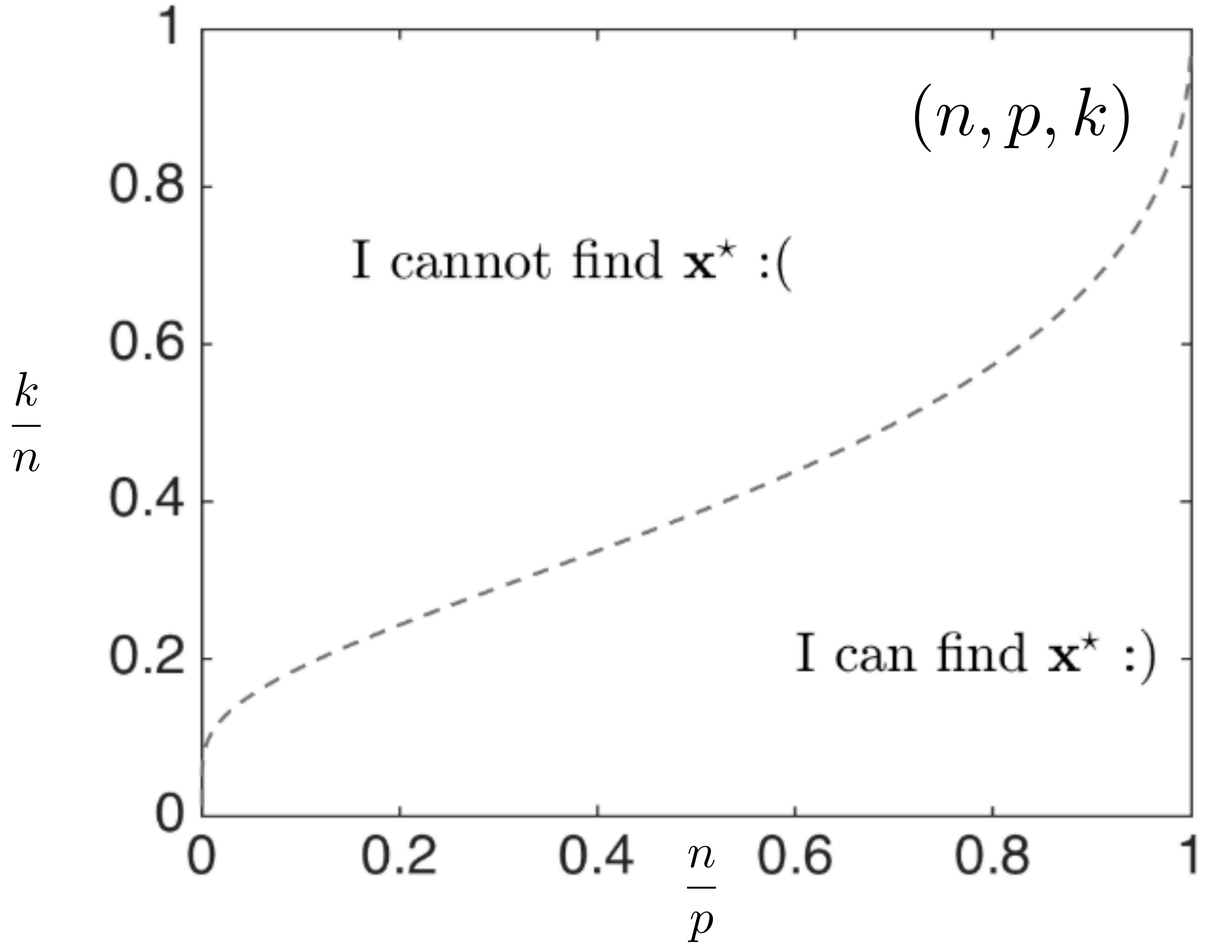
- Q: "What happens if we overshoot sparsity level?" (Demo)
- A: "We actually get denser and denser solutions"
- Q: "Is there any non-provable tweaks?"
- A: "Problem-dependent strategies"

# Heuristic for $k$ on linear regression

(More like an upper bound for sparsity level)



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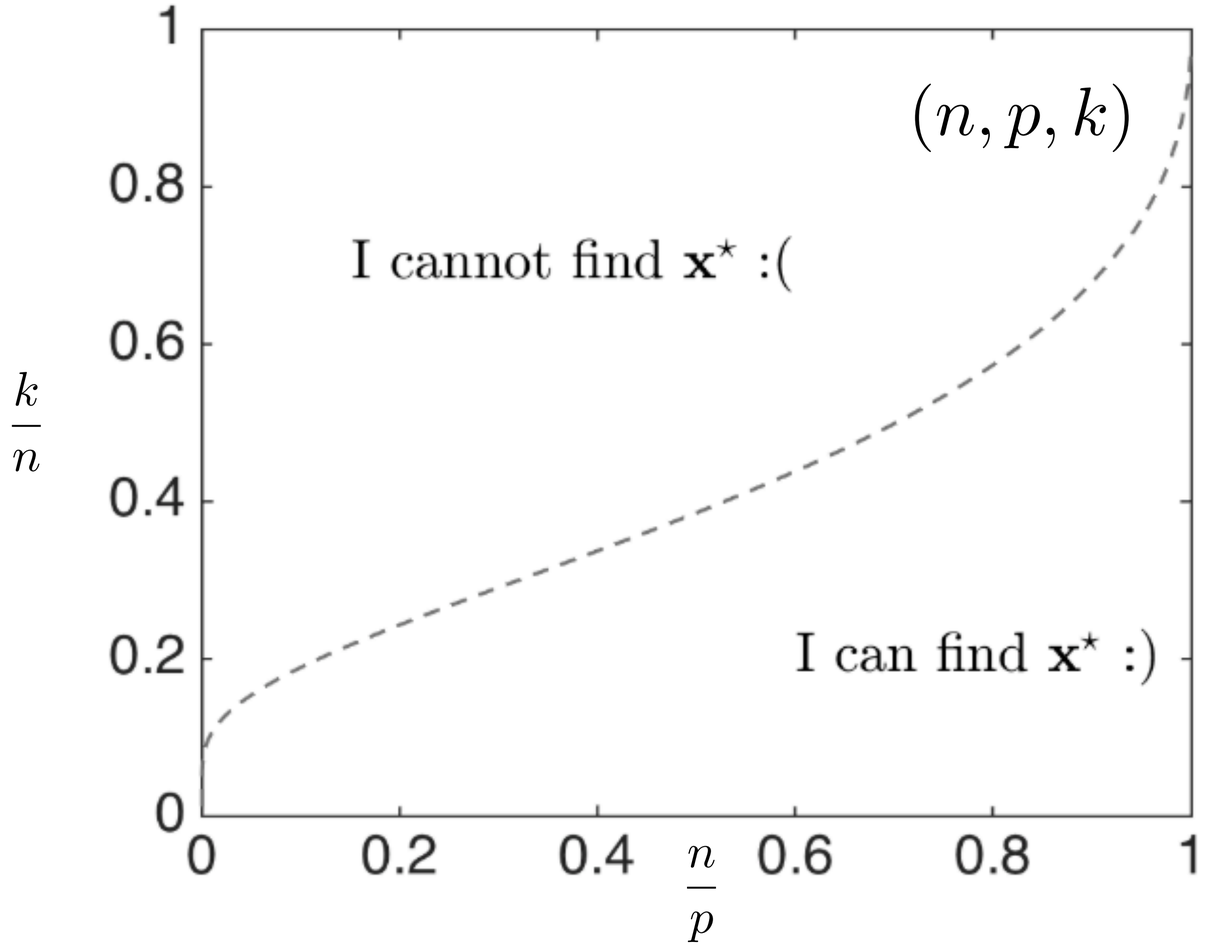


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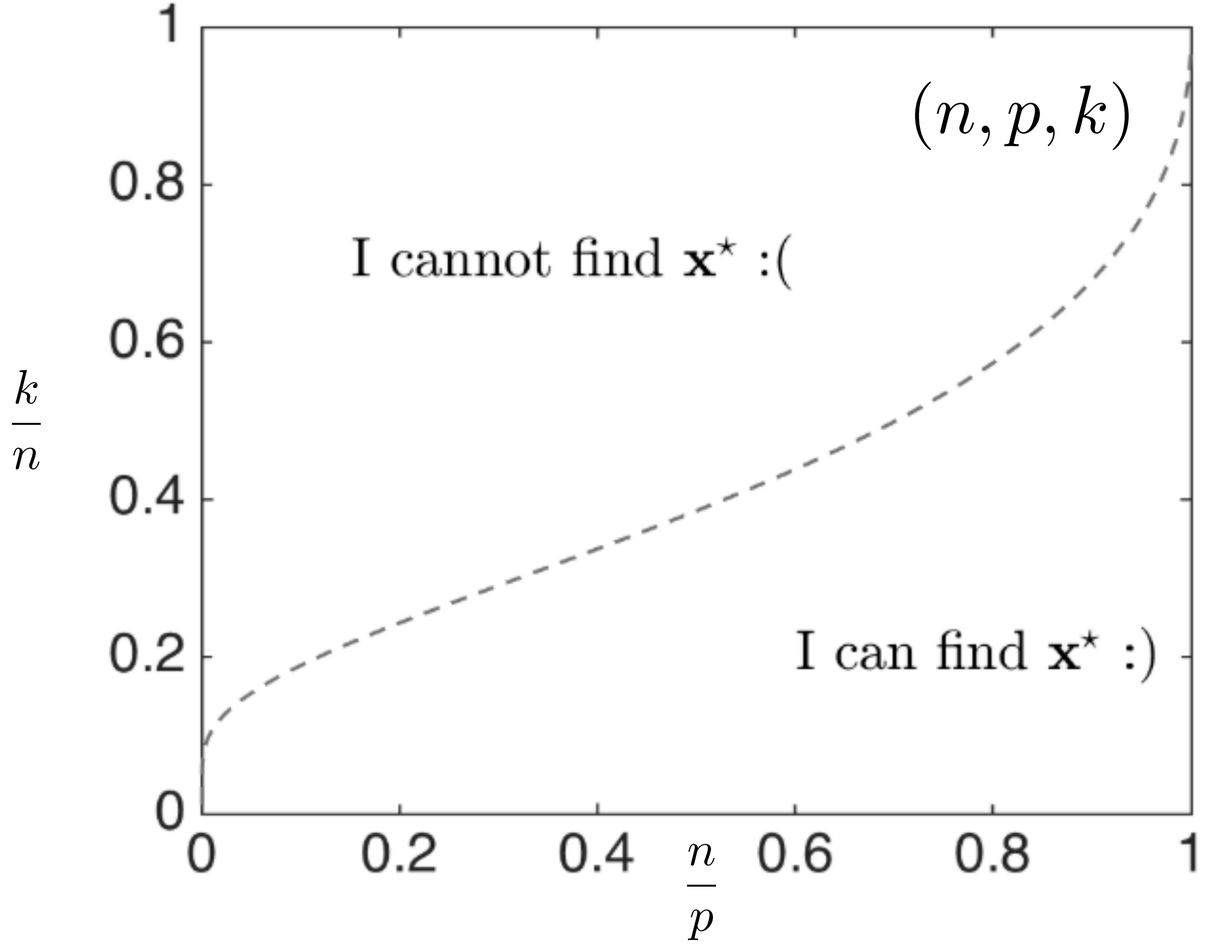


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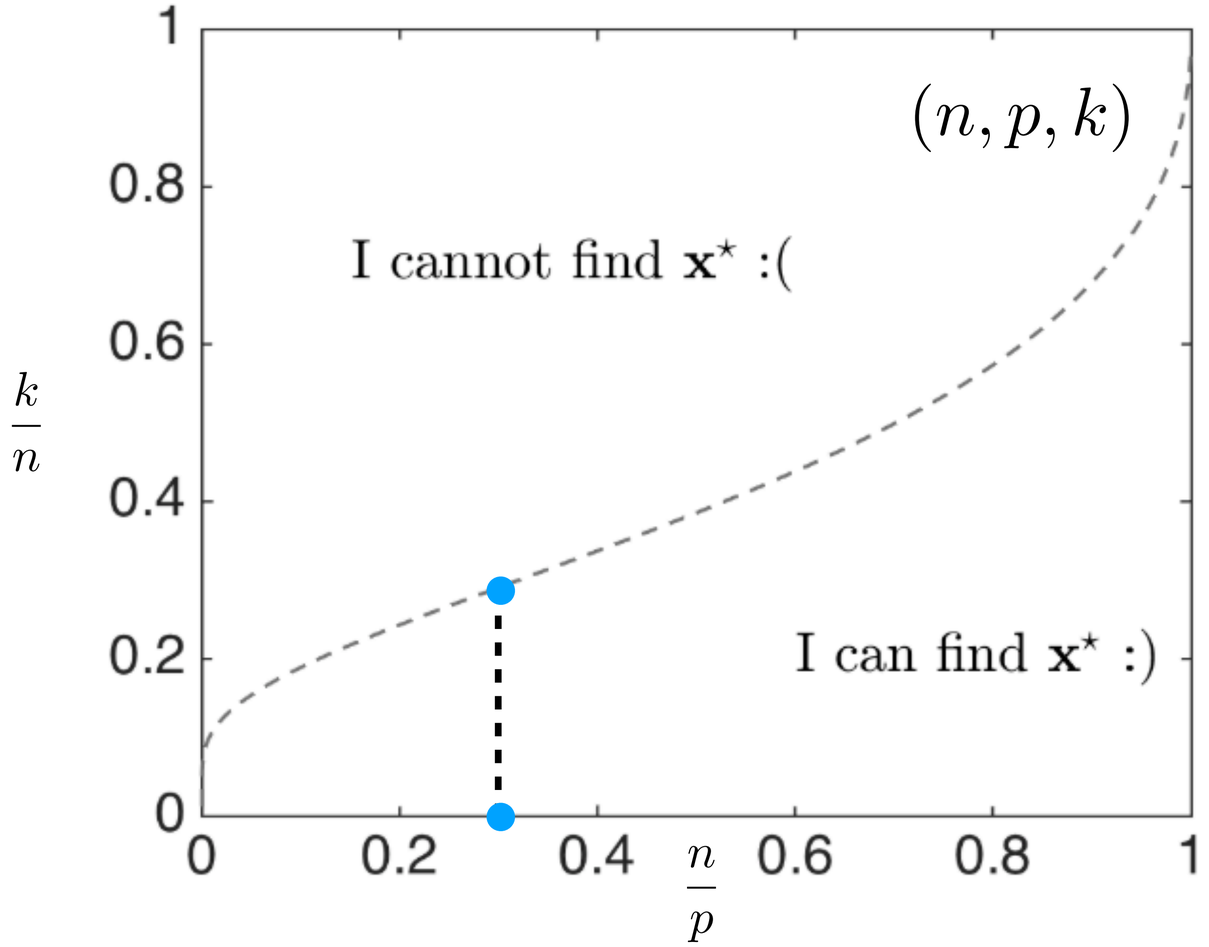


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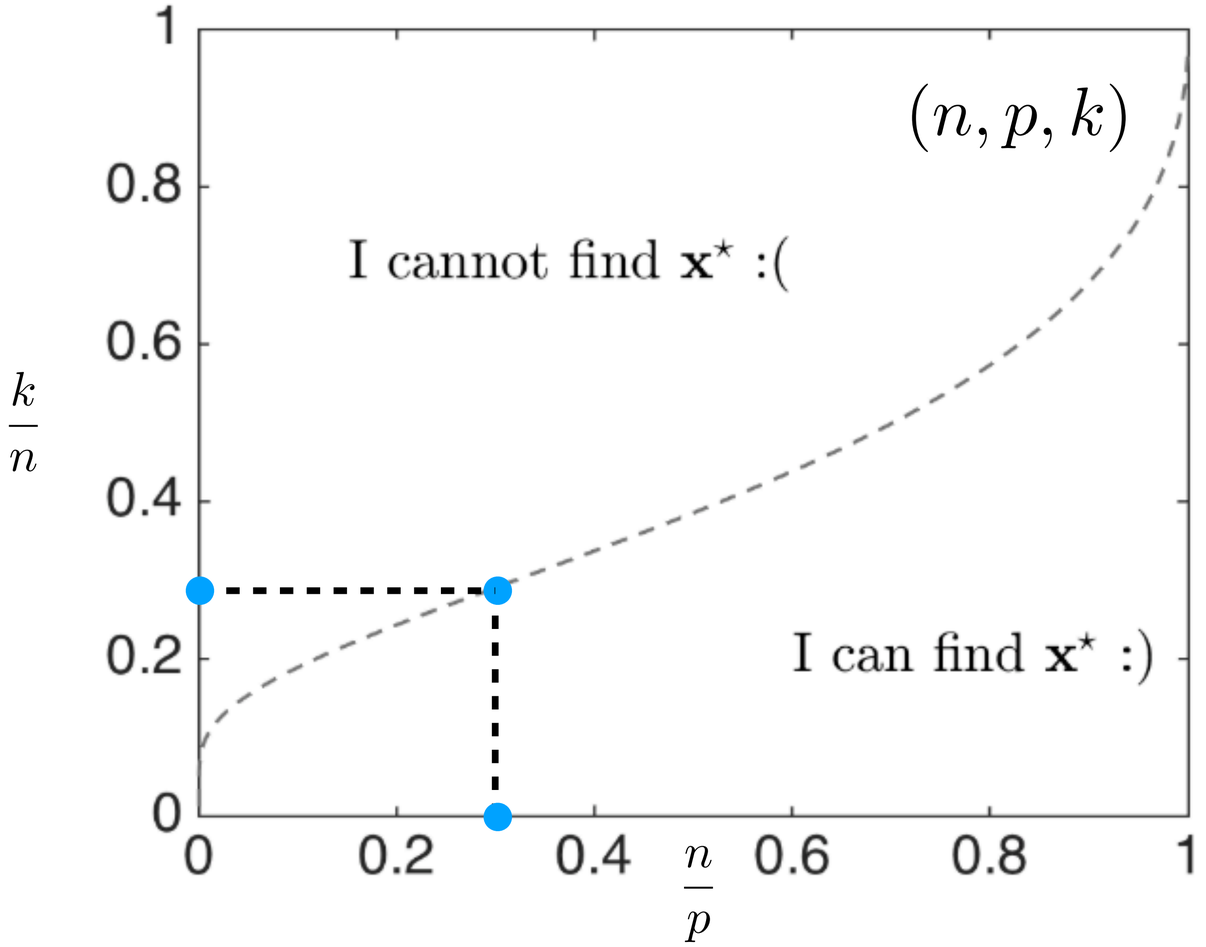


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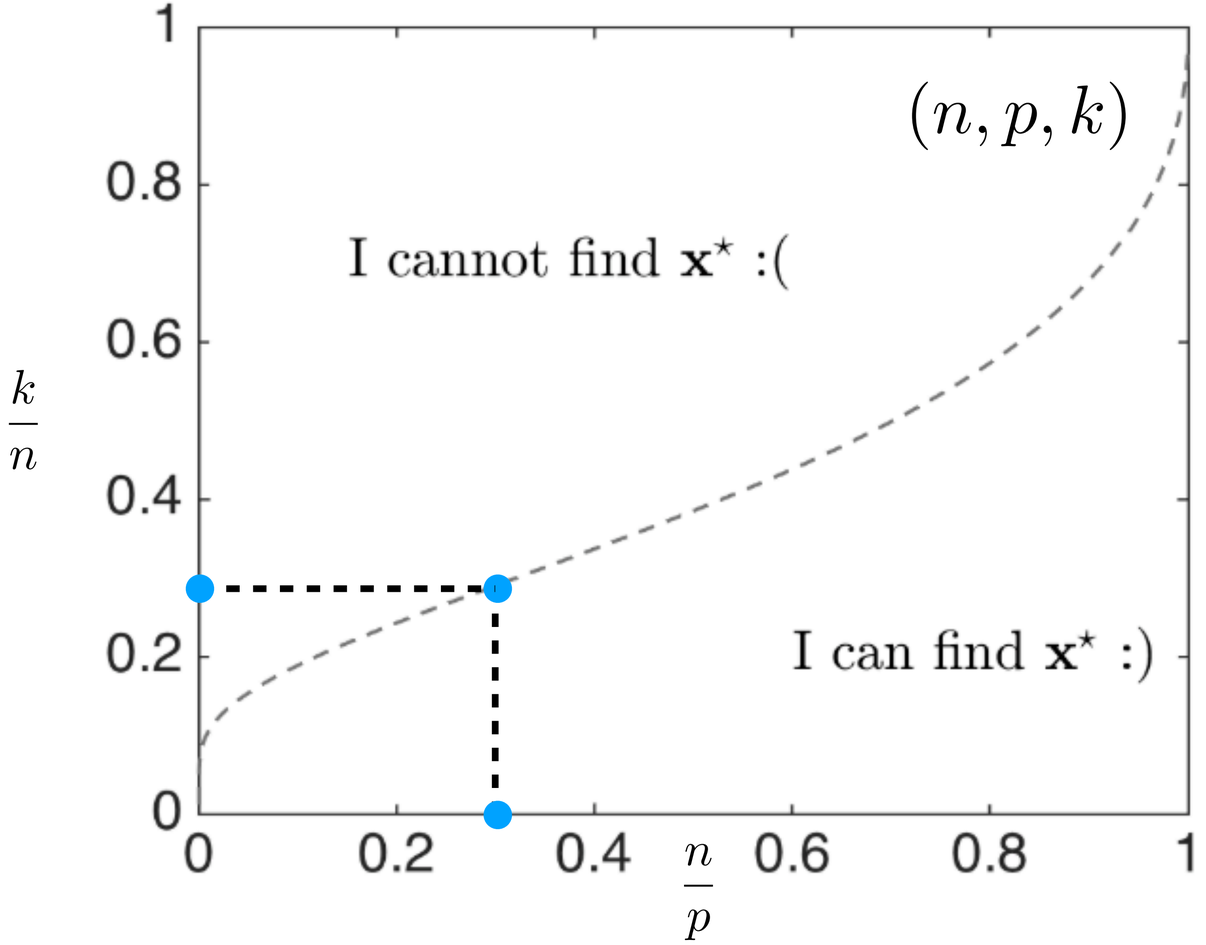


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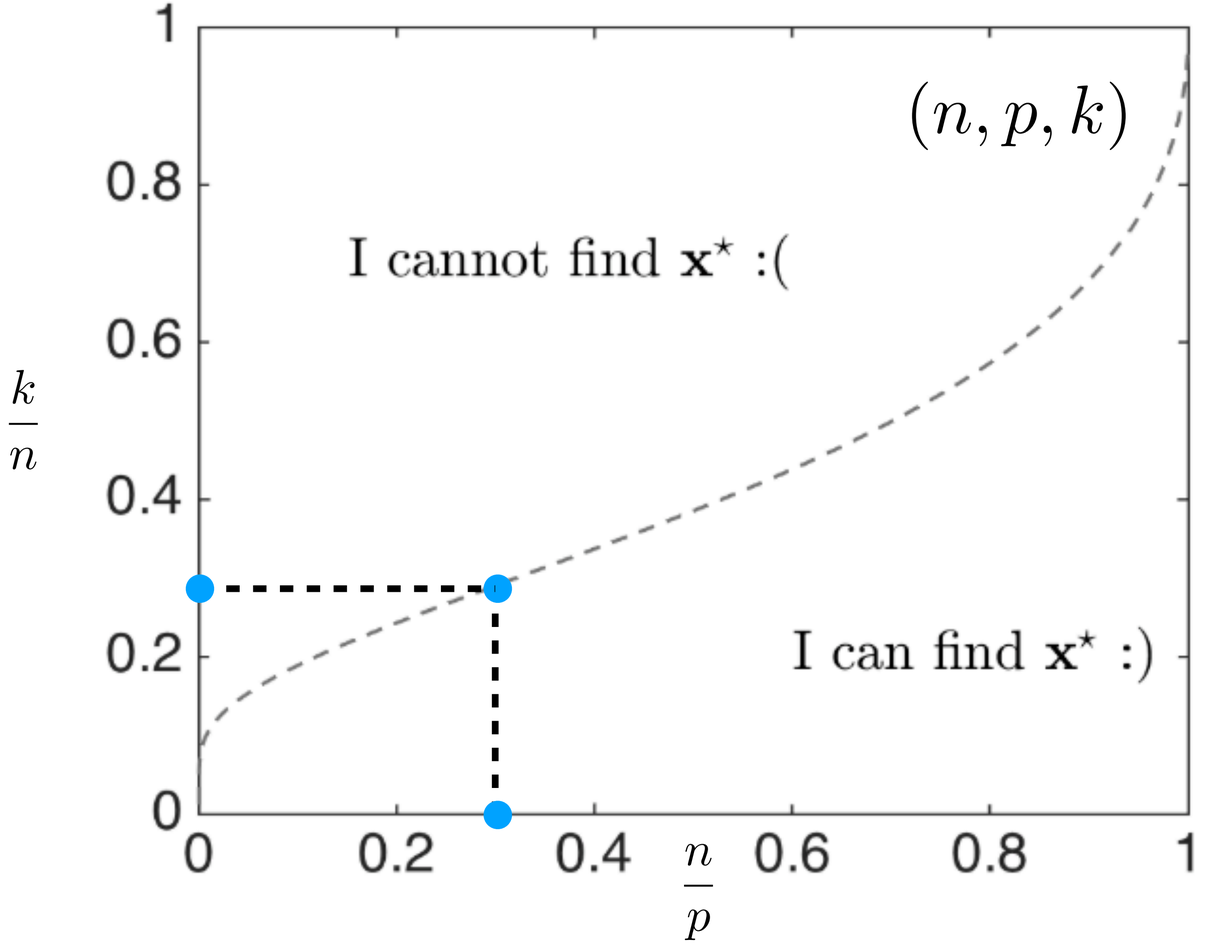


- We know $(n, p)$
- Map these values in the phase transition plot
- The curve indicates what levels might work
- Run algorithm, take results, interpret: if it works, good; otherwise, do binary search over sparsity level

"All these sound interesting.. but do they extend to other objectives? And how are they related with what we discussed so far?"

(Lipschitz gradient continuity, strong convexity, Hessians, etc..)

# A different view of RIP

- Reminder

$$(1-\delta)\|x_1 - x_2\|_2^2 \leq \|A(x_1 - x_2)\|_2^2 \leq (1+\delta)\|x_1 - x_2\|_2^2,$$
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- When objective has Lipschitz continuous gradients and is strongly convex:

$$\mu I \preceq \nabla^2 f(x) \preceq LI$$

# Restricted smoothness and strong convexity

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  Let's say $x \in \mathcal{C} \subseteq \mathbb{R}^p$

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(Note that this $L$ is different from that of general smoothness)

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- Restricted strong convexity

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|_2^2, \quad \forall x, y \in \mathcal{C}$$

(Note that this $\mu$ is different from that of general strong convexity)

# Examples

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$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & \frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \exp\left(-y_i a_i^\top x\right)\right) + \frac{\lambda}{2}\|x\|_2^2 \\ \text{s.t.} \quad & \|x\|_0 \leq k \end{aligned}$$

$y_i \in \{\pm 1\}$

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and restricted smoothness with constant:

$$L = (\lambda_{\max}\left(A^\top A, k\right) + \lambda)$$

(For more information, see "Gradient hard thresholding pursuit")

# Examples

- Graphical model selection (under Gaussian assumptions)

Whiteboard

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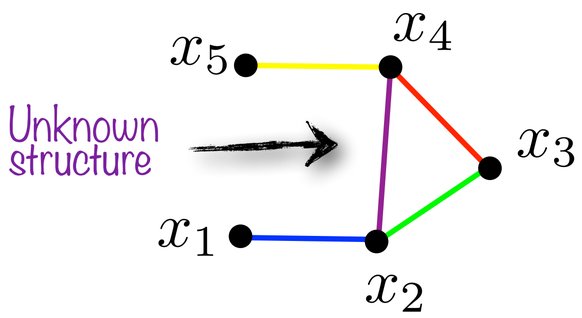

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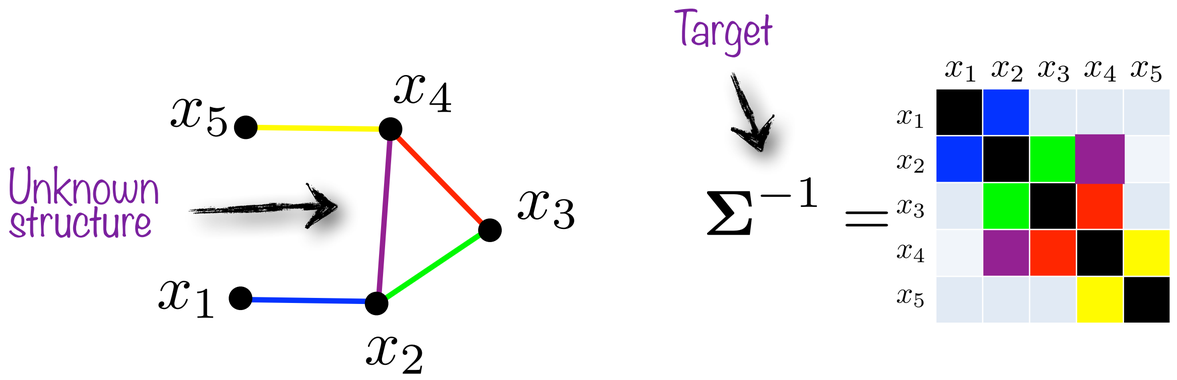

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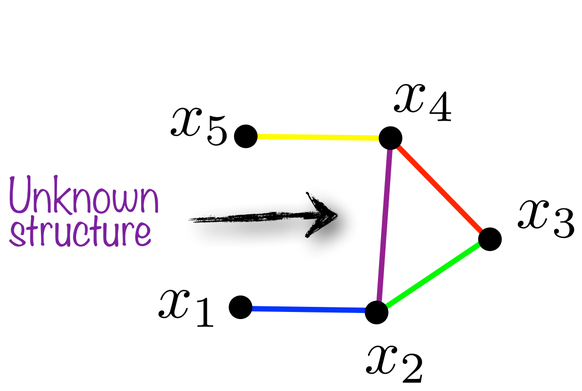


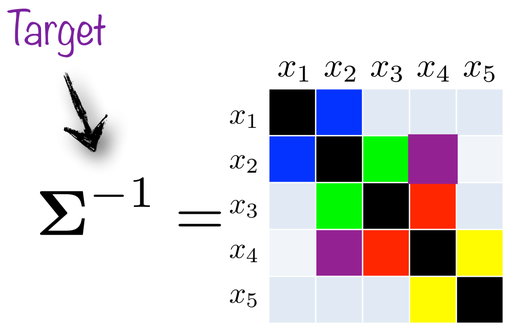


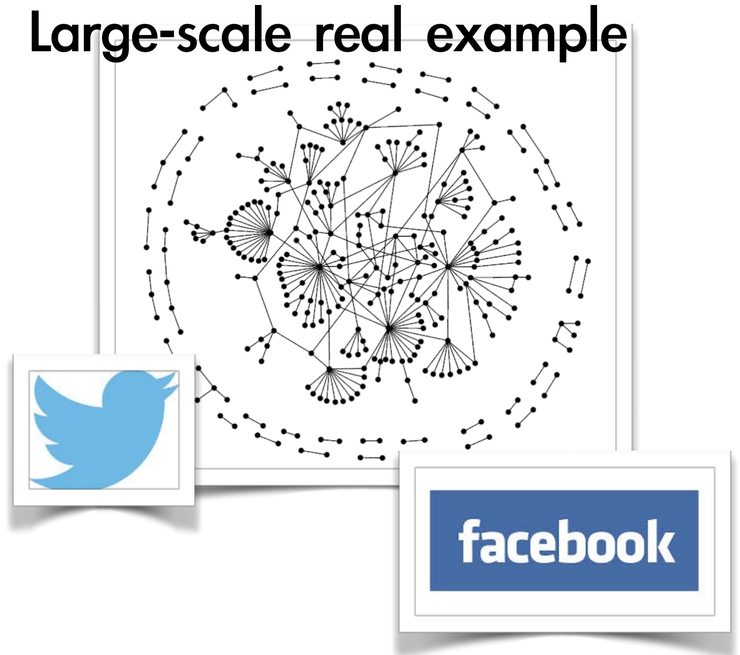

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**Optimization problem 2**

Non-convex

$$\min_{\Theta \in \mathbb{R}^{p \times p}} \quad -\log\det(\Theta) + \mathrm{Tr}\left(\widehat{\Sigma}\Theta\right)$$

$$\text{s.t.} \quad \|\Theta\|_0 \leq k$$

# Beyond plain sparsity

- Our discussion so far holds for discrete structures beyond sparsity:

  Block-sparsity, overlapping block-sparsity, dispersive models, tree sparsity, graph-sparsity, etc..

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  Block-sparsity, overlapping block-sparsity, dispersive models, tree sparsity, graph-sparsity, etc..

- As long as the projection onto the combinatorial constraint can be computed efficiently:

$$\begin{aligned} \min_{x \in \mathbb{R}^p} \quad & \|x - y\|_2^2 \\ \text{s.t.} \quad & x \in \mathcal{C} \end{aligned}$$

- Various extensions include inexact projections, greedy approaches, and there are connections with (sub/super)modular optimization

# Interlude: Statistics in Data Science

- We will use the example of RIP

  (The goal is to highlight that we cannot have data science without using a diverse set of tools; IMHO, ML is not an research area "fallen from the sky": it is just a cool combination of optimization, statistics, coding and linear algebra)

- Disclaimer: this is not a complete introduction to concentration inequalities

  (How many would be interested in learning about concentration inequalities (as a course)? )

# Papers to review – due next Tuesday

(Select one of the following papers)

- "CoSaMP: Iterative signal recovery from incomplete and inaccurate samples", Needell et al., 2009.

- "Compressed sensing using generative models", Bora et al., 2017.

- "A Nearly-Linear Time Framework for Graph-Structured Sparsity", Hegde et al., 2015.

- "Deep image prior", Ulyanov et al., 2018.

# Conclusion

- This lecture considers **sparse model selection** in Data Science applications
- While there are rigorous and efficient methods also in the convex domain we followed the **non-convex path** of hard thresholding methods
- We discussed some global convergence guarantees, and highlighted the importance of hyper-parameter tuning

# Next lecture

- We will consider the case of **low-rank recovery**, natural extension of sparsity - there, we have different ways to exploit non-convexity