COMP 545: Advanced topics in optimization
From simple to complex ML systems

Lecture 7

Overview

- In the previous lecture, we:
  - Considered **low-rank model selection** in Data Science applications
  - Followed the **non-convex path,** beyond hard thresholding methods
  - Discussed some global convergence guarantees (under proper initialization assumptions) and mentioned some open questions

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- For the next 2-3 lectures, we will worry about the **landscape** of such non-convex scenaria:
  - We will discuss about **types of stationary points**, focus on **saddle points** and study some of their properties
  - We will introduce conditions that allow **escaping from saddle points**
  - We will study matrix sensing as a test case, and how to prove "no spurious local minima" arguments

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- Non-convex continuous optimization = NP-hard in general

(Specifically can be polynomially solvable)

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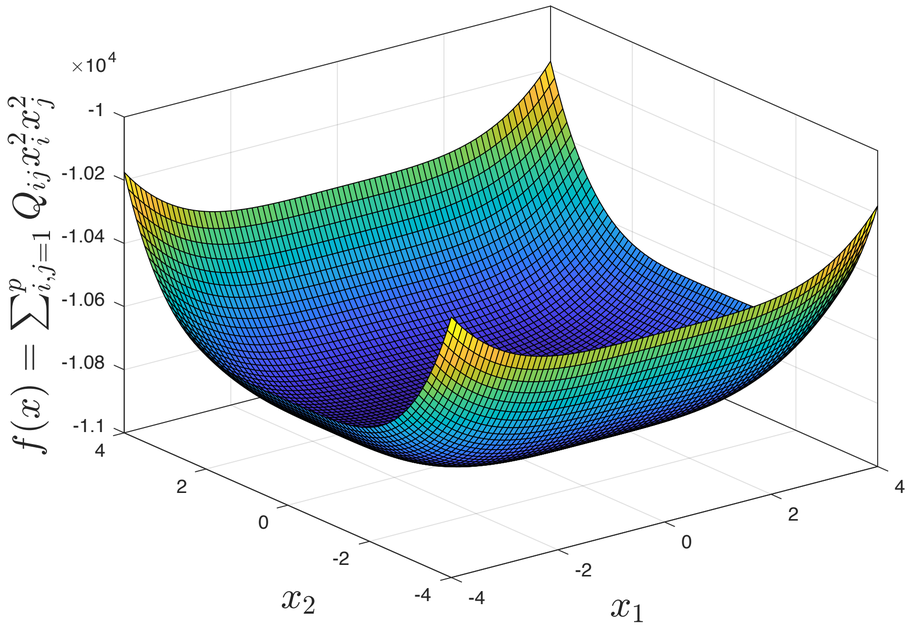

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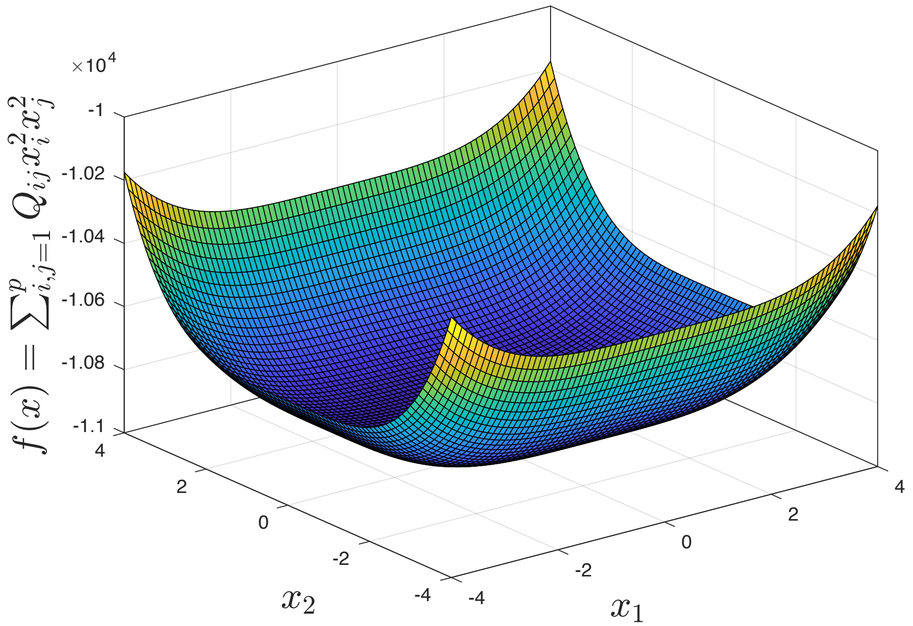

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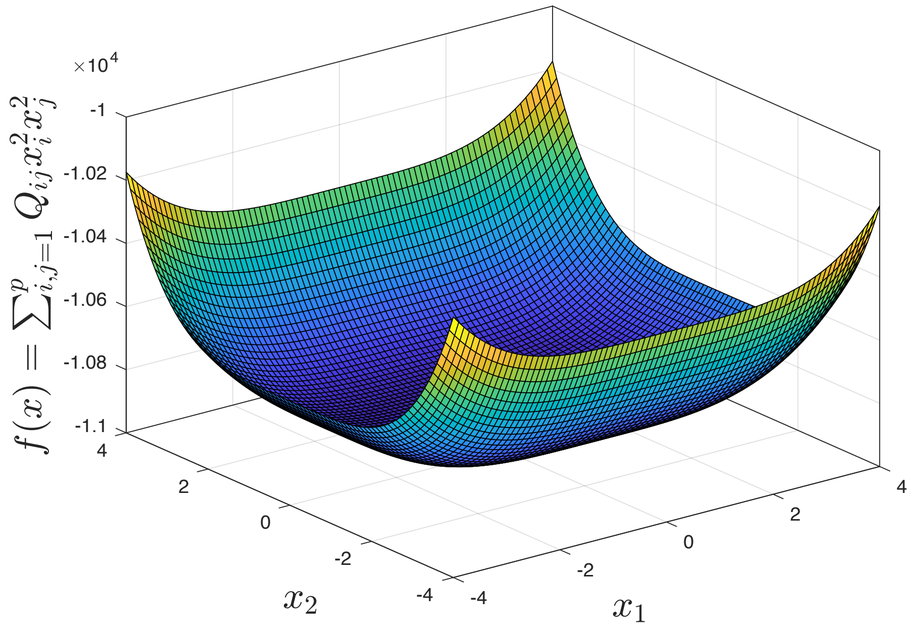

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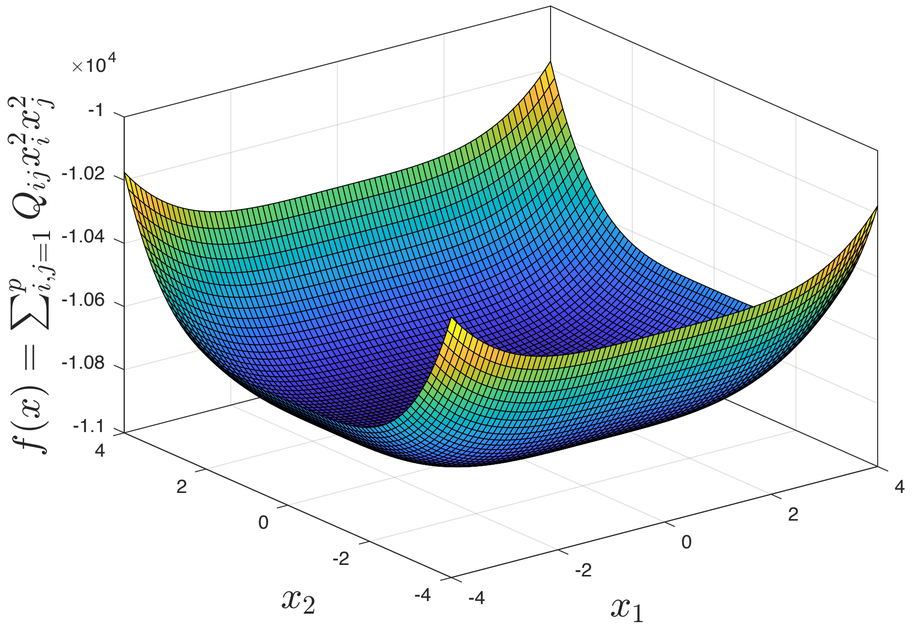

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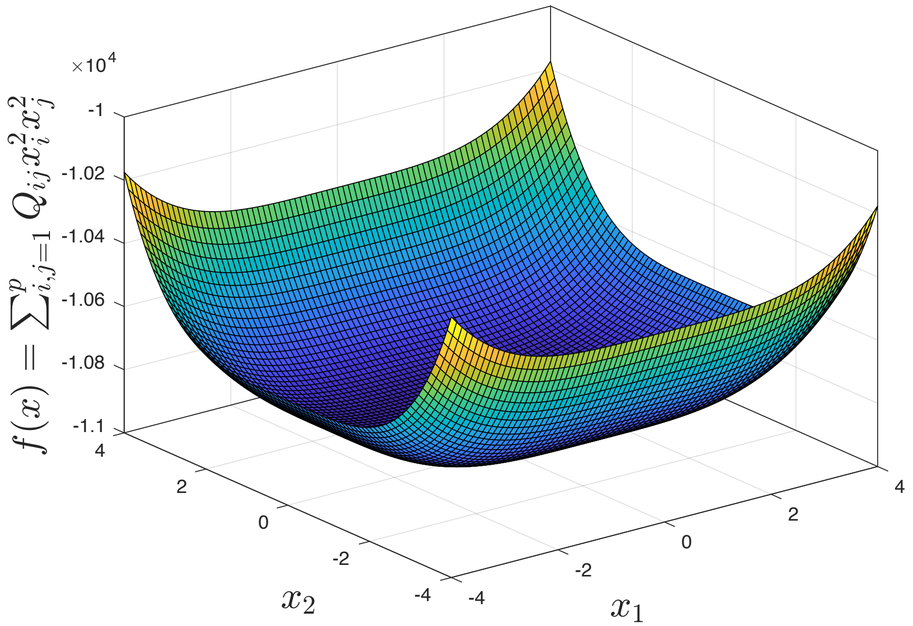

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  - Thus, $x = 0$ is global min.

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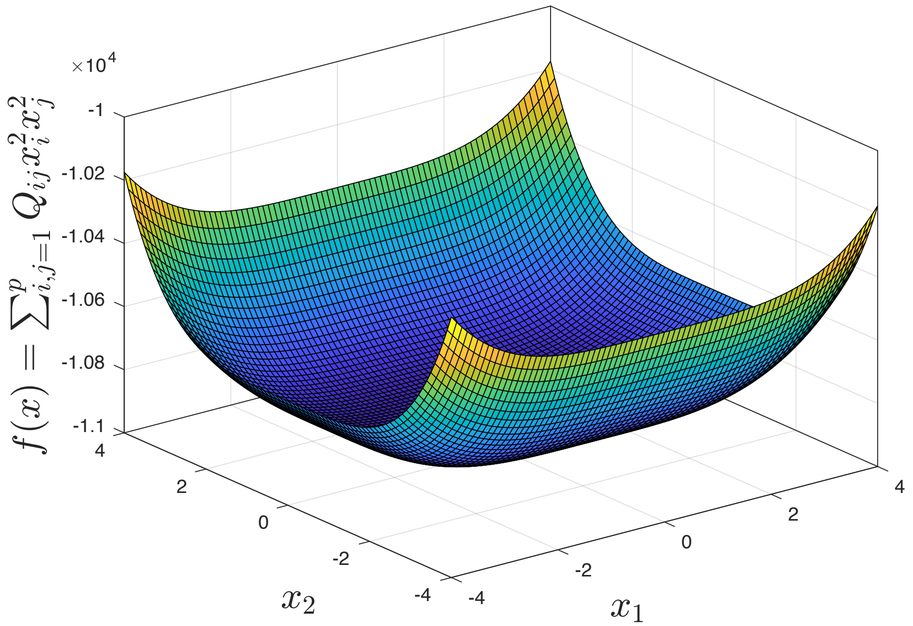

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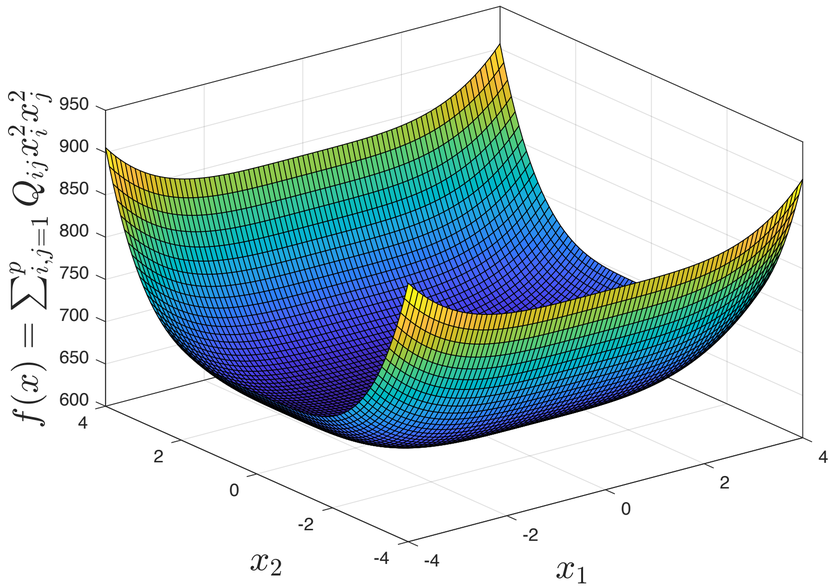

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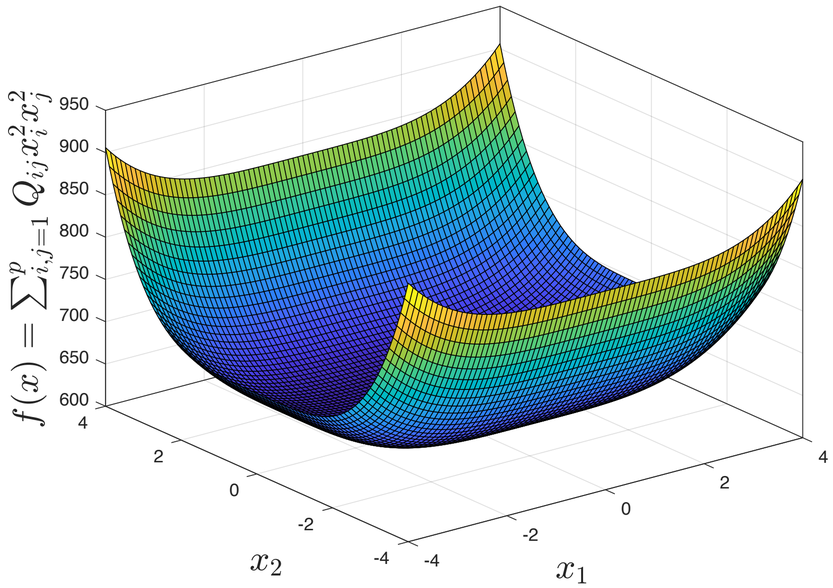

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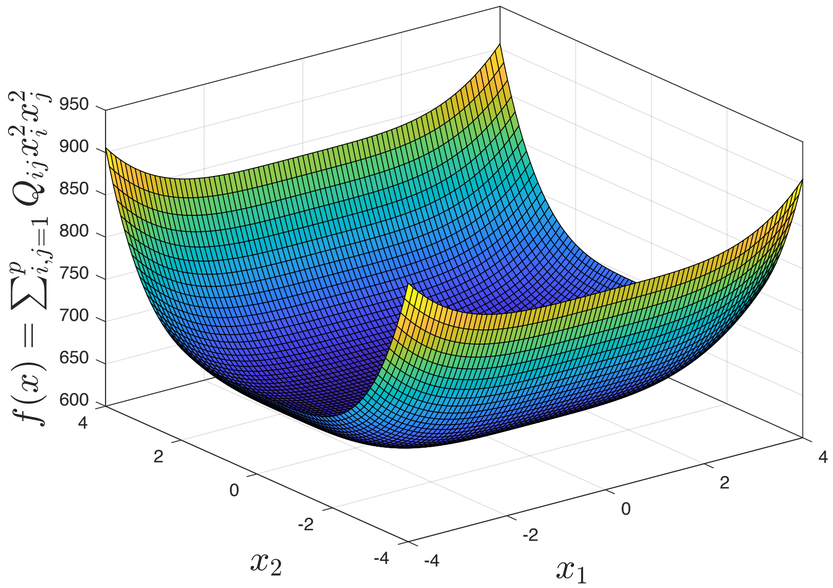

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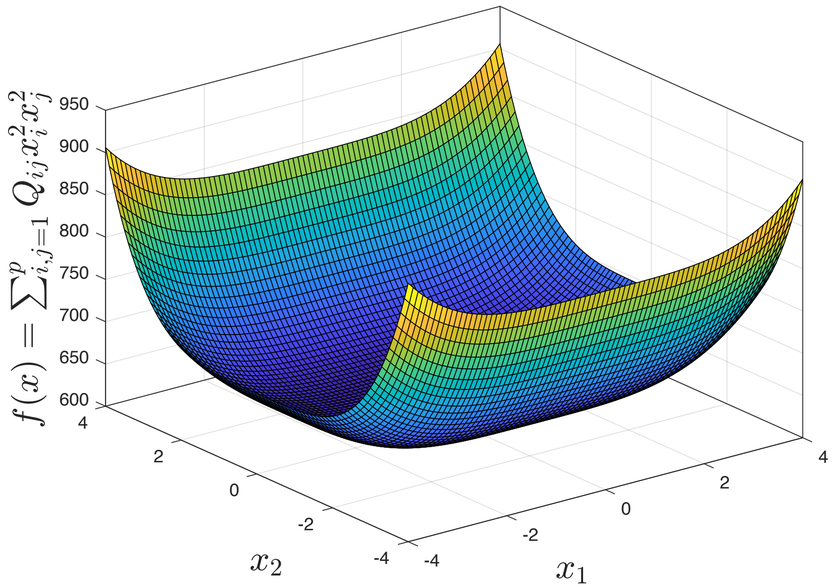

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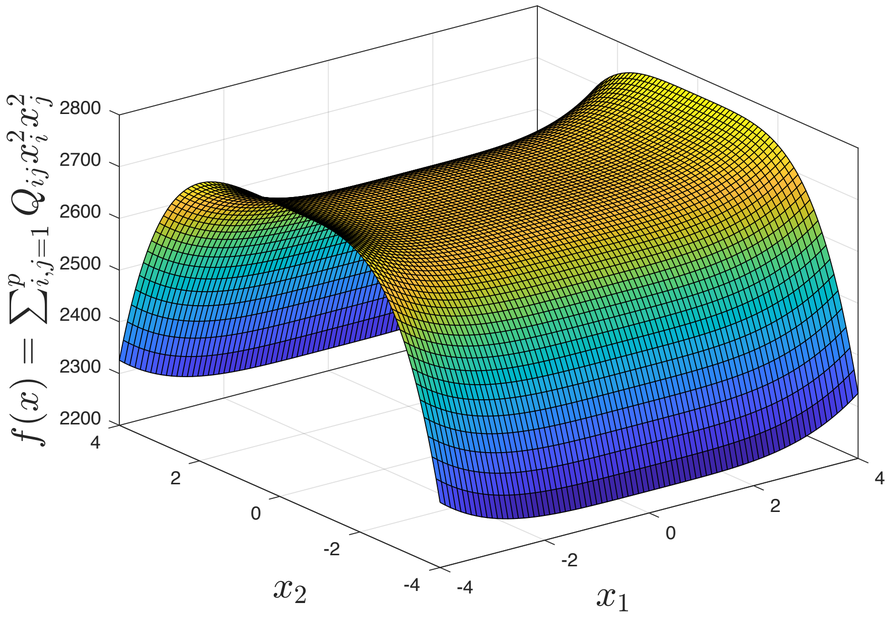

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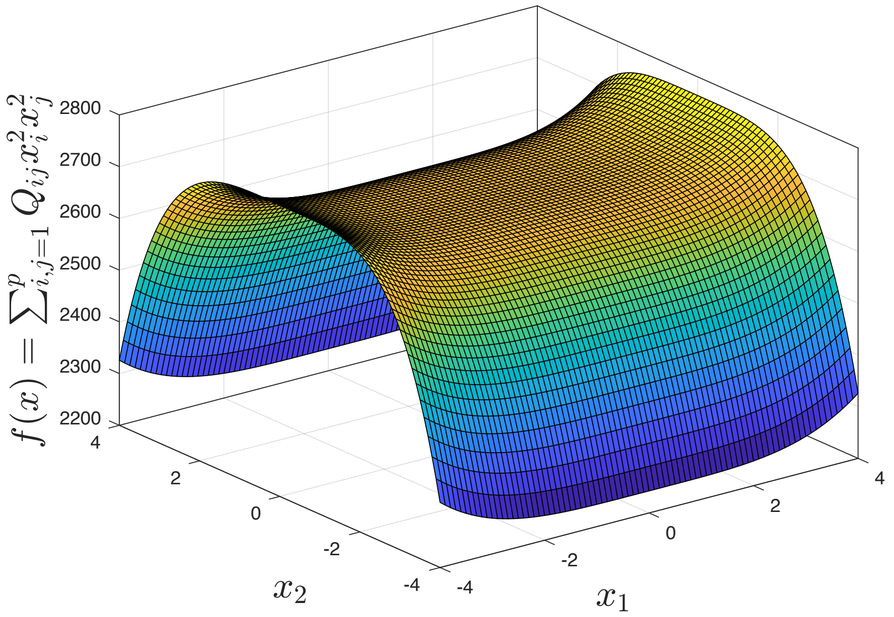

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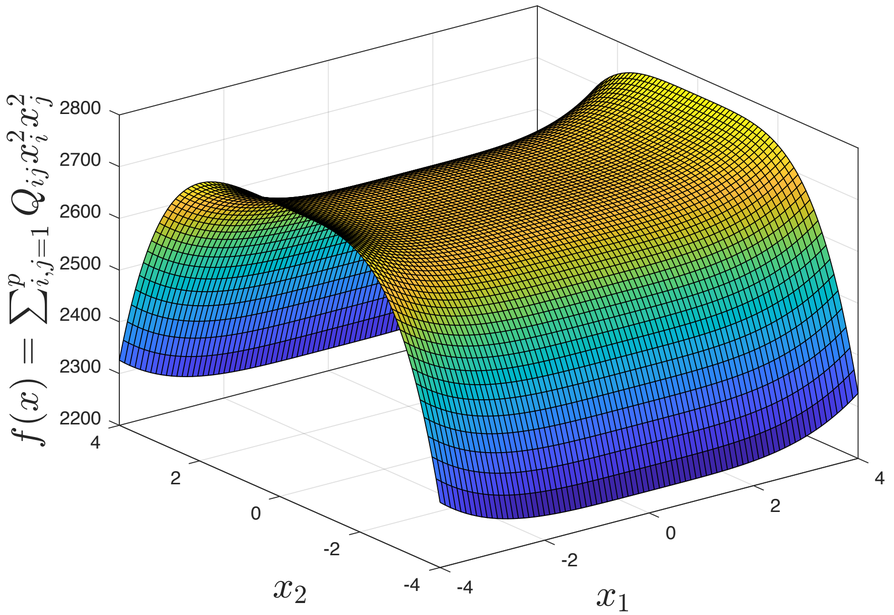

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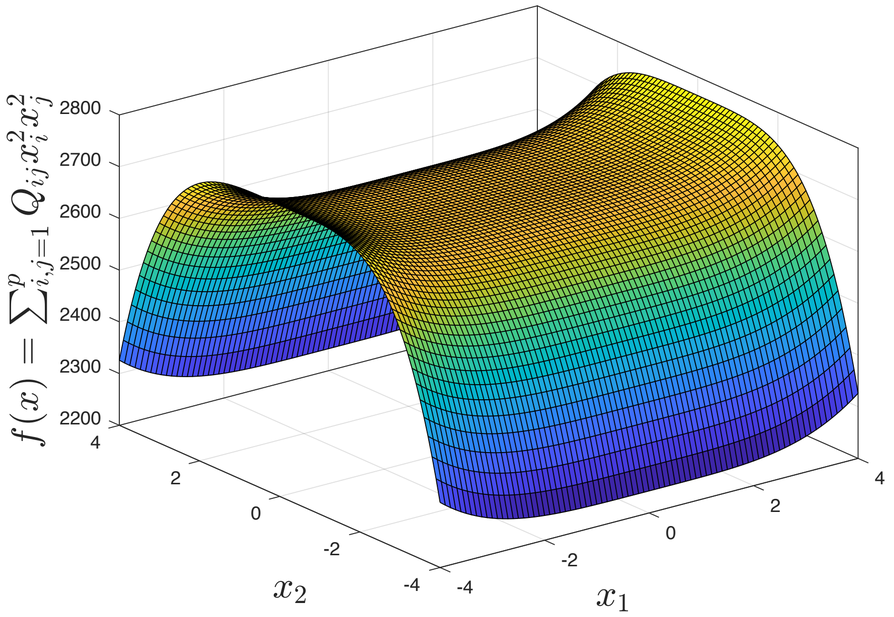

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  - Thus, zero is a minimum, maximum or saddle point

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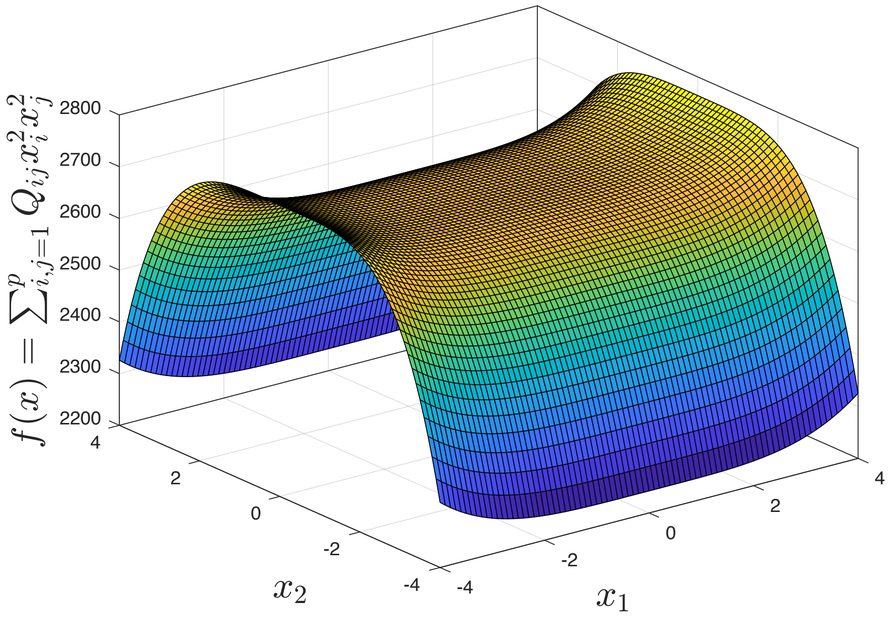

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- This is equivalent to checking if Q is not co-positive: NP-hard!

(equivalent to finding plant cliques in graphs)

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- Not the only example: QCQP, matrix completion/matrix sensing, etc.
  tensor (matrix) decompositions

Flash back: GD and types of critical points

(also called stationary points)

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As t increases, the objective function decreases,
Until the point where the gradient has close to zero energy

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- No guarantees on the type of critical point we converge to

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and $f(x^\star) \leq f(x),\ \forall x$ $(f(x^\star) \geq f(x),\ \forall x)$

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- Saddle points:

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- Global minima/global maxima: ← Desirable but not easily attainable!

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- "The loss surfaces of multilinear networks", Chromanska et al., 2014

> *"We conjecture that both simulated annealing and SGD converge to the band of low critical points, and that all critical points found there are local minima of high quality measured by the test error. This emphasizes a major difference between large- and small-size networks where for the latter poor quality local minima have non-zero probability of being recovered."*

(Not the only work on this subject)

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- Why would this be true in practice?
  - Different random seeds lead to different models with similar performance

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- Saddle points:

Often stall the convergence to a better point

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– "Identifying and attacking the saddle point problem in high-dimensional non-convex optimization.", Dauphin et al., 2014

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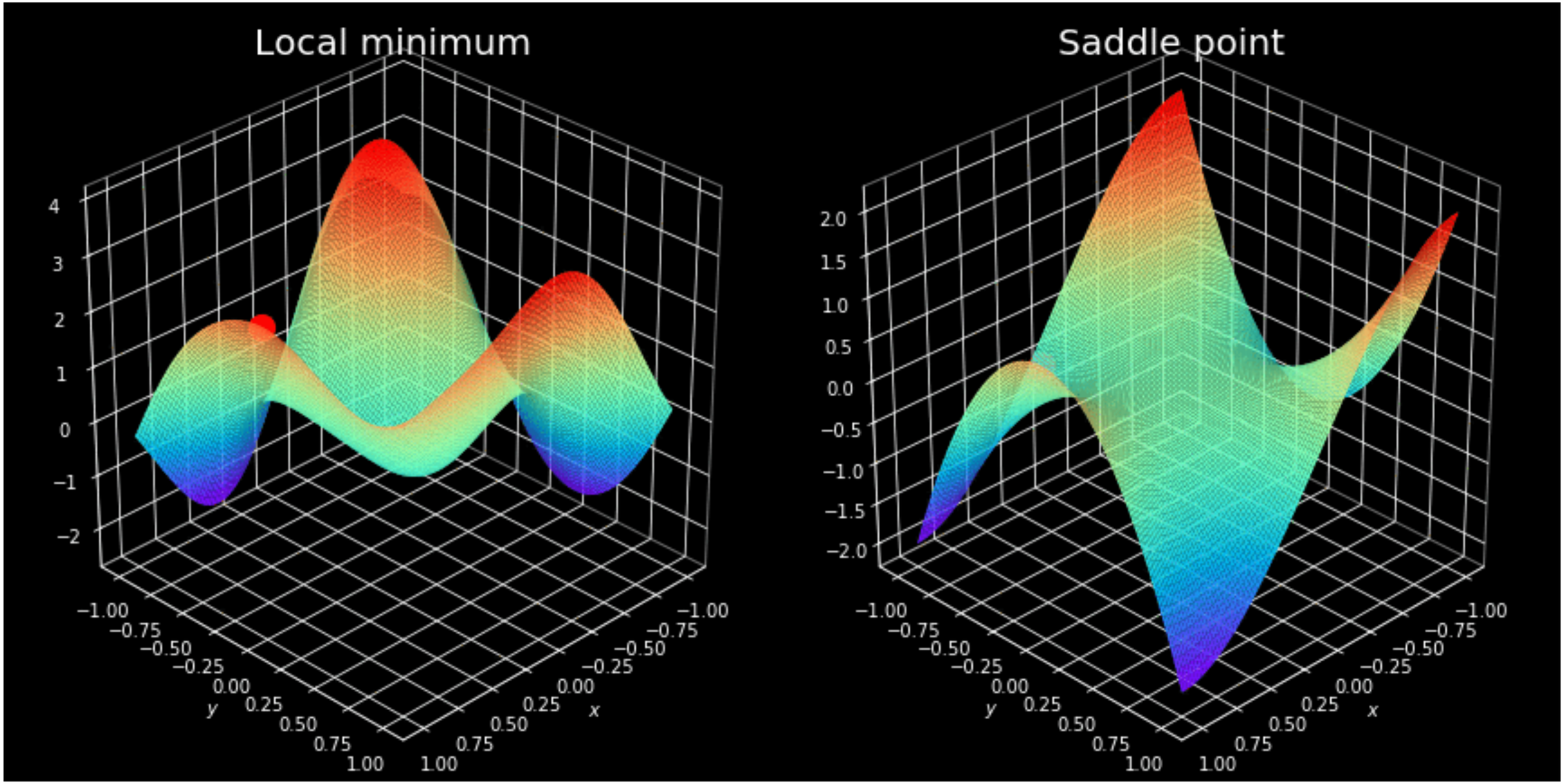

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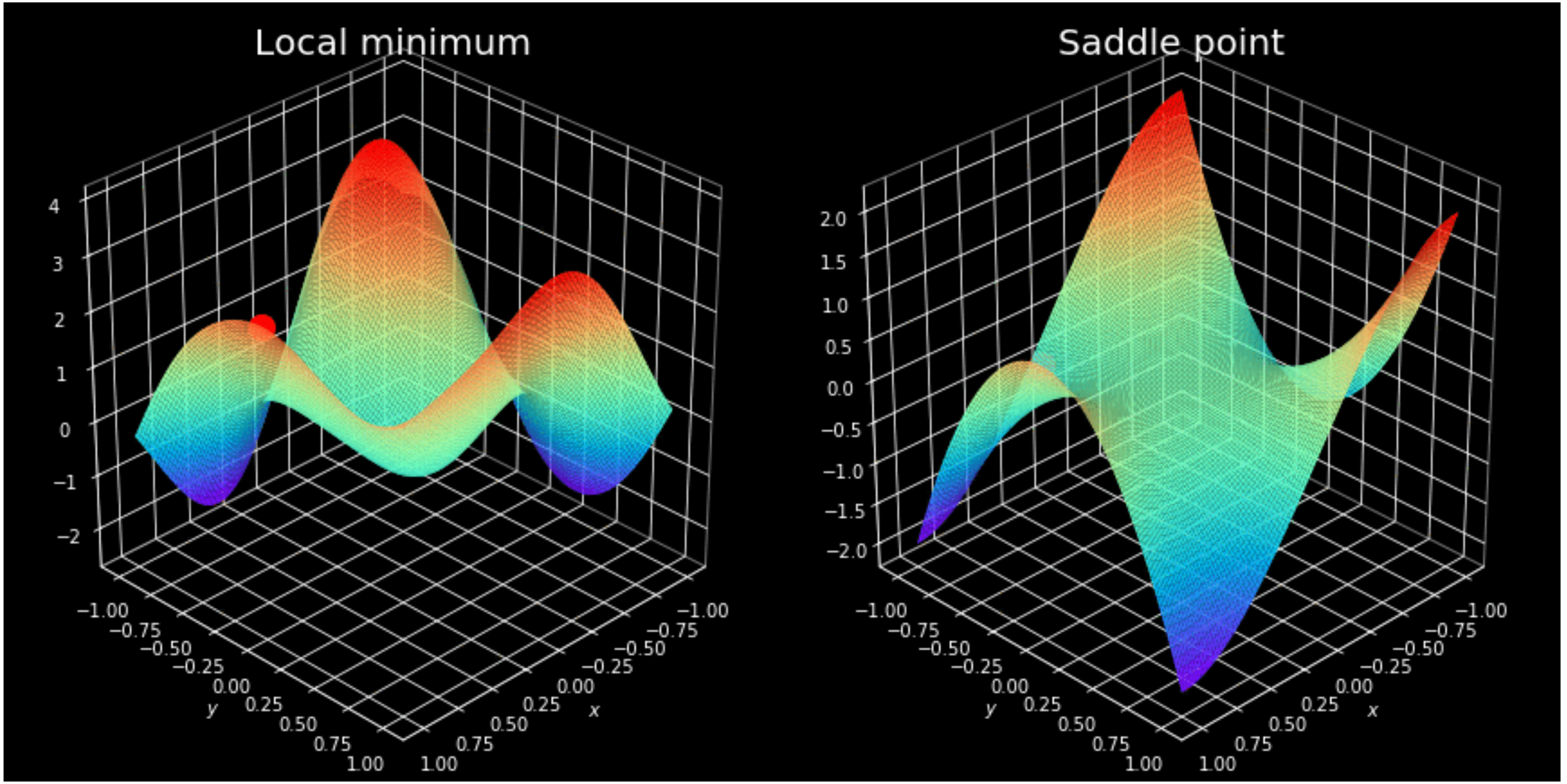

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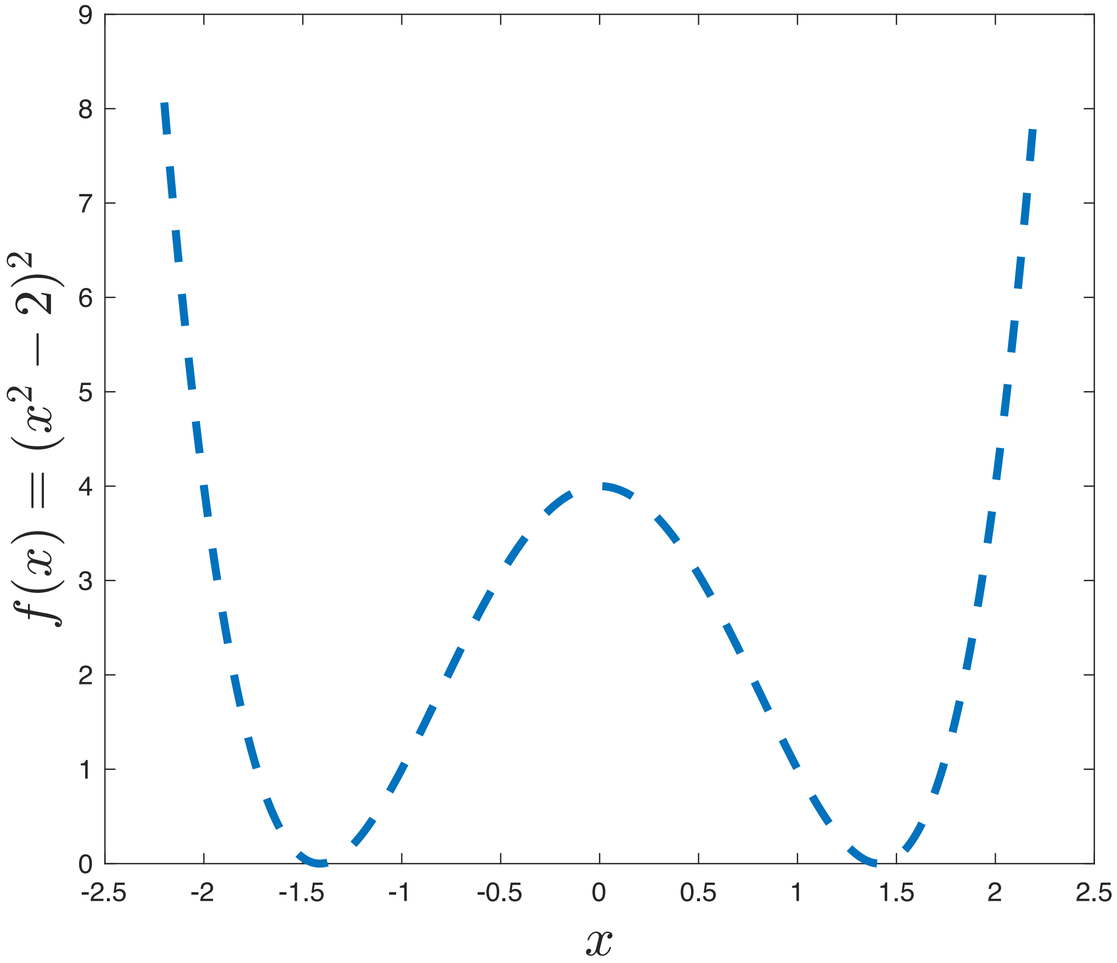
How many saddle points could be there?

- Toy example #1: $f(x) = (x^2 - 2)^2$



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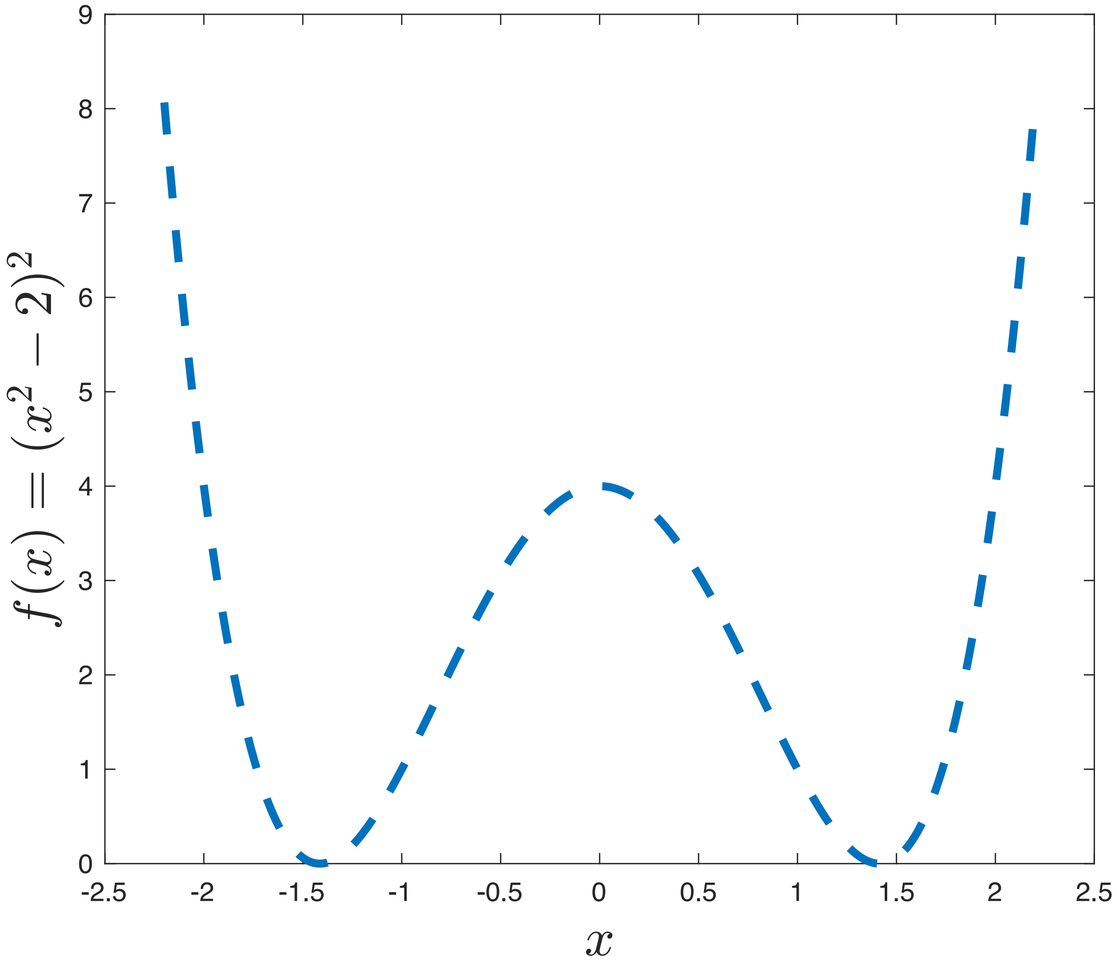

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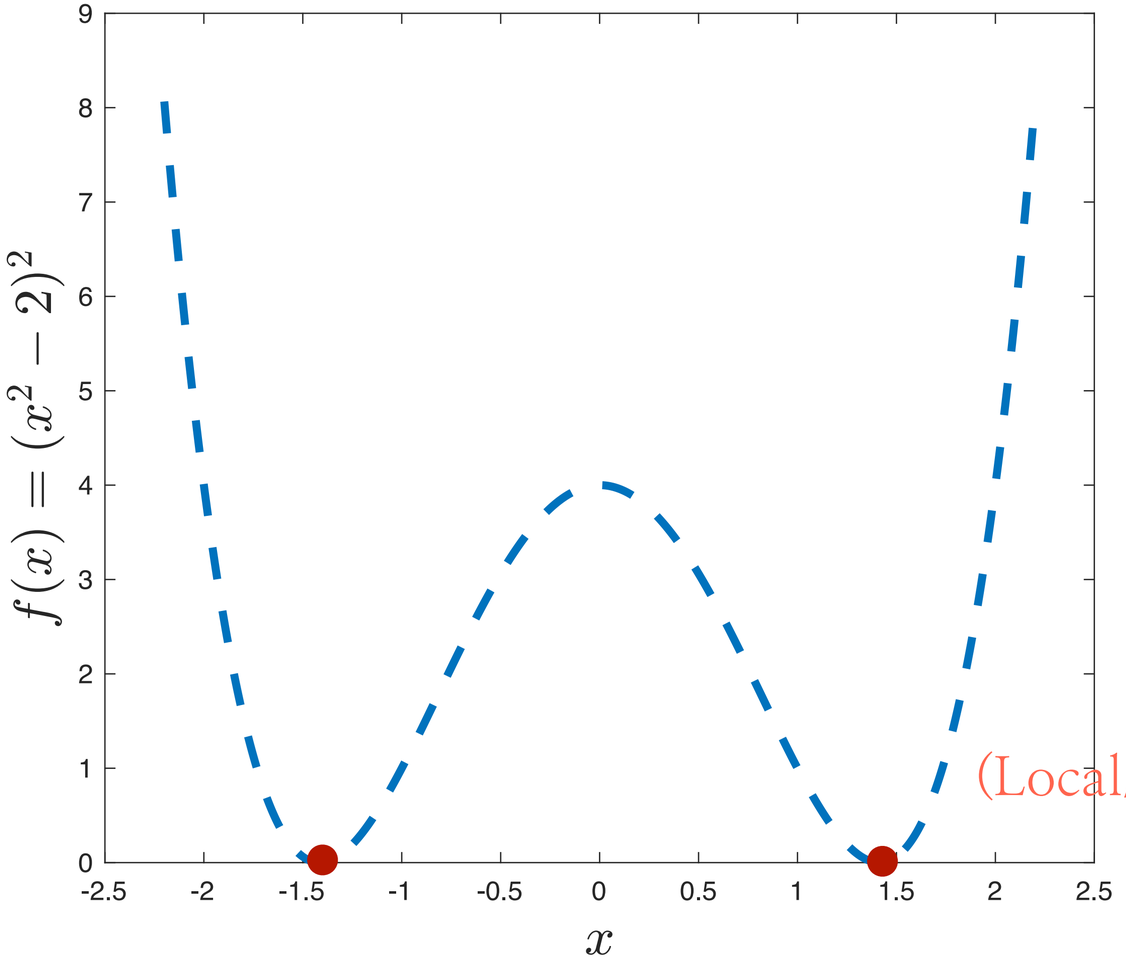

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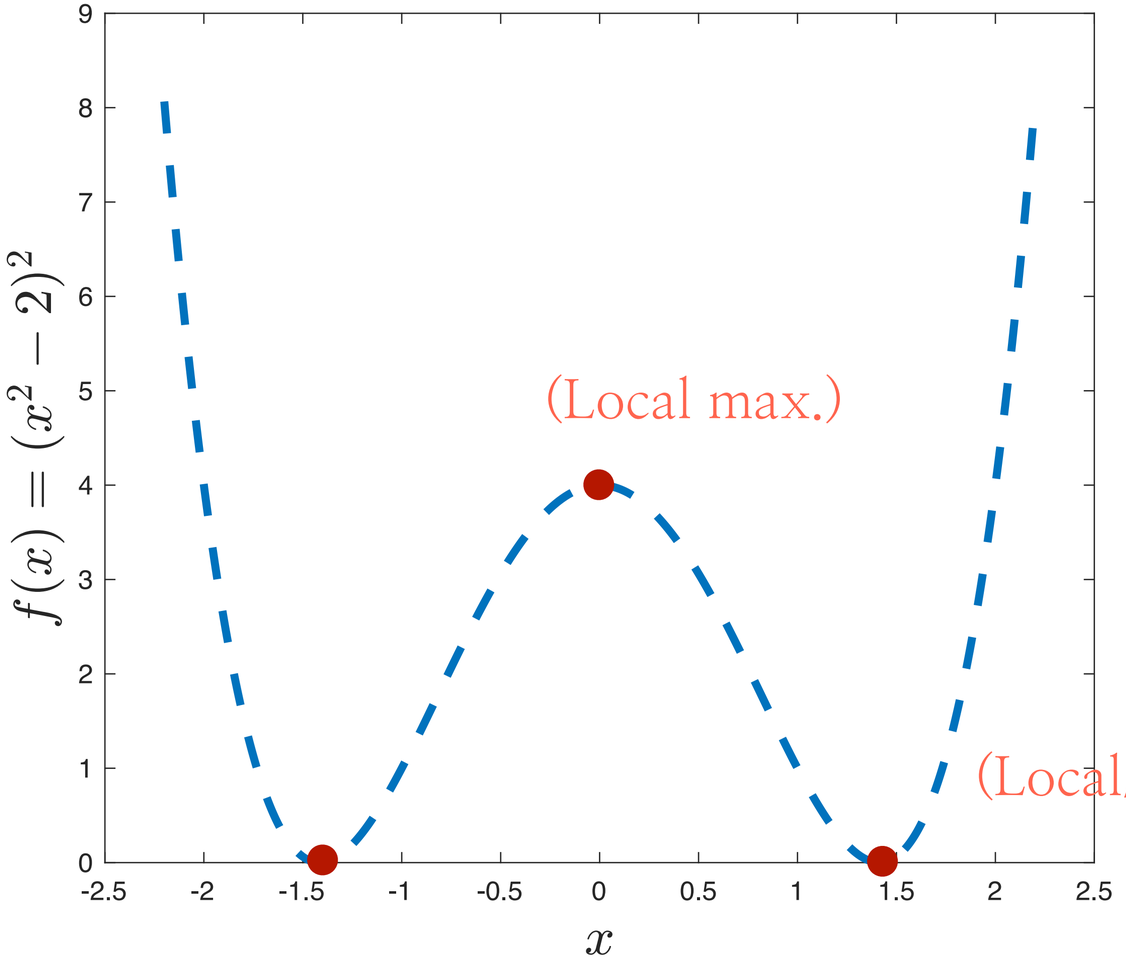

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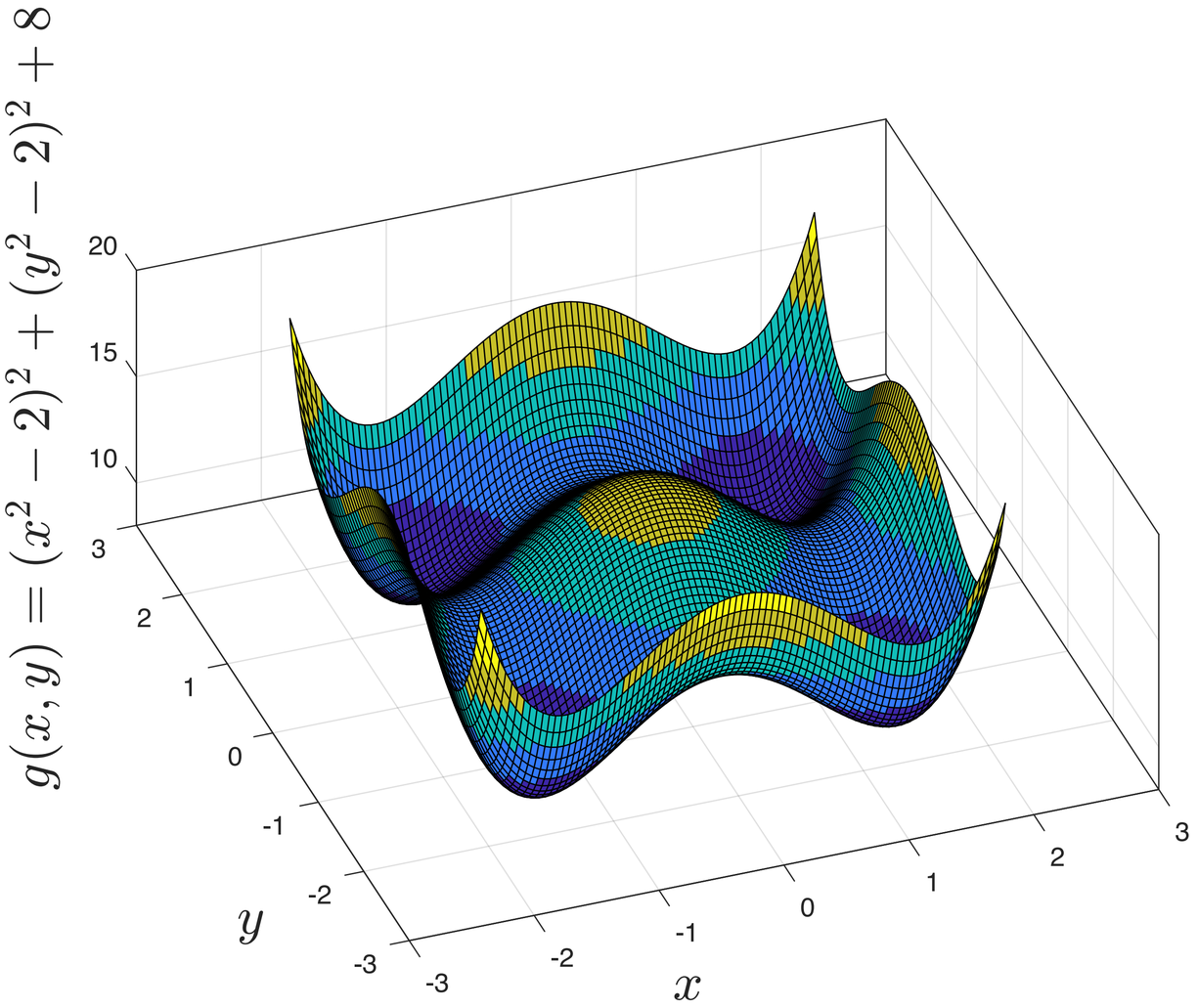

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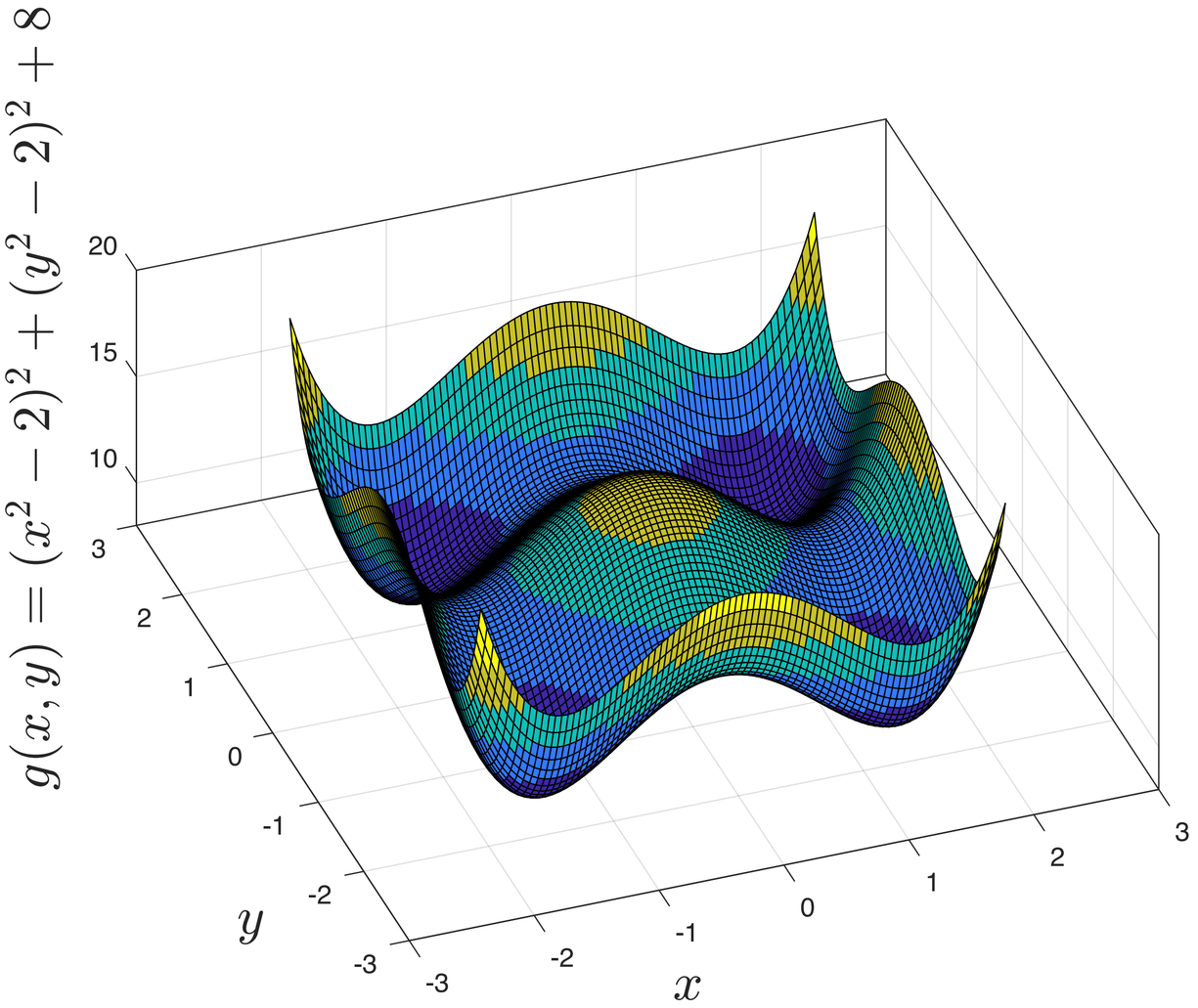

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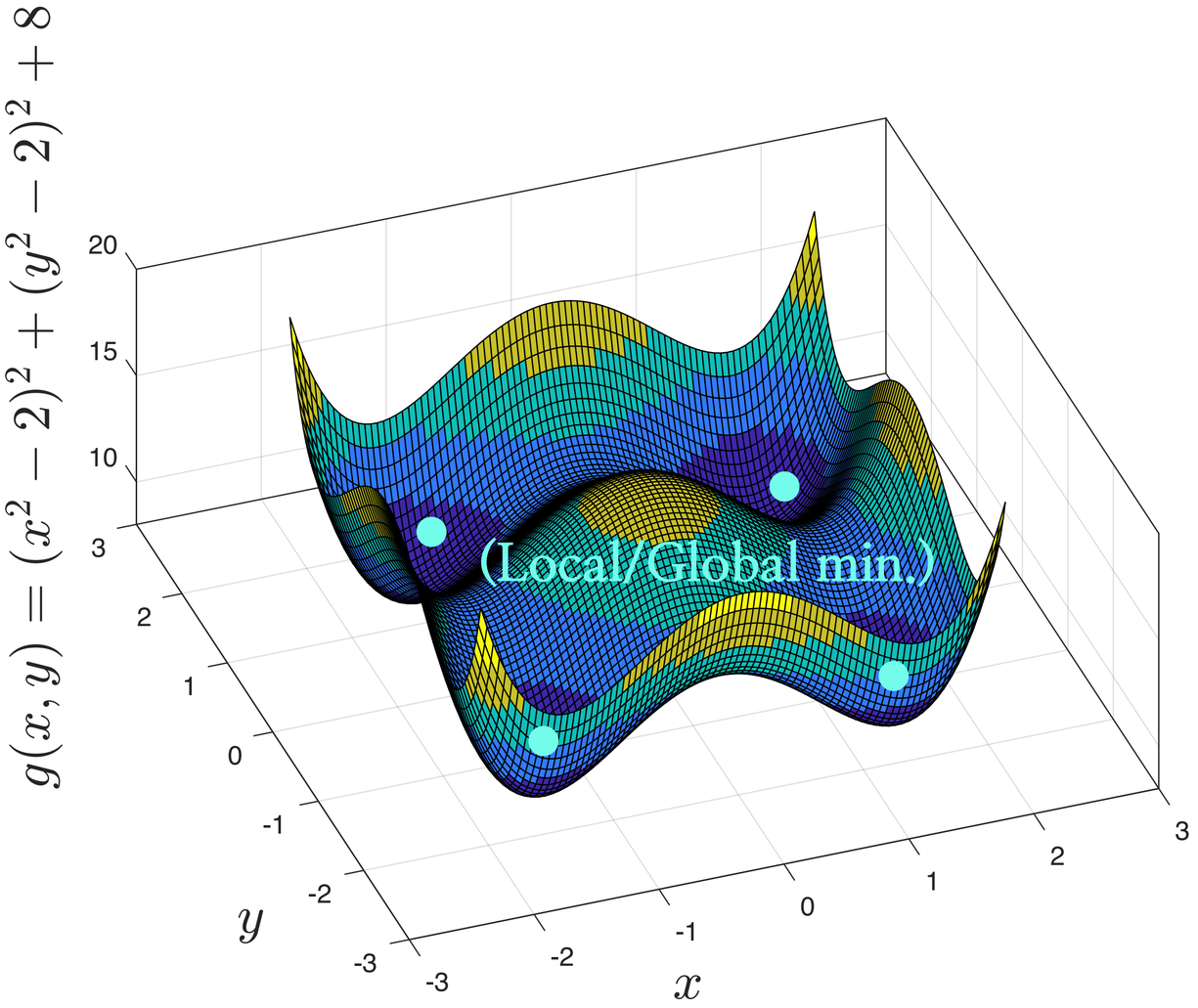

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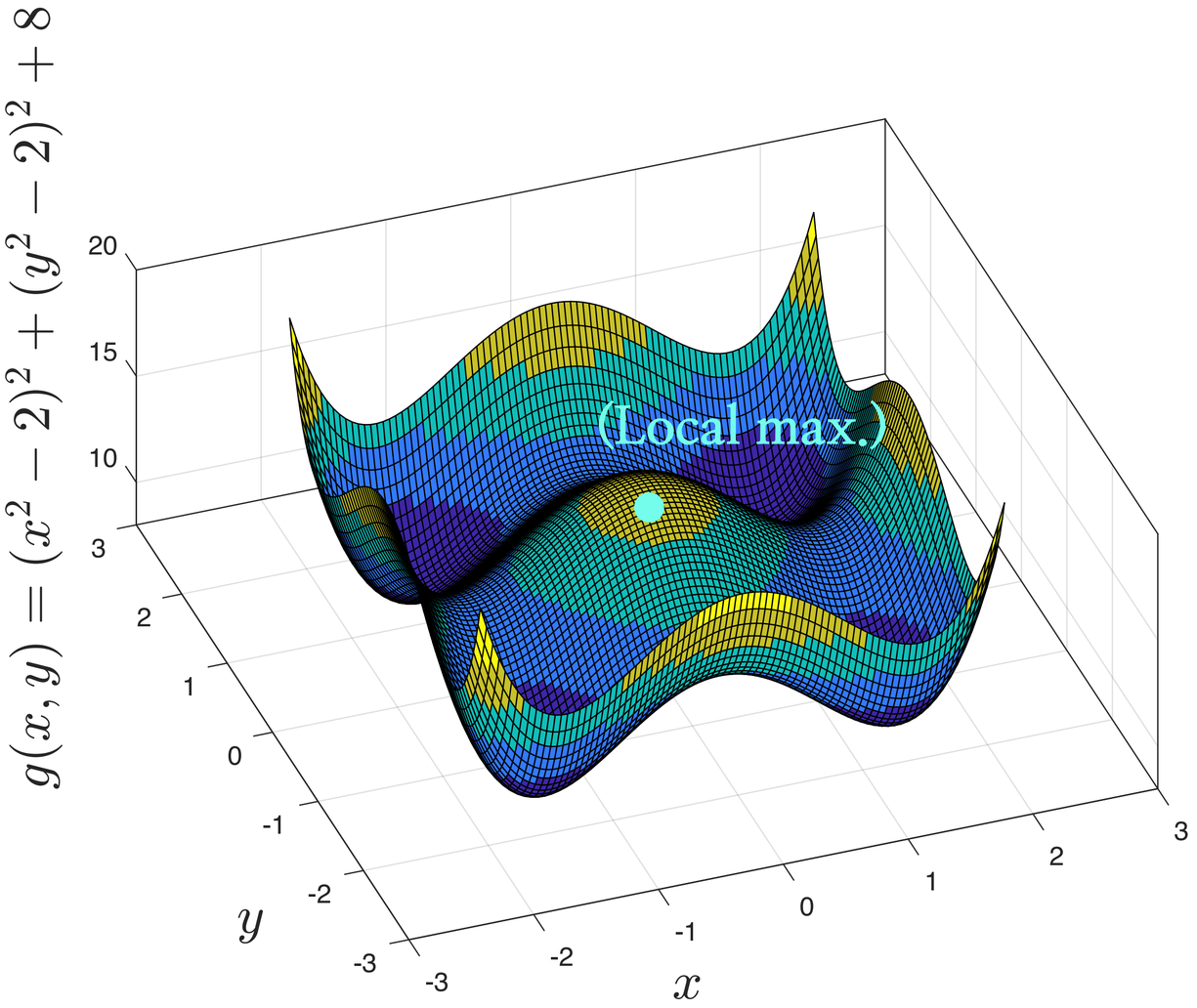

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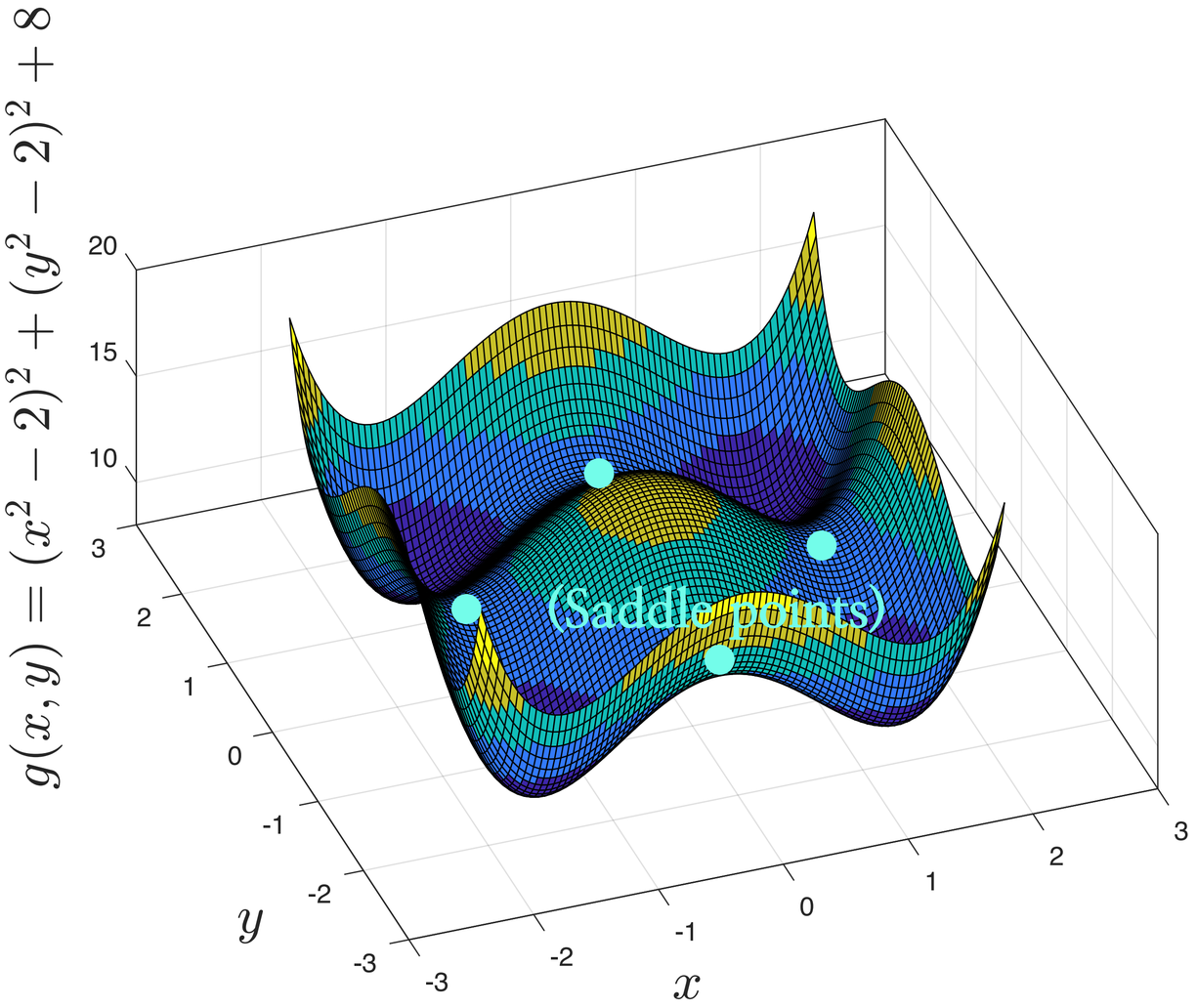

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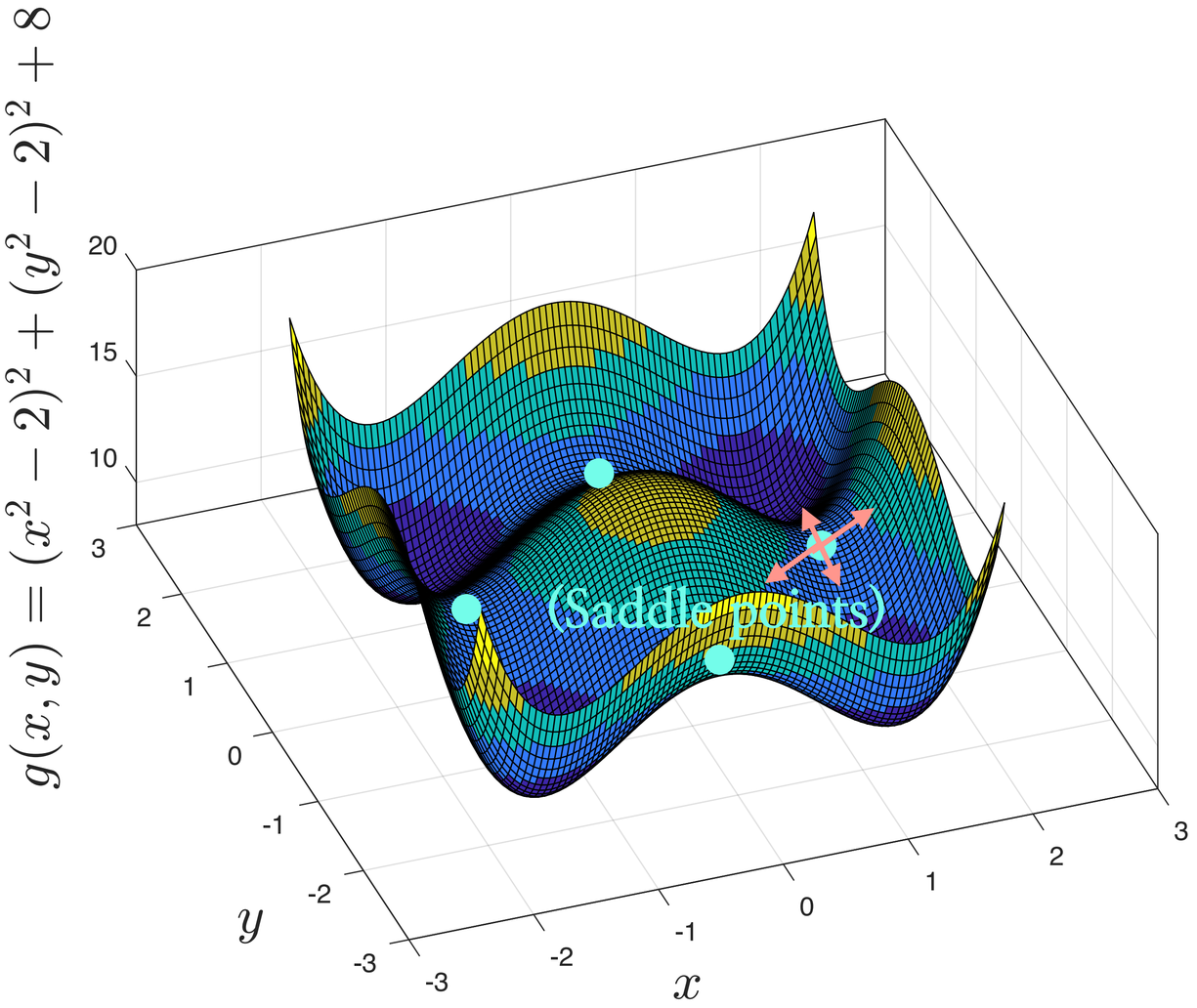

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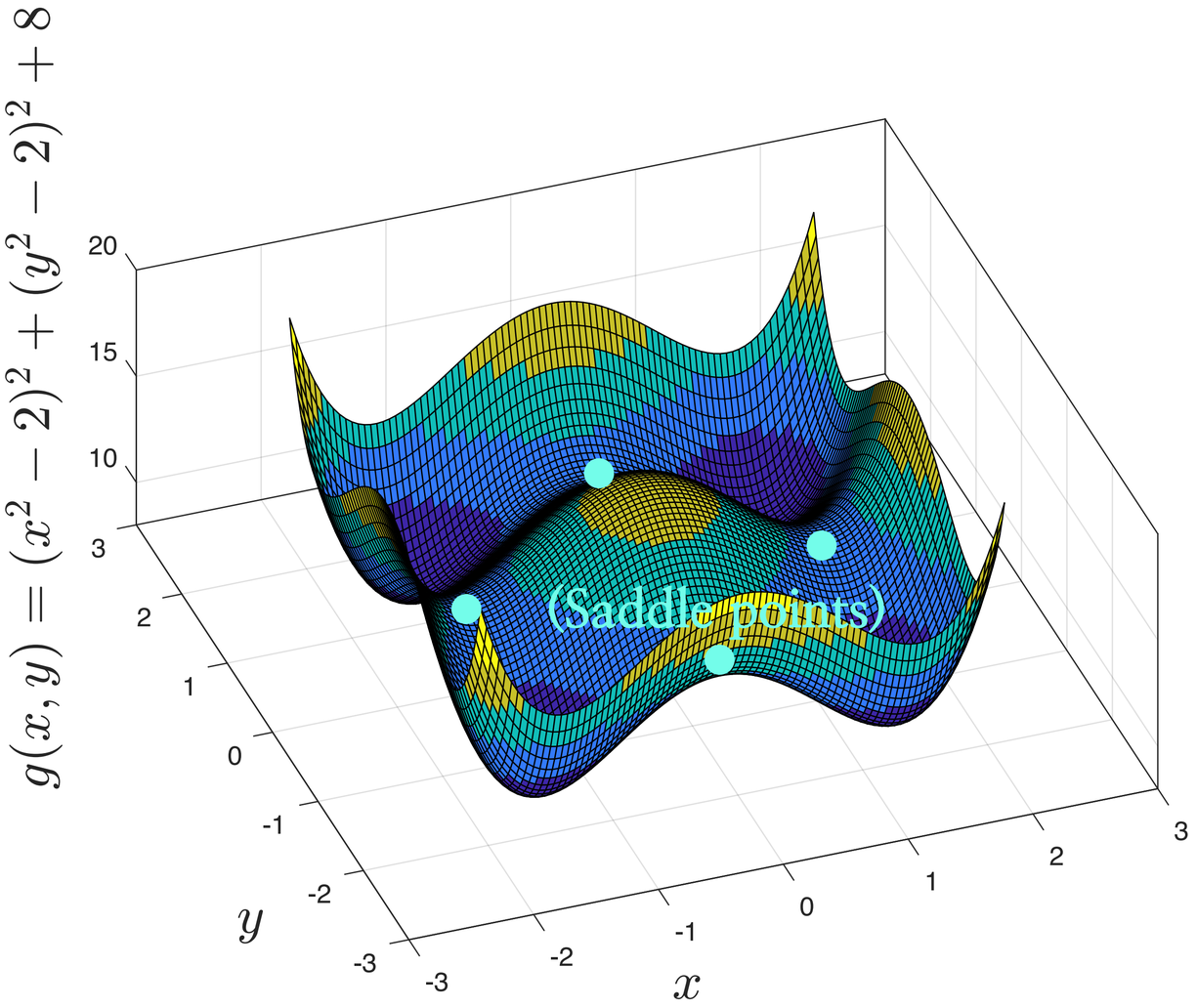

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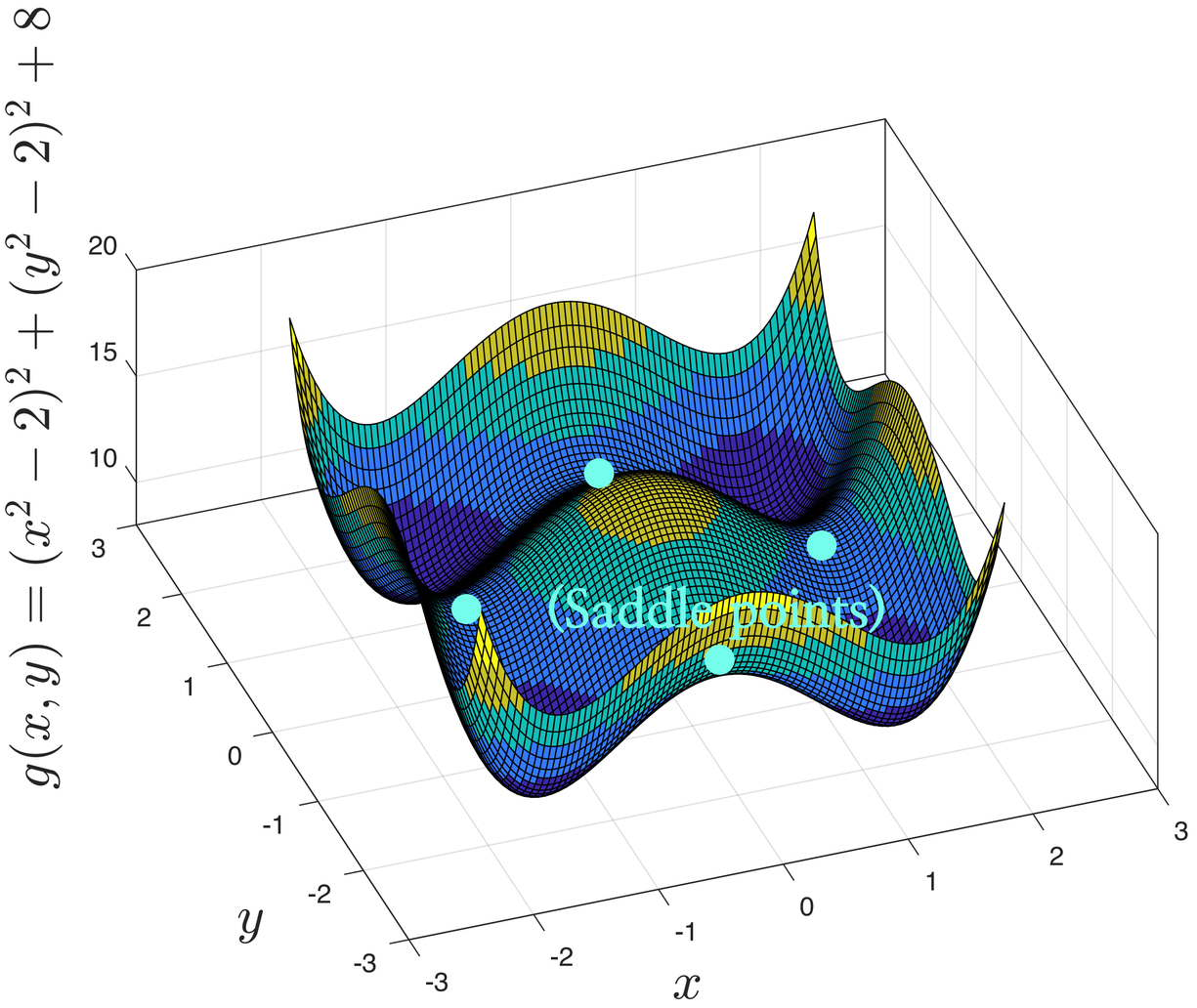
- "Does it generalize?"

Yes! From 2D to 3D, we get 4 local minima, and 8 saddle points!



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2. How does methods such as gradient descent behave in practice?

3. Are there conditions that indicate that always there is a way-out of saddle points?

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By second-order Taylor's expansion:

$$f(x + \eta u) \approx f(x) + \frac{\eta^2}{2} \left\langle \nabla^2 f(x) u, u \right\rangle > f(x) \qquad \text{(Local min.)}$$

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4. **Positive, negative and zero eigenvalues: general saddle point**

(Does this ring a bell?)

What can we hope for at saddle points?

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- Thus, we need to characterize the # of steps we might require to escape

Strict saddle property and functions

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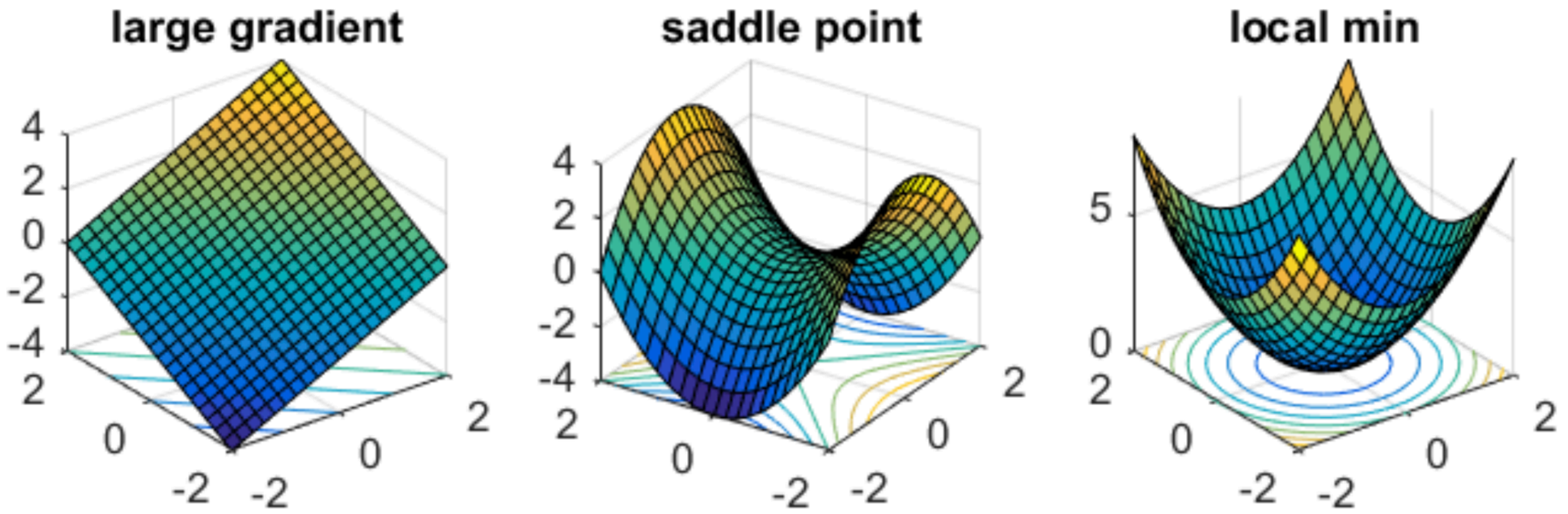

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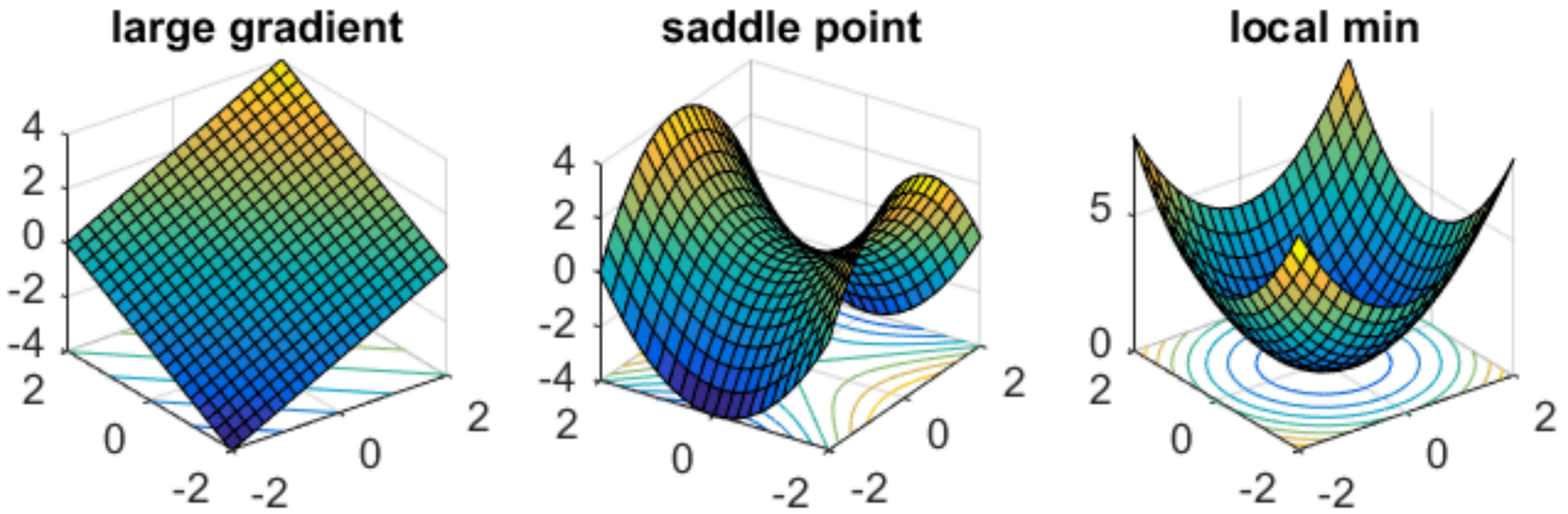

– Tensor decomposition, dictionary learning, phase retrieval, matrix sensing..

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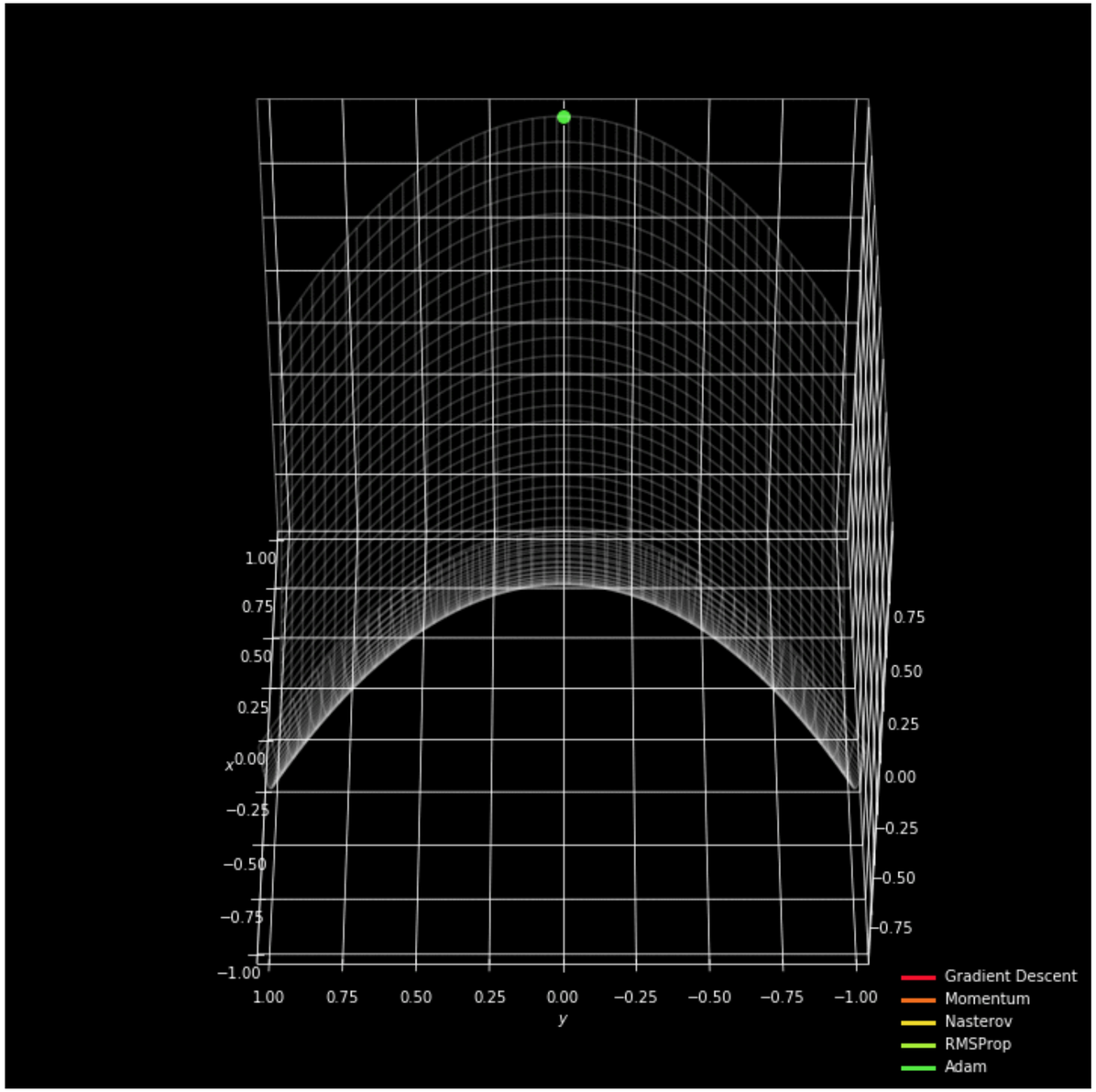

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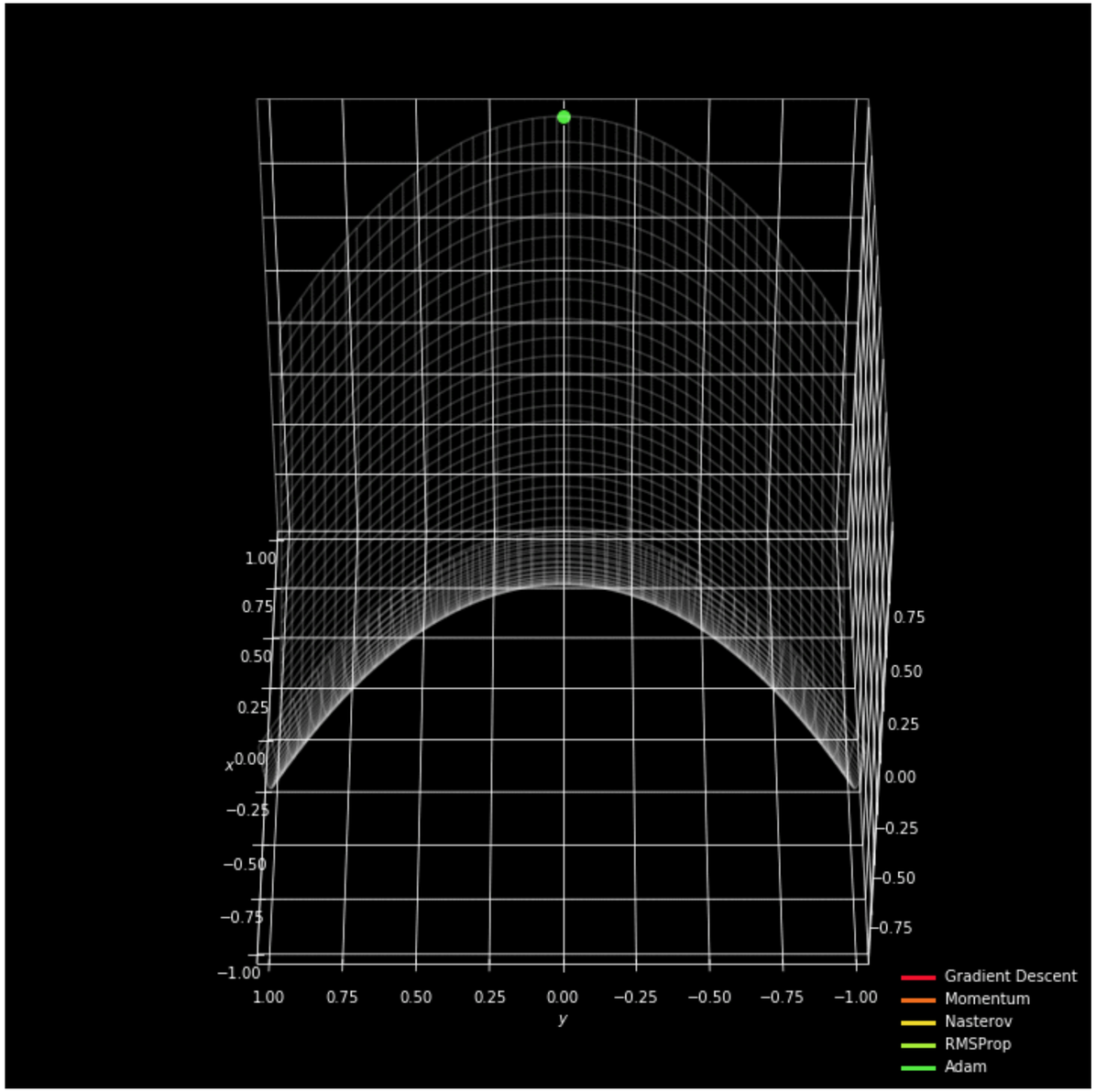

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(Noisy gradient descent)

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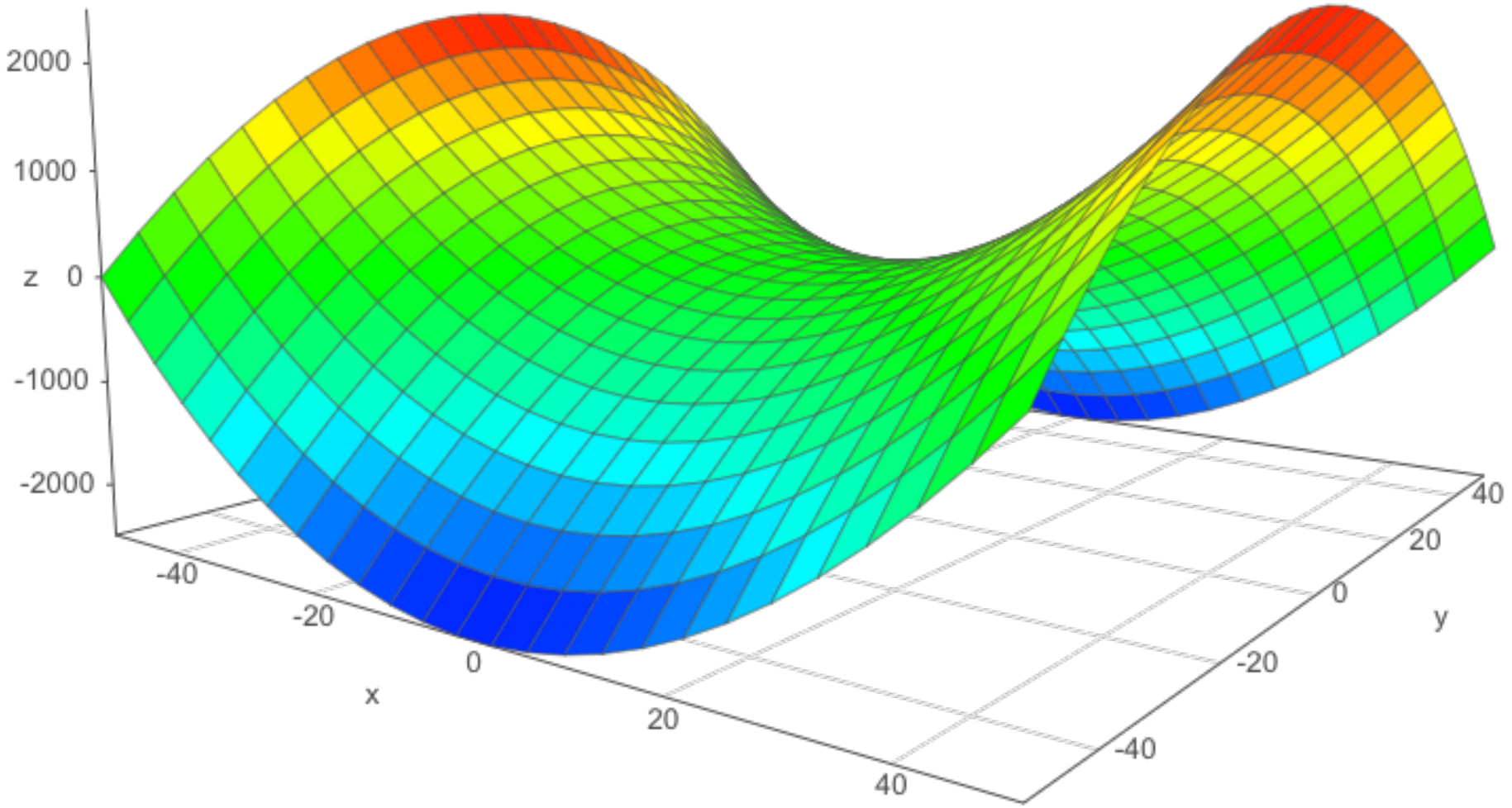

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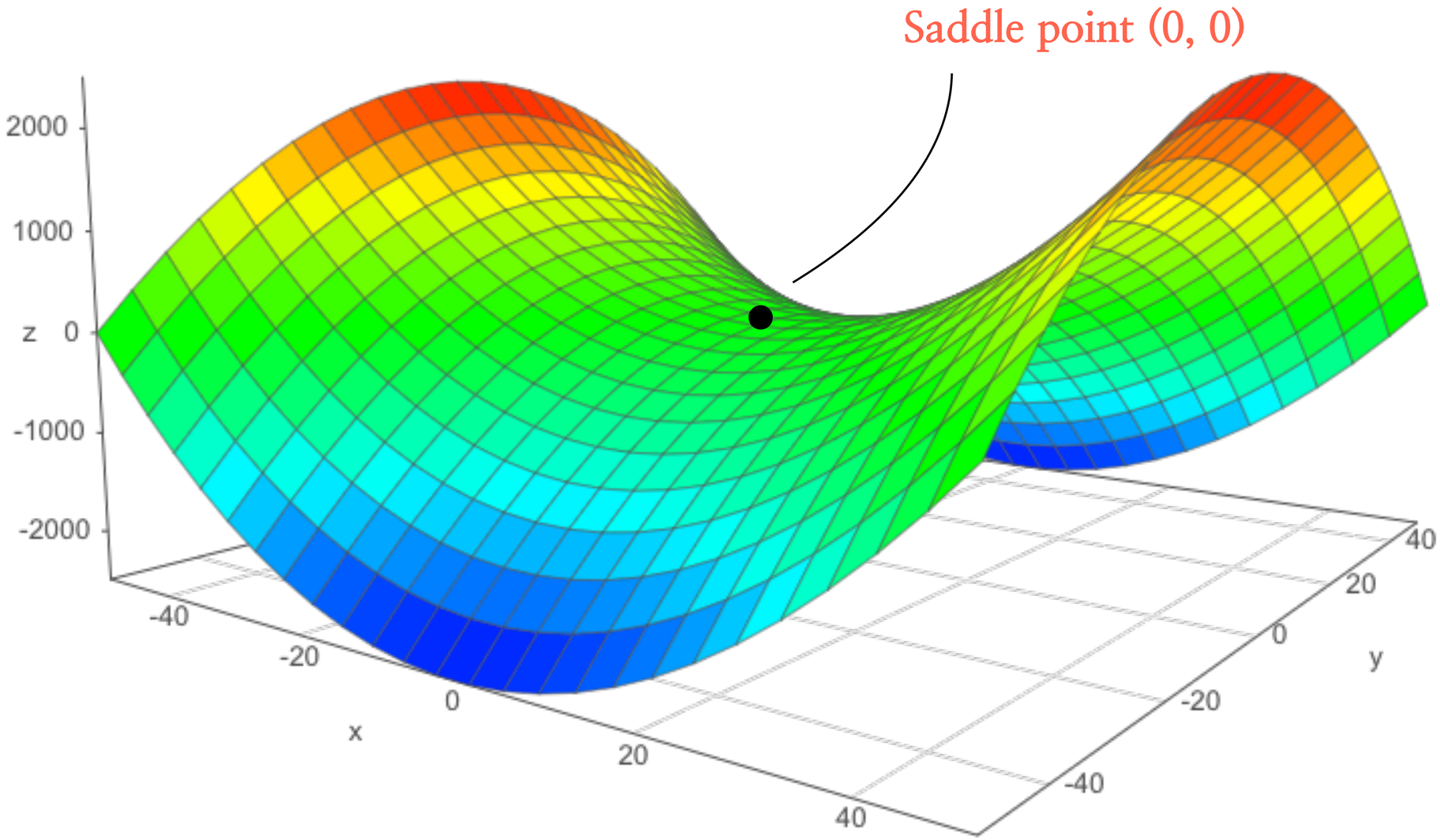

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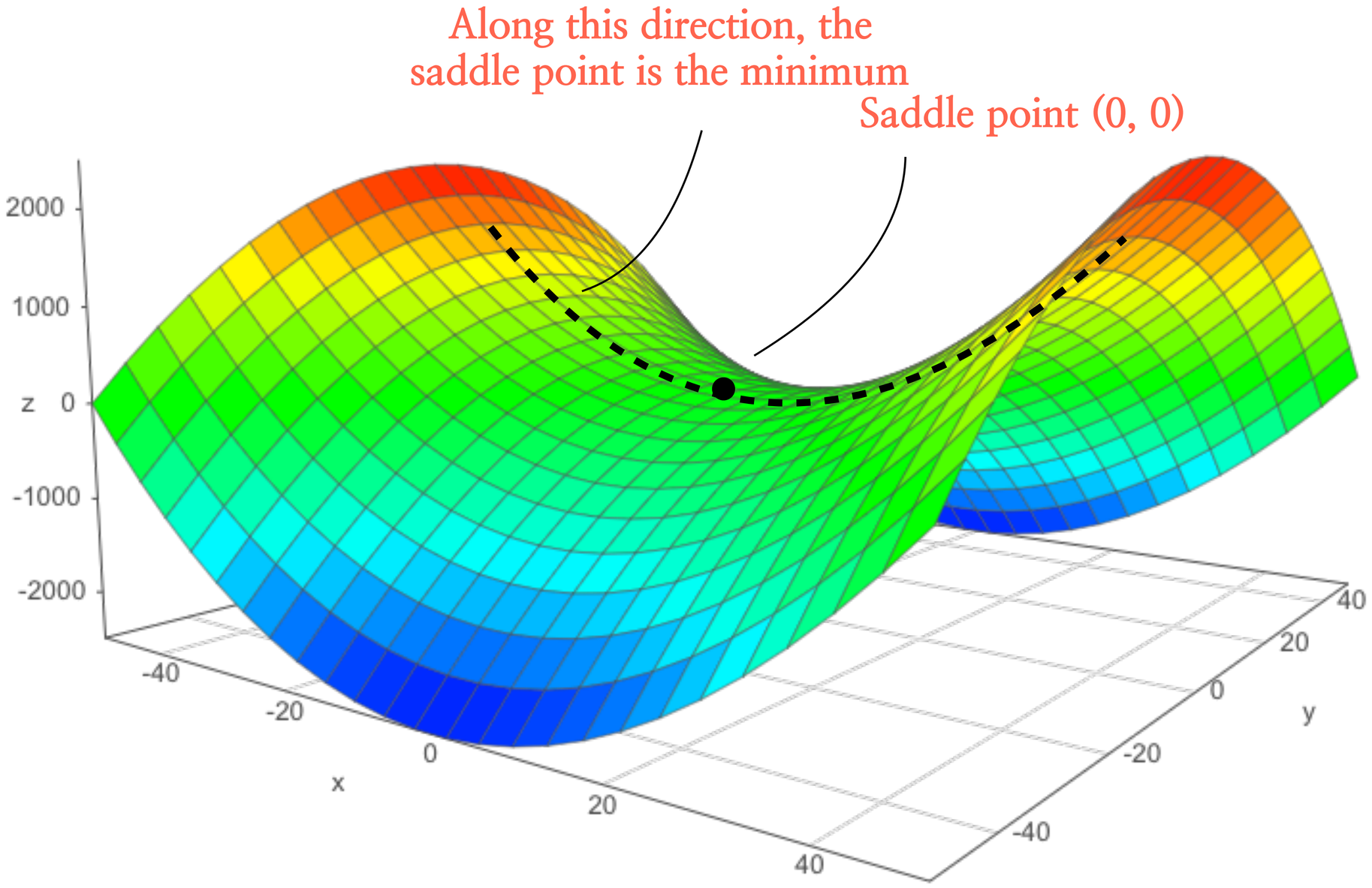

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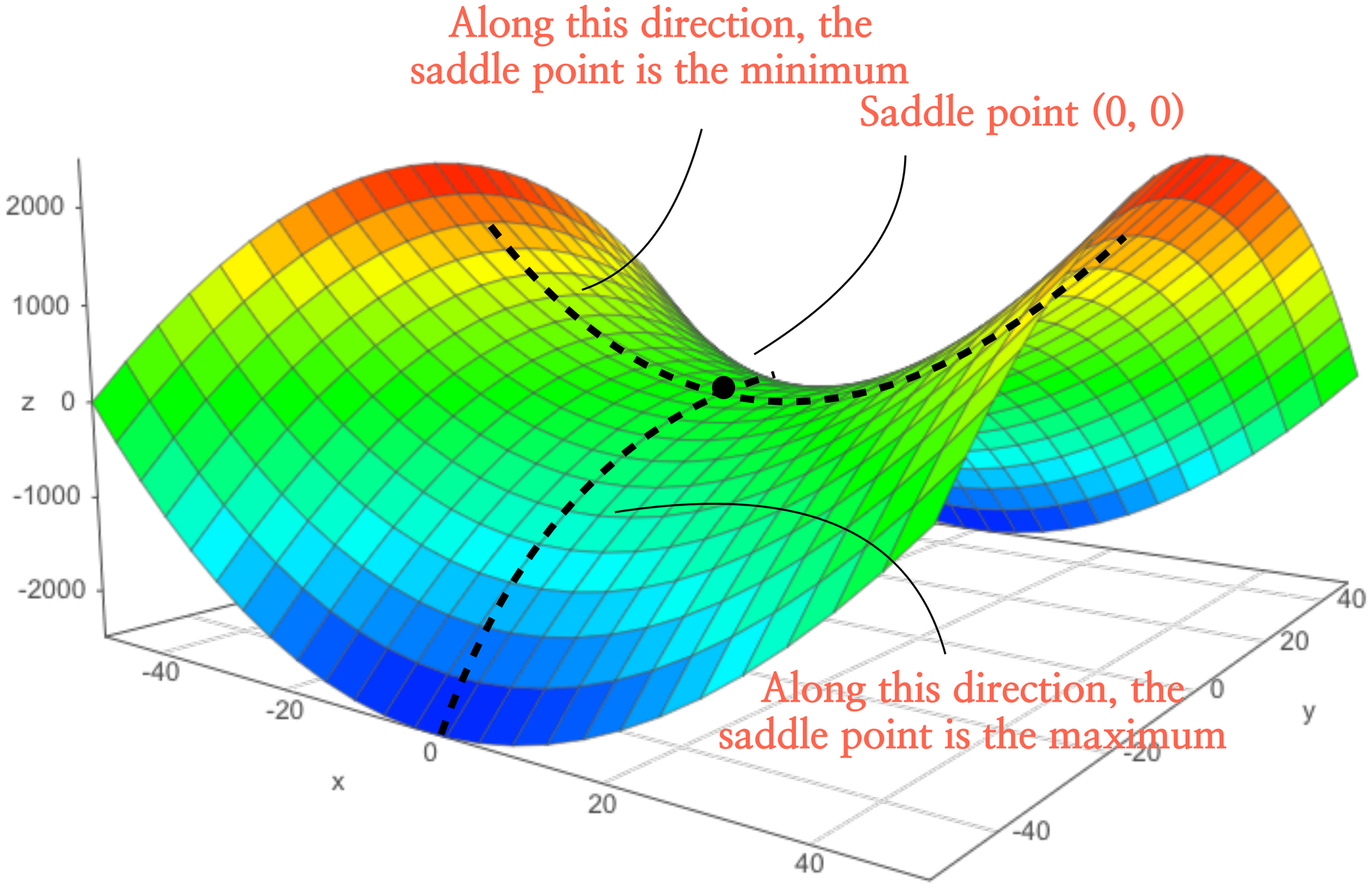

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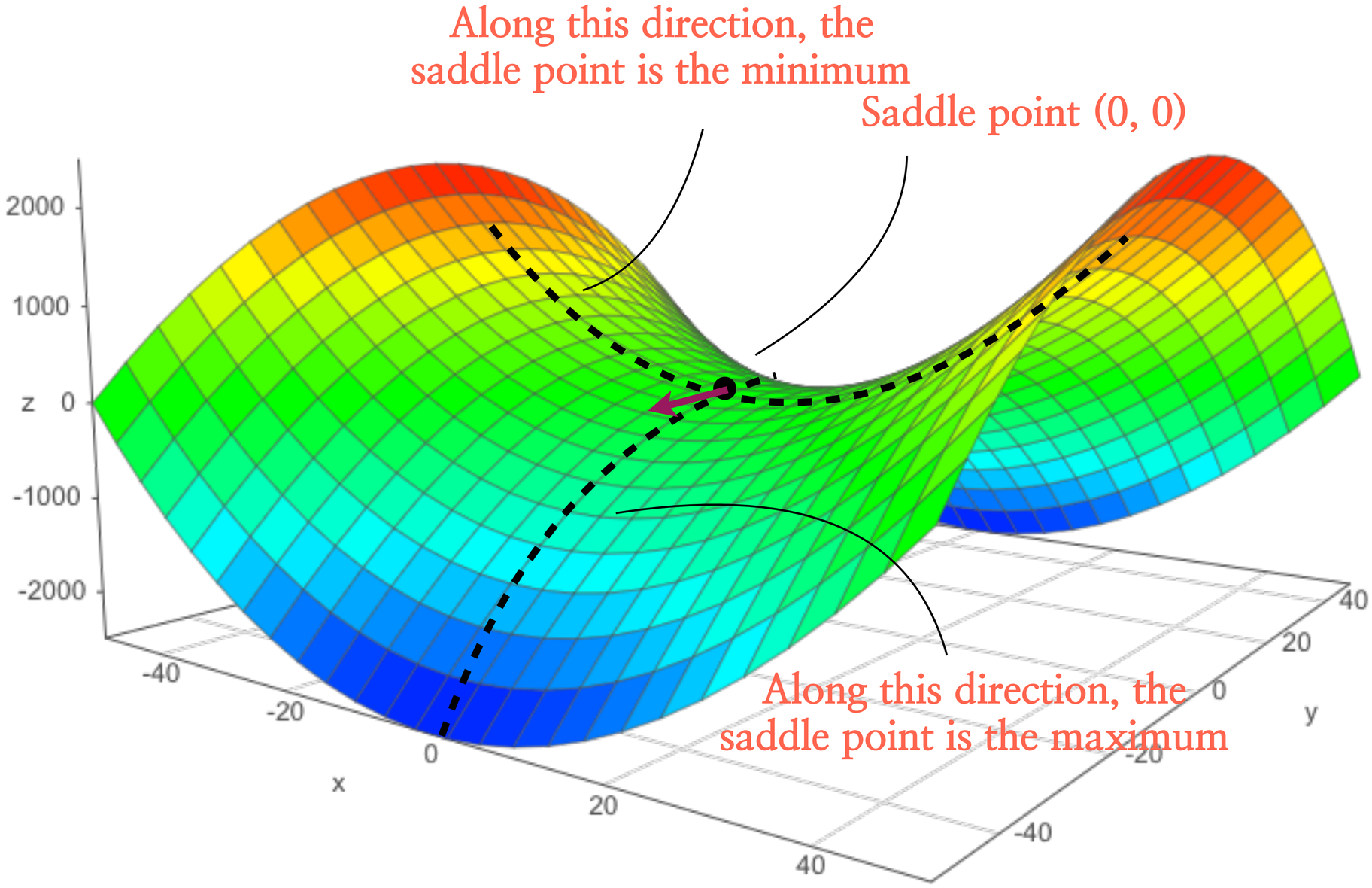

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("Noisy" stochastic gradient descent – stochasticity is not a problem, but a feature)

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- Formal result:

> ``*With probability* $1-\delta$ *, after* $t \geq \log \frac{p}{\eta^2} \cdot \log \frac{2}{\delta}$ *iterations, noisy gradient descent converges close to a local minimum* $x^\star$ *such that* $\|x_t - x^\star\|_2 \leq \epsilon$*. Here,*
>
> $$\eta < \min \left\{ \frac{\epsilon^2}{\log(1/\epsilon\delta)}, \ \frac{\mu}{L^2}, \ \frac{\xi^2}{\log(1/\xi\delta)}, \ \frac{1}{(L+\rho)^2}, \ \frac{\alpha^2}{L} \right\}$$
>
> *where* ρ *is the Hessian-Lipschitz constant.*``

How does noisy gradient descent perform?

- Informal result:

> ``*Noisy gradient descent finds a local minimum of a function that satisfies the strict saddle property in polynomial time*``

(This justifies that the method is not hopeless in practice)

- Formal result:

> ``*With probability* $1-\delta$ *, after* $t \geq \log \frac{p}{\eta^2} \cdot \log \frac{2}{\delta}$ *iterations, noisy gradient descent converges close to a local minimum* $x^\star$ *such that* $\|x_t - x^\star\|_2 \leq \epsilon$*. Here,*
>
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- Overall, total runtime could be up to $O(p^3)$ — differently, $\widetilde{O}(1/\epsilon^4)$ iters.

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A different perspective

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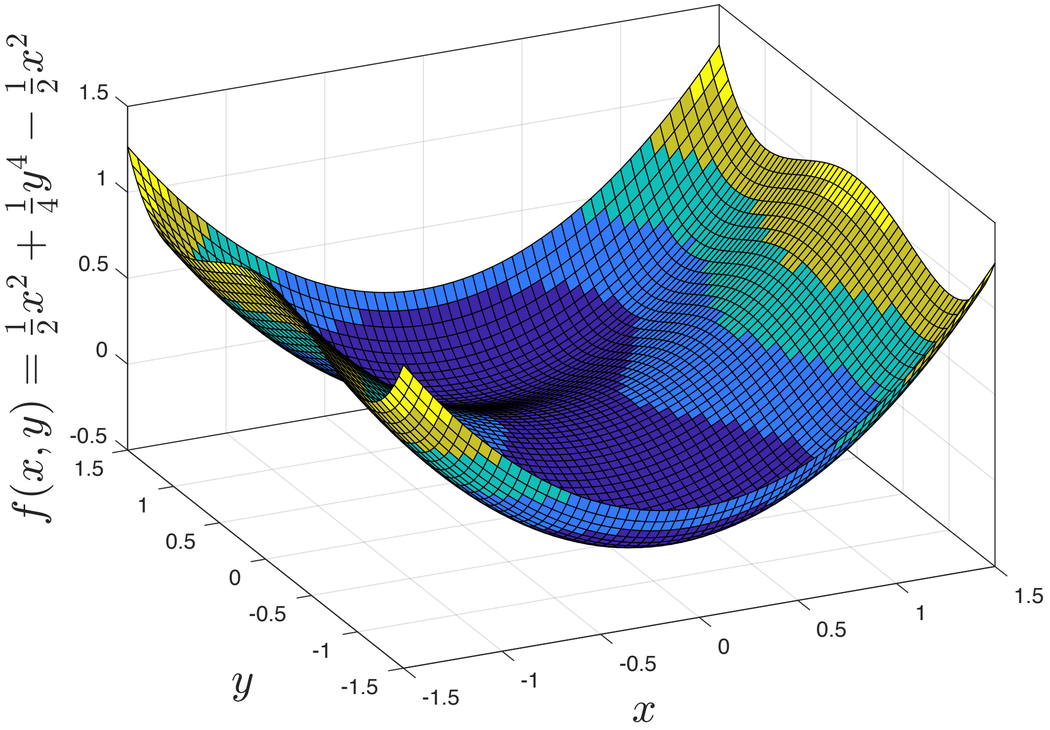

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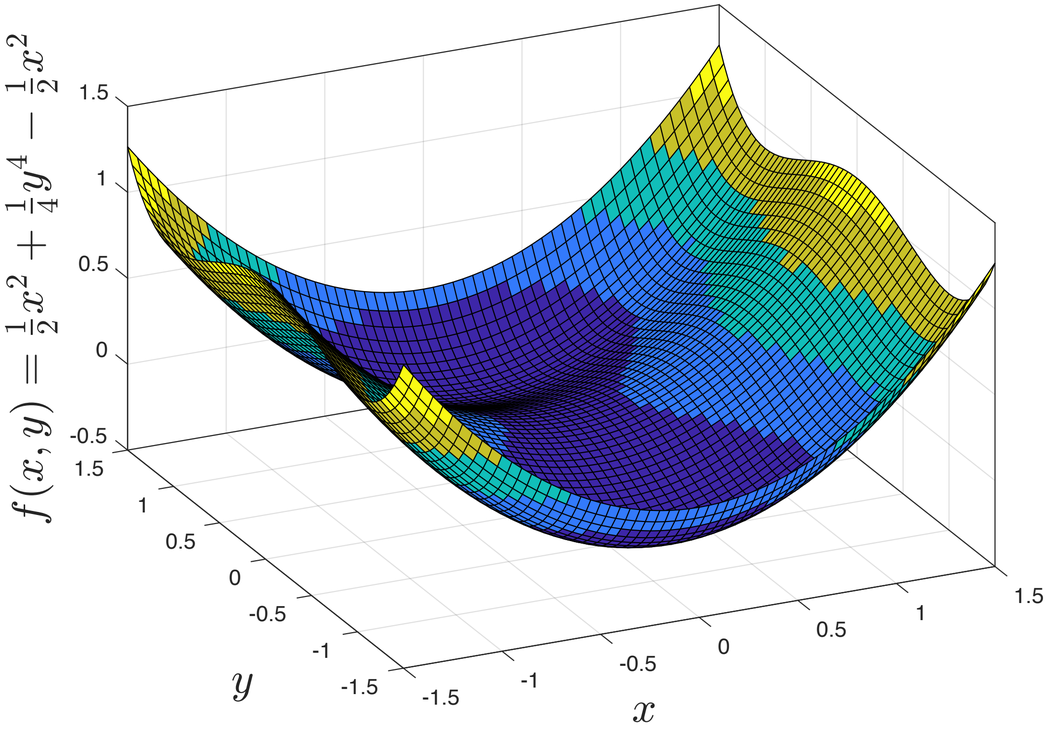

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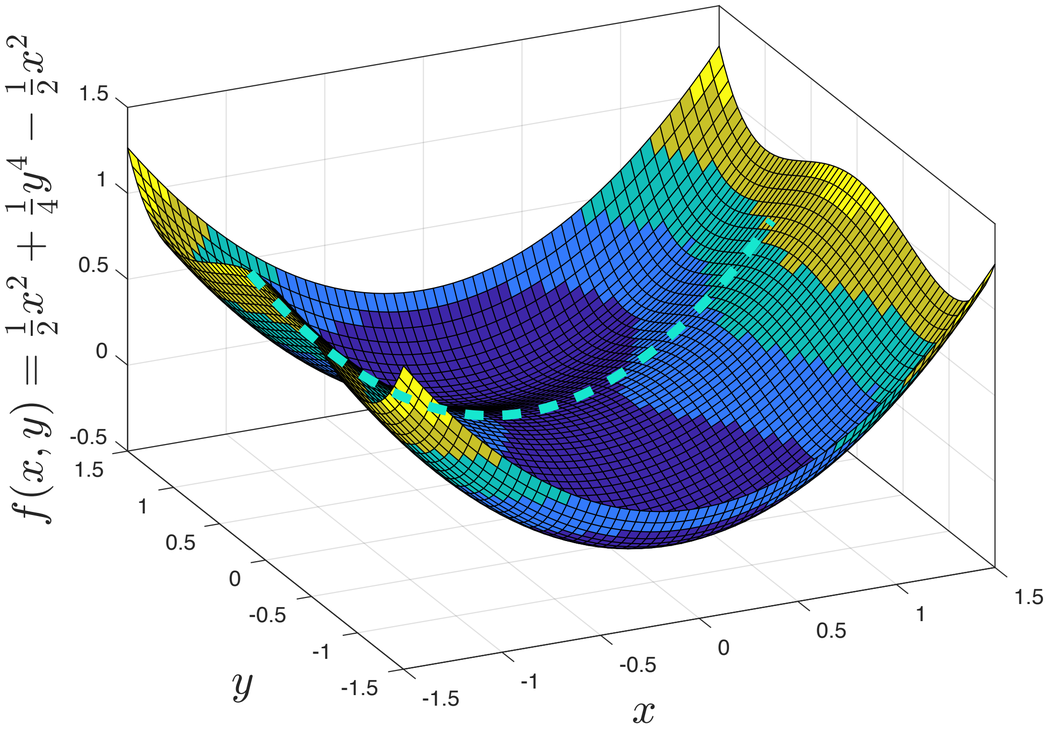

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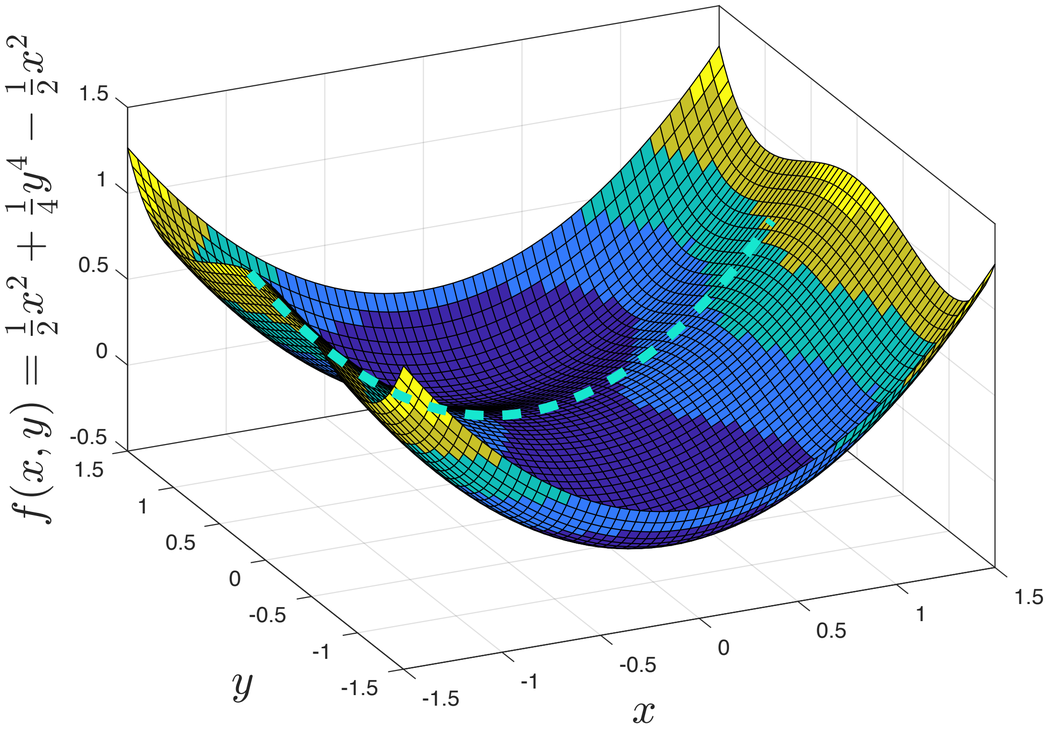

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  Saddle point Local minima

- Any initialization of the form $\begin{bmatrix}x\\0\end{bmatrix}$ (cyan) leads to convergence to the saddle point

- But, **any other initialization converges to local minimizer!**

  (With random initialization, this happens with prob. 1)

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- In practice: **if you pick any random initial point, you are safe not to converge to a saddle point**

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(This is what we have (partially) discussed so far)

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Whiteboard

- Similar results have been proven for: phase retrieval, matrix completion, dictionary recovery, semidefinite programming (SDPs), signal recovery from quadratic measurements, ..

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    and compare with global minima (to show potential equivalence)
- For saddles, identify a (negative) upper bound for

$$\lambda_{\min}\left(\nabla^2 f(x)\right)$$

Papers to review - due next Tuesday

(Select one of the following papers)

- "The loss surfaces of multilinear networks", Choromanska et al, 2014.

- "Identifying and attacking the saddle point problem in high-dimensional non-convex optimization", Dauphin et al., 2014.

- "Gradient descent only converges to minimizers", Lee et al., 2016.

- "When are non convex problems not scary?", Sun et al., 2016.

Conclusion

- We discussed about **types of stationary points**, focus on **saddle points** and study some of their properties

- We introduced conditions that allow **escaping from saddle points**

- We studied (overview) matrix sensing as a test case, and how to prove "no spurious local minima" arguments