Some additional and useful continuous random variables

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The normal distribution

Definition

Let $m \in \mathbb{R}$, $\sigma \in \mathbb{R}_+^*$ and X a continuous random variable. We say that X follows the normal distribution of parameters m and σ and we denote $X \hookrightarrow N(m, \sigma)$, if its probability density function is given by:

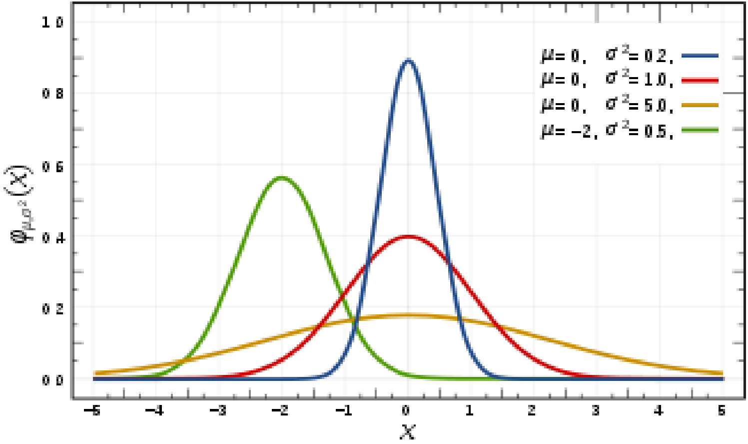
$$f_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

When $m = 0$ and $\sigma = 1$, X is said to follow the standard normal distribution $N(0, 1)$.

Essentials

- Let $X \sim N(m, \sigma^2)$ for some $m \in \mathbb{R}$ and $\sigma^2 > 0$
  - $$\mathbb{E}(X) = m \text{ and } \mathbb{V}(X) = \sigma^2$$
  - $$\frac{X - m}{\sigma} \sim N(0, 1)$$
- If X_1 and X_2 are **independent**, $X_1 \sim N(m_1\,,\,\sigma_1^2)$ and $X_2 \sim N(m_2\,,\,\sigma_2^2)$. Then
$$X_1 + X_2 \sim N(m_1 + m_2\,,\,\sigma_1^2 + \sigma_2^2)$$
.

the normal distribution



Multi-variate normal distribution

Definition

A random vector $X^T = (X_1, \cdots, X_n)$ is said to follow a multi-variate normal distribution if for any vector $\alpha^T = (\alpha_1, \cdots, \alpha_n)$ of real numbers, the random variable scalar product

$$\alpha^T \cdot X = \sum_{i=1}^{n} \alpha_i X_i$$

follows a normal distribution.

Note:

1. Actually the vector X is seen as an $n \times 1$ matrix and X^T denotes its transpose matrix.
2. Be careful even if all components are normally distributed this doesn't implies that the vector X is multi-variate normal. For instance consider the vector (Z, Z) for $Z \sim N(0, 1)$.

The covariance matrix

Let $X^T = (X_1, \cdots, X_n)$ be a given random vector we associate its covariance matrix Σ_X

$$\begin{bmatrix} \sigma_1^2 & \cdots & \cdots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & \sigma_2^2 & \cdots & Cov(X_2, X_n) \\ \vdots & & \ddots & \vdots \\ Cov(X_n, X_1) & \cdots & \cdots & \sigma_n^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_1 \sigma_n \rho_{1n} \\ \sigma_2 \sigma_1 \rho_{21} & \sigma_2^2 & \sigma_2 \sigma_n \rho_{2n} \\ \vdots & & \vdots \\ \sigma_n \sigma_1 \rho_{n1} & \cdots & \sigma_n^2 \end{bmatrix}$$

where if Z_1 and Z_2 are two random variables $Cov(Z_1, Z_2) = \mathbb{E}\Big[\Big(Z_1 - \mathbb{E}(Z_1)\Big)\Big(Z_2 - \mathbb{E}(Z_2)\Big)\Big]$ and

$$\rho_{12} = \rho_{Z_1 Z_2} = \frac{Cov(Z_1, Z_2)}{\sigma_{Z_1} \sigma_{Z_2}}$$

A characterization Theorem

Theorem

Let $X^T = (X_1, \cdots, X_n)$ be a random vector that follows a multi-variate normal. It has a joint density function if and only if its covariance matrix Σ_X is non-singular, in this case the joint density function is given by

$$f_X(x_1, \cdots, x_n) = \frac{1}{2\pi\sqrt{\det(\Sigma_X)}} \exp\left\{-\frac{1}{2}\big(x - \mathbb{E}(X)\big)^T \cdot \Sigma_X^{-1} \cdot \big(x - \mathbb{E}(X)\big)\right\}$$

where $x = \begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix}$ *and* $\mathbb{E}(X) = \begin{vmatrix} \mathbb{E}(X_1) \\ \vdots \\ \mathbb{E}(X_n) \end{vmatrix}$ *is the expectation vector of X.*

We denote

$$X \sim N\Big(\mathbb{E}(X), \Sigma_X\Big)$$

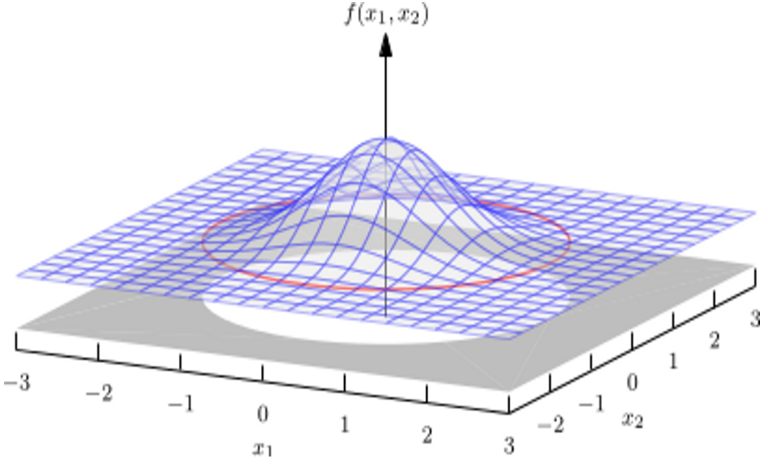
Exercise

1. Show that if (X, Y) follows a bivariate normal distribution its joint density function is given by:

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{(1-\rho_{XY}^2)}} \exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\left(\frac{x-\mathbb{E}(X)}{\sigma_X}\right)^2 -2\rho_{XY}\frac{(x-\mathbb{E}(X))(y-\mathbb{E}(Y))}{\sigma_X\sigma_Y} + \left(\frac{y-\mathbb{E}(Y)}{\sigma_Y}\right)^2\right]\right\}$$

2. Deduce that if (X, Y) follows a bivariate normal distribution, X and Y are independent if and only if $\rho_{XY} = 0$ which means that X and Y are not linearly correlated.

Bivariate normal distribution



The conditional expectation

Theorem

Let (X, Y) be a given random pair that follows a bivariate normal distribution, then

$$\mathbb{E}(Y|X) = \mathbb{E}(Y) + \rho_{XY}\frac{\sigma_Y}{\sigma_X}\Big(X - \mathbb{E}(X)\Big)$$

χ^2_d-distribution

Definition

Let $U_1, U_2, \cdots, U_d$, d independent variables that follow all the standard normal distribution $N(0,1)$. We call the "Chi-square" distribution with d degrees of freedom and denote χ^2_d the distribution of the sum:

$$U_1^2 + U_2^2 + \cdots + U_d.$$

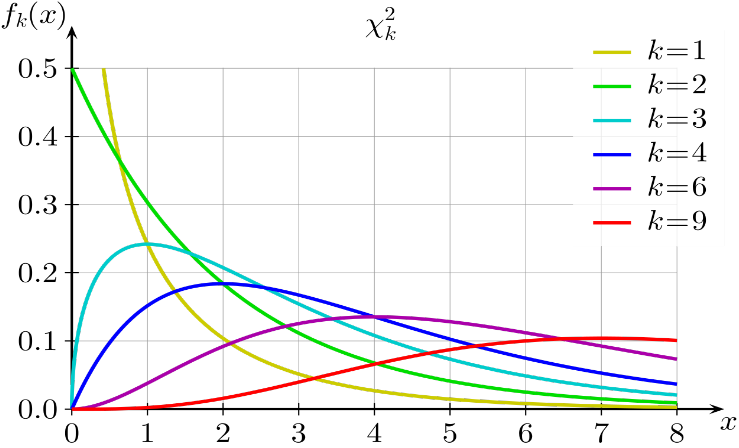
One can prove that

$$\mathbb{E}(\chi^2_2) = d, \quad \mathbb{V}(\chi^2_2) = 2d$$

and that the density function is given by

$$g_d(x) = \frac{1}{2^{d/2}\Gamma(\frac{d}{2})} \exp\left(\frac{-x}{2}\right) x^{d-\frac{1}{2}} \mathbf{1}_{\mathbb{R}_+}(x).$$

χ^2_d-distribution



Case d=1

Let $U \hookrightarrow N(0,1)$ and $x \geq 0$, the CDF of U^2 is given by

$$\begin{aligned} F_{U^2}(x) &= \mathbb{P}\{U^2 \leq x\} \\ &= \mathbb{P}\{-\sqrt{x} \leq U \leq \sqrt{x}\} \\ &= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{e^{-t^2/2}}{\sqrt{2\pi}} t = 2\int_0^{\sqrt{x}} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \end{aligned}$$

this leads to

$$g_1(x) = \frac{\partial F_{U^2}}{\partial x}(x) = 2\frac{e^{-\sqrt{x}^2/2}}{\sqrt{2\pi}} \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2\pi}} e^{-x/2} \frac{1}{\sqrt{x}} \mathbf{1}_{\mathbb{R}_+}(x)$$

Case d=2

Let U_1 and U_2 be two independent r.v that follow each the standard normal distribution $N(0,1)$ then their joint distribution is bivariate normal and is: $f(u_1,u_2) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}(u_1^2+u_2^2)\right\}$

then, for $x \geq 0$, the CDF of $U_1^2+U_2^2$ equals

$$\begin{aligned}
F_{U_1^2+U_2^2}(x) &= \mathbb{P}\{U_1^2+U_2^2 \leq x\} \\
&= \frac{1}{2\pi}\iint_{\{u_1^2+u_2^2\leq x\}} \exp\left\{-\frac{1}{2}(u_1^2+u_2^2)\right\} du_1 du_2 \\
&= \frac{1}{2\pi}\int_0^{\sqrt{x}}\int_0^{2\pi} re^{-\frac{r^2}{2}} dr d\theta = \int_0^{\sqrt{x}} re^{-\frac{r^2}{2}} dr \\
&= \int_0^{\sqrt{x}} -d\left(e^{-\frac{r^2}{2}}\right) = \left(1-e^{x/2}\right)\mathbf{1}_{\mathbb{R}_+}(x)
\end{aligned}$$

it follows that $\chi_2^2 = \mathcal{E}(1/2)$.

The $\mathcal{T}_d$ distribution of Student

Let Y and U two independent random variables that follow respectively χ^2_d and the standard normal distribution $N(0,1)$. We define the Student random variable with d degrees of freedom, $\mathcal{T}_d$ by:

$$\frac{U}{\sqrt{\frac{Y}{d}}}.$$

Its probability density function is given by:

$$f_d(x) = \frac{1}{\sqrt{d\pi}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \left(1 + \frac{x^2}{d}\right)^{-\frac{d+1}{2}}$$

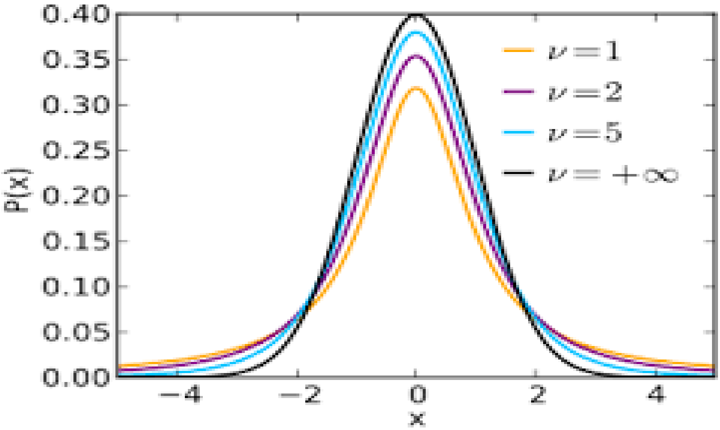
It is easy to show that for $d > 1$:

$$\mathbb{E}(T_d) = 0$$

and that for $d > 2$

$$\mathbb{V}(T_d) = \frac{d}{d-2}.$$

$\mathcal{T}$ – *distribution*



Exercise

1. Suppose $d = 1$ show that $\mathcal{T}_1$ follows the Cauchy distribution given by:

$$f_1(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

2. Deduce that for $d = 1$ the Student distribution has no expectation and has an infinite variance.
3. Suppose now $d = 2$, prove the expression of the density function:

$$f_2(x) = \frac{1}{2\sqrt{2}}\left(1+\frac{x^2}{2}\right)^{-\frac{3}{2}}$$

The F-distribution

Definition

Let A and B two independent random variable that follows respectively $\chi^2_{d_1}$ and $\chi^2_{d_2}$ for d_1 and d_2 two given integers. Then the random variable

$$F = \frac{A/d_1}{B/d_2}$$

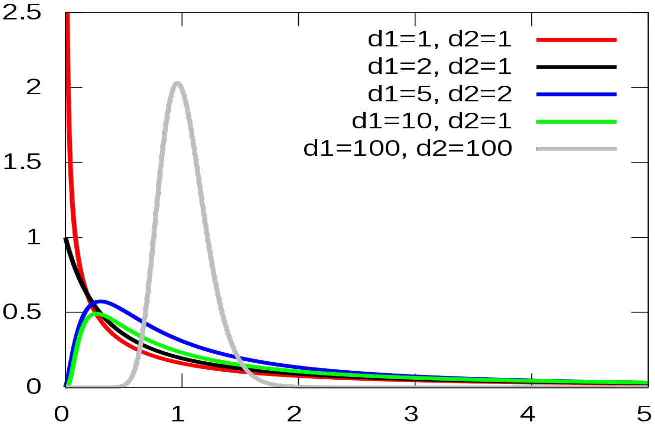
defines the so-called Fisher distribution of degrees of freedom d_1 and d_2. We denote

$$\frac{A/d_1}{B/d_2} = F \sim \mathcal{F}(d_1, d_2).$$

If $F \sim \mathcal{F}(d_1, d_2)$ then its statistical parameters are given by

- $\mathbb{E}(F) = \dfrac{d_1}{d_2 - 2}$ for $d_2 > 2$
- $\mathbb{V}(F) = \dfrac{2d_2^2(d_1 + d_2 - 2)}{d_1(d_2 - 2)^2(d_2 - 4)}$

The F-distribution



The Gamma distribution

A continuous random variable X follows a Gamma distribution of parameter $r > 0$ if its density function has the form

$$f_X(x) = \frac{1}{\Gamma(r)} x^{r-1} e^{-x},$$

where the function Gamma is defined, for any $r > 0$, by

$$\Gamma(r) = \int_0^{+\infty} e^{-x} x^{r-1} \, dx.$$

Exercise

1. Check that for any $r > 0$, $\Gamma(r+1) = r\Gamma(r)$.
2. Deduce that $\Gamma(n+1) = n!$.
3. Prove
   - $\lim\limits_{0^+} \Gamma(r) = +\infty$.
   - $\lim\limits_{+\infty} \Gamma(r) = +\infty$.

The parameters of the Gamma distribution

- **Expectation**

$$\begin{aligned}\mathbb{E}(X) &= \frac{1}{\Gamma(r)}\int_0^{+\infty} x^r e^{-x}dx \\ &= \frac{\Gamma(r+1)}{\Gamma(r)} = r\end{aligned}$$

- **Variance**

$$\begin{aligned}\mathbb{V}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= \frac{1}{\Gamma(r)}\int_0^{+\infty} x^{r+1} e^{-x}dx - r^2 \\ &= \frac{\Gamma(r+2)}{\Gamma(r)} - r^2 = r(r+1) - r^2 = r\end{aligned}$$

The Gamma sum

Let X_1 and X_2 be two **independent** random variables that follow respectively the Gamma distributions with parameters r_1 and r_2.

Then the random variable sum $X_1 + X_2$ follows also a Gamma distribution with parameter $r_1 + r_2$.

The Gamma distribution

Γ-distribution