Continuous Random Variable

anis.rezguii@gmail.com

INSAT
Mathematics Department

February 10, 2022

Table of contents

Expected learning outcomes

After this lecture the student should:

- Understand the concept of continuous random variable and the difference between them and the discrete ones.
- Know most usual examples of continuous random variable.
- be able to chose which model is adequate for a given situation.

Continuous random variable

Definition

Let $(\Omega, \mathbb{P})$ be a probability space and X be a mapping, $X : \Omega \longrightarrow \mathbb{R}$. X is a continuous random variable if there exists a positive piecewise[a] *continuous function f_X such that:*

1. $\displaystyle\int_{-\infty}^{+\infty} f_X(x)dx = 1,$
2. *for any $-\infty \leq a \leq b \leq +\infty$ we have*

$$\mathbb{P}\{X \in [a, b]\} = \int_a^b f_X(x)dx.$$

The function f_X is called the Probability Density Function (PDF).

[a] continuous everywhere except for a finite number of points

Note

- If X is a continuous random variable with a PDF f_X we denote
$$X \hookrightarrow f_X(x)dx.$$
- We also say that X follows the distribution characterized by its PDF f_X.

Example: The uniform distribution

Let $a < b$ two real numbers and, X a continuous random variable is following the uniform distribution if its PDF is given by:

$$f_X = \begin{cases} \dfrac{1}{b-a} & \text{on } [a,b] \\ 0 & \text{if not.} \end{cases} = \frac{1}{b-a}\chi_{[a,b]}(x).$$

It is easy to check that f_X is a positive piecewise continuous function and that it satisfies

$$\int_{\mathbb{R}} f_X(x)dx = \frac{1}{b-a}\int_a^b dx = 1.$$

We denote:

$$X \hookrightarrow \mathcal{U}([a,b]).$$

Note

By definition of a continuous random variable, X, it is clear that for any real number $a \in \mathbb{R}$:

$$\mathbb{P}\{X = a\} = \int_a^a f_X(x)dx = 0.$$

We say that a continuous random variable doesn't charge points, however a discrete random variable charges points !

Cumulative distribution function

Definition

Let X be a continuous random variable with a PDF given by f_X. We associate to X its Cumulative Distribution Function (CDF) F_X, defined by:

$$\begin{array}{rccl} F_X : & \mathbb{R} & \longrightarrow & [0,1] \\ & x & \longmapsto & \mathbb{P}\{X \leq x\} = \int_{-\infty}^{x} f_X(t)dt. \end{array}$$

The example of the uniform distribution

Suppose X following the uniform distribution, $X \hookrightarrow \mathcal{U}([a,b])$, its CDF is defined by:

$$F_{\mathcal{U}([a,b])}(x) = \int_{-\infty}^{x} \chi_{[a,b]}(t)dt.$$

The example of the uniform distribution

Suppose X following the uniform distribution, $X \hookrightarrow \mathcal{U}([a,b])$, its CDF is defined by:

$$F_{\mathcal{U}([a,b])}(x) = \int_{-\infty}^{x} \chi_{[a,b]}(t)dt.$$

Check that:

$$F_{\mathcal{U}([a,b])}(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } x \in [a,b] \\ 1 & \text{if } x > b. \end{cases}$$

Note

1. The CDF, F_X, of a given continuous random variable X satisfies:
   i. It is non-decreasing.
   ii. $\lim_{x \to +\infty} F_X(x) = 1$ and $\lim_{x \to -\infty} F_X(x) = 0$.
2. The CDF of a given continuous random variable, X, is differentiable wherever the associated PDF is, and we have:
$$\frac{dF_X(x)}{dx} = f_X(x).$$
3. The latter relation shows that the CDF determines the PDF, so, we can say that the CDF and the PDF are simply equivalent.

Exercise

Let $X \hookrightarrow ke^{-|x|}dx$ be a continuous random variable.

1. Evaluate k.
2. Determine its cumulative distribution function, F_X.
3. Deduce the probability density function, f_Y, of $Y = X^2$.

The mathematical expectation

Definition

Let X be a continuous random variable $X \hookrightarrow f_X(x)dx$, its mathematical expectation is the quantity:

$$\mathbb{E}(X) = \int_{\mathbb{R}} x f_X(x)\, dx.$$

Notes

1. If any random experiment is repeated many times we expect, in the "usual" cases, that most of X's values are "close" to the mathematical expectation $\mathbb{E}(X)$. Thus the mathematical expectation of a d.r.v is considered as a parameter of central tendency.
2. If $X = c$ is constant, then its expectation $\mathbb{E}(X) = c$.
3. $\mathbb{E}(\lambda X + c) = \lambda\mathbb{E}(X) + c$ for any real numbers λ and c.
4. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, then the mathematical expectation of the new random variable $Y = g(X)$ is given by:

$$\mathbb{E}(g(X)) = \int_{+\infty}^{-\infty} g(x) f_X(x) dx.$$

Examples

1. Let $X \hookrightarrow \mathcal{U}([a,b])$ then its expectation is:

$$\mathbb{E}(X) = \frac{a+b}{2}.$$

2. Let $X \hookrightarrow \frac{1}{2}e^{-|x|}dx$, then its expectation is:

$$\mathbb{E}(X) = 0.$$

The Markov Inequality

We have exactly the same result as for discrete random variables. The Markov inequality gives a rough estimation of some probabilities by the only knowledge of the expectation.

Theorem

Let $(\Omega, \Sigma, \mathbb{P})$ *be a probability space,* X *a positive d.r.v and* λ *a positive real number. Then*

$$\mathbb{P}\{X \geq \lambda\} \leq \frac{\mathbb{E}(X)}{\lambda}.$$

The Markov Inequality

We have exactly the same result as for discrete random variables. The Markov inequality gives a rough estimation of some probabilities by the only knowledge of the expectation.

Theorem

Let $(\Omega, \Sigma, \mathbb{P})$ *be a probability space,* X *a positive d.r.v and* λ *a positive real number. Then*

$$\mathbb{P}\{X \geq \lambda\} \leq \frac{\mathbb{E}(X)}{\lambda}.$$

Note: The only knowledge of the expectation doesn't determine, in general, the probability distribution of a continuous random variable.

The variance

Definition

Let X *be a c.r.v with as a PDF* $f_X(x)$*. Its variance, denoted by* $\mathbb{V}(X)$ *or* σ_X^2*, is the following non negative quantity:*

$$\mathbb{V}(X) = \sigma_X^2 = \int_{\mathbb{R}} (x - \mathbb{E}(X))^2 f_X(x) dx.$$

Its standard deviation is

$$\sigma_X = \sqrt{\mathbb{V}(X)}.$$

Notes

1. We expect, after many repetitions of the experiment, and in the "usual" cases, that most of X's values lies in the interval $[\mathbb{E}(X) - \sigma_X, \mathbb{E}(X) + \sigma_X]$.

2. The variance and the standard deviation are considered to be parameters of dispersion which means that they give idea about how the X's values, in case of repetitions, are spread out around the expectation.

3. If the variance is finite we have the following new formulation:

$$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2.$$

The Tchebychev Inequality

Theorem

Let $(\Omega, \Sigma, \mathbb{P})$ *be a probability space, and* X *a c.r.v, then for any* $\lambda > 0$ *we have*

$$\mathbb{P}\{|X - E(X)| \geq \lambda\} \leq \frac{\mathbb{V}(X)}{\lambda^2}.$$

Note

Tchebychev inequality is often stated as follows:

$$\mathbb{P}\{|X - E(X)| \geq n\sigma\} \leq \frac{1}{n^2}$$

which can be read

$$\mathbb{P}\left\{X \in [\mathbb{E}(X) - n\sigma, \mathbb{E}(X) + n\sigma]\right\} > 1 - \frac{1}{n^2}$$

If for example we take $n = 6$ we obtain that for any random variable X, which means for any phenomenon we have the following estimation:

$$\mathbb{P}\left\{X \in [\mathbb{E}(X) - 6\sigma, \mathbb{E}(X) + 6\sigma]\right\} > 1 - \frac{1}{n^2} = 97.22\%$$

.

The example of the uniform distribution

Let $X \hookrightarrow \mathcal{U}([a,b])$ then:

$$\begin{aligned}\mathbb{V}(X) &= E(X^2) - E(X)^2 \\ &= \int_a^b x^2 dx - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{(b-a)^2}{12}\end{aligned}$$

The normal distribution

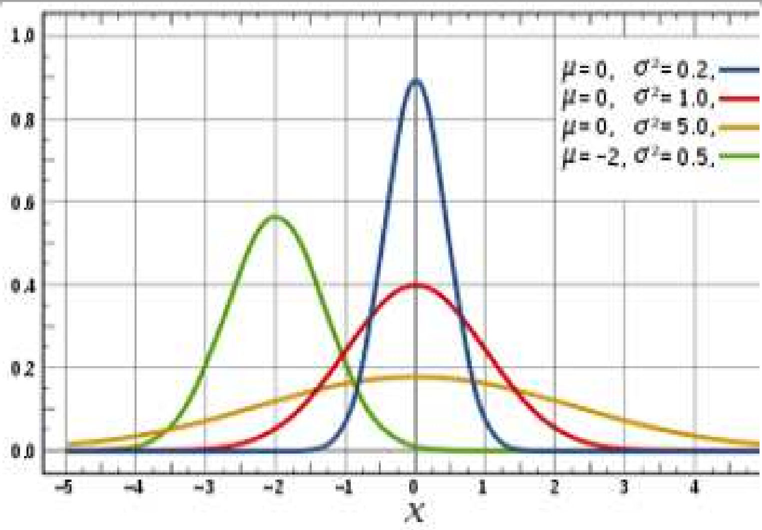
Definition

Let $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+^*$ and X a continuous random variable. We say that X follows a normal distribution of parameters μ and σ and we denote $X \hookrightarrow N(\mu, \sigma)$, if its probability density function is given by:

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

If $\mu = 0$ and $\sigma = 1$, X is said to follow the standard normal distribution $N(0, 1)$.

Shape of its p.d.f curve



Expectation and Variance

Suppose $X \hookrightarrow N(0,1)$ then:

$$\mathbb{E}(X) = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} x e^{-\frac{x^2}{2}} dx = 0$$

and

$$\begin{aligned}
\mathbb{V}(X) &= \mathbb{E}(X^2) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx \\
&= \frac{-2}{\sqrt{2\pi}} \int_0^{+\infty} x d(e^{-\frac{x^2}{2}}) \\
&= \frac{-2}{\sqrt{2\pi}} \left[x e^{-\frac{x^2}{2}} \right]_0^{+\infty} + \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{x^2}{2}} dx \\
&= 1.
\end{aligned}$$

The cumulative distribution function

Suppose Z following the standard normal distribution, $Z \hookrightarrow N(0,1)$, then its CDF is given by:

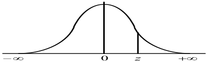
$$\phi(x) = \mathbb{P}\{Z \leq x\} = \mathbb{P}\left\{Z \in]-\infty, x]\right\} = \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \frac{dt}{\sqrt{2\pi}}$$

- It is easy to check that:

$$\phi(x) + \phi(-x) = 1.$$

- ϕ has no explicit formula, to get its values we usually use value's table.

NORMAL DISTRIBUTION TABLE



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Use of the value's table

1. Evaluate $\phi(x) = \mathbb{P}\{Z \leq x\}$ for a given $x \in \mathbb{R}$:
   - $\phi(0) = 0.5$
   - $\phi(1.64) = 0.9495$
2. Find out x such that $\phi(x) = p$ for a given percentage p:
   - $p = 95\%$, $x = 1.64$
   - $p = 99\%$, $x = 2.33$
3. Evaluate $\mathbb{P}\{-x \leq z \leq x\}$ for a given $x \in \mathbb{R}$:
   Note that $\mathbb{P}\{-x \leq z \leq x\} = \phi(x) - \phi(-x) = 2\phi(x) - 1$
   - For $x = 1.64$, $\mathbb{P}\{-1.64 \leq z \leq 1.64\} = 0.899$
   - For $x = 2.5$, $\mathbb{P}\{-2.5 \leq z \leq 2.5\} = 0.9876$

Linear transformations

The normal distribution is stable under linear transformation:

- If $Z \hookrightarrow N(\mu, \sigma)$ then

$$\frac{Z-\mu}{\sigma} \hookrightarrow N(0,1)$$

- If $Z \hookrightarrow N(0,1)$ then

$$\sigma \times Z + \mu \hookrightarrow N(\mu, \sigma).$$

The case of non-standard normal distribution

Suppose that $Z \hookrightarrow N(10, 2)$.

1. Evaluate $\mathbb{P}\{Z \leq x\}$:
   - $\mathbb{P}\{Z \leq 4\} = \mathbb{P}\{\frac{Z-10}{2} \leq \frac{4-10}{2}\} = \phi(-3) = 0.13\%$
   - $\mathbb{P}\{Z \leq 11\} = \mathbb{P}\{\frac{Z-11}{2} \leq \frac{11-10}{2}\} = \phi(0.5) = 69.15\%$
2. Find out x such that $\mathbb{P}\{Z \leq x\} = p$ for a given percentage p:
   - $p = 95\%$, $x = 13.289$
   - $p = 99\%$, $x = 14.652$

Example: Quality Control

We control two dimensions of pieces produced in a factory. The two dimensions x and y should measure respectively $650\ mm$ and $830\ mm$. We tolerate on each dimension an error of $\pm 0.1mm$. Measurements of a sample of pieces give the records below:

dimension	mean	standard deviation
x	650.01mm	0.05mm
y	830.02mm	0.06mm

Suppose that the theoretical models of the two dimensions are both normally distributed.

1. Evaluate the theoretical percentage such that the dimension x is acceptable.
2. Do the same thing for the dimension$y$.
3. Evaluate the theoretical percentage of the defect pieces.

Solution

1. The dimension x is acceptable if $x \in [649.9\,;\,650.1]$. The theoretical distribution of x is normal $N(650,01\,;\,0.05)$. Then

$$\begin{aligned}\mathbb{P}\{649.9 \leq x \leq 650.1\} &= \mathbb{P}\left\{\frac{649.9-650.01}{0.05} \leq \frac{x-\bar{x}}{\sigma_x} \leq \frac{650.1-650.01}{0.05}\right\} \\ &= \mathbb{P}\left\{-2.2 \leq \frac{x-\bar{x}}{\sigma_x} \leq 1.8\right\}\end{aligned}$$

or $\dfrac{x-\bar{x}}{\sigma_x} \hookrightarrow N(0,1)$ so

$$\begin{aligned}\mathbb{P}\{649.9 \leq x \leq 650.1\} &= \phi(1,8) - \phi(-2,2) \\ &= \phi(1,8) + \phi(2,2) - 1 = 95\%\end{aligned}$$

Still solution

2. We do the same for the dimension y:

$$\mathbb{P}\{829.9 \leq y \leq 830.1\} = 88\%.$$

3. A piece is good if both dimensions x and y are acceptable, so we get:

$$\mathbb{P}\{649.9 \leq x \leq 650.1\ ,\ 829.9 \leq y \leq 830.1\} = 95\% \times 88\% = 84\%.$$

The percentage of defect pieces is then:

$$100\% - 84\% = 16\%.$$

Exponential distribution

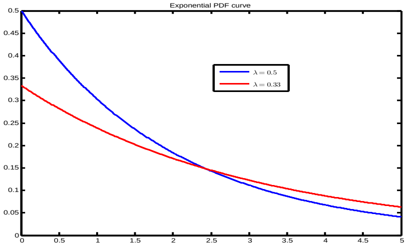
Definition

Let X be a continuous random variable and $\lambda \in \mathbb{R}_+^*$ a given parameter. X is said to follow an exponential distribution if its probability density function is given by:

$$f_X = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \in \mathbb{R}_+ \\ 0 & \text{otherwise} \end{cases}$$

we denote $X \hookrightarrow \mathcal{E}(\lambda)$.

The PDF curve



The exponential cumulative distribution function

Let $X \hookrightarrow \mathcal{E}(\lambda)$, it is easy to check that the cumulative distribution function of X is given by:

$$F_X = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \in \mathbb{R}_+ \\ 0 & \text{elsewhere} \end{cases}$$

Expectation and variance

Let $X \hookrightarrow \mathcal{E}(\lambda)$ then

1. $$\mathbb{E}(X) = \frac{1}{\lambda}.$$

2. $$\mathbb{V}(X) = \frac{1}{\lambda^2}.$$

The Gamma distribution

Definition

A continuous random variable X follows a Gamma distribution of parameter $r > 0$ if its density function has the form

$$f_X(x) = \frac{1}{\Gamma(r)} x^{r-1} e^{-x},$$

where the function Gamma is defined, for any $r > 0$, by

$$\Gamma(r) = \int_0^{+\infty} e^{-x} x^{r-1}\, dx.$$

Exercise

1. Check that for any $r > 0$, $\Gamma(r+1) = r\Gamma(r)$.
2. Deduce that $\Gamma(n+1) = n!$.
3. Prove that
   - $\lim\limits_{0^+} \Gamma(r) = +\infty$.
   - $\lim\limits_{+\infty} \Gamma(r) = +\infty$.

The parameters of the Gamma distribution

- **Expectation**

$$\begin{aligned}\mathbb{E}(X) &= \frac{1}{\Gamma(r)}\int_0^{+\infty} x^r e^{-x}dx \\ &= \frac{\Gamma(r+1)}{\Gamma(r)} = r\end{aligned}$$

- **Variance**

$$\begin{aligned}\mathbb{V}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\ &= \frac{1}{\Gamma(r)}\int_0^{+\infty} x^{r+1} e^{-x}dx - r^2 \\ &= \frac{\Gamma(r+2)}{\Gamma(r)} - r^2 = r(r+1) - r^2 = r\end{aligned}$$

The Chisquare distribution

Definition

Let $U_1, U_2, \cdots, U_d$, d independent variables that follow all the standard normal distribution $N(0,1)$. We call the "Chisquare" distribution with d degrees of freedom and denote χ^2_d the distribution of the sum:

$$U_1^2 + U_2^2 + \cdots + U_d^2.$$

- The CDF of the chisquare distribution has no explicit formulae, to get its values we use either a statistical software or a values table.
- One can prove that

$$\mathbb{E}(\chi^2_d) = d, \quad \mathbb{V}(\chi^2_d) = 2d$$

- The PDF of the chisquare distribution is given by:

$$g_d(x) = \frac{1}{2^{d/2}\Gamma(\frac{d}{2})} \exp\left(\frac{-x}{2}\right) x^{d-\frac{1}{2}} \mathbf{1}_{\mathbb{R}_+}(x).$$

Case d=1

Let $U \hookrightarrow N(0,1)$ and $x \geq 0$, the CDF of U^2 is given by

$$\begin{aligned} F_{U^2}(x) &= \mathbb{P}\{U^2 \leq x\} \\ &= \mathbb{P}\{-\sqrt{x} \leq U \leq \sqrt{x}\} \\ &= \int_{-\sqrt{x}}^{\sqrt{x}} \frac{e^{-t^2/2}}{\sqrt{2\pi}} t = 2\int_0^{\sqrt{x}} \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt \end{aligned}$$

this leads to

$$g_1(x) = \frac{\partial F_{U^2}}{\partial x}(x) = 2\frac{e^{-\sqrt{x}^2/2}}{\sqrt{2\pi}} \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2\pi}} e^{-x/2} \frac{1}{\sqrt{x}} \mathbf{1}_{\mathbb{R}_+}(x)$$

Case d=2

Let U_1 and U_2 two independent r.v such that $U_1,\ U_2 \hookrightarrow N(0,1)$, then

$$\begin{aligned}
F_{U_1^2+U_2^2}(x) &= \mathbb{P}\{U_1^2 + U_2^2 \leq x\} \\
&= \frac{1}{2\pi} \iint_{\{u_1^2+u_2^2 \leq x\}} \exp\left\{-\frac{1}{2}(u_1^2+u_2^2)\right\} du_1 du_2 \\
&= \frac{1}{2\pi} \int_0^{\sqrt{x}} \int_0^{2\pi} r e^{-\frac{r^2}{2}} dr d\theta = \int_0^{\sqrt{x}} r e^{-\frac{r^2}{2}} dr \\
&= \int_0^{\sqrt{x}} -d\left(e^{-\frac{r^2}{2}}\right) = \left(1 - e^{x/2}\right) \mathbf{1}_{\mathbb{R}_+}(x)
\end{aligned}$$

it follows that $\chi_2^2 = \mathcal{E}(1/2)$.

The Beta distribution

Definition

Let $\alpha,\ \beta > 0$, the Beta distribution, $Beta(\alpha,\beta)$ is defined by its PDF as follows:

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

where $B(\alpha,\beta)$ denotes for the Beta function:

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

- The CDF of the Beta distribution has no explicit formulae, to get its values we use either a statistical software or a values table.
- One can prove that

$$\mathbb{E}(Beta(\alpha, \beta)) = \frac{\alpha}{\alpha + \beta}$$

and

$$\mathbb{V}(Beta(\alpha, \beta)) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

Relation with the Binomial distribution

Suppose X following the Beta distribution $Beta(\alpha, \beta)$ with α and β integers. Then:

$$\mathbb{P}\{X \leq x\} = \mathbb{P}\{\mathcal{B}(\alpha+\beta-1, x) \geq \alpha\}$$

where $\mathcal{B}(\alpha+\beta-1, x)$ is the binomial distribution with as a trials number $\alpha+\beta-1$ and a success probability x.