All the rules we know

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Chapter 1.A

Definition 1.1 (Complex Numbers).
A **complex number** is an ordered pair (a, b), where $a, b \in \mathbb{R}$, but we will write this as $a + bi$.

The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

Addition and multiplication on $\mathbb{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

here $a, b, c, d \in \mathbb{R}$.

Property 1.3 (Properties of real arithmetic).
Intentionally restricted to $\mathbb{R}$ for the sake of the 1.A exercises. Equivalent properties can be derived for $\mathbb{C}$.

commutativity

$\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{R}$;

associativity

$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{R}$;

identities

$\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{R}$;

additive inverse

for every $\alpha \in \mathbb{R}$, there exists a unique $\beta \in \mathbb{R}$ such that $\alpha + \beta = 0$;

multiplicative inverse

for every $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{R}$ such that $\alpha\beta = 1$;

distributive property

$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{R}$.

Definition 1.5 ($-\alpha$, subtraction, $1/\alpha$, division).
Let $\alpha, \beta \in \mathbb{C}$.

Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0.$$

Subtraction on $\mathbb{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha).$$

For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1.$$

Division on $\mathbb{C}$ is defined by

$$\beta/\alpha = \beta(1/\alpha).$$

Notation 1.6 ($\mathbb{F}$).
$\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.8 (list, length).
Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \ldots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

Notation 1.10 (notation: n).
n represents a positive integer.

Definition 1.11 ($\mathbb{F}^n$, coordinate).
$\mathbb{F}^n$ is the set of all lists of length n of elements of F:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) \mid x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}.$$

For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} **coordinate** of $(x_1, \ldots, x_n)$.

Definition 1.13 (addition in $\mathbb{F}^n$).
Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

Definition 1.15 (0).
Let 0 denote the list of length n whose coordinates are all 0:

$$(0, \ldots, 0).$$

Definition 1.17 (additive inverse in $\mathbb{F}^n$).
For $x \in \mathbb{F}^n$, the **additive inverse** of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0.$$

In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$.

Definition 1.18 (scalar multiplication in $\mathbb{F}^n$).
The **product** of a number λ and a vector in $\mathbb{F}^n$ is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n);$$

Here $\lambda \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Chapter 1.B

Definition 1.19 (addition, scalar multiplication).
An **addition** on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.

A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

Definition 1.20 (vector space).
A **vector space over** $\mathbb{F}$ is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity

$u + v = v + u$ for all $u, v \in V$;

associativity

$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$;

additive identity

there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;

additive inverse

for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;

multiplicative identity

$1v = v$ for all $v \in V$;

distributive property

$a(u+v) = au + av$ and $(a+b)v = av + bv$ for all $a, b \in \mathbb{F}$ and all $u, v \in V$.

Definition 1.21 (vector, point).
Elements of a vector space are called **vectors** or **points**.

Definition 1.22 (real vector space, complex vector space).
A vector space over $\mathbb{R}$ is called a **real vector space**.

A vector space over $\mathbb{C}$ is called a **complex vector space**.

Notation 1.24 ($\mathbb{F}^S$).
If S is a set, $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.

For $f, g \in \mathbb{F}^S$ the **sum** $f + g \in \mathbb{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in S$.

For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the **product** $\lambda f \in \mathbb{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

Notation 1.28 ($-v, w - v$).
Let $v, w \in V$. Then

1. $-v$ denotes the additive inverse of v;
2. $w - v$ is defined to be $w + (-v)$.

Notation 1.29 (V).
V denotes a vector space over $\mathbb{F}$.

The following rules can all derived from the definition of a vector space.

Result 1.26 (unique additive identity).
A vector space has a unique additive identity.

Result 1.27 (unique additive inverse).
Every element in a vector space has a unique additive inverse.

Result 1.30 (the number 0 times a vector).
$0v = 0$ for every $v \in V$.

Result 1.31 (a number times the vector 0).
$a0 = 0$ for every $a \in \mathbb{F}$.

Result 1.32 (the number -1 times a vector).
$(-1)v = -v$ for every $v \in V$.

Chapter 1.C

Definition 1.33 (subspace)**.**
A subset U of V is called a **subspace** of V if U is also a vector space with the same additive identity, addition, and scalar multiplication as on V.

Definition 1.36 (sum of subspaces)**.**
Suppose $V_1, \ldots, V_m$ are subspaces of V. The **sum** of $V_1, \ldots, V_m$, denoted by $V_1 + \cdots + V_m$, is the set of all possible sums of elements of $V_1, \ldots, V_m$. More precisely:

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m \mid v_1 \in V_1, \ldots, v_m \in V_m\}.$$

Definition 1.41 (direct sum, $\oplus$)**.**
Suppose $V_1, \ldots, V_m$ are subspaces of V.

1. The sum $V_1 + \cdots + V_m$ is called a **direct sum** if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$.
2. If $V_1 + \cdots + V_m$ is a direct sum, then $V_1 \oplus \cdots \oplus V_m$ denotes $V_1 + \cdots + V_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

The following rules can all be derived from the definitions.

Result 1.34 (conditions for a subspace)**.**
A subset U of V is a subspace of V if and only if U satisfies the following three conditions.

additive identity

$0 \in U$.

closed under addition

$u, w \in U$ implies $u + w \in U$.

closed under scalar multiplication

$a \in \mathbb{F}$ and $u \in U$ implies $au \in U$.

Result 1.40 (sum of subspaces is the smallest containing subspace)**.**
Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is the smallest subspace of V containing $V_1, \ldots, V_m$.

Result 1.45 (condition for a direct sum)**.**
Suppose $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as a sum $v_1 + \cdots + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0.

Result 1.46 (direct sum of two subspaces)**.**
Suppose U and W are subspaces of V. Then

$$U + W \text{ is a direct sum} \iff U \cap W = \{0\}.$$

Chapter 2.A

Notation 2.1 (list of vectors).
We write lists of vectors without surrounding parentheses.

Definition 2.2 (linear combination).
A **linear combination** of a list $v_1, \dots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \dots, a_m \in \mathbb{F}$.

Definition 2.4 (span).
The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the **span** of $v_1, \dots, v_m$, denoted by $\operatorname{span}(v_1, \dots, v_m)$. In other words

$$\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{F}\}.$$

The span of the empty list $(\ )$ is defined to be $\{0\}$.

Definition 2.7 (spans).
If $\operatorname{span}(v_1, \dots, v_m)$ equals V, we say that the list $v_1, \dots, v_m$ **spans** V.

Definition 2.9 (finite-dimensional vector space).
A vector space is called **finite-dimensional** if some list of vectors in it spans the space.

Definition 2.10 (polynomial, $\mathcal{P}(\mathbb{F})$).

1. A function $p : \mathbb{F} \to \mathbb{F}$ is called a **polynomial** with coefficients in $\mathbb{F}$ if there exist $a_0, \dots, a_m \in \mathbb{F}$ such that

   $$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

   for all $z \in \mathbb{F}$.
2. $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$.

Definition 2.11 (degree of a polynomial, $\deg\ p$).

1. A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have **degree** m if there exist scalars $a_0, a_1, \dots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that for every $z \in \mathbb{F}$, we have

   $$p(z) = a_0 + a_1 z + \cdots + a_m z^m.$$
2. The polynomial that is identically 0 is said to have degree $-\infty$.
3. The degree of polynomial p is denoted by $\deg\ p$.

Notation 2.12 ($\mathcal{P}_m(\mathbb{F})$).
For m a nonnegative integer, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

Definition 2.13 (infinite-dimensional vector space).
A vector space is called **infinite-dimensional** if it is not finite-dimensional.

Definition 2.15 (linearly independent).

1. A list $v_1, \dots, v_m$ of vectors in V is called **linearly independent** if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that makes

   $$a_v v_1 + \cdots + a_m v_m = 0$$

   is $a_1 = \cdots = a_m = 0$.
2. The empty list $(\ )$ is also declared to be linearly independent.

Definition 2.17 (linearly dependent).

1. A list of vectors V is called **linearly dependent** if it is not linearly independent.
2. In other words, a list $v_1, \dots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

The following can be derived from the definitions.

Result 2.6 (span is the smallest containing subspace).
The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Lemma 2.19 (linear dependence lemma).
Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, 2, \dots, m\}$ such that

$$v_k \in \operatorname{span}(v_1, \dots, v_{k-1}).$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \dots, v_m$, then the span of the remaining list equals $\operatorname{span}(v_1, \dots, v_m)$.

Result 2.22 (length of linearly independent list $\leq$ length of spanning list).
In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Result 2.25 (finite-dimensional subspaces).
Every subspace of a finite-dimensional vector space is finite-dimensional.

Chapter 2.B

Definition 2.26 (basis).
A **basis** of V is a list of vectors in V that is linearly dependent and spans V.

Result 2.28 (criterion for basis).
A list $v_1, \dots, v_n$ of vectors of V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \dots, a_n \in \mathbb{F}$.

Result 2.30 (every spanning list contains a basis).
Every spanning list in a vector space can be reduced to a basis of the vector space.

Corollary 2.31 (basis of finite-dimensional vector space).
Every finite-dimensional vector space has a basis.

Result 2.32 (every linearly independent list extends to a basis).
Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Corollary 2.33 (every subspace of V is part of a direct sum equal to V).
Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Chapter 2.C

Definition 2.35 (dimension, $\dim V$).
The **dimension** of a finite-dimensional vector space is the length of any basis of the vector space.

The dimension of a finite-dimensional vector space V is denoted by $\dim V$.

Result 2.34 (basis length does not depend on basis).
Any two bases of a finite-dimensional vector space have the same length.

Result 2.37 (dimension of a subspace).
If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Result 2.38 (linearly independent list of the right length is a basis).
Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

Corollary 2.39 (subspace of full dimension equals the whole space).
Suppose that V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Then $U = V$.

Result 2.42 (spanning list of the right length is a basis).
Suppose V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V.

Result 2.43 (dimension of sum).
If V_1 and V_2 are subspaces of a finite-dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Result Page 48 (Sets and vector spaces).

sets	**vector spaces**
S is a finite set	V is a finite-dimensional vector space
$\#S$	$\dim V$
for subsets S_1, S_2 of S, the union $S_1 \cup S_2$ is the smallest subset of S containing S_1 and S_2	for subspaces V_1, V_2 of V, the sum $V_1 + V_2$ is the smallest subspace of V containing V_1 and V_2
$\#(S_1 \cup S_2) = \#S_1 + \#S_2 - \#(S_1 \cap S_2)$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$
$\#(S_1 \cup S_2) = \#S_1 + \#S_2 \iff \#(S_1 \cap S_2) =$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 \iff \dim(V_1 \cap V_2) =$
$S_1 \cup \cdots \cup S_m$ is a disjoint union $\iff \#(S_1 \cup \cdots \cup S_M) = \#S_1 + \cdots + \#S_M$	$V_1 \cdots + V_m$ is a direct sum $\iff \dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m$

Chapter 3.A

Definition 3.1 (linear map)**.**
A **linear map** from V to W is a function $T : V \to W$ with the following properties.

additivity

$T(u + v) = Tu + Tv$ for all $u, v \in V$.

homogeneity

$T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$.

Notation 3.2 ($\mathcal{L}(V, W), \mathcal{L}(V)$)**.**

1. The set of linear maps from V to W is denoted by $\mathcal{L}(V, W)$.
2. The set of linear maps from V to V is denoted by $\mathcal{L}(V)$. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Definition 3.5 (addition and scalar multiplication on $\mathcal{L}(V, W)$)**.**
Suppose $S, T \in \mathcal{L}(V, w)$ and $\lambda \in \mathbb{F}$. The **sum** $S + T$ and the **product** λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

Definition 3.7 (product of linear maps)**.**
If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the **product** $ST \in \mathcal{L}(U, V)$ is defined by

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

Lemma 3.4 (linear map lemma)**.**
Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_k = w_k$$

for each $k = 1, \ldots, n$.

Result 3.6 ($\mathcal{L}(V, W)$ is a vector space)**.**
With the operations of addition and scalar multiplication as in Definition 3.5, $\mathcal{L}(V, W)$ is a vector space.

Result 3.8 (algebraic properties of products of linear maps)**.**

associativity

$(T_1T_2)T_3 = T_1(T_2T_3)$ whenever T_1, T_2 and T_3 are linear maps such that the products make sense (meaning T_3 maps into the domain of T_2 and T_2 maps into the domain of T_1).

identity

$TI = IT = T$ whenever $T \in \mathcal{L}(V, W)$; here the first I is the identity operator on V and the second I is the identity operator on W.

distributive properties

$(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$ whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

Result 3.10 (linear maps take 0 to 0)**.**
Suppose T is a linear map from V to W. Then $T(0) = 0$.

Result Ex. 3A, 13 (Linear maps on a subspace can be extended to a map on the whole vector space)**.**
Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, V)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in E$.

Chapter 3.B

Definition 3.11 (null space, null T)**.**
For $T \in \mathcal{L}(V, W)$, the **null space** of T, denoted by null T, is the subset of V consisting of those vectors that T maps to 0:

$$\operatorname{null} T = \{v \in V \mid Tv = 0\}.$$

Definition 3.14 (injective)**.**
A function $T : V \to W$ is called **injective** if $Tu = Tv$ implies $u = v$.

Definition 3.16 (range)**.**
For $T \in \mathcal{L}(V, W)$, the **range** of T is the subset W consisting of those vectors that are equal to Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv \mid v \in V\}.$$

Definition 3.19 (surjective)**.**
A function $T : V \to W$ is called **surjective** if its range equals W.

Result 3.13 (the null space is a subspace)**.**
Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

Result 3.15 (injectivity $\Leftrightarrow$ null space equals $\{0\}$)**.**
Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

Result 3.18 (the range is a subspace)**.**
If $T \in \mathcal{L}(V, W)$, then the range T is a subspace of W.

Theorem 3.21 (fundamental theorem of linear maps)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Result 3.22 (linear map to a lower-dimensional space is not injective)**.**
Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

Result 3.24 (linear map to a higher-dimensional space is not surjective)**.**
Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Result 3.26 (homogeneous system of linear equations)**.**
A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Result 3.28 (inhomogeneous systems of linear equations)**.**
An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Chapter 3.C

Definition 3.29 (matrix, $A_{j,k}$)**.**
Suppose m and n are nonnegative integers. An m-by-n **matrix** A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

Definition 3.31 (matrix of a linear map, $\mathcal{M}(T)$)**.**
Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The **matrix of** T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the basis $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ is used.

Definition 3.34 (matrix addition)**.**
The **sum of two matrices of the same size** is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{pmatrix}.$$

Definition 3.36 (scalar multiplication of a matrix)**.**
The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}.$$

Notation 3.39 ($\mathbb{F}^{m,n}$)**.**
For m and n positive integers, the set of all m-by-n matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{m,n}$.

Definition 3.41 (matrix multiplication)**.**
Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then AB is defined to be the m-by-p matrix whose entry in row j, column k, is given by the equation

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r}B_{r,k}.$$

Thus the entry in row j, column k, of AB is computed by taking row i of A and column k of B, multiplying together corresponding entries, and then summing.

Notation 3.44 ($A_{j,\cdot}, A_{\cdot,k}$)**.**
Suppose A is an m-by-n matrix.

1. If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
2. If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

Definition 3.52 (column rank, row rank)**.**
Suppose A is an m-by-n matrix with entries in $\mathbb{F}$.

1. The **column rank** of A is the dimension of the span of the columns of A in $\mathbb{F}^{m,1}$.
2. The **row rank** of A is the dimension of the span of the rows of A in $\mathbb{F}^{1,n}$.

Definition 3.54 (transpose, A^{t})**.**
The **transpose** of a matrix A, denoted by A^{t}, is the matrix obtained from A by interchanging rows and columns. Specifically, if A is an m-by-n matrix, then A^{t} is an n-by-m matrix whose entries are given by the equation

$$(A^{\mathrm{t}})_{k,j} = A_{j,k}.$$

Definition 3.58 (rank)**.**
The **rank** of a matrix $A \in \mathbb{F}^{m,n}$ is the column rank of A.

Result 3.35 (matrix of the sum of linear maps).
Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Result 3.38 (the matrix of a scalar times a linear map).
Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

Result 3.40 ($\dim \mathbb{F}^{m,n} = mn$).
Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbb{F}^{m,n}$ is a vector space of dimension mn.

Result 3.43 (matrix of product of linear maps).
If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Result 3.46 (entry of matrix product equals row times column).
Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{j,k} = A_{j,\cdot}B_{\cdot,k}$$

if $1 \leq j \leq m$ and $1 \leq k \leq p$. In other words, the entry in row j, column k, of AB equals (row j of A) times (column k of B).

Result 3.48 (column of matrix product equals matrix times column).
Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{\cdot,k} = AB_{\cdot,k}$$

if $1 \leq k \leq p$. In other words, column k of AB equals A times column k of B.

Result Ex. 3C, 8 (row of matrix product equals matrix times row).
Suppose A is an m-by-n matrix and B is an n-by-p matrix. Then

$$(AB)_{j,\cdot} = A_{j,\cdot}B$$

if $1 \leq j \leq m$. In other words, row j of AB equals row j of A times B.

This is the row version of Result 3.48.

Result 3.50 (linear combination of columns).
Suppose A is an m-by-n matrix and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n-by-1 matrix. Then

$$Ab = b_1 A_{\cdot,1} + \cdots + b_n A_{\cdot,n}.$$

In other words, Ab is a linear combination of the columns of A, with the scalars that multiply the columns coming from b.

Result Ex. 3C, 9 (linear combination of rows).
Suppose $a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is a 1-by-n matrix and B is an n-by-p matrix. Then

$$aB = a_1 B_{1,\cdot} + \cdots + a_n B_{n,\cdot}.$$

In other words, aB is a linear combination of the rows of B, with the scalars that multiply the rows coming from a.

This is the row version of Result 3.50.

Result 3.51 (matrix multiplication as linear combinations of columns).
Suppose C is an m-by-c matrix and R is a c-by-n matrix.

(a) If $k \in \{1, \ldots, n\}$, then column k of CR is a linear combination of the columns of C, with the coefficients of this linear combination coming from columns k of R.

(b) If $j \in \{1, \ldots, m\}$, then row j of CR is a linear combination of the rows of R, with the coefficients of this linear combination coming from row j of C.

Result Ex. 3C, 14 (transpose is a linear map).
Suppose m and n are positive integers. Then the function $A \mapsto A^{\mathrm{t}}$ is a linear map from $\mathbb{F}^{m,n}$ to $\mathbb{F}^{n,m}$.

In other words $(A+B)^{\mathrm{t}} = A^{\mathrm{t}} + B^{\mathrm{t}}$, $(\lambda A)^{\mathrm{t}} = \lambda A^{\mathrm{t}}$ for all m-by-n matrices A, B and all $\lambda \in \mathbb{F}$.

Result Ex. 3C, 15 (The transpose of the product is the product of the transposes in the opposite order).
If A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^{\mathrm{t}} = C^{\mathrm{t}} A^{\mathrm{t}}.$$

Result 3.56 (column-row factorisation).
Suppose A is an m-by-n matrix with entries in $\mathbb{F}$ and column rank $c \geq 1$. Then there exist an m-by-c matrix C and a c-by-n matrix R, both with entries in $\mathbb{F}$, such that $A = CR$.

Result 3.57 (column rank equals row rank).
Suppose $A \in \mathbb{F}^{m,n}$. Then the column rank of A equals the row rank of A.

Chapter 3.D

Definition 3.59 (invertible, inverse).

1. A linear map $T \in \mathcal{L}(V, W)$ is called **invertible** if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity operator on V and TS equals the identity operator on W.
2. A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an **inverse** of T (note that the first I is the identity operator on V and the second I is the identity operator on W.

Notation 3.61 (T^{-1}).
If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

Definition 3.69 (isomorphism, isomorphic).

1. An **isomorphism** is an invertible linear map.
2. Two vector spaces are called **isomorphic** if there is an isomorphism from one vector space onto the other one.

Definition 3.73 (matrix of a vector, $\mathcal{M}(v)$).
Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The **matrix of** V with respect to this basis is the n-by-1 matrix

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

where $b_1, \ldots, b_n$ are the scalars such that

$$v = b_1 v_1 + \cdots + b_n v_n.$$

Definition 3.79 (identity matrix, I).
Suppose n is a positive integer. The n-by-n matrix

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

with 1's on the diagonal (the entries where the row number equals the column number) and 0's elsewhere is called the **identity matrix** and is denoted by I.

Definition 3.80 (invertible, inverse, A^{-1}).
A square matrix A is called **invertible** if there is a square matrix B of the same size such that $AB = BA = I$; we call B the **inverse** of A and denote it by A^{-1}.

Notation Ex. 3D, 9 (Restriction of a map to a subset).
If $T : V \to W$ and $U \subseteq V$ then the **restriction** $T|_U$ of T to U is the function $T : U \to W$ whose domain is U, with $T|_U$ defined by

$$T|_U(u) = T(u)$$

for every $u \in U$.

Result 3.60 (inverse is unique).
An invertible linear map has a unique inverse.

Result 3.63 (invertibility $\iff$ injectivity and surjectivity).
A linear map is invertible if and only if it is injective and surjective.

Result 3.65 (injectivity is equivalent to surjectivity (if $\dim V = \dim W < \infty$)).
Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then

$$T \text{ is invertible } \iff T \text{ is injective } \iff T \text{ is surjective.}$$

Result 3.68 ($ST = I \iff TS = I$ (on vector spaces of the same dimension)).
Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(V, W)$, and $T \in \mathcal{L}(W, V)$. Then $ST = I$ if and only if $TS = I$.

Result 3.70 (dimension shows whether vector spaces are isomorphic).
Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Result 3.71 ($\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$ are isomorphic).
Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then M is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.

Result 3.72 ($\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$).
Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W).$$

Result 3.75 ($\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(Tv_k)$).
Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $1 \le k \le n$. Then the k^{th} column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(Tv_k)$.

Result 3.76 (linear maps act like matrix multiplication).
Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1 \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v).$$

Result 3.78 (dimension of range T equals column rank of $\mathcal{M}(T)$).
Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$.

Result 3.81 (matrix of product of linear maps).
Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. If $u_1, \ldots, u_m$ is a basis of U, $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_p$ is a basis for W, then

$$\begin{aligned} \mathcal{M}(ST, (u_1, \ldots, u_m), &(w_1, \ldots, w_p)) = \\ &\mathcal{M}(S, (v_1, \ldots, v_n), (w_1, \ldots, w_p)) \\ &\mathcal{M}(U, (u_1, \ldots, u_m), (v_1, \ldots, v_n)). \end{aligned}$$

Result 3.82 (matrix of identity operator with respect to two bases)**.**
Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then the matrices $\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$ and $\mathcal{M}(I, (v_1, \ldots, v_n), (u_1, \ldots, u_n))$ are invertible, and each is the inverse of the other.

Result 3.84 (change-of-basis formula)**.**
Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let $A = \mathcal{M}(T, (u_1, \ldots, u_n))$ and $B = \mathcal{M}(T, (v_1, \ldots, v_n))$ and $C = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1 \ldots, v_n))$. Then

$$A = C^{-1}BC.$$

Result 3.86 (matrix of inverse equals inverse of matrix)**.**
Suppose that $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$ is invertible. Then $\mathcal{M}(T^{-1}) = (\mathcal{M}(T))^{-1}$, where both matrices are with respect to the basis $v_1, \ldots, v_n$.

Chapter 3.E

Definition 3.87 (product of vector spaces).
Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbb{F}$.

1. The **product** $V_1 \times \cdots \times V_m$ is defined by
$$V_1 \times \cdots \times V_m = \{(v_1, \ldots, v_m) \mid v_1 \in V_1, \ldots, v_m \in V_m\}.$$
2. Addition on $V_1 \times \cdots \times V_m$ is defined by
$$(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m).$$
3. Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by
$$\lambda(v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m).$$

Notation 3.95 ($v + U$).
Suppose $v \in V$ and $U \subseteq V$. Then $v + U$ is the subset of V defined by
$$v + U = \{v + u \mid u \in U\}$$

Definition 3.97 (translate).
For $v \in V$ and U a subset of V, the set $v + U$ is said to be a **translate** of U.

Definition 3.99 (quotient space, V/U).
Suppose U is a subspace of V. The **quotient space** V/U is the set of all translates of U. Thus
$$V/U = \{v + U \mid v \in V\}.$$

Definition 3.102 (addition and scalar multiplication on V/U).
Suppose U is a subspace of V. The **addition** and **scalar multiplication** are defined on V/U by
$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$
for all $v, w \in V$ and all $\lambda \in \mathbb{F}$.

Definition 3.104 (quotient map, π).
Suppose U is a subspace of V. The **quotient map** $\pi : V \to V/U$ is the linear map defined by
$$\pi(v) = v + U$$
for each $v \in V$.

Notation 3.106 ($\widetilde{T}$).
Suppose $T \in \mathcal{L}(V, W)$. Define $\widetilde{T} : V/(\operatorname{null} T) \to W$ by
$$\widetilde{T}(v + \operatorname{null} T) = Tv.$$

Notation (composition of linear maps, $\circ$).
If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the **composition** notation $S \circ T$ is an alternative way of writing the product ST of S and T, as given in Definition 3.7.

Result 3.89 (product of vector spaces is a vector space).
Suppose $V_1, \ldots, V_n$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

Result 3.92 (dimension of a product is the sum of dimensions).
Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and
$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

Result 3.93 (products and direct sums).
Suppose that $V_1, \ldots, V_m$ are subspaces of V. Define a linear map $\Gamma : V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by
$$\Gamma(v_1, \ldots, v_m) = v_1 + \ldots + v_m.$$
Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

Result 3.94 (a sum is a direct sum if and only if dimensions add up).
Suppose V is finite-dimensional and $V_1, \ldots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if
$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m.$$

Result 3.101 (two translates of a subspace are equal or disjoint).
Suppose U is a subspace of V and $v, w \in V$. Then
$$v - w \in U \iff v + U = w + U \iff (v + U) \cap (w + U) \neq \emptyset.$$

Result 3.103 (quotient space is a vector space).
Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined in Definition 3.102, is a vector space.

Result 3.105 (dimension of quotient space).
Suppose V is finite-dimensional and U is a subspace of V. Then
$$\dim V/U = \dim U - \dim V.$$

Result 3.107 (null space and range of $\widetilde{T}$).
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\widetilde{T} \circ \pi = T$, where π is the quotient map of V onto $V/(\operatorname{null} T)$;

(b) $\widetilde{T}$ is injective;

(c) $\operatorname{range} \widetilde{T} = \operatorname{range} T$;

(d) $V/(\operatorname{null} T)$ and $\operatorname{range} T$ are isomorphic vector spaces.

Result Ex. 3E, 18 (Direct sum of a quotient).
Suppose U is a subspace of V such that V/U is finite-dimensional. Then there exists a finite-dimensional subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

Chapter 3.F

Definition 3.108 (linear functional)**.**
A **linear functional** on V is a linear map from V to $\mathbb{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

Definition 3.110 (dual space V')**.**
The **dual space** of V, denoted by V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

Definition 3.112 (dual basis)**.**
If $v_1, \ldots, v_n$ is a basis of V, then the **dual basis** of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Definition 3.118 (dual map, T')**.**
Suppose $T \in \mathcal{L}(V, W)$. The **dual map** of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by

$$T'(\varphi) = \varphi \circ T.$$

Definition 3.121 (annihilator, U^0)**.**
For $U \subseteq V$, the **annihilator** of U, denoted by U^0, is defined by

$$U^0 = \{\varphi \in V' \mid \varphi(u) = 0 \text{ for all } u \in U\}.$$

Result 3.111 ($\dim V' = \dim V$)**.**
Suppose V is finite-dimensional. Then V' is also finite-dimensional and

$$\dim V' = \dim V.$$

Result 3.114 (dual basis gives coefficients for linear combination)**.**
Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_i, \ldots, \varphi_n$ is the dual basis. Then for each $v \in V$

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

Result 3.116 (dual basis is a basis of the dual space)**.**
Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V'.

Result 3.120 (algebraic properties of dual maps)**.**
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$;

(c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

Result 3.124 (the annihilator is a subspace)**.**
Suppose $U \subseteq V$. Then U^0 is a subspace of V'.

Result 3.125 (dimension of the annihilator)**.**
If V is finite-dimensional and U is a subspace of V then

$$\dim U^0 = \dim V - \dim U.$$

Result 3.127 (condition for the annihilator to equal $\{0\}$ or the whole space)**.**
If V is finite-dimensional and U is a subspace of V then

(a) $U^0 = \{0\} \iff U = V$;

(b) $U^0 = V' \iff U = \{0\}$.

Result 3.128 (the null space of T')**.**
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T' = (\operatorname{range} T)^0$.

Suppose further that V and W are finite-dimensional. Then

(b) $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim W$.

Result 3.129 (T surjective is equivalent to T' injective)**.**
If V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$ then

$$T \text{ is surjective } \iff T' \text{ is injective.}$$

Result 3.130 (the range of T')**.**
If V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$ then

(a) $\dim \operatorname{range} T' = \dim \operatorname{range} T$;

(b) $\operatorname{range} T' = (\operatorname{null} T)^0$.

Result 3.131 (T injective is equivalent to T' surjective)**.**
If V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$ then

$$T \text{ is injective } \iff T' \text{ is surjective.}$$

Result 3.132 (matrix of T' is transpose of matrix of T)**.**
If V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$ then

$$\mathcal{M}(T') = (\mathcal{M}(T))^{\mathrm{t}}.$$

Chapter 4

Definition 4.1 (real part, $\operatorname{Re} z$, imaginary part, $\operatorname{Im} z$).
Suppose $z = a + bi$, where a and b are real numbers.

1. The **real part** of z, denoted $\operatorname{Re} z$, is defined by $\operatorname{Re} z = a$.
2. The **imaginary part** of z, denoted by $\operatorname{Im} z$, is defined by $\operatorname{Im} z = b$.

Definition 4.2 (complex conjugate, $\bar{z}$, absolute value, $|z|$).
Suppose $z \in \mathbb{C}$.

1. The **complex conjugate** of $z \in \mathbb{C}$, denoted by $\bar{z}$, is defined by
$$\bar{z} = \operatorname{Re} z - (\operatorname{Im} z)i.$$
2. The **absolute value** of a complex number z, denoted by $|z|$, is defined by
$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

Property 4.4 (properties of complex numbers).
Suppose $w, z \in \mathbb{C}$. Then the following equalities and inequalities hold.

sum of z and $\bar{z}$

$$z + \bar{z} = 2 \operatorname{Re} z.$$

difference of z and $\bar{z}$

$$z - \bar{z} = 2(\operatorname{Im} z)i.$$

product of z and $\bar{z}$

$$z\bar{z} = |z|^2.$$

additivity and multiplicativity of complex conjugate

$$\overline{w + z} = \bar{w} + \bar{z} \text{ and } \overline{wz} = \bar{w}\bar{z}.$$

double complex conjugate

$$\bar{\bar{z}} = z.$$

real and imaginary parts are bounded by $|z|$

$$|\operatorname{Re} z| \leq |z| \text{ and } |\operatorname{Im} z| \leq |z|.$$

absolute value of the complex conjugate

$$|\bar{z}| = |z|.$$

multiplicativity of absolute value

$$|wz| = |w||z|.$$

triangle inequality

$$|w + z| \leq |w| + |z|.$$

Definition 4.5 (zero of a polynomial).
A number $\lambda \in \mathbb{F}$ is called a **zero** (or **root**) of a polynomial $p \in \mathcal{P}(\mathbb{F})$ if
$$p(\lambda) = 0.$$

Result 4.6 (each zero of a polynomial corresponds to a degree-one factor).
Suppose m is a positive integer and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial of degree m. Suppose $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ if and only if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ of degree $m - 1$ such that
$$p(z) = (z - \lambda)q(z)$$
for every $z \in \mathbb{F}$.

Result 4.8 (degree m implies at most m zeros).
Suppose m is a positive integer and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial of degree m. Then p has at most m zeros in $\mathbb{F}$.

Result 4.9 (division algorithm for polynomials).
Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that
$$p = sq + r$$
and $\deg r < \deg s$.

Result 4.12 (fundamental theorem of algebra, first version).
Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Result 4.13 (fundamental theorem of algebra, second version).
If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorisation (except for the order of the factors) of the form
$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$
where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

Result 4.14 (polynomials with real coefficients have nonreal zeros in pairs).
Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is $\bar{\lambda}$.

Result 4.15 (factorisation of a quadratic polynomial).
Suppose $b, c \in \mathbb{R}$. Then there is a polynomial factorisation of the form
$$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$$
with $\lambda_1, \lambda_2 \in \mathbb{R}$ if and only if $b^2 \geq 4c$.

Result 4.16 (factorisation of a polynomial over $\mathbb{R}$).
Suppose $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial. Then p has a unique factorisation (except for the order of the factors) of the form
$$p(x) = c(x-\lambda_1) \cdots (x-\lambda_m)(x^2+b_1x+c_1) \cdots (x^2+b_Mx+c_M).$$
where $c, \lambda_1, \ldots, \lambda_m, b_1, \ldots, b_M, c_1, \ldots, c_M \in \mathbb{R}$ with $b_k^2 < 4c_k$ for each k.

Chapter 5.A

Definition 5.1 (operator)**.**
A linear map from a vector space to itself is called an *operator*.

Definition 5.2 (invariant subspace)**.**
Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called **invariant** under T if $Tu \in U$ for every $u \in U$.

Definition 5.5 (eigenvalue)**.**
Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Definition 5.8 (eigenvector)**.**
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Notation 5.13 (T^m)**.**
Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

1. $T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m \text{ times}}$.
2. T^0 is defined to be the identity operator I on V.
3. If T is invertible with inverse T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by
$$T^{-m} = (T^{-1})^m.$$

Notation 5.14 ($p(T)$)**.**
Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$. Then $p(T)$ is the operator V defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

Definition 5.16 (product of polynomials)**.**
If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z)$$

for all $z \in \mathbb{F}$.

Result 5.7 (equivalent conditions to be an eigenvalue)**.**
Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Then the following are equivalent.

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Result 5.11 (linearly independent eigenvectors)**.**
Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.

Result 5.12 (operator cannot have more eigenvalues than dimension of vector space)**.**
Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Result 5.17 (multiplicative properties)**.**
Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$. Then

(a) $(pq)(T) = p(T)q(T)$;

(b) $p(T)q(T) = q(T)p(T)$.

Result 5.18 (null space and range of $p(T)$ are invariant under T)**.**
Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then $\operatorname{null} p(T)$ and $\operatorname{range} p(T)$ are invariant under T.

Chapter 5.B

Definition 5.21 (monic polynomial)**.**
A **monic polynomial** is a polynomial whose highest-degree coefficient equals 1.

Definition 5.24 (minimal polynomial)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the **minimial polynomial** of T is the unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$.

Result 5.19 (existence of eigenvalues)**.**
Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.

Result 5.22 (existence, uniqueness, and degree of minimal polynomial)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$. Furthermore, $\deg p \leq \dim V$.

Result 5.27 (eigenvalues are the zeros of the minimal polynomial)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$,

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.

(b) If V is a complex vector space, then the minimal polynomial of T has the form

$$(z - \lambda_1) \cdots (z - \lambda_m),$$

where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.

Result 5.29 ($q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial)**.**
Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial T.

Result 5.31 (minimal polynomial of a restriction operator)**.**
Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$.

Result 5.32 (T not invertible $\iff$ constant term of minimal polynomial of T is 0)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the minimal polynomial of T is 0.

Result 5.33 (even-dimensional null space)**.**
Suppose $\mathbb{F} = \mathbb{R}$ and V is finite-dimensional. Suppose also that $T \in \mathcal{L}(V)$ and $b, c \in \mathbb{R}$ with $b^2 < 4c$. Then $\dim (T^2 + bT + cI)$ is an even number.

Result 5.34 (operators on odd-dimensional vector spaces have eigenvalues)**.**
Every operator of an odd-dimensional vector space has an eigenvalue.

Chapter 5.C

Definition 5.35 (matrix of an operator, $\mathcal{M}(T)$)**.**
Suppose $T \in \mathcal{L}(V)$. The **matrix of** T with respect to a basis $v_1, \ldots, v_n$ of V is the n-by-n matrix

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

The notation $\mathcal{M}(T, (v, 1, \ldots, v_n))$ is used if the basis is not clear from the context.

Definition 5.37 (diagonal of a matrix)**.**
The **diagonal** of a square matrix consists of the entries on the line from the upper left corner to the bottom right corner.

Definition 5.38 (upper-triangular matrix)**.**
A square matrix is called **upper triangular** if all entries below the diagonal are 0.

Result 5.39 (conditions for upper-triangular matrix)**.**
Suppose $T \in \mathcal{L}(V)$ and $v_1 \ldots, v_n$ is a basis for V. Then the following are equivalent.

(a) The matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular.

(b) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

(c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

Result 5.40 (equation satisfied by operator with upper–triangular matrix)**.**
Suppose $T \in \mathcal{L}(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then

$$(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0.$$

Result 5.41 (determination of eigenvalues from upper-triangular matrix)**.**
Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

Result 5.44 (necessary and sufficient condition to have an upper-triangular matrix)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.

Result 5.47 (if $\mathbb{F} = \mathbb{C}$, then every operator on V has an upper triangular matrix)**.**
Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

Chapter 5.D

Definition 5.48 (diagnoal matrix)**.**
A **diagonal matrix** is a square matrix that is 0 everywhere except possibly on the diagonal.

Definition 5.50 (diagonalizable)**.**
An operator V is called **diagonizable** if the operator has a diagonal matrix with respect to some basis V.

Definition 5.52 (eigenspace, $E(\lambda, T)$)**.**
Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **eigenspace** of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) = \text{null}(T - \lambda I) = \{v \in V \mid Tv = \lambda v\}.$$

Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

Definition 5.66 (Gershgorin disks)**.**
Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Let A denote the matrix of T with respect to this basis. A **Gershgorin disk** of T with respect to the basis $v_1, \ldots, v_n$ is a set of the form

$$\left\{ z \in \mathbb{F} \mid |z - A_{j,j}| \leq \sum_{\substack{k=1 \\ k \neq j}}^{n} |A_{j,k}| \right\},$$

where $j \in \{1, \ldots, n\}$.

Result 5.54 (sum of eigenspaces is a direct sum)**.**
Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

Result 5.55 (conditions equivalent to diagonalizability)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Result 5.58 (enough eigenvalues implies diagonalizability)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

Result 5.62 (necessary and sufficient condition for diagonalizability)**.**
Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is diagonalizable if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of distinct numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.

Result 5.65 (restriction of diagonalizable operator to invariant subspace)**.**
Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Then $T|_U$ is a diagonalizable operator on U.

Theorem 5.67 (Gershgorin disk theorem)**.**
Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then each eigenvalue of T is contained in some Gershgorin disk of T with respect to the basis $v_1, \ldots, v_n$.

Chapter 5.E

Definition 5.71 (commute)**.**

1. Two operators S and T on the same vector space **commute** if $ST = TS$.
2. Two square matrices A and B of the same size **commute** if $AB = BA$.

Result 5.74 (commuting operators correspond to commuting matrices)**.**
Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then S and T commute if and only if $\mathcal{M}(S, (v_1, \ldots, v_n))$ and $\mathcal{M}(T, (v_1, \ldots, v_n))$ commute.

Result 5.75 (eigenspace is invariant under commuting operator)**.**
Suppose $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbb{F}$. Then $E(\lambda, S)$ is invariant under T.

Result 5.76 (simultaneous diagonalizability $\iff$ commutativity)**.**
Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if the two operators commute.

Result 5.78 (common eigenvector for commuting operators)**.**
Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.

Result 5.80 (commuting operators are simultaneously upper triangularizable)**.**
Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then there is a basis of V with respect to which both S and T have upper-triangular matrices.

Result 5.81 (eigenvalues of sum and product of commuting operators)**.**
Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then

1. every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T,
2. every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.

Chapter 6.A

Definition 6.1 (dot product)**.**
For $x, y \in \mathbb{R}^n$, the **dot product** of x and y, denoted $x \cdot y$, is defined by
$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$
where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Notation 6.1$\frac{1}{2}$ (complex nonnegative)**.**
For $\lambda \in \mathbb{C}$, the notation $\lambda \geq 0$ means λ is real and nonnegative.

Definition 6.2 (inner product)**.**
An **inner product** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v \rangle \in \mathbb{F}$ and has the following properties.

positivity

$\langle v, v \rangle \geq 0$ for all $v \in V$.

definiteness

$\langle v, v \rangle = 0$ if and only if $v = 0$.

additivity in first slot

$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.

homogeneity in first slot

$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbb{F}$ and all $u, v \in V$.

conjugate symmetry

$\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$.

Definition 6.4 (inner product space)**.**
An **inner product space** is a vector space V along with an inner product V.

Notation 6.5 (V, W)**.**
For chapters 6 and 7, V and W denote inner product spaces over F.

Definition 6.7 (norm, $\|v\|$)**.**
For $v \in V$, the **norm**, denoted by $\|v\|$, is defined by
$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Definition 6.10 (orthogonal)**.**
Two vectors $u, v \in V$ are called **orthogonal** if $\langle u, v \rangle = 0$.

Result 6.6 (basic properties of an inner product)**.**

(a) For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v \rangle$ is a linear map from V to $\mathbb{F}$.

(b) $\langle 0, v \rangle = 0$ for every $v \in V$.

(c) $\langle v, 0 \rangle = 0$ for every $v \in V$.

(d) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbb{F}$ and all $u, v \in V$.

Result 6.9 (basic properties of the norm)**.**
Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbb{F}$.

Result Ex. 6A, 15 (angle between vectors in $\mathbb{R}^2$)**.**
If $u, v \in \mathbb{R}^2$ are non-zero then
$$\langle u, v \rangle = \|u\| \, \|v\| \cos \theta,$$
where θ is the angle between u and v.

Result 6.11 (orthogonality and 0)**.**

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

Theorem 6.12 (Pythagorean theorem)**.**
Suppose $u, v \in V$. If u and v are orthogonal, then
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Result 6.13 (an orthogonal decomposition)**.**
Suppose $u, v \in V$, with $v \neq 0$. Set $c = \frac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \frac{\langle u, v \rangle}{\|v\|^2} v$. Then
$$u = cv + w \quad \text{and} \quad \langle w, v \rangle = 0.$$

Theorem 6.14 (Cauchy-Schwarz inequality)**.**
Suppose $u, v \in V$. Then
$$|\langle u, v \rangle| \leq \|u\| \, \|v\|.$$
This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

Theorem 6.17 (triangle inequality)**.**
Suppose $u, v \in V$. Then
$$\|u + v\| \leq \|u\| + \|v\|.$$
This inequality is an equality if and only if one of u, v is a nonnegative real multiple of the other.

Result Ex. 6A, 20 (reverse triangle inequality)**.**
If $u, v \in V$, then
$$\Big| \|u\| - \|v\| \Big| \leq \|u - v\|.$$

Result 6.21 (parallelogram inequality)**.**
Suppose $u, v \in V$. Then
$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Chapter 6.B

Definition 6.22 (orthonormal).

1. A list of vectors is called **orthonormal** if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
2. In other words, a list $e_1, \ldots, e_m$ of vectors in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

for all $j, k \in \{1, \ldots, m\}$.

Definition 6.27 (orthonormal basis).
An **orthonormal basis** of V is an orthonormal list of vectors in V that is also a basis of V.

Definition 6.39 (linear functional, dual space, V').

1. A **linear functional** on V is a linear map from V to F.
2. The **dual space** of V, denoted by V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

Result 6.24 (norm of an orthonormal linear combination).
Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors V. Then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$.

Corollary 6.25 (orthonormal lists are linearly independent).
Every orthonormal list of vectors is linearly independent.

Result Ex. 6B, 1 (equivalence of square norm sums implies orthonormality).
Suppose $e_1, \ldots, e_m$ is a list of vectors in V such that

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$. Then $e_1, \ldots, e_m$ is an orthonormal list.

Result 6.26 (Bessel's inequality).
Suppose $e_1 \ldots, e_m$ is an orthonormal list of vectors in V. If $v \in V$ then

$$|\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \leq \|v\|^2 .$$

Result 6.28 (orthonormal lists of the right length are orthonormal bases).
Suppose V is finite-dimensional. Then every orthonormal list of vectors in V of length $\dim V$ is an orthonormal basis of V.

Result 6.30 (writing a vector as a linear combination of an orthonormal basis).
Suppose $e_1, \ldots, e_n$ is in orthonormal basis V and $u, v \in V$. Then

(a) $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$,

(b) $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$,

(c) $\langle u, v \rangle = \langle u, e_1 \rangle \overline{\langle v, e_1 \rangle} + \cdots + \langle u, e_n \rangle \overline{\langle v, e_n \rangle}$.

Result 6.32 (Gram-Schmidt procedure).
Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $f_1 = v_1$. For $k = 2, \ldots, m$, define f_k inductively by

$$f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}.$$

For each $k = 1, \ldots, m$, let $e_k = \frac{f_k}{\|f_k\|}$. Then $e_1, \ldots, e_m$ is an orthonormal list of vectors V such that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$$

for each $k = 1, \ldots, m$.

Result 6.35 (existence of orthonormal basis).
Every finite-dimensional inner product space has an orthonormal basis.

Result 6.36 (every orthonormal list extends to an orthonormal basis).
Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Result 6.37 (upper-triangular matrix with respect to some orthonormal basis).
Suppose V is finite-dimentional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.

Theorem 6.38 (Schur's theorem).
Every operator on a finite-dimensional complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.

Theorem 6.42 (Riesz representation theorem).
Suppose V is finite-dimensional and φ is a linear functional on V. Then there is a unique vector $v \in V$ such that

$$\varphi(u) = \langle u, v \rangle$$

for every $u \in V$.

Chapter 6.C

Definition 6.46 (orthogonal complement, $U^\perp$).
If U is a subset of V, then the **orthogonal complement** of U, denoted by $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{v \in V \mid \langle u, v\rangle = 0 \text{ for every } u \in U\}.$$

Definition 6.55 (orthogonal projection, P_U).
Suppose U is a finite-dimensional subspace of V. The **orthogonal projection** of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For each $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then $P_U v = u$.

Definition 6.68 (pseudoinverse, $T^\dagger$).
Suppose that V is finite-dimensional and $T \in \mathcal{L}(V, W)$. The **pseudoinverse** $T^\dagger \in \mathcal{L}(W, V)$ of T is the linear map from W to V defined by

$$T^\dagger w = (T|_{(\operatorname{null} T)^\perp})^{-1} P_{\operatorname{range} T} w$$

for each $w \in W$.

Result 6.48 (properties of orthogonal complement).

(a) If U is a subset of V, then $U^\perp$ is a subspace of V.

(b) $\{0\}^\perp = V$.

(c) $V^\perp = \{0\}$.

(d) If U is a subset of V, then $U \cap U^\perp \subseteq \{0\}$.

(e) If G and H are subsets of V and $G \subseteq H$, then $H^\perp \subseteq G^\perp$.

Result 6.49 (direct sum of a subspace and its orthogonal complement).
Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^\perp$$

Result 6.51 (dimension of orthogonal complement).
Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U^\perp = \dim V - \dim U.$$

Result 6.52 (orthogonal complement of the orthogonal complement).
Suppose U is a fintite-dimensional subspace of V. Then

$$U = \left(U^\perp\right)^\perp.$$

Result 6.54 ($U^\perp = \{0\} \iff U = V$ (for U a finite-dimensional subspace of V)).
Suppose U is a finite-dimensional subspace of V. Then

$$U^\perp = \{0\} \iff U = V.$$

Result 6.57 (properties of orthogonal projection P_U).

(a) $P_U \in \mathcal{L}(V)$;

(b) $P_U u = u$ for every $u \in U$;

(c) $P_U w = 0$ for every $w \in U^\perp$;

(d) $\operatorname{range} P_U = U$;

(e) $\operatorname{null} P_U = U^\perp$;

(f) $v - P_U v \in U^\perp$ for every $v \in V$;

(g) $P_U^2 = P_U$;

(h) $\|P_U v\| \le \|v\|$ for every $v \in V$;

(i) if $e_1, \ldots, e_m$ is an orthonormal basis of U and $v \in V$, then
$$P_U v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m.$$

Theorem 6.58 (Riesz representation theorem, revisited).
Suppose V is finite-dimensional. For each $v \in V$, define $\varphi_v \in V'$ by
$$\varphi_v(u) = \langle u, v\rangle$$
for each $u \in V$. Then $v \to \varphi_v$ is a one-to-one function from V to V'.

Result 6.61 (minimising distance to a subspace).
Suppose U is a finite-dimensional subspace of V, $v \in V$ and $u \in U$. Then
$$\|v - P_U v\| \le \|v - u\|.$$
Furthermore, the inequality above is an equality if and only if $u = P_U v$.

Result 6.67 (restriction of a linear map to obtain a one–to-one and onto map).
Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $T|_{(\operatorname{null} T)^\perp}$ is a one-to-one map of $(\operatorname{null} T)^\perp$ onto $\operatorname{range} T$.

Result 6.69 (algebraic properties of the pseudoinverse).
Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) If T is invertible, then $T^\dagger = T^{-1}$.

(b) $TT^\dagger = P_{\operatorname{range} T}$ = the orthogonal projection of W onto $\operatorname{range} T$.

(c) $T^\dagger T = P_{(\operatorname{null} T)^\perp}$ = the orthogonal projection of V onto $(\operatorname{null} T)^\perp$.

Result 6.70 (pseudoinverse provides best approximate solution or best solution).
Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and $b \in W$.

(a) If $x \in V$, then
$$\left\|T(T^\dagger b) - b\right\| \le \|Tx - b\|,$$
with equality if and only if $x \in T^\dagger b + \operatorname{null} T$.

(b) If $x \in T^\dagger b + \operatorname{null} T$, then
$$\left\|T^\dagger b\right\| \le \|x\|,$$
with equality if and only if $x = T^\dagger b$.

Chapter 7.A

Definition 7.1 (adjoint, T^*).
Suppose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w\rangle = \langle v, T^*w\rangle$$

for every $v \in V$ and every $w \in W$.

Definition 7.7 (conjugate transpose, A^*).
The **conjugate transpose** of an m-by-n matrix A is the n-by-m matrix A^* obtained by interchanging the rows and columns and then taking the complex conjugate of each entry. In other words if $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then

$$(A^*)_{j,k} = \overline{A_{k,j}}.$$

Definition 7.10 (self-adjoint).
An operator $T \in \mathcal{L}(V)$ is called **self-adjoint** if $T = T^*$.

Definition 7.18 (normal).

1. An operator on an inner product space is called **normal** if it commutes with its adjoint.
2. In other words, $T \in \mathcal{L}(V)$ is normal if $TT^* = T^*T$.

Result 7.4 (adjoint of a linear map is a linear map).
If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.

Result 7.5 (properties of the adjoint).
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $(S + T)^* = S^* + T^*$ for all $S \in \mathcal{L}(V, W)$;

(b) $(\lambda T)^* = \bar{\lambda}T^*$ for all $\lambda \in \mathbb{F}$;

(c) $(T^*)^* = T$;

(d) $(ST)^* = T^*S^*$ for all $S \in \mathcal{L}(W, U)$ where U is a finite-dimensional inner product space over $\mathbb{F}$;

(e) $I^* = I$, where I is the identity operator on V;

(f) if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Result 7.6 (null space and range of T^*).
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\operatorname{null} T^* = (\operatorname{range} T)^\perp$;

(b) $\operatorname{range} T^* = (\operatorname{null} T)^\perp$;

(c) $\operatorname{null} T = (\operatorname{range} T^*)^\perp$;

(d) $\operatorname{range} T = (\operatorname{null} T^*)^\perp$.

Result 7.9 (matrix of T^* equals conjugate transpose of matrix of T).
Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Then $\mathcal{M}(T^*, (f_1, \ldots, f_m), (e_1, \ldots, e_n))$ is the conjugate transpose of $\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$. In other words,

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*.$$

Result 7.12 (eigenvalues of self-adjoint operators).
Every eigenvalue of a self-adjoint operator is real.

Result 7.13 (Tv is orthogonal to v for all $v \iff T = 0$ (assuming $\mathbb{F} = \mathbb{C}$)).
Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$\langle Tv, v\rangle = 0 \text{ for every } v \in V \iff T = 0.$$

Result 7.14 ($\langle Tv, v\rangle$ is real for all $v \iff T$ is self-adjoint (assuming $\mathbb{F} = \mathbb{C}$)).
Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then

$$T \text{ is self-adjoint} \iff \langle Tv, v\rangle \in \mathbb{R} \text{ for every } v \in V.$$

Result 7.16 (T self-adjoint and $\langle Tv, v\rangle = 0$ for all $v \iff T = 0$).
Suppose T is a self-adjoint operator on V. Then

$$\langle Tv, v\rangle = 0 \text{ for every } v \in V \iff T = 0.$$

Result 7.20 (T is normal if and only if Tv and T^*v have the same norm).
Suppose $T \in \mathcal{L}(V)$. Then

$$T \text{ is normal} \iff \|Tv\| = \|T^*v\| \text{ for ever } v \in V.$$

Result 7.21 (range, null space and eigenvectors of a normal operator).
Suppose $T \in \mathcal{L}(V)$ is normal. Then

(a) $\operatorname{null} T = \operatorname{null} T^*$;

(b) $\operatorname{range} T = \operatorname{range} T^*$;

(c) $V = \operatorname{null} T \oplus \operatorname{range} T$;

(d) $T - \lambda I$ is normal for every $\lambda \in \mathbb{F}$;

(e) if $v \in V$ and $\lambda \in \mathbb{F}$, then $Tv = \lambda v$ if and only if $T^*v = \bar{\lambda}v$.

Result 7.22 (orthogonal eigenvectors for normal operators).
Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

Result 7.23 (T is normal $\iff$ the real and imaginary parts of T commute).
Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$.

Chapter 7.B

Result 7.26 (invertible quadratic expressions)**.**
Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$. Then

$$T^2 + bT + cI$$

is an invertible operator.

Result 7.27 (minimal polynomial of self-adjoint operator)**.**
Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Then the minimal polynomial of T equals $(z-\lambda_1)\cdots(z-\lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$.

Theorem 7.29 (real spectral theorem)**.**
Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is self-adjoint.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

Theorem 7.31 (complex spectral theorem)**.**
Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is normal.

(b) T has a diagonal matrix with respect to some orthonormal basis of V.

(c) V has an orthonormal basis consisting of eigenvectors of T.

Chapter 7.C

Definition 7.34 (positive operator)**.**
An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$.

Definition 7.36 (square root)**.**
An operator R is called a **square root** of an operator T if $R^2 = T$.

Notation 7.40 ($\sqrt{T}$)**.**
For T a positive operator, $\sqrt{T}$ denotes the unique positive square root of T.

Result 7.38 (characterisation of positive operators)**.**
Let $T \in \mathcal{L}(V)$. Then the following are equivalent.

(a) T is a positive operator.

(b) T is self-adjoint and all eigenvalues of T are nonnegative.

(c) With respect to some orthonormal basis of V, the matrix of T is a diagonal matrix with only nonnegative numbers on the diagonal.

(d) T has a positive square root.

(e) T has a self-adjoint square root.

(f) $T = R^*R$ for some $R \in \mathcal{L}(V)$.

Result 7.39 (each positive operator has only one positive square root)**.**
Every positive operator on V has a unique positive square root.

Result 7.43 (T positive and $\langle Tv, v \rangle = 0 \Longrightarrow Tv = 0$)**.**
Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.

Chapter 7.D

Definition 7.44 (isometry).
A linear map $S \in \mathcal{L}(V, W)$ is called an **isometry** if

$$\|Sv\| = \|v\|$$

for every $v \in V$. In other words, a linear map is an isometry if it preserves norms.

Definition 7.51 (unitary operator).
An operator $S \in \mathcal{L}(V)$ is called **unitary** if S is an invertible isometry.

Definition 7.56 (unitary matrix).
An n-by-n matrix is called **unitary** if its columns form an orthonormal list in $\mathbb{F}^n$.

Definition 7.62 (positive definite).
A matrix $B \in \mathbb{F}^{n,n}$ is called **positive definite** if $B^* = B$ and

$$\langle Bx, x\rangle > 0$$

for every nonzero $x \in \mathbb{F}^n$.

Result 7.49 (characterisation of isometries).
Suppose $S \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthnormal basis of W. Then the following are equivalent.

(a) S is an isometry.

(b) $S^*S = I$.

(c) $\langle Su, Sv\rangle = \langle u, v\rangle$ for all $u, v \in V$.

(d) $Se_1, \ldots, Se_n$ is an orthonromal list in W.

(e) The columns of $\mathcal{M}(S, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$ form an orthonormal list in $\mathbb{F}^m$ with respect to the Euclidean inner product.

Result 7.53 (characterisation of unitary operators).
Suppose $S \in \mathcal{L}(V)$. Suppose $e_1, \ldots, e_n$ is an orthnormal basis of V. Then the following are equivalent.

(a) S is a unitary operator.

(b) $S * S = SS* = I$.

(c) S is invertible and $S^{-1} = S^*$.

(d) $Se_1, \ldots, Se_n$ is an orthonormal basis of V.

(e) The rows of $\mathcal{M}(S, (e_1, \ldots, e_n))$ form an orthonormal basis of $\mathbb{F}^n$ with respect to the Euclidean inner product.

(f) S^* is a unitary operator.

Result 7.54 (eigenvalues of unitary operators have absolute value 1).
Suppose λ is an eigenvalue of a unitary operator. Then $|\lambda| = 1$.

Result 7.55 (description of unitary operators on complex inner product spaces).
Suppose $\mathbb{F} = \mathbb{C}$ and $S \in \mathcal{L}(V)$. Then the following are equivalent.

(a) S is a unitary operator.

(b) There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have asbolute value 1.

Result 7.57 (characterisation of unitary matrices).
Suppose Q is an n-bu-n matrix. Then the following are equivalent.

(a) Q is a unitary matrix.

(b) The rows of Q form an orthonormal list in $\mathbb{F}^n$.

(c) $\|Qv\| = \|v\|$ for every $v \in \mathbb{F}^n$

(d) $Q^*Q = QQ^* = I$, the n-by-n matrix with 1's on the diagonal and 0's elsewhre.

Result 7.58 (QR factorisation).
Suppose A is a square matrix with linearly independent columns. Then there exist unique matrices Q and R such that Q is unitary, R is upper triangular with only positive numbers on its diagonal, and

$$A = QR.$$

Result 7.61 (positive invertible operator).
A self-adjoint operator $T \in \mathcal{L}(VB)$ is a positive invertible operator if and only if $\langle Tv, v\rangle > 0$ for every nonzero $v \in V$.

Result 7.63 (Cholesky factorisation).
Suppose B is a positive definite matrix. Then there exists a unique upper-triangular matrix R with only positive numbers on its diagonal such that

$$B = R^*R.$$

Chapter 7.E

Definition 7.65 (singular values)**.**
Suppose $T \in \mathcal{L}(V, W)$. The **singular values** of T are the nonnegative square roots of the eigenvalues of T^*T, listed in decreasing order, each included as many times as the dimension of the corresponding eigenspace of T^*T.

Definition 7.74 (diagonal matrix)**.**
An M-by-N matrix A is called a **diagonal mtarix** if all entries of the matrix are 0 except possibly $A_{k,k}$ for $k = 1, \ldots, \min\{M, N\}$.

Result 7.64 (properties of T^*T)**.**
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) T^*T is a positive operator on V;

(b) $\operatorname{null} T^*T = \operatorname{null} T$;

(c) $\operatorname{range} T^*T = \operatorname{range} T^*$;

(d) $\dim \operatorname{range} T = \dim \operatorname{range} T^* = \dim \operatorname{range} T^*T$.

Result 7.68 (role of positive singular valuse)**.**
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) T is injective $\iff$ 0 is not a singular value of T;

(b) the number of positive singular values of T equals $\dim \operatorname{range} T$;

(c) T is surjective $\iff$ number of positive singular values of T equals $\dim W$.

Result 7.69 (isometries characterised by having all singular values equal to 1)**.**
Suppose that $S \in \mathcal{L}(V, W)$. Then

$$S \text{ is an isometry} \iff \text{all singular values of } S \text{ equal } 1.$$

Result 7.70 (singular value decomposition)**.**
Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Then there exist orthonormal lists $e_1, \ldots, e_m$ in V and $f_1, \ldots . f_m$ in W such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdot + s_m \langle v, e_m \rangle f_m \tag{7.71}$$

for every $v \in V$.

Result 7.75 (singular value decomposition of adjoint and pseudoinverse)**.**
Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Suppose $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ are orthonormal lists in V and W such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_m \langle v, e_m \rangle f_m \tag{7.76}$$

for every $v \in V$. Then

$$T^*w = s_1 \langle w, f_1 \rangle e_1 + \cdots + s_m \langle w, f_m \rangle e_m \tag{7.77}$$

and

$$T^\dagger w = \frac{\langle w, f_1 \rangle}{s_1} e_1 + \cdots \frac{\langle w, f_m \rangle}{s_m} e_m \tag{7.78}$$

for every $w \in W$.

Result 7.80 (matrix versoin of SVG)**.**
Suppose A is an M-by-N matrix of rank $m \geq 1$. Then there exist an M-by-m matrix B with orthonormal columns, an m-by-m diagonal matrix D with positive numbers on the diagonal, and an n-by-m matrix C with orthonormal columns such that

$$A = BDC^*.$$

Chapter 7.F

Definition 7.86 (norm of a linear map, $\|\cdot\|$)**.**
Suppose $T \in \mathcal{L}(V, w)$. Then the **norm** of T, deonted by $\|T\|$, is defined by

$$\|T\| = \max\{\|Tv\| \mid v \in V \text{ and } \|v\| \leq 1\}.$$

Result 7.82 (upper bound for $\|Tv\|$)**.**
Suppose $T \in \mathcal{L}(V, W)$. Let s_1 be the largest singular value of T. Then

$$\|Tv\| \leq s_v \|v\|$$

for all $v \in V$.

Result 7.87 (basic properties of norms of linear maps)**.**
Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\|T\| \geq 0$;

(b) $\|t\| = 0 \Longleftrightarrow T = 0$;

(c) $\|\lambda T\| = |\lambda| \|T\|$ for all $\lambda \in \mathbb{F}$;

(d) $\|S + T\| \leq \|T\| + \|T\|$ for all $S \in \mathcal{L}(V, W)$.

Result 7.91 (norm of the adjoint)**.**
Suppose $T \in \mathcal{L}(V, W)$. Then $\|T^*\| = \|T\|$.

Result 7.92 (best approximation by linear map whose range has dimension $\leq k$)**.**
Suppose $T \in \mathcal{L}(V, W)$ and $s_1 \geq \cdots \geq s_m$ are the positive singular values of T. Suppose $1 \leq k < m$. Then

$$\min\{\|T - S\| \mid S \in \mathcal{L}(V, W) \text{ and } \dim \operatorname{range} S \leq k\} = s_{k+1}.$$

Furthermore, if

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_k \langle v, e_k \rangle f_k$$

is a singular value decomposition of T and $T_k \in \mathcal{L}(V, W)$ is defined by

$$T_k v = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_k \langle v, e_k \rangle f_k$$

for each $v \in V$, then $\dim \operatorname{range} T_k = k$ and $\|T - T_k\| = s_{k+1}$.

Result 7.93 (polar decomposition)**.**
Suppose $T \in \mathcal{L}(V)$. Then there exists a unitary operator $S \in \mathcal{L}(V)$ such that

$$T = S\sqrt{T^*T}.$$

Definition 7.95 (ball, B).
The **ball** in V of radius 1 centered on 0, denoted by B, is defined by

$$B = \{v \in V \mid \|v\| < 1\}.$$

Definition 7.96 (ellipsoid, $E(s_1 f_1, \ldots, s_n f_n)$, principal axes).
Suppose that $f_1, \ldots, f_m$ is an orthonormal basis of V and $s_1, \ldots, s_n$ are positive numbers. The **elliposoid** $E(s_1 f_1, \ldots, s_n f_n)$ with **principal axes** $s_1, f_1, \ldots, s_n f_n$ is defined by

$$E(s_1 f_1, \ldots, s_n f_n) = \left\{ v \in V \mid \frac{|\langle v, f_1\rangle|^2}{s_1^2} + \cdots + \frac{|\langle v, f_n\rangle|^2}{s_n^2} < 1 \right\}.$$

Notation 7.98 ($T(\Omega)$).
For T a function defined on V and $\Omega \subseteq V$, define $T(\Omega)$ by

$$T(\Omega) = \{Tv \mid v \in \Omega\}.$$

Definition 7.102 ($P(v_1, \ldots, v_n)$, parallelepiped).
Suppose $v_1, \ldots, v_n$ is a basis for V. Let

$$P(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots + a_n v_n \mid a_1, \ldots, a_n \in (0, 1)\}.$$

A **parallelepiped** is a set of the form $u + P(v_1, \ldots, v_n)$ for some $u \in V$. The vectors $v_1, \ldots, v_n$ are called the **edges** of this parallelepiped.

Definition 7.105 (box).
A **box** in V is a set of the form

$$u + P(r_1 e_1, \ldots, r_n e_n),$$

where $u \in V$ and $r_1, \ldots, r_n$ are positive numbers and $e_1, \ldots, e_n$ is an orthonormal basis of V.

Definition 7.108 (volume of a box).
Suppose $\mathbb{F} = \mathbb{R}$. If $u \in V$ and $r_1, \ldots, r_n$ are positive numbers and $e_1, \ldots, e_n$ is an orthonormal basis of V, then

$$\text{volume}(u + P(r_1 e_1, \ldots, r_n e_n)) = r_1 \times \cdots \times r_n.$$

Definition 7.109 (volume).
Suppose $\mathbb{F} = \mathbb{R}$ and $\Omega \subseteq V$. Then the **volume** of Ω, denoted by volume Ω, is approximately the sum of the volumes of a collection of disjoint boxes that approximate Ω.

Result 7.99 (invertible operator takes ball to ellipsoid).
Suppose $T \in \mathcal{L}(V)$ is invertible. Then T maps the ball B in V onto an ellipsoid in V.

Result 7.101 (invertible operator takes ellipsoids to ellipsoids).
Suppose $T \in \mathcal{L}(V)$ is invertible and E is an ellipsoid in V. Then $T(E)$ is an ellipsoid in V.

Result 7.104 (invertible operator takes parallelepipeds to parallelepipeds).
Suppose $u \in V$ and $v_1, \ldots, v_n$ is a basis of V. Suppose $T \in \mathcal{L}(V)$ is invertible. Then

$$T(u + P(v_1, \ldots, v_n)) = Tu + P(Tv_1, \ldots, Tv_n).$$

Result 7.107 (every invertible operator takes some boxes to boxes).
Suppose $T \in L(V)$ is invertible. Suppose T has singular value decomposition

$$Tv = s_1 \langle v, e_1\rangle f_1 + \cdots + s_n \langle v, e_n\rangle f_n,$$

where $s_1, \ldots, s_n$ are singular values of T and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of V and the equation above holds for all $v \in V$. Then T maps the box $u + P(r_1 e_1, \ldots, r_n e_n)$ onto the box $Tu + P(r_1 s_1 f_1, \ldots, r_n s_n f_n)$ for all positive numbers $r_1, \ldots, r_n$ and all $u \in V$.

Result 7.111 (volume changes by a factor of the product of the singular values).
Suppose $\mathbb{F} = \mathbb{R}, T \in \mathcal{L}(V)$ is invertible, and $\Omega \subseteq V$. Then

$$\text{volume}\, T(\Omega) = (\text{product of singular values of } T)(\text{volume}\, \Omega).$$