Vector Space Bijection

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Bijections of Vectors Spaces

Result 1.0 (Bijections of Vectors Spaces are Vector Spaces)**.**
Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$ and X a set such that

$$f : V \to X$$

is a bijection (one-to-one and onto). Then we can understand X as a vector space over $\mathbb{F}$ by setting

$$\begin{aligned} + &: X \times X \to X \\ \times &: \mathbb{F} \times X \to X \end{aligned}$$

such that

$$\begin{aligned} v + w &\mapsto f(f^{-1}(v) + f^{-1}(w)) \\ av &\mapsto f(af^{-1}(v)) \end{aligned}$$

Proof. We need to demonstrate that the set X satisfies the definition for a vector space (Definition 1.20 in Axler 4th edition): commutativity, associativity, additive identity, additive inverse, multiplicative identity and the distributive property.

In the steps below, where it says "by bijectivity of f" we're making use of the fact that f^{-1} is the inverse of f, so $f^{-1} \circ f = 1$ (the identity map), hence we can add or remove cases of $f(f^{-1}(x))$ at will.

Commutativity

Let $u, v \in X$. Then

$$\begin{aligned} u + v &= f(f^{-1}(u) + f^{-1}(v)) && \text{by definition of } +, \\ &= f(f^{-1}(v) + f^{-1}(u)) && \text{by commutativity in } V, \\ &= v + u. \end{aligned}$$

Associativity

Let $u, v, w \in X$. Then

$$\begin{aligned} (u + v) + w &= f(f^{-1}(f(f^{-1}(u) + f^{-1}(v))) + f^{-1}(w)) && \text{by definition of } +, \\ &= f((f^{-1}(u) + f^{-1}(v)) + f^{-1}(w)) && \text{by bijectivity of } f, \\ &= f(f^{-1}(u) + (f^{-1}(v) + f^{-1}(w))) && \text{by associativity in } V, \\ &= f(f^{-1}(u) + f^{-1}(f(f^{-1}(v) + f^{-1}(w)))) && \text{by bijectivity of } f, \\ &= u + (v + w) && \text{by definition of } +. \end{aligned}$$

Let $a, b \in \mathbb{F}$ and $v \in V$. Then

$$\begin{aligned} (ab)v &= f((ab)f^{-1}(v)) && \text{by definition of } \times, \\ &= f(a(bf^{-1}(v))) && \text{by associativity in } V, \\ &= f(af^{-1}(f(bf^{-1}(v)))) && \text{by bijectivity of } f, \\ &= f(af^{-1}(bv)) && \text{by definition of } \times, \\ &= a(bv) && \text{by definition of } \times. \end{aligned}$$

Additive identity

Since V is a vector space we know that there is an additive identity $0_V \in V$. Set $0_X = f^{-1}(0_V)$. We claim that 0_X is the additive identity in X:

$$\begin{aligned}
v + 0_X &= f(f^{-1}(v) + f^{-1}(0_X)) && \text{by definition of } +, \\
&= f(f^{-1}(v) + 0_V) && \text{by definition of } 0_X, \\
&= f(f^{-1}(v) && \text{since } 0_V \text{ is the additive identity in } V, \\
&= v && \text{by bijectivity of } f.
\end{aligned}$$

Additive inverse

For $v \in X$ we know that $f^{-1}(v) \in V$. Since V is a vector space we know that $f^{-1}(v)$ has an additive inverse $-f^{-1}(v) \in V$. Set $w = f(-f^{-1}(v)) \in X$. Then

$$\begin{aligned}
v + w &= f(f^{-1}(v) + f^{-1}(f(-f^{-1}(v))) && \text{by definition of } +, \\
&= f(f^{-1}(v) - f^{-1}(v)) && \text{by bijectivity of } f, \\
&= f(0_V) && \text{by additiive inverse in } V, \\
&= 0_X && \text{by definition of } 0_X.
\end{aligned}$$

Multiplicative identity

Let $v \in X$. Then

$$\begin{aligned}
1v &= f(1 \times f^{-1}(v)) && \text{by definition of } \times, \\
&= f(f^{-1}(v)) && \text{by multiplicative identity in } V, \\
&= v && \text{by bijectivity of } f.
\end{aligned}$$

Distributive property

Let $a \in \mathbb{F}$ and $u, v \in X$. Then

$$\begin{aligned}
a(u + v) &= f(af^{-1}(u + v)) && \text{by definition of } \times, \\
&= f(\ f^{-1}(f(f^{-1}(u) + f^{-1}(v)))) && \text{by definition of } +, \\
&= f(a(f^{-1}(u) + f^{-1}(v))) && \text{by bijectivity of } f, \\
&= f(af^{-1}(u) + af^{-1}(v)) && \text{by distributivity in } V, \\
&= f(f^{-1}(f(af^{-1}(u))) + f^{-1}(f(af^{-1}(v)))) && \text{by bijectivity of } f, \\
&= f(f^{-1}(au) + f^{-1}(av)) && \text{by definition of } \times, \\
&= au + av && \text{by definition of } +.
\end{aligned}$$

Let $a, b \in \mathbb{F}$ and $v \in X$. Then

$$\begin{aligned}
(a + b)v &= f((a + b)f^{-1}(v)) && \text{by definition of } \times, \\
&= f(af^{-1}(v) + bf^{-1}(v)) && \text{by distributivity in } V, \\
&= f(f^{-1}(f(af^{-1}(v))) + f^{-1}(f(bf^{-1}(v)))) && \text{by bijectivity of } f, \\
&= f(f^{-1}(av) + f^{-1}(bv)) && \text{by definition of } \times, \\
&= av + bv && \text{by definition of } +.
\end{aligned}$$

Since all of the properties are met it follows that X is a vector space when $+$ and $\times$ are as defined. $\square$

Unit Sphere as a Vector Space

The above tells us that if we can find a bijection from a vector space to any other set, then we can also represent the set as a vector space. Let's define S^2 to be the unit sphere with centre $(0, 0, 0)$ like this:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The sets S^2 and $\mathbb{R}^2$ have the same cardinality so we can find a bijection between them which we *could* use to impose a vector space structure on S^2.

However, as already noted during the session there is no bijection between $\mathbb{R}^2$ and a sphere that preserves continuity. A bijection that is continuous and with a continuous inverse is called a **homeomorphism**[1] and is what we'd ideally want for the vector space on the sphere to be useful.

On the other hand we *can* find a homeomorphsim between $\mathbb{R}^2$ and a sphere with a single point (usually one of the poles) removed. One example is the stereographic projection. Embed $\mathbb{R}^2$ into $\mathbb{R}^3$ as the plane

$$\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}.$$

Then $S^2 \setminus \{(0, 0, 1)\}$ is the sphere with the north pole removed. Let $\mathcal{M} = S^2 \setminus \{(0, 0, 1)\}$.

We can find a homeomorphism between $\mathbb{R}^2$ and $\mathcal{M}$ as follows. For any point $(x, y, 0) \in \mathbb{R}^2$ draw a line from the north pole of the sphere through the point $(x, y, 0)$ on the plane. The line will intersect the sphere at exactly one point (other than the north pole). The point at which it crosses is the point on the sphere that $(x, y, 0)$ maps to.

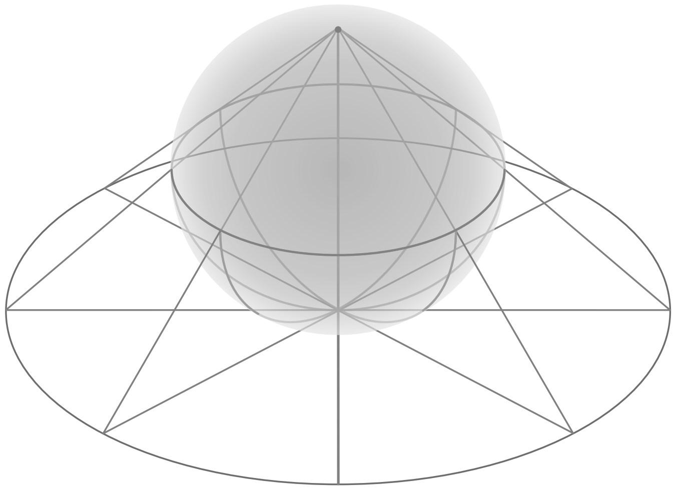

Figure 1: Stereographic projection between the plane $\mathbb{R}^2$ and the north-poleless-sphere $\mathcal{M}$. Image from Wikipedia, Mark Howison and CheChe, CC BY-SA 4.0 Deed[2].

Using this approach every point on the $\mathbb{R}^2$ plane maps uniquely to a point on $\mathcal{M}$ and any point on $\mathcal{M}$ maps uniquely to a point on $\mathbb{R}^2$.

We can define the mapping $f : \mathbb{R}^2 \to \mathcal{M}$ and its inverse $f^{-1} : \mathcal{M} \to \mathbb{R}^2$ as follows.

$$\begin{aligned} f(x, y) &\mapsto \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right), \\ f^{-1}(x, y, z) &\mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right). \end{aligned}$$

For the derivation see Wikipedia[3].

Since f is a bijection we can consider $\mathcal{M}$, the unit sphere minus a pole, as being a vector space by defining vector addition and scalar multiplication as earlier. So let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathcal{M} \subset \mathbb{R}^3$ and $a \in \mathbb{R}$ then

$$\begin{aligned} (x_1, y_1, z_1) + (x_2, y_2, z_2) &\mapsto f(f^{-1}((x_1, y_1, z_1)) + f^{-1}((x_2, y_2, z_2))) \\ a(x_1, y_1, z_1) &\mapsto f(af^{-1}((x_1, y_1, z_1))). \end{aligned}$$

If we want (I'm not sure we do) we can expand these out by working through all of the functions and inverse functions. The formula for addition of vectors in X then becomes the following.

$$\begin{aligned} (x_1, y_1, z_1) + (x_2, y_2, z_2) = \Bigg(&\frac{2(x_1(1 - z_2) + x_2(1 - z_1)(1 - z_1)^2(1 - z_2)^2}{(1 - z_1)^2(1 - z_2)^2 + (x_1(1 - z_2) + x_2(1 - z_1)^2 + (y_1(1 - z_2) = y_2(1 - z_1))^2)}, \\ &\frac{2(y_1(1 - z_2) + y_2(1 - z_1)(1 - z_1)^2(1 - z_2)^2}{(1 - z_1)^2(1 - z_2)^2 + (x_1(1 - z_2) + x_2(1 - z_1)^2 + (y_1(1 - z_2) = y_2(1 - z_1))^2)}, \\ &\frac{(x_1(1 - z_2) + x_2(1 - z_1))^2 + (y_1(1 - z_2) + y_2(1 - z_1))^2 - (1 - z_1)^2(1 - z_2)^2}{(x_1(1 - z_2) + x_2(1 - z_1))^2 + (y_1(1 - z_2) + y_2(1 - z_1))^2 + (1 - z_1)^2(1 - z_2)^2} \Bigg). \end{aligned}$$

[1] https://en.wikipedia.org/w/index.php?title=Homeomorphism&oldid=1185280979
[2] https://commons.wikimedia.org/wiki/File:Stereographic_projection_in_3D.svg
[3] https://en.wikipedia.org/w/index.php?title=Stereographic_projection&oldid=1183460764

While the formula for scalar multiplication is the following.

$$a \times (x, y, z) = \left(\frac{2ax(1-z)}{(1-z)^2 + a^2x^2 + a^2y^2}, \frac{2ay(1-z)}{(1-z)^2 + a^2x^2 + a^2y^2}, \frac{a^2x^2 + a^2y^2 - (1-z)^2}{a^2x^2 + a^2y^2 + (1-z)^2} \right)$$

In these two equations addition, multiplication and division are all operations on the underlying field $\mathbb{R}$. These remind me of the formulas for addition and scalar multiplication in $\mathbb{C}$ from Axler (Definition 1.1): although they don't look much like addition and multiplication they still satisfy all of properties from Definition 1.20.

Since $\{(1,0),(0,1)\}$ is a basis for $\mathbb{R}^2$ we can find a basis for $\mathcal{M}$ as:

$$\begin{aligned}
\{f((1,0)), f((0,1))\} &= \left\{ \left(\frac{2\times 1}{1+1^2+0^2}, \frac{2\times 0}{1+1^2+0^2}, \frac{-1+1^2+0^2}{1+1^2+0^2} \right), \left(\frac{2\times 0}{1+0^2+1^2}, \frac{2\times 1}{1+0^2+1^2}, \frac{-1+0^2+1^2}{1+0^2+1^2} \right) \right\} \\
&= \left\{ \left(\frac{2}{2}, \frac{0}{2}, \frac{0}{2} \right), \left(\frac{0}{2}, \frac{2}{2}, \frac{0}{2} \right) \right\} \\
&= \left\{ (1,0,0), (0,1,0) \right\}.
\end{aligned}$$

The additive identity element in $\mathcal{M}$ is:

$$\begin{aligned}
f((0,0)) &= \left(\frac{2\times 0}{1+0^2+0^2}, \frac{2\times 0}{1+0^2+0^2}, \frac{-1+0^2+0^2}{1+0^2+0^2} \right) \\
&= (0,0,-1).
\end{aligned}$$

It's worth noting that all three of these are (thankfully) points on the unit sphere.