All we know about vector spaces

21 October 2023

Set theory

Sets

Set	A collection of things, called its *elements* (or *members*).
$x \in X$	"x is an element of set X."
$X \cup Y$	The **union** of X and Y: all elements of X together with all elements of Y.
$X \cap Y$	The **intersection** of X and Y; its elements are those that are elements of both X and Y.
$X \setminus Y$	The **set difference**: all elements of X that are not elements of Y. Sometimes $X - Y$.
$X \subset Y$	"Every element of X is an element of Y." X is a *subset* of Y.
$\{x \in X \mid P(x)\}$	The set of those elements of X satisfying the predicate P (*i.e.*, for which $P(x)$ is true).
$\emptyset$	The **empty set**—the set with no members.

Pairs and tuples

Pair	An ordered list of two things: like a set but the order matters and they can be the same thing. For example: $(1, 2)$.
n**-tuple**	An ordered list of n things. For example: $(x_1, x_2, \ldots, x_n)$.
$X \times Y$	The **Cartesian product** of X and Y is the set of all pairs (x, y) where $x \in X$ and $y \in Y$.
X^n	The set of all n-tuples of elements of X. The Cartesian product of X with itself n times.

Maps

Map	A rule which assigns, to every element of a set (called the *domain*), an element of another set (called the *codomain*). Sometimes called a *function*, especially when the codomain is numbers.
$f\colon X \to Y$	"f is a map from X to Y."
$f\colon x \mapsto y$	"Specifically, f maps the element $x \in X$ to the element $y \in Y$."
$f \circ g$	The map g followed by the map f.
Injection	A map, $f\colon X \to Y$, is *injective* (also *one-to-one*) if no more than one element of X maps to a particular element of Y.
Surjection	A map, $f\colon X \to Y$, is *surjective* (also *onto*) if every element of Y is mapped to by some element of X.
Bijection	A map, $f\colon X \to Y$, is *bijective* if it is both injective and surjective (also "one-to-one and onto").
$X \cong Y$	There exists a bijection between sets X and Y. Equivantly: X and Y are *isomorphic as sets*.

Useful maps

Operator A *binary operator* on a set $\mathbf{X}$ is a map $\star\colon \mathbf{X} \times \mathbf{X} \to \mathbf{X}$. That is, a binary operator takes two elements of $\mathbf{X}$ and returns an element of $\mathbf{X}$. Operators are typically written in "infix" notation, $x \star y$, rather than in a function notation, $\star(x, y)$.

Associative A binary operator, $\star$, is *associative* if

$$(a \star b) \star c = a \star (b \star c).$$

Almost all operators you will meet are associative. Since the order in which the operations are carried out doesn't affect the result, the parentheses are often omitted, as in $a \star b \star c$.

Commutative A binary operator, $\star$, is *commutative* if

$$a \star b = b \star a.$$

Numbers

$\mathbf{N}$	The *natural numbers:* $0, 1, 2, \ldots$.
$\mathbf{Z}$	The *integers:* $\ldots, -2, -1, 0, 1, 2, \ldots$.
$\mathbf{Q}$	The *rationals:* All numbers that can be written as m/n where m and n are integers.
$\mathbf{R}$	The *reals:* The rationals and "all the numbers in between."
$\mathbf{R}^+$	The non-negative reals.
$\mathbf{C}$	The *complex numbers:* Numbers of the form $a + bi$, where a and b are real numbers and $i^2 = -1$.

Vector spaces

Vector space A *real vector space* is a set, V, together with: (i) a commutative, associative, binary operator, $+$, on V; (ii) a map, $\cdot\colon \mathbf{R}\times V \to V$; and (iii) a distinguished element $0 \in V$, such that:

1. $v + 0 = v$ for all $v \in V$;
2. For any $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = 0$;
3. $1 \cdot v = v$ for all $v \in V$;
4. $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$ for all $\alpha, \beta \in \mathbf{R}$ and all $v \in V$.
5. $\alpha \cdot (v + w) = \alpha \cdot v + \alpha \cdot w$ and $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$ for all $\alpha, \beta \in \mathbf{R}$ and $v, w \in V$.

Replacing $\mathbf{R}$ with $\mathbf{C}$ in the above we obtain a *complex vector space*.

Vector An element of a vector space.

Subspace A subset of a vector space that is itself a vector space with respect to the addition and scalar multiplication inherited from the larger space.
Equivalently: A subset, $U \subset V$ of a vector space, V, is a *subspace* if, for all $u, v \in U$ and number α, the combination $u + \alpha \cdot v$ is also in U.

Examples of vector spaces

$\mathbf{R}^n$ (or $\mathbf{C}^n$) The set of n-tuples of $\mathbf{R}$ (or $\mathbf{C}$), together with the operation of "element-wise" addition:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) \stackrel{\text{def}}{=} (x_1 + y_1, \ldots, x_n + y_n)$$

and multiplication by $\mathbf{R}$ (or $\mathbf{C}$):

$$\lambda \cdot (x_1, \ldots, x_n) \stackrel{\text{def}}{=} (\lambda x_1, \ldots, \lambda x_n)$$

$\mathbf{R}^X$ (or $\mathbf{C}^X$) For X a set, $\mathbf{R}^X$ denotes the set of all functions $X \to \mathbf{R}$, together with the operation of "pointwise addition":

$$(f + g)(x) \stackrel{\text{def}}{=} f(x) + g(x)$$

and "pointwise multiplication by $\mathbf{R}$":

$$(\alpha f)(x) \stackrel{\text{def}}{=} \alpha f(x).$$

(The notation "$(f + g)(x)$" means "the function $f + g$, where $+$ is addition of functions, evaluated at the point x.")

$\mathcal{P}_m(\mathbf{R})$ The set of polynomials of degree at most m. Thus, $f \in \mathcal{P}_m(\mathbf{R})$ implies $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ for some m.

$\mathcal{P}(\mathbf{R})$ The set of all polynomials of finite degree.

Combining vector spaces

Sum (This definition is not commonplace.) For $U_1, U_2, \ldots, U_n$ subspaces of V, their sum is the set of all sums of vectors from the U_i:

$$U_1 + \cdots + U_n \stackrel{\text{def}}{=} \{v_1 + \cdots + v_n \mid v_i \in U_i\}.$$

Equivalently, it is the set of all sums from $\cup_i U_i$ (since sums from within the same U_i are already elements of that U_i).
Equivalently, it is the span of $\cup_i U_i$.

Direct sum (Version 1: This definition is from Axler.) The sum of subspaces, $U_1 + \cdots + U_n$, is called a *direct sum* if the U_i satisfy the following property: if $v_1 + \cdots + v_n = 0$, where $v_i \in U_i$, then each of the v_i is 0.
If the sum of the U_i is a direct sum, it is written $U_1 \oplus \cdots \oplus U_n$.

Direct sum (Version 2.) Suppose $U_1, \ldots, U_n$ are vector spaces (not necessarily subspaces of some other space). Their *direct sum*, $U_1 \oplus \cdots \oplus U_n$, is:

1. The set $U_1 \times \cdots \times U_n$; together with
2. Addition and scalar multiplication given by

$$(u_1, \ldots, u_n) + \alpha \cdot (v_1, \ldots, v_n) = (u_1 + \alpha \cdot v_1, \ldots, u_n + \alpha \cdot v_n).$$

Note that there are natural injective maps $U_i \to U_1 \oplus \cdots \oplus U_n$ which "embed" the U_i as subspaces of the direct sum.

Linear independence and bases

Span Let $v_1, \dots, v_n$ be vectors in a real vector space, V. The *span* of this set is the subspace given by

$$\{\alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_1, \dots \alpha_n \in \mathbf{R}\}$$

(and likewise for a complex vector space).

Linear independence Vectors $v_1, \dots, v_n \in V$ are *linearly independent* if $\alpha_1 v_1 + \cdots + \alpha_n v_m = 0$ implies $\alpha_1 = \cdots = \alpha_n = 0$.

Basis (Of a vector space, V.) A collection of vectors that (a) spans V; (b) is linearly independent.

Dimension (Of a vector-space, V.) The number of elements of any basis of V. (Noting that any two bases of V have the same cardinality.)

Linear maps

Linear map A map, $T\colon V \to W$ (where V and W are vector spaces) such that $T(u + v) = T(u) + T(v)$ and $T(\alpha u) = \alpha T(u)$ for all $u, v \in V$.

$\mathcal{L}(V, W)$ The set of all linear maps $V \to W$ with the vector space structure given by $(S + \alpha T)(v) \equiv S(v) + \alpha T(v)$ for any $S, T \in V$.

$\mathbf{1}_V$ The identity map $\mathbf{1}_V\colon V \to V$ where $\mathbf{1}_V\colon v \mapsto v$.

$\circ$ Composition of linear maps. For linear maps $T\colon V \to W$ and $S\colon W \to X$, their composition $S \circ T$ is that linear map given by $(S \circ T)(v) \equiv S(T(v))$.
Composition is associative and the identity map is a left and right identity.

Image Of a linear map, that subspace of the codomain that is mapped to by *some* element of the domain. Sometimes called the *range*. However, range used to mean "codomain" so the term can be ambiguous.

Null space (Or "kernel".) Of a linear map, that subspace of the domain whose image is the subspace containing only the zero vector.

Inverse For $T : V \to W$ a linear map, the *inverse* (if it exists) is the linear map T^{-1} such that $TT^{-1} = \mathbf{1}_V$ and $T^{-1}T = \mathbf{1}_W$.

Isomorphism An isomorphism of vector spaces V and W is an invertible, linear map between the two. If an isophormism exists the vector spaces are said to be *isomorphic*.

Quotient space

$a + U$ The set $\{a + u \mid u \in U\}$. What you get by "translating" U by a.

Coset (Or "translate," in Axler's terminology.) For U a subspace of V, a coset of U in V is the set $a + U$ for some $a \in V$.
Two cosets are either identical or disjoint.
Every vector in V lies in one and only one coset.

Quotient Of a vector space V by a subspace U:

1. The set of all cosets of U; together with
2. The vector space structure given by

$$(a + U) + \alpha \cdot (b + U) = (a + \alpha \cdot b) + U,$$

noting that this formula does not depend on which representatives a and b are chosen.

V/U The quotient of V by a subspace U.

Quotient map The map, $\pi\colon V \to V/U$, which takes $v \in V$ to its coset $v + U$.

Dual space

Dual space The dual of a real (respectively, complex) vector space V is the vector space $V^* = \mathcal{L}(V, \mathbf{R})$ (respectively, $V^* = \mathcal{L}(V, \mathbf{C})$). An element of V^* is sometimes called a *linear functional*.

Dual basis For $(e_{(1)}, \dots, e_{(n)})$ a basis of V, the dual basis is that basis $(f^{(1)}, \dots, f^{(n)})$ of V^* such that

$$f^{(i)}(e_{(j)}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Dual map For $\phi \colon U \to V$ a linear map, its dual (sometimes called its transpose), is that map $\phi^* \colon V^* \to U^*$ such that, for all $u \in U$ and $\tilde{v} \in V^*$,

$$\tilde{v}(\phi(u)) = (\phi^*(\tilde{v}))(u).$$

Annihilator For $U \subset V$ a subset of V, the annihilator of U is that subspace $U^0 \subset V^*$ of the dual of V given by

$$U^0 = \{\tilde{u} \in V^* \mid \tilde{u}(w) = 0 \text{ for all } w \in U\}.$$

Operators I

Operator A linear map from a vector space to itself.

Invariant subspace A subspace $U \subset V$ is *invariant* under operator T if $Tu \in U$ for all $u \in U$.

Minimal polynomial Of an operator, T on a finite-dimensional vector space over field $\mathbf{F}$. The (unique) monic polynomial $p \in \mathcal{P}(\mathbf{F})$ such that $p(T) = 0$. ("Monic" means that the coefficient of the highest-degree term is 1.)

Matrices

Matrix An $m \times n$ matrix A is $m \times n$ numbers, A_{ij}, indexed by $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. (Axler writes the elements as $A_{i,j}$.)
Conventionally, the matrix is written as a rectangular array of the numbers, with A_{ij} being written in the ith row (starting at the top) and jth column (starting from the left).
If $T \colon V \to W$ is a linear map and $\vec{e}_j \in V$ a basis of V and $\vec{f}_i \in W$ a basis of W, the **matrix of** T is given by:

$$T(\vec{e}_j) = \sum_i T_{ij} \vec{f}_i.$$

(That is, T_{ij} is the ith component of $T(\vec{e}_j)$ with respect to the basis $\vec{f}_i$.)

$\mathbf{R}^{m,n}$ The set of all $m \times n$ matrices over $\mathbf{R}$ (or $\mathbf{C}$, *mutatis mutandis*).
Addition of matrices and multiplication of matrices by numbers is defined elementwise:

$$(A + B)_{ij} \stackrel{\text{def}}{=} A_{ij} + B_{ij}$$
$$(\lambda A)_{ij} \stackrel{\text{def}}{=} \lambda A_{ij}.$$

Under these definitions, $\mathbf{R}^{m,n}$ is a vector space. (The zero element of this space is the matrix $0_{ij} = 0$.)
"Multiplication" of matrices is defined between an $l \times m$ and and $m \times n$ matrix, the result being an $l \times n$ matrix:

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}.$$

Transpose The transpose of A_{ij} is the matrix A_{ji}.

Rank The rank of a matrix $A_{ij} \in \mathbf{R}^{m,n}$ is the dimension of the span of the n vectors $\vec{c}_j \in \mathbf{R}^m$ where $\vec{c}_j = A_{ij}$ (that is, the $\vec{c}_j$ are the "columns" of A_{ij}).
The dimension of the span of the "rows" of A_{ij} is also the rank.

Upper triangular Of a matrix, having zero for all entries below the diagonal.

Diagonal Of a matrix, having zero for all entries except on the diagonal.

Eigenvalues

Eigenvalue Of an operator, T. A number λ such that there exists a vector $v \neq 0$ with $Tv = \lambda v$.

Eigenvector Of an operator, T. A non-zero vector, v, such that $Tv = \lambda v$ for some number λ.

Inner product spaces

Inner product On a real or complex vector space, V, a conjugate-symmetric, positive-definite map $V \times V \to \mathbf{F}$, written $\langle v, w\rangle$ for $v, w \in \mathbf{F}$, which is linear in its first argument (and conjugate linear in its second). That is:

$$\langle v, w\rangle \geqslant 0 \qquad \text{(positive)}$$
$$\langle v, v\rangle = 0 \implies v = 0 \qquad \text{(definite)}$$
$$\langle v, w\rangle = \overline{\langle w, v\rangle} \qquad \text{(conj. symm.)}$$
$$\langle v, x + \lambda y\rangle = \langle v, x\rangle + \lambda\langle v, y\rangle \qquad \text{(linear)}$$

An **inner product space** is a vector space, V, together with an inner product on V.

Norm (Given an inner product) the norm of v is

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

Most authors give an independent definition of a norm, from which it follows that the above construction gives rise to a norm, in those authors' sense.
The *Cauchy-Schwarz inequality* is that $|\langle v, w\rangle| \leqslant \|v\|\|w\|$ with the inequality saturated only if v and w are colinear. The *triangle inequality* is that $\|v + w\| \leqslant \|v\| + \|w\|$.

Orthogonal Describes two vectors, v and w, having the property that $\langle v, w\rangle = 0$.

Orthogonality

Orthonormal Of a list of vectors, having the property that every vector in the list has norm 1 and is orthogonal to every other vector in the list. Orthonormal lists are linearly independent.

Orthonomal basis A basis that is orthonormal.

Gram-Schmidt procedure The Gram-Schmidt procedure is an algorithm for constructing, from a list of vectors, an orthonormal list having the same span as the original list.

Orthogonal complement Of a subset $U \subset V$, the set, $U^{\perp}$, of all vectors in V that are orthogonal to every vector in U. Usually U is a subspace. When U is a finite-dimensional subspace, $(U^{\perp})^{\perp} = U$.

Orthogonal projection Of a subspace U of V, the operator P_U define by $P_U v = u$ where $v = u + w$ and $u \in U$ and $w \in W^{\perp}$.

Pseudoinverse For V finite-dimensional and $T \in \mathcal{L}(V, W)$. The pseudoinverse $T^{\dagger} \in \mathcal{L}(W, V)$ is

$$T^{\dagger} w = (T|_{(\operatorname{null} T)_{\perp}})^{-1} P_{\operatorname{range} T} w.$$

Assorted results

Schur's theorem Every operator on a finite-dimensional, complex inner product space has an upper-triangular matrix with respect to some orthonormal basis.

Reisz representation theorem For finite-dimensional V, let $\phi \in V'$ be a linear functional on V. Then there exists a unique $v \in V$ such that

$$\phi(u) = \langle u, v\rangle$$

for every $u \in V$.

Operators II

Commuting Operators A and B commute if $AB - BA = 0$.

Adjoint