Ex. 3.23 (b)

Here we wish to show that the correlation between each x_j and its residual is equal to $\lambda(\alpha)$.
e.g.

$$Cor(x_j, y-u(\alpha)) = \frac{(1-\alpha)\lambda}{\sqrt{(1-\alpha)^2 + \frac{\alpha(2-\alpha)}{N} RSS}}$$

Now $Cor(X, Y) = \dfrac{Cov(X, Y)}{\sigma_x \cdot \sigma_Y}$

and $\sigma_x = \sqrt{\dfrac{\langle X, X\rangle}{N}}$ and $Cov(X, Y) = \dfrac{\langle X, Y\rangle}{N}$

So,

$$Cor(x_j, y-u(\alpha)) = \frac{\overset{①}{1/N \langle x_j, y-u(\alpha)\rangle}}{\underset{②}{\sqrt{\frac{\langle x_j, x_j\rangle}{N}}} \cdot \underset{③}{\sqrt{\frac{\langle y-u(\alpha), y-u(\alpha)\rangle}{N}}}}$$

Now we know from Part (a) that ① is equal to $(1-\alpha)\lambda$.
We have been told that all variables have mean zero and standard deviation 1, so $\langle x_j, x_j\rangle = N$ and thus ② $\Rightarrow \sqrt{N/N} = 1$

This just leaves ③ unsolved and give us that:

$$Cor(x_j, y-u(\alpha)) = \frac{(1-\alpha)\lambda}{\sqrt{\frac{\langle y-u(\alpha), y-u(\alpha)\rangle}{N}}}$$

Thus all that is left to show is that:

$$\frac{\langle y-u(\alpha), y-u(\alpha)\rangle}{N} = (1-\alpha)^2 + \frac{\alpha(2-\alpha)}{N} RSS$$

Firstly we will need to solve the following equation to use later:

$\langle X\hat{\beta}, \; y - X\hat{\beta} \rangle$

$= \hat{\beta}^T X^T y - \hat{\beta}^T X^T X \hat{\beta}$

$= \hat{\beta}^T X^T y - \hat{\beta}^T X^T X (X^T X)^{-1} X^T y$ (expanding $\hat{\beta}$)

$= \hat{\beta}^T X^T y - \hat{\beta}^T X^T y = 0$

Thus, $\langle X\hat{\beta}, y - X\hat{\beta} \rangle = 0$ (*)

Now returning to the main problem, we wish to solve:

$$\tfrac{1}{N} \langle y - \alpha X\hat{\beta}, y - \alpha X\hat{\beta} \rangle$$

$= \tfrac{1}{N} \left(\langle y, y \rangle - 2\alpha \langle y, X\hat{\beta} \rangle + \alpha^2 \langle X\hat{\beta}, X\hat{\beta} \rangle \right)$

$= \tfrac{1}{N} \left(N - 2\alpha \langle y, X\hat{\beta} \rangle + \alpha^2 \langle y + X\hat{\beta} - y, X\hat{\beta} \rangle \right)$

($y^T y = N$ since y has mean zero, standard deviation 1)

$= \tfrac{1}{N} \left(N - 2\alpha \langle y, X\hat{\beta} \rangle + \alpha^2 \langle y, X\hat{\beta} \rangle + \alpha^2 \langle X\hat{\beta} - y, X\hat{\beta} \rangle \right)$

Notice that for vectors A, B and constant γ we have that $\langle \gamma A, B \rangle = \gamma \langle A, B \rangle$.

Thus $\langle X\hat{\beta} - y, X\hat{\beta} \rangle = (-1) \langle y - X\hat{\beta}, X\hat{\beta} \rangle = 0$ (by *)

so $\Rightarrow \tfrac{1}{N} \left(N - 2\alpha \langle y, X\hat{\beta} \rangle + \alpha^2 \langle y, X\hat{\beta} \rangle + 0 \right)$

$= \tfrac{1}{N} \left(N + (\alpha^2 - 2\alpha) \langle y, X\hat{\beta} \rangle \right)$

$= \tfrac{1}{N} \left(N + (\alpha^2 - 2\alpha) \langle y, X\hat{\beta} + y - y \rangle \right)$

$= \tfrac{1}{N} \left(N + (2\alpha - \alpha^2) \langle y, y - X\hat{\beta} - y \rangle \right)$

$= \tfrac{1}{N} \left(N + \overset{①}{(2\alpha - \alpha^2) \langle y, y - X\hat{\beta} \rangle} + \overset{②}{(2\alpha - \alpha^2) \langle y, -y \rangle} \right)$

Solving ① we obtain:

$$\langle y, y - X\hat{\beta} \rangle = \langle y - X\hat{\beta} + X\hat{\beta}, y - X\hat{\beta} \rangle$$

$$= \langle y - X\hat{\beta}, y - X\hat{\beta} \rangle + \langle X\hat{\beta}, y - X\hat{\beta} \rangle$$

$$= \underset{\text{(by definition)}}{RSS} + \underset{\text{(by } *)}{0}$$

and Solving ②:

$$\langle y, -y \rangle = (-1)\langle y, y \rangle = -N$$

and Putting this all together we get:

$$1/N \left(N + (2\alpha - \alpha^2) RSS + (\alpha^2 - 2\alpha) N \right)$$

$$= \frac{\alpha(2-\alpha)}{N} RSS + \alpha^2 - 2\alpha + 1$$

$$= \frac{\alpha(2-\alpha)}{N} RSS + (1-\alpha)^2 \quad \text{(as required)}$$

and Putting it all together we obtain that:

$$Cor(x_j, y - u(\alpha)) = \frac{(1-\alpha)\lambda}{\sqrt{(1-\alpha)^2 + \frac{\alpha(2-\alpha)}{N} RSS}}$$