Chapter 12

Rational Expressions and Equations

What You'll Learn

- **Lesson 12-1** Solve problems involving inverse variation.
- **Lessons 12-2, 12-3, 12-4, 12-6, and 12-7** Simplify, add, subtract, multiply, and divide rational expressions.
- **Lesson 12-5** Divide polynomials.
- **Lesson 12-8** Simplify mixed expressions and complex fractions.
- **Lesson 12-9** Solve rational equations.

Key Vocabulary

- inverse variation (p. 642)
- rational expression (p. 648)
- excluded values (p. 648)
- complex fraction (p. 684)
- extraneous solutions (p. 693)

Why It's Important

Performing operations on rational expressions is an important part of working with equations. For example, knowing how to divide rational expressions and polynomials can help you simplify complex expressions. You can use this process to determine the number of flags that a marching band can make from a given amount of material. *You will divide rational expressions and polynomials in Lessons 12-4 and 12-5.*

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 12.

For Lesson 12-1 — Solve Proportions

Solve each proportion. *(For review, see Lesson 3-6.)*

1. $\frac{y}{9} = \frac{-7}{16}$ **2.** $\frac{4}{x} = \frac{2}{10}$ **3.** $\frac{3}{15} = \frac{1}{n}$ **4.** $\frac{x}{8} = \frac{0.21}{2}$

5. $\frac{1.1}{0.6} = \frac{8.47}{n}$ **6.** $\frac{9}{8} = \frac{y}{6}$ **7.** $\frac{2.7}{3.6} = \frac{8.1}{a}$ **8.** $\frac{0.19}{2} = \frac{x}{24}$

For Lesson 12-2 — Greatest Common Factor

Find the greatest common factor for each pair of monomials. *(For review, see Lesson 9-1.)*

9. 30, 42 **10.** $60r^2, 45r^3$ **11.** $32m^2n^3, 12m^2n$ **12.** $14a^2b^2, 18a^3b$

For Lessons 12-3 through 12-8 — Factor Polynomials

Factor each polynomial. *(For review, see Lessons 9-2 and 9-3.)*

13. $3c^2d - 6c^2d^2$ **14.** $6mn + 15m^2$ **15.** $x^2 + 11x + 24$

16. $x^2 + 4x - 45$ **17.** $2x^2 + x - 21$ **18.** $3x^2 - 12x + 9$

For Lesson 12-9 — Solve Equations

Solve each equation. *(For review, see Lessons 3-4, 3-5, and 9-3.)*

19. $3x - 2 = -5$ **20.** $5x - 8 - 3x = (2x - 3)$ **21.** $\frac{m+9}{5} = \frac{m-10}{11}$ **22.** $\frac{5+x}{x-3} = \frac{14}{10}$

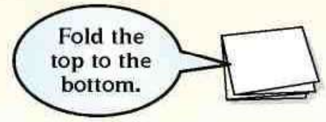
23. $\frac{7n-1}{6} = 5$ **24.** $\frac{4t-5}{-9} = 7$ **25.** $x^2 - x - 56 = 0$ **26.** $x^2 + 2x = 8$

FOLDABLES™ Study Organizer

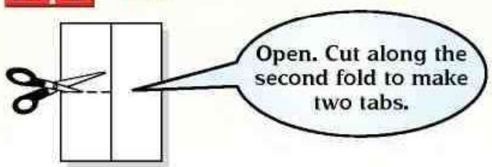
Make this Foldable to help you organize information about rational expressions and equations. Begin with a sheet of plain $8\frac{1}{2}$" by 11" paper.

Step 1 Fold in Half

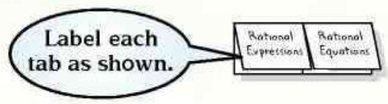
Step 2 Fold Again



Step 3 Cut



Step 4 Label



Reading and Writing As you read and study the chapter, write notes and examples under each tab. Use this Foldable to apply what you learned about simplifying rational expressions and solving rational equations in Chapter 12.

12-1 Inverse Variation

What You'll Learn

- Graph inverse variations.
- Solve problems involving inverse variation.

Vocabulary

- inverse variation
- product rule

How is inverse variation related to the gears on a bicycle?

The number of revolutions of the pedals made when riding a bicycle at a constant speed varies inversely as the gear ratio of the bicycle. In other words, as the gear ratio *decreases*, the revolutions per minute (rpm) *increase*. This is why when pedaling up a hill, shifting to a lower gear allows you to pedal with less difficulty.

Pedaling Rates to Maintain Speed of 10 mph

Gear Ratio	Rate
117.8	89.6
108.0	97.8
92.6	114.0
76.2	138.6
61.7	171.2
49.8	212.0
40.5	260.7

Study Tip

Look Back
To review **direct variation**, see Lesson 5-2.

GRAPH INVERSE VARIATION Recall that some situations in which y increases as x increases are *direct variations*. If y varies directly as x, we can represent this relationship with an equation of the form $y = kx$, where $k \neq 0$. However, in the application above, as one value increases the other value decreases. When the product of two values remains constant, the relationship forms an **inverse variation**. We say *y varies inversely as x* or *y is inversely proportional to x*.

Key Concept — *Inverse Variation*

y varies inversely as x if there is some nonzero constant k such that $xy = k$.

Example 1 Graph an Inverse Variation

DRIVING **The time t it takes to travel a certain distance varies inversely as the rate r at which you travel. The equation $rt = 250$ can be used to represent a person driving 250 miles. Complete the table and draw a graph of the relation.**

r (mph)	5	10	15	20	25	30	35	40	45	50
t (hours)										

Study Tip

Inverse Variation Problems
Note that to solve some inverse variation problems, there are two steps: first finding the value of k, and then using this value to find a specific value of x or y.

Solve for $r = 5$.

$rt = 250$ Original equation

$5t = 250$ Replace r with 5.

$t = \frac{250}{5}$ Divide each side by 5.

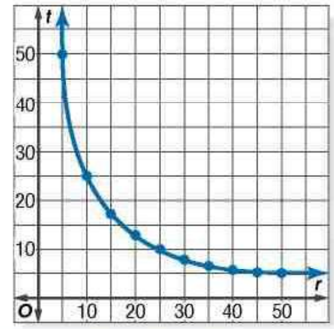
$t = 50$ Simplify.

Solve the equation for the other values of r.

r (mph)	5	10	15	20	25	30	35	40	45	50
t (hours)	50	25	16.67	12.5	10	8.33	7.14	6.25	5.56	5

Next, graph the ordered pairs: (5, 50), (10, 25), (15, 16.67), (20, 12.5), (25, 10), (30, 8.33), (35, 7.14), (40, 6.25), (45, 5.56), and (50, 5).

The graph of an inverse variation is not a straight line like the graph of a direct variation. As the rate r increases, the time t that it takes to travel the same distance decreases.



Graphs of inverse variations can also be drawn using negative values of x.

Example 2 Graph an Inverse Variation

Graph an inverse variation in which y varies inversely as x and $y = 15$ when $x = 6$.

Solve for k.

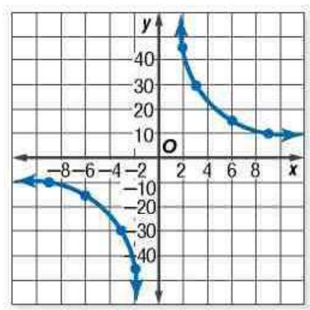
$$xy = k \quad \text{Inverse variation equation}$$

$$(6)(15) = k \quad x = 6, y = 15$$

$$90 = k \quad \text{The constant of variation is 90.}$$

Choose values for x and y whose product is 90.

x	y
−9	−10
−6	−15
−3	−30
−2	−45
0	undefined
2	45
3	30
6	15
9	10



USE INVERSE VARIATION If (x_1, y_1) and (x_2, y_2) are solutions of an inverse variation, then $x_1y_1 = k$ and $x_2y_2 = k$.

$$x_1y_1 = k \text{ and } x_2y_2 = k$$

$$x_1y_1 = x_2y_2 \quad \text{Substitute } x_2y_2 \text{ for } k.$$

The equation $x_1y_1 = x_2y_2$ is called the **product rule** for inverse variations. You can use this equation to form a proportion.

$$x_1y_1 = x_2y_2 \quad \text{Product rule for inverse variations}$$

$$\frac{x_1y_1}{x_2y_1} = \frac{x_2y_2}{x_2y_1} \quad \text{Divide each side by } x_2y_1.$$

$$\frac{x_1}{x_2} = \frac{y_2}{y_1} \quad \text{Simplify.}$$

Study Tip

Proportions
Notice that the proportion for inverse variations is different from the proportion for direct variation, $\frac{x_1}{x_2} = \frac{y_1}{y_2}$.

You can use the product rule or a proportion to solve inverse variation problems.

Example 3 Solve for x

If y varies inversely as x and $y = 4$ when $x = 7$, find x when $y = 14$.

Let $x_1 = 7$, $y_1 = 4$, and $y_2 = 14$. Solve for x_2.

Method 1 Use the product rule.

$x_1y_1 = x_2y_2$ Product rule for inverse variations

$7 \cdot 4 = x_2 \cdot 14$ $x_1 = 7$, $y_1 = 4$, and $y_2 = 14$

$\frac{28}{14} = x_2$ Divide each side by 14.

$2 = x_2$ Simplify.

Method 2 Use a proportion.

$\frac{x_1}{x_2} = \frac{y_2}{y_1}$ Proportion for inverse variations

$\frac{7}{x_2} = \frac{14}{4}$ $x_1 = 7$, $y_1 = 4$, and $y_2 = 14$

$28 = 14x_2$ Cross multiply.

$2 = x_2$ Divide each side by 14.

Both methods show that $x = 2$ when $y = 14$.

Example 4 Solve for y

If y varies inversely as x and $y = -6$ when $x = 9$, find y when $x = 6$.

Use the product rule.

$x_1y_1 = x_2y_2$ Product rule for inverse variations

$9 \cdot (-6) = 6y_2$ $x_1 = 9$, $y_1 = -6$, and $x_2 = 6$

$\frac{-54}{6} = y_2$ Divide each side by 6.

$-9 = y_2$ Simplify.

Thus, $y = -9$ when $x = 6$.

Inverse variation is often used in real-world situations.

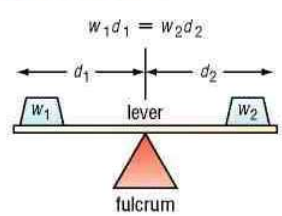
Study Tip

Levers

A lever is a bar with a pivot point called the *fulcrum*. For a lever to balance, the lesser weight must be positioned farther from the fulcrum.

Example 5 Use Inverse Variation to Solve a Problem

PHYSICAL SCIENCE When two objects are balanced on a lever, their distances from the fulcrum are inversely proportional to their weights. In other words, the greater the weight, the less distance it should be from the fulcrum in order to maintain balance. If an 8-kilogram weight is placed 1.8 meters from the fulcrum, how far should a 12-kilogram weight be placed from the fulcrum in order to balance the lever?



Let $w_1 = 8$, $d_1 = 1.8$, and $w_2 = 12$. Solve for d_2.

$w_1d_1 = w_2d_2$ Original equation

$8 \cdot 1.8 = 12d_2$ $w_1 = 8$, $d_1 = 1.8$, and $w_2 = 12$

$\frac{14.4}{12} = d_2$ Divide each side by 12.

$1.2 = d_2$ Simplify.

The 12-kilogram weight should be placed 1.2 meters from the fulcrum.

Check for Understanding

Concept Check

1. **OPEN ENDED** **Write** an equation showing an inverse variation with a constant of 8.
2. **Compare and contrast** direct variation and indirect variation equations and graphs.
3. **Determine** which situation is an example of inverse variation. Explain.
   a. Emily spends \$2 each day for snacks on her way home from school. The total amount she spends each week depends on the number of days school was in session.
   b. A business donates \$200 to buy prizes for a school event. The number of prizes that can be purchased depends upon the price of each prize.

Guided Practice

Graph each variation if y varies inversely as x.

4. $y = 24$ when $x = 8$
5. $y = -6$ when $x = -2$

Write an inverse variation equation that relates x and y. Assume that y varies inversely as x. Then solve.

6. If $y = 12$ when $x = 6$, find y when $x = 8$.
7. If $y = -8$ when $x = -3$, find y when $x = 6$.
8. If $y = 2.7$ when $x = 8.1$, find x when $y = 5.4$.
9. If $x = \frac{1}{2}$ when $y = 16$, find x when $y = 32$.

Application

10. **MUSIC** The length of a violin string varies inversely as the frequency of its vibrations. A violin string 10 inches long vibrates at a frequency of 512 cycles per second. Find the frequency of an 8-inch string.

Practice and Apply

Homework Help

For Exercises	See Examples
11–16	1, 2
17–28	3, 4
29–37	5

Extra Practice

See page 846.

Graph each variation if y varies inversely as x.

11. $y = 24$ when $x = -8$
12. $y = 3$ when $x = 4$
13. $y = 5$ when $x = 15$
14. $y = -4$ when $x = -12$
15. $y = 9$ when $x = 8$
16. $y = 2.4$ when $x = 8.1$

Write an inverse variation equation that relates x and y. Assume that y varies inversely as x. Then solve.

17. If $y = 12$ when $x = 5$, find y when $x = 3$.
18. If $y = 7$ when $x = -2$, find y when $x = 7$.
19. If $y = 8.5$ when $x = -1$, find x when $y = -1$.
20. If $y = 8$ when $x = 1.55$, find x when $y = -0.62$.
21. If $y = 6.4$ when $x = 4.4$, find x when $y = 3.2$.
22. If $y = 1.6$ when $x = 0.5$, find x when $y = 3.2$.
23. If $y = 4$ when $x = 4$, find y when $x = 7$.
24. If $y = -6$ when $x = -2$, find y when $x = 5$.
25. Find the value of y when $x = 7$ if $y = 7$ when $x = \frac{2}{3}$.
26. Find the value of y when $x = 32$ if $y = 16$ when $x = \frac{1}{2}$.
27. If $x = 6.1$ when $y = 4.4$, find x when $y = 3.2$.
28. If $x = 0.5$ when $y = 2.5$, find x when $y = 20$.

29. **GEOMETRY** A rectangle is 36 inches wide and 20 inches long. How wide is a rectangle of equal area if its length is 90 inches?

30. **MUSIC** The pitch of a musical note varies inversely as its wavelength. If the tone has a pitch of 440 vibrations per second and a wavelength of 2.4 feet, find the pitch of a tone that has a wavelength of 1.6 feet.

31. **COMMUNITY SERVICE** Students at Roosevelt High School are collecting canned goods for a local food pantry. They plan to distribute flyers to homes in the community asking for donations. Last year, 12 students were able to distribute 1000 flyers in nine hours. How long would it take if 15 students hand out the same number of flyers this year?

TRAVEL **For Exercises 32 and 33, use the following information.**
The Zalinski family can drive the 220 miles to their cabin in 4 hours at 55 miles per hour. Son Jeff claims that they could save half an hour if they drove 65 miles per hour, the speed limit.

32. How long will it take the family if they drive 65 miles per hour?

33. How much time would be saved driving at 65 miles per hour?

CHEMISTRY **For Exercises 34–36, use the following information.**
Boyle's Law states that the volume of a gas V varies inversely with applied pressure P.

34. Write an equation to show this relationship.

35. Pressure on 60 cubic meters of a gas is raised from 1 atmosphere to 3 atmospheres. What new volume does the gas occupy?

36. A helium-filled balloon has a volume of 22 cubic meters at sea level where the air pressure is 1 atmosphere. The balloon is released and rises to a point where the air pressure is 0.8 atmosphere. What is the volume of the balloon at this height?

37. **ART** Anna is designing a mobile to suspend from a gallery ceiling. A chain is attached eight inches from the end of a bar that is 20 inches long. On the shorter end of the bar is a sculpture weighing 36 kilograms. She plans to place another piece of artwork on the other end of the bar. How much should the second piece of art weigh if she wants the bar to be balanced?

More About. . .

Art
American sculptor Alexander Calder was the first artist to use mobiles as an art form.
Source: www.infoplease.com

CRITICAL THINKING **For Exercises 38 and 39, assume that y varies inversely as x.**

38. If the value of x is doubled, what happens to the value of y?

39. If the value of y is tripled, what happens to the value of x?

40. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How is inverse variation related to the gears on a bicycle?

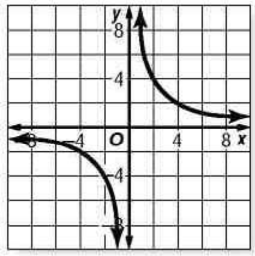
Include the following in your answer:
- an explanation of how shifting to a lower gear ratio affects speed and the pedaling rate on a certain bicycle if a rider is pedaling 73.4 revolutions per minute while traveling at 15 miles per hour, and
- an explanation why the gear ratio affects the pedaling speed of the rider.

Standardized Test Practice

41. Determine the constant of variation if y varies inversely as x and $y = 4.25$ when $x = -1.3$.

Ⓐ -3.269 Ⓑ -5.525 Ⓒ -0.306 Ⓓ -2.950

42. Identify the graph of $xy = k$ if $x = -2$ when $y = -4$.

Ⓐ


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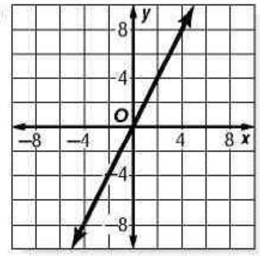

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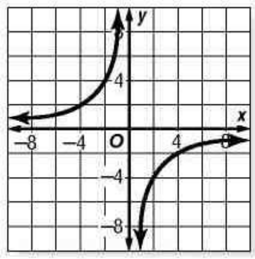

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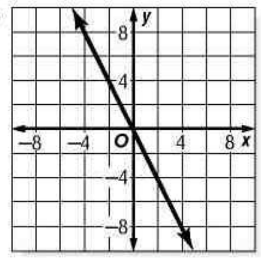

Maintain Your Skills

Mixed Review

For each triangle, find the measure of the indicated angle to the nearest degree. *(Lesson 11-7)*

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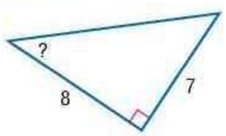

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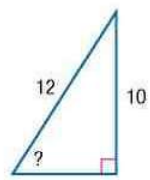

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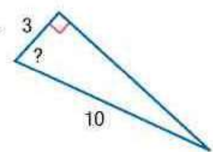


For each set of measures given, find the measures of the missing sides if $\triangle ABC \sim \triangle DEF$. *(Lesson 11-6)*

46. $a = 3, b = 10, c = 9, d = 12$

47. $b = 8, c = 4, d = 21, e = 28$

48. **MUSIC** Two musical notes played at the same time produce harmony. The closest harmony is produced by frequencies with the greatest GCF. A, C, and C sharp have frequencies of 220, 264, and 275, respectively. Which pair of these notes produce the closest harmony? *(Lesson 9-1)*

Solve each equation. *(Lesson 8-6)*

49. $7(2y - 7) = 5(4y + 1)$

50. $w(w + 2) = 2w(w - 3) + 16$

Solve each system of inequalities by graphing. *(Lesson 7-5)*

51. $y \le 3x - 5$
$y > -x + 1$

52. $y \ge 2x + 3$
$2y \ge -5x - 14$

53. $x + y \le 1$
$x - y \le -3$
$y \ge 0$

54. $3x - 2y \ge -16$
$x + 4y < 4$
$5x - 8y < -8$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the greatest common factor for each set of monomials.**
*(To review **greatest common factors**, see Lesson 9-1.)*

55. 36, 15, 45

56. 48, 60, 84

57. 210, 330, 150

58. $17a, 34a^2$

59. $12xy^2, 18x^2y^3$

60. $12pr^2, 40p^4$

12-2 Rational Expressions

What You'll Learn

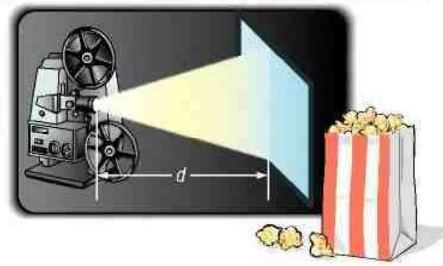
- Identify values excluded from the domain of a rational expression.
- Simplify rational expressions.

Vocabulary

- rational expression
- excluded values

How can a rational expression be used in a movie theater?

The intensity I of an image on a movie screen is inversely proportional to the square of the distance d between the projector and the screen. Recall from Lesson 12-1 that this can be represented by the equation $I = \frac{k}{d^2}$, where k is a constant.



EXCLUDED VALUES OF RATIONAL EXPRESSIONS The expression $\frac{k}{d^2}$ is an example of a rational expression. A **rational expression** is an algebraic fraction whose numerator and denominator are polynomials.

Because a rational expression involves division, the denominator may not have a value of zero. Any values of a variable that result in a denominator of zero must be excluded from the domain of that variable. These are called **excluded values** of the rational expression.

Example 1 One Excluded Value

State the excluded value of $\frac{5m + 3}{m - 6}$.

Exclude the values for which $m - 6 = 0$.

$m - 6 = 0$ The denominator cannot equal 0.

$m = 6$ Add 6 to each side.

Therefore, m cannot equal 6.

To determine the excluded values of a rational expression, you may be able to factor the denominator first.

Example 2 Multiple Excluded Values

State the excluded values of $\frac{x^2 - 5}{x^2 - 5x + 6}$.

Exclude the values for which $x^2 - 5x + 6 = 0$.

$x^2 - 5x + 6 = 0$ The denominator cannot equal zero.

$(x - 2)(x - 3) = 0$ Factor.

Use the Zero Product Property to solve for x.

$x - 2 = 0$ or $x - 3 = 0$

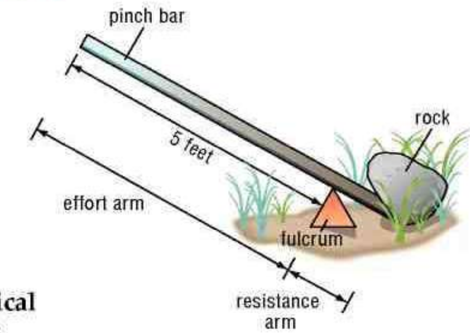
$x = 2$ $\quad$ $x = 3$

Therefore, x cannot equal 2 or 3.

You can use rational expressions to solve real-world problems.

Example 3 Use Rational Expressions

LANDSCAPING **Kenyi is helping his parents landscape their yard and needs to move some large rocks. He plans to use a 6-foot bar as a lever. He positions it as shown at the right.**



a. The mechanical advantage of a lever is $\frac{L_E}{L_R}$, where L_E is the length of the effort arm and L_R is the length of the resistance arm. Calculate the mechanical advantage of the lever Kenyi is using.

Let b represent the length of the bar and e represent the length of the effort arm. Then $b - e$ represents the length of the resistance arm.

Use the expression for mechanical advantage to write an expression for the mechanical advantage in this situation.

$$\frac{L_E}{L_R} = \frac{e}{b-e} \quad L_E = e, L_R = b - e$$

$$= \frac{5}{6-5} \quad e = 5, b = 6$$

$$= 5 \quad \text{Simplify.}$$

The mechanical advantage is 5.

b. The force placed on the rock is the product of the mechanical advantage and the force applied to the end of the lever. If Kenyi can apply a force of 180 pounds, what is the greatest weight he can lift with the lever?

Since the mechanical advantage is 5, Kenyi can lift $5 \cdot 180$ or 900 pounds with this lever.

Career Choices

Landscape Architect

Landscape architects plan the location of structures, roads, and walkways as well as the arrangement of flowers, trees, and shrubs in a variety of settings.

Source: U.S. Bureau of Labor and Statistics

Online Research For information about a career as a landscape architect, visit: www.algebra1.com/careers

SIMPLIFY RATIONAL EXPRESSIONS Simplifying rational expressions is similar to simplifying fractions with numbers. To simplify a rational expression, you must eliminate any common factors of the numerator and denominator. To do this, use their greatest common factor (GCF). Remember that $\frac{ab}{ac} = \frac{a}{a} \cdot \frac{b}{c}$ and $\frac{a}{a} = 1$. So, $\frac{ab}{ac} = 1 \cdot \frac{b}{c}$ or $\frac{b}{c}$.

Example 4 Expression Involving Monomials

Simplify $\frac{-7a^2b^3}{21a^5b}$.

$$\frac{-7a^2b^3}{21a^5b} = \frac{(7a^2b)(-b^2)}{(7a^2b)(3a^3)} \quad \text{The GCF of the numerator and denominator is } 7a^2b.$$

$$= \frac{\overset{1}{\cancel{(7a^2b)}}(-b^2)}{\underset{1}{\cancel{(7a^2b)}}(3a^3)} \quad \text{Divide the numerator and denominator by } 7a^2b.$$

$$= \frac{-b^2}{3a^3} \quad \text{Simplify.}$$

www.algebra1.com/extra_examples

You can use the same procedure to simplify a rational expression in which the numerator and denominator are polynomials.

Study Tip

Simplest Form
When a rational expression is in simplest form, the numerator and denominator have no common factors other than 1 or −1.

Example 5 Expressions Involving Polynomials

Simplify $\frac{x^2 - 2x - 15}{x^2 - x - 12}$.

$\frac{x^2 - 2x - 15}{x^2 - x - 12} = \frac{(x + 3)(x - 5)}{(x + 3)(x - 4)}$ Factor.

$= \frac{\cancel{(x + 3)}^{1}(x - 5)}{\cancel{(x + 3)}_{1}(x - 4)}$ Divide the numerator and denominator by the GCF, $x + 3$.

$= \frac{x - 5}{x - 4}$ Simplify.

It is important to determine the excluded values of a rational expression using the original expression rather than the simplified expression.

Example 6 Excluded Values

Simplify $\frac{3x - 15}{x^2 - 7x + 10}$. State the excluded values of x.

$\frac{3x - 15}{x^2 - 7x + 10} = \frac{3(x - 5)}{(x - 2)(x - 5)}$ Factor.

$= \frac{3\cancel{(x - 5)}^{1}}{(x - 2)\cancel{(x - 5)}_{1}}$ Divide the numerator and denominator by the GCF, $x - 5$.

$= \frac{3}{x - 2}$ Simplify.

Exclude the values for which $x^2 - 7x + 10$ equals 0.

$x^2 - 7x + 10 = 0$ The denominator cannot equal zero.

$(x - 5)(x - 2) = 0$ Factor.

$x = 5$ or $x = 2$ Zero Product Property

CHECK Verify the excluded values by substituting them into the original expression.

$\frac{3x - 15}{x^2 - 7x + 10} = \frac{3(5) - 15}{5^2 - 7(5) + 10}$ $x = 5$

$= \frac{15 - 15}{25 - 35 + 10}$ Evaluate.

$= \cancel{\frac{0}{0}}$ Simplify.

$\frac{3x - 15}{x^2 - 7x + 10} = \frac{3(2) - 15}{2^2 - 7(2) + 10}$ $x = 2$

$= \frac{6 - 15}{4 - 14 + 10}$ Evaluate.

$= \cancel{\frac{-9}{0}}$ Simplify.

The expression is undefined when $x = 5$ and $x = 2$. Therefore, $x \neq 5$ and $x \neq 2$.

Check for Understanding

Concept Check

1. **Describe** how you would determine the values to be excluded from the expression $\frac{x+3}{x^2+5x+6}$.
2. **OPEN ENDED** Write a rational expression involving one variable for which the excluded values are -4 and -7.
3. **Explain** why -2 may not be the only excluded value of a rational expression that simplifies to $\frac{x-3}{x+2}$.

Guided Practice

State the excluded values for each rational expression.

4. $\frac{4a}{3+a}$
5. $\frac{x^2-9}{2x+6}$
6. $\frac{n+5}{n^2+n-20}$

Simplify each expression. State the excluded values of the variables.

7. $\frac{56x^2y}{70x^3y^2}$
8. $\frac{x^2-49}{x+7}$
9. $\frac{x+4}{x^2+8x+16}$
10. $\frac{x^2-2x-3}{x^2-7x+12}$
11. $\frac{a^2+4a-12}{a^2+2a-8}$
12. $\frac{2x^2-x-21}{2x^2-15x+28}$
13. Simplify $\frac{b^2-3b-4}{b^2-13b+36}$. State the excluded values of b.

Application

AQUARIUMS **For Exercises 14 and 15, use the following information.**
Jenna has guppies in her aquarium. One week later, she adds four neon fish.

14. Write an expression that represents the fraction of neon fish in the aquarium.
15. Suppose that two months later the guppy population doubles, she still has four neons, and she buys 5 different tropical fish. Write an expression that shows the fraction of neons in the aquarium after the other fish have been added.

Practice and Apply

Homework Help

For Exercises	See Examples
16–23	1, 2
24–27	4
28–41	5, 6
42–54	3

Extra Practice
See page 846.

State the excluded values for each rational expression.

16. $\frac{m+3}{m-2}$
17. $\frac{3b}{b+5}$
18. $\frac{3n+18}{n^2-36}$
19. $\frac{2x-10}{x^2-25}$
20. $\frac{a^2-2a+1}{a^2+2a-3}$
21. $\frac{x^2-6x+9}{x^2+2x-15}$
22. $\frac{n^2-36}{n^2+n-30}$
23. $\frac{25-x^2}{x^2+12x+35}$

Simplify each expression. State the excluded values of the variables.

24. $\frac{35yz^2}{14y^2z}$
25. $\frac{14a^3b^2}{42ab^3}$
26. $\frac{64qr^2s}{16q^2rs}$
27. $\frac{9x^2yz}{24xyz^2}$
28. $\frac{7a^3b^2}{21a^2b+49ab^3}$
29. $\frac{3m^2n^3}{36mn^3-12m^2n^2}$
30. $\frac{x^2+x-20}{x+5}$
31. $\frac{z^2+10z+16}{z+2}$
32. $\frac{4x+8}{x^2+6x+8}$
33. $\frac{2y-4}{y^2+3y-10}$
34. $\frac{m^2-36}{m^2-5m-6}$
35. $\frac{a^2-9}{a^2+6a-27}$
36. $\frac{x^2+x-2}{x^2-3x+2}$
37. $\frac{b^2+2b-8}{b^2-20b+64}$
38. $\frac{x^2-x-20}{x^3+10x^2+24x}$
39. $\frac{n^2-8n+12}{n^3-12n^2+36n}$
40. $\frac{4x^2-6x-4}{2x^2-8x+8}$
41. $\frac{3m^2+9m+6}{4m^2+12m+8}$

COOKING **For Exercises 42–45, use the following information.**

The formula $t = \frac{40(25 + 1.85a)}{50 - 1.85a}$ relates the time t in minutes that it takes to cook an average-size potato in an oven that is at an altitude of a thousands of feet.

42. What is the value of a for an altitude of 4500 feet?

43. Calculate the time is takes to cook a potato at an altitude of 3500 feet.

44. About how long will it take to cook a potato at an altitude of 7000 feet?

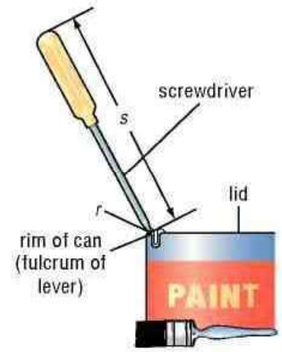
45. The altitude in Exercise 44 is twice that of Exercise 43. How do your cooking times compare for those two altitudes?

WebQuest

You can use a rational expression to determine how an amusement park can finance a new roller coaster. Visit www.algebra1.com/webquest to continue work on your WebQuest project.

PHYSICAL SCIENCE **For Exercises 46–48, use the following information.**

To pry the lid off a paint can, a screwdriver that is 17.5 centimeters long is used as a lever. It is placed so that 0.4 centimeter of its length extends inward from the rim of the can.



46. Write an equation that can be used to calculate the mechanical advantage.

47. What is the mechanical advantage?

48. If a force of 6 pounds is applied to the end of the screwdriver, what is the force placed on the lid?

FIELD TRIPS **For Exercises 49–52, use the following information.**

Mrs. Hoffman's art class is taking a trip to the museum. A bus that can seat up to 56 people costs \$450 for the day, and group rate tickets at the museum cost \$4 each.

49. If there are no more than 56 students going on the field trip, write an expression for the total cost for n students to go to the museum.

50. Write a rational expression that could be used to calculate the cost of the trip per student.

51. How many students must attend in order to keep the cost under \$15 per student?

52. How would you change the expression for cost per student if the school were to cover the cost of two adult chaperones?

More About...

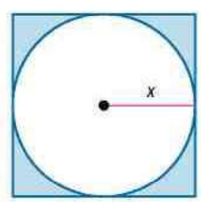
Farming

Although the amount of farmland in the United States is declining, crop production has increased steadily due in part to better irrigation practices.

Source: U.S. Department of Agriculture

FARMING **For Exercises 53 and 54, use the following information.**

Some farmers use an irrigation system that waters a circular region in a field. Suppose a square field with sides of length $2x$ is irrigated from the center of the square. The irrigation system can reach a radius of x.



53. Write an expression that represents the fraction of the field that is irrigated.

54. Calculate the percent of the field that is irrigated to the nearest whole percent.

55. CRITICAL THINKING Two students graphed the following equations on their calculators.

$$y = \frac{x^2 - 16}{x - 4} \qquad y = x + 4$$

They were surprised to see that the graphs appeared to be identical.

a. Explain why the graphs appear to be the same.

b. Explain how and why the graphs are different.

56. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can a rational expression be used in a movie theater?

Include the following in your answer:

- a description of how you determine the excluded values of a rational expression, and
- an example of another real-world situation that could be described using a rational expression.

Standardized Test Practice

57. Which expression is written in simplest form?

(A) $\dfrac{x^2 + 3x + 2}{x^2 + x - 2}$ (B) $\dfrac{3x - 3}{2x^2 - 2}$

(C) $\dfrac{x^2 + 7x}{x^2 + 3x - 4}$ (D) $\dfrac{2x^2 - 5x - 3}{x^2 + x - 12}$

58. In which expression are 1 and 5 excluded values?

(A) $\dfrac{x^2 + 6x + 5}{x^2 - 3x + 2}$ (B) $\dfrac{x^2 - 3x + 2}{x^2 - 6x + 5}$

(C) $\dfrac{x^2 - 6x + 5}{x^2 - 3x + 2}$ (D) $\dfrac{x^2 - 3x + 2}{x^2 + 6x + 5}$

Maintain Your Skills

Mixed Review

Write an inverse variation equation that relates x and y. Assume that y varies inversely as x. Then solve. *(Lesson 12-1)*

59. If $y = 6$ when $x = 10$, find y when $x = -12$.

60. If $y = 16$ when $x = \frac{1}{2}$, find x when $y = 32$.

61. If $y = -2.5$ when $x = 3$, find y when $x = -8$.

Use a calculator to find the measure of each angle to the nearest degree. *(Lesson 11-7)*

62. $\sin N = 0.2347$

63. $\cos B = 0.3218$

64. $\tan V = 0.0765$

65. $\sin A = 0.7011$

Solve each equation. Check your solution. *(Lesson 11-3)*

66. $\sqrt{a + 3} = 2$

67. $\sqrt{2z + 2} = z - 3$

68. $\sqrt{13 - 4p} - p = 8$

69. $\sqrt{3r^2 + 61} = 2r + 1$

Find the next three terms in each geometric sequence. *(Lesson 10-7)*

70. $1, 3, 9, 27, \ldots$

71. $6, 24, 96, 384, \ldots$

72. $\frac{1}{4}, -\frac{1}{2}, 1, -2, \ldots$

73. $4, 3, \frac{9}{4}, \frac{27}{16}, \ldots$

74. **GEOMETRY** Find the area of a rectangle if the length is $2x + y$ units and the width is $x + y$ units. *(Lesson 8-7)*

Getting Ready for the Next Lesson

BASIC SKILL **Complete.**

75. 84 in. = ____ ft

76. 4.5 m = ____ cm

77. 4 h 15 min = ____ s

78. 18 mi = ____ ft

79. 3 days = ____ h

80. 220 mL = ____ L

Graphing Calculator Investigation

A Follow-Up of Lesson 12-2

Rational Expressions

When simplifying rational expressions, you can use a TI-83 Plus graphing calculator to support your answer. If the graphs of the original expression and the simplified expression coincide, they are equivalent. You can also use the graphs to see excluded values.

Simplify $\frac{x^2 - 25}{x^2 + 10x + 25}$.

Step 1 *Factor the numerator and denominator.*

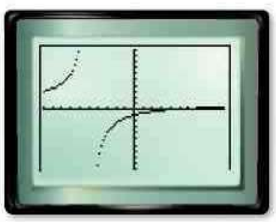
- $\frac{x^2 - 25}{x^2 + 10x + 25} = \frac{(x - 5)(x + 5)}{(x + 5)(x + 5)}$

  $= \frac{(x - 5)}{(x + 5)}$ When $x = -5$, $x + 5 = 0$. Therefore, x cannot equal -5 because you cannot divide by zero.

Step 2 *Graph the original expression.*

- Set the calculator to **Dot** mode.
- Enter $\frac{x^2 - 25}{x^2 + 10x + 25}$ as **Y1** and graph.

KEYSTROKES: MODE ▼ ▼ ▼ ▼ ▶ ENTER Y= (X,T,θ x^2 − 25) ÷ (X,T,θ x^2 + 10 X,T,θ + 25) ZOOM 6



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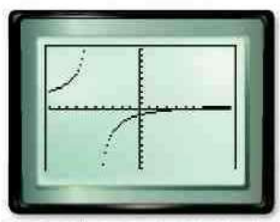
Step 3 *Graph the simplified expression.*

- Enter $\frac{(x - 5)}{(x + 5)}$ as **Y2** and graph.

KEYSTROKES: Y= ▼ ▼ (X,T,θ − 5) ÷ (X,T,θ + 5) GRAPH



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Since the graphs overlap, the two expressions are equivalent.

Exercises

Simplify each expression. Then verify your answer graphically. Name the excluded values.

1. $\frac{3x + 6}{x^2 + 7x + 10}$ **2.** $\frac{x^2 - 9x + 8}{x^2 - 16x + 64}$ **3.** $\frac{5x^2 + 10x + 5}{3x^2 + 6x + 3}$

4. Simplify the rational expression $\frac{2x - 9}{4x^2 - 18x}$ and answer the following questions using the **TABLE** menu on your calculator.

a. How can you use the **TABLE** function to verify that the original expression and the simplified expression are equivalent?

b. How does the **TABLE** function show you that an x value is an excluded value?

www.algebra1.com/other_calculator_keystrokes

12-3 Multiplying Rational Expressions

What You'll Learn

- Multiply rational expressions.
- Use dimensional analysis with multiplication.

How can you multiply rational expressions to determine the cost of electricity?

There are 25 lights around a patio. Each light is 40 watts, and the cost of electricity is 15 cents per kilowatt-hour. You can use the expression below to calculate the cost of using the lights for h hours.

$$25\ \cancel{\text{lights}} \cdot h\ \cancel{\text{hours}} \cdot \frac{40\ \cancel{\text{watts}}}{\cancel{\text{light}}} \cdot \frac{1\ \cancel{\text{kilowatt}}}{1000\ \cancel{\text{watts}}} \cdot \frac{15\ \cancel{\text{cents}}}{1\ \cancel{\text{kilowatt}} \cdot \cancel{\text{hour}}} \cdot \frac{1\ \text{dollar}}{100\ \cancel{\text{cents}}}$$

From this point on, you may assume that no denominator of a rational expression has a value of zero.

MULTIPLY RATIONAL EXPRESSIONS The multiplication expression above is similar to the multiplication of rational expressions. Recall that to multiply rational numbers expressed as fractions, you multiply numerators and multiply denominators. You can use this same method to multiply rational expressions.

Example 1 Expressions Involving Monomials

a. Find $\frac{5ab^3}{8c^2} \cdot \frac{16c^3}{15a^2b}$.

Method 1 Divide by the greatest common factor after multiplying.

$$\frac{5ab^3}{8c^2} \cdot \frac{16c^3}{15a^2b} = \frac{80ab^3c^3}{120a^2bc^2}$$
← Multiply the numerators.
← Multiply the denominators.

$$= \frac{\overset{1}{\cancel{40abc^2}}(2b^2c)}{\underset{1}{\cancel{40abc^2}}(3a)}$$
The GCF is $40abc^2$.

$$= \frac{2b^2c}{3a}$$
Simplify.

Method 2 Divide by the common factors before multiplying.

$$\frac{5ab^3}{8c^2} \cdot \frac{16c^3}{15a^2b} = \frac{\overset{1\,1\,b^2}{\cancel{5}\cancel{a}\cancel{b^3}}}{\underset{1\,1}{\cancel{8}\cancel{c^2}}} \cdot \frac{\overset{2\,c}{\cancel{16}\cancel{c^3}}}{\underset{3\,a\,1}{\cancel{15}\cancel{a^2}\cancel{b}}}$$
Divide by common factors 5, 8, a, b, and c^2.

$$= \frac{2b^2c}{3a}$$
Multiply.

b. Find $\frac{12xy^2}{45mp^2} \cdot \frac{27m^3p}{40x^3y}$.

$$\frac{12xy^2}{45mp^2} \cdot \frac{27m^3p}{40x^3y} = \frac{\overset{3\,1\,y}{\cancel{12}\cancel{x}\cancel{y^2}}}{\underset{5\,1\,p}{\cancel{45}\cancel{m}\cancel{p^2}}} \cdot \frac{\overset{3\,m^2\,1}{\cancel{27}\cancel{m^3}\cancel{p}}}{\underset{10\,x^2\,1}{\cancel{40}\cancel{x^3}\cancel{y}}}$$
Divide by common factors 4, 9, x, y, m, and p.

$$= \frac{9m^2y}{50x^2p}$$
Multiply.

Sometimes you must factor a quadratic expression before you can simplify a product of rational expressions.

Example 2 Expressions Involving Polynomials

a. Find $\frac{x-5}{x} \cdot \frac{x^2}{x^2-2x-15}$.

$$\frac{x-5}{x} \cdot \frac{x^2}{x^2-2x-15} = \frac{x-5}{x} \cdot \frac{x^2}{(x-5)(x+3)}$$ Factor the denominator.

$$= \frac{\overset{x}{\cancel{x^2}}\overset{1}{\cancel{(x-5)}}}{\underset{1}{\cancel{x}}\underset{1}{\cancel{(x-5)}}(x+3)}$$ The GCF is $x(x-5)$.

$$= \frac{x}{x+3}$$ Simplify.

b. Find $\frac{a^2+7a+10}{a+1} \cdot \frac{3a+3}{a+2}$.

$$\frac{a^2+7a+10}{a+1} \cdot \frac{3a+3}{a+2} = \frac{(a+5)(a+2)}{a+1} \cdot \frac{3(a+1)}{a+2}$$ Factor the numerators.

$$= \frac{3(a+5)\overset{1}{\cancel{(a+2)}}\overset{1}{\cancel{(a+1)}}}{\underset{1}{\cancel{(a+1)}}\underset{1}{\cancel{(a+2)}}}$$ The GCF is $(a+1)(a+2)$.

$$= \frac{3(a+5)}{1}$$ Multiply.

$$= 3a + 15$$ Simplify.

Study Tip

Look Back

To review **dimensional analysis**, see Lesson 3-8.

DIMENSIONAL ANALYSIS When you multiply fractions that involve units of measure, you can divide by the units in the same way that you divide by variables. Recall that this process is called dimensional analysis.

Example 3 Dimensional Analysis

OLYMPICS In the 2000 Summer Olympics in Sydney, Australia, Maurice Green of the United States won the gold medal for the 100-meter sprint. His winning time was 9.87 seconds. What was his speed in kilometers per hour? Round to the nearest hundredth.

$$\frac{100 \text{ meters}}{9.87 \text{ seconds}} \cdot \frac{1 \text{ kilometer}}{1000 \text{ meters}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}}$$

$$= \frac{100 \cancel{\text{ meters}}}{9.87 \cancel{\text{ seconds}}} \cdot \frac{1 \text{ kilometer}}{1000 \cancel{\text{ meters}}} \cdot \frac{60 \cancel{\text{ seconds}}}{1 \cancel{\text{ minute}}} \cdot \frac{60 \cancel{\text{ minutes}}}{1 \text{ hour}}$$

$$= \frac{\overset{1}{\cancel{100}} \cdot 1 \cdot 60 \cdot 60 \text{ kilometers}}{9.87 \cdot \underset{10}{\cancel{1000}} \cdot 1 \cdot 1 \text{ hours}}$$

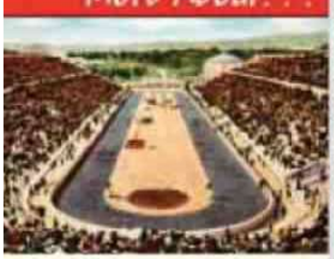
$$= \frac{60 \cdot 60 \text{ kilometers}}{9.87 \cdot 10 \text{ hours}}$$ Simplify.

$$= \frac{3600 \text{ kilometers}}{98.7 \text{ hours}}$$ Multiply.

$$= \frac{36.47 \text{ kilometers}}{1 \text{ hour}}$$ Divide numerator and denominator by 98.7.

His speed was 36.47 kilometers per hour.

More About. . .



Olympics

American sprinter Thomas Burke won the 100-meter dash at the first modern Olympics in Athens, Greece, in 1896 in 12.0 seconds.

Source: www.olympics.org

Check for Understanding

Concept Check

1. **OPEN ENDED** Write two rational expressions whose product is $\frac{2}{x}$.
2. **Explain** why $-\frac{x+6}{x-5}$ is not equivalent to $\frac{-x+6}{x-5}$.
3. **FIND THE ERROR** Amiri and Hoshi multiplied $\frac{x-3}{x+3}$ and $\frac{4x}{x^2-4x+3}$.

Amiri

$$\frac{x-3}{x+3}\cdot\frac{4x}{x^2-4x+3}$$
$$=\frac{\cancel{(x-3)}4x}{(x+3)\cancel{(x-3)}(x-1)}$$
$$=\frac{4x}{(x+3)(x-1)}$$

Hoshi

$$\frac{x-3}{x+3}\cdot\frac{4x}{x^2-4x+3}$$
$$=\frac{\cancel{x}-\cancel{3}}{\cancel{x}+\cancel{3}}\cdot\frac{\cancel{4x}}{x^2-\cancel{4x}+3}$$
$$=\frac{1}{x^2+3}$$

Who is correct? Explain your reasoning.

Guided Practice

Find each product.

4. $\frac{64y^2}{5y}\cdot\frac{5y}{8y}$
5. $\frac{15s^2t^3}{12st}\cdot\frac{16st^2}{10s^3t^3}$
6. $\frac{m+4}{3m}\cdot\frac{4m^2}{(m+4)(m+5)}$
7. $\frac{x^2-4}{2}\cdot\frac{4}{x-2}$
8. $\frac{n^2-16}{n+4}\cdot\frac{n+2}{n^2+-8n+16}$
9. $\frac{x-5}{x^2-7x+10}\cdot\frac{x^2+x-6}{5}$
10. Find $\frac{24\text{ feet}}{1\text{ second}}\cdot\frac{60\text{ seconds}}{1\text{ minute}}\cdot\frac{60\text{ minutes}}{1\text{ hour}}\cdot\frac{1\text{ mile}}{5280\text{ feet}}$.

Application

11. **SPACE** The moon is about 240,000 miles from Earth. How many days would it take a spacecraft to reach the moon if it travels at an average of 100 miles per minute?

Practice and Apply

Homework Help

For Exercises	See Examples
12–15	1
16–27	2
28–37	3

Extra Practice

See page 847.

Find each product.

12. $\frac{8}{x^2}\cdot\frac{x^4}{4x}$
13. $\frac{10r^3}{6n^3}\cdot\frac{42n^2}{35r^3}$
14. $\frac{10y^3z^2}{6wx^3}\cdot\frac{12w^2x^2}{25y^2z^4}$
15. $\frac{3a^2b}{2gh}\cdot\frac{24g^2h}{15ab^2}$
16. $\frac{(x-8)}{(x+8)(x-3)}\cdot\frac{(x+4)(x-3)}{(x-8)}$
17. $\frac{(n-1)(n+1)}{(n+1)}\cdot\frac{(n-4)}{(n-1)(n+4)}$
18. $\frac{(z+4)(z+6)}{(z-6)(z+1)}\cdot\frac{(z+1)(z-5)}{(z+3)(z+4)}$
19. $\frac{(x-1)(x+7)}{(x-7)(x-4)}\cdot\frac{(x-4)(x+10)}{(x+1)(x+10)}$
20. $\frac{x^2-25}{9}\cdot\frac{x+5}{x-5}$
21. $\frac{y^2-4}{y^2-1}\cdot\frac{y+1}{y+2}$
22. $\frac{1}{x^2+x-12}\cdot\frac{x-3}{x+5}$
23. $\frac{x-6}{x^2+4x-32}\cdot\frac{x-4}{x+2}$
24. $\frac{x+3}{x+4}\cdot\frac{x}{x^2+7x+12}$
25. $\frac{n}{n^2+8n+15}\cdot\frac{2n+10}{n^2}$
26. $\frac{b^2+12b+11}{b^2-9}\cdot\frac{b+9}{b^2+20b+99}$
27. $\frac{a^2-a-6}{a^2-16}\cdot\frac{a^2+7a+12}{a^2+4a+4}$

Find each product.

28. $\frac{2.54 \text{ centimeters}}{1 \text{ inch}} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{3 \text{ feet}}{1 \text{ yard}}$

29. $\frac{60 \text{ kilometers}}{1 \text{ hour}} \cdot \frac{1000 \text{ meters}}{1 \text{ kilometer}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}} \cdot \frac{1 \text{ minutes}}{60 \text{ seconds}}$

30. $\frac{32 \text{ feet}}{1 \text{ second}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}}$

31. $10 \text{ feet} \cdot 18 \text{ feet} \cdot 3 \text{ feet} \cdot \frac{1 \text{ yard}^3}{27 \text{ feet}^3}$

32. **DECORATING** Alani's bedroom is 12 feet wide and 14 feet long. What will it cost to carpet her room if the carpet costs $18 per square yard?

33. **EXCHANGE RATES** While traveling in Canada, Johanna bought some gifts to bring home. She bought 2 T-shirts that cost $21.95 (Canadian). If the exchange rate at the time was 1 U.S. dollar for 1.37 Canadian dollars, how much did Johanna spend in U.S. dollars?

Online Research **Data Update** Visit www.algebra1.com/data_update to find the most recent exchange rate between the United States and Canadian currency. How much does a $21.95 (Canadian) purchase cost in U.S. dollars?

More About. . .

Exchange Rates

A system of floating exchange rates among international currencies was established in 1976. It was needed because the old system of basing a currency's value on gold had become obsolete.

Source: www.infoplease.com

34. **CITY MAINTENANCE** Street sweepers can clean 3 miles of streets per hour. A city owns 2 street sweepers, and each sweeper can be used for three hours before it comes in for an hour to refuel. How many miles of streets can be cleaned in 18 hours on the road?

TRAFFIC **For Exercises 35–37, use the following information.**
During rush hour one evening, traffic was backed up for 13 miles along a particular stretch of freeway. Assume that each vehicle occupied an average of 30 feet of space in a lane and that the freeway has three lanes.

35. Write an expression that could be used to determine the number of vehicles involved in the backup.

36. How many vehicles are involved in the backup?

37. Suppose that there are 8 exits along this stretch of freeway, and if it takes each vehicle an average of 24 seconds to exit the freeway. Approximately how many hours will it take for all the vehicles in the backup to exit?

38. **CRITICAL THINKING** Identify the expressions that are equivalent to $\frac{x}{y}$. Explain why the expressions are equivalent.

a. $\frac{x+3}{y+3}$ **b.** $\frac{3-x}{3-y}$ **c.** $\frac{3x}{3y}$ **d.** $\frac{x^3}{y^3}$ **e.** $\frac{n^3x}{n^3y}$

39. ***WRITING IN MATH*** Answer the question that was posed at the beginning of the lesson.

How can you multiply rational expressions to determine the cost of electricity?

Include the following in your answer:
- an expression that you could use to determine the cost of using 60-watt light bulbs instead of 40-watt bulbs, and
- an example of a real-world situation in which you must multiply rational expressions.

Standardized Test Practice

40. Which expression is the product of $\frac{13xyz}{4x^2y}$ and $\frac{8x^2z^2}{2y^3}$?

(A) $\frac{13xy^3}{z^3}$ (B) $\frac{13xz^2}{y^3}$ (C) $\frac{13xyz}{z^3}$ (D) $\frac{13xz^3}{y^3}$

41. Identify the product of $\frac{4a+4}{a^2+a}$ and $\frac{a^2}{3a-3}$.

(A) $\frac{4a}{3(a-1)}$ (B) $\frac{4a}{3}$ (C) $\frac{4a}{3(a+1)}$ (D) $\frac{4a^2}{3(a-1)}$

Maintain Your Skills

Mixed Review

State the excluded values for each rational expression. *(Lesson 12-2)*

42. $\frac{s+6}{s^2-36}$ 43. $\frac{a^2-25}{a^2+3a-10}$ 44. $\frac{x+3}{x^2+6x+9}$

Write an inverse variation equation that relates x and y. Assume that y varies inversely as x. Then solve. *(Lesson 12-1)*

45. If $y = 9$ when $x = 8$, find x when $y = 6$.
46. If $y = 2.4$ when $x = 8.1$, find y when $x = 3.6$.
47. If $y = 24$ when $x = -8$, find y when $x = 4$.
48. If $y = 6.4$ when $x = 4.4$, find x when $y = 3.2$.

Simplify. Assume that no denominator is equal to zero. *(Lesson 8-2)*

49. $\frac{-7^{12}}{7^9}$ 50. $\frac{20p^6}{8p^8}$ 51. $\frac{24a^3b^4c^7}{6a^6c^2}$

Solve each inequality. Then check your solution. *(Lesson 6-2)*

52. $\frac{g}{8} < \frac{7}{2}$ 53. $3.5r \geq 7.35$ 54. $\frac{9k}{4} > \frac{3}{5}$

55. **FINANCE** The total amount of money Antonio earns mowing lawns and doing yard work varies directly with the number of days he works. At one point, he earned \$340 in 4 days. At this rate, how long will it take him to earn \$935? *(Lesson 5-2)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Factor each polynomial.**
*(To review **factoring polynomials**, see Lessons 9-3 through 9-6.)*

56. $x^2 - 3x - 40$ 57. $n^2 - 64$ 58. $x^2 - 12x + 36$
59. $a^2 + 2a - 35$ 60. $2x^2 - 5x - 3$ 61. $3x^3 - 24x^2 + 36x$

Practice Quiz 1

Lessons 12-1 through 12-3

Graph each variation if y varies inversely as x. *(Lesson 12-1)*

1. $y = 28$ when $x = 7$ 2. $y = -6$ when $x = 9$

Simplify each expression. *(Lesson 12-2)*

3. $\frac{28a^2}{49ab}$ 4. $\frac{y+3y^2}{3y+1}$ 5. $\frac{b^2-3b-4}{b^2-13b+36}$ 6. $\frac{3n^2+5n-2}{3n^2-13n+4}$

Find each product. *(Lesson 12-3)*

7. $\frac{3m^2}{2m} \cdot \frac{18m^2}{9m}$ 8. $\frac{5a+10}{10x^2} \cdot \frac{4x^3}{a^2+11a+18}$ 9. $\frac{4n+8}{n^2-25} \cdot \frac{n-5}{5n+10}$ 10. $\frac{x^2-x-6}{x^2-9} \cdot \frac{x^2+7x+12}{x^2+4x+4}$

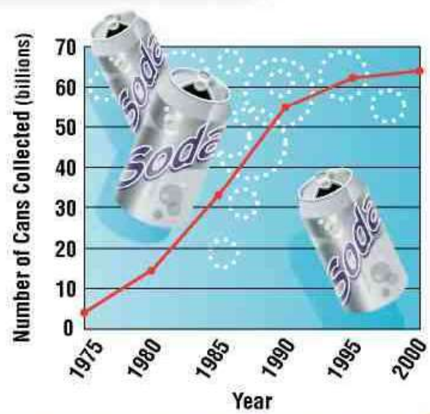
12-4 Dividing Rational Expressions

What You'll Learn

- Divide rational expressions.
- Use dimensional analysis with division.

How can you determine the number of aluminum soft drink cans made each year?

Most soft drinks come in aluminum cans. Although more cans are used today than in the 1970s, the demand for new aluminum has declined. This is due in large part to the great number of cans that are recycled. In recent years, approximately 63.9 billion cans were recycled annually. This represents $\frac{5}{8}$ of all cans produced.



DIVIDE RATIONAL EXPRESSIONS Recall that to divide rational numbers expressed as fractions you multiply by the reciprocal of the divisor. You can use this same method to divide rational expressions.

Example 1 Expression Involving Monomials

Find $\frac{5x^2}{7} \div \frac{10x^3}{21}$.

$$\frac{5x^2}{7} \div \frac{10x^3}{21} = \frac{5x^2}{7} \cdot \frac{21}{10x^3} \quad \text{Multiply by } \frac{21}{10x^3}\text{, the reciprocal of } \frac{10x^3}{21}.$$

$$= \frac{\overset{1}{\cancel{5x^2}}}{\underset{1}{\cancel{7}}} \cdot \frac{\overset{3}{\cancel{21}}}{\underset{2x}{\cancel{10x^3}}} \quad \text{Divide by common factors 5, 7, and } x^2.$$

$$= \frac{3}{2x} \quad \text{Simplify.}$$

Example 2 Expression Involving Binomials

Find $\frac{n+1}{n+3} \div \frac{2n+2}{n+4}$.

$$\frac{n+1}{n+3} \div \frac{2n+2}{n+4} = \frac{n+1}{n+3} \cdot \frac{n+4}{2n+2} \quad \text{Multiply by } \frac{n+4}{2n+2}\text{, the reciprocal of } \frac{2n+2}{n+4}.$$

$$= \frac{n+1}{n+3} \cdot \frac{n+4}{2(n+1)} \quad \text{Factor } 2n+2.$$

$$= \frac{\overset{1}{\cancel{n+1}}}{n+3} \cdot \frac{n+4}{2(\underset{1}{\cancel{n+1}})} \quad \text{The GCF is } n+1.$$

$$= \frac{n+4}{2(n+3)} \text{ or } \frac{n+4}{2n+6} \quad \text{Simplify.}$$

Often the quotient of rational expressions involves a divisor that is a binomial.

Study Tip

Multiplicative Inverse

As with rational numbers, dividing rational expressions involves multiplying by the inverse. Remember that the inverse of $a + 2$ is $\frac{1}{a+2}$.

Example 3 Divide by a Binomial

Find $\frac{5a + 10}{a + 5} \div (a + 2)$.

$$\frac{5a+10}{a+5} \div (a+2) = \frac{5a+10}{a+5} \cdot \frac{1}{(a+2)} \quad \text{Multiply by } \frac{1}{(a+2)}\text{, the reciprocal of } (a+2).$$

$$= \frac{5(a+2)}{a+5} \cdot \frac{1}{(a+2)} \quad \text{Factor } 5a + 10.$$

$$= \frac{5\cancel{(a+2)}^{1}}{a+5} \cdot \frac{1}{\cancel{(a+2)}_{1}} \quad \text{The GCF is } a + 2.$$

$$= \frac{5}{a+5} \quad \text{Simplify.}$$

Sometimes you must factor a quadratic expression before you can simplify the quotient of rational expressions.

Example 4 Expression Involving Polynomials

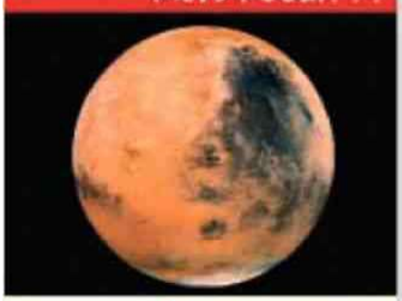
Find $\frac{m^2 + 3m + 2}{4} \div \frac{m + 2}{m + 1}$.

$$\frac{m^2+3m+2}{4} \div \frac{m+2}{m+1} = \frac{m^2+3m+2}{4} \cdot \frac{m+1}{m+2} \quad \text{Multiply by the reciprocal, } \frac{m+1}{m+2}.$$

$$= \frac{(m+1)(m+2)}{4} \cdot \frac{m+1}{m+2} \quad \text{Factor } m^2 + 3m + 2.$$

$$= \frac{(m+1)\cancel{(m+2)}^{1}}{4} \cdot \frac{m+1}{\cancel{m+2}_{1}} \quad \text{The GCF is } m + 2.$$

$$= \frac{(m+1)^2}{4} \quad \text{Simplify.}$$



More About...

Space

The first successful Mars probe was the Mariner 4, which arrived at Mars on July 14, 1965.

Source: NASA

DIMENSIONAL ANALYSIS You can divide rational expressions that involve units of measure by using dimensional analysis.

Example 5 Dimensional Analysis

SPACE In November, 1996, NASA launched the Mars Global Surveyor. It took 309 days for the orbiter to travel 466,000,000 miles from Earth to Mars. What was the speed of the spacecraft in miles per hour? Round to the nearest hundredth.

Use the formula for rate, time, and distance.

$$rt = d \quad \text{rate} \cdot \text{time} = \text{distance}$$

$$r \cdot 309 \text{ days} = 466{,}000{,}000 \text{ mi} \quad t = 309 \text{ days}, d = 466{,}000{,}000$$

$$r = \frac{466{,}000{,}000 \text{ mi}}{309 \text{ days}} \quad \text{Divide each side by 309 days.}$$

$$= \frac{466{,}000{,}000 \text{ miles}}{309 \cancel{\text{days}}} \cdot \frac{1 \cancel{\text{day}}}{24 \text{ hours}} \quad \text{Convert days to hours.}$$

$$= \frac{466{,}000{,}000 \text{ miles}}{7416 \text{ hours}} \text{ or about } \frac{62{,}837.11 \text{ miles}}{1 \text{ hour}}$$

Thus, the spacecraft traveled at a rate of about 62,837.11 miles per hour.

Check for Understanding

Concept Check

1. **OPEN ENDED** Write two rational expressions whose quotient is $\frac{5z}{xy}$.
2. **Tell** whether the following statement is *always*, *sometimes*, or *never* true. Explain your reasoning.
   Every real number has a reciprocal.
3. **Explain** how to calculate the mass in kilograms of one cubic meter of a substance whose density is 2.16 grams per cubic centimeter.

Guided Practice

Find each quotient.

4. $\frac{10n^3}{7} \div \frac{5n^2}{21}$
5. $\frac{2a}{a+3} \div \frac{a+7}{a+3}$
6. $\frac{3m-15}{m+4} \div \frac{m-5}{6m+24}$
7. $\frac{2x+6}{x+5} \div (x+3)$
8. $\frac{k+3}{k^2+4k+4} \div \frac{2k+6}{k+2}$
9. $\frac{2x-4}{x^2+11x+18} \div \frac{x+1}{x^2+5x+6}$
10. Express 85 kilometers per hour in meters per second.
11. Express 32 pounds per square foot in square inches.

Application

12. **COOKING** Latisha was making candy using a two-quart pan. As she stirred the mixture, she noticed that the pan was about $\frac{2}{3}$ full. If each piece of candy has a volume of about $\frac{3}{4}$ ounce, approximately how many pieces of candy will Latisha make?

Practice and Apply

Homework Help

For Exercises	See Examples
13–18	1
19–22	3
23, 24	2
29–36	4
25–28, 37–41	5

Extra Practice

See page 847.

Find each quotient.

13. $\frac{a^2}{b^2} \div \frac{a}{b^3}$
14. $\frac{n^4}{p^2} \div \frac{n^2}{p^3}$
15. $\frac{4x^3}{y^4} \div \frac{8x^2}{y^2}$
16. $\frac{10m^2}{7n^2} \div \frac{25m^4}{14n^3}$
17. $\frac{x^2y^3z}{s^2t^2} \div \frac{x^2yz^3}{s^3t^2}$
18. $\frac{a^4bc^3}{g^2h^3} \div \frac{ab^2c^2}{g^3h^3}$
19. $\frac{b^2-9}{4b} \div (b-3)$
20. $\frac{m^2-16}{5m} \div (m+4)$
21. $\frac{3k}{k+1} \div (k-2)$
22. $\frac{5d}{d-3} \div (d+1)$
23. $\frac{3x+12}{4x-18} \div \frac{2x+8}{x+4}$
24. $\frac{4a-8}{2a-6} \div \frac{2a-4}{a-4}$

Complete.

25. 24 yd^3 = ____ ft^3
26. 0.35 m^3 = ____ cm^3
27. 330 ft/s = ____ mi/h
28. 1730 plants/km^2 = ____ plants/m^2
29. What is the quotient when $\frac{2x+6}{x+5}$ is divided by $\frac{2}{x+5}$?
30. Find the quotient when $\frac{m-8}{m+7}$ is divided by $m^2 - 7m - 8$.

Find each quotient.

31. $\dfrac{x^2 + 2x + 1}{2} \div \dfrac{x + 1}{x - 1}$

32. $\dfrac{n^2 + 3n + 2}{4} \div \dfrac{n + 1}{n + 2}$

33. $\dfrac{a^2 + 8a + 16}{a^2 - 6a + 9} \div \dfrac{2a + 8}{3a - 9}$

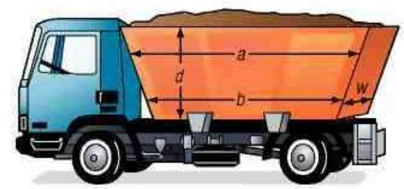
34. $\dfrac{b + 2}{b^2 + 4b + 4} \div \dfrac{2b + 4}{b + 4}$

35. $\dfrac{x^2 + x - 2}{x^2 + 5x + 6} \div \dfrac{x^2 + 2x - 3}{x^2 + 7x + 12}$

36. $\dfrac{x^2 + 2x - 15}{x^2 - x - 30} \div \dfrac{x^2 - 3x - 18}{x^2 - 2x - 24}$

37. **TRIATHLONS** Irena is training for an upcoming triathlon and plans to run 12 miles today. Jorge offered to ride his bicycle to help her maintain her pace. If Irena wants to keep a steady pace of 6.5 minutes per mile, how fast should Jorge ride in miles per hour?

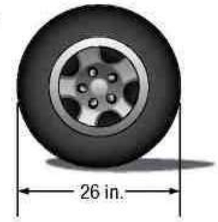
CONSTRUCTION **For Exercises 38 and 39, use the following information.**
A construction supervisor needs to determine how many truckloads of earth must be removed from a site before a foundation can be poured. The bed of the truck has the shape shown at the right.



38. Use the formula $V = \dfrac{d(a + b)}{2} \cdot w$ to write an equation involving units that represents the volume of the truck bed in cubic yards if $a = 18$ feet, $b = 15$ feet, $w = 9$ feet, and $d = 5$ feet.

39. There are 20,000 cubic yards of earth that must be removed from the excavation site. Write an equation involving units that represents the number of truckloads that will be required to remove all of the earth. Then solve the equation.

TRUCKS **For Exercises 40 and 41, use the following information.**
The speedometer of John's truck uses the revolutions of his tires to calculate the speed of the truck.



40. How many revolutions per minute do the tires make when the truck is traveling at 55 miles per hour?

41. Suppose John buys tires with a diameter of 30 inches. When the speedometer reads 55 miles per hour, the tires would still revolve at the same rate as before. However, with the new tires, the truck travels a different distance in each revolution. Calculate the actual speed when the speedometer reads 55 miles per hour.

42. **CRITICAL THINKING** Which expression is *not* equivalent to the reciprocal of $\dfrac{x^2 - 4y^2}{x + 2y}$? Justify your answer.

a. $\dfrac{1}{\dfrac{x^2 - 4y^2}{x + 2y}}$ b. $\dfrac{-1}{2y - x}$ c. $\dfrac{1}{x - 2y}$ d. $\dfrac{1}{x} - \dfrac{1}{2y}$

SCULPTURE **For Exercises 43 and 44, use the following information.**
A sculptor had a block of marble in the shape of a cube with sides x feet long. A piece that was $\frac{1}{2}$ foot thick was chiseled from the bottom of the block. Later, the sculptor removed a piece $\frac{3}{4}$ foot wide from the side of the marble block.

43. Write a rational expression that represents the volume of the block of marble that remained.

44. If the remaining marble was cut into ten pieces weighing 85 pounds each, write an expression that represents the weight of the original block of marble.

More About. . .

Triathlons

The Ironman Championship Triathlon held in Hawaii consists of a 2.4-mile swim, a 112-mile bicycle ride, and a 26.2-mile run.

Source: www.infoplease.com

www.algebra1.com/self_check_quiz

45. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you determine the number of aluminum soft drink cans made each year?

Include the following in your answer:

- a rational expression that will give the amount of new aluminum needed to produce x aluminum cans today when $\frac{5}{8}$ of the cans are recycled and 33 cans are produced from a pound of aluminum.

Standardized Test Practice

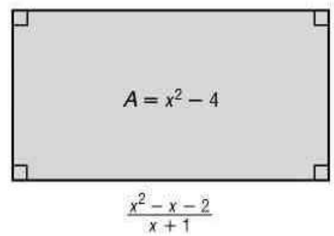
46. Which expression is the quotient of $\frac{3b}{5c}$ and $\frac{18b}{15c}$?

(A) $\frac{18b^2}{15c^2}$ (B) $\frac{1}{2}$ (C) $\frac{18b}{15c}$ (D) 2

47. Which expression could be used for the width of the rectangle?

(A) $x - 2$ (B) $(x + 2)(x - 2)^2$

(C) $x + 2$ (D) $(x + 2)(x - 2)$



Maintain Your Skills

Mixed Review

Find each product. *(Lesson 12-3)*

48. $\frac{x - 5}{x^2 - 7x + 10} \cdot \frac{x - 2}{1}$

49. $\frac{x^2 + 3x - 10}{x^2 + 8x + 15} \cdot \frac{x^2 + 5x + 6}{x^2 + 4x + 4}$

50. $\frac{x + 4}{4y} \cdot \frac{16y}{x^2 + 7x + 12}$

51. $\frac{x^2 + 8x + 15}{x + y} \cdot \frac{7x + 14y}{x + 3}$

Simplify each expression. *(Lesson 12-2)*

52. $\frac{c - 6}{c^2 - 12c + 36}$

53. $\frac{25 - x^2}{x^2 + x - 30}$

54. $\frac{a + 3}{a^2 + 4a + 3}$

55. $\frac{n^2 - 16}{n^2 - 8n + 16}$

Solve each equation. Check your solutions. *(Lesson 9-6)*

56. $3y^2 = 147$

57. $9x^2 - 24x = -16$

58. $a^2 + 225 = 30a$

59. $(n + 6)^2 = 14$

Find the degree of each polynomial. *(Lesson 8-4)*

60. $13 + \frac{1}{8}$

61. $z^3 - 2z^2 + 3z - 4$

62. $a^5b^2c^3 + 6a^3b^3c^2$

Solve each inequality. Then check your solution. *(Lesson 6-2)*

63. $6 \le 0.8g$

64. $-15b < -28$

65. $-0.049 \le 0.07x$

66. $\frac{3}{7}h + \frac{3}{49}$

67. $\frac{12r}{-4} > \frac{3}{20}$

68. $\frac{y}{6} \ge \frac{1}{2}$

69. **MANUFACTURING** Tanisha's Sporting Equipment manufactures tennis racket covers at the rate of 3250 each month. How many tennis racket covers will the company manufacture by the end of the year? *(Lesson 5-3)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Simplify.** *(To review* ***dividing monomials****, see Lesson 8-2.)*

70. $\frac{6x^2}{x^4}$

71. $\frac{5m^4}{25m}$

72. $\frac{18a^3}{45a^5}$

73. $\frac{b^6c^3}{b^3c^6}$

74. $\frac{12x^3y^2}{28x^4y}$

75. $\frac{7x^4z^2}{z^3}$

Reading Mathematics

Rational Expressions

Several concepts need to be applied when reading rational expressions.

- A fraction bar acts as a grouping symbol, where the entire numerator is divided by the entire denominator.

Example 1 $\frac{6x + 4}{10}$

It is correct to read the expression as *the quantity six x plus four divided by ten.*

It is incorrect to read the expression as *six x divided by ten plus four*, or *six x plus four divided by ten.*

- If a fraction consists of two or more terms divided by a one-term denominator, the denominator divides each term.

Example 2 $\frac{6x + 4}{10}$

It is correct to write $\frac{6x+4}{10} = \frac{6x}{10} + \frac{4}{10}$.

$$= \frac{3x}{5} + \frac{2}{5} \quad \text{or} \quad \frac{3x+2}{5}$$

It is also correct to write $\frac{6x+4}{10} = \frac{2(3x+2)}{2 \cdot 5}$.

$$= \frac{\cancel{2}(3x+2)}{\cancel{2} \cdot 5} \quad \text{or} \quad \frac{3x+2}{5}$$

It is incorrect to write $\frac{6x+4}{10} = \frac{\overset{3x}{\cancel{6x}}+4}{\underset{5}{\cancel{10}}} = \frac{3x+4}{5}$.

Reading to Learn

Write the verbal translation of each rational expression.

1. $\frac{m+2}{4}$

2. $\frac{3x}{x-1}$

3. $\frac{a+2}{a^2+8}$

4. $\frac{x^2-25}{x+5}$

5. $\frac{x^2-3x+18}{x-2}$

6. $\frac{x^2+2x-35}{x^2-x-20}$

Simplify each expression.

7. $\frac{3x+6}{9}$

8. $\frac{4n-12}{8}$

9. $\frac{5x^2-25x}{10x}$

10. $\frac{x+3}{x^2+7x+12}$

11. $\frac{x+y}{x^2+2xy+y^2}$

12. $\frac{x^2-16}{x^2-8x+16}$

12-5 Dividing Polynomials

What You'll Learn

- Divide a polynomial by a monomial.
- Divide a polynomial by a binomial.

How is division used in sewing?

Marching bands often use intricate marching routines and colorful flags to add interest to their shows. Suppose a partial roll of fabric is used to make flags. The original roll was 36 yards long, and $7\frac{1}{2}$ yards of the fabric were used to make a banner for the band. Each flag requires $1\frac{1}{2}$ yards of fabric. The expression $\dfrac{36 \text{ yards} - 7\frac{1}{2} \text{ yards}}{1\frac{1}{2} \text{ yards}}$ can be used to represent the number of flags that can be made using the roll of fabric.

DIVIDE POLYNOMIALS BY MONOMIALS To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example 1 Divide a Binomial by a Monomial

Find $(3r^2 - 15r) \div 3r$.

$$
\begin{aligned}
(3r^2 - 15r) \div 3r &= \frac{3r^2 - 15r}{3r} && \text{Write as a rational expression.}\\
&= \frac{3r^2}{3r} - \frac{15r}{3r} && \text{Divide each term by } 3r.\\
&= \frac{\overset{r}{\cancel{3r^2}}}{\underset{1}{\cancel{3r}}} - \frac{\overset{5}{\cancel{15r}}}{\underset{1}{\cancel{3r}}} && \text{Simplify each term.}\\
&= r - 5 && \text{Simplify.}
\end{aligned}
$$

Example 2 Divide a Polynomial by a Monomial

Find $(n^2 + 10n + 12) \div 5n$.

$$
\begin{aligned}
(n^2 + 10n + 12) \div 5n &= \frac{n^2 + 10n + 12}{5n} && \text{Write as a rational expression.}\\
&= \frac{n^2}{5n} + \frac{10n}{5n} + \frac{12}{5n} && \text{Divide each term by } 5n.\\
&= \frac{\overset{n}{\cancel{n^2}}}{\underset{1}{\cancel{5n}}} + \frac{\overset{2}{\cancel{10n}}}{\underset{1}{\cancel{5n}}} + \frac{12}{5n} && \text{Simplify each term.}\\
&= \frac{n}{5} + 2 + \frac{12}{5n} && \text{Simplify.}
\end{aligned}
$$

DIVIDE POLYNOMIALS BY BINOMIALS

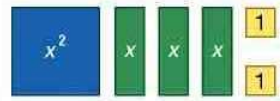
You can use algebra tiles to model some quotients of polynomials.

Algebra Activity

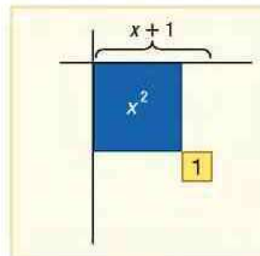
Dividing Polynomials

Use algebra tiles to find $(x^2 + 3x + 2) \div (x + 1)$.

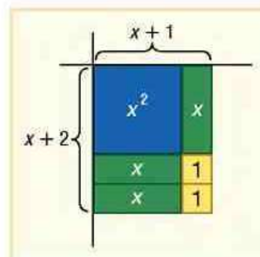
Step 1 Model the polynomial $x^2 + 3x + 2$.



Step 2 Place the x^2 tile at the corner of the product mat. Place one of the 1 tiles as shown to make a length of $x + 1$.



Step 3 Use the remaining tiles to make a rectangular array.



The width of the array, $x + 2$, is the quotient.

Model and Analyze

Use algebra tiles to find each quotient.

1. $(x^2 + 3x - 4) \div (x - 1)$

2. $(x^2 - 5x + 6) \div (x - 2)$

3. $(x^2 - 16) \div (x + 4)$

4. $(2x^2 - 4x - 6) \div (x - 3)$

5. Describe what happens when you try to model $(3x^2 - 4x + 3) \div (x + 2)$. What do you think the result means?

Recall from Lesson 12-4 that when you factor, some divisions can be performed easily.

Example 3 Divide a Polynomial by a Binomial

Find $(s^2 + 6s - 7) \div (s + 7)$.

$$
\begin{aligned}
(s^2 + 6s - 7) \div (s + 7) &= \frac{s^2 + 6s - 7}{(s + 7)} && \text{Write as a rational expression.} \\
&= \frac{(s + 7)(s - 1)}{(s + 7)} && \text{Factor the numerator.} \\
&= \frac{\overset{1}{\cancel{(s + 7)}}(s - 1)}{\underset{1}{\cancel{(s + 7)}}} && \text{Divide by the GCF.} \\
&= s - 1 && \text{Simplify.}
\end{aligned}
$$

In Example 3 the division could be performed easily by dividing by common factors. However, when you cannot factor, you can use a long division process similar to the one you use in arithmetic.

Example 4 Long Division

Find $(x^2 + 3x - 24) \div (x - 4)$.

The expression $x^2 + 3x - 24$ cannot be factored, so use long division.

Step 1 Divide the first term of the dividend, x^2, by the first term of the divisor, x.

$$\begin{array}{rll} & x & \quad x^2 \div x = x \\ x - 4 \,\big)\!\! & \overline{x^2 + 3x - 24} & \\ & \underline{(-)\ x^2 - 4x} & \quad \text{Multiply } x \text{ and } x - 4. \\ & 7x & \quad \text{Subtract.} \end{array}$$

Step 2 Divide the first term of the partial dividend, $7x - 24$, by the first term of the divisor, x.

$$\begin{array}{rll} & x + 7 & \quad 7x \div x = 7 \\ x - 4 \,\big)\!\! & \overline{x^2 + 3x - 24} & \\ & \underline{(-)\ x^2 - 4x} & \\ & 7x - 24 & \quad \text{Subtract and bring down the 24.} \\ & \underline{(-)\ 7x - 28} & \quad \text{Multiply 7 and } x - 4. \\ & 4 & \quad \text{Subtract.} \end{array}$$

The quotient of $(x^2 + 3x - 24) \div (x - 4)$ is $x + 7$ with a remainder of 4, which can be written as $x + 7 + \frac{4}{x - 4}$. Since there is a nonzero remainder, $x - 4$ is not a factor of $x^2 + 3x - 24$.

When the dividend is an expression like $a^3 + 8a - 21$, there is no a^2 term. In such situations, you must rename the dividend using 0 as the coefficient of the missing terms.

Example 5 Polynomial with Missing Terms

Find $(a^3 + 8a - 24) \div (a - 2)$.

Rename the a^2 term using a coefficient of 0.

$(a^3 + 8a - 24) \div (a - 2) = (a^3 + 0a^2 + 8a - 24) \div (a - 2)$

$$\begin{array}{rll} & a^2 + 2a + 12 & \\ a - 2 \,\big)\!\! & \overline{a^3 + 0a^2 + 8a - 24} & \\ & \underline{(-)\ a^3 - 2a^2} & \quad \text{Multiply } a^2 \text{ and } a - 2. \\ & 2a^2 + 8a & \quad \text{Subtract and bring down } 8a. \\ & \underline{(-)\ 2a^2 - 4a} & \quad \text{Multiply } 2a \text{ and } a - 2. \\ & 12a - 24 & \quad \text{Subtract and bring down 24.} \\ & \underline{(-)\ 12a - 24} & \quad \text{Multiply 12 and } a - 2. \\ & 0 & \quad \text{Subtract.} \end{array}$$

Therefore, $(a^3 + 8a - 24) \div (a - 2) = a^2 + 2a + 12$.

Study Tip

Factors

When the remainder in a division problem is 0, the divisor is a factor of the dividend.

Check for Understanding

Concept Check

1. **Choose** the divisors of $2x^2 - 9x + 9$ that result in a remainder of 0.
   a. $x + 3$ b. $x - 3$ c. $2x - 3$ d. $2x + 3$
2. **Explain** the meaning of a remainder of zero in a long division of a polynomial by a binomial.
3. **OPEN ENDED** Write a third-degree polynomial that includes a zero term. Rewrite the polynomial so that it can be divided by $x + 5$ using long division.

Guided Practice

Find each quotient.

4. $(4x^3 + 2x^2 - 5) \div 2x$
5. $\dfrac{14a^2b^2 + 35ab^2 + 2a^2}{7a^2b^2}$
6. $(n^2 + 7n + 12) \div (n + 3)$
7. $(r^2 + 12r + 36) \div (r + 9)$
8. $\dfrac{4m^3 + 5m - 21}{2m - 3}$
9. $(2b^2 + 3b - 5) \div (2b - 1)$

Application

10. **ENVIRONMENT** The equation $C = \dfrac{120{,}000p}{1 - p}$ models the cost C in dollars for a manufacturer to reduce the pollutants by a given percent, written as p in decimal form. How much will the company have to pay to remove 75% of the pollutants it emits?

Practice and Apply

Homework Help

For Exercises	See Examples
11–14	1, 2
15–18, 23, 24	3
19–22, 25, 26	4
27–30	5

Extra Practice

See page 847.

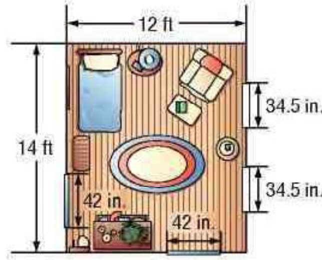
Find each quotient.

11. $(x^2 + 9x - 7) \div 3x$
12. $(a^2 + 7a - 28) \div 7a$
13. $\dfrac{9s^3t^2 - 15s^2t + 24t^3}{3s^2t^2}$
14. $\dfrac{12a^3b + 16ab^3 - 8ab}{4ab}$
15. $(x^2 + 9x + 20) \div (x + 5)$
16. $(x^2 + 6x - 16) \div (x - 2)$
17. $(n^2 - 2n - 35) \div (n + 5)$
18. $(s^2 + 11s + 18) \div (s + 9)$
19. $(z^2 - 2z - 30) \div (z + 7)$
20. $(a^2 + 4a - 22) \div (a - 3)$
21. $(2r^2 - 3r - 35) \div (r - 5)$
22. $(3p^2 + 20p + 11) \div (p + 6)$
23. $\dfrac{3t^2 + 14t - 24}{3t - 4}$
24. $\dfrac{12n^2 + 36n + 15}{2n + 5}$
25. $\dfrac{3x^3 + 8x^2 + x - 7}{x + 2}$
26. $\dfrac{20b^3 - 27b^2 + 13b - 3}{4b - 3}$
27. $\dfrac{6x^3 - 9x^2 + 6}{2x - 3}$
28. $\dfrac{9g^3 + 5g - 8}{3g - 2}$
29. Determine the quotient when $6n^3 + 5n^2 + 12$ is divided by $2n + 3$.
30. What is the quotient when $4t^3 + 17t^2 - 1$ is divided by $4t + 1$?

LANDSCAPING **For Exercises 31 and 32, use the following information.**
A heavy object can be lifted more easily using a lever and fulcrum. The amount that can be lifted depends upon the length of the lever, the placement of the fulcrum, and the force applied. The expression $\dfrac{W(L - x)}{x}$ represents the weight of an object that can be lifted if W pounds of force are applied to a lever L inches long with the fulcrum placed x inches from the object.

31. Suppose Leyati, who weighs 150 pounds, uses all of his weight to lift a rock using a 60-inch lever. Write an expression that could be used to determine the heaviest rock he could lift if the fulcrum is x inches from the rock.
32. Use the expression to find the weight of a rock that could be lifted by a 210-pound man using a six-foot lever placed 20 inches from the rock.

33. **DECORATING** Anoki wants to put a decorative border 3 feet above the floor around his bedroom walls. If the border comes in 5-yard rolls, how many rolls of border should Anoki buy?



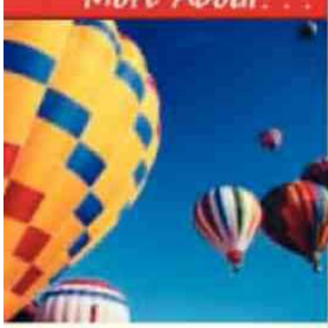
PIZZA **For Exercises 34 and 35, use the following information.**

The expression $\frac{\pi d^2}{64}$ can be used to determine the number of slices of a round pizza with diameter d.

34. Write a formula to calculate the cost per slice of a pizza s that costs C dollars.

35. Copy and complete the table below. Which size pizza offers the best price per slice?

Size	10-inch	14-inch	18-inch
Price	$4.99	$8.99	$12.99
Number of slices			
Cost per slice			



More About. . .

Science

When air is heated it is less dense than the air surrounding it, and the heated air rises. This is why a hot air balloon is able to fly.

Source: www.howstuffworks.com

SCIENCE **For Exercises 36–38, use the following information.**

The *density* of a material is its mass per unit volume.

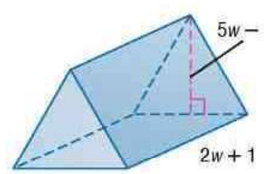
Material	Mass (g)	Volume (cm^3)
aluminum	4.15	1.54
gold	2.32	0.12
silver	6.30	0.60
steel	7.80	1.00
iron	15.20	1.95
copper	2.48	0.28
blood	4.35	4.10
lead	11.30	1.00
brass	17.90	2.08
concrete	40.00	20.00

36. Determine the densities for the materials listed in the table.

37. Make a line plot of the densities computed in Exercise 36. Use densities rounded to the nearest whole number.

38. Interpret the line plot made in Exercise 37.

39. **GEOMETRY** The volume of a prism with a triangular base is $10w^3 + 23w^2 + 5w - 2$. The height of the prism is $2w + 1$, and the height of the triangle is $5w - 1$. What is the measure of the base of the triangle? $\left(Hint: V = \frac{1}{2}Bh\right)$



CRITICAL THINKING **Find the value of k in each situation.**

40. k is an integer and there is no remainder when $x^2 + 7x + 12$ is divided by $x + k$.

41. When $x^2 + 7x + k$ is divided by $x + 2$, there is a remainder of 2.

42. $x + 7$ is a factor of $x^2 - 2x - k$.

43. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

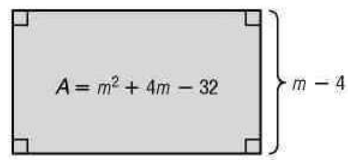
How is division used in sewing?

Include the following in your answer:

- a description showing that $\frac{36 \text{ yards} - 7\frac{1}{2} \text{ yards}}{1\frac{1}{2} \text{ yards}}$ and $\frac{36 \text{ yards}}{1\frac{1}{2} \text{ yards}} - \frac{7\frac{1}{2} \text{ yards}}{1\frac{1}{2} \text{ yards}}$ result in the same answer, and
- a convincing explanation to show that $\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}$.

Standardized Test Practice

44. Which expression represents the length of the rectangle?



Ⓐ $m + 7$ Ⓑ $m - 8$
Ⓒ $m - 7$ Ⓓ $m + 8$

45. What is the quotient of $x^3 + 5x - 20$ divided by $x - 3$?

Ⓐ $x^2 - 3x + 14 + \frac{22}{x - 3}$ Ⓑ $x^2 + 3x + 14 + \frac{22}{x - 3}$
Ⓒ $x^2 + 8x + \frac{4}{x - 3}$ Ⓓ $x^2 + 3x - 14 + \frac{22}{x - 3}$

Maintain Your Skills

Mixed Review

Find each quotient. *(Lesson 12-4)*

46. $\frac{x^2 + 5x + 6}{x^2 - x - 12} \div \frac{x + 2}{x^2 + x - 20}$

47. $\frac{m^2 + m - 6}{m^2 + 8m + 15} \div \frac{m^2 - m - 2}{m^2 + 9m + 20}$

Find each product. *(Lesson 12-3)*

48. $\frac{b^2 + 19b + 84}{b - 3} \cdot \frac{b^2 - 9}{b^2 + 15b + 36}$

49. $\frac{z^2 + 16z + 39}{z^2 + 9z + 18} \cdot \frac{z + 5}{z^2 + 18z + 65}$

Simplify. Then use a calculator to verify your answer. *(Lesson 11-2)*

50. $3\sqrt{7} - \sqrt{7}$

51. $\sqrt{72} + \sqrt{32}$

52. $\sqrt{12} - \sqrt{18} + \sqrt{48}$

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*. *(Lesson 9-6)*

53. $d^2 - 3d - 40$

54. $x^2 + 8x + 16$

55. $t^2 + t + 1$

56. **BUSINESS** Jorge Martinez has budgeted \$150 to have business cards printed. A card printer charges \$11 to set up each job and an additional \$6 per box of 100 cards printed. What is the greatest number of cards Mr. Martinez can have printed? *(Lesson 6-3)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each sum.**
*(To review **addition of polynomials**, see Lesson 8-5.)*

57. $(6n^2 - 6n + 10m^3) + (5n - 6m^3)$

58. $(3x^2 + 4xy - 2y^2) + (x^2 + 9xy + 4y^2)$

59. $(a^3 - b^3) + (-3a^3 - 2a^2b + b^2 - 2b^3)$

60. $(2g^3 + 6h) + (-4g^2 - 8h)$

12-6 Rational Expressions with Like Denominators

What You'll Learn

- Add rational expressions with like denominators.
- Subtract rational expressions with like denominators.

How can you use rational expressions to interpret graphics?

The graphic at the right shows the number of credit cards Americans have. To determine what fraction of those surveyed have no more than two credit cards, you can use addition. Remember that percents can be written as fractions with denominators of 100.

$$\underbrace{\frac{22}{100}}_{\text{No credit cards}} \underbrace{+}_{\text{plus}} \underbrace{\frac{33}{100}}_{\text{one or two credit cards}} \underbrace{=}_{\text{equals}} \underbrace{\frac{55}{100}}_{\text{no more than two credit cards.}}$$

Thus, $\frac{55}{100}$ or 55% of those surveyed have no more than two credit cards.

ADD RATIONAL EXPRESSIONS Recall that to add fractions with like denominators you add the numerators and then write the sum over the common denominator. You can add rational expressions with like denominators in the same way.

Example 1 Numbers in Denominator

Find $\frac{3n}{12} + \frac{7n}{12}$.

$\frac{3n}{12} + \frac{7n}{12} = \frac{3n + 7n}{12}$ The common denominator is 12.

$= \frac{10n}{12}$ Add the numerators.

$= \frac{\overset{5}{\cancel{10}}n}{\underset{6}{\cancel{12}}}$ Divide by the common factor, 2.

$= \frac{5n}{6}$ Simplify.

Sometimes the denominators of rational expressions are binomials. As long as each rational expression has exactly the same binomial as its denominator, the process of adding is the same.

Example 2 Binomials in Denominator

Find $\frac{2x}{x+1} + \frac{2}{x+1}$.

$\frac{2x}{x+1} + \frac{2}{x+1} = \frac{2x+2}{x+1}$ The common denominator is $x + 1$.

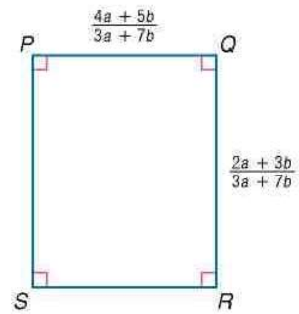
$= \frac{2(x+1)}{x+1}$ Factor the numerator.

$= \frac{2\cancel{(x+1)}^{1}}{\cancel{x+1}_{1}}$ Divide by the common factor, $x + 1$.

$= \frac{2}{1}$ or 2 Simplify.

Example 3 Find a Perimeter

GEOMETRY **Find an expression for the perimeter of rectangle $PQRS$.**



$P = 2\ell + 2w$ Perimeter formula

$= 2\left(\frac{4a+5b}{3a+7b}\right) + 2\left(\frac{2a+3b}{3a+7b}\right)$ $\ell = \frac{4a+5b}{3a+7b}$, $w = \frac{2a+3b}{3a+7b}$

$= \frac{2(4a+5b)+2(2a+3b)}{3a+7b}$ The common denominator is $3a + 7b$.

$= \frac{8a+10b+4a+6b}{3a+7b}$ Distributive Property

$= \frac{12a+16b}{3a+7b}$ Combine like terms.

$= \frac{4(3a+4b)}{3a+7b}$ Factor.

The perimeter can be represented by the expression $\frac{4(3a+4b)}{3a+7b}$.

SUBTRACT RATIONAL EXPRESSIONS To subtract rational expressions with like denominators, subtract the numerators and write the difference over the common denominator. Recall that to subtract an expression, you add its additive inverse.

Study Tip

Common Misconception

Adding the additive inverse will help you avoid the following error in the numerator.

$\cancel{(3x+4)-(x-1) = 3x+4-x-1.}$

Example 4 Subtract Rational Expressions

Find $\frac{3x+4}{x-2} - \frac{x-1}{x-2}$.

$\frac{3x+4}{x-2} - \frac{x-1}{x-2} = \frac{(3x+4)-(x-1)}{x-2}$ The common denominator is $x - 2$.

$= \frac{(3x+4)+[-(x-1)]}{x-2}$ The additive inverse of $(x - 1)$ is $-(x - 1)$.

$= \frac{3x+4-x+1}{x-2}$ Distributive Property

$= \frac{2x+5}{x-2}$ Simplify.

Sometimes you must express a denominator as its additive inverse to have like denominators.

Example 5 Inverse Denominators

Find $\frac{2m}{m-9} + \frac{4m}{9-m}$.

The denominator $9 - m$ is the same as $-(-9 + m)$ or $-(m - 9)$. Rewrite the second expression so that it has the same denominator as the first.

$$\frac{2m}{m-9} + \frac{4m}{9-m} = \frac{2m}{m-9} + \frac{4m}{-(m-9)} \qquad 9 - m = -(m-9)$$

$$= \frac{2m}{m-9} - \frac{4m}{m-9} \qquad \text{Rewrite using like denominators.}$$

$$= \frac{2m-4m}{m-9} \qquad \text{The common denominator is } m-9.$$

$$= \frac{-2m}{m-9} \qquad \text{Subtract.}$$

Check for Understanding

Concept Check

1. **OPEN ENDED** Write two rational expressions with a denominator of $x + 2$ that have a sum of 1.
2. **Describe** how adding rational expressions with like denominators is similar to adding fractions with like denominators.
3. **Compare and contrast** two rational expressions whose sum is 0 with two rational expressions whose difference is 0.
4. **FIND THE ERROR** Russell and Ginger are finding the difference of $\frac{7x+2}{4x-3}$ and $\frac{x-8}{3-4x}$.

Russell

$$\frac{7x+2}{4x-3} - \frac{x-8}{3-4x} = \frac{7x+2}{4x-3} + \frac{x-8}{4x-3}$$
$$= \frac{7x+x+2-8}{4x-3}$$
$$= \frac{8x-6}{4x-3}$$
$$= \frac{2(4x-3)}{4x-3}$$
$$= 2$$

Ginger

$$\frac{7x+2}{4x-3} - \frac{x-8}{3-4x} = \frac{-2-7x}{3-4x} - \frac{x-8}{3-4x}$$
$$= \frac{-2+8-7x-x}{3-4x}$$
$$= \frac{-6-8x}{3-4x}$$
$$= \frac{-2(3-4x)}{3-4x}$$
$$= -2$$

Who is correct? Explain your reasoning.

Guided Practice

Find each sum.

5. $\frac{a+2}{4} + \frac{a-2}{4}$
6. $\frac{3x}{x+1} + \frac{3}{x+1}$
7. $\frac{2-n}{n-1} + \frac{1}{n-1}$
8. $\frac{4t-1}{1-4t} + \frac{2t+3}{1-4t}$

Find each difference.

9. $\frac{5a}{12} - \frac{7a}{12}$
10. $\frac{7}{n-3} - \frac{4}{n-3}$
11. $\frac{3m}{m-2} - \frac{6}{2-m}$
12. $\frac{x^2}{x-y} - \frac{y^2}{x-y}$

Application

13. **SCHOOL** Most schools create daily attendance reports to keep track of their students. Suppose that one day, out of 960 students, 45 were absent due to illness, 29 were participating in a wrestling tournament, 10 were excused to go to their doctors, and 12 were at a music competition. What fraction of the students were absent from school on this day?

Practice and Apply

Homework Help

For Exercises	See Examples
14–17	1
18–25, 27, 42, 43, 45, 46	2, 3
28–35, 38, 39, 44, 47, 48	4
26, 36, 37	5

Extra Practice

See page 848.

Find each sum.

14. $\frac{m}{3} + \frac{2m}{3}$

15. $\frac{12z}{7} + \frac{-5z}{7}$

16. $\frac{x+3}{5} + \frac{x+2}{5}$

17. $\frac{n-7}{2} + \frac{n+5}{2}$

18. $\frac{2y}{y+3} + \frac{6}{y+3}$

19. $\frac{3r}{r+5} + \frac{15}{r+5}$

20. $\frac{k-5}{k-1} + \frac{4}{k-1}$

21. $\frac{n-2}{n+3} + \frac{-1}{n+3}$

22. $\frac{4x-5}{x-2} + \frac{x+3}{x-2}$

23. $\frac{2a+3}{a-4} + \frac{a-2}{a-4}$

24. $\frac{5s+1}{2s+1} + \frac{3s-2}{2s+1}$

25. $\frac{9b+3}{2b+6} + \frac{5b+4}{2b+6}$

26. What is the sum of $\frac{12x-7}{3x-2}$ and $\frac{9x-5}{2-3x}$?

27. Find the sum of $\frac{11x-5}{2x+5}$ and $\frac{11x+12}{2x+5}$.

Find each difference.

28. $\frac{5x}{7} - \frac{3x}{7}$

29. $\frac{4n}{3} - \frac{2n}{3}$

30. $\frac{x+4}{5} - \frac{x+2}{5}$

31. $\frac{a+5}{6} - \frac{a+3}{6}$

32. $\frac{2}{x+7} - \frac{-5}{x+7}$

33. $\frac{4}{z-2} - \frac{-6}{z-2}$

34. $\frac{5}{3x-5} - \frac{3x}{3x-5}$

35. $\frac{4}{7m-2} - \frac{7m}{7m-2}$

36. $\frac{2x}{x-2} - \frac{2x}{2-x}$

37. $\frac{5y}{y-3} - \frac{5y}{3-y}$

38. $\frac{8}{3t-4} - \frac{6t}{3t-4}$

39. $\frac{15x}{5x+1} - \frac{-3}{5x+1}$

40. Find the difference of $\frac{10a-12}{2a-6}$ and $\frac{6a}{6-2a}$.

41. What is the difference of $\frac{b-15}{2b+12}$ and $\frac{-3b+8}{2b+12}$?

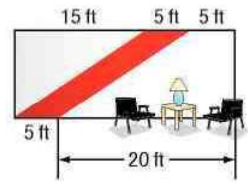
42. **POPULATION** The United States population in 1998 is described in the table. Use this information to write the fraction of the population that is 80 years or older.

Age	Number of People
0–19	77,525,000
20–39	79,112,000
40–59	68,699,000
60–79	35,786,000
80–99	8,634,000
100+	61,000

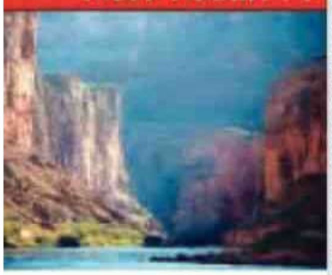
Source: *Statistical Abstract of the United States*

43. **CONSERVATION** The freshman class chose to plant spruce and pine trees at a wildlife sanctuary for a service project. Some students can plant 140 trees on Saturday, and others can plant 20 trees after school on Monday and again on Tuesday. Write an expression for the fraction of the trees that could be planted on these days if n represents the number of spruce trees and there are twice as many pine trees.

44. **GEOMETRIC DESIGN** A student center is a square room that is 25 feet wide and 25 feet long. The walls are 10 feet high and each wall is painted white with a red diagonal stripe as shown. What fraction of the walls are painted red?



More About. . .



Hiking

Due to its popularity, the Grand Canyon is one of the most threatened natural areas in the United States.

Source: The Wildlife Foundation

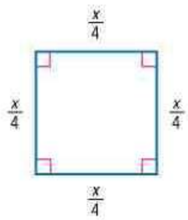
HIKING **For Exercises 45 and 46, use the following information.**
A tour guide recommends that hikers carry a gallon of water on hikes to the bottom of the Grand Canyon. Water weighs 62.4 pounds per cubic foot, and one cubic foot of water contains 7.48 gallons.

45. Tanika plans to carry two 1-quart bottles and four 1-pint bottles for her hike. Write a rational expression for this amount of water written as a fraction of a cubic foot.

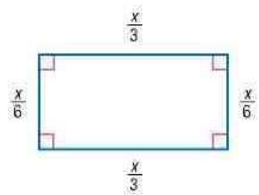
46. How much does this amount of water weigh?

GEOMETRY **For Exercises 47 and 48, use the following information.**
Each figure has a perimeter of x units.

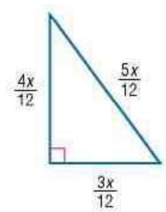
a.



b.



c.



47. Find the ratio of the area of each figure to its perimeter.

48. Which figure has the greatest ratio?

49. **CRITICAL THINKING** Which of the following rational numbers is not equivalent to the others?

a. $\frac{3}{2-x}$ **b.** $\frac{-3}{x-2}$ **c.** $-\frac{3}{2-x}$ **d.** $-\frac{3}{x-2}$

50. ***WRITING IN MATH*** Answer the question that was posed at the beginning of the lesson.

How can you use rational expressions to interpret graphics?

Include the following in your answer:
- an explanation of how the numbers in the graphic relate to rational expressions, and
- a description of how to add two rational expressions whose denominators are $3x - 4y$ and $4y - 3x$.

Standardized Test Practice

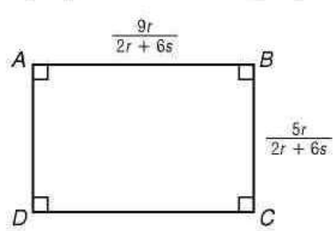
51. Find $\frac{k+2}{k-7} + \frac{-3}{k-7}$.

Ⓐ $\frac{k-1}{k-7}$ Ⓑ $\frac{k-5}{k-7}$ Ⓒ $\frac{k+1}{k-7}$ Ⓓ $\frac{k+5}{k-7}$

52. Which is an expression for the perimeter of rectangle $ABCD$?

Ⓐ $\frac{14r}{2r+6s}$ Ⓑ $\frac{14r}{r+3s}$

Ⓒ $\frac{14r}{r+6s}$ Ⓓ $\frac{28r}{r+3s}$



CONTENTS

Maintain Your Skills

Mixed Review

Find each quotient. *(Lessons 12-4 and 12-5)*

53. $\frac{x^3 - 7x + 6}{x - 2}$

54. $\frac{56x^3 + 32x^2 - 63x - 36}{7x + 4}$

55. $\frac{b^2 - 9}{4b} \div (b - 3)$

56. $\frac{x}{x + 2} \div \frac{x^2}{x^2 + 5x + 6}$

Factor each trinomial. *(Lesson 9-3)*

57. $a^2 + 9a + 14$

58. $p^2 + p - 30$

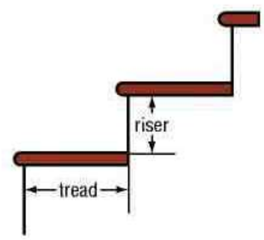
59. $y^2 - 11yz + 28z^2$

Find each sum or difference. *(Lesson 8-5)*

60. $(3x^2 - 4x) - (7 - 9x)$

61. $(5x^2 - 6x + 14) + (2x^2 + 3x + 8)$

62. CARPENTRY When building a stairway, a carpenter considers the ratio of riser to tread. If each stair being built is to have a width of 1 foot and a height of 8 inches, what will be the slope of the stairway?



Getting Ready for the Next Lesson

BASIC SKILL **Find the least common multiple for each set of numbers.**

63. 4, 9, 12

64. 7, 21, 5

65. 6, 12, 24

66. 45, 10, 6

67. 5, 6, 15

68. 8, 9, 12

69. 16, 20, 25

70. 36, 48, 60

71. 9, 16, 24

Practice Quiz 2

Lessons 12-4 through 12-6

Find each quotient. *(Lessons 12-4 and 12-5)*

1. $\frac{a}{a + 3} \div \frac{a + 11}{a + 3}$

2. $\frac{4z + 8}{z + 3} \div (z + 2)$

3. $\frac{(2x - 1)(x - 2)}{(x - 2)(x - 3)} \div \frac{(2x - 1)(x + 5)}{(x - 3)(x - 1)}$

4. $(9xy^2 - 15xy + 3) \div 3xy$

5. $(2x^2 - 7x - 16) \div (2x + 3)$

6. $\frac{y^2 - 19y + 9}{y - 4}$

Find each sum or difference. *(Lesson 12-6)*

7. $\frac{2}{x + 7} + \frac{5}{x + 7}$

8. $\frac{2m}{m + 3} - \frac{-6}{m + 3}$

9. $\frac{5x - 1}{3x + 2} - \frac{2x - 1}{3x + 2}$

10. MUSIC Suppose the record shown played for 16.5 minutes on one side and the average of the radii of the grooves on the record was $3\frac{3}{4}$ inches. Write an expression involving units that represents how many inches the needle passed through the grooves while the record was being played. Then evaluate the expression.

33 revolutions per minute

12-7 Rational Expressions with Unlike Denominators

What You'll Learn

- Add rational expressions with unlike denominators.
- Subtract rational expressions with unlike denominators.

Vocabulary

- least common multiple (LCM)
- least common denominator (LCD)

How can rational expressions be used to describe elections?

The President of the United States is elected every four years, and senators are elected every six years. A certain senator is elected in 2004, the same year as a presidential election, and is reelected in subsequent elections. In what year is the senator's reelection the same year as a presidential election?

ADD RATIONAL EXPRESSIONS The number of years in which a specific senator's election coincides with a presidential election is related to the common multiples of 4 and 6. The least number of years that will pass until the next election for both a specific senator and the President is the least common multiple of these numbers. The **least common multiple (LCM)** is the least number that is a common multiple of two or more numbers.

Example 1 LCM of Monomials

Find the LCM of $15m^2b^3$ and $18mb^2$.

Find the prime factors of each coefficient and variable expression.

$15m^2b^3 = 3 \cdot 5 \cdot m \cdot m \cdot b \cdot b \cdot b$

$18mb^2 = 2 \cdot 3 \cdot 3 \cdot m \cdot b \cdot b$

Use each prime factor the greatest number of times it appears in any of the factorizations.

$15m^2b^3 = 3 \cdot 5 \cdot m \cdot m \cdot b \cdot b \cdot b$

$18mb^2 = 2 \cdot 3 \cdot 3 \cdot m \cdot b \cdot b$

LCM $= 2 \cdot 3 \cdot 3 \cdot 5 \cdot m \cdot m \cdot b \cdot b \cdot b$ or $90m^2b^3$

Example 2 LCM of Polynomials

Find the LCM of $x^2 + 8x + 15$ and $x^2 + x - 6$.

Express each polynomial in factored form.

$x^2 + 8x + 15 = (x + 3)(x + 5)$

$x^2 + x - 6 = (x - 2)(x + 3)$

Use each factor the greatest number of times it appears.

LCM $= (x - 2)(x + 3)(x + 5)$

Recall that to add fractions with unlike denominators, you need to rename the fractions using the least common multiple (LCM) of the denominators, known as the **least common denominator (LCD)**.

Key Concept *Add Rational Expressions*

Use the following steps to add rational expressions with unlike denominators.

Step 1 Find the LCD.

Step 2 Change each rational expression into an equivalent expression with the LCD as the denominator.

Step 3 Add just as with rational expressions with like denominators.

Step 4 Simplify if necessary.

Example 3 Monomial Denominators

Find $\frac{a+1}{a} + \frac{a-3}{3a}$.

Factor each denominator and find the LCD.

$a = a$

$3a = 3 \cdot a$

LCD $= 3a$

Since the denominator of $\frac{a-3}{3a}$ is already $3a$, only $\frac{a+1}{a}$ needs to be renamed.

$$\frac{a+1}{a} + \frac{a-3}{3a} = \frac{3(a+1)}{3(a)} + \frac{a-3}{3a} \quad \text{Multiply } \frac{a+1}{a} \text{ by } \frac{3}{3}.$$

$$= \frac{3a+3}{3a} + \frac{a-3}{3a} \quad \text{Distributive Property}$$

$$= \frac{3a+3+a-3}{3a} \quad \text{Add the numerators.}$$

$$= \frac{4\cancel{a}^{1}}{3\cancel{a}_{1}} \quad \text{Divide out the common factor } a.$$

$$= \frac{4}{3} \quad \text{Simplify.}$$

Example 4 Polynomial Denominators

Find $\frac{y-2}{y^2+4y+4} + \frac{y-2}{y+2}$.

$$\frac{y-2}{y^2+4y+4} + \frac{y-2}{y+2} = \frac{y-2}{(y+2)^2} + \frac{y-2}{y+2} \quad \text{Factor the denominators.}$$

$$= \frac{y-2}{(y+2)^2} + \frac{y-2}{y+2} \cdot \frac{y+2}{y+2} \quad \text{The LCD is } (y+2)^2.$$

$$= \frac{y-2}{(y+2)^2} + \frac{y^2-4}{(y+2)^2} \quad (y-2)(y+2) = y^2-4$$

$$= \frac{y-2+y^2-4}{(y+2)^2} \quad \text{Add the numerators.}$$

$$= \frac{y^2+y-6}{(y+2)^2} \text{ or } \frac{(y-2)(y+3)}{(y+2)^2} \quad \text{Simplify.}$$

SUBTRACT RATIONAL EXPRESSIONS As with addition, to subtract rational expressions with unlike denominators, you must first rename the expressions using a common denominator.

Example 5 Binomials in Denominators

Find $\frac{4}{3a - 6} - \frac{a}{a + 2}$.

$$\frac{4}{3a-6} - \frac{a}{a+2} = \frac{4}{3(a-2)} - \frac{a}{a+2}$$ Factor.

$$= \frac{4(a+2)}{3(a-2)(a+2)} - \frac{3a(a-2)}{3(a+2)(a-2)}$$ The LCD is $3(a + 2)(a - 2)$.

$$= \frac{4(a+2) - 3a(a-2)}{3(a-2)(a+2)}$$ Subtract the numerators.

$$= \frac{4a + 8 - 3a^2 + 6a}{3(a-2)(a+2)}$$ Multiply.

$$= \frac{-3a^2 + 10a + 8}{3(a-2)(a+2)} \text{ or } -\frac{3a^2 - 10a - 8}{3(a-2)(a+2)}$$ Simplify.

Standardized Test Practice

(A) (B) (C) (D)

Example 6 Polynomials in Denominators

Multiple-Choice Test Item

Find $\frac{h - 2}{h^2 + 4h + 4} - \frac{h - 4}{h^2 - 4}$.

(A) $\frac{2h - 12}{(h-2)(h+2)^2}$

(B) $\frac{-2h + 12}{(h-2)(h+2)^2}$

(C) $\frac{2h - 12}{(h-2)^2(h+2)}$

(D) $\frac{-2h + 12}{(h-2)(h+2)}$

The Princeton Review

Test-Taking Tip

Examine all of the answer choices carefully. Look for differences in operations, positive and negative signs, and exponents.

Read the Test Item

The expression $\frac{h-2}{h^2+4h+4} - \frac{h-4}{h^2-4}$ represents the difference of two rational expressions with unlike denominators.

Solve the Test Item

Step 1 Factor each denominator and find the LCD.

$h^2 + 4h + 4 = (h + 2)^2$

$h^2 - 4 = (h + 2)(h - 2)$

The LCD is $(h - 2)(h + 2)^2$.

Step 2 Change each rational expression into an equivalent expression with the LCD. Then subtract.

$$\frac{h-2}{(h+2)^2} - \frac{h-4}{(h+2)(h-2)} = \frac{(h-2)}{(h+2)^2} \cdot \frac{(h-2)}{(h-2)} - \frac{(h-4)}{(h+2)(h-2)} \cdot \frac{(h+2)}{(h+2)}$$

$$= \frac{(h-2)(h-2)}{(h+2)^2(h-2)} - \frac{(h-4)(h+2)}{(h+2)^2(h-2)}$$

$$= \frac{h^2 - 4h + 4}{(h+2)^2(h-2)} - \frac{h^2 - 2h - 8}{(h+2)^2(h-2)}$$

$$= \frac{(h^2 - 4h + 4) - (h^2 - 2h - 8)}{(h+2)^2(h-2)}$$

$$= \frac{h^2 - h^2 - 4h + 2h + 4 + 8}{(h+2)^2(h-2)}$$

$$= \frac{-2h + 12}{(h-2)(h+2)^2}$$

The correct answer is B.

Check for Understanding

Concept Check

1. **Describe** how to find the LCD of two rational expressions with unlike denominators.
2. **Explain** how to rename rational expressions using their LCD.
3. **OPEN ENDED** Give an example of two rational expressions in which the LCD is equal to twice the denominator of one of the expressions.

Guided Practice

Find the LCM for each pair of expressions.

4. $5a^2, 7a$
5. $2x - 4, 3x - 6$
6. $n^2 + 3n - 4, (n - 1)^2$

Find each sum.

7. $\frac{6}{5x} + \frac{7}{10x^2}$
8. $\frac{a}{a - 4} + \frac{4}{a + 4}$
9. $\frac{2y}{y^2 - 25} + \frac{y + 5}{y - 5}$
10. $\frac{a + 2}{a^2 + 4a + 3} + \frac{6}{a + 3}$

Find each difference.

11. $\frac{3z}{6w^2} - \frac{z}{4w}$
12. $\frac{4a}{2a + 6} - \frac{3}{a + 3}$
13. $\frac{b + 8}{b^2 - 16} - \frac{1}{b - 4}$
14. $\frac{x}{x - 2} - \frac{3}{x^2 + 3x - 10}$

Standardized Test Practice

A B C D

15. Find $\frac{2y}{y^2 + 7y + 12} + \frac{y + 2}{y + 4}$.

(A) $\frac{y^2 + 5y + 6}{(y + 4)(y + 3)}$
(B) $\frac{y^2 + 2y + 6}{(y + 4)(y + 3)}$
(C) $\frac{y^2 + 7y + 6}{(y + 4)(y + 3)}$
(D) $\frac{y^2 - 5y + 6}{(y + 4)(y + 3)}$

Practice and Apply

Homework Help

For Exercises	See Examples
16, 17, 54–57	1
18–21	2
22–25	3
26–37	4
38–49	5
50–53	6

Extra Practice

See page 848.

Find the LCM for each pair of expressions.

16. a^2b, ab^3
17. $7xy, 21x^2y$
18. $x - 4, x + 2$
19. $2n - 5, n + 2$
20. $x^2 + 5x - 14, (x - 2)^2$
21. $p^2 - 5p - 6, p + 1$

Find each sum.

22. $\frac{3}{x^2} + \frac{5}{x}$
23. $\frac{2}{a^3} + \frac{7}{a^2}$
24. $\frac{7}{6a^2} + \frac{5}{3a}$
25. $\frac{3}{7m} + \frac{4}{5m^2}$
26. $\frac{3}{x + 5} + \frac{4}{x - 4}$
27. $\frac{n}{n + 4} + \frac{3}{n - 3}$
28. $\frac{7a}{a + 5} + \frac{a}{a - 2}$
29. $\frac{6x}{x - 3} + \frac{x}{x + 1}$
30. $\frac{5}{3x - 9} + \frac{3}{x - 3}$
31. $\frac{m}{3m + 2} + \frac{2}{9m + 6}$
32. $\frac{-3}{5 - a} + \frac{5}{a^2 - 25}$
33. $\frac{18}{y^2 - 9} + \frac{-7}{3 - y}$
34. $\frac{x}{x^2 + 2x + 1} + \frac{1}{x + 1}$
35. $\frac{2x + 1}{(x - 1)^2} + \frac{x - 2}{x^2 + 3x - 4}$
36. $\frac{x^2}{4x^2 - 9} + \frac{x}{(2x + 3)^2}$
37. $\frac{a^2}{a^2 - b^2} + \frac{a}{(a - b)^2}$

Find each difference.

38. $\frac{7}{3x} - \frac{3}{6x^2}$

39. $\frac{4}{15x^2} - \frac{5}{3x}$

40. $\frac{11x}{3y^2} - \frac{7x}{6y}$

41. $\frac{5a}{7x} - \frac{3a}{21x^2}$

42. $\frac{x^2 - 1}{x + 1} - \frac{x^2 + 1}{x - 1}$

43. $\frac{k}{k + 5} - \frac{3}{k - 3}$

44. $\frac{k}{2k + 1} - \frac{2}{k + 2}$

45. $\frac{m - 1}{m + 1} - \frac{4}{2m + 5}$

46. $\frac{2x}{x^2 - 5x} - \frac{-3x}{x - 5}$

47. $\frac{-3}{a - 6} - \frac{-6}{a^2 - 6a}$

48. $\frac{n}{5 - n} - \frac{3}{n^2 - 25}$

49. $\frac{3a + 2}{6 - 3a} - \frac{a + 2}{a^2 - 4}$

50. $\frac{3x}{x^2 + 3x + 2} - \frac{3x - 6}{x^2 + 4x + 4}$

51. $\frac{5a}{a^2 + 3a - 4} - \frac{a - 1}{a^2 - 1}$

52. $\frac{x^2 + 4x - 5}{x^2 - 2x - 3} - \frac{2}{x + 1}$

53. $\frac{m - 4}{m^2 + 8m + 16} - \frac{m + 4}{m - 4}$

54. **MUSIC** A music director wants to form a group of students to sing and dance at community events. The music they will sing is 2-part, 3-part, or 4-part harmony. The director would like to have the same number of voices on each part. What is the least number of students that would allow for an even distribution on all these parts?

55. **CHARITY** Maya, Makalla, and Monya can walk one mile in 12, 15, and 20 minutes respectively. They plan to participate in a walk-a-thon to raise money for a local charity. Sponsors have agreed to pay $2.50 for each mile that is walked. What is the total number of miles the girls would walk in one hour and how much money would they raise?

56. **PET CARE** Kendra takes care of pets while their owners are out of town. One week she has three dogs that all eat the same kind of dog food. The first dog eats a bag of food every 12 days, the second dog eats a bag every 15 days, and the third dog eats a bag every 16 days. How many bags of food should Kendra buy for one week?

More About...

Pet Care

Kell, an English Mastiff owned by Tom Scott of the United Kingdom, is the heaviest dog in the world. Weighing in at 286 pounds, Kell eats a high protein diet of eggs, goat's milk, and beef.

Source: *The Guinness Book of Records*

57. **AUTOMOBILES** Car owners need to follow a regular maintenance schedule to keep their cars running safely and efficiently. The table shows several items that should be performed on a regular basis. If all of these items are performed when a car's odometer reads 36,000 miles, what would be the car's mileage reading the next time all of the items should be performed?

Inspection or Service	Frequency
engine oil and oil filter change	every 3000 miles (about 3 months)
transmission fluid level check	every oil change
brake system inspection	every oil change
chassis lubrication	every 6000 miles
power steering pump fluid level check	every 6000 miles
tire and wheel rotation and inspection	every 15,000 miles

58. **CRITICAL THINKING** Janelle says that a shortcut for adding fractions with unlike denominators is to add the cross products for the numerator and write the denominator as the product of the denominators. She gives the following example.

$$\frac{2}{7} + \frac{5}{8} = \frac{2 \cdot 8 + 5 \cdot 7}{7 \cdot 8} = \frac{51}{56}$$

Explain why Janelle's method will always work or provide a counterexample to show that it does not always work.

59. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.

How can rational expressions be used to describe elections?

Include the following in your answer:

- an explanation of how to determine the least common multiple of two or more rational expressions, and
- if a certain senator is elected in 2006, when is the next election in which the senator and a President will be elected?

Standardized Test Practice

60. What is the least common denominator of $\frac{6}{a^2 - 2ab + b^2}$ and $\frac{6}{a^2 - b^2}$?

Ⓐ $(a - b)^2$ Ⓑ $(a - b)(a + b)$

Ⓒ $(a + b)^2$ Ⓓ $(a - b)^2(a + b)$

61. Find $\frac{x - 4}{(2 - x)^2} - \frac{x - 5}{x^2 + x - 6}$.

Ⓐ $\frac{8x - 22}{(x + 3)(x - 2)^2}$ Ⓑ $\frac{x^2 - 2x - 17}{(x - 2)(x + 3)}$

Ⓒ $\frac{6x - 22}{(x + 3)(x - 2)^2}$ Ⓓ $\frac{22 - 6x}{(x + 3)(x - 2)}$

Maintain Your Skills

Mixed Review

Find each sum. *(Lesson 12-6)*

62. $\frac{3m}{2m + 1} + \frac{3}{2m + 1}$ 63. $\frac{4x}{2x + 3} + \frac{5}{2x + 3}$ 64. $\frac{2y}{y - 3} + \frac{5}{3 - y}$

Find each quotient. *(Lesson 12-5)*

65. $\frac{b^2 + 8b - 20}{b - 2}$ 66. $\frac{t^3 - 19t + 9}{t - 4}$ 67. $\frac{4m^2 + 8m - 19}{2m + 7}$

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lesson 9-4)*

68. $2x^2 + 10x + 8$ 69. $5r^2 + 7r - 6$ 70. $16p^2 - 4pq - 30q^2$

71. **BUDGETING** JoAnne Paulsen's take-home pay is \$1782 per month. She spends \$525 on rent, \$120 on groceries, and \$40 on gas. She allows herself 5% of the remaining amount for entertainment. How much can she spend on entertainment each month? *(Lesson 3-9)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each quotient.**

*(To review **dividing rational expressions**, see Lesson 12-4.)*

72. $\frac{x}{2} \div \frac{3x}{5}$ 73. $\frac{a^2}{5b} \div \frac{4a}{10b^2}$ 74. $\frac{x + 7}{x} \div \frac{x + 7}{x + 3}$

75. $\frac{3n}{2n + 5} \div \frac{12n^2}{2n + 5}$ 76. $\frac{3x}{x + 2} \div (x - 1)$ 77. $\frac{x^2 + 7x + 12}{x + 6} \div (x + 3)$

12-8 Mixed Expressions and Complex Fractions

What You'll Learn

- Simplify mixed expressions.
- Simplify complex fractions.

Vocabulary

- mixed expression
- complex fraction

How are rational expressions used in baking?

Katelyn bought $2\frac{1}{2}$ pounds of chocolate chip cookie dough. If the average cookie requires $1\frac{1}{2}$ ounces of dough, the number of cookies that Katelyn can bake can be found by simplifying the expression $\dfrac{2\frac{1}{2}\text{ pounds}}{1\frac{1}{2}\text{ ounces}}$.

SIMPLIFY MIXED EXPRESSIONS Recall that a number like $2\frac{1}{2}$ is a mixed number because it contains the sum of an integer, 2, and a fraction, $\frac{1}{2}$. An expression like $3 + \frac{x+2}{x-3}$ is called a **mixed expression** because it contains the sum of a monomial, 3, and a rational expression, $\frac{x+2}{x-3}$. Changing mixed expressions to rational expressions is similar to changing mixed numbers to improper fractions.

Example 1 Mixed Expression to Rational Expression

Simplify $3 + \frac{6}{x+3}$.

$$3 + \frac{6}{x+3} = \frac{3(x+3)}{x+3} + \frac{6}{x+3} \quad \text{The LCD is } x + 3.$$

$$= \frac{3(x+3)+6}{x+3} \quad \text{Add the numerators.}$$

$$= \frac{3x+9+6}{x+3} \quad \text{Distributive Property}$$

$$= \frac{3x+15}{x+3} \quad \text{Simplify.}$$

SIMPLIFY COMPLEX FRACTIONS If a fraction has one or more fractions in the numerator or denominator, it is called a **complex fraction**. You simplify an algebraic complex fraction in the same way that you simplify a numerical complex fraction.

numerical complex fraction

$$\frac{\frac{8}{3}}{\frac{7}{5}} = \frac{8}{3} \div \frac{7}{5}$$
$$= \frac{8}{3} \cdot \frac{5}{7}$$
$$= \frac{40}{21}$$

algebraic complex fraction

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d}$$
$$= \frac{a}{b} \cdot \frac{d}{c}$$
$$= \frac{ad}{bc}$$

Key Concept — Simplifying a Complex Fraction

Any complex fraction $\frac{\frac{a}{b}}{\frac{c}{d}}$, where $b \neq 0$, $c \neq 0$, and $d \neq 0$, can be expressed as $\frac{ad}{bc}$.

Example 2 Complex Fraction Involving Numbers

BAKING Refer to the application at the beginning of the lesson. How many cookies can Katelyn make with $2\frac{1}{2}$ pounds of chocolate chip cookie dough?

To find the total number of cookies, divide the amount of cookie dough by the amount of dough needed for each cookie.

$$\frac{2\frac{1}{2}\text{ pounds}}{1\frac{1}{2}\text{ ounces}} = \frac{2\frac{1}{2}\cancel{\text{ pounds}}}{1\frac{1}{2}\cancel{\text{ ounces}}} \cdot \frac{16\cancel{\text{ ounces}}}{1\cancel{\text{ pound}}}$$ Convert pounds to ounces. Divide by common units.

$$= \frac{16 \cdot 2\frac{1}{2}}{1\frac{1}{2}}$$ Simplify.

$$= \frac{\frac{16}{1} \cdot \frac{5}{2}}{\frac{3}{2}}$$ Express each term as an improper fraction.

$$= \frac{\frac{80}{2}}{\frac{3}{2}}$$ Multiply in the numerator.

$$= \frac{80 \cdot 2}{2 \cdot 3}$$ $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

$$= \frac{160}{6} \text{ or } 26\frac{2}{3}$$ Simplify.

Katelyn can make 27 cookies.

Study Tip

Fraction Bar

Recall that when applying the order of operations, a fraction bar serves as a grouping symbol. Simplify the numerator and denominator of a complex fraction before proceeding with division.

Example 3 Complex Fraction Involving Monomials

Simplify $\frac{\frac{x^2y^2}{a}}{\frac{x^2y}{a^3}}$.

$$\frac{\frac{x^2y^2}{a}}{\frac{x^2y}{a^3}} = \frac{x^2y^2}{a} \div \frac{x^2y}{a^3}$$ Rewrite as a division sentence.

$$= \frac{x^2y^2}{a} \cdot \frac{a^3}{x^2y}$$ Rewrite as multiplication by the reciprocal.

$$= \frac{\overset{1}{\cancel{x^2}}\overset{y}{\cancel{y^2}}}{\underset{1}{\cancel{a}}} \cdot \frac{\overset{a^2}{\cancel{a^3}}}{\underset{1}{\cancel{x^2}}\underset{1}{\cancel{y}}}$$ Divide by common factors x^2, y, and a.

$$= a^2y$$ Simplify.

Example 4 Complex Fraction Involving Polynomials

Simplify $\dfrac{a - \frac{15}{a-2}}{a+3}$.

The numerator contains a mixed expression. Rewrite it as a rational expression first.

$$\frac{a - \frac{15}{a-2}}{a+3} = \frac{\frac{a(a-2)}{a-2} - \frac{15}{a-2}}{a+3}$$ The LCD of the fractions in the numerator is $a - 2$.

$$= \frac{\frac{a^2 - 2a - 15}{a-2}}{a+3}$$ Simplify the numerator.

$$= \frac{\frac{(a+3)(a-5)}{a-2}}{a+3}$$ Factor.

$$= \frac{(a+3)(a-5)}{a-2} \div (a+3)$$ Rewrite as a division sentence.

$$= \frac{(a+3)(a-5)}{a-2} \cdot \frac{1}{a+3}$$ Multiply by the reciprocal of $a + 3$.

$$= \frac{\overset{1}{\cancel{(a+3)}}(a-5)}{a-2} \cdot \frac{1}{\underset{1}{\cancel{a+3}}}$$ Divide by the GCF, $a + 3$.

$$= \frac{a-5}{a-2}$$ Simplify.

Check for Understanding

Concept Check

1. **Describe** the similarities between mixed numbers and mixed rational expressions.

2. **OPEN ENDED** Give an example of a complex fraction and show how to simplify it.

3. **FIND THE ERROR** Bolton and Lian found the LCD of $\frac{4}{2x+1} - \frac{5}{x+1} + \frac{2}{x-1}$.

Bolton

$\frac{4}{2x+1} - \frac{5}{x+1} + \frac{2}{x-1}$

LCD: $(2x+1)(x+1)(x-1)$

Lian

$\frac{4}{2x+1} - \frac{5}{x+1} + \frac{2}{x-1}$

LCD: $2(x+1)(x-1)$

Who is correct? Explain your reasoning.

Guided Practice **Write each mixed expression as a rational expression.**

4. $3 + \frac{4}{x}$

5. $7 + \frac{5}{6y}$

6. $\frac{a-1}{3a} + 2a$

Simplify each expression.

7. $\dfrac{3\frac{1}{2}}{4\frac{3}{4}}$

8. $\dfrac{\frac{x^3}{y^2}}{\frac{y^3}{x}}$

9. $\dfrac{\frac{x-y}{a+b}}{\frac{x^2-y^2}{a^2-b^2}}$

Application **10. ENTERTAINMENT** The student talent committee is arranging the performances for their holiday pageant. The first-act performances and their lengths are shown in the table. What is the average length of each performance?

Holiday Pageant Line-Up	
Performance	**Length (min)**
A	7
B	$4\frac{1}{2}$
C	$6\frac{1}{2}$
D	$8\frac{1}{4}$
E	$10\frac{1}{5}$

Practice and Apply

Homework Help

For Exercises	See Examples
11–22, 35	1
23–26, 37–40	2
27–32, 36	3
33, 34	4

Extra Practice

See page 848.

Write each mixed expression as a rational expression.

11. $8 + \frac{3}{n}$

12. $4 + \frac{5}{a}$

13. $2x + \frac{x}{y}$

14. $6z + \frac{2z}{w}$

15. $2m - \frac{4 + m}{m}$

16. $3a - \frac{a + 1}{2a}$

17. $b^2 + \frac{a - b}{a + b}$

18. $r^2 + \frac{r - 4}{r + 3}$

19. $5n^2 - \frac{n + 3}{n^2 - 9}$

20. $3s^2 - \frac{s + 1}{s^2 - 1}$

21. $(x - 5) + \frac{x + 2}{x - 3}$

22. $(p + 4) + \frac{p + 1}{p - 4}$

Simplify each expression.

23. $\dfrac{5\frac{3}{4}}{7\frac{2}{3}}$

24. $\dfrac{8\frac{2}{7}}{4\frac{4}{5}}$

25. $\dfrac{\frac{a}{b^3}}{\frac{a^2}{b}}$

26. $\dfrac{\frac{n^3}{m^2}}{\frac{n^2}{m^2}}$

27. $\dfrac{\frac{x + 4}{y - 2}}{\frac{x^2}{y^2}}$

28. $\dfrac{\frac{s^3}{t^2}}{\frac{s + t}{s - t}}$

29. $\dfrac{\frac{y^2 - 1}{y^2 + 3y - 4}}{y + 1}$

30. $\dfrac{\frac{a^2 - 2a - 3}{a^2 - 1}}{a - 3}$

31. $\dfrac{\frac{n^2 + 2n}{n^2 + 9n + 18}}{\frac{n^2 - 5n}{n^2 + n - 30}}$

32. $\dfrac{\frac{x^2 + 4x - 21}{x^2 - 9x + 18}}{\frac{x^2 + 3x - 28}{x^2 - 10x + 24}}$

33. $\dfrac{x - \frac{15}{x - 2}}{x - \frac{20}{x - 1}}$

34. $\dfrac{n + \frac{35}{n + 12}}{n - \frac{63}{n - 2}}$

35. What is the quotient of $b + \frac{1}{b}$ and $a + \frac{1}{a}$?

36. What is the product of $\frac{2b^2}{5c}$ and the quotient of $\frac{4b^3}{2c}$ and $\frac{7b^3}{8c^2}$?

37. PARTIES The student council is planning a party for the school volunteers. There are five 66-ounce bottles of soda left from a recent dance. When poured over ice, $5\frac{1}{2}$ ounces of soda fills a cup. How many servings of soda can they get from the bottles they have?

ACOUSTICS **For Exercises 38 and 39, use the following information.**
If a vehicle is moving toward you at v miles per hour and blowing its horn at a frequency of f, then you hear the horn as if it were blowing at a frequency of h. This can be defined by the equation $h = \frac{f}{1 - \frac{v}{s}}$, where s is the speed of sound, approximately 760 miles per hour.

38. Simplify the complex fraction in the formula.

39. Suppose a truck horn blows at 370 cycles per second and is moving toward you at 65 miles per hour. Find the frequency of the horn as you hear it.

40. **POPULATION** According to the 2000 Census, New Jersey was the most densely populated state, and Alaska was the least densely populated state. The population of New Jersey was 8,414,350, and the population of Alaska was 626,932. The land area of New Jersey is about 7419 square miles, and the land area of Alaska is about 570,374 square miles. How many more people were there per square mile in New Jersey than in Alaska?

41. **BICYCLES** When air is pumped into a bicycle tire, the pressure P required varies inversely as the volume of the air V and is given by the equation $P = \frac{k}{V}$. If the pressure is 30 lb/in^2 when the volume is $1\frac{2}{3}$ cubic feet, find the pressure when the volume is $\frac{3}{4}$ cubic feet.

42. **CRITICAL THINKING** Which expressions are equivalent to 0?

a. $\dfrac{a}{1 - \frac{3}{a}} + \dfrac{a}{\frac{3}{a} - 1}$ b. $\dfrac{a - \frac{1}{3}}{b} - \dfrac{a + \frac{1}{3}}{b}$ c. $\dfrac{\frac{1}{2} + 2a}{b - 1} - \dfrac{2a + \frac{1}{2}}{1 - b}$

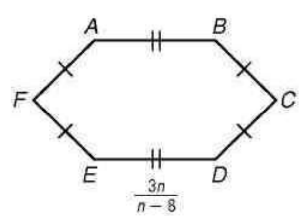
43. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are rational expressions used in baking?

Include the following in your answer:
- an example of a situation in which you would divide a measurement by a fraction when cooking, and
- an explanation of the process used to simplify a complex fraction.

Standardized Test Practice

44. The perimeter of hexagon $ABCDEF$ is 12. Which expression can be used to represent the measure of $\overline{BC}$?



Ⓐ $\dfrac{6n - 96}{n - 8}$ Ⓑ $\dfrac{9n - 96}{n - 8}$

Ⓒ $\dfrac{6n - 96}{4n - 32}$ Ⓓ $\dfrac{9n - 96}{4n - 32}$

45. Express $\dfrac{\frac{6mn}{5p}}{\frac{24n^2}{20mp}}$ in simplest form.

Ⓐ $\dfrac{n}{m^2}$ Ⓑ $\dfrac{1}{n}$ Ⓒ $\dfrac{m^2}{n}$ Ⓓ $\dfrac{36n^3}{25p^2}$

CONTENTS

Maintain Your Skills

Mixed Review

Find each sum. *(Lesson 12-7)*

46. $\frac{12x}{4y^2} + \frac{8}{6y}$

47. $\frac{a}{a-b} + \frac{b}{2b+3a}$

48. $\frac{a+3}{3a^2-10a-8} + \frac{2a}{a^2-8a+16}$

49. $\frac{n-4}{(n-2)^2} + \frac{n-5}{n^2+n-6}$

Find each difference. *(Lesson 12-6)*

50. $\frac{7}{x^2} - \frac{3}{x^2}$

51. $\frac{x}{(x-3)^2} - \frac{3}{(x-3)^2}$

52. $\frac{2}{t^2-t-2} - \frac{t}{t^2-t-2}$

53. $\frac{2n}{n^2+2n-24} - \frac{8}{n^2+2n-24}$

54. **BIOLOGY** Ana is working on a biology project for her school's science fair. For her experiment, she needs to have a certain type of bacteria that doubles its population every hour. Right now Ana has 1000 bacteria. If Ana does not interfere with the bacteria, predict how many there will be in ten hours. *(Lesson 10-6)*

Solve each equation by factoring. Check your solutions. *(Lesson 9-5)*

55. $s^2 = 16$

56. $9p^2 = 64$

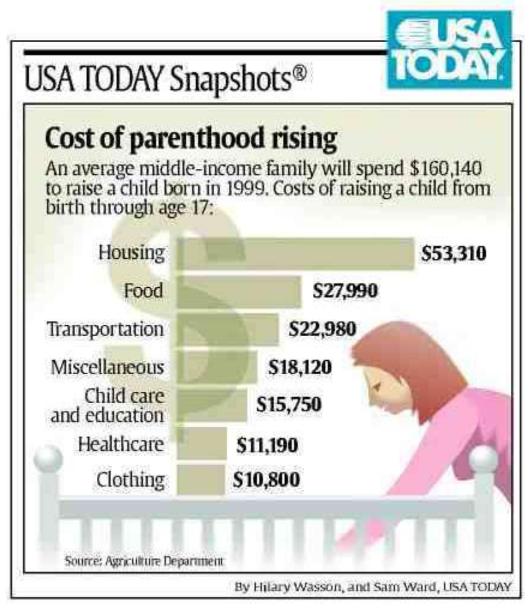
57. $z^3 - 9z = 45 - 5z^2$

FAMILIES **For Exercises 58–60, refer to the graph.** *(Lesson 8-3)*

58. Write each number in the graph using scientific notation.

59. How many times as great is the amount spent on food as the amount spent on clothing? Express your answer in scientific notation.

60. What percent of the total amount is spent on housing?



TELEPHONE RATES **For Exercises 61 and 62, use the following information.** *(Lesson 5-4)*

A 15-minute call to Mexico costs \$3.39. A 24-minute call costs \$4.83.

61. Write a linear equation to find the total cost C of an m-minute call.

62. Find the cost of a 9-minute call.

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Solve each equation.**
*(To review **solving equations**, see Lessons 3-2 through 3-4.)*

63. $-12 = \frac{x}{4}$

64. $1.8 = g - 0.6$

65. $\frac{3}{4}n - 3 = 9$

66. $7x^2 = 28$

67. $3.2 = \frac{-8+n}{-7}$

68. $\frac{-3n-(-4)}{-6} = -9$

12-9 Solving Rational Equations

What You'll Learn

- Solve rational equations.
- Eliminate extraneous solutions.

Vocabulary

- rational equations
- work problems
- rate problems
- extraneous solutions

How are rational equations important in the operation of a subway system?

The Washington, D.C., Metrorail is one of the safest subway systems in the world, serving a population of more than 3.5 million. It is vital that a rail system of this size maintain a consistent schedule. Rational equations can be used to determine the exact positions of trains at any given time.

Washington Metropolitan Area Transit Authority

Train	Distance
Red Line	19.4 mi
Orange Line	24.14 mi
Blue Line	19.95 mi
Green Line	20.59 mi
Yellow Line	9.46 mi

SOLVE RATIONAL EQUATIONS **Rational equations** are equations that contain rational expressions. You can use cross products to solve rational equations, but only when both sides of the equation are single fractions.

Example 1 Use Cross Products

Solve $\frac{12}{x+5} = \frac{4}{(x+2)}$.

$\frac{12}{x+5} = \frac{4}{(x+2)}$	Original equation
$12(x+2) = 4(x+5)$	Cross multiply.
$12x + 24 = 4x + 20$	Distributive Property
$8x = -4$	Add $-4x$ and -24 to each side.
$x = -\frac{4}{8}$ or $-\frac{1}{2}$	Divide each side by 8.

Another method you can use to solve rational equations is to multiply each side of the equation by the LCD to eliminate fractions.

Example 2 Use the LCD

Solve $\frac{n-2}{n} - \frac{n-3}{n-6} = \frac{1}{n}$.

$\frac{n-2}{n} - \frac{n-3}{n-6} = \frac{1}{n}$	Original equation
$n(n-6)\left(\frac{n-2}{n} - \frac{n-3}{n-6}\right) = n(n-6)\left(\frac{1}{n}\right)$	The LCD is $n(n-6)$.
$\left(\frac{\overset{1}{\cancel{n}}(n-6)}{1} \cdot \frac{n-2}{\underset{1}{\cancel{n}}}\right) - \left(\frac{n\overset{1}{\cancel{(n-6)}}}{1} \cdot \frac{n-3}{\underset{1}{\cancel{n-6}}}\right) = \frac{\overset{1}{\cancel{n}}(n-6)}{1} \cdot \frac{1}{\underset{1}{\cancel{n}}}$	Distributive Property
$(n-6)(n-2) - n(n-3) = n - 6$	Simplify.
$(n^2 - 8n + 12) - (n^2 - 3n) = n - 6$	Multiply.
$n^2 - 8n + 12 - n^2 + 3n = n - 6$	Subtract.
$-5n + 12 = n - 6$	Simplify.
$-6n = -18$	Subtract 12 from each side.
$n = 3$	Divide each side by -6.

A rational equation may have more than one solution.

Example 3 Multiple Solutions

Solve $\frac{-4}{a+1} + \frac{3}{a} = 1$.

$$\frac{-4}{a+1} + \frac{3}{a} = 1 \quad \text{Original equation}$$

$$a(a+1)\left(\frac{-4}{a+1} + \frac{3}{a}\right) = a(a+1)(1) \quad \text{The LCD is } a(a+1).$$

$$\left(\frac{a\cancel{(a+1)}^{1}}{1} \cdot \frac{-4}{\cancel{a+1}_{1}}\right) + \left(\frac{\cancel{a}^{1}(a+1)}{1} \cdot \frac{3}{\cancel{a}_{1}}\right) = a(a+1) \quad \text{Distributive Property}$$

$$-4a + 3a + 3 = a^2 + a \quad \text{Simplify.}$$

$$-a + 3 = a^2 + a \quad \text{Add like terms.}$$

$$0 = a^2 + 2a - 3 \quad \text{Set equal to 0.}$$

$$0 = (a+3)(a-1) \quad \text{Factor.}$$

$$a + 3 = 0 \quad \text{or} \quad a - 1 = 0$$

$$a = -3 \qquad\qquad a = 1$$

CHECK Check by substituting each value in the original equation.

$$\frac{-4}{a+1} + \frac{3}{a} = 1 \qquad\qquad \frac{-4}{a+1} + \frac{3}{a} = 1$$

$$\frac{-4}{-3+1} + \frac{3}{-3} \stackrel{?}{=} 1 \quad a = -3 \qquad\qquad \frac{-4}{1+1} + \frac{3}{1} \stackrel{?}{=} 1 \quad a = 1$$

$$2 + (-1) \stackrel{?}{=} 1 \qquad\qquad -2 + 3 \stackrel{?}{=} 1$$

$$1 = 1 \qquad\qquad 1 = 1$$

The solutions are 1 or −3.

Study Tip

Look Back
To review **solving quadratic equations by factoring**, see Lessons 9-4 through 9-7.

Rational equations can be used to solve **work problems**.

Example 4 Work Problem

LAWN CARE **Abbey has a lawn care service. One day she asked her friend Jamal to work with her. Normally, it takes Abbey two hours to mow and trim Mrs. Harris' lawn. When Jamal worked with her, the job took only 1 hour and 20 minutes. How long would it have taken Jamal to do the job himself?**

Explore Since it takes Abbey two hours to do the yard, she can finish $\frac{1}{2}$ the job in one hour. The amount of work Jamal can do in one hour can be represented by $\frac{1}{t}$. To determine how long it takes Jamal to do the job, use the formula Abbey's work + Jamal's work = 1 completed yard.

Plan The time that both of them worked was $1\frac{1}{3}$ hours. Each rate multiplied by this time results in the amount of work done by each person.

Solve

$$\underbrace{\frac{1}{2}\left(\frac{4}{3}\right)}_{\text{Abbey's work}} \quad \underbrace{+}_{\text{plus}} \quad \underbrace{\frac{1}{t}\left(\frac{4}{3}\right)}_{\text{Jamal's work}} \quad \underbrace{=}_{\text{equals}} \quad \underbrace{1}_{\text{total work.}}$$

$$\frac{4}{6} + \frac{4}{3t} = 1 \quad \text{Multiply.}$$

(continued on the next page)

Study Tip

Work Problems
When solving work problems, remember that each term should represent the portion of a job completed in one unit of time.

$$6t\left(\frac{4}{6} + \frac{4}{3t}\right) = 6t \cdot 1 \quad \text{The LCD is } 6t.$$

$$\overset{1}{\cancel{6t}}\left(\frac{4}{\underset{1}{\cancel{6}}}\right) + \overset{2}{\cancel{6t}}\left(\frac{4}{\underset{1}{\cancel{3t}}}\right) = 6t \quad \text{Distributive Property}$$

$$4t + 8 = 6t \quad \text{Simplify.}$$

$$8 = 2t \quad \text{Add } -4t \text{ to each side.}$$

$$4 = t \quad \text{Divide each side by 2.}$$

Examine The time that it would take Jamal to do the yard by himself is four hours. This seems reasonable because the combined efforts of the two took longer than half of Abbey's usual time.

Rational equations can also be used to solve **rate problems**.

Example 5 Rate Problem

TRANSPORTATION Refer to the application at the beginning of the lesson. The Yellow Line runs between Huntington and Mt. Vernon Square. Suppose one train leaves Mt. Vernon Square at noon and arrives at Huntington 24 minutes later, and a second train leaves Huntington at noon and arrives at Mt. Vernon Square 28 minutes later. At what time do the two trains pass each other?

Study Tip

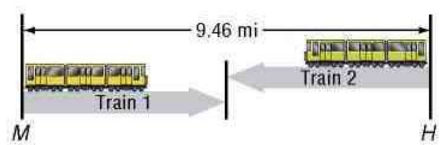
Rate Problems

You can solve rate problems, also called *uniform motion problems*, more easily if you first make a drawing.

Determine the rates of both trains. The total distance is 9.46 miles.

Train 1 $\frac{9.46 \text{ mi}}{24 \text{ min}}$ **Train 2** $\frac{9.46 \text{ mi}}{28 \text{ min}}$

Next, since both trains left at the same time, the time both have traveled when they pass will be the same. And since they started at opposite ends of the route, the sum of their distances is equal to the total route, 9.46 miles.



	r	t	d
Train 1	$\frac{9.46 \text{ mi}}{24 \text{ min}}$	t min	$\frac{9.46t}{24}$ mi
Train 2	$\frac{9.46 \text{ mi}}{28 \text{ min}}$	t min	$\frac{9.46t}{28}$ mi

$$\frac{9.46t}{24} + \frac{9.46t}{28} = 9.46 \quad \text{The sum of the distances is 9.46.}$$

$$168\left(\frac{9.46t}{24} + \frac{9.46t}{28}\right) = 168 \cdot 9.46 \quad \text{The LCD is 168.}$$

$$\frac{\overset{7}{\cancel{168}}}{1} \cdot \frac{9.46t}{\underset{1}{\cancel{24}}} + \frac{\overset{6}{\cancel{168}}}{1} \cdot \frac{9.46t}{\underset{1}{\cancel{28}}} = 1589.28 \quad \text{Distributive Property}$$

$$66.22t + 56.76t = 1589.28 \quad \text{Simplify.}$$

$$122.98t = 1589.28 \quad \text{Add.}$$

$$t = 12.92 \quad \text{Divide each side by 122.98.}$$

The trains passed at about 12.92 or about 13 minutes after leaving their stations, which is 12:13 P.M.

EXTRANEOUS SOLUTIONS Multiplying each side of an equation by the LCD of two rational expressions can yield results that are not solutions to the original equation. Recall that such solutions are called **extraneous solutions**.

Example 6 No Solution

Solve $\frac{3x}{x-1} + \frac{6x-9}{x-1} = 6$.

$$\frac{3x}{x-1} + \frac{6x-9}{x-1} = 6 \quad \text{Original equation}$$

$$(x-1)\left(\frac{3x}{x-1} + \frac{6x-9}{x-1}\right) = (x-1)6 \quad \text{The LCD is } x-1.$$

$$\cancel{(x-1)}^{1}\left(\frac{3x}{\cancel{x-1}_{1}}\right) + \cancel{(x-1)}^{1}\left(\frac{6x-9}{\cancel{x-1}_{1}}\right) = (x-1)6 \quad \text{Distributive Property}$$

$$3x + 6x - 9 = 6x - 6 \quad \text{Simplify.}$$

$$9x - 9 = 6x - 6 \quad \text{Add like terms.}$$

$$3x = 3 \quad \text{Add 9 to each side.}$$

$$x = 1 \quad \text{Divide each side by 3.}$$

Since 1 is an excluded value for x, the number 1 is an extraneous solution. Thus, the equation has no solution.

Rational equations can have both valid solutions and extraneous solutions.

Example 7 Extraneous Solution

Solve $\frac{2n}{1-n} + \frac{n+3}{n^2-1} = 1$.

$$\frac{2n}{1-n} + \frac{n+3}{n^2-1} = 1$$

$$\frac{2n}{1-n} + \frac{n+3}{(n-1)(n+1)} = 1$$

$$-\frac{2n}{n-1} + \frac{n+3}{(n-1)(n+1)} = 1$$

$$(n-1)(n+1)\left(-\frac{2n}{n-1} + \frac{n+3}{(n-1)(n+1)}\right) = (n-1)(n+1)1$$

$$\cancel{(n-1)}^{1}(n+1)\left(-\frac{2n}{\cancel{n-1}_{1}}\right) + \cancel{(n-1)}^{1}\cancel{(n+1)}^{1}\left(\frac{n+3}{\cancel{(n-1)}_{1}\cancel{(n+1)}_{1}}\right) = (n-1)(n+1)$$

$$-2n(n+1) + (n+3) = n^2 - 1$$

$$-2n^2 - 2n + n + 3 = n^2 - 1$$

$$-3n^2 - n + 4 = 0$$

$$3n^2 + n - 4 = 0$$

$$(3n+4)(n-1) = 0$$

$3n + 4 = 0$ or $n - 1 = 0$

$n = -\frac{4}{3}$ $\quad$ $n = 1$

The number 1 is an extraneous solution, since 1 is an excluded value for n. Thus, $-\frac{4}{3}$ is the solution of the equation.

Check for Understanding

Concept Check

1. **OPEN ENDED** Explain why the equation $n + \frac{1}{n-1} = \frac{1}{n-1} + 1$ has no solution.
2. **Write** an expression to represent the amount of work Aminta can do in h hours if it normally takes her 3 hours to change the oil and tune up her car.
3. **Find a counterexample** for the following statement.
   The solution of a rational equation can never be zero.

Guided Practice

Solve each equation. State any extraneous solutions.

4. $\frac{2}{x} = \frac{3}{x+1}$
5. $\frac{7}{a-1} = \frac{5}{a+3}$
6. $\frac{3x}{5} + \frac{3}{2} = \frac{7x}{10}$
7. $\frac{x+1}{x} + \frac{x+4}{x} = 6$
8. $\frac{5}{k+1} - \frac{7}{k} = \frac{1}{k+1}$
9. $\frac{x+2}{x-2} - \frac{2}{x+2} = \frac{-7}{3}$

Application

10. **BASEBALL** Omar has 32 hits in 128 times at bat. He wants his batting average to be .300. His current average is $\frac{32}{128}$ or .250. How many at bats does Omar need to reach his goal if he gets a hit in each of his next b at bats?

Practice and Apply

Homework Help

For Exercises	See Examples
11–14	1
15–19, 21, 23, 26, 27	2
22, 24, 25	3
29–34	4, 5
20, 28	6, 7

Extra Practice
See page 849.

Solve each equation. State any extraneous solutions.

11. $\frac{4}{a} = \frac{3}{a-2}$
12. $\frac{3}{x} = \frac{1}{x-2}$
13. $\frac{x-3}{x} = \frac{x-3}{x-6}$
14. $\frac{x}{x+1} = \frac{x-6}{x-1}$
15. $\frac{2n}{3} + \frac{1}{2} = \frac{2n-3}{6}$
16. $\frac{5}{4} + \frac{3y}{2} = \frac{7y}{6}$
17. $\frac{a-1}{a+1} - \frac{2a}{a-1} = -1$
18. $\frac{7}{x^2-5x} + \frac{3}{5-x} = \frac{4}{x}$
19. $\frac{4x}{2x+3} - \frac{2x}{2x-3} = 1$
20. $\frac{5}{5-p} - \frac{p^2}{p-5} = -2$
21. $\frac{a}{3a+6} - \frac{a}{5a+10} = \frac{2}{5}$
22. $\frac{c}{c-4} - \frac{6}{4-c} = c$
23. $\frac{2b-5}{b-2} - 2 = \frac{3}{b+2}$
24. $\frac{7}{k-3} - \frac{1}{2} = \frac{3}{k-4}$
25. $\frac{x^2-4}{x-2} + x^2 = 4$
26. $\frac{2n}{n-1} + \frac{n-5}{n^2-1} = 1$
27. $\frac{3z}{z^2-5z+4} = \frac{2}{z-4} + \frac{3}{z-1}$
28. $\frac{4}{m^2-8m+12} = \frac{m}{m-2} + \frac{1}{m-6}$

29. **QUIZZES** Each week, Mandy's algebra teacher gives a 10-point quiz. After 5 weeks, Mandy has earned a total of 36 points for an average of 7.2 points per quiz. She would like to raise her average to 9 points. On how many quizzes must she score 10 points in order to reach her goal?

BOATING For Exercises 30 and 31, use the following information.
Jim and Mateo live across a lake from each other at a distance of about 3 miles. Jim can row his boat to Mateo's house in 1 hour and 20 minutes. Mateo can drive his motorboat the same distance in a half hour.

30. If they leave their houses at the same time and head toward each other, how long will it be before they meet?
31. How far from the nearest shore will they be when they meet?
32. **CAR WASH** Ian and Nadya can each wash a car and clean its interior in about 2 hours, but Chris needs 3 hours to do the work. If the three work together, how long will it take to clean seven cars?

SWIMMING POOLS For Exercises 33 and 34, use the following information.
The pool in Kara's backyard is cleaned and ready to be filled for the summer. It measures 15 feet long and 10 feet wide with an average depth of 4 feet.

33. What is the volume of the pool?

34. How many gallons of water will it take to fill the pool? ($1\text{ ft}^3 = 7.5$ gal)

35. **CRITICAL THINKING** Solve $\dfrac{\frac{x+3}{x-2} \cdot \frac{x^2+x-2}{x+5}}{x-1} + 2 = 0$.

36. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are rational equations important in the operation of a subway system?

Include the following in your answer:

- an explanation of how rational equations can be used to approximate the time that trains will pass each other if they leave distant stations and head toward each other.

Standardized Test Practice

37. What is the value of a in the equation $\frac{a-2}{a} - \frac{a-3}{a-6} = \frac{1}{a}$?

(A) 3 (B) 2 (C) 6 (D) 0

38. Which value is an extraneous solution of $\frac{-1}{n+2} = \frac{n^2-7n-8}{3n^2+2n-8}$?

(A) 6 (B) 2 (C) -1 (D) -2

Maintain Your Skills

Mixed Review

Simplify each expression. *(Lesson 12-8)*

39. $\dfrac{\frac{x^2+8x+15}{x^2+x-6}}{\frac{x^2+2x-15}{x^2-2x-3}}$

40. $\dfrac{\frac{a^2-6a+5}{a^2+13a+42}}{\frac{a^2-4a+3}{a^2+3a-18}}$

41. $\dfrac{x+2+\frac{2}{x+5}}{x+6+\frac{6}{x+1}}$

Find each difference. *(Lesson 12-7)*

42. $\frac{3}{2m-3} - \frac{m}{6-4m}$

43. $\frac{y}{y^2-2y+1} - \frac{1}{y-1}$

44. $\frac{a+2}{a^2-9} - \frac{2a}{6a^2-17a-3}$

Factor each polynomial. *(Lesson 9-2)*

45. $20x - 8y$

46. $14a^2b + 21ab^2$

47. $10p^2 - 12p + 25p - 30$

48. **CHEMISTRY** One solution is 50% glycol, and another is 30% glycol. How much of each solution should be mixed to make a 100-gallon solution that is 45% glycol? *(Lesson 7-2)*

WebQuest Internet Project

Building the Best Roller Coaster

It is time to complete your project. Use the information and data you have gathered about the building and financing of a roller coaster to prepare a portfolio or Web page. Be sure to include graphs and/or tables in the presentation.

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Chapter 12 Study Guide and Review

Vocabulary and Concept Check

complex fraction (p. 684)
excluded values (p. 648)
extraneous solutions (p. 693)
inverse variation (p. 642)
least common multiple (p. 678)
least common denominator (p. 679)
mixed expression (p. 684)
product rule (p. 643)
rate problem (p. 692)
rational equation (p. 690)
rational expression (p. 648)
work problem (p. 691)

State whether each sentence is *true* or *false*. If false, replace the underlined expression to make a true sentence.

1. A <u>mixed</u> expression is a fraction whose numerator and denominator are polynomials.
2. The complex fraction $\frac{\frac{4}{5}}{\frac{2}{3}}$ can be simplified as $\underline{\frac{6}{5}}$.
3. The equation $\frac{x}{x-1} + \frac{2x-3}{x-1} = 2$ has an extraneous solution of $\underline{1}$.
4. The mixed expression $6 - \frac{a-2}{a+3}$ can be rewritten as $\underline{\frac{5a+16}{a+3}}$.
5. The least common multiple for $(x^2 - 144)$ and $(x + 12)$ is $\underline{x + 12}$.
6. The excluded values for $\frac{4x}{x^2 - x - 12}$ are $\underline{-3 \text{ and } 4}$.

Lesson-by-Lesson Review

12-1 Inverse Variation

See pages 642–647.

Concept Summary

- The product rule for inverse variations states that if (x_1, y_1) and (x_2, y_2) are solutions of an inverse variation, then $x_1y_1 = k$ and $x_2y_2 = k$.
- You can use $\frac{x_1}{x_2} = \frac{y_2}{y_1}$ to solve problems involving inverse variation.

Example If y varies inversely as x and $y = 24$ when $x = 30$, find x when $y = 10$.

$\frac{x_1}{x_2} = \frac{y_2}{y_1}$ Proportion for inverse variations

$\frac{30}{x_2} = \frac{10}{24}$ $x_1 = 30$, $y_1 = 24$, and $y_2 = 10$

$720 = 10x_2$ Cross multiply.

$72 = x_2$ Thus, $x = 72$ when $y = 10$.

Exercises **Write an inverse variation equation that relates x and y. Assume that y varies inversely as x. Then solve.** *See Examples 3 and 4 on page 644.*

7. If $y = 28$ when $x = 42$, find y when $x = 56$.
8. If $y = 15$ when $x = 5$, find y when $x = 3$.
9. If $y = 18$ when $x = 8$, find x when $y = 3$.
10. If $y = 35$ when $x = 175$, find y when $x = 75$.

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CONTENTS

12-2 Rational Expressions

See pages 648–653.

Concept Summary

- Excluded values are values of a variable that result in a denominator of zero.

Example **Simplify $\frac{x+4}{x^2+12x+32}$. State the excluded values of x.**

$$\frac{x+4}{x^2+12x+32} = \frac{\cancel{x+4}^{\,1}}{\cancel{(x+4)}_{1}(x+8)} \quad \text{Factor.}$$

$$= \frac{1}{x+8} \quad \text{Simplify.}$$

The expression is undefined when $x = -4$ and $x = -8$.

Exercises **Simplify each expression.** *See Example 5 on page 650.*

11. $\frac{3x^2y}{12xy^3z}$ 12. $\frac{n^2-3n}{n-3}$ 13. $\frac{a^2-25}{a^2+3a-10}$ 14. $\frac{x^2+10x+21}{x^3+x^2-42x}$

12-3 Multiplying Rational Expressions

See pages 655–659.

Concept Summary

- Multiplying rational expressions is similar to multiplying rational numbers.

Example **Find $\frac{1}{x^2+x-12} \cdot \frac{x-3}{x+5}$.**

$$\frac{1}{x^2+x-12} \cdot \frac{x-3}{x+5} = \frac{1}{(x+4)\cancel{(x-3)}_{1}} \cdot \frac{\cancel{x-3}^{\,1}}{x+5} \quad \text{Factor.}$$

$$= \frac{1}{(x+4)(x+5)} \quad \text{Simplify.}$$

Exercises **Find each product.** *See Examples 1–3 on pages 655 and 656.*

15. $\frac{7b^2}{9} \cdot \frac{6a^2}{b}$ 16. $\frac{5x^2y}{8ab} \cdot \frac{12a^2b}{25x}$ 17. $(3x+30) \cdot \frac{10}{x^2-100}$

18. $\frac{3a-6}{a^2-9} \cdot \frac{a+3}{a^2-2a}$ 19. $\frac{x^2+x-12}{x+2} \cdot \frac{x+4}{x^2-x-6}$ 20. $\frac{b^2+19b+84}{b-3} \cdot \frac{b^2-9}{b^2+15b+36}$

12-4 Dividing Rational Expressions

See pages 660–664.

Concept Summary

- Divide rational expressions by multiplying by the reciprocal of the divisor.

Example **Find $\frac{y^2-16}{y^2-64} \div \frac{y+4}{y-8}$.**

$$\frac{y^2-16}{y^2-64} \div \frac{y+4}{y-8} = \frac{y^2-16}{y^2-64} \cdot \frac{y-8}{y+4} \quad \text{Multiply by the reciprocal of } \frac{y+4}{y-8}.$$

$$= \frac{(y-4)\cancel{(y+4)}^{\,1}}{\cancel{(y-8)}_{1}(y+8)} \cdot \frac{\cancel{y-8}^{\,1}}{\cancel{y+4}_{1}} \text{ or } \frac{y-4}{y+8} \quad \text{Simplify.}$$

Exercises **Find each quotient.** *See Examples 1–4 on pages 660 and 661.*

21. $\frac{p^3}{2q} \div \frac{p^2}{4q}$

22. $\frac{y^2}{y+4} \div \frac{3y}{y^2-16}$

23. $\frac{3y-12}{y+4} \div (y^2 - 6y + 8)$

24. $\frac{2m^2+7m-15}{m+5} \div \frac{9m^2-4}{3m+2}$

12-5 Dividing Polynomials

See pages 666–671.

Concept Summary

- To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.
- To divide a polynomial by a binomial, use long division.

Example **Find $(x^3 - 2x^2 - 22x + 21) \div (x - 3)$.**

$$
\begin{array}{rll}
 & x^2 \quad + x - 19 & \\
x - 3 \,\big) & x^3 - 2x^2 - 22x + 21 & \\
 & \underline{(-)x^3 - 3x^2} & \text{Multiply } x^2 \text{ and } x-3. \\
 & x^2 - 22x & \text{Subtract.} \\
 & \underline{(-)x^2 - 3x} & \text{Multiply } x \text{ and } x-3. \\
 & -19x + 21 & \text{Subtract.} \\
 & \underline{(-)-19x + 57} & \text{Multiply } -19 \text{ and } x-3. \\
 & -36 & \text{Subtract.}
\end{array}
$$

The quotient is $x^3 + x - 19 - \frac{36}{x-3}$.

Exercises **Find each quotient.** *See Examples 1–5 on pages 666–668.*

25. $(4a^2b^2c^2 - 8a^3b^2c + 6abc^2) \div 2ab^2$

26. $(x^3 + 7x^2 + 10x - 6) \div (x + 3)$

27. $\frac{x^3 - 7x + 6}{x - 2}$

28. $(48b^2 + 8b + 7) \div (12b - 1)$

12-6 Rational Expressions with Like Denominators

See pages 672–677.

Concept Summary

- Add (or subtract) rational expressions with like denominators by adding (or subtracting) the numerators and writing the sum (or difference) over the denominator.

Example **Find $\frac{m^2}{m+4} - \frac{16}{m+4}$.**

$\frac{m^2}{m+4} - \frac{16}{m+4} = \frac{m^2-16}{m+4}$ Subtract the numerators.

$= \frac{(m-4)\cancel{(m+4)}^{1}}{\cancel{m+4}_{1}}$ or $m - 4$ Factor.

Exercises **Find each sum or difference.** *See Examples 1–4 on pages 672 and 673.*

29. $\frac{m+4}{5} + \frac{m-1}{5}$

30. $\frac{-5}{2n-5} + \frac{2n}{2n-5}$

31. $\frac{a^2}{a-b} + \frac{-b^2}{a-b}$

32. $\frac{7a}{b^2} - \frac{5a}{b^2}$

33. $\frac{2x}{x-3} - \frac{6}{x-3}$

34. $\frac{m^2}{m-n} - \frac{2mn-n^2}{m-n}$

12-7 Rational Expressions with Unlike Denominators

See pages 678–683.

Concept Summary

- Rewrite rational expressions with unlike denominators using the least common denominator (LCD). Then add or subtract.

Example Find $\frac{x}{x+3} + \frac{5}{x-2}$.

$$\frac{x}{x+3} + \frac{5}{x-2} = \frac{x-2}{x-2} \cdot \frac{x}{x+3} + \frac{x+3}{x+3} \cdot \frac{5}{x-2}$$ The LCD is $(x+3)(x-2)$.

$$= \frac{x^2-2x}{(x+3)(x-2)} + \frac{5x+15}{(x+3)(x-2)}$$ Multiply.

$$= \frac{x^2+3x+15}{(x+3)(x-2)}$$ Add.

Exercises **Find each sum or difference.** *See Examples 3–5 on pages 679 and 680.*

35. $\frac{2c}{3d^2} + \frac{3}{2cd}$ **36.** $\frac{r^2+21r}{r^2-9} + \frac{3r}{r+3}$ **37.** $\frac{3a}{a-2} + \frac{5a}{a+1}$

38. $\frac{7n}{3} - \frac{9n}{7}$ **39.** $\frac{7}{3a} - \frac{3}{6a^2}$ **40.** $\frac{2x}{2x+8} - \frac{4}{5x+20}$

12-8 Mixed Expressions and Complex Fractions

See pages 684–689.

Concept Summary

- Write mixed expressions as rational expressions in the same way as mixed numbers are changed to improper fractions.
- Simplify complex fractions by writing them as division problems.

Example Simplify $\dfrac{y - \frac{40}{y-3}}{y+5}$.

$$\frac{y - \frac{40}{y-3}}{y+5} = \frac{\frac{y(y-3)}{(y-3)} - \frac{40}{y-3}}{y+5}$$ The LCD in the numerator is $y-3$.

$$= \frac{\frac{y^2-3y-40}{y-3}}{y+5}$$ Add in the numerator.

$$= \frac{y^2-3y-40}{y-3} \div (y+5)$$ Rewrite as a division sentence.

$$= \frac{y^2-3y-40}{y-3} \cdot \frac{1}{y+5}$$ Multiply by the reciprocal of $y+5$.

$$= \frac{(y-8)\overset{1}{\cancel{(y+5)}}}{y-3} \cdot \frac{1}{\underset{1}{\cancel{y+5}}} \text{ or } \frac{y-8}{y-3}$$ Factor.

Exercises **Write each mixed expression as a rational expression.**
See Example 1 on page 684.

41. $4 + \frac{x}{x-2}$ **42.** $2 - \frac{x+2}{x^2-4}$ **43.** $3 + \frac{x^2+y^2}{x^2-y^2}$

Chapter 12 For More ...
• Extra Practice, see pages 846–849.
• Mixed Problem Solving, see page 864.

Simplify each expression. *See Examples 3 and 4 on pages 685 and 686.*

44. $\dfrac{\frac{x^2}{y^3}}{\frac{3x}{9y^2}}$ **45.** $\dfrac{5+\frac{4}{a}}{\frac{a}{2}-\frac{3}{4}}$ **46.** $\dfrac{y+9-\frac{6}{y+4}}{y+4+\frac{2}{y+1}}$

12-9 Solving Rational Equations

See pages 690–695.

Concept Summary

- Use cross products to solve rational equations with a single fraction on each side of the equal sign.
- Multiply every term of a more complicated rational equation by the LCD to eliminate fractions.

Example Solve $\dfrac{5n}{6}+\dfrac{1}{n-2}=\dfrac{n+1}{3(n-2)}$.

$$\frac{5n}{6}+\frac{1}{n-2}=\frac{n+1}{3(n-2)} \quad \text{Original equation}$$

$$6(n-2)\left(\frac{5n}{6}+\frac{1}{n-2}\right)=6(n-2)\frac{n+1}{3(n-2)} \quad \text{The LCD is } 6(n-2)$$

$$\frac{\overset{1}{\cancel{6}}(n-2)(5n)}{\underset{1}{\cancel{6}}}+\frac{6\overset{1}{\cancel{(n-2)}}}{\underset{1}{\cancel{n-2}}}=\frac{\overset{2}{\cancel{6}}\overset{1}{\cancel{(n-2)}}(n+1)}{\underset{1}{\cancel{3}}\underset{1}{\cancel{(n-2)}}} \quad \text{Distributive Property}$$

$$(n-2)(5n)+6=2(n+1) \quad \text{Simplify.}$$

$$5n^2-10n+6=2n+2 \quad \text{Multiply.}$$

$$5n^2-12n+4=0 \quad \text{Subtract.}$$

$$(5n-2)(n-2)=0 \quad \text{Factor.}$$

$$n=\frac{2}{5} \text{ or } n=2$$

CHECK Let $n=\frac{2}{5}$.

$$\frac{\frac{2}{5}+1}{3\left(\frac{2}{5}-2\right)} \stackrel{?}{=} \frac{5\left(\frac{2}{5}\right)}{6}+\frac{1}{\frac{2}{5}-2}$$

$$-\frac{7}{24}=-\frac{7}{24} \checkmark$$

Let $n=2$.

$$\frac{2+1}{3(2-2)} \stackrel{?}{=} \frac{5(2)}{6}+\frac{1}{2-2}$$

$$\frac{3}{3(0)} \stackrel{?}{=} \frac{10}{6}+\frac{1}{0}$$

When you check the value 2, you get a zero in the denominator. So, 2 is an extraneous solution.

Exercises **Solve each equation. State any extraneous solutions.**
See Examples 6 and 7 on page 693.

47. $\dfrac{4x}{3}+\dfrac{7}{2}=\dfrac{7x}{12}-\dfrac{1}{4}$ **48.** $\dfrac{11}{2x}-\dfrac{2}{3x}=\dfrac{1}{6}$

49. $\dfrac{2}{3r}-\dfrac{3r}{r-2}=-3$ **50.** $\dfrac{x-2}{x}-\dfrac{x-3}{x-6}=\dfrac{1}{x}$

51. $\dfrac{3}{x^2+3x}+\dfrac{x+2}{x+3}=\dfrac{1}{x}$ **52.** $\dfrac{1}{n+4}-\dfrac{1}{n-1}=\dfrac{2}{n^2+3n-4}$

Chapter 12 Practice Test

Vocabulary and Concepts

Choose the letter that best matches each algebraic expression.

1. $\dfrac{\frac{a}{b}}{\frac{x}{y}}$ **2.** $3 - \dfrac{a+1}{a-1}$ **3.** $\dfrac{2}{x^2 + 2x - 4}$

a. complex fraction
b. rational expression
c. mixed expression

Skills and Applications

Write an inverse variation equation that relates x and y. Assume that y varies inversely as x. Then solve.

4. If $y = 21$ when $x = 40$, find y when $x = 84$.

5. If $y = 22$ when $x = 4$, find x when $y = 16$.

Simplify each expression. State the excluded values of the variables.

6. $\dfrac{5 - 2m}{6m - 15}$ **7.** $\dfrac{3 + x}{2x^2 + 5x - 3}$ **8.** $\dfrac{4c^2 + 12c + 9}{2c^2 - 11c - 21}$

9. $\dfrac{1 - \frac{9}{t}}{1 - \frac{81}{t^2}}$ **10.** $\dfrac{\frac{5}{6} + \frac{u}{t}}{\frac{2u}{t} - 3}$ **11.** $\dfrac{x + 4 + \frac{5}{x-2}}{x + 6 + \frac{15}{x-2}}$

Perform the indicated operations.

12. $\dfrac{2x}{x-7} - \dfrac{14}{x-7}$

13. $\dfrac{n+3}{2n-8} \cdot \dfrac{6n-24}{2n+1}$

14. $(10m^2 + 9m - 36) \div (2m - 3)$

15. $\dfrac{x^2 + 4x - 32}{x + 5} \cdot \dfrac{x - 3}{x^2 - 7x + 12}$

16. $\dfrac{z^2 + 2z - 15}{z^2 + 9z + 20} \div (z - 3)$

17. $\dfrac{4x^2 + 11x + 6}{x^2 - x - 6} \div \dfrac{x^2 + 8x + 16}{x^2 + x - 12}$

18. $(10z^4 + 5z^3 - z^2) \div 5z^3$

19. $\dfrac{y}{7y + 14} + \dfrac{6}{6 - 3y}$

20. $\dfrac{x+5}{x+2} + 6$

21. $\dfrac{x^2 - 1}{x + 1} - \dfrac{x^2 + 1}{x - 1}$

Solve each equation. State any extraneous solutions.

22. $\dfrac{2n}{n-4} - 2 = \dfrac{4}{n+5}$

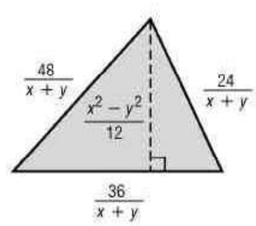
23. $\dfrac{3}{x^2 + 5x + 6} - \dfrac{7}{x+3} = -\dfrac{x-1}{x+2}$

24. FINANCE Barrington High School is raising money for Habitat for Humanity by doing lawn work for friends and neighbors. Scott can rake a lawn and bag the leaves in 5 hours, while Kalyn can do it in 3 hours. If Scott and Kalyn work together, how long will it take them to rake a lawn and bag the leaves?

25. STANDARDIZED TEST PRACTICE Which expression can be used to represent the area of the triangle?

Ⓐ $\frac{1}{2}(x - y)$ Ⓑ $\frac{3}{2}(x - y)$

Ⓒ $\frac{1}{4}(x - y)$ Ⓓ $\frac{108}{x + y}$



Chapter 12 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. A cylindrical container is 8 inches in height and has a radius of 2.5 inches. What is the volume of the container to the nearest cubic inch? (*Hint*: $V = \pi r^2 h$) (Lesson 3-8)

Ⓐ 63 Ⓑ 126

Ⓒ 150 Ⓓ 157

2. Which function includes all of the ordered pairs in the table? (Lesson 4-8)

x	−3	−1	1	3	5
y	10	4	−2	−8	−14

Ⓐ $y = -2x$ Ⓑ $y = -3x + 1$

Ⓒ $y = 2x - 4$ Ⓓ $y = 3x + 1$

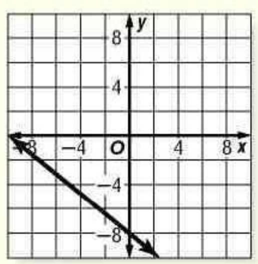
3. Which equation describes the graph below? (Lesson 5-4)

Ⓐ $4x - 5y = 40$

Ⓑ $4x + 5y = -40$

Ⓒ $4x + 5y = -8$

Ⓓ $rx - 5y = 10$



4. Which equation represents the line that passes through $(-12, 5)$ and has a slope of $-\frac{1}{4}$? (Lesson 5-5)

Ⓐ $x + 4y = 8$ Ⓑ $-x + 4y = 20$

Ⓒ $-4x + y = 65$ Ⓓ $x + 4y = 5$

The Princeton Review Test-Taking Tip

Questions 2, 4, 8 Sometimes sketching the graph of a function can help you to see the relationship between *x* and *y* and answer the question.

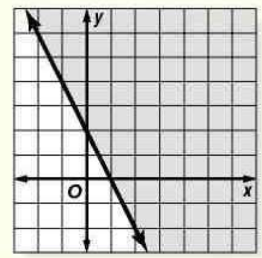
5. Which inequality represents the shaded region? (Lesson 6-6)

Ⓐ $y \le -\frac{1}{2}x - 2$

Ⓑ $y \ge -\frac{1}{2}x + 2$

Ⓒ $y \le -2x + 2$

Ⓓ $y \ge -2x + 2$



6. Which ordered pair is the solution of the following system of equations? (Lesson 7-4)

$$3x + y = -2$$
$$-2x + y = 8$$

Ⓐ $(-6, 16)$ Ⓑ $(-2, 4)$

Ⓒ $(-3, 2)$ Ⓓ $(2, -8)$

7. The length of a rectangular door is 2.5 times its width. If the area of the door is 9750 square inches, which equation will determine the width w of the door? (Lesson 8-1)

Ⓐ $w^2 + 2.5w = 9750$

Ⓑ $2.5w^2 = 9750$

Ⓒ $2.5w^2 + 9750 = 0$

Ⓓ $7w = 9750$

8. A scientist monitored a 144-gram sample of a radioactive substance, which decays into a nonradioactive substance. The table shows the amount, in grams, of the radioactive substance remaining at intervals of 20 hours. How many grams of the radioactive substance are likely to remain after 100 hours? (Lessons 10-6 and 10-7)

Time (h)	0	20	40	60	80	100
Mass (g)	144	72	36			

Ⓐ 1 g Ⓑ 2.25 g

Ⓒ 4.5 g Ⓓ 9 g

Aligned and verified by The Princeton Review

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

9. A family drove an average of 350 miles per day during three days of their trip. They drove 360 miles on the first day and 270 miles on the second day. How many miles did they drive on the third day? (Lesson 3-4)

10. The area of the rectangular playground at Hillcrest School is 750 square meters. The length of the playground is 5 meters greater than its width. What are the length and width of the playground in meters? (Lesson 9-5)

11. Use the Quadratic Formula or factoring to determine whether the graph of $y = 16x^2 + 24x + 9$ intersects the x-axis in zero, one, or two points. (Lesson 10-4)

12. Express $\frac{x^2 - 9}{x^3 + x} \cdot \frac{3x}{x - 3}$ as a quotient of two polynomials written in simplest form. (Lesson 11-3)

13. Express the following quotient in simplest form. (Lesson 11-4)

$$\frac{x}{x + 4} \div \frac{4x}{x^2 - 16}$$

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

Ⓐ **the quantity in Column A is greater,**

Ⓑ **the quantity in Column B is greater,**

Ⓒ **the two quantities are equal, or**

Ⓓ **the relationship cannot be determined from the information given.**

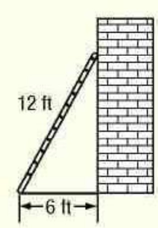
	Column A	Column B
14.	$x = \frac{1}{4}, y = 4$	
	$\frac{1}{x^2 - 2x}$	$\frac{1}{y^2 - 2y}$
		(Lesson 1-3)
15.	$\sqrt{500} - \sqrt{20} + \sqrt{180} - \sqrt{720}$	$\sqrt{125} - \sqrt{45}$
		(Lesson 11-2)
16.	the excluded value of a in $\frac{16a - 24}{32a}$	the excluded value of b in $\frac{5b + 3}{b + 6}$
		(Lesson 12-2)
17.	$5 + \frac{3x}{x + 1}$	$\dfrac{\frac{24y + 15}{3}}{\frac{6y + 6}{6}}$
		(Lesson 12-8)

Part 4 Open Ended

Record your answers on a sheet of paper. Show your work.

18. A 12-foot ladder is placed against the side of a building so that the bottom of the ladder is 6 feet from the base of the building. (Lesson 12-1)

a. Suppose the bottom of the ladder is moved closer to the base of the building. Does the height that the ladder reaches increase or decrease?

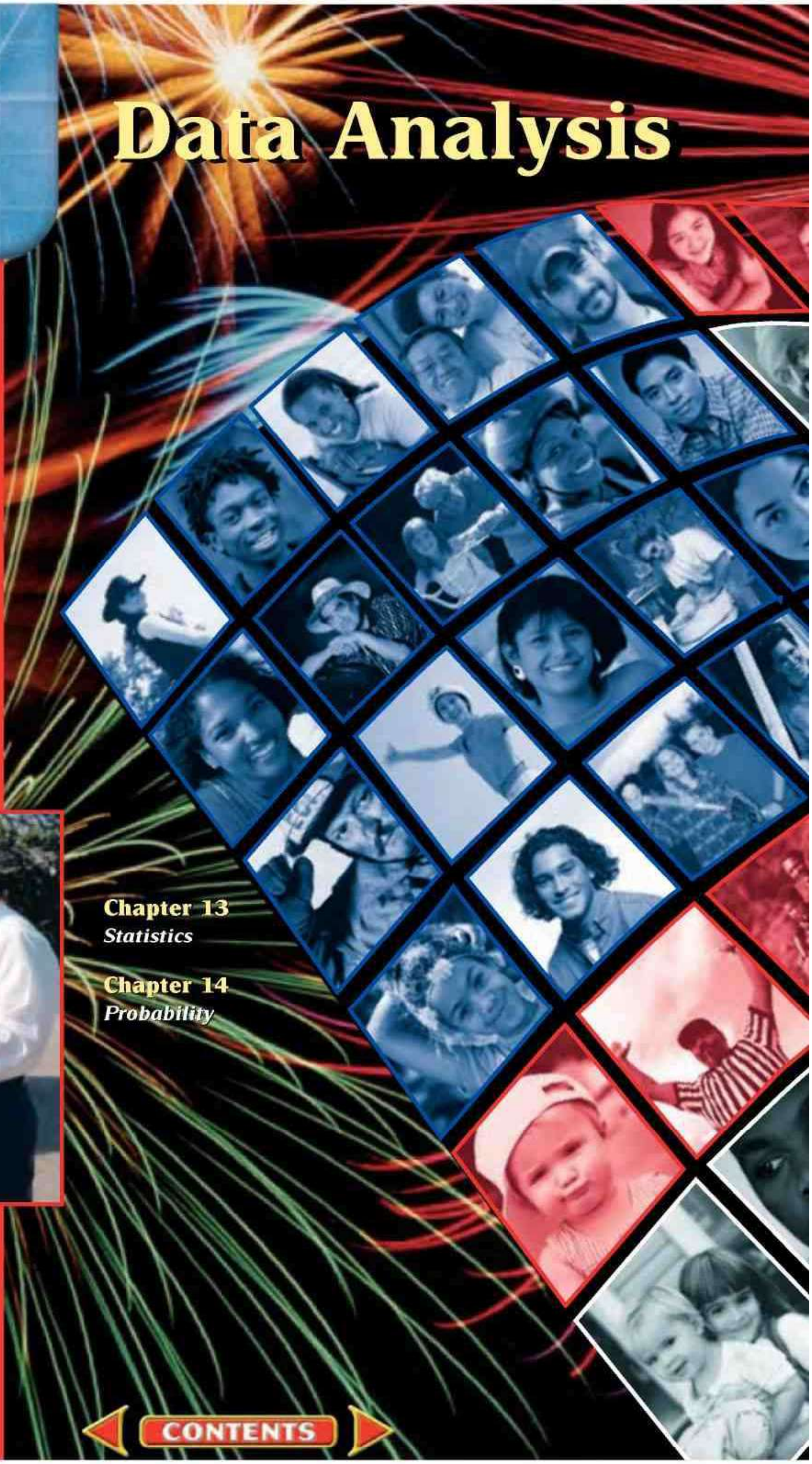


b. What conclusion can you make about the height the ladder reaches and the distance between the bottom of the ladder and the base of the building?

c. Does this relationship form an inverse proportion? Explain your reasoning.

UNIT 5

Data Analysis



Collecting and analyzing data allows you to make decisions and predictions about the future. In this unit, you will learn about statistics and probability.

WebQuest Internet Project

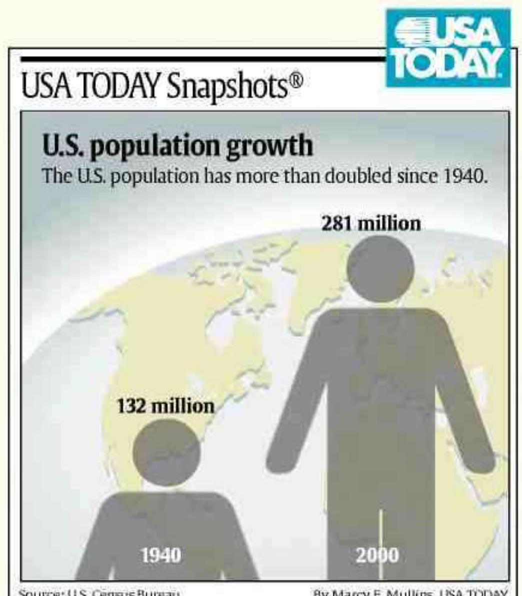
America Counts!

The U.S. government has been counting each person in the country since its first Census following independence was taken in 1790. Befitting the first Census of the 21st century, the Census Bureau allowed Census 2000 questionnaires to be completed electronically for the first time. In this project, you will see how data analysis can be used to compare statistics about a state of your choice to other states in the United States.

Log on to www.algebra1.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 5.

Lesson	13-5	14-2
Page	742	766



Chapter

13 Statistics

What You'll Learn

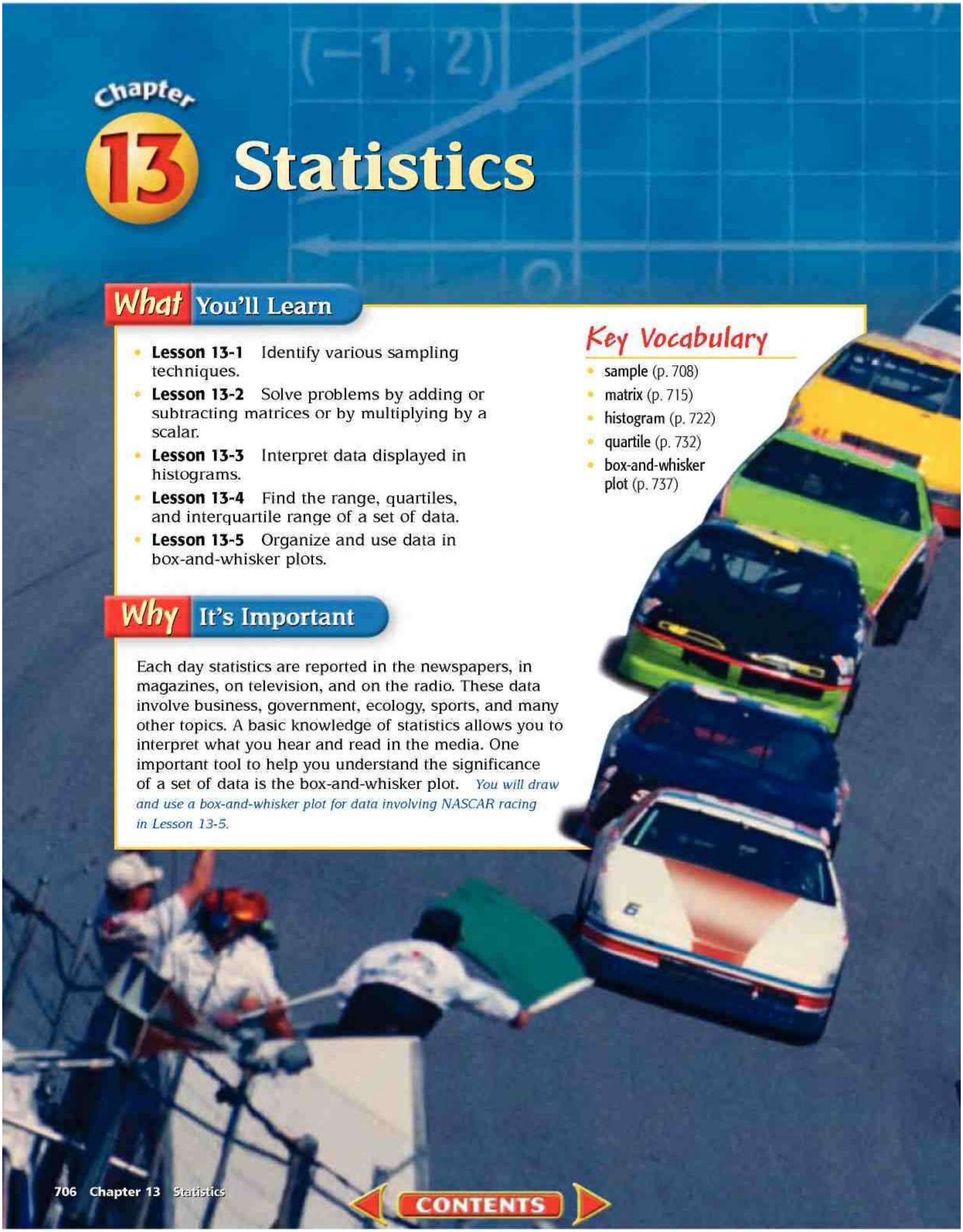
- **Lesson 13-1** Identify various sampling techniques.
- **Lesson 13-2** Solve problems by adding or subtracting matrices or by multiplying by a scalar.
- **Lesson 13-3** Interpret data displayed in histograms.
- **Lesson 13-4** Find the range, quartiles, and interquartile range of a set of data.
- **Lesson 13-5** Organize and use data in box-and-whisker plots.

Key Vocabulary

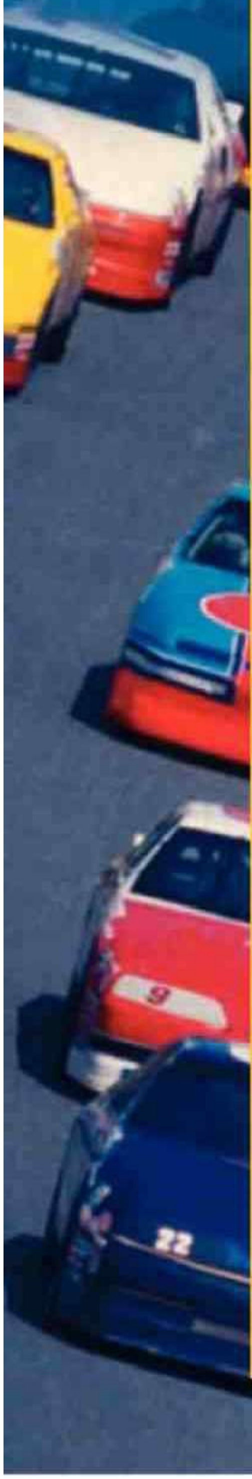
- **sample** (p. 708)
- **matrix** (p. 715)
- **histogram** (p. 722)
- **quartile** (p. 732)
- **box-and-whisker plot** (p. 737)

Why It's Important

Each day statistics are reported in the newspapers, in magazines, on television, and on the radio. These data involve business, government, ecology, sports, and many other topics. A basic knowledge of statistics allows you to interpret what you hear and read in the media. One important tool to help you understand the significance of a set of data is the box-and-whisker plot. *You will draw and use a box-and-whisker plot for data involving NASCAR racing in Lesson 13-5.*



CONTENTS



Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 13.

For Lesson 13-1 — Use Logical Reasoning

Find a counterexample for each statement. *(For review, see Lesson 1-7.)*

1. If $a + b = c$, then $a < c$.

2. If a flower is a rose, then it is red.

3. If Tara obeys the speed limit, then she will drive 45 miles per hour or less.

4. If a number is even, then it is divisible by 4.

For Lesson 13-4 — Find the Median

Find the median for each set of data. *(For review, see pages 818 and 819.)*

5. 1, 7, 9, 15, 25, 59, 63

6. 0, 10, 2, 2, 9, 5, 4, 2, 8, 3, 8, 7, 3

7. 726, 411, 407, 407, 395, 355, 317, 235, 218, 211

For Lesson 13-5 — Graph Numbers on a Number Line

Graph each set of numbers on a number line. *(For review, see Lesson 2-1.)*

8. {7, 9, 10, 13, 14}

9. {15, 17.5, 19, 20.5, 23}

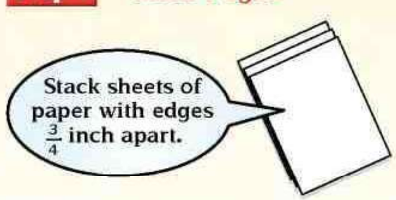
10. {3.2, 4.8, 5.0, 5.7, 6.1}

11. {2.3, 2.8, 3.1, 3.7, 4.5}

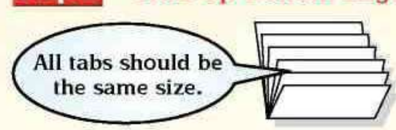
FOLDABLES™ Study Organizer

Make this Foldable to help you organize information about statistics. Begin with three sheets of plain $8\frac{1}{2}"$ by 11" paper.

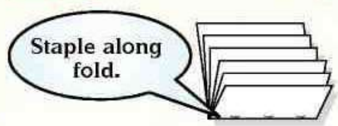
Step 1 Stack Pages



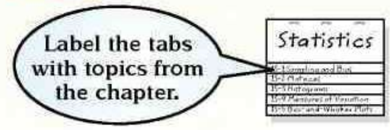
Step 2 Fold Up Bottom Edges



Step 3 Crease and Staple



Step 4 Turn and Label



Reading and Writing As you read and study the chapter, use each page to write notes and examples.

13-1 Sampling and Bias

What You'll Learn

- Identify various sampling techniques.
- Recognize a biased sample.

Why is sampling important in manufacturing?

Manufacturing music CDs involves burning, or recording, copies from a master. However, not every burn is successful. It is costly and time-consuming to check every CD that is burned. Therefore, in order to monitor production, some CDs are picked at random and checked for defects.

Vocabulary

- sample
- population
- census
- random sample
- simple random sample
- stratified random sample
- systematic random sample
- biased sample
- convenience sample
- voluntary response sample

SAMPLING TECHNIQUES When you wish to make an investigation, there are four ways that you can collect data.

- **published data** Use data that are already in a source like a newspaper or book.
- **observational study** Watch naturally occurring events and record the results.
- **experiment** Conduct an experiment and record the results.
- **survey** Ask questions of a group of people and record the results.

When performing an experiment or taking a survey, researchers often choose a sample. A **sample** is some portion of a larger group, called the **population**, selected to represent that group. If all of the units within a population are included, it is called a **census**. Sample data are often used to estimate a characteristic within an entire population, such as voting preferences prior to elections.

Population	Sample
all of the light bulbs manufactured on a production line	24 light bulbs selected from the production line
all of the water in a swimming pool	a test tube of water from the pool
all of the people in the United States	1509 people from throughout the United States

A **random sample** of a population is selected so that it is representative of the entire population. The sample is chosen without any preference. There are several ways to pick a random sample.

Key Concept — Random Samples

Type	Definition	Example
Simple Random Sample	A simple random sample is a sample that is as likely to be chosen as any other from the population.	The 26 students in a class are each assigned a different number from 1 to 26. Then three of the 26 numbers are picked at random.
Stratified Random Sample	In a stratified random sample, the population is first divided into similar, nonoverlapping groups. A simple random sample is then selected from each group.	The students in a school are divided into freshman, sophomores, juniors, and seniors. Then two students are randomly selected from each group of students.
Systematic Random Sample	In a systematic random sample, the items are selected according to a specified time or item interval.	Every 2 minutes, an item is pulled off the assembly line. or Every twentieth item is pulled off the assembly line.

Example 1 Classify a Random Sample

ECOLOGY Ten lakes are selected randomly from a list of all public-access lakes in Minnesota. Then 2 liters of water are drawn from 20 feet deep in each of the ten lakes.

a. Identify the sample and suggest a population from which it was selected.

The sample is ten 2-liter containers of lake water, one from each of 10 lakes. The population is lake water from all of the public-access lakes in Minnesota.

b. Classify the sample as *simple, stratified,* or *systematic*.

This is a simple random sample. Each of the ten lakes was equally likely to have been chosen from the list.

More About. . .

Ecology

More than one twentieth of the area of Minnesota is covered by inland lakes. The largest lake is Red Lake, which covers 430 square miles.

Source: *World Book Encyclopedia*

BIASED SAMPLE Random samples are unbiased. In a **biased sample**, one or more parts of a population are favored over others.

Example 2 Identify Sample as Biased or Unbiased

Identify each sample as *biased* or *unbiased*. Explain your reasoning.

a. MANUFACTURING Every 1000th bolt is pulled from the production line and measured for length.

The sample is chosen using a specified time interval. This is an unbiased sample because it is a systematic random sample.

b. MUSIC Every tenth customer in line for a certain rock band's concert tickets is asked about his or her favorite rock band.

The sample is a biased sample because customers in line for concert tickets are more likely to name the band giving the concert as a favorite band.

Two popular forms of samples that are often biased include convenience samples and voluntary response samples.

Key Concept — Biased Samples

Type	Definition	Example
Convenience Sample	A convenience sample includes members of a population that are easily accessed.	To check spoilage, a produce worker selects 10 apples from the top of the bin. The 10 apples are unlikely to represent all of the apples in the bin.
Voluntary Response Sample	A voluntary response sample involves only those who want to participate in the sampling.	A radio call-in show records that 75% of its 40 callers voiced negative opinions about a local football team. Those 40 callers are unlikely to represent the entire local population. Volunteer callers are more likely to have strong opinions and are typically more negative than the entire population.

Example 3 Identify and Classify a Biased Sample

BUSINESS The travel account records from 4 of the 20 departments in a corporation are to be reviewed. The accountant states that the first 4 departments to voluntarily submit their records will be reviewed.

a. Identify the sample and suggest a population from which it was selected.

The sample is the travel account records from 4 departments in the corporation. The population is the travel account records from all 20 departments in the corporation.

b. **Classify the sample as *convenience* or *voluntary response*.**

Since the departments voluntarily submit their records, this is a voluntary response sample.

Example 4 Identify the Sample

NEWS REPORTING For an article in the school paper, Rafael needs to determine whether students in his school believe that an arts center should be added to the school. He polls 15 of his friends who sing in the choir. Twelve of them think the school needs an arts center, so Rafael reports that 80% of the students surveyed support the project.

a. **Identify the sample.**

The sample is a group of students from the choir.

b. **Suggest a population from which the sample was selected.**

The population for the survey is all of the students in the school.

c. **State whether the sample is *unbiased* (random) or *biased*. If unbiased, classify it as *simple, stratified,* or *systematic*. If biased, classify it as *convenience* or *voluntary response*.**

The sample was not randomly selected from the entire student body. So the reported support is not likey to be representative of the student body. The sample is biased. Since Rafael polled only his friends, it is a convenience sample.

Check for Understanding

Concept Check

1. **Describe** how the following three types of sampling techniques are similar and how they are different.
   - simple random sample
   - stratified random sample
   - systematic random sample
2. **Explain** the difference between a convenience sample and a voluntary response sample.
3. **OPEN ENDED** Give an example of a biased sample.

Guided Practice

Identify each sample, suggest a population from which it was selected, and state whether it is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple, stratified,* or *systematic*. If biased, classify as *convenience* or *voluntary response*.

4. **NEWSPAPERS** The local newspaper asks readers to write letters stating their preferred candidate for mayor.
5. **SCHOOL** A teacher needs a sample of work from 4 students in her first-period math class to display at the school open house. She selects the work of the first 4 students who raise their hands.
6. **BUSINESS** A hardware store wants to assess the strength of nails it sells. Store personnel select 25 boxes at random from among all of the boxes on the shelves. From each of the 25 boxes, they select one nail at random and subject it to a strength test.
7. **SCHOOL** A class advisor hears complaints about an incorrect spelling of the school name on pencils sold at the school store. The advisor goes to the store and asks Namid to gather a sample of pencils and look for spelling errors. Namid grabs the closest box of pencils and counts out 12 pencils from the top of the box. She checks the pencils, returns them to the box, and reports the results to the advisor.

Practice and Apply

Homework Help

For Exercises	See Examples
8–28	1–4

Extra Practice

See page 849.

Identify each sample, suggest a population from which it was selected, and state whether it is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple*, *stratified*, or *systematic*. If biased, classify as *convenience* or *voluntary response*.

8. **SCHOOL** Pieces of paper with the names of 3 sophomores are drawn from a hat containing identical pieces of paper with all sophomores' names.

9. **FOOD** Twenty shoppers outside a fast-food restaurant are asked to name their preferred cola among two choices.

10. **RECYCLING** An interviewer goes from house to house on weekdays between 9 A.M. and 4 P.M. to determine how many people recycle.

11. **POPULATION** A state is first divided into its 86 counties and then 10 people from each county are chosen at random.

12. **SCOOTERS** A scooter manufacturer is concerned about quality control. The manufacturer checks the first 5 scooters off the line in the morning and the last 5 off the line in the afternoon for defects.

13. **SCHOOL** To determine who will speak for her class at the school board meeting, Ms. Finchie used the numbers appearing next to her students' names in her grade book. She writes each of the numbers on an identical piece of paper and shuffles the pieces of papers in a box. Without seeing the contents of the box, one student draws 3 pieces of paper from the box. The students whose numbers match the numbers chosen will speak for the class.

14. **FARMING** An 8-ounce jar was filled with corn from a storage silo by dipping the jar into the pile of corn. The corn in the jar was then analyzed for moisture content.

15. **COURT** The gender makeup of district court judges in the United States is to be estimated from a sample. All judges are grouped geographically by federal reserve districts. Within each of the 11 federal reserve districts, all judges' names are assigned a distinct random number. In each district, the numbers are then listed in order. A number between 1 and 20 inclusive is selected at random, and the judge with that number is selected. Then every 20th name after the first selected number is also included in the sample.

16. **TELEVISION** A television station asks its viewers to share their opinions about a proposed golf course to be built just outside the city limits. Viewers can call one of two 900-numbers. One number represents a "yes" vote, and the other number represents a "no" vote.

17. **GOVERNMENT** To discuss leadership issues shared by all United States Senators, the President asks 4 of his closest colleagues in the Senate to meet with him.

18. **FOOD** To sample the quality of the Bing cherries throughout the produce department, the produce manager picks up a handful of cherries from the edge of one case and checks to see if these cherries are spoiled.

19. **MANUFACTURING** During the manufacture of high-definition televisions, units are checked for defects. Within the first 10 minutes of a work shift, a television is randomly chosen from the line of completed sets. For the rest of the shift, every 15th television on the line is checked for defects.

More About. . .

Food

Michigan leads the nation in cherry production by growing about 219 million pounds of cherries per year.

Source: *World Book Encyclopedia*

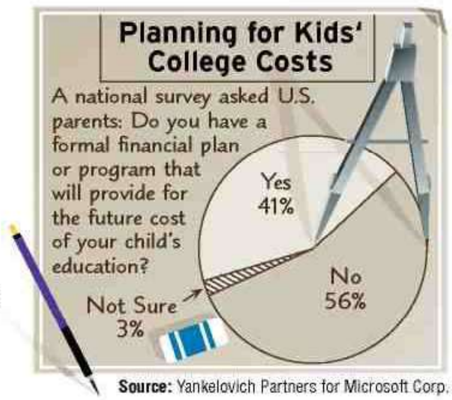
Identify each sample, suggest a population from which it was selected, and state whether it is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple, stratified,* or *systematic*. If biased, classify as *convenience* or *voluntary response*.

20. **BUSINESS** To get reaction about a benefits package, a company uses a computer program to randomly pick one person from each of its departments.

21. **MOVIES** A magazine is trying to determine the most popular actor of the year. It asks its readers to mail the name of their favorite actor to the magazine's office.

COLLEGE For Exercises 22 and 23, use the following information.

The graph at the right reveals that 56% of survey respondents did not have a formal financial plan for a child's college tuition.



Source: Yankelovich Partners for Microsoft Corp.

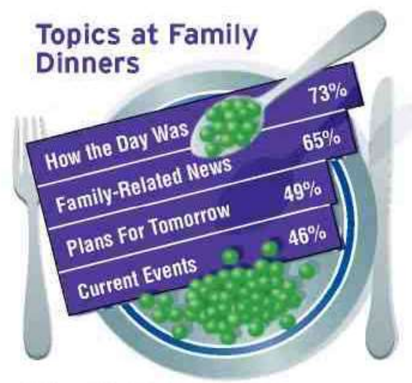
22. Write a statement to describe what you do know about the sample.

23. What additional information would you like to have about the sample to determine whether the sample is biased?

24. **SCHOOL** Suppose you want to sample the opinion of the students in your school about a new dress code. Describe an unbiased way to conduct your survey.

25. **ELECTIONS** Suppose you are running for mayor of your city and want to know if you are likely to be elected. Describe an unbiased way to poll the voters.

26. **FAMILY** Study the graph at the right. Describe the information that is revealed in the graph. What information is there about the type or size of the sample?



Source: National Pork Producers Council

27. **FARMING** Suppose you are a farmer and want to know if your tomato crop is ready to harvest. Describe an unbiased way to determine whether the crop is ready to harvest.

28. **MANUFACTURING** Suppose you want to know whether the infant car seats manufactured by your company meet the government standards for safety. Describe an unbiased way to determine whether the seats meet the standards.

29. **CRITICAL THINKING** The following is a proposal for surveying a stratified random sample of the student body.

    Divide the student body according to those who are on the basketball team, those who are in the band, and those who are in the drama club. Then take a simple random sample from each of the three groups. Conduct the survey using this sample.

    Study the proposal. Describe its strengths and weaknesses. Is the sample a stratified random sample? Explain.

30. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

Why is sampling important in manufacturing?

Include the following in your answer:
- an unbiased way to pick which CDs to check, and
- a biased way to pick which CDs to check.

Standardized Test Practice

31. To predict the candidate who will win the seat in city council, which method would give the newspaper the most accurate result?
   (A) Ask every 5th person that passes a reporter in the mall.
   (B) Use a list of registered voters and call every 20th person.
   (C) Publish a survey and ask readers to reply.
   (D) Ask reporters at the newspaper.

32. A cookie manufacturer plans to make a new type of cookie and wants to know if people will buy these cookies. For accurate results, which method should they use?
   (A) Ask visitors to their factory to evaluate the cookie.
   (B) Place a sample of the new cookie with their other cookies, and ask people to answer a questionnaire about the cookie.
   (C) Take samples to a school, and ask students to raise their hands if they like the cookie.
   (D) Divide the United States into 6 regions. Then pick 3 cities in each region at random, and conduct a taste test in each of the 18 cities.

Maintain Your Skills

Mixed Review

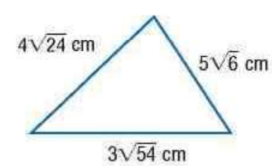
Solve each equation. *(Lesson 12-9)*

33. $\frac{10}{3y} - \frac{5}{2y} = \frac{1}{4}$ 34. $\frac{3}{r+4} - \frac{1}{r} = \frac{1}{r}$ 35. $\frac{1}{4m} + \frac{2m}{m-3} = 2$

Simplify. *(Lesson 12-8)*

36. $\dfrac{2 + \frac{5}{x}}{\frac{x}{3} + \frac{5}{6}}$ 37. $\dfrac{a + \frac{35}{a+12}}{a+7}$ 38. $\dfrac{\frac{t^2-4}{t^2+5t+6}}{t-2}$

39. **GEOMETRY** What is the perimeter of $\triangle ABC$? *(Lesson 11-2)*



Solve each equation by using the Quadratic Formula. Approximate any irrational roots to the nearest tenth. *(Lesson 10-4)*

40. $x^2 - 6x - 40 = 0$ 41. $6b^2 + 15 = -19b$ 42. $2d^2 = 9d + 3$

Find each product. *(Lesson 8-7)*

43. $(y + 5)(y + 7)$ 44. $(c - 3)(c - 7)$ 45. $(x + 4)(x - 8)$

Getting Ready for the Next Lesson

BASIC SKILL **Find each sum or difference.**

46. $4.5 + 3.8$ 47. $16.9 + 7.21$ 48. $3.6 + 18.5$
49. $7.6 - 3.8$ 50. $18 - 4.7$ 51. $13.2 - 0.75$

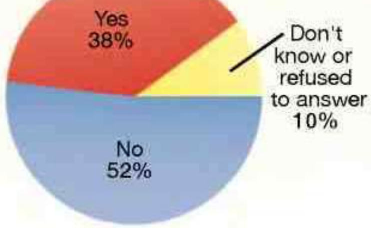
Reading Mathematics

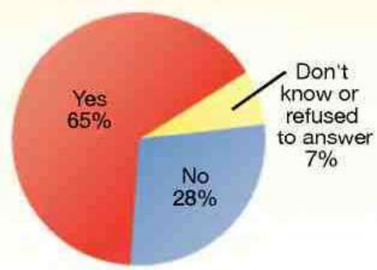
Survey Questions

Even though taking a random sample eliminates bias or favoritism in the choice of a sample, questions may be worded to influence people's thoughts in a desired direction. Two different surveys on Internet sales tax had different results.

Question 1
Should there be sales tax on purchases made on the Internet?

Question 2
Do you think people should or should not be required to pay the same sales tax for purchases made over the Internet that they would if they had bought the item in person at a local store?





Notice the difference in Questions 1 and 2. Question 2 includes more information. Pointing out that customers pay sales tax for items bought at a local store may give the people answering the survey a reason to answer "yes." Asking the question in that way probably led people to answer the way they did.

Because they are random samples, the results of both of these surveys are accurate. However, the results could be used in a misleading way by someone with an interest in the issue. For example, an Internet retailer would prefer to state the results of Question 1. Be sure to think about survey questions carefully to make sure that you interpret the results correctly.

Reading to Learn

For Exercises 1–2, tell whether each question is likely to bias the results. Explain your reasoning.

1. On a survey on environmental issues:
   - **a.** "Due to diminishing resources, should a law be made to require recycling?"
   - **b.** "Should the government require citizens to participate in recycling efforts?"
2. On a survey on education:
   - **a.** "Should schools fund extracurricular sports programs?"
   - **b.** "The budget of the River Valley School District is short of funds. Should taxes be raised in order for the district to fund extracurricular sports programs?"
3. Suppose you want to determine whether to serve hamburgers or pizza at the class party.
   - **a.** Write a survey question that would likely produce biased results.
   - **b.** Write a survey question that would likely produce unbiased results.

13-2 Introduction to Matrices

What You'll Learn

- Organize data in matrices.
- Solve problems by adding or subtracting matrices or by multiplying by a scalar.

Vocabulary

- matrix
- dimensions
- row
- column
- element
- scalar multiplication

How are matrices used to organize data?

To determine the best type of aircraft to use for certain flights, the management of an airline company considers the following aircraft operating statistics.

Aircraft	Number of Seats	Airborne Speed (mph)	Possible Flight Distance (miles)	Fuel per Hour (gallons)	Operating Cost per Hour (dollars)
B747-100	462	512	2297	3517	7224
DC-10-10	297	496	1402	2311	5703
MD-11	259	527	3073	2464	6539
A300-600	228	475	1372	1505	4783

Source: Air Transport Association of America

The table has rows and columns of information. When we concentrate only on the numerical information, we see an array with 4 rows and 5 columns.

$$\begin{bmatrix} 462 & 512 & 2297 & 3517 & 7224 \\ 297 & 496 & 1402 & 2311 & 5703 \\ 259 & 527 & 3073 & 2464 & 6539 \\ 228 & 475 & 1372 & 1505 & 4783 \end{bmatrix}$$

This array of numbers is called a matrix.

ORGANIZE DATA IN MATRICES If you have ever used a spreadsheet program on the computer, you have worked with matrices. A **matrix** is a rectangular arrangement of numbers in rows and columns. A matrix is usually described by its **dimensions**, or the number of **rows** and **columns**, with the number of rows stated first. Each entry in a matrix is called an **element**.

Example 1 Name Dimensions of Matrices

State the dimensions of each matrix. Then identify the position of the circled element in each matrix.

a. $[11 \quad 15 \quad 24]$

This matrix has 1 row and 3 columns. Therefore, it is a 1-by-3 matrix.

The circled element is in the first row and the second column.

b. $\begin{bmatrix} -4 & 2 \\ 0 & 1 \\ 3 & -6 \end{bmatrix}$

This matrix has 3 rows and 2 columns. Therefore, it is a 3-by-2 matrix.

The circled element is in the third row and the first column.

Two matrices are *equal* only if they have the same dimensions and each element of one matrix is equal to the corresponding element in the other matrix.

$$\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix} \qquad \begin{bmatrix} 4 & 8 \\ 1 & -3 \end{bmatrix} \neq \begin{bmatrix} 4 & 8 & 0 \\ 1 & -3 & 0 \end{bmatrix}$$

MATRIX OPERATIONS If two matrices have the same dimensions, you can add or subtract them. To do this, add or subtract corresponding elements of the two matrices.

Example 2 Add Matrices

If $A = \begin{bmatrix} 3 & -4 & 7 \\ -1 & 6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -4 & -2 \\ 1 & 6 & -3 \end{bmatrix}$, **and** $C = \begin{bmatrix} 3 & 6 \\ -4 & 5 \end{bmatrix}$, **find each sum. If the sum does not exist, write *impossible*.**

a. $A + B$

$$A + B = \begin{bmatrix} 3 & -4 & 7 \\ -1 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 7 & -4 & -2 \\ 1 & 6 & -3 \end{bmatrix} \quad \text{Substitution}$$

$$= \begin{bmatrix} 3+7 & -4+(-4) & 7+(-2) \\ -1+1 & 6+6 & 0+(-3) \end{bmatrix} \quad \text{Definition of matrix addition}$$

$$= \begin{bmatrix} 10 & -8 & 5 \\ 0 & 12 & -3 \end{bmatrix} \quad \text{Simplify.}$$

b. $B + C$

$$B + C = \begin{bmatrix} 7 & -4 & -2 \\ 1 & 6 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -4 & 5 \end{bmatrix} \quad \text{Substitution}$$

Since B is a 2-by-3 matrix and C is a 2-by-2 matrix, the matrices do not have the same dimensions. Therefore, it is impossible to add these matrices.

More About. . .

College Football

Each year the National Football Foundation awards the MacArthur Bowl to the number one college football team. The bowl is made of about 400 ounces of silver and represents a stadium with rows of seats.

Source: *ESPN Information Please® Sports Almanac*

Addition and subtraction of matrices can be used to solve real-world problems.

Example 3 Subtract Matrices

COLLEGE FOOTBALL **The Division I-A college football teams with the five best records during the 1990s are listed below.**

	Overall Record		
	Wins	Losses	Ties
Florida State	109	13	1
Nebraska	108	16	1
Marshall	114	25	0
Florida	102	22	1
Tennessee	99	22	2

	Bowl Record		
	Wins	Losses	Ties
Florida State	8	2	0
Nebraska	5	5	0
Marshall	2	1	0
Florida	5	4	0
Tennessee	6	4	0

Use subtraction of matrices to determine the regular season records of these teams during the decade.

$$\begin{bmatrix} 109 & 13 & 1 \\ 108 & 16 & 1 \\ 114 & 25 & 0 \\ 102 & 22 & 1 \\ 99 & 22 & 2 \end{bmatrix} - \begin{bmatrix} 8 & 2 & 0 \\ 5 & 5 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 0 \\ 6 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 109-8 & 13-2 & 1-0 \\ 108-5 & 16-5 & 1-0 \\ 114-2 & 25-1 & 0-0 \\ 102-5 & 22-4 & 1-0 \\ 99-6 & 22-4 & 2-0 \end{bmatrix}$$

$$= \begin{bmatrix} 101 & 11 & 1 \\ 103 & 11 & 1 \\ 112 & 24 & 0 \\ 97 & 18 & 1 \\ 93 & 18 & 2 \end{bmatrix}$$

CONTENTS

So, the regular season records of the teams can be described as follows.

$$\begin{array}{l} \\ \\ \textbf{Florida State} \\ \textbf{Nebraska} \\ \textbf{Marshall} \\ \textbf{Florida} \\ \textbf{Tennessee} \end{array} \begin{array}{c} \textbf{Regular Season Record} \\ \begin{array}{ccc} \textbf{Wins} & \textbf{Losses} & \textbf{Ties} \end{array} \\ \begin{bmatrix} 101 & 11 & 1 \\ 103 & 11 & 1 \\ 112 & 24 & 0 \\ 97 & 18 & 1 \\ 93 & 18 & 2 \end{bmatrix} \end{array}$$

You can multiply any matrix by a constant called a *scalar*. This is called **scalar multiplication**. When scalar multiplication is performed, each element is multiplied by the scalar and a new matrix is formed.

Key Concept — Scalar Multiplication of a Matrix

$$m\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ma & mb & mc \\ md & me & mf \end{bmatrix}$$

Example 4 Perform Scalar Multiplication

If $T = \begin{bmatrix} -4 & 2 \\ 0 & 1 \\ 3 & -6 \end{bmatrix}$, find $3T$.

$3T = 3\begin{bmatrix} -4 & 2 \\ 0 & 1 \\ 3 & -6 \end{bmatrix}$ Substitution

$= \begin{bmatrix} 3(-4) & 3(2) \\ 3(0) & 3(1) \\ 3(3) & 3(-6) \end{bmatrix}$ Definition of scalar multiplication

$= \begin{bmatrix} -12 & 6 \\ 0 & 3 \\ 9 & -18 \end{bmatrix}$ Simplify.

Check for Understanding

Concept Check

1. **Describe** the difference between a 2-by-4 matrix and a 4-by-2 matrix.

2. **OPEN ENDED** Write two matrices whose sum is $\begin{bmatrix} 0 & 4 & 5 & -3 \\ 1 & -1 & 4 & 9 \end{bmatrix}$.

3. **FIND THE ERROR** Hiroshi and Estrella are finding $-5\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}$.

Hiroshi	Estrella
$-5\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 10 & 5 \end{bmatrix}$	$-5\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -15 \\ 10 & -25 \end{bmatrix}$

Who is correct? Explain your reasoning.

Guided Practice

State the dimensions of each matrix. Then, identify the position of the circled element in each matrix.

4. $\begin{bmatrix} 4 & 0 & 2 \\ 5 & -1 & -3 \\ 6 & 2 & 7 \end{bmatrix}$

5. $\begin{bmatrix} 3 & -3 & 1 & 9 \end{bmatrix}$

6. $\begin{bmatrix} 5 \\ 2 \\ 1 \\ -3 \end{bmatrix}$

7. $\begin{bmatrix} 0.6 & 4.2 \\ -1.7 & 1.05 \\ 0.625 & -2.1 \end{bmatrix}$

If $A = \begin{bmatrix} 20 & -10 \\ 12 & 19 \end{bmatrix}$, $B = \begin{bmatrix} 15 & 14 \\ -10 & 6 \end{bmatrix}$, and $C = \begin{bmatrix} -5 & 7 \end{bmatrix}$, find each sum, difference, or product. If the sum or difference does not exist, write *impossible*.

8. $A + C$ **9.** $B - A$ **10.** $2A$ **11.** $-4C$

Application

PIZZA SALES **For Exercises 12–16, use the following tables that list the number of pizzas sold at Sylvia's Pizza one weekend.**

FRIDAY	Small	Medium	Large
Thin Crust	12	10	3
Thick Crust	11	8	8
Deep Dish	14	8	10

SATURDAY	Small	Medium	Large
Thin Crust	13	12	11
Thick Crust	1	5	10
Deep Dish	8	11	2

SUNDAY	Small	Medium	Large
Thin Crust	11	8	6
Thick Crust	1	8	11
Deep Dish	10	15	11

12. Create a matrix for each day's data. Name the matrices F, R, and N, respectively.

13. Does F equal R? Explain.

14. Create matrix T to represent $F + R + N$.

15. What does T represent?

16. Which type of pizza had the most sales during the entire weekend?

Practice and Apply

Homework Help

For Exercises	See Examples
17–26	1
27–38	2–4
39–48	3

Extra Practice

See page 849.

State the dimensions of each matrix. Then, identify the position of the circled element in each matrix.

17. $\begin{bmatrix} 2 & 1 \\ 5 & -8 \end{bmatrix}$

18. $\begin{bmatrix} -36 & 3 \\ 25 & -1 \\ 11 & 14 \end{bmatrix}$

19. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

20. $\begin{bmatrix} -3 & 56 & -21 \\ 60 & 112 & -65 \end{bmatrix}$

21. $\begin{bmatrix} -4 & 0 & -2 \\ 5 & 1 & 12 \\ -6 & 3 & -7 \end{bmatrix}$

22. $\begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 1 & 5 \\ -1 & 7 \end{bmatrix}$

23. $\begin{bmatrix} -5 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$

24. $\begin{bmatrix} -6 & 3 \\ 5 & -4 \end{bmatrix}$

25. Create a 2-by-3 matrix with 2 in the first row and first column and 5 in the second row and second column. The rest of the elements should be ones.

26. Create a 3-by-2 matrix with 8 in the second row and second column and 4 in the third row and second column. The rest of the elements should be zeros.

If $A = \begin{bmatrix} -1 & 5 & 9 \\ 0 & -4 & -2 \\ 3 & 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -12 & 7 & -16 \\ 5 & 10 & 13 \\ 20 & 11 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 34 & 91 & 63 \\ 81 & 79 & 60 \end{bmatrix}$, **and**

$D = \begin{bmatrix} -52 & 9 & 70 \\ -49 & -8 & 45 \end{bmatrix}$, **find each sum, difference, or product. If the sum or difference does not exist, write *impossible*.**

27. $A + B$	**28.** $C + D$	**29.** $C - D$	**30.** $B - A$
31. $5A$	**32.** $2C$	**33.** $A + C$	**34.** $B + D$
35. $2B + A$	**36.** $4A - B$	**37.** $2C - 3D$	**38.** $5D + 2C$

FOOD **For Exercises 39–41, use the table that shows the nutritional value of food.**

Food	Calories	Protein (grams)	Fat (grams)	Saturated Fat (grams)
Fish Stick	70	6	3	0.8
Vegetable Soup (1 cup)	70	2	2	0.3
Soft Drink (12 oz)	160	0	0	0
Chocolate-Chip Cookie	185	2	11	3.9

Source: U.S.Department of Agriculture

39. If $F = [70 \quad 6 \quad 3 \quad 0.8]$ is a matrix representing the nutritional value of a fish stick, create matrices V, S, and C to represent vegetable soup, soft drink, and chocolate chip cookie, respectively.

40. Suppose Lakeisha has two fish sticks for lunch. Write a matrix representing the nutritional value of the fish sticks.

41. Suppose Lakeisha has two fish sticks, a cup of vegetable soup, a 12-ounce soft drink, and a chocolate chip cookie. Write a matrix representing the nutritional value of her lunch.

FUND-RAISING **For Exercises 42–44, use the table that shows the last year's sales of T-shirts for the student council fund-raiser.**

Color	XS	S	M	L	XL
Red	18	28	32	24	21
White	24	30	45	47	25
Blue	17	19	26	30	28

42. Create a matrix to show the number of T-shirts sold last year according to size and color. Label this matrix N.

43. The student council anticipates a 20% increase in T-shirt sales this year. What value of the scalar r should be used so that rN results in a matrix that estimates the number of each size and color T-shirts needed this year?

44. Calculate rN, rounding appropriately, to show estimates for this year's sales.

CONTENTS

FOOTBALL For Exercises 45–48, use the table that shows the passing performance of four National Football League quarterbacks.

1999 Regular Season

Quarterback	Attempts	Completions	Passing Yards	Touchdowns	Interceptions
Peyton Manning	533	331	4135	26	15
Rich Gannon	515	304	3840	24	14
Kurt Warner	499	325	4353	41	13
Steve Beuerlein	571	343	4436	36	15

2000 Regular Season

Quarterback	Attempts	Completions	Passing Yards	Touchdowns	Interceptions
Peyton Manning	571	357	4413	33	15
Rich Gannon	473	284	3430	28	11
Kurt Warner	347	235	3429	21	18
Steve Beuerlein	533	324	3730	19	18

Source: ESPN

45. Create matrix A for the 1999 data and matrix B for the 2000 data.

46. What are the dimensions of each matrix in Exercise 45?

47. Calculate $T = A + B$.

48. What does matrix T represent?

49. CRITICAL THINKING Suppose M and N are each 3-by-3 matrices. Determine whether each statement is *sometimes*, *always*, or *never* true.

a. $M = N$

b. $M + N = N + M$

c. $M - N = N - M$

d. $5M = M$

e. $M + N = M$

f. $5M = N$

50. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are matrices used to organize data?

Include the following in your answer:

- a comparison of a table and a matrix, and
- description of some real-world data that could be organized in a matrix.

Standardized Test Practice

51. Which of the following is equal to $\begin{bmatrix} 3 & 4 & 5 \\ -6 & -1 & 8 \end{bmatrix}$?

Ⓐ $\begin{bmatrix} -1 & 8 & 3 \\ -4 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 2 \\ 2 & -1 & -2 \end{bmatrix}$

Ⓑ $\begin{bmatrix} 7 & -1 & 2 \\ 3 & 4 & -5 \end{bmatrix} + \begin{bmatrix} -4 & -3 & 3 \\ -3 & -5 & -3 \end{bmatrix}$

Ⓒ $\begin{bmatrix} 1 & -3 & 5 \\ 7 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 7 & 0 \\ -13 & 1 & 8 \end{bmatrix}$

Ⓓ $\begin{bmatrix} 5 & 9 & -2 \\ 3 & 7 & 5 \end{bmatrix} + \begin{bmatrix} -2 & -5 & -3 \\ 3 & -8 & 3 \end{bmatrix}$

52. Suppose M and N are each 2-by-2 matrices. If $M + N = M$, which of the following is true?

Ⓐ $N = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Ⓑ $N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ⓒ $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ⓓ $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Graphing Calculator

MATRIX OPERATIONS **You can use a graphing calculator to perform matrix operations. Use the EDIT command on the MATRX menu of a TI-83 Plus to enter each of the following matrices.**

$$A = \begin{bmatrix} 7.9 & 5.4 & -6.8 \\ -5.9 & 4.4 & -7.7 \end{bmatrix}, B = \begin{bmatrix} -7.2 & -5.8 & 9.1 \\ 4.3 & -8.4 & 5.3 \end{bmatrix}, C = \begin{bmatrix} 9.8 & -1.2 & 5.2 \\ -7.8 & 5.1 & -9.0 \end{bmatrix}$$

Use these stored matrices to find each sum, difference, or product.

53. $A + B$ **54.** $C - B$ **55.** $B + C - A$ **56.** $1.8A$ **57.** $0.4C$

Maintain Your Skills

Mixed Review

PRINTING **For Exercises 58 and 59, use the following information.**
To determine the quality of calendars printed at a local shop, the last 10 calendars printed each day are examined. *(Lesson 13-1)*

58. Identify the sample.

59. State whether it is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple*, *stratified*, or *systematic*. If biased, classify as *convenience* or *voluntary response*.

Solve each equation. *(Lesson 12-9)*

60. $\frac{-4}{a+1} + \frac{3}{a} = 1$ **61.** $\frac{3}{x} + \frac{4x}{x-3} = 4$ **62.** $\frac{d+3}{d+5} + \frac{2}{d-9} = \frac{5}{2d+10}$

Find the *n*th term of each geometric sequence. *(Lesson 10-7)*

63. $a_1 = 4, n = 5, r = 3$ **64.** $a_1 = -2, n = 3, r = 7$ **65.** $a_1 = 4, n = 5, r = -2$

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lesson 9-3)*

66. $b^2 + 7b + 12$ **67.** $a^2 + 2ab - 3b^2$ **68.** $d^2 + 8d - 15$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **For Exercises 69 and 70, use the graph that shows the amount of money in Megan's savings account.** *(To review **interpreting graphs**, see Lesson 1-9.)*

69. Describe what is happening to Megan's bank balance. Give possible reasons why the graph rises and falls at particular points.

70. Describe the elements in the domain and range.

Practice Quiz 1

Lessons 13-1 and 13-2

Identify each sample, suggest a population from which it was selected, and state whether it is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple*, *stratified*, or *systematic*. If biased, classify as *convenience* or *voluntary response*. *(Lesson 13-1)*

1. Every other household in a neighborhood is surveyed to determine how to improve the neighborhood park.

2. Every other household in a neighborhood is surveyed to determine the favorite candidate for the state's governor.

Find each sum, difference, or product. *(Lesson 13-2)*

3. $\begin{bmatrix} -8 & 3 \\ -4 & -9 \end{bmatrix} + \begin{bmatrix} 5 & -7 \\ -1 & 0 \end{bmatrix}$ **4.** $\begin{bmatrix} -9 & 6 & 4 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 7 & -2 & 8 \\ 5 & -3 & 1 \end{bmatrix}$ **5.** $3\begin{bmatrix} 8 & -3 & -4 & 5 \\ 6 & -1 & 2 & 10 \end{bmatrix}$

13-3 Histograms

What You'll Learn

- Interpret data displayed in histograms.
- Display data in histograms.

Vocabulary

- frequency table
- histogram
- measurement classes
- frequency

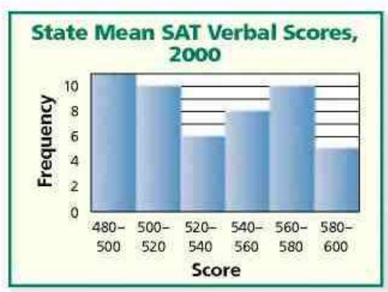
How are histograms used to display data?

A **frequency table** shows the frequency of events. The frequency table below shows the number of states with the mean SAT verbal and mathematics scores in each score interval. The data are from the 1999–2000 school year.

SAT Scores		
Score Interval	Verbal	Mathematics
	Number of States	Number of States
$480 \leq s < 500$	11	5
$500 \leq s < 520$	10	18
$520 \leq s < 540$	6	5
$540 \leq s < 560$	8	10
$560 \leq s < 580$	10	5
$580 \leq s < 600$	5	5
$600 \leq s < 620$	0	2

Source: The College Board

The distribution of the mean scores on the SAT verbal exam is displayed in the graph.



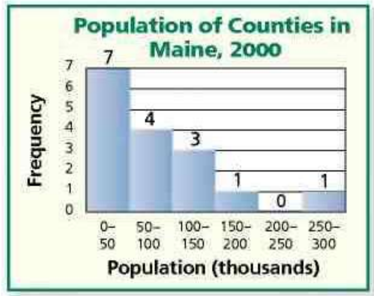
INTERPRET DATA IN HISTOGRAMS The graph above is called a histogram. A **histogram** is a bar graph in which the data are organized into equal intervals. In the histogram above, the horizontal axis shows the range of data values separated into **measurement classes**, and the vertical axis shows the number of values, or the **frequency**, in each class. Consider the histogram shown below.

Source: *The World Almanac*

A histogram is a visual summary of a frequency table.

Example 1 Determine Information from a Histogram

GEOGRAPHY **Answer each question about the histogram shown below.**



a. In what measurement class does the median occur?

First, add the frequencies to determine the number of counties in Maine.

$7 + 4 + 3 + 1 + 0 + 1 = 16$

There are 16 counties, so the middle data value is between the 8th and 9th data values. Both the 8th and 9th data values are located in the 50–100 thousand measurement class. Therefore, the median occurs in the 50–100 thousand measurement class.

b. Describe the distribution of the data.

- Only two counties have populations above 150 thousand. It is likely that these counties contain the largest cities in Maine.
- There is a gap in the 200–250 thousand measurement class.
- Most of the counties have populations below 150 thousand.
- As population increases, the histogram shows that the number of counties decreases. We say that the distribution is *skewed*, or pulled in one direction away from the center. This distribution is *skewed to the left* because the majority of the data are located at the low end of the scale.

Study Tip

Look Back
To review **median**, see pages 818 and 819.

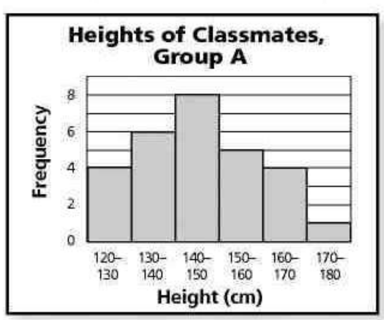
You can sometimes use the appearances of histograms to compare data.

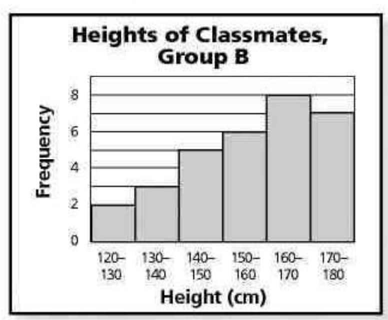
Standardized Test Practice

Example 2 Compare Data in Histograms

Multiple-Choice Test Item

Which group of students has a greater median height?





(A) Group A
(B) Group B
(C) The medians are about the same.
(D) cannot be determined

(continued on the next page)

www.algebra1.com/extra_examples

The Princeton Review

Test-Taking Tip

When answering a test question involving a graph, always read the labels on the graph carefully.

Read the Test Item

You have two histograms depicting the heights of two groups of students. You are asked to determine which group of students has a greater median height.

Solve the Test Item

Study the histograms carefully. The measurement classes and the frequency scales are the same for each histogram. The distribution for Group A is somewhat *symmetrical* in shape, while the distribution for Group B is *skewed to the right*. This would indicate that Group B has the greater median height. To check this assumption, locate the measurement class of each median.

Group A

$4 + 6 + 8 + 5 + 4 + 1 = 28$

The median is between the 14th and 15th data values. The median is in the 140–150 measurement class.

Group B

$2 + 3 + 5 + 6 + 8 + 7 = 31$

The median is the 16th data value. The median is in the 150–160 measurement class.

This confirms that Group B has the greater median height. The answer is B.

DISPLAY DATA IN A HISTOGRAM

Data from a list or a frequency table can be used to create a histogram.

Example 3 Create a Histogram

SCHOOL Create a histogram to represent the following scores for a 50-point mathematics test.

40, 34, 38, 23, 41, 39, 39, 34, 43, 44, 32, 44, 41, 39,
22, 47, 36, 25, 41, 30, 28, 37, 39, 33, 30, 40, 28

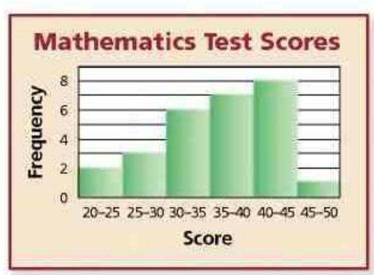
Step 1 Identify the greatest and least values in the data set.
The test scores range from 22 to 47 points.

Step 2 Create measurement classes of equal width.
For these data, use measurement classes from 20 to 50 with a 5-point interval for each class.

Step 3 Create a frequency table using the measurement classes.

Score Intervals	Tally	Frequency
$20 \leq s < 25$	II	2
$25 \leq s < 30$	III	3
$30 \leq s < 35$	~~IIII~~ I	6
$35 \leq s < 40$	~~IIII~~ II	7
$40 \leq s < 45$	~~IIII~~ III	8
$45 \leq s < 50$	I	1

Step 4 Draw the histogram.
Use the measurement classes to determine the scale for the horizontal axis and the frequency values to determine the scale for the vertical axis. For each measurement class, draw a rectangle as wide as the measurement class and as tall as the frequency for the class. Label the axes and include a descriptive title for the histogram.



CONTENTS

Check for Understanding

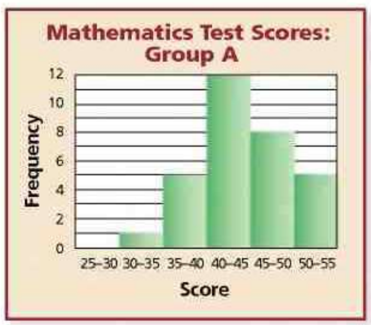
Concept Check

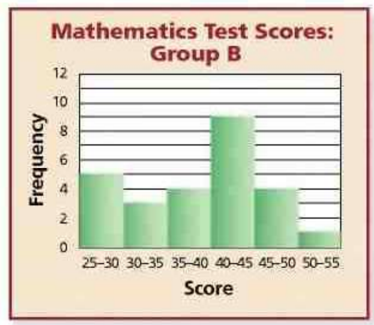
1. **Describe** how to create a histogram.
2. **Write** a compound inequality to represent all of the values v included in a 50–60 measurement class.
3. **OPEN ENDED** Write a set of data whose histogram would be skewed to the left.

Guided Practice

MONEY For Exercises 4 and 5, use the following histogram that shows the amount of money spent by several families during a holiday weekend.

4. In what measurement class does the median occur?
5. Describe the distribution of the data.





SCHOOL For Exercises 6 and 7, use the following histograms.

6. Compare the medians of the two data sets.
7. Compare and describe the overall shape of each distribution of data.
8. **AIR TRAVEL** The busiest U.S. airports as determined by the number of passengers arriving and departing are listed below. Create a histogram.

Passenger Traffic at U.S. Airports, 2000			
Airport	Passengers (millions)	Airport	Passengers (millions)
Atlanta (Hartsfield)	80	Minneapolis/St. Paul	37
Chicago (O'Hare)	72	Phoenix (Sky Harbor)	36
Los Angeles	68	Detroit	36
Dallas/Fort Worth	61	Houston (George Bush)	35
San Francisco	41	Newark	34
Denver	39	Miami	34
Las Vegas (McCarran)	37	New York (JFK)	33

Source: Airports Council International

Online Research **Data Update** What are the current busiest airports? Visit www.algebra1.com/data_update to get statistics on airports.

Standardized Test Practice

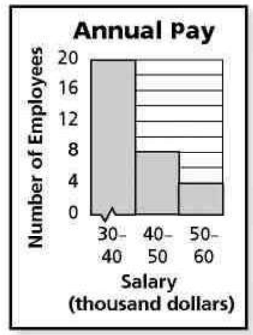
9. Which statement about the graph at the right is *not* correct?

   Ⓐ The data are skewed to the left.

   Ⓑ The median is in the 40–50 thousand measurement class.

   Ⓒ There are 32 employees represented by the graph.

   Ⓓ The width of each measurement class is \$10 thousand.



Practice and Apply

Homework Help

For Exercises	See Examples
10, 11	1
12, 13	2
14–20	3

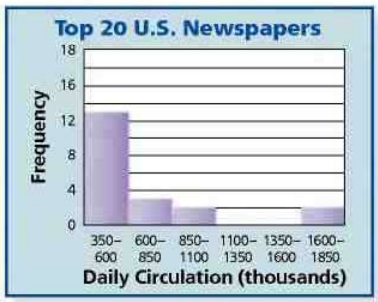
Extra Practice

See page 850.

For each histogram, answer the following.

- **In what measurement class does the median occur?**
- **Describe the distribution of the data.**

10.



Source: *Editor & Publisher International Yearbook*

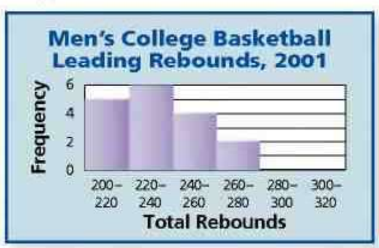
11.

Source: *USA TODAY*

For each pair of histograms, answer the following.

- **Compare the medians of the two data sets.**
- **Compare and describe the overall shape of each distribution of data.**

12.



Source: *USA TODAY*

Source: *USA TODAY*

13.

Source: *The World Almanac*

Source: *The World Almanac*

Create a histogram to represent each data set.

14. Students' semester averages in a mathematics class: 96.53, 95.96, 94.25, 93.58, 91.91, 90.33, 90.27, 90.11, 89.30, 89.06, 88.33, 88.30, 87.43, 86.67, 86.31, 84.21, 83.53, 82.30, 78.71, 77.51, 73.83

15. Number of raisins found in a snack-size box: 54, 59, 55, 109, 97, 59, 102, 68, 104, 63, 101, 59, 59, 96, 58, 57, 63, 57, 94, 61, 104, 62, 58, 59, 102, 60, 54, 58, 53, 78

More About...

Baseball

The New York Yankees won the 2000 World Series and had the largest payroll of all major league teams that year.

Source: *USA TODAY*

BASEBALL **For Exercises 16 and 17, use the following table.**

Payrolls for Major League Baseball Teams in 2000					
Team	**Payroll (millions)**	**Team**	**Payroll (millions)**	**Team**	**Payroll (millions)**
Yankees	$112	Orioles	$59	White Sox	$37
Braves	$94	Tigers	$59	Reds	$36
Red Sox	$91	Rockies	$56	Phillies	$36
Dodgers	$90	Padres	$55	Athletics	$32
Mets	$82	Blue Jays	$54	Pirates	$29
Indians	$77	Giants	$54	Expos	$28
Diamondbacks	$74	Angels	$53	Brewers	$26
Cardinals	$73	Devil Rays	$51	Marlins	$25
Rangers	$62	Astros	$51	Royals	$24
Mariners	$62	Cubs	$50	Twins	$15

Source: *USA TODAY*

16. Create a histogram to represent the payroll data.

17. On your histogram, locate and label the median team payroll.

ELECTIONS **For Exercises 18–20, use the following table.**

Percent of Eligible Voters Who Voted in the 2000 Presidential Election									
State	**Percent**	**State**	**Percent**	**State**	**Percent**	**State**	**Percent**	**State**	**Percent**
MN	68.75	WY	59.70	OH	55.76	MD	51.56	AR	47.79
ME	67.34	CT	58.40	ID	54.46	NJ	51.04	NM	47.40
AK	66.41	SD	58.24	RI	54.29	FL	50.65	SC	46.49
WI	66.07	MI	57.52	LA	54.24	NC	50.28	WV	45.74
VT	63.98	MO	57.49	KS	54.07	AL	49.99	CA	44.09
NH	62.33	WA	56.95	PA	53.66	IN	49.44	GA	43.84
MT	61.52	MA	56.92	IL	52.79	NY	49.42	NV	43.81
IA	60.71	CO	56.78	UT	52.61	TN	49.19	TX	43.15
OR	60.63	NE	56.44	VA	52.05	OK	48.76	AZ	42.26
ND	60.63	DE	56.22	KY	51.59	MS	48.57	HI	40.48

Source: *USA TODAY*

18. Determine the median of the data.

19. Create a histogram to represent the data.

20. Write a sentence or two describing the distribution of the data.

21. **RESEARCH** Choose your favorite professional sport. Use the Internet or other reference to find how many games each team in the appropriate league won last season. Use this information to create a histogram. Describe your histogram.

22. **CRITICAL THINKING** Create a histogram with a gap between 20 and 40, one item in the 50–55 measurement class, and the median value in the 50–55 measurement class.

23. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

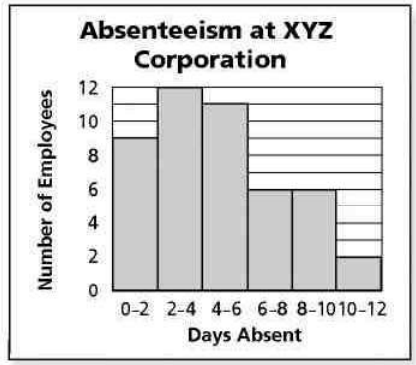
How are histograms used to display data?

Include the following in your answer:

- the advantage of the histogram over the frequency table, and
- a histogram depicting the distribution of the mean scores on the SAT mathematics exam.

Standardized Test Practice

For Exercises 24 and 25, use the information in the graph.



24. How many employees are represented in the graph?

Ⓐ 38 Ⓑ 40
Ⓒ 46 Ⓓ 48

25. In which measurement class is the median of the data located?

Ⓐ 2–4 Ⓑ 4–6
Ⓒ 6–8 Ⓓ 8–10

Graphing Calculator

HISTOGRAMS You can use a graphing calculator to create histograms. On a TI-83 Plus, enter the data in L1. In the STAT PLOT menu, turn on Plot 1 and select the histogram. Define the viewing window and press GRAPH.

Use a graphing calculator to create a histogram for each set of data.

26. 5, 5, 6, 7, 9, 4, 10, 12, 13, 8, 15, 16, 13, 8

27. 12, 14, 25, 30, 11, 35, 41, 47, 13, 18, 58, 59, 42, 13, 18

28. 124, 83, 81, 130, 111, 92, 178, 179, 134, 92, 133, 145, 180, 144

29. 2.2, 2.4, 7.5, 9.1, 3.4, 5.1, 6.3, 1.8, 2.8, 3.7, 8.6, 9.5, 3.6, 3.7, 5.0

Maintain Your Skills

Mixed Review

If $A = \begin{bmatrix} -2 & 3 & 7 \\ 0 & -4 & 6 \\ 1 & -5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -8 & 1 & -1 \\ 2 & 3 & -7 \end{bmatrix}$, **and** $C = \begin{bmatrix} 7 & -5 & 2 \\ 0 & 0 & 3 \\ -1 & 4 & 6 \end{bmatrix}$, **find each sum, difference, or product. If the sum or difference does not exist, write *impossible*.** *(Lesson 13-2)*

30. $A + B$ 31. $C - A$ 32. $2B$ 33. $-5A$

34. **MANUFACTURING** Every 15 minutes, a CD player is taken off the assembly line and tested. State whether this sample is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple*, *stratified*, or *systematic*. If biased, classify as *convenience* or *voluntary response*. *(Lesson 13-1)*

Find each quotient. Assume that no denominator has a value of 0. *(Lesson 12-4)*

35. $\frac{s}{s+7} \div \frac{s-5}{s+7}$

36. $\frac{2m^2+7m-15}{m+2} \div \frac{2m-3}{m^2+5m+6}$

Solve each equation. Check your solution. *(Lesson 11-3)*

37. $\sqrt{y+3} + 5 = 9$ 38. $\sqrt{x-2} = x - 4$ 39. $13 = \sqrt{2w-5}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the median for each set of data.**
*(To review **median**, see pages 818 and 819.)*

40. 2, 4, 7, 9, 12, 15

41. 10, 3, 17, 1, 8, 6, 12, 15

42. 7, 19, 9, 4, 7, 2

43. 2.1, 7.4, 13.9, 1.6, 5.21, 3.901

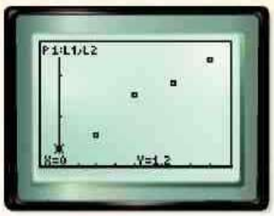
Graphing Calculator Investigation

A Follow-Up of Lesson 13-3

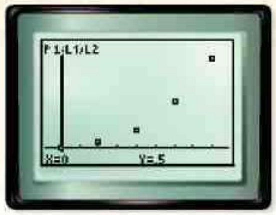
Curve Fitting

If there is a constant increase or decrease in data values, there is a linear trend. If the values are increasing or decreasing more and more rapidly, there may be a quadratic or exponential trend. The curvature of a quadratic trend tends to appear more gradual. Below are three scatter plots, each showing a different trend.

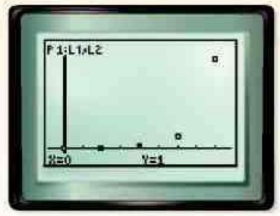
Linear Trend



Quadratic Trend



Exponential Trend



With a TI-83 Plus, you can use the LinReg, QuadReg, and ExpReg functions to find the appropriate regression equation that best fits the data.

FARMING **A study is conducted in which groups of 25 corn plants are given a different amount of fertilizer and the gain in height after a certain time is recorded. The table below shows the results.**

Fertilizer (mg)	0	20	40	60	80
Gain in Height (in.)	6.48	7.35	8.73	9.00	8.13

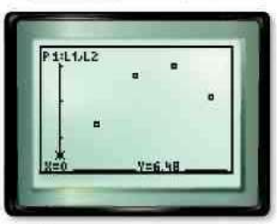
Step 1 *Make a scatter plot.*

- Enter the fertilizer in L1 and the height in L2.

  KEYSTROKES: *Review entering a list on page 204.*

- Use STAT PLOT to graph the scatter plot.

  KEYSTROKES: *Review statistical plots on page 204. Use* ZOOM *9 to graph.*



[−8, 88] scl: 5 by [6.0516, 9.4284] scl: 1

The graph appears to be a quadratic regression.

Step 2 *Find the quadratic regression equation.*

- Select **QuadReg** on the STAT CALC menu.

  KEYSTROKES: STAT ▶ 5 ENTER

The equation is in the form $y = ax^2 + bx + c$.

The equation is about $y = -0.0008x^2 + 0.1x + 6.3$.

R^2 is the **coefficient of determination**. The closer R^2 is to 1, the better the model. To choose a quadratic or exponential model, fit both and use the one with the R^2 value closer to 1.

www.algebra1.com/other_calculator_keystrokes

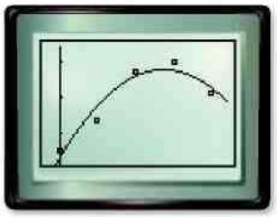
(continued on the next page)

Graphing Calculator Investigation

Step 3 *Graph the quadratic regression equation.*

- Copy the equation to the Y= list and graph.

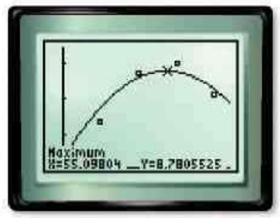
KEYSTROKES: [Y=] [VARS] 5 [▶] [▶] [ZOOM] 9



Step 4 *Predict using the equation.*

- Find the amount of fertilizer that produces the maximum gain in height.

On average, about 55 milligrams of the fertilizer produces the maximum gain.



Exercises

Plot each set of data points. Determine whether to use a *linear*, *quadratic*, or *exponential* regression equation. State the coefficient of determination.

1.

x	y
0.0	2.98
0.2	1.46
0.4	0.90
0.6	0.51
0.8	0.25
1.0	0.13

2.

x	y
1	25.9
2	22.2
3	20.0
4	19.3
5	18.2
6	15.9

3.

x	y
10	35
20	50
30	70
40	88
50	101
60	120

4.

x	y
1	3.67
3	5.33
5	6.33
7	5.67
9	4.33
11	2.67

TECHNOLOGY **The cost of cellular phone use is expected to decrease. For Exercises 5–9, use the graph at the right.**

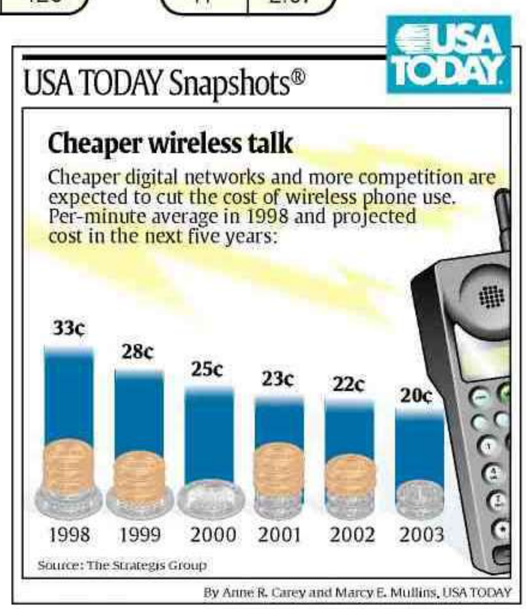
5. Make a scatter plot of the data.

6. Find an appropriate regression equation, and state the coefficient of determination.

7. Use the regression equation to predict the expected cost in 2004.

8. Do you believe that your regression equation is appropriate for a year beyond the range of data, such as 2020? Explain.

9. What model may be more appropriate for predicting cost beyond 2003?



CONTENTS

13-4 Measures of Variation

What You'll Learn

- Find the range of a set of data.
- Find the quartiles and interquartile range of a set of data.

Vocabulary

- range
- measures of variation
- quartiles
- lower quartile
- upper quartile
- interquartile range
- outlier

How is variation used in weather?

The average monthly temperatures for three U.S. cities are given. Which city shows the greatest change in monthly highs?

To answer this question, find the difference between the greatest and least values in each data set.

Buffalo: $80.2 - 30.2 = 50.0$
Honolulu: $88.7 - 80.1 = 8.6$
Tampa: $90.2 - 69.8 = 20.4$

Buffalo shows the greatest change.

Average Monthly High Temperatures (°F)

Month	Buffalo	Honolulu	Tampa
January	30.2	80.1	69.8
February	31.6	80.5	71.4
March	41.7	81.6	76.6
April	54.2	82.8	81.7
May	66.1	84.7	87.2
June	75.3	86.5	89.5
July	80.2	87.5	90.2
August	77.9	88.7	90.2
September	70.8	88.5	89.0
October	59.4	86.9	84.3
November	47.1	84.1	77.7
December	35.3	81.2	72.1

Source: www.stormfax.com

RANGE The difference between the greatest and the least monthly high temperatures is called the **range** of the temperatures.

Key Concept — Definition of Range

The range of a set of data is the difference between the greatest and the least values of the set.

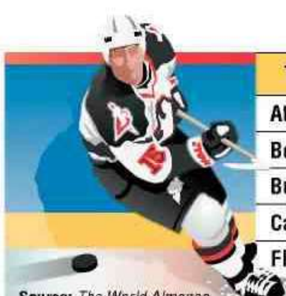
Study Tip

Look Back
To review **mean**, **median**, and **mode**, see pages 818 and 819.

The mean, median, and mode describe the central tendency of a set of data. The range of a set of data is a measure of the spread of the data. Measures that describe the spread of the values in a set of data are called **measures of variation**.

Example 1 Find the Range

HOCKEY The number of wins for each team in the Eastern Conference of the NHL for the 1999–2000 season are listed below. Find the range of the data.



Team	Wins	Team	Wins	Team	Wins
Atlanta	14	Montreal	35	Philadelphia	45
Boston	24	New Jersey	45	Pittsburgh	37
Buffalo	35	N.Y. Islanders	24	Tampa Bay	19
Carolina	37	N.Y. Rangers	29	Toronto	45
Florida	43	Ottawa	41	Washington	44

Source: *The World Almanac*

The greatest number of wins is 45, and the least number of wins is 14. Since $45 - 14 = 31$, the range of the number of wins is 31.

Study Tip

Reading Math

The abbreviations LQ and UQ are often used to represent the lower quartile and upper quartile, respectively.

QUARTILES AND INTERQUARTILE RANGE In a set of data, the **quartiles** are values that separate the data into four equal subsets, each containing one fourth of the data. Statisticians often use Q_1, Q_2, and Q_3 to represent the three quartiles. Remember that the median separates the data into two equal parts. Q_2 is the median. Q_1 is the **lower quartile**. It divides the lower half of the data into two equal parts. Likewise Q_3 is the **upper quartile**. It divides the upper half of the data into two equal parts. The difference between the upper and lower quartiles is the **interquartile range** (IQR).

median

1 1 2 4 6 7 7 8 9 10 12 13 17 17 18

Q_1 (4), Q_2 (8), Q_3 (13)

$Q_3 - Q_1 = \text{IQR}$

Key Concept *Definition of Interquartile Range*

The difference between the upper quartile and the lower quartile of a set of data is called the interquartile range. It represents the middle half, or 50%, of the data in the set.

Example 2 Find the Quartiles and the Interquartile Range

GEOGRAPHY **The areas of the original 13 states are listed in the table. Find the median, the lower quartile, the upper quartile, and the interquartile range of the areas.**

Explore You are given a table with the areas of the original 13 states. You are asked to find the median, the lower quartile, the upper quartile, and the interquartile range.

State	Area (thousand square miles)
Connecticut	6
Delaware	2
Georgia	59
Maryland	12
Massachusetts	11
New Hampshire	9
New Jersey	9
New York	54
North Carolina	54
Pennsylvania	46
Rhode Island	2
South Carolina	32
Virginia	43

Source: www.infoplease.com

Plan First, list the areas from least to greatest. Then find the median of the data. The median will divide the data into two sets of data. To find the upper and lower quartiles, find the median of each of these sets of data. Finally, subtract the lower quartile from the upper quartile to find the interquartile range.

Solve

median

2 2 6 9 9 11 12 32 43 46 54 54 59

$Q_1 = \frac{6 + 9}{2}$ or 7.5 $\qquad$ $Q_3 = \frac{46 + 54}{2}$ or 50

The median is 12 thousand square miles.

The lower quartile is 7.5 thousand square miles, and the upper quartile is 50 thousand square miles.

The interquartile range is 50 − 7.5 or 42.5 thousand square miles.

Examine Check to make sure that the numbers are listed in order. Since 7.5, 12, and 50 divide the data into four equal parts, the lower quartile, median, and upper quartile are correct.

In a set of data, a value that is much less or much greater than the rest of the data is called an **outlier**. An outlier is defined as any element of a set of data that is at least 1.5 interquartile ranges less than the lower quartile or greater than the upper quartile.

IQR = 13 − 9 or 4

Q_1 Q_2 Q_3

1 8 9 10 10 11 12 13 13 15 27

outlier | ← 9 − 1.5(4) = 3 | | 13 + 1.5(4) = 19 → | outlier

Example 3 Identify Outliers

Study Tip

Look Back
To review **stem-and-leaf plots**, see Lesson 2-5.

Identify any outliers in the following set of data.

Stem	Leaf
1	[2 2 7
2	3 $\boxed{3\ 3}$ 4 4 5 6] [6 8 8 9
3	$\boxed{0\ 1}$ 4 6
4	0 6] *1 | 2 = 12*

Step 1 **Find the quartiles.**

The brackets group the values in the lower half and the values in the upper half. The boxes are used to find the lower quartile and the upper quartile.

$Q_1 = \frac{23 + 23}{2}$ or 23 $\qquad$ $Q_3 = \frac{30 + 31}{2}$ or 30.5

Step 2 **Find the interquartile range.**

The interquartile range is 30.5 − 23 or 7.5.

Step 3 **Find the outliers, if any.**

An outlier must be 1.5(7.5) less than the lower quartile, 23, or 1.5(7.5) greater than the upper quartile, 30.5.

$23 - 1.5(7.5) = 11.75$ $\qquad$ $30.5 + 1.5(7.5) = 41.75$

There are no values less than 11.75. Since $46 > 41.75$, 46 is the only outlier.

Check for Understanding

Concept Check

1. **OPEN ENDED** Find a counterexample for the following statement.
   If the range of data set 1 is greater than the range of data set 2, then the interquartile range of data set 1 will be greater than the interquartile range of data set 2.
2. **Describe** how the mean is affected by an outlier.
3. **FIND THE ERROR** Alonso and Sonia are finding the range of this set of data: 28, 30, 32, 36, 40, 41, 43.

Alonso	Sonia
43 − 28 = 15	The range is all numbers
The range is 15.	between 28 and 43, inclusive.

Who is correct? Explain your reasoning.

Guided Practice

Find the range, median, lower quartile, upper quartile, and interquartile range of each set of data. Identify any outliers.

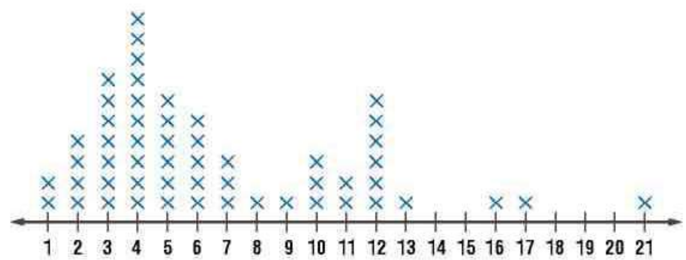
4. 85, 77, 58, 69, 62, 73, 25, 82, 67, 77, 59, 75, 69, 76

5.

Stem	Leaf
7	3 7 8
8	0 0 3 5 7
9	4 6 8
10	0 1 8
11	1 9 *7|3 = 7.3*

Application

LITTLE LEAGUE For Exercises 6–10, use the following information.
The number of runs scored by the winning team in the Little League World Series each year from 1947 to 2000 are given in the line plot below.



Source: *The World Almanac*

6. What is the range of the data?

7. What is the median of the data?

8. What is the lower quartile and upper quartile of the data?

9. What is the interquartile range of the data?

10. Name any outliers.

Practice and Apply

Homework Help

For Exercises	See Examples
11–18	1–3
19, 24, 29	1
20–22, 25–27, 30, 31	2
23, 28, 32	3

Extra Practice

See page 850.

Find the range, median, lower quartile, upper quartile, and interquartile range of each set of data. Identify any outliers.

11. 85, 77, 58, 69, 62, 73, 55, 82, 67, 77, 59, 92, 75

12. 28, 42, 37, 31, 34, 29, 44, 28, 38, 40, 39, 42, 30

13. 30.8, 29.9, 30.0, 31.0, 30.1, 30.5, 30.7, 31.0

14. 2, 3.4, 5.3, 3, 1, 3.2, 4.9, 2.3

15.

Stem	Leaf
5	3 6 8
6	5 8
7	0 3 7 7 9
8	1 4 8 8 9
9	9 *5|3 = 53*

16.

Stem	Leaf
19	3 5 5
20	2 2 5 8
21	5 8 8 9 9 9
22	0 1 7 8 9
23	2 *19|3 = 193*

17.

Stem	Leaf
5	0 3 7 9
6	1 3 4 5 5 6
7	1 5 6 6 9
8	1 2 3 5 8
9	2 5 6 9
10	
11	7 *5|0 = 5.0*

18.

Stem	Leaf
0	0 2 3
1	1 7 9
2	2 3 5 6
3	3 4 4 5 9
4	0 7 8 8
5	
6	8 *0|2 = 0.2*

More About. . .

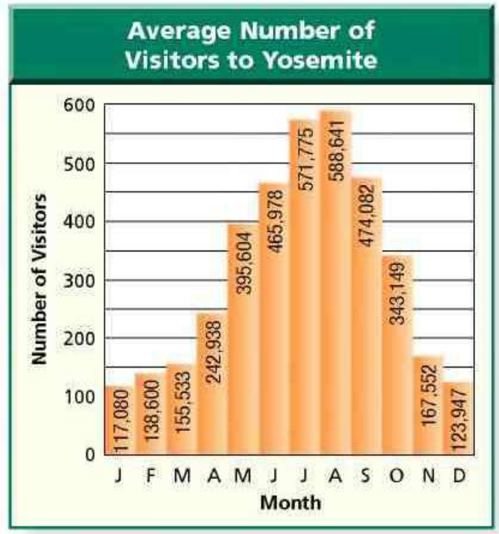
National Parks

Yosemite National Park boasts of sparkling lakes, mountain peaks, rushing streams, and beautiful waterfalls. It has about 700 miles of hiking trails.

Source: *World Book Encyclopedia*

NATIONAL PARKS For Exercises 19–23, use the graph at the right.

19. What is the range of the visitors per month?
20. What is the median number of visitors per month?
21. What are the lower quartile and the upper quartile of the data?
22. What is the interquartile range of the data?
23. Name any outliers.



Source: *USA TODAY*

NUTRITION For Exercises 24–28, use the following table.

Calories for One Serving of Vegetables					
Vegetable	Calories	Vegetable	Calories	Vegetable	Calories
Asparagus	14	Carrots	28	Lettuce	9
Avocado	304	Cauliflower	10	Onion	60
Bell pepper	20	Celery	17	Potato	89
Broccoli	25	Corn	66	Spinach	9
Brussels sprouts	60	Green beans	30	Tomato	35
Cabbage	17	Jalapeno peppers	13	Zucchini	17

Source: *Vitality*

24. What is the range of the data?
25. What is the median of the data?
26. What are the lower quartile and the upper quartile of the data?
27. What is the interquartile range of the data?
28. Identify any outliers.

BRIDGES For Exercises 29–33, use the following information and the double stem-and-leaf plot at the right.

The main span of cable-stayed bridges and of steel-arch bridges in the United States are given in the stem-and-leaf plot.

29. Find the ranges for each type of bridge.
30. Find the quartiles for each type of bridge.
31. Find the interquartile ranges for each type of bridge.
32. Identify any outliers.
33. Compare the ranges and interquartile ranges of the two types of bridges. What can you conclude from these statistics?

Cable-Stayed	Stem	Steel-Arch
6 4 3	6	
9 8 6 5 1 1	7	3 8 8
0	8	0 0 2 3 4
5 2	9	0 1 1 8 8 9
8 2 0 0	10	0 3 8
2	11	0 0
8 2 0	12	0 6
5 2 0	13	
9 0	14	
	15	
3	16	5
3|6 = 630 feet	17	0 7|3 = 730 feet

Source: *The World Almanac*

www.algebra1.com/self_check_quiz

34. **CRITICAL THINKING** Trey measured the length of each classroom in his school. He then calculated the range, median, lower quartile, upper quartile, and interquartile range of the data. After his calculations, he discovered that the tape measure he had used started at the 2-inch mark instead of at the 0-inch mark. All of his measurements were 2 inches greater than the actual lengths of the rooms. How will the values that Trey calculated change? Explain your reasoning.

35. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How is variation used in weather?

Include the following in your answer:
- the meaning of the range and interquartile range of temperatures for a city, and
- the average highs for your community with the appropriate measures of variation.

Standardized Test Practice

36. What is the range of the following set of data?
53, 57, 62, 48, 45, 65, 40, 42, 55

(A) 11 (B) 25 (C) 53 (D) 65

37. What is the median of the following set of data?
7, 8, 14, 3, 2, 1, 24, 18, 9, 15

(A) 8.5 (B) 10.1 (C) 11.5 (D) 23

Maintain Your Skills

Mixed Review

38. Create a histogram to represent the following data. *(Lesson 13-3)*
36, 43, 61, 45, 37, 41, 32, 46, 60, 38, 35, 64, 46, 47, 30, 38, 48, 39

State the dimensions of each matrix. Then identify the position of the circled element in each matrix. *(Lesson 13-2)*

39. $[5 \quad -3 \quad 6]$ 40. $\begin{bmatrix} 3 & 1 \\ 2 & 9 \\ 4 & 3 \end{bmatrix}$ 41. $\begin{bmatrix} 4 & 2 & -1 & 3 \\ 5 & 9 & 0 & 2 \end{bmatrix}$

Simplify each rational expression. State the excluded values of the variables. *(Lesson 12-2)*

42. $\frac{15a}{39a^2}$ 43. $\frac{t-3}{t^2-7t+12}$ 44. $\frac{m-3}{m^2-9}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Graph each set of numbers on a number line.**
(To review ***number lines***, *see Lesson 2-1.)*

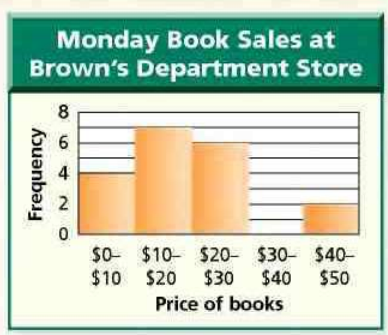
45. {4, 7, 8, 10, 11} 46. {13, 17, 22, 23, 27} 47. {30, 35, 40, 50, 55}

Practice Quiz 2

Lessons 13-3 and 13-4

For Exercises 1–2, use the histogram at the right. *(Lesson 13-3)*

1. In what measurement class does the median occur?
2. Describe the distribution of the data.



For Exercises 3–5, use the following set of data. *(Lesson 13-4)*
1050, 1175, 835, 1075, 1025, 1145, 1100, 1125, 975, 1005, 1125, 1095, 1075, 1055

3. Find the range of the data.
4. Find the median, the lower quartile, the upper quartile, and interquartile range of the data.
5. Identify any outliers of the data.

13-5 Box-and-Whisker Plots

What You'll Learn

- Organize and use data in box-and-whisker plots.
- Organize and use data in parallel box-and-whisker plots.

Vocabulary

- box-and-whisker plot
- extreme values

How are box-and-whisker plots used to display data?

Everyone should eat a number of calcium-rich foods each day. Selected foods and the amount of calcium in a serving are listed in the table. To create a box-and-whisker plot of the data, you need to find the quartiles of the data.

Calcium-Rich Foods

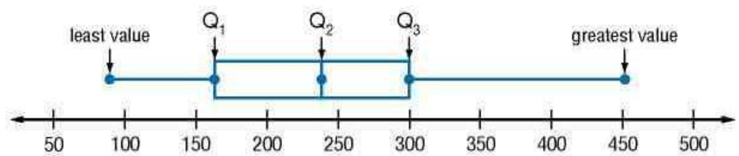
Food (serving size)	Calcium (milligrams)
Plain Yogurt, Nonfat (8 oz)	452
Plain Yogurt, Low-fat (8 oz)	415
Skim Milk (8 oz)	302
1% Milk (8 oz)	300
Whole Milk (8 oz)	291
Swiss Cheese (1 oz)	272
Tofu (4 oz)	258
Sardines (2 oz)	217
Cheddar Cheese (1 oz)	204
Collards (4 oz)	179
American Cheese (1 oz)	163
Frozen Yogurt with Fruit (4 oz)	154
Salmon (2 oz)	122
Broccoli (4 oz)	89

Source: *Vitality*

89 122 154 163 179 204 217 258 272 291 300 302 415 452

$Q_1 = 163$ $\qquad Q_2 = \frac{217 + 258}{2}$ or 237.5 $\qquad Q_3 = 300$

This information can be displayed on a number line as shown below.



Study Tip

Reading Math
Box-and-whisker plots are sometimes called *box plots*.

BOX-AND-WHISKER PLOTS Diagrams such as the one above are called **box-and-whisker plots**. The length of the box represents the interquartile range. The line inside the box represents the median. The lines or *whiskers* represent the values in the lower fourth of the data and the upper fourth of the data. The bullets at each end are the **extreme values**. In the box-and-whisker plot above, the least value (LV) is 89, and the greatest value (GV) is 452.

If a set of data has outliers, these data points are represented by bullets. The whisker representing the lower data is drawn from the box to the least value that is not an outlier. The whisker representing the upper data is drawn from the box to the greatest value that is not an outlier.

Example 1 Draw a Box-and-Whisker Plot

ECOLOGY **The amount of rain in Florida from January to May is crucial to its ecosystems. The following is a list of the number of inches of rain in Florida during this crucial period for the years 1990 to 2000.**

14.03, 30.11, 16.03, 19.61, 18.15, 16.34, 20.43, 18.46, 22.24, 12.70, 8.25

a. Draw a box-and-whisker plot for these data.

Step 1 **Determine the quartiles and any outliers.**

Order the data from least to greatest. Use this list to determine the quartiles.

8.25, 12.70, 14.03, 16.03, 16.34, 18.15, 18.46, 19.61, 20.43, 22.24, 30.11

14.03 ↑ Q_1 18.15 ↑ Q_2 20.43 ↑ Q_3

Determine the interquartile range.

$IQR = 20.43 - 14.03$ or 6.4

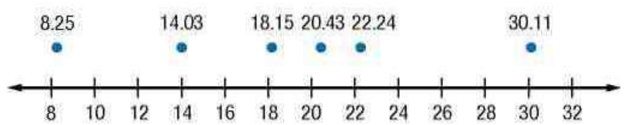
Check to see if there are any outliers.

$14.03 - 1.5(6.4) = 4.43$ $\quad$ $20.43 + 1.5(6.4) = 30.03$

Any numbers less than 4.43 or greater than 30.03 are outliers. The only outlier is 30.11.

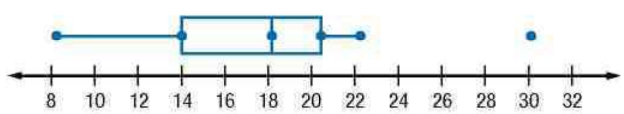
Step 2 **Draw a number line.**

Assign a scale to the number line that includes the extreme values. Above the number line, place bullets to represent the three quartile points, any outliers, the least number that is *not* an outlier, and the greatest number that is *not* an outlier.



Step 3 **Complete the box-and-whisker plot.**

Draw a box to designate the data between the upper and lower quartiles. Draw a vertical line through the point representing the median. Draw a line from the lower quartile to the least value that is *not* an outlier. Draw a line from the upper quartile to the greatest value that is *not* an outlier.



b. What does the box-and-whisker plot tell about the data?

Notice that the whisker and the box for the top half of the data is shorter than the whisker and box for the lower half of the data. Therefore, except for the outlier, the upper half of the data are less spread out than the lower half of the data.

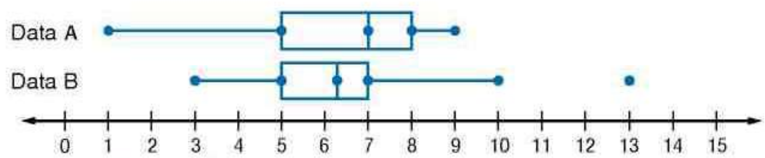
More About...

Ecology

The ecosystem of the Everglades in Florida is unique. The Everglades National Park is a subtropical swamp area covering 1,506,499 acres.

Source: *World Book Encyclopedia*

PARALLEL BOX-AND-WHISKER PLOTS Two sets of data can be compared by drawing parallel box-and-whisker plots such as the one shown below.



Example 2 Draw Parallel Box-and-Whisker Plots

WEATHER **Jalisa Thompson has job offers in Fresno, California, and Brownsville, Texas. Since she likes both job offers, she decides to compare the temperatures of each city.**

Average Monthly High Temperatures (°F)												
Month	Jan.	Feb.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Fresno	54.1	61.7	66.6	75.1	84.2	92.7	98.6	96.7	90.1	79.7	64.7	53.7
Brownsville	68.9	72.2	78.4	84.0	87.8	91.0	93.3	93.6	90.4	85.3	78.3	71.7

Source: www.stormfax.com

a. Draw a parallel box-and-whisker plot for the data.

Determine the quartiles and outliers for each city.

Fresno

53.7, 54.1, 61.7, 64.7, 66.6, 75.1, 79.7, 84.2, 90.1, 92.7, 96.7, 98.6

↑ ↑ ↑

$Q_1 = 63.2$ $Q_2 = 77.4$ $Q_3 = 91.4$

Brownsville

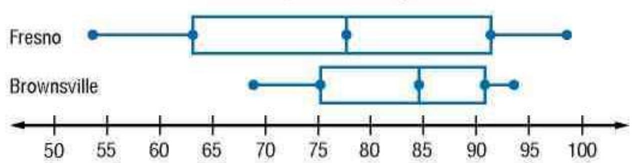
68.9, 71.7, 72.2, 78.3, 78.4, 84.0, 85.3, 87.8, 90.4, 91.0, 93.3, 93.6

↑ ↑ ↑

$Q_1 = 75.25$ $Q_2 = 84.65$ $Q_3 = 90.7$

Neither city has any outliers.

Draw the box-and-whisker plots using the same number line.



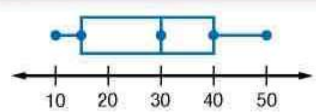
b. Use the parallel box-and-whisker plots to compare the data.

The range of temperatures in Fresno is much greater than in Brownsville. Except for the fourth quartile, Brownville's average temperatures appear to be as high or higher than Fresno's.

Check for Understanding

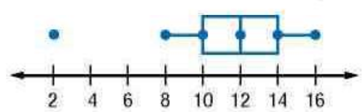
Concept Check

1. **Describe** the data represented by the box-and-whisker plot at the right. Include the extreme values, the quartiles, and any outliers.



2. **Explain** how to determine the scale of the number line in a box-and-whisker plot.

3. **OPEN ENDED** Write a set of data that could be represented by the box-and-whisker plot at the right.



Guided Practice

Draw a box-and-whisker plot for each set of data.

4. 30, 28, 24, 24, 22, 22, 21, 17, 16, 15

5. 64, 69, 65, 71, 66, 66, 74, 67, 68, 67

www.algebra1.com/extra_examples

CONTENTS

Draw a parallel box-and-whisker plot for each pair of data. Compare the data.

6. A: 22, 18, 22, 17, 32, 24, 31, 26, 28
B: 28, 30, 45, 23, 24, 32, 30, 27, 27

7. A: 8, 15.5, 14, 14, 24, 19, 16.7, 15, 11.4, 16
B: 18, 14, 15.8, 9, 12, 16, 20, 16, 13, 15

Application **CHARITY** **For Exercises 8 and 9, use the information in the table below.**

Top Ten Charities	
Charity	**Private Contributions (millions)**
Salvation Army	\$1397
YMCA of the U.S.A.	\$693
American Red Cross	\$678
American Cancer Society	\$620
Fidelity Investments Charitable Gift Fund	\$573
Lutheran Services in America	\$559
United Jewish Communities	\$524
America's Second Harvest	\$472
Habitat for Humanity International	\$467
Harvard University	\$452

Source: *The Chronicle of Philanthropy*

8. Make a box-and-whisker for the data.

9. Write a brief description of the data distribution.

Practice and Apply

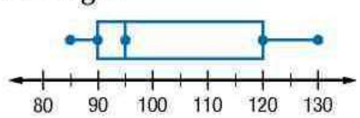
Homework Help

For Exercises	See Examples
10–19	1
20–27	2

Extra Practice

See page 850.

For Exercises 10–13, use the box-and-whisker plot at the right.



10. What is the range of the data?

11. What is the interquartile range of the data?

12. What fractional part of the data is less than 90?

13. What fractional part of the data is greater than 95?

Draw a box-and-whisker plot for each set of data.

14. 15, 8, 10, 1, 3, 2, 6, 5, 4, 27, 1

15. 20, 2, 12, 5, 4, 16, 17, 7, 6, 16, 5, 0, 5, 30

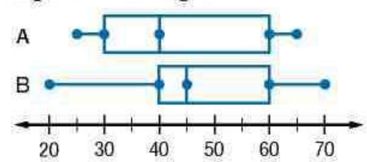
16. 4, 1, 1, 1, 10, 15, 4, 5, 27, 5, 14, 10, 6, 2, 2, 5, 8

17. 51, 27, 55, 54, 69, 60, 39, 46, 46, 53, 81, 23

18. 15.1, 9.0, 8.5, 5.8, 6.2, 8.5, 10.5, 11.5, 8.8, 7.6

19. 1.3, 1.2, 14, 1.8, 1.6, 5.7, 1.3, 3.7, 3.3, 2, 1.3, 1.3, 7.7, 8.5, 2.2

For Exercises 20–23, use the parallel box-and-whisker plot at the right.



20. Which set of data contains the least value?

21. Which set of data contains the greatest value?

22. Which set of data has the greatest interquartile range?

23. Which set of data has the greatest range?

Draw a parallel box-and-whisker plot for each pair of data. Compare the data.

24. A: 15, 17, 22, 28, 32, 40, 16, 24, 26, 38, 19
B: 24, 32, 25, 27, 37, 29, 30, 30, 28, 31, 27

25. A: 50, 45, 47, 55, 51, 58, 49, 51, 51, 48, 47
B: 40, 41, 48, 39, 41, 41, 38, 37, 35, 37, 45

26. A: 1.5, 3.8, 4.2, 3.5 4.1, 4.4, 4.1, 4.0, 4.0, 3.9
B: 6.8, 4.2, 7.6, 5.5, 12.2, 6.7, 7.1, 4.8

27. A: 4.4, 4.5, 4.6, 4.5, 4.4, 4.4, 4.1, 4.9, 2.9
B: 5.1, 4.9, 4.2, 3.9, 4.5, 4.1, 4.3, 4.5, 5.2

PROFESSIONAL SPORTS **For Exercises 28 and 29, use the table at the right.**

28. Draw a box-and-whisker plot for the data.

29. What does the box-and-whisker plot tell about the data?

Professional Athletes

Professional Sport	Average Length of Career (years)
Bowling	17
Surfing	10
Hockey	5.5
Baseball	4.5
Basketball	4.5
Tennis	4
Football	3.5
Boxing	3.5

Source: *Men's Health Fitness Special*

More About. . .

Life Expectancy

A newborn resident of the United States has a life expectancy of 77 years, while a newborn resident of Canada has a life expectancy of 79 years.

Source: UNICEF

RACING **For Exercises 30 and 31, use the following list of earnings in thousands from the November 2000 NAPA 500 NASCAR Race at the Atlanta Motor Speedway.**

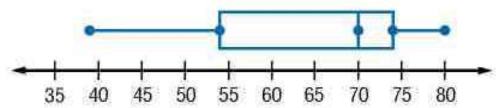
$181, $100, $98, $89, $76, $58, $60; $58; $55, $57, $54, $64, $44, $39, $66, $52, $56, $38, $56, $51, $49, $38, $50, $48, $48, $40, $36, $36, $39, $36, $47, $36, $47, $38, $35, $46, $35, $55, $46, $55, $45, $43, $35

Source: *USA TODAY*

30. Draw a box-and-whisker plot for the data. Identify any outliers.

31. Determine whether the top half of the data or the bottom half of the data are more dispersed. Explain.

LIFE EXPECTANCY **For Exercises 32–35, use the box-and-whisker plot depicting the UNICEF life expectancy data for 171 countries.**



32. Estimate the range and the interquartile range.

33. Determine whether the top half of the data or the bottom half of the data are more dispersed. Explain.

34. State three different intervals of ages that contain half the data.

35. Jamie claims that the number of data values is greater in the interval 54 years to 70 years than the number of data values in the interval 70 years to 74 years. Is Jamie correct? Explain.

SOCCER **For Exercises 36–38, use the following list of top 50 lifetime scores for all players in Division 1 soccer leagues in the United States from 1922 to 1999.**

253, 223, 193, 189, 152, 150, 138, 137, 135, 131, 131, 129, 128, 126, 124, 119, 118, 108, 107, 102, 101, 100, 96, 92, 87, 83, 82, 81, 80, 78, 78, 76, 74, 74, 73, 73, 72, 71, 69, 68, 67, 65, 64, 63, 63, 63, 62, 61, 61, 61

Source: www.internetsoccer.com

36. Draw a box-and-whisker plot for the data.

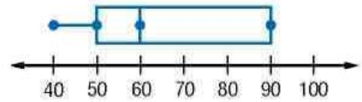
37. Create a histogram to represent the data.

38. Compare and contrast the box-and-whisker plot and the histogram.

Web Quest

A box-and-whisker plot of population densities will help you compare the states. Visit www.algebra1.com/webquest to continue work on your WebQuest project.

39. **CRITICAL THINKING** Write a set of data that could be represented by the box-and-whisker plot at the right.



40. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.

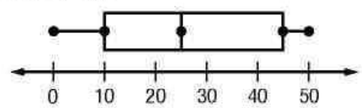
How are box-and-whisker plots used to display data?

Include the following in your answer:

- a sample of a box-and-whisker plot showing what each part of the plot represents, and
- a box-and-whisker plot representing data found in a newspaper or magazine.

Standardized Test Practice

For Exercises 41 and 42, use the box-and-whisker plot below.



41. What is the median of the data?

Ⓐ 0 Ⓑ 10
Ⓒ 25 Ⓓ 45

42. Which interval represents 75% of the data?

Ⓐ 0–25 Ⓑ 10–45 Ⓒ 25–50 Ⓓ 0–45

Maintain Your Skills

Mixed Review

For Exercises 43 and 44, use the following data.

13, 32, 45, 45, 54, 55, 58, 67, 82, 93

43. Find the range, median, lower quartile, upper quartile, and interquartile range of the data. Identify any outliers. *(Lesson 13-4)*

44. Create a histogram to represent the data. *(Lesson 13-3)*

Find each sum or difference. *(Lesson 12-7)*

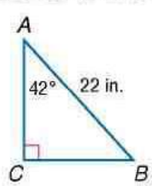
45. $\frac{3}{y-3} - \frac{y}{y+4}$ 46. $\frac{2}{r+3} + \frac{3}{r-2}$ 47. $\frac{w}{5w+2} - \frac{4}{15w+6}$

Find each product. Assume that no denominator has a value of 0. *(Lesson 12-3)*

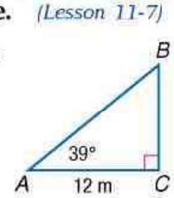
48. $\frac{7a^2}{5} \cdot \frac{15}{14a}$ 49. $\frac{6r+3}{r+6} \cdot \frac{r^2+9r+18}{2r+1}$

Solve each right triangle. State the side length to the nearest tenth and the angle measures to the nearest degree. *(Lesson 11-7)*

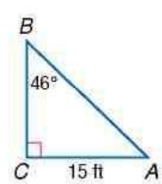
50.



51.



52.



Solve each equation by completing the square. Approximate any irrational roots to the nearest tenth. *(Lesson 10-3)*

53. $a^2 - 7a + 6 = 0$ 54. $x^2 - 6x + 2 = 0$ 55. $t^2 + 8t - 18 = 0$

Find each sum or difference. *(Lesson 8-5)*

56. $(7p^2 - p - 7) - (p^2 + 11)$ 57. $(3a^2 - 8) + (5a^2 + 2a + 7)$

Algebra Activity

A Follow-Up of Lesson 13-5

Investigating Percentiles

When data are arranged in order from least to greatest, you can describe the data using percentiles. A **percentile** is the point below which a given percent of the data lies. For example, 50% of the data falls below the median. So the median is the 50th percentile for the data.

To determine a percentile, a cumulative frequency table can be used. In a **cumulative frequency table**, the frequencies are accumulated for each item.

Collect the Data

A student's score on the SAT is one factor that some colleges consider when selecting applicants. The tables below show the raw scores from a sample math SAT test for 160 juniors in a particular school. For raw scores, the highest possible score is 800 and the lowest is 200.

Table 1: Frequency Table

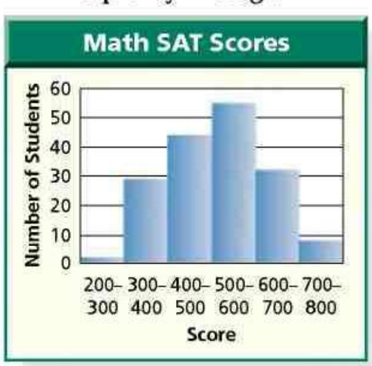
Math SAT Scores	Number of Students
200–300	2
300–400	19
400–500	44
500–600	55
600–700	32
700–800	8

Table 2: Cumulative Frequency Table

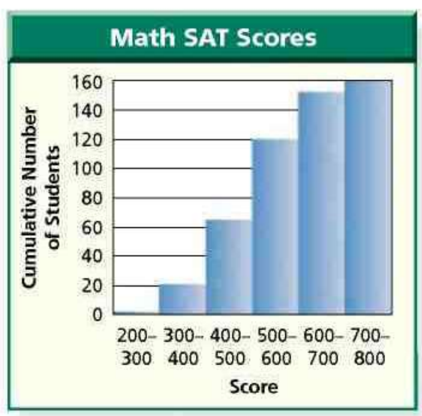
Math SAT Scores	Number of Students	Cumulative Number of Students
200–300	2	2
300–400	19	21
400–500	44	65
500–600	55	120
600–700	32	152
700–800	8	160

The data in each table can be displayed in a histogram.

Frequency Histogram



Cumulative Frequency Histogram



Analyze the Data

1. Examine the data in the two tables. Explain how the numbers in Column 3 of Table 2 are determined.

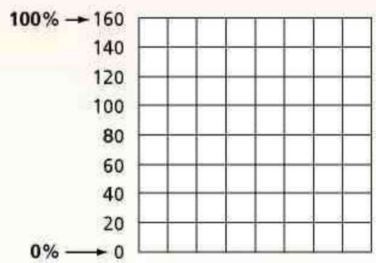
(continued on the next page)

2. Describe the similarities and differences between the two histograms.
3. Which histogram do you prefer for displaying these data? Explain your choice.

Make a Conjecture

Sometimes colleges are not interested in your raw score. They are interested in the percentile. Your percentile indicates what percent of all test-takers scored just as well or lower than you.

4. Use the histogram for Table 2. Place percentile labels on the vertical axis. For example, write 100% next to 160 and 0% next to 0. Now label 25%, 50%, and 75%. What numbers of students correspond to 25%, 50%, and 75%?



5. Suppose a college is interested in students with scores in the 90th percentile. Using the histogram, move up along the vertical axis to the 90th percentile. Then move right on the horizontal axis to find the score. What is an estimate for the score that represents the 90th percentile?
6. For a more accurate answer, use a proportion to find 90% of the total number of students. (Recall that the total number of students is 160.)
7. If a student is to be in the 90th percentile, in what interval will the score lie?

Extend the Activity

For Exercises 8–10, use the following information.

The weights of 45 babies born at a particular hospital during the month of January are shown below.

9 lb 1 oz	8 lb 2 oz	7 lb 2 oz	10 lb 0 oz	4 lb 4 oz
5 lb 0 oz	7 lb 6 oz	7 lb 8 oz	11 lb 2 oz	6 lb 1 oz
3 lb 8 oz	8 lb 0 oz	7 lb 5 oz	9 lb 15 oz	6 lb 1 oz
7 lb 10 oz	6 lb 9 oz	6 lb 15 oz	7 lb 10 oz	8 lb 0 oz
5 lb 15 oz	8 lb 3 oz	8 lb 1 oz	7 lb 12 oz	7 lb 8 oz
7 lb 7 oz	6 lb 14 oz	7 lb 13 oz	8 lb 0 oz	7 lb 14 oz
5 lb 10 oz	8 lb 5 oz	6 lb 12 oz	8 lb 8 oz	7 lb 11 oz
8 lb 15 oz	9 lb 3 oz	5 lb 14 oz	6 lb 8 oz	8 lb 8 oz
7 lb 4 oz	7 lb 10 oz	8 lb 1 oz	7 lb 8 oz	7 lb 10 oz

8. Make a cumulative frequency table for the data.
9. Make a cumulative frequency histogram for the data.
10. Find the weight for a baby in the 80th percentile.

Chapter 13

Study Guide and Review

Vocabulary and Concept Check

biased sample (p. 709)
box-and-whisker plot (p. 737)
census (p. 708)
column (p. 715)
convenience sample (p. 709)
dimensions (p. 715)
element (p. 715)
extreme value (p. 737)
frequency (p. 722)
frequency table (p.722)
histogram (p. 722)
interquartile range (p. 732)
lower quartile (p. 732)
matrix (p. 715)
measurement classes (p. 722)
measures of variation (p. 731)
outlier (p. 733)
population (p. 708)
quartiles (p. 732)
random sample (p. 708)
range (p. 731)
row (p. 715)
sample (p. 708)
scalar multiplication (p. 717)
simple random sample (p. 708)
stratified random sample (p. 708)
systematic random sample (p. 708)
upper quartile (p. 732)
voluntary response sample (p. 709)

Choose the correct term from the list above that best completes each statement.

1. A(n) ______ is a sample that is as likely to be chosen as any other from the population.
2. Measures that describe the spread of the values in a set of data are called ______.
3. Each ______ separates a data set into four sets with equal number of members.
4. In a(n) ______, the items are selected according to a specified time or item interval.
5. A(n) ______ has a systematic error within it so that certain populations are favored.
6. In a(n) ______, the population is first divided into similar, nonoverlapping groups.
7. The ______ is found by subtracting the lower quartile from the upper quartile.
8. A(n) ______ involves only those who want to participate in the sampling.
9. An extreme value that is much less or greater than the rest of the data is a(n) ______.
10. The ______ is the difference between the greatest and least values of a data set.

Lesson-by-Lesson Review

13-1 Sampling and Bias

See pages 708–713.

Concept Summary

- Samples are used to represent a larger group called a population.
- Simple random sample, stratified random sample, and systematic random sample are types of unbiased, or random, samples.
- Convenience sample and voluntary response sample are types of biased samples.

Example **GOVERNMENT To determine whether voters support a new trade agreement, 5 people from the list of registered voters in each state and the District of Columbia are selected at random. Identify the sample, suggest a population from which it was selected, and state whether the sample is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple*, *stratified*, or *systematic*. If biased, classify as *convenience* or *voluntary response*.**

Since $5 \times 51 = 255$, the sample is 255 registered voters in the United States. The population is all of the registered voters in the United States.

The sample is unbiased. It is an example of a stratified random sample.

Exercises **Identify the sample, suggest a population from which it was selected, and state whether it is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple*, *stratified*, or *systematic*. If biased, classify the sample as *convenience* or *voluntary response*.** *See Examples 1–3 on pages 709 and 710.*

11. **SCIENCE** A laboratory technician needs a sample of results of chemical reactions. She selects test tubes from the first 8 experiments performed on Tuesday.
12. **CANDY BARS** To ensure that all of the chocolate bars are the appropriate weight, every 50th bar on the conveyor belt in the candy factory is removed and weighed.

13-2 Introduction to Matrices

See pages 715–721.

Concept Summary

- A matrix can be used to organize data and make data analysis more convenient.
- Equal matrices must have the same dimensions and corresponding elements are equal.
- Matrices with the same dimensions can be added or subtracted.
- Each element of a matrix can be multiplied by a number called a scalar.

Example **If $R = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$, $S = \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix}$, and $T = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, find each sum. If it does not exist, write *impossible*.**

a. $R + S$

$$R + S = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + (-1) & 2 + 3 \\ -1 + 0 & 3 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ -1 & 4 \end{bmatrix}$$

b. $S + T$

$$S + T = \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Since S is a 2×2 matrix and T is a 2×1 matrix, the matrices do not have the same dimensions. Therefore, it is impossible to add these matrices.

Exercises **If $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 4 \\ -1 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & -1 \\ -1 & -2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$, find each sum, difference, or product. If the sum or difference does not exist, write *impossible*.** *See Examples 3 and 4 on pages 716 and 717.*

13. $A + B$
14. $3B$
15. $-2D$
16. $C - D$
17. $C + D$
18. $B + C$
19. $5A$
20. $A - D$
21. $C + 3D$
22. $2A - B$

13-3 Histograms

See pages 722–728.

Concept Summary

- A histogram can illustrate the information in a frequency table.
- The distribution of the data can be determined from a histogram.

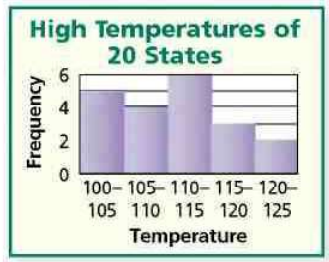
Example **Create a histogram to represent the following high temperatures in twenty states.**

118 122 117 105 114 115 122 102 103 110

110 112 106 109 100 103 110 108 111 102

Since the temperatures range from 100 to 122, use measurement classes from 100 to 125 with 5 degree intervals. First create a frequency table and then draw the histogram.

Temperature Intervals	Tally	Frequency
$100 \le d < 105$	~~IIII~~	5
$105 \le d < 110$	IIII	4
$110 \le d < 115$	~~IIII~~ I	6
$115 \le d < 120$	III	3
$120 \le d < 125$	II	2



Exercises **Create a histogram to represent each data set.** *See Example 3 on page 724.*

23. the number of cellular minutes used last month by employees of a company

122 150 110 290 145 330 300 210 95 101 106 289 219
105 302 29 288 154 235 168 55 84 92 175 180

24. the number of cups of coffee consumed per customer at a snack shop between 6 A.M. and 8 A.M.

0 2 0 2 1 3 2 1 2 3 0 2 2 1 0 2 1 3 0 1 2 2
3 2 1 0 1 2 1 0 2 2 2 1 1 2 1 2 0 3 1 0 0 1

13-4 Measures of Variation

See pages 731–736.

Concept Summary

- The range of the data set is the difference between the greatest and the least values of the set and describes the spread of the data.
- The interquartile range is the difference between the upper and lower quartiles of a set of data. It is range of the middle half of the data.
- Outliers are values that are much less than or much greater than the rest of the data.

Example **Find the range, median, lower quartile, upper quartile, and interquartile range of the set of data below. Identify any outliers.**
25, 20, 30, 24, 22, 26, 28, 29, 19

Order the set of data from least to greatest.

19 20 22 24 25 26 28 29 30

Q_1 (between 20 and 22), Q_2 (25), Q_3 (between 28 and 29)

The range is 30 − 19 or 11.

The median is the middle number, 25.

The lower quartile is $\frac{20 + 22}{2}$ or 21.

The upper quartile is $\frac{28 + 29}{2}$ or 28.5.

The interquartile range is 28.5 − 21 or 7.5.

The outliers would be less than 21 − 1.5(7.5) or 9.75 and greater than 28.5 + 1.5(7.5) or 39.25. There are no outliers.

Chapter 13 For More ...

- Extra Practice, see pages 849–850.
- Mixed Problem Solving, see page 865.

Exercises **Find the range, median, lower quartile, upper quartile, and interquartile range of each set of data. Identify any outliers.**
See Examples 1–3 on pages 731–733.

25. 30, 90, 40, 70, 50, 100, 80, 60

26. 3, 3.2, 45, 7, 2, 1, 3.4, 4, 5.3, 5, 78, 8, 21, 5

27. 85, 77, 58, 69, 62, 73, 55, 82, 67, 77, 59, 92, 75, 69, 76

28. 111.5, 70.7, 59.8, 68.6, 63.8, 254.8, 64.3, 82.3, 91.7, 88.9, 110.5, 77.1

13-5 Box-and-Whisker Plots

See pages 737–742.

Concept Summary

- The vertical rule in the box of a box-and-whisker plot represents the median.
- The box of a box-and-whisker plot represents the interquartile range.
- The bullets at each end of a box-and-whisker plot are the extremes.
- Parallel box-and-whisker plots can be used to compare data.

Example **The following high temperatures (°F) were recorded during a two-week cold spell in St. Louis. Draw a box-and-whisker plot of the temperatures.**

20	**2**	**12**	**5**	**4**	**16**	**17**
7	**6**	**16**	**5**	**0**	**5**	**30**

Order the data from least to greatest.

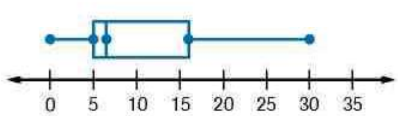
0 2 4 5 5 5 6 7 12 16 16 17 20 30

Q_1 = 5, $Q_2 = \frac{6+7}{2}$ or 6.5, Q_3 = 16

The interquartile range is 16 − 5 or 11. Check to see if there are any outliers.

$5 - 1.5(11) = -11.5$ $\quad$ $16 + 1.5(11) = 32.5$

There are no outliers.



Exercises **Draw a box-and-whisker plot for each set of data.**
See Example 1 on page 738.

29. The number of Calories in a serving of French fries at 13 restaurants are 250, 240, 220, 348, 199, 200, 125, 230, 274, 239, 212, 240, and 327.

30. Mrs. Lowery's class has the following scores on their math tests.
60, 70, 70, 75, 80, 85, 85, 90, 95, 100

31. The average daily temperatures on a beach in Florida for each month of one year are 52.4, 55.2, 61.1, 67.0, 73.4, 79.1, 81.6, 81.2, 78.1, 69.8, 61.9, and 55.1.

Chapter 13 Practice Test

Vocabulary and Concepts

In a matrix, identify each item described.

1. a vertical set of numbers
2. an entry in a matrix
3. a horizontal set of numbers
4. a constant multiplied by each element in the matrix
5. number of rows and columns

- **a.** element
- **b.** column
- **c.** row
- **d.** dimensions
- **e.** scalar

Skills and Applications

Identify the sample, suggest a population from which it was selected, and state whether it is *unbiased* (random) or *biased*. If unbiased, classify the sample as *simple*, *stratified*, or *systematic*. If biased, classify as *convenience* or *voluntary response*.

6. **DOGS** A veterinarian needs a sample of dogs in his kennel to be tested for fleas. She selects the first 5 dogs who run from the pen.
7. **LIBRARIES** A librarian wants to sample book titles checked out on Wednesday. He randomly chooses a book for each hour that the library is open.

If $W = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -1 \\ 2 & -2 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 4 & 2 & -1 \\ -2 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, $Y = \begin{bmatrix} 3 & -2 & 1 \\ -1 & -2 & 4 \end{bmatrix}$, **and** $Z = \begin{bmatrix} 3 & 1 & 6 \\ 4 & -1 & -1 \end{bmatrix}$, **find each sum, difference, or product. If the sum or difference does not exist, write *impossible*.**

8. $W + X$ **9.** $Y - Z$ **10.** $3X$ **11.** $-2Z$ **12.** $2W - Z$ **13.** $Y - 2Z$

Create a histogram to represent each data set.

14. 68 71 74 90 81 72 71 69 65 92 75 69 71 73 73
68 74 80 83 70 80 74 74 70 71

15. 10 40 50 52 22 50 60 90 41 51 90 40 75 63 53

Find the range, median, lower quartile, upper quartile, and interquartile range for each set of data. Identify any outliers.

16. 1055, 1075, 1095, 1125, 1005, 975, 1125, 1100, 1145, 1025, 1075

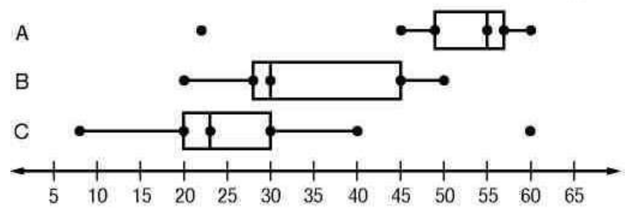
17. 0.4, 0.2, 0.5, 0.9, 0.3, 0.4, 0.5, 1.9, 0.5, 0.7, 0.8, 0.6, 0.2, 0.1, 0.4

Draw a box-and-whisker plot for each set of data.

18. 1, 3, 2, 2, 1, 9, 4, 6, 1, 10, 1, 4, 5, 10, 1, 3, 6

19. 14, 18, 9, 9, 12, 22, 16, 12, 14, 16, 15, 13, 9, 10, 11, 12

20. **STANDARDIZED TEST PRACTICE** Which box-and-whisker plot has the greatest interquartile range?



Ⓐ A Ⓑ B Ⓒ C Ⓓ They all have the same interquartile range.

Chapter 13 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. Which equation represents a line perpendicular to the graph of $y = 4x - 6$? (Lesson 5-6)

Ⓐ $y = \frac{1}{4}x + \frac{1}{6}$ Ⓑ $y = -\frac{1}{4}x + 2$

Ⓒ $y = -4x + 6$ Ⓓ $y = 4x + 6$

2. A certain number is proportional to another number in the ratio 3:5. If 8 is subtracted from the sum of the numbers, the result is 32. What is the greater number? (Lesson 7-2)

Ⓐ 15 Ⓑ 25

Ⓒ 35 Ⓓ 40

3. The expression $(x - 8)^2$ is equivalent to (Lesson 8-8)

Ⓐ $x^2 - 64$. Ⓑ $x^2 - 16x + 64$.

Ⓒ $x^2 + 16x + 64$. Ⓓ $x^2 + 64$.

4. What is the least y value of the graph of $y = x^2 - 4$? (Lesson 10-1)

Ⓐ 2 Ⓑ 0

Ⓒ -2 Ⓓ -4

5. The expression $3\sqrt{72} - 3\sqrt{2}$ is equivalent to (Lesson 11-2)

Ⓐ $3\sqrt{70}$. Ⓑ $3\sqrt{2}$.

Ⓒ $15\sqrt{2}$. Ⓓ $5\sqrt{2}$.

6. A 12-meter flagpole casts a 9-meter shadow. At the same time, the building next to it casts a 27-meter shadow. How tall is the building? (Lesson 11-6)

Ⓐ 20.25 m Ⓑ 36 m

Ⓒ 40 m Ⓓ 84 m

7. Students are conducting a poll at Cedar Grove High School to determine whether to change the school colors. Which would be the best place to find an unbiased sample of students who represent the entire student body? (Lesson 13-1)

Ⓐ a football practice

Ⓑ a freshmen class party

Ⓒ a Spanish class

Ⓓ the cafeteria

8. A Mars year is longer than an Earth year because Mars takes longer to orbit the Sun. The table shows a person's age in both Earth years and Mars years. The data represent which kind of function? (Lesson 13-3)

Earth	10	20	30	40	50
Mars	5.3	10.6	15.9	21.2	26.5

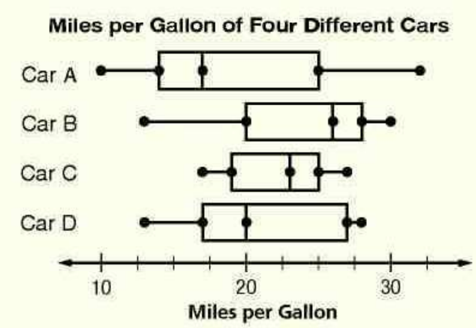
Ⓐ linear function

Ⓑ quadratic function

Ⓒ exponential function

Ⓓ rational function

Use the box-and-whisker plot for Questions 9 and 10.



9. Which car shows the least variation in miles per gallon? (Lesson 13-5)

Ⓐ A Ⓑ B Ⓒ C Ⓓ D

10. Which car model has the highest median miles per gallon? (Lesson 13-5)

Ⓐ A Ⓑ B Ⓒ C Ⓓ D

Aligned and verified by The Princeton Review

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

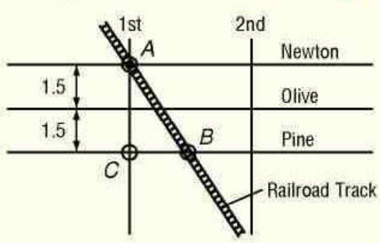
11. Factor $x^3 + 8x^2 + 16x$. (Lesson 9-3)

12. Solve $6x^2 + x - 2 = 0$ by factoring. (Lesson 9-4)

13. Simplify $\sqrt{4\sqrt[3]{27}}$. (Lesson 11-1)

14. Maren can do a job in 4 hours. Juliana can do the same job in 6 hours. Suppose Juliana works on the job for 2 hours and then is joined by Maren. Find the number of hours it will take both working together to finish the job. (Lesson 11-4)

15. The map below shows train tracks cutting across a grid of city streets. Newton Street is 1.5 miles apart from Olive Street, Olive Street is 1.5 miles apart from Pine Street, and the three streets are parallel to each other. If the distance between points A and B is 5 miles, then what is the distance in miles between points B and C? (Lesson 12-4)



The Princeton Review Test-Taking Tip

Questions 14 and 15 If a problem seems difficult, don't panic. Reread the question slowly and carefully. Always ask yourself, "What have I been asked to find?" and, "What information will help me find the answer?"

www.algebra1.com/standardized_test

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- **(A) the quantity in Column A is greater,**
- **(B) the quantity in Column B is greater,**
- **(C) the two quantities are equal, or**
- **(D) the relationship cannot be determined from the information given.**

	Column A	Column B
16.	the sum of the next three terms of the arithmetic sequence $-250, 83, 416, \ldots$	the 67th term of the arithmetic sequence $-49, 2, 53, \ldots$
17.	the root of $y = -0.25x^2 + x - 1$	the sum of the roots of $b = 3a^2 - 5a + 2$
18.	the value of x if $\frac{3x+1}{14} = \frac{x}{4}$	the value of y if $\frac{45}{4y+1} = \frac{10}{y}$

(16: Lesson 4-7) (17: Lesson 10-4) (18: Lesson 12-9)

Part 4 Open Ended

Record your answers on a sheet of paper. Show your work.

19. Construct a histogram for the following data. Use intervals of 40–50, 50–60, 60–70, 70–80, 80–90, and 90–100. (Lesson 13-3)
45, 62, 78, 84, 63, 73, 68, 91, 65, 80, 71, 87, 85, 77, 78, 80, 83, 87, 90, 91

20. In Exercise 19, what percent of the data lies within the tallest bar? (Lesson 13-3)

21. Draw a box-and-whisker plot of the following test scores. (Lesson 13-5)
24, 38, 47, 22, 40, 36, 25, 48, 30, 32, 45, 41, 34, 39, 40, 47, 40, 38, 42, 49, 45

Chapter

14 Probability

What You'll Learn

- **Lesson 14-1** Count outcomes using the Fundamental Counting Principle.
- **Lesson 14-2** Determine probabilities using permutations and combinations.
- **Lesson 14-3** Find probabilities of compound events.
- **Lesson 14-4** Use probability distributions.
- **Lesson 14-5** Use probability simulations.

Key Vocabulary

- **permutation** (p. 760)
- **combination** (p. 762)
- **compound event** (p. 769)
- **theoretical probability** (p. 782)
- **experimental probability** (p. 782)

Why It's Important

The United States Senate forms committees to focus on different issues. These committees are made up of senators from various states and political parties. There are many ways these committees could be formed. *You will learn how to find the number of possible committees in Lesson 14-2.*

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 14.

For Lessons 14-2 through 14-5 — Find Simple Probabilities

Determine the probability of each event if you randomly select a cube from a bag containing 6 red cubes, 3 blue cubes, 4 yellow cubes, and 1 green cube.
(For review, see Lesson 2-6.)

1. $P(\text{red})$ **2.** $P(\text{blue})$ **3.** $P(\text{yellow})$ **4.** $P(\text{not red})$

For Lesson 14-2 — Multiply Fractions

Find each product. *(For review, see pages 800 and 801.)*

5. $\frac{4}{5} \cdot \frac{3}{4}$ **6.** $\frac{5}{12} \cdot \frac{6}{11}$ **7.** $\frac{7}{20} \cdot \frac{4}{19}$

8. $\frac{4}{32} \cdot \frac{7}{32}$ **9.** $\frac{13}{52} \cdot \frac{4}{52}$ **10.** $\frac{56}{100} \cdot \frac{24}{100}$

For Lesson 14-4 — Write Decimals as Percents

Write each decimal as a percent. *(For review, see pages 804 and 805.)*

11. 0.725 **12.** 0.148 **13.** 0.4 **14.** 0.0168

For Lesson 14-5 — Write Fractions as Percents

Write each fraction as a percent. Round to the nearest tenth. *(For review, see pages 804 and 805.)*

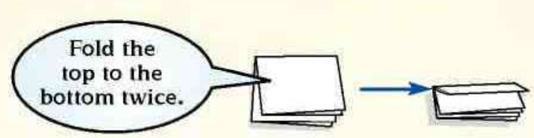
15. $\frac{7}{8}$ **16.** $\frac{33}{80}$ **17.** $\frac{107}{125}$ **18.** $\frac{625}{1024}$

FOLDABLES™ Study Organizer

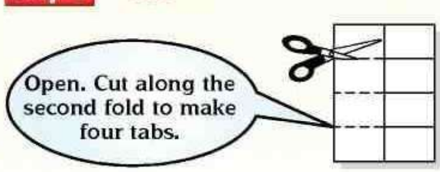
Make this Foldable to help you organize what you learn about probability. Begin with a sheet of plain $8\frac{1}{2}$" by 11" paper.

Step 1 Fold in Half

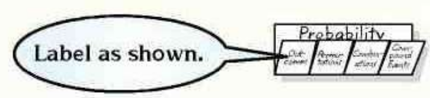
Step 2 Fold Again in Fourths



Step 3 Cut



Step 4 Label



Reading and Writing As you read and study the chapter, write notes and examples for each concept under the tabs.

14-1 Counting Outcomes

What You'll Learn

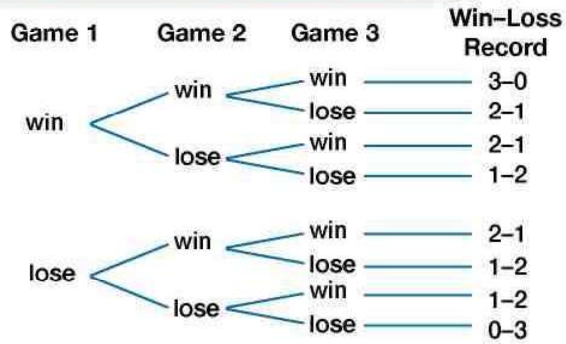
- Count outcomes using a tree diagram.
- Count outcomes using the Fundamental Counting Principle.

Vocabulary

- tree diagram
- sample space
- event
- Fundamental Counting Principle
- factorial

How are possible win–loss records counted in football?

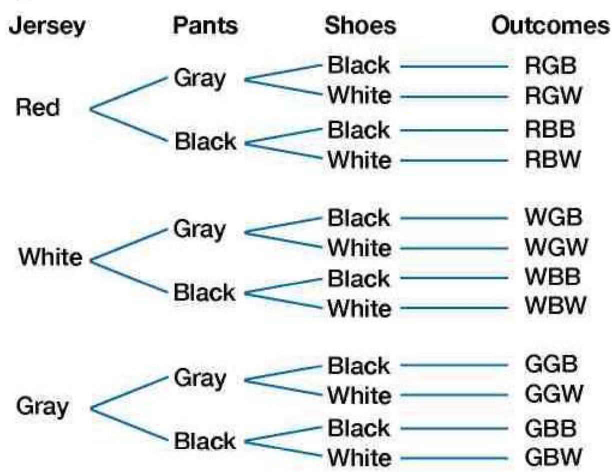
The championship in the Atlantic Coast Conference is decided by the number of conference wins. If there is a tie in conference wins, then the team with more nonconference wins is champion. If Florida State plays 3 nonconference games, the diagram at the right shows the different records they could have for those games.



TREE DIAGRAMS One method used for counting the number of possible outcomes is to draw a **tree diagram**. The last column of a tree diagram shows all of the possible outcomes. The list of all possible outcomes is called the **sample space**, while any collection of one or more outcomes in the sample space is called an **event**.

Example 1 Tree Diagram

A football team uses red jerseys for road games, white jerseys for home games, and gray jerseys for practice games. The team uses gray or black pants, and black or white shoes. Use a tree diagram to determine the number of possible uniforms.



The tree diagram shows that there are 12 possible uniforms.

THE FUNDAMENTAL COUNTING PRINCIPLE The number of possible uniforms in Example 1 can also be found by multiplying the number of choices for each item. If the team can choose from 3 different colored jerseys, 2 different colored pants, and 2 different colored pairs of shoes, there are $3 \cdot 2 \cdot 2$ or 12 possible uniforms. This example illustrates the **Fundamental Counting Principle**.

Key Concept — *Fundamental Counting Principle*

If an event M can occur in m ways and is followed by an event N that can occur in n ways, then the event M followed by event N can occur in $m \cdot n$ ways.

Example 2 Fundamental Counting Principle

The Uptown Deli offers a lunch special in which you can choose a sandwich, a side dish, and a beverage. If there are 10 different sandwiches, 12 different side dishes, and 7 different beverages from which to choose, how many different lunch specials can you order?

Multiply to find the number of lunch specials.

$$\underbrace{10}_{\text{sandwich choices}} \cdot \underbrace{12}_{\text{side dish choices}} \cdot \underbrace{7}_{\text{beverage choices}} = \underbrace{840}_{\text{number of specials}}$$

The number of different lunch specials is 840.

Example 3 Counting Arrangements

Mackenzie is setting up a display of the ten most popular video games from the previous week. If she places the games side-by-side on a shelf, in how many different ways can she arrange them?

The number of ways to arrange the games can be found by multiplying the number of choices for each position.

- Mackenzie has ten games from which to choose for the first position.
- After choosing a game for the first position, there are nine games left from which to choose for the second position.
- There are now eight choices for the third position.
- This process continues until there is only one choice left for the last position.

Let n represent the number of arrangements.

$n = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 3,628,800

There are 3,628,800 different ways to arrange the video games.

The expression $n = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ used in Example 3 can be written as 10! using a **factorial**.

Key Concept — *Factorial*

- **Words** The expression $n!$, read n factorial, where n is greater than zero, is the product of all positive integers beginning with n and counting backward to 1.
- **Symbols** $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$
- **Example** $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 120

By definition, $0! = 1$.

Example 4 Factorial

Find the value of each expression.

a. 6!

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ Definition of factorial

$= 720$ Simplify.

b. 10!

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ Definition of factorial

$= 3{,}628{,}800$ Simplify.

Example 5 Use Factorials to Solve a Problem

ROLLER COASTERS Zach and Kurt are going to an amusement park. They cannot decide in which order to ride the 12 roller coasters in the park.

a. How many different orders can they ride all of the roller coasters if they ride each once?

Use a factorial.

$12! = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ Definition of factorial

$= 479{,}001{,}600$ Simplify.

There are 479,001,600 ways in which Zach and Kurt can ride all 12 roller coasters.

b. If they only have time to ride 8 of the roller coasters, how many ways can they do this?

Use the Fundamental Counting Principle to find the sample space.

$s = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ Fundamental Counting Principle

$= 19{,}958{,}400$ Simplify.

There are 19,958,400 ways for Zach and Kurt to ride 8 of the roller coasters.

More About...

Roller Coasters

In 2000, there were 646 roller coasters in the United States.

Type	Number
Wood	118
Steel	445
Inverted	35
Stand Up	10
Suspended	11
Wild Mouse	27

Source: Roller Coaster Database

Check for Understanding

Concept Check

1. **OPEN ENDED** Give an example of an event that has $7 \cdot 6$ or 42 outcomes.
2. **Draw** a tree diagram to represent the outcomes of tossing a coin three times.
3. **Explain** what the notation 5! means.

Guided Practice

For Exercises 4–6, suppose the spinner at the right is spun three times.

4. Draw a tree diagram to show the sample space.
5. How many outcomes are possible?
6. How many outcomes involve both green and blue?
7. Find the value of 8!.

Application

8. **SCHOOL** In a science class, each student must choose a lab project from a list of 15, write a paper on one of 6 topics, and give a presentation about one of 8 subjects. How many different ways can students choose to do their assignments?

Practice and Apply

Homework Help

For Exercises	See Examples
9, 10, 19	1
11–14	4
15–18, 20–22	2, 3, 5

Extra Practice

See page 851.

Draw a tree diagram to show the sample space for each event. Determine the number of possible outcomes.

9. earning an A, B, or C in English, Math, and Science classes
10. buying a computer with a choice of a CD-ROM, a CD recorder, or a DVD drive, one of 2 monitors, and either a printer or a scanner

Find the value of each expression.

11. 4!
12. 7!
13. 11!
14. 13!

15. Three dice, one red, one white, and one blue are rolled. How many outcomes are possible?

16. How many outfits are possible if you choose one each of 5 shirts, 3 pairs of pants, 3 pairs of shoes, and 4 jackets?

17. **TRAVEL** Suppose four different airlines fly from Seattle to Denver. Those same four airlines and two others fly from Denver to St. Louis. If there are no direct flights from Seattle to St. Louis, in how many ways can a traveler book a flight from Seattle to St. Louis?

COMMUNICATIONS **For Exercises 18 and 19, use the following information.**
A new 3-digit area code is needed in a certain area to accommodate new telephone numbers.

18. If the first digit must be odd, the second digit must be a 0 or a 1, and the third digit can be anything, how many area codes are possible?
19. Draw a tree diagram to show the different area codes using 4 or 5 for the first digit, 0 or 1 for the second digit, and 7, 8, or 9 for the third digit.

SOCCER **For Exercises 20–22, use the following information.**
The Columbus Crew are playing the D.C. United in a best three-out-of-five championship soccer series.

20. What are the possible outcomes of the series?
21. How many outcomes require only four games be played to determine the champion?
22. How many ways can D.C. United win the championship?

23. **CRITICAL THINKING** To get to and from school, Tucker can walk, ride his bike, or get a ride with a friend. Suppose that one week he walked 60% of the time, rode his bike 20% of the time, and rode with his friend 20% of the time. How many outcomes represent this situation? Assume that he returns home the same way that he went to school.

24. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

    How are possible win–loss records counted in football?

    Include the following in your answer:
    - a few sentences describing how a tree diagram can be used to count the wins and losses of a football team, and
    - a demonstration of how to find the number of possible outcomes for a team that plays 4 home games.

Standardized Test Practice

25. Evaluate 9!.

(A) 362,880 (B) 40,320 (C) 36 (D) 8

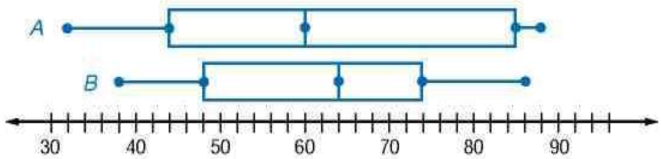
26. A car manufacturer offers a sports car in 4 different models with 6 different option packages. Each model is available in 12 different colors. How many different possibilities are available for this car?

(A) 96 (B) 144 (C) 288 (D) 384

Maintain Your Skills

Mixed Review

For Exercises 27–30, use box-and-whisker plots A and B. *(Lesson 13-5)*



27. Determine the least value, greatest value, lower quartile, upper quartile, and median for each plot.
28. Which set of data contains the least value?
29. Which plot has the smaller interquartile range?
30. Which plot has the greater range?

For Exercises 31–34, use the stem-and-leaf plot. *(Lesson 13-4)*

31. Find the range of the data.
32. What is the median?
33. Determine the upper quartile, lower quartile, and interquartile range of the data.
34. Identify any outliers.

Stem	Leaf
3	0 1 4 5
4	4 4 8
5	6 9
6	6 8
7	1 6
8	0 1
9	

Find each sum or difference. *(Lesson 12-7)*

35. $\frac{2x + 1}{3x - 1} + \frac{x + 4}{x - 2}$

36. $\frac{4n}{2n + 6} + \frac{3}{n + 3}$

37. $\frac{3z + 2}{3z - 6} - \frac{z + 2}{z^2 - 4}$

38. $\frac{m - n}{m + n} - \frac{1}{m^2 - n^2}$

Study Tip

Deck of Cards
In this text, a *standard deck of cards* always means a deck of 52 playing cards. There are 4 suits—clubs (black), diamonds (red), hearts (red), and spades (black)—with 13 cards in each suit.

Solve each equation. *(Lesson 11-3)*

39. $5\sqrt{2n^2 - 28} = 20$

40. $\sqrt{5x^2 - 7} = 2x$

41. $\sqrt{x + 2} = x - 4$

Solve each equation by completing the square. Round to the nearest tenth if necessary. *(Lesson 10-3)*

42. $b^2 - 6b + 4 = 0$

43. $n^2 + 8n - 5 = 0$

44. $x^2 - 11x - 17 = 0$

45. $2p^2 + 10p + 3 = 0$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **One card is drawn at random from a standard deck of cards. Find each probability.** *(To review **simple probability**, see Lesson 2-6.)*

46. $P(10)$

47. $P(\text{ace})$

48. $P(\text{red } 5)$

49. $P(\text{queen of clubs})$

50. $P(\text{even number})$

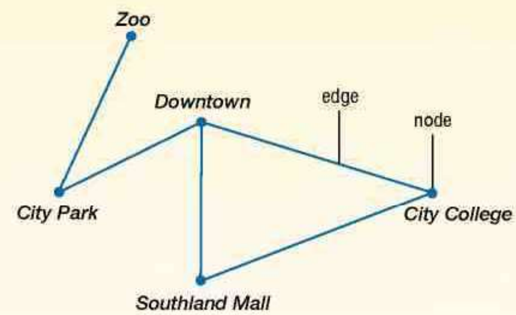
51. $P(3 \text{ or king})$

Algebra Activity

A Follow-Up of Lesson 14-1

Finite Graphs

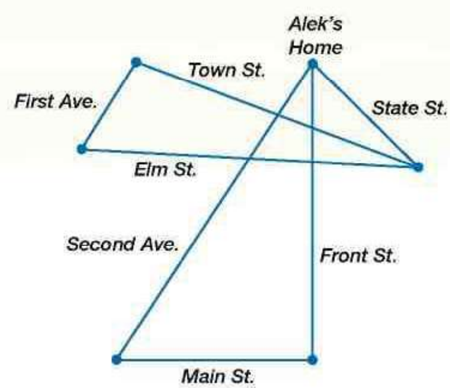
The City Bus Company provides daily bus service between City College and Southland Mall, City College and downtown, downtown and Southland Mall, downtown and City Park, and City Park and the zoo. The daily routes can be represented using a **finite graph** like the one at the right.



The graph is called a **network**, and each point on the graph is called a **node**. The paths connecting the nodes are called **edges**. A network is said to be **traceable** if all of the nodes can be connected, and each edge can be covered exactly once when the graph is used.

Collect the Data

The graph represents the streets on Alek's newspaper route. To get his route completed as quickly as possible, Alek would like to ride his bike down each street only once.



- Copy the graph onto your paper.
- Beginning at Alek's home, trace over his route without lifting your pencil. Remember to trace each edge only once.
- Compare your graph with those of your classmates.

Analyze the Data

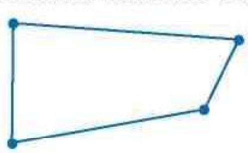
1. Is Alek's route traceable? If so, describe his route.

2. Is there more than one traceable route that begins at Alek's house? If so, how many?

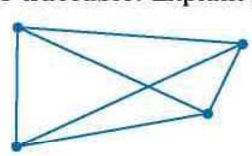
3. Suppose it does not matter where Alek starts his route. How many traceable routes are possible now?

Determine whether each graph is traceable. Explain your reasoning.

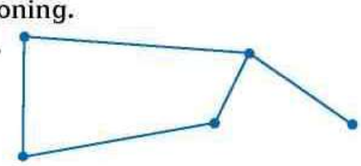
4.



5.



6.



7. The campus for Centerburgh High School has five buildings built around the edge of a circular courtyard. There is a sidewalk between each pair of buildings.

a. Draw a graph of the campus.

b. Is the graph traceable?

c. Suppose that there is not a sidewalk between the pairs of adjacent buildings. Is it possible to reach all five buildings without walking down any sidewalk more than once?

8. Make a conjecture for a rule to determine whether a graph is traceable.

14-2 Permutations and Combinations

What You'll Learn

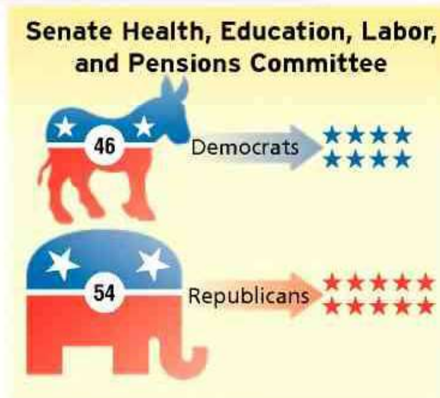
- Determine probabilities using permutations.
- Determine probabilities using combinations.

Vocabulary

- permutation
- combination

How can combinations be used to form committees?

The United States Senate forms various committees by selecting senators from both political parties. The Senate Health, Education, Labor, and Pensions Committee of the 106th Congress was made up of 10 Republican senators and 8 Democratic senators. How many different ways could the committee have been selected? The members of the committee were selected in no particular order. This is an example of an arrangement called a combination.



PERMUTATIONS An arrangement or listing in which order or placement is important is called a **permutation**.

Example 1 Tree Diagram Permutation

EMPLOYMENT The manager of a coffee shop needs to hire two employees, one to work at the counter and one to work at the drive-through window. Katie, Bob, Alicia, and Jeremiah all applied for a job. How many possible ways are there for the manager to place the applicants?

Use a tree diagram to show the possible arrangements.

Counter	Drive-Through	Outcomes
Katie (K)	Bob	KB
	Alicia	KA
	Jeremiah	KJ
Bob (B)	Katie	BK
	Alicia	BA
	Jeremiah	BJ
Alicia (A)	Jeremiah	AJ
	Katie	AK
	Bob	AB
Jeremiah (J)	Katie	JK
	Bob	JB
	Alicia	JA

There are 12 different ways for the 4 applicants to hold the 2 positions.

Study Tip

Common Misconception

When arranging two objects A and B using a permutation, the arrangement AB is different from the arrangement BA.

In Example 1, the positions are in a specific order, so each arrangement is unique. The symbol ${}_4P_2$ denotes the number of permutations when arranging 4 applicants in 2 positions. You can also use the Fundamental Counting Principle to determine the number of permutations.

$$
\begin{aligned}
{}_4P_2 &= \underbrace{4}_{\text{ways to choose first employee}} \cdot \underbrace{3}_{\text{ways to choose second employee}} \\
&= 4 \cdot 3 \cdot \frac{2 \cdot 1}{2 \cdot 1} && \frac{2 \cdot 1}{2 \cdot 1} = 1 \\
&= \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} && \text{Multiply.} \\
&= \frac{4!}{2!} && 4 \cdot 3 \cdot 2 \cdot 1 = 4!,\ 2 \cdot 1 = 2!
\end{aligned}
$$

In general, ${}_nP_r$ is used to denote the number of permutations of n objects taken r at a time.

Key Concept — Permutation

- **Words** The number of permutations of n objects taken r at a time is the quotient of $n!$ and $(n - r)!$.
- **Symbols** ${}_nP_r = \frac{n!}{(n - r)!}$

Example 2 Permutation

Find ${}_{10}P_6$.

$$
\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!} && \text{Definition of } {}_nP_r \\
{}_{10}P_6 &= \frac{10!}{(10-6)!} && n = 10, r = 6 \\
{}_{10}P_6 &= \frac{10!}{4!} && \text{Subtract.} \\
{}_{10}P_6 &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}^{1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}_{1}} && \text{Definition of factorial} \\
{}_{10}P_6 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \text{ or } 151{,}200 && \text{Simplify.}
\end{aligned}
$$

There are 151,200 permutations of 10 objects taken 6 at a time.

Permutations are often used to find the probability of events occurring.

Example 3 Permutation and Probability

A word processing program requires a user to enter a 7-digit registration code made up of the digits 1, 2, 4, 5, 6, 7, and 9. Each number has to be used, and no number can be used more than once.

a. How many different registration codes are possible?

Since the order of the numbers in the code is important, this situation is a permutation of 7 digits taken 7 at a time.

$$
\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!} && \text{Definition of permutation} \\
{}_7P_7 &= \frac{7!}{(7-7)!} && n = 7, r = 7 \\
{}_7P_7 &= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} \text{ or } 5040 && \text{Definition of factorial}
\end{aligned}
$$

There are 5040 possible codes with the digits 1, 2, 4, 5, 6, 7, and 9.

Study Tip

Permutations
The number of permutations of n objects taken n at a time is $n!$.
${}_nP_n = n!$

b. What is the probability that the first three digits of the code are even numbers?

Use the Fundamental Counting Principle to determine the number of ways for the first three digits to be even.

- There are 3 even digits and 4 odd digits.
- The number of choices for the first three digits, if they are even, is $3 \cdot 2 \cdot 1$.
- The number of choices for the remaining odd digits is $4 \cdot 3 \cdot 2 \cdot 1$.
- The number of favorable outcomes is $3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 144. There are 144 ways for this event to occur out of the 5040 possible permutations.

$$P(\text{first 3 digits even}) = \frac{144}{5040} \begin{array}{l} \leftarrow \text{number of favorable outcomes} \\ \leftarrow \text{number of possible outcomes} \end{array}$$

$$= \frac{1}{35} \quad \text{Simplify.}$$

The probability that the first three digits of the code are even is $\frac{1}{35}$ or about 3%.

Study Tip

Look Back

To review **probability**, see Lesson 2-6.

COMBINATIONS An arrangement or listing in which order is not important is called a **combination**. For example, if you are choosing 2 salad ingredients from a list of 10, the order in which you choose the ingredients does not matter.

Key Concept — *Combination*

- **Words** The number of combinations of n objects taken r at a time is the quotient of $n!$ and $(n - r)!r!$.
- **Symbols** ${}_nC_r = \frac{n!}{(n - r)!r!}$

Standardized Test Practice

Example 4 Combination

Multiple-Choice Test Item

The students of Mr. DeLuca's homeroom had to choose 4 out of the 7 people who were nominated to serve on the Student Council. How many different groups of students could be selected?

(A) 840	(B) 210
(C) 35	(D) 24

Read the Test Item

The order in which the students are chosen does not matter, so this situation represents a combination of 7 people taken 4 at a time.

Solve the Test Item

$${}_nC_r = \frac{n!}{(n - r)!r!} \quad \text{Definition of combination}$$

$${}_7C_4 = \frac{7!}{(7 - 4)!4!} \quad n = 7, r = 4$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}^{1}}{3 \cdot 2 \cdot 1 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}_{1}} \quad \text{Definition of factorial}$$

$$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} \text{ or } 35 \quad \text{Simplify.}$$

There are 35 different groups of students that could be selected. Choice C is correct.

The Princeton Review

Test-Taking Tip

Read each question carefully to determine whether the situation involves a permutation or a combination. Often, the answer choices include examples of both.

Combinations and the products of combinations can be used to determine probabilities.

Example 5 Use Combinations

SCHOOL A science teacher at Sunnydale High School needs to choose 12 students out of 16 to serve as peer tutors. A group of 7 seniors, 5 juniors, and 4 sophomores have volunteered to be tutors.

a. How many different ways can the teacher choose 12 students?

The order in which the students are chosen does not matter, so we must find the number of combinations of 16 students taken 12 at a time.

$${}_nC_r = \frac{n!}{(n-r)!r!} \quad \text{Definition of combination}$$

$${}_{16}C_{12} = \frac{16!}{(16-12)!12!} \quad n = 16, r = 12$$

$$= \frac{16!}{4!12!} \quad 16 - 12 = 4$$

$$= \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot \overset{1}{\cancel{12!}}}{4! \cdot \underset{1}{\cancel{12!}}} \quad \text{Divide by the GCF, 12!.}$$

$$= \frac{43{,}680}{24} \text{ or } 1820 \quad \text{Simplify.}$$

There are 1820 ways to choose 12 students out of 16.

b. If the students are chosen randomly, what is the probability that 4 seniors, 4 juniors, and 4 sophomores will be selected?

There are three questions to consider.

- How many ways can 4 seniors be chosen from 7?
- How many ways can 4 juniors be chosen from 5?
- How many ways can 4 sophomores be chosen from 4?

Using the Fundamental Counting Principle, the answer can be determined with the product of the three combinations.

$$\underbrace{({}_7C_4)}_{\text{ways to choose 4 seniors out of 7}} \cdot \underbrace{({}_5C_4)}_{\text{ways to choose 4 juniors out of 5}} \cdot \underbrace{({}_4C_4)}_{\text{ways to choose 4 sophomores out of 4}}$$

$$({}_7C_4)({}_5C_4)({}_4C_4) = \frac{7!}{(7-4)!4!} \cdot \frac{5!}{(5-4)!4!} \cdot \frac{4!}{(4-4)!4!} \quad \text{Definition of combination}$$

$$= \frac{7!}{3!4!} \cdot \frac{5!}{1!4!} \cdot \frac{4!}{0!4!} \quad \text{Simplify.}$$

$$= \frac{7 \cdot 6 \cdot 5}{3!} \cdot \frac{5}{1} \quad \text{Divide by the GCF, 4!.}$$

$$= 175 \quad \text{Simplify.}$$

There are 175 ways to choose this particular combination out of 1820 possible combinations.

$$P(\text{4 seniors, 4 juniors, 4 sophomores}) = \frac{175}{1820} \quad \begin{matrix} \leftarrow \text{number of favorable outcomes} \\ \leftarrow \text{number of possible outcomes} \end{matrix}$$

$$= \frac{5}{52} \quad \text{Simplify.}$$

The probability that the science teacher will randomly select 4 seniors, 4 juniors, and 4 sophomores is $\frac{5}{52}$ or about 10%.

Study Tip

Combinations
The number of combinations of n objects taken n at a time is 1. ${}_nC_n = 1$

Check for Understanding

Concept Check

1. **OPEN ENDED** Describe the difference between a permutation and a combination. Then give an example of each.

2. **Demonstrate** and explain why ${}_nC_r = 1$ whenever $n = r$. What does ${}_nP_r$ always equal when $n = r$?

3. **FIND THE ERROR** Eric and Alisa are taking a trip to Washington, D.C. Their tour bus stops at the Lincoln Memorial, the Jefferson Memorial, the Washington Monument, the White House, the Capitol Building, the Supreme Court, and the Pentagon. Both are finding the number of ways they can choose to visit 5 of these 7 sites.

Eric	Alisa
${}_7C_5 = \frac{7!}{2!}$ or 2520	${}_7C_5 = \frac{7!}{2!\,5!}$ or 21

Who is correct? Explain your reasoning.

Guided Practice

Determine whether each situation involves a *permutation* or *combination*. Explain your reasoning.

4. choosing 6 books from a selection of 12 for summer reading
5. choosing digits for a personal identification number

Evaluate each expression.

6. ${}_8P_5$
7. ${}_7C_5$
8. $({}_{10}P_5)({}_3P_2)$
9. $({}_6C_2)({}_4C_3)$

For Exercises 10–12, use the following information.
The digits 0 through 9 are written on index cards. Three of the cards are randomly selected to form a 3-digit code.

10. Does this situation represent a permutation or a combination? Explain.
11. How many different codes are possible?
12. What is the probability that all 3 digits will be odd?

Standardized Test Practice

13. A diner offers a choice of two side items from the list with each entrée. How many ways can two items be selected?

Ⓐ 15 Ⓑ 28
Ⓒ 30 Ⓓ 56

Side Items

French fries	mixed vegetables
baked potato	rice pilaf
cole slaw	baked beans
small salad	applesauce

Practice and Apply

Determine whether each situation involves a *permutation* or *combination*. Explain your reasoning.

14. team captains for the soccer team
15. three mannequins in a display window
16. a hand of 10 cards from a selection of 52
17. the batting order of the New York Yankees

Homework Help

For Exercises	See Examples
14–21, 34 36, 40	1, 4
22–33, 35, 37–39, 41–49	2, 3, 5

Extra Practice
See page 851.

18. first place and runner-up winners for the table tennis tournament
19. a selection of 5 DVDs from a group of eight
20. selection of 2 candy bars from six equally-sized bars
21. the selection of 2 trombones, 3 clarinets, and 2 trumpets for a jazz combo

Evaluate each expression.

22. ${}_{12}P_3$
23. ${}_{4}P_1$
24. ${}_{6}C_6$
25. ${}_{7}C_3$
26. ${}_{15}C_3$
27. ${}_{20}C_8$
28. ${}_{15}P_3$
29. ${}_{16}P_5$
30. $({}_{7}P_7)({}_{7}P_1)$
31. $({}_{20}P_2)({}_{16}P_4)$
32. $({}_{3}C_2)({}_{7}C_4)$
33. $({}_{8}C_5)({}_{5}P_5)$

SOFTBALL **For Exercises 34 and 35, use the following information.**
The manager of a softball team needs to prepare a batting lineup using her nine starting players.

34. Is this situation a permutation or a combination?
35. How many different lineups can she make?

More About. . .

Softball

The game of softball was developed in 1888 as an indoor sport for practicing baseball during the winter months.

Source: www.encyclopedia.com

SCHOOL **For Exercises 36–39, use the following information.**
Mrs. Moyer's class has to choose 4 out of 12 people to serve on an activity committee.

36. Does the selection of the students involve a permutation or a combination? Explain.
37. How many different groups of students could be selected?
38. Suppose the students are selected for the positions of chairperson, activities planner, activity leader, and treasurer. How many different groups of students could be selected?
39. What is the probability that any one of the students is chosen to be the chairperson?

GAMES **For Exercises 40–42, use the following information.**
In your turn of a certain game, you roll five dice at the same time.

40. Do the outcomes of rolling the five dice represent a permutation or a combination? Explain.
41. How many outcomes are possible?
42. What is the probability that all five dice show the same number on one roll?

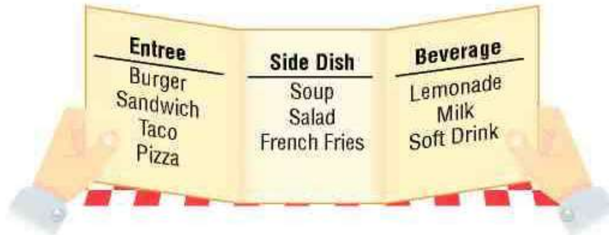
BUSINESS **For Exercises 43 and 44, use the following information.**
There are six positions available in the research department of a software company. Of the applicants, 15 are men and 10 are women.

43. In how many ways could 4 men and 2 women be chosen if each were equally qualified?
44. What is the probability that five women would be selected if the positions were randomly filled?

TRACK **For Exercises 45 and 46, use the following information.**
Central High School is competing against West High School at a track meet. Each team entered 4 girls to run the 1600-meter event. The top three finishers are awarded medals.

45. How many different ways can the runners place first, second, and third?
46. If all eight runners have an equal chance of placing, what is the probability that the first and second place finishers are from West and the third place finisher is from Central?

DINING **For Exercises 47–49, use the following information.**
For lunch in the school cafeteria, you can select one item from each category to get the daily combo.



47. Find the number of possible meal combinations.

48. If a side dish is chosen at random, what is the probability that a student will choose soup?

49. What is the probability that a student will randomly choose a sandwich and soup?

CRITICAL THINKING **For Exercises 50 and 51, use the following information.**
Larisa is trying to solve a word puzzle. She needs to arrange the letters H, P, S, T, A, E, and O into a two-word arrangement.

50. How many different arrangements of the letters can she make?

51. Assuming that each arrangement has an equal chance of occurring, what is the probability that she will form the words *tap shoe* on her first try?

WebQuest

You can use permutations and combinations to analyze data on U.S. schools. Visit www.algebra1.com/webquest to continue work on your WebQuest project.

SWIMMING **For Exercises 52–54, use the following information.**
A swimming coach plans to pick four swimmers out of a group of 6 to form the 400-meter freestyle relay team.

52. How many different teams can he form?

53. The coach must decide in which order the four swimmers should swim. He timed the swimmers in each possible order and chose the best time. How many relays did the four swimmers have to swim so that the coach could collect all of the data necessary?

54. If Tomás is chosen to be on the team, what is the probability that he will swim in the third leg?

55. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can combinations be used to form committees?

Include the following in your answer:

- a few sentences explaining why forming a Senate committee is a combination, and
- an explanation of how to find the number of ways to select the committee if committee positions are based upon seniority.

Standardized Test Practice

56. There are 12 songs on a CD. If 10 songs are played randomly and each song is played once, how many arrangements are there?

Ⓐ 479,001,600 Ⓑ 239,500,800 Ⓒ 66 Ⓓ 1

57. Julie remembered that the 4 digits of her locker combination were 4, 9, 15, and 22, but not their order. What is the maximum number of attempts Julie could make before her locker opened?

Ⓐ 4 Ⓑ 16 Ⓒ 24 Ⓓ 256

CONTENTS

Maintain Your Skills

Mixed Review

58. The Sanchez family acts as a host family for a foreign exchange student during each school year. It is equally likely that they will host a girl or a boy. How many different ways can they host boys and girls over the next four years? *(Lesson 14-1)*

STATISTICS For Exercises 59–62, use the table at the right. *(Lesson 13-5)*

59. Make a box-and-whisker plot of the data.

60. What is the range of the data?

61. Identify the lower and upper quartiles.

62. Name any outliers.

Highest Paying Occupations in America

Occupation	Median Salary
Physician	$148,000
Dentist	$93,000
Lobbyist	$91,300
Management Consultant	$61,900
Lawyer	$60,500
Electrical Engineer	$59,100
School Principal	$57,300
Aeronautical Engineer	$56,700
Airline Pilot	$56,500
Civil Engineer	$55,800

Source: U.S. Bureau of Labor Statistics

Online Research **Data Update** For current data on the highest-paying occupations, visit: www.algebra1.com/data_update

Simplify each expression. *(Lesson 12-2)*

63. $\frac{x+3}{x^2+6x+9}$ **64.** $\frac{x^2-49}{x^2-2x-35}$ **65.** $\frac{n^2-n-20}{n^2+9n+20}$

Find the distance between each pair of points whose coordinates are given. Express answers in simplest radical form and as decimal approximations rounded to the nearest hundredth if necessary. *(Lesson 11-5)*

66. $(12, 20), (16, 34)$ **67.** $(-18, 7), (2, 15)$ **68.** $(-2, 5), \left(-\frac{1}{2}, 3\right)$

Solve each equation by using the Quadratic Formula. Approximate irrational roots to the nearest hundredth. *(Lesson 10-4)*

69. $m^2 + 4m + 2 = 0$ **70.** $2s^2 + s - 15 = 0$ **71.** $2n^2 - n = 4$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each sum or difference.**
*(To review **fractions**, see pages 798 and 799.)*

72. $\frac{8}{52} + \frac{4}{52}$ **73.** $\frac{7}{32} + \frac{5}{8}$ **74.** $\frac{5}{15} + \frac{6}{15} - \frac{2}{15}$

75. $\frac{15}{24} + \frac{11}{24} - \frac{3}{4}$ **76.** $\frac{2}{3} + \frac{15}{36} - \frac{1}{4}$ **77.** $\frac{16}{25} + \frac{3}{10} - \frac{1}{4}$

Practice Quiz 1

Lessons 14-1 and 14-2

Find the number of outcomes for each event. *(Lesson 14-1)*

1. A die is rolled and two coins are tossed.

2. A certain model of mountain bike comes in 5 sizes, 4 colors, with regular or off-road tires, and with a choice of 1 of 5 accessories.

Find each value. *(Lesson 14-2)*

3. ${}_{13}C_8$ **4.** ${}_{9}P_6$

5. A flower bouquet has 5 carnations, 6 roses, and 3 lilies. If four flowers are selected at random, what is the probability of selecting two roses and two lilies? *(Lesson 14-2)*

Reading Mathematics

Mathematical Words and Related Words

You may have noticed that many words used in mathematics contain roots of other words and are closely related to other English words. You can use the more familiar meanings of these related words to better understand mathematical meanings.

The table shows two mathematical terms along with related words and their meanings as well as additional notes.

Mathematical Term and Meaning	Related Words and Meanings	Notes
combination A combination is a selection of distinct objects from a group of objects, where the order in which they were selected does not matter.	*combine* (n): a harvesting machine that performs many functions *binary*: a base-two numerical system	*Combine* originally meant to put just two things together; it now means to put any number of things together.
permutation A permutation is an arrangement of distinct objects from a group of objects, where the arrangement is in a certain order.	*mutation*: a change in genes or other characteristics *commute*: to change places; for example, $2 + 5 = 5 + 2$	

Notice how the meanings of the related words can give an insight to the meanings of the mathematical terms.

Reading to Learn

1. Do the related words of combination and permutation help you to remember their mathematical meanings? Explain.

2. What is a similarity and a difference between the mathematical meanings of combination and permutation?

3. **RESEARCH** Use the Internet or other reference to find the mathematical meaning of the word *factorial* and meanings of at least two related words. How are these meanings connected?

4. **RESEARCH** Use the Internet or other reference to find the meanings of the word *probability* and its Latin origins *probus* and *probare*. Compare the three.

14-3 Probability of Compound Events

What You'll Learn

- Find the probability of two independent events or dependent events.
- Find the probability of two mutually exclusive or inclusive events.

Vocabulary

- simple event
- compound event
- independent events
- dependent events
- complements
- mutually exclusive
- inclusive

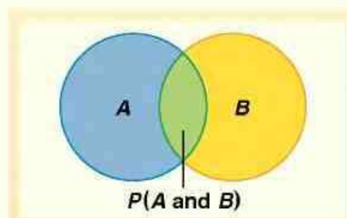
How are probabilities used by meteorologists?

The weather forecast for the weekend calls for rain. By using the probabilities for both days, we can find other probabilities for the weekend. What is the probability that it will rain on both days? only on Saturday? Saturday or Sunday?

INDEPENDENT AND DEPENDENT EVENTS A single event, like rain on Saturday, is called a **simple event**. Suppose you wanted to determine the probability that it will rain both Saturday and Sunday. This is an example of a **compound event**, which is made up of two or more simple events. The weather on Saturday does not affect the weather on Sunday. These two events are called **independent events** because the outcome of one event does not affect the outcome of the other.

Key Concept — Probability of Independent Events

- **Words** If two events, A and B, are independent, then the probability of both events occurring is the product of the probability of A and the probability of B.
- **Symbols** $P(A \text{ and } B) = P(A) \cdot P(B)$
- **Model**



Example 1 Independent Events

Refer to the application above. Find the probability that it will rain on Saturday and Sunday.

$$P(A \text{ and } B) = P(A) \cdot P(B) \qquad \text{Definition of independent events}$$

$$P(\text{Saturday and Sunday}) = \underbrace{P(\text{Saturday})}_{} \cdot \underbrace{P(\text{Sunday})}_{}$$

$$= 0.4 \cdot 0.8 \qquad 40\% = 0.4 \text{ and } 80\% = 0.8$$

$$= 0.32 \qquad \text{Multiply.}$$

The probability that it will rain on Saturday and Sunday is 32%.

When the outcome of one event affects the outcome of another event, the events are **dependent events**. For example, drawing a card from a deck, not returning it, then drawing a second card are dependent events because the drawing of the second card is dependent on the drawing of the first card.

Key Concept — *Probability of Dependent Events*

- **Words** If two events, A and B, are dependent, then the probability of both events occurring is the product of the probability of A and the probability of B after A occurs.
- **Symbols** $P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$

Example 2 Dependent Events

A bag contains 8 red marbles, 12 blue marbles, 9 yellow marbles, and 11 green marbles. Three marbles are randomly drawn from the bag and not replaced. Find each probability if the marbles are drawn in the order indicated.

a. *P*(red, blue, green)

The selection of the first marble affects the selection of the next marble since there is one less marble from which to choose. So, the events are dependent.

First marble: $P(\text{red}) = \frac{8}{40}$ or $\frac{1}{5}$ ← number of red marbles / ← total number of marbles

Second marble: $P(\text{blue}) = \frac{12}{39}$ or $\frac{4}{13}$ ← number of blue marbles / ← number of marbles remaining

Third marble: $P(\text{green}) = \frac{11}{38}$ ← number of green marbles / ← number of marbles remaining

$$\begin{aligned} P(\text{red, blue, green}) &= \underbrace{P(\text{red})} \cdot \underbrace{P(\text{blue})} \cdot \underbrace{P(\text{green})} \\ &= \frac{1}{5} \cdot \frac{4}{13} \cdot \frac{11}{38} && \text{Substitution} \\ &= \frac{44}{2470} \text{ or } \frac{22}{1235} && \text{Multiply.} \end{aligned}$$

The probability of drawing red, blue, and green marbles is $\frac{22}{1235}$.

b. *P*(blue, yellow, yellow)

Notice that after selecting a yellow marble, not only is there one fewer marble from which to choose, there is also one fewer yellow marble.

$$\begin{aligned} P(\text{blue, yellow, yellow}) &= \underbrace{P(\text{blue})} \cdot \underbrace{P(\text{yellow})} \cdot \underbrace{P(\text{yellow})} \\ &= \frac{12}{40} \cdot \frac{9}{39} \cdot \frac{8}{38} && \text{Substitution} \\ &= \frac{864}{59{,}280} \text{ or } \frac{18}{1235} && \text{Multiply.} \end{aligned}$$

The probability of drawing a blue and then two yellow marbles is $\frac{18}{1235}$.

Study Tip

More Than Two Dependent Events
Notice that the formula for the probability of dependent events can be applied to more than two events.

c. *P*(red, yellow, *not* green)

Since the marble that is not green is selected after the first two marbles, there are 29 − 2 or 27 marbles that are not green.

$$\begin{aligned} P(\text{red, yellow, } \textit{not} \text{ green}) &= \underbrace{P(\text{red})} \cdot \underbrace{P(\text{yellow})} \cdot \underbrace{P(\text{not green})} \\ &= \frac{8}{40} \cdot \frac{9}{39} \cdot \frac{27}{38} \\ &= \frac{1944}{59{,}280} \text{ or } \frac{81}{2470} \end{aligned}$$

The probability of drawing a red, a yellow, and *not* a green marble is $\frac{81}{2470}$.

Study Tip

Reading Math
A complement is one of two parts that make up a whole.

In part **c** of Example 2, the events for drawing a marble that is green and for drawing a marble that is *not* green are called **complements**. Consider the probabilities for drawing the third marble.

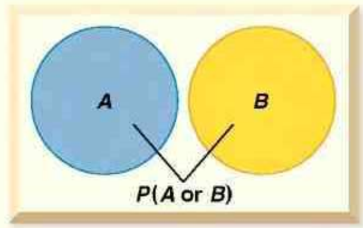
$$\underbrace{\frac{11}{38}}_{P(\text{green})} + \underbrace{\frac{27}{38}}_{P(\textit{not}\text{ green})} = \underbrace{1}_{\text{sum of probabilities}}$$

This is always true for any two complementary events.

MUTUALLY EXCLUSIVE AND INCLUSIVE EVENTS Events that cannot occur at the same time are called **mutually exclusive**. Suppose you want to find the probability of rolling a 2 *or* a 4 on a die. Since a die cannot show both a 2 and a 4 at the same time, the events are mutually exclusive.

Key Concept — *Mutually Exclusive Events*

- **Words** If two events, A and B, are mutually exclusive, then the probability that either A or B occurs is the sum of their probabilities.
- **Symbols** $P(A \text{ or } B) = P(A) + P(B)$
- **Model**



Example 3 Mutually Exclusive Events

During a magic trick, a magician randomly draws one card from a standard deck of cards. What is the probability that the card drawn is a heart or a diamond?

Since a card cannot be both a heart and a diamond, the events are mutually exclusive.

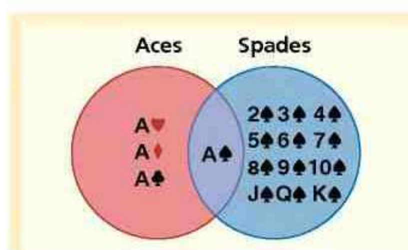
$P(\text{heart}) = \frac{13}{52}$ or $\frac{1}{4}$ ← number of hearts / ← total number of cards

$P(\text{diamond}) = \frac{13}{52}$ or $\frac{1}{4}$ ← number of diamonds / ← total number of cards

$$\begin{aligned} P(\text{heart or diamond}) &= \underbrace{P(\text{heart})}_{\frac{1}{4}} + \underbrace{P(\text{diamond})}_{\frac{1}{4}} && \text{Definition of mutually exclusive events} \\ &= \frac{1}{4} + \frac{1}{4} && \text{Substitution} \\ &= \frac{2}{4} \text{ or } \frac{1}{2} && \text{Add.} \end{aligned}$$

The probability of drawing a heart or a diamond is $\frac{1}{2}$.

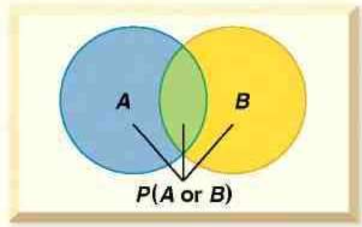
Suppose you wanted to find the probability of randomly selecting an ace or a spade from a standard deck of cards. Since it is possible to draw a card that is both an ace and a spade, these events are not mutually exclusive. They are called **inclusive** events.



If the formula for the probability of mutually exclusive events is used, the probability of drawing an ace of spades is counted twice, once for an ace and once for a spade. To correct this, you must subtract the probability of drawing the ace of spades from the sum of the individual probabilities.

Key Concept — *Probability of Inclusive Events*

- **Words** If two events, A and B, are inclusive, then the probability that either A or B occurs is the sum of their probabilities decreased by the probability of both occurring.
- **Model**



- **Symbols** $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example 4 *Inclusive Events*

GAMES **In the game of bingo, balls or tiles are numbered 1 through 75. These numbers correspond to columns on a bingo card. The numbers 1 through 15 can appear in the B column, 16 through 30 in the I column, 31 through 45 in the N column, 46 through 60 in the G column, and 61 through 75 in the O column. A number is selected at random. What is the probability that it is a multiple of 4 or is in the O column?**

Since the numbers 64, 68, and 72 are multiples of 4 and they can be in the O column, these events are inclusive.

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ Definition of inclusive events

$P(\text{multiple of 4 or O column})$

$$= \underbrace{P(\text{multiple of 4})} + \underbrace{P(\text{O column})} - \underbrace{P(\text{multiple of 4 and O column})}$$

$$= \frac{18}{75} + \frac{15}{75} - \frac{3}{75} \quad \text{Substitution}$$

$$= \frac{18 + 15 - 3}{75} \quad \text{LCD is 75.}$$

$$= \frac{30}{75} \text{ or } \frac{2}{5} \quad \text{Simplify.}$$

The probability of a number being a multiple of 4 or in the O column is $\frac{2}{5}$ or 40%.

Check for Understanding

Concept Check

1. **Explain** the difference between a simple event and a compound event.
2. **Find a counterexample** for the following statement.
   If two events are independent, then the probability of both events occurring is less than 1.
3. **OPEN ENDED** Explain how dependent events are different than independent events. Give specific examples in your explanation.

4. **FIND THE ERROR** On the school debate team, 6 of the 14 girls are seniors, and 9 of the 20 boys are seniors. Chloe and Amber are both seniors on the team. Each girl calculated the probability that either a girl or a senior would randomly be selected to argue a position at a state debate.

Chloe	Amber
P(girl or senior)	P(girl or senior)
$= \frac{14}{34} + \frac{15}{34} - \frac{6}{34}$	$= \frac{6}{34} + \frac{15}{34} - \frac{14}{34}$
$= \frac{23}{34}$	$= \frac{7}{34}$

Who is correct? Explain your reasoning.

Guided Practice

A bin contains 8 blue chips, 5 red chips, 6 green chips, and 2 yellow chips. Find each probability.

5. drawing a red chip, replacing it, then drawing a green chip
6. selecting two yellow chips without replacement
7. choosing green, then blue, then red, replacing each chip after it is drawn
8. choosing green, then blue, then red without replacing each chip

A student is selected at random from a group of 12 male and 12 female students. There are 3 male students and 3 female students from each of the 9th, 10th, 11th, and 12th grades. Find each probability.

9. P(9th or 12th grader)
10. P(10th grader or female)
11. P(male or female)
12. P(male or not 11th grader)

Application

BUSINESS For Exercises 13–15, use the following information.
Mr. Salyer is a buyer for an electronics store. He received a shipment of 5 DVD players in which one is defective. He randomly chose 3 of the DVD players to test.

13. Determine whether choosing the DVD players are independent or dependent events.
14. What is the probability that he selected the defective player?
15. Suppose the defective player is one of the three that Mr. Salyer tested. What is the probability that the last one tested was the defective one?

Practice and Apply

Homework Help

For Exercises	See Examples
16–19, 24, 25, 28–31	2
20–23, 32–34	1
26, 27, 41, 44, 45	4
36–40, 42, 43, 46, 47	3

Extra Practice

See page 851.

A bag contains 2 red, 6 blue, 7 yellow, and 3 orange marbles. Once a marble is selected, it is not replaced. Find each probability.

16. P(2 orange)
17. P(blue, then red)
18. P(2 yellows in a row then orange)
19. P(blue, then yellow, then red)

A die is rolled and a spinner like the one at the right is spun. Find each probability.

20. P(3 and D)
21. P(an odd number and a vowel)
22. P(a prime number and A)
23. P(2 and A, B, or C)

Raffle tickets numbered 1 through 30 are placed in a box. Tickets for a second raffle numbered 21 to 48 are placed in another box. One ticket is randomly drawn from each box. Find each probability.

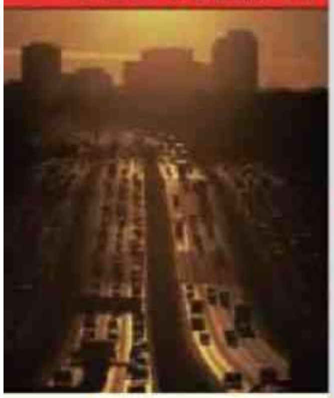
24. Both tickets are even.

25. Both tickets are greater than 20 and less than 30.

26. The first ticket is greater than 10, and the second ticket is less than 40 or odd.

27. The first ticket is greater than 12 or prime, and the second ticket is a multiple of 6 or a multiple of 4.

More About. . .



Safety

In the U.S., 60% of carbon monoxide emissions come from transportation sources. The largest contributor is highway motor vehicles. In urban areas, motor vehicles can contribute more than 90%.

Source: U.S. Environmental Protection Agency

SAFETY **For Exercises 28–31, use the following information.**
A carbon monoxide detector system uses two sensors, A and B. If carbon monoxide is present, there is a 96% chance that sensor A will detect it, a 92% chance that sensor B will detect it, and a 90% chance that both sensors will detect it.

28. Draw a Venn diagram that illustrates this situation.

29. If carbon monoxide is present, what is the probability that it will be detected?

30. What is the probability that carbon monoxide would go undetected?

31. Do sensors A and B operate independently of each other? Explain.

BIOLOGY **For Exercises 32–34, use the table and following information.**
Each person carries two types of genes for eye color. The gene for brown eyes (B) is dominant over the gene for blue eyes (b). That is, if a person has one gene for brown eyes and the other for blue, that person will have brown eyes. The Punnett square at the right shows the genes for two parents.

	B	b
B	BB	Bb
b	Bb	bb

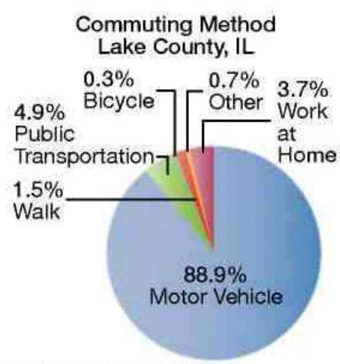
32. What is the probability that any child will have blue eyes?

33. What is the probability that the couple's two children both have brown eyes?

34. Find the probability that the first or the second child has blue eyes.

35. **RESEARCH** Use the Internet or other reference to investigate various blood types. Use this information to determine the probability of a child having blood type O if the father has blood type A(Ai) and the mother has blood type B(Bi).

TRANSPORTATION **For Exercises 36 and 37, use the graph and the following information.**
The U.S. Census Bureau conducted an American Community Survey in Lake County, Illinois. The circle graph at the right shows the survey results of how people commute to work.



Source: U.S. Census Bureau

36. If a person from Lake County was chosen at random, what is the probability that he or she uses public transportation or walks to work?

37. If offices are being built in Lake County to accommodate 400 employees, what is the minimum number of parking spaces an architect should plan for the parking lot?

ECONOMICS **For Exercises 38–40, use the table below that compares the total number of hourly workers who earned the minimum wage of $5.15 with those making less than minimum wage.**

Number of Hourly Workers (thousands)			
Age (years)	Total	At $5.15	Below $5.15
16–24	15,793	1145	2080
25+	55,287	970	2043

Source: U.S. Bureau of Labor Statistics

38. If an hourly worker was chosen at random, what is the probability that he or she earned minimum wage? less than minimum wage?

39. What is the probability that a randomly-chosen hourly worker earned less than or equal to minimum wage?

40. If you randomly chose an hourly worker from each age group, which would you expect to have earned no more than minimum wage? Explain.

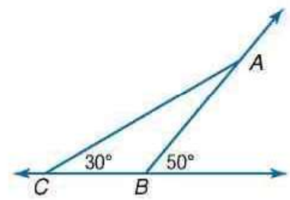
More About. . .

Economics

The first federal minimum wage was set in 1938 at $0.25 per hour. That was the equivalent of $3.05 in 2000.

Source: U.S. Department of Labor

GEOMETRY **For Exercises 41–43, use the figure and the following information.**
Two of the six angles in the figure are chosen at random.

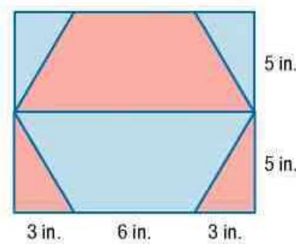


41. What is the probability of choosing an angle inside $\triangle ABC$ or an obtuse angle?

42. What is the probability of selecting a straight angle or a right angle inside $\triangle ABC$?

43. Find the probability of picking a 20° angle or a 130° angle.

A dart is thrown at a dartboard like the one at the right. If the dart can land anywhere on the board, find the probability that it lands in each of the following.



44. a triangle or a red region

45. a trapezoid or a blue region

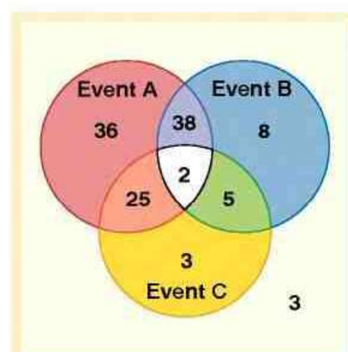
46. a blue triangle or a red triangle

47. a square or a hexagon

CRITICAL THINKING **For Exercises 48–51, use the following information.**
A sample of high school students were asked if they:

A) drive a car to school,
B) are involved in after-school activities, or
C) have a part-time job.

The results of the survey are shown in the Venn diagram.



48. How many students were surveyed?

49. How many students said that they drive a car to school?

50. If a high school student is chosen at random, what is the probability that he or she does all three?

51. What is the probability that a randomly-chosen student drives a car to school or is involved in after-school activities or has a part-time job?

52. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are probabilities used by meteorologists?

Include the following in your answer:

- a few sentences about how compound probabilities can be used to predict the weather, and
- assuming that the events are independent, the probability that it will rain either Saturday or Sunday if there is a 30% chance of rain on Saturday and a 50% chance of rain Sunday.

Standardized Test Practice

53. A bag contains 8 red marbles, 5 blue marbles, 4 green marbles, and 7 yellow marbles. Five marbles are randomly drawn from the bag and not replaced. What is the probability that the first three marbles drawn are red?

Ⓐ $\frac{1}{27}$ Ⓑ $\frac{28}{1771}$ Ⓒ $\frac{7}{253}$ Ⓓ $\frac{7}{288}$

54. Yolanda usually makes 80% of her free throws. What is the probability that she will make at least one free throw in her next three attempts?

Ⓐ 99.2% Ⓑ 51.2% Ⓒ 38.4% Ⓓ 9.6%

Maintain Your Skills

Mixed Review

CIVICS **For Exercises 55 and 56, use the following information.**
The Stratford town council wants to form a 3-person parks committee. Five people have applied to be on the committee. *(Lesson 14-2)*

55. How many committees are possible?

56. What is the probability of any one person being selected if each has an equal chance?

57. **BUSINESS** A real estate developer built a strip mall with seven different-sized stores. Ten small businesses have shown interest in renting space in the mall. The developer must decide which business would be best suited for each store. How many different arrangements are possible? *(Lesson 14-1)*

Find each sum or difference. *(Lesson 13-2)*

58. $\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix}$

59. $\begin{bmatrix} -4 & -5 \\ 8 & 8 \end{bmatrix} - \begin{bmatrix} -9 & -7 \\ 4 & 9 \end{bmatrix}$

60. Find the quotient of $\frac{2m^2 + 7m - 15}{m + 5}$ and $\frac{9m^2 - 4}{3m + 2}$. *(Lesson 12-4)*

Simplify. *(Lesson 11-1)*

61. $\sqrt{45}$

62. $\sqrt{128}$

63. $\sqrt{40b^4}$

64. $\sqrt{120a^3b}$

65. $3\sqrt{7} \cdot 6\sqrt{2}$

66. $\sqrt{3}(\sqrt{3} + \sqrt{6})$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Express each fraction as a decimal. Round to the nearest thousandth.** *(To review* ***expressing fractions as decimals****, see pages 804 and 805.)*

67. $\frac{9}{24}$

68. $\frac{2}{15}$

69. $\frac{63}{128}$

70. $\frac{5}{52}$

71. $\frac{8}{36}$

72. $\frac{11}{38}$

73. $\frac{81}{2470}$

74. $\frac{18}{1235}$

75. $\frac{128}{3570}$

14-4 Probability Distributions

What You'll Learn

- Use random variables to compute probability.
- Use probability distributions to solve real-world problems.

Vocabulary

- random variable
- probability distribution
- probability histogram

How can a pet store owner use a probability distribution?

The owner of a pet store asked customers how many pets they owned. The results of this survey are shown in the table.

Number of Pets	Number of Customers
0	3
1	37
2	33
3	18
4	9

RANDOM VARIABLES AND PROBABILITY A **random variable** is a variable whose value is the numerical outcome of a random event. In the situation above, we can let the random variable X represent the number of pets owned. Thus, X can equal 0, 1, 2, 3, or 4.

Example 1 Random Variable

Refer to the application above.

a. Find the probability that a randomly-chosen customer has 2 pets.

There is only one outcome in which there are 2 pets owned, and there are 100 survey results.

$$P(X = 2) = \frac{\text{2 pets owned}}{\text{customers surveyed}}$$
$$= \frac{33}{100}$$

The probability that a randomly-chosen customer has 2 pets is $\frac{33}{100}$ or 33%.

b. Find the probability that a randomly-chosen customer has at least 3 pets.

There are $18 + 9$ or 27 outcomes in which a customer owns at least 3 pets.

$$P(X \geq 3) = \frac{27}{100}$$

The probability that a randomly-chosen customer owns at least 3 pets is $\frac{27}{100}$ or 27%.

Study Tip

Reading Math
The notation $P(X = 2)$ means the same as P(2 pets), the probability of a customer having 2 pets.

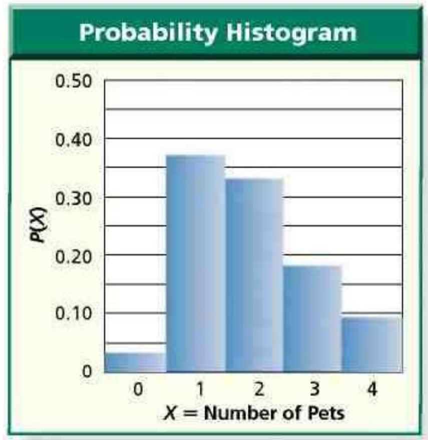
PROBABILITY DISTRIBUTIONS The probability of every possible value of the random variable X is called a **probability distribution**.

Key Concept — *Properties of Probability Distributions*

1. The probability of each value of X is greater than or equal to 0 and less than or equal to 1.
2. The probabilities of all of the values of X add up to 1.

The probability distribution for a random variable can be given in a table or in a **probability histogram**. The probability distribution and a probability histogram for the application at the beginning of the lesson are shown below.

Probability Distribution Table	
X = Number of Pets	$P(X)$
0	0.03
1	0.37
2	0.33
3	0.18
4	0.09



Example 2 Probability Distribution

CARS **The table shows the probability distribution of the number of vehicles per household for the Columbus, Ohio, area.**

Vehicles per Household Columbus, OH	
X = Number of Vehicles	Probability
0	0.10
1	0.42
2	0.36
3+	0.12

Source: U.S. Census Bureau

a. Show that the distribution is valid.

Check to see that each property holds.

1. For each value of X, the probability is greater than or equal to 0 and less than or equal to 1.
2. $0.10 + 0.42 + 0.36 + 0.12 = 1$, so the probabilities add up to 1.

b. What is the probability that a household has fewer than 2 vehicles?

Recall that the probability of a compound event is the sum of the probabilities of each individual event.

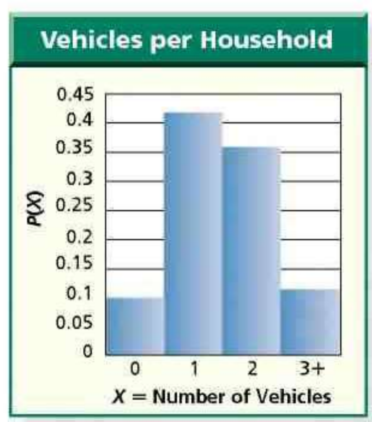
The probability of a household having fewer than 2 vehicles is the sum of the probability of 0 vehicles and the probability of 1 vehicle.

$P(X < 2) = P(X = 0) + P(X = 1)$ Sum of individual probabilities

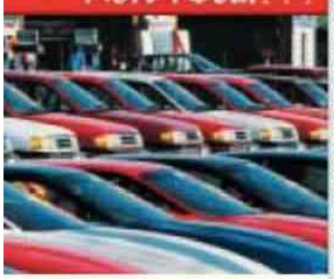
$= 0.10 + 0.42$ or 0.52 $P(X = 0) = 0.10$, $P(X = 1) = 0.42$

c. Make a probability histogram of the data.

Draw and label the vertical and horizontal axes. Remember to use equal intervals on each axis. Include a title.



More About...



Cars

In 1900, there were 8000 registered cars in the United States. By 1998, there were over 131 million registered cars. This is an increase of more than 1,637,400%.

Source: *The World Almanac*

CONTENTS

Check for Understanding

Concept Check

1. **List** the conditions that must be satisfied to have a valid probability distribution.
2. **Explain** why the probability of tossing a coin three times and getting 1 head and 2 tails is the same as the probability of getting 1 tail and 2 heads.
3. **OPEN ENDED** Describe a situation that could be displayed in a probability histogram.

Guided Practice

For Exercises 4–6, use the table that shows the possible sums when rolling two dice and the number of ways each sum can be found.

Sum of Two Dice	2	3	4	5	6	7	8	9	10	11	12
Ways to Achieve Sum	1	2	3	4	5	6	5	4	3	2	1

4. Draw a table to show the sample space of all possible outcomes.
5. Find the probabilities for $X = 4$, $X = 5$, and $X = 6$.
6. What is the probability that the sum of two dice is greater than 6 on three separate rolls?

Application

GRADES For Exercises 7–9, use the table that shows a class's grade distribution, where A = 4.0, B = 3.0, C = 2.0, D = 1.0, and F = 0.

X = Grade	0	1.0	2.0	3.0	4.0
Probability	0.05	0.10	0.40	0.40	0.05

7. Show that the probability distribution is valid.
8. What is the probability that a student passes the course?
9. What is the probability that a student chosen at random from the class receives a grade of B or better?

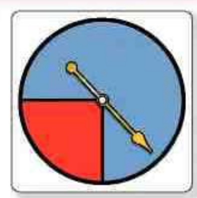
Practice and Apply

Homework Help

For Exercises	See Examples
10, 11, 14, 18	1
12, 13, 15–17, 19–22	2

Extra Practice
See page 852.

For Exercises 10–13, the spinner shown is spun three times.



10. Write the sample space with all possible outcomes.
11. Find the probability distribution X, where X represents the number of times the spinner lands on blue for $X = 0$, $X = 1$, $X = 2$, and $X = 3$.
12. Make a probability histogram.
13. Do all possible outcomes have an equal chance of occurring? Explain.

SALES For Exercises 14–17, use the following information.
A music store manager takes an inventory of the top 10 CDs sold each week. After several weeks, the manager has enough information to estimate sales and make a probability distribution table.

Number of Top 10 CDs Sold Each Week	0–100	101–200	201–300	301–401	401–500
Probability	0.10	0.15	0.40	0.25	0.10

14. Define a random variable and list its values.
15. Show that this is a valid probability distribution.
16. In a given week, what is the probability that fewer than 400 CDs sell?
17. In a given week, what is the probability that more than 200 CDs sell?

EDUCATION **For Exercises 18–20, use the table that shows the education level of persons aged 25 and older in the United States.**

X = Level of Education	Probability
Some High School	0.167
High School Graduate	0.333
Some College	0.173
Associate's Degree	0.075
Bachelor's Degree	0.170
Advanced Degree	0.082

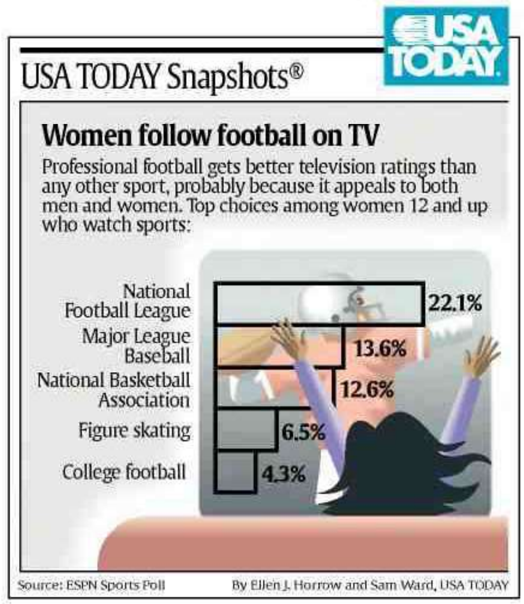
Source: U.S. Census Bureau

18. If a person was randomly selected, what is the probability that he or she completed at most some college?

19. Make a probability histogram of the data.

20. Explain how you can find the probability that a randomly selected person has earned at least a bachelor's degree.

SPORTS **For Exercises 21 and 22, use the graph that shows the sports most watched by women on TV.**



21. Determine whether this is a valid probability distribution. Justify your answer.

22. Based on the graph, in a group of 35 women how many would you expect to say they watch figure skating?

23. CRITICAL THINKING Suppose a married couple has children until they have a girl. Let the random variable X represent the number of children in their family.

a. Calculate the probability distribution for $X = 1, 2, 3,$ and 4.

b. Find the probability that the couple will have more than 4 children.

24. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.

How can a pet store owner use a probability distribution?

Include the following in your answer:

- a sentence or two describing how to create a probability distribution, and
- an explanation of how the store owner could use a probability distribution to establish a frequent buyer program.

Standardized Test Practice

25. The table shows the probability distribution for the number of heads when four coins are tossed. What is the probability that there are no more than two heads showing on a random toss?

X = Number of Heads	0	1	2	3	4
Probability $P(X)$	0.0625	0.25	0.375	0.25	0.0625

(A) 0.6875 (B) 0.375 (C) 0.875 (D) 0.3125

26. On a random roll of two dice, what is the probability that the sum of the numbers showing is less than 5?

(A) 0.08 (B) 0.17 (C) 0.11 (D) 0.28

Maintain Your Skills

Mixed Review

A card is drawn from a standard deck of 52 cards. Find each probability. *(Lesson 14-3)*

27. P(ace or 10) **28.** P(3 or diamond) **29.** P(odd number or spade)

Evaluate. *(Lesson 14-2)*

30. ${}_{10}C_7$ **31.** ${}_{12}C_5$ **32.** $({}_6P_3)({}_5P_3)$

Let $A = \begin{bmatrix} 1 & 4 \\ 5 & 7 \end{bmatrix}$ **and** $B = \begin{bmatrix} -3 & 0 \\ -2 & 5 \end{bmatrix}$. *(Lesson 13-2)*

33. Find $A + B$. **34.** Find $B - A$.

Write an inverse variation equation that relates x and y. Assume that y varies inversely as x. Then solve. *(Lesson 12-1)*

35. If $y = -2.4$ when $x = -0.6$, find y when $x = 1.8$.

36. If $y = 4$ when $x = -1$, find x when $y = -3$.

Simplify each expression. *(Lesson 11-2)*

37. $3\sqrt{8} + 7\sqrt{2}$ **38.** $2\sqrt{3} + \sqrt{12}$ **39.** $3\sqrt{7} - 2\sqrt{28}$

SAVINGS For Exercises 40–42, use the following information.
Selena is investing her $900 tax refund in a certificate of deposit that matures in 4 years. The interest rate is 8.25% compounded quarterly. *(Lesson 10-6)*

40. Determine the balance in the account after 4 years.

41. Her friend Monique invests the same amount of money at the same interest rate, but her bank compounds interest monthly. Determine how much she will have after 4 years.

42. Which type of compounding appears more profitable? Explain.

Getting Ready for the Next Lesson

PREREQUISITE SKILL Write each fraction as a percent rounded to the nearest whole number. *(To review **writing fractions as percents**, see pages 804 and 805.)*

43. $\frac{16}{80}$ **44.** $\frac{20}{52}$ **45.** $\frac{30}{114}$

46. $\frac{57}{120}$ **47.** $\frac{72}{340}$ **48.** $\frac{54}{162}$

Practice Quiz 2

Lessons 14-3 and 14-4

For Exercises 1–3, use the probability distribution for the number of people in a household. *(Lesson 14-4)*

1. Show that the probability distribution is valid.

2. If a household is chosen at random, what is the probability that 4 or more people live in it?

3. Make a histogram of the data.

American Households

X = Number of People	Probability
1	0.25
2	0.32
3	0.18
4	0.15
5	0.07
6	0.02
7+	0.01

Source: U.S. Census Bureau

A ten-sided die, numbered 1 through 10, is rolled. Find each probability.

4. P(odd or greater than 4)

5. P(less than 3 or greater than 7)

14-5 Probability Simulations

What You'll Learn

- Use theoretical and experimental probability to represent and solve problems involving uncertainty.
- Perform probability simulations to model real-world situations involving uncertainty.

Vocabulary

- theoretical probability
- experimental probability
- relative frequency
- empirical study
- simulation

How can probability simulations be used in health care?

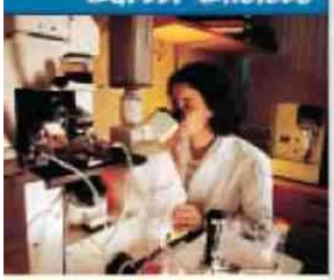
A pharmaceutical company is developing a new medication to treat a certain heart condition. Based on similar drugs, researchers at the company expect the new drug to work successfully in 70% of patients.

To test the drug's effectiveness, the company performs three clinical studies. Each study involves 100 volunteers who use the drug for six months. The results of the studies are shown in the table.

Study Of New Medication

Result	Study 1	Study 2	Study 3
Expected Success Rate	70%	70%	70%
Condition Improved	61%	74%	67%
No Improvement	39%	25%	33%
Condition Worsened	0%	1%	0%

Career Choices



Medical Researcher

Many medical researchers conduct research to advance knowledge of living organisms, including viruses and bacteria.

Online Research
For information about a career as a medical researcher, visit: www.algebra1.com/careers

THEORETICAL AND EXPERIMENTAL PROBABILITY The probability we have used to describe events in previous lessons is theoretical probability. **Theoretical probabilities** are determined mathematically and describe what should happen. In the situation above, the expected success rate of 70% is a theoretical probability.

A second type of probability we can use is **experimental probability**, which is determined using data from tests or experiments. Experimental probability is the ratio of the number of times an outcome occurred to the total number of events or trials. This ratio is also known as the **relative frequency**.

$$\text{experimental probability} = \frac{\text{frequency of an outcome}}{\text{total number of trials}}$$

Example 1 Experimental Probability

MEDICAL RESEARCH Refer to the application at the beginning of the lesson. What is the experimental probability that the drug was successful for a patient in Study 1?

In Study 1, the drug worked successfully in 61 of the 100 patients.

$$\text{experimental probability} = \frac{61}{100} \quad \begin{matrix} \leftarrow \text{frequency of successes} \\ \leftarrow \text{total number of patients} \end{matrix}$$

The experimental probability of Study 1 is $\frac{61}{100}$ or 61%.

It is often useful to perform an experiment repeatedly, collect and combine the data, and analyze the results. This is known as an **empirical study**.

Example 2 Empirical Study

Refer to the application at the beginning of the lesson. What is the experimental probability that the drug was successful for all three studies?

The number of successful outcomes of the three studies was 61 + 74 + 67 or 202 out of the 300 total patients.

experimental probability $= \frac{202}{300}$ or $\frac{101}{150}$

The experimental probability of the three studies was $\frac{101}{150}$ or about 67%.

PERFORMING SIMULATIONS A method that is often used to find experimental probability is a simulation. A **simulation** allows you to use objects to act out an event that would be difficult or impractical to perform.

Algebra Activity

Simulations

Collect the Data

- Roll a die 20 times. Record the value on the die after each roll.
- Determine the experimental probability distribution for X, the value on the die.
- Combine your results with the rest of the class to find the experimental probability distribution for X given the new number of trials.
  (20 · the number of students in your class)

Analyze the Data

1. Find the theoretical probability of rolling a 2.
2. Find the theoretical probability of rolling a 1 or a 6.
3. Find the theoretical probability of rolling a value less than 4.
4. Compare the experimental and theoretical probabilities. Which pair of probabilities was closer to each other: your individual probabilities or your class's probabilities?
5. Suppose each person rolls the die 50 times. Explain how this would affect the experimental probabilities for the class.

Make a Conjecture

6. What can you conclude about the relationship between the number of experiments in a simulation and the experimental probability?

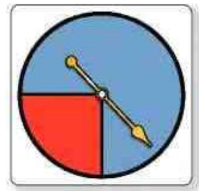
You can conduct simulations of the outcomes for many problems by using one or more objects such as dice, coins, marbles, or spinners. The objects you choose should have the same number of outcomes as the number of possible outcomes of the problem, and all outcomes should be equally likely.

Example 3 Simulation

In one season, Malcolm made 75% of the field goals he attempted.

a. What could be used to simulate his kicking a field goal? Explain.

You could use a spinner like the one at the right, where 75% of the spinner represents making a field goal.



b. Describe a way to simulate his next 8 attempts.

Spin the spinner once to simulate a kick. Record the result, then repeat this 7 more times.

Example 4 Theoretical and Experimental Probability

DOGS **Ali raises purebred dogs. One of her dogs is expecting a litter of four puppies, and Ali would like to figure out the most likely mix of male and female puppies. Assume that $P(\text{male}) = P(\text{female}) = \frac{1}{2}$.**

a. What objects can be used to model the possible outcomes of the puppies?

Each puppy can be male or female, so there are $2 \cdot 2 \cdot 2 \cdot 2$ or 16 possible outcomes for the litter. Use a simulation that also has 2 outcomes for each of 4 events. One possible simulation would be to toss four coins, one for each puppy, with heads representing female and tails representing male.

Study Tip

Alternative Simulation

You could also create a spinner with two even parts and spin it 4 times to simulate the outcomes of the puppies.

b. Find the theoretical probability that there will be two female and two male puppies.

There are 16 possible outcomes, and the number of combinations that have two female and two male puppies is ${}_4C_2$ or 6. So the theoretical probability is $\frac{6}{16}$ or $\frac{3}{8}$.

c. The results of a simulation Ali performed are shown in the table below. What is the experimental probability that there will be three male puppies?

Outcomes	Frequency
4 female, 0 male	3
3 female, 1 male	13
2 female, 2 male	18
1 female, 3 male	12
0 female, 4 male	4

Ali performed 50 trials and 12 of those resulted in three males. So, the experimental probability is $\frac{12}{50}$ or 24%.

d. How does the experimental probability compare to the theoretical probability of a litter with three males?

Theoretical probability

$$P(\text{3 males}) = \frac{{}_4C_3}{16} \quad \begin{array}{l} \leftarrow \text{combinations with 3 male puppies} \\ \leftarrow \text{possible outcomes} \end{array}$$

$$= \frac{4}{16} \text{ or } 25\% \quad \text{Simplify.}$$

The experimental probability, 24%, is very close to the theoretical probability.

Check for Understanding

Concept Check

1. **Explain** why it is useful to carry out an empirical study when calculating experimental probabilities.
2. **Analyze** the relationship between the theoretical and experimental probability of an event as the number of trials in a simulation increases.
3. **OPEN ENDED** Describe a situation that could be represented by a simulation. What objects would you use for this experiment?
4. **Tell** whether the theoretical probability and the experimental probability of an event are *sometimes*, *always*, or *never* the same.

Guided Practice

5. So far this season, Rita has made 60% of her free throws. Describe a simulation that could be used to predict the outcome of her next 25 free throws.

For Exercises 6–8, roll a die 25 times and record your results.

6. Based on your results, what is the probability of rolling a 3?
7. Based on your results, what is the probability of rolling a 5 or an odd number?
8. Compare your results to the theoretical probabilities.

Application

ASTRONOMY **For Exercises 9–12, use the following information.**
Enrique is writing a report about meteorites and wants to determine the probability that a meteor reaching Earth's surface hits land. He knows that 70% of Earth's surface is covered by water. He places 7 blue marbles and 3 brown marbles in a bag to represent hitting water $\left(\frac{7}{10}\right)$ and hitting land $\left(\frac{3}{10}\right)$. He draws a marble from the bag, records the color, and then replaces the marble. The table shows the results of his experiment.

Blue	Brown
56	19

9. Did Enrique choose an appropriate simulation for his research? Explain.
10. What is the theoretical probability that a meteorite reaching Earth's surface hits land?
11. Based on his results, what is the probability that a meteorite hits land?
12. Using the experimental probability, how many of the next 500 meteorites that strike Earth would you expect to hit land?

Practice and Apply

Homework Help

For Exercises	See Examples
13–16	3
17–21, 25–31	4
22–24	1, 2

Extra Practice
See page 852.

13. What could you use to simulate the outcome of guessing on 15 true-false questions?
14. There are 12 cans of cola, 8 cans of diet cola, and 4 cans of root beer in a cooler. What could be used for a simulation determining the probability of randomly picking any one type of soft drink?

For Exercises 15 and 16, use the following information.
Central City Mall is randomly giving each shopper one of 12 different gifts during the holidays.

15. What could be used to perform a simulation of this situation? Explain your choice.
16. How could you use this simulation to model the next 100 gifts handed out?

For Exercises 17 and 18, toss 3 coins, one at a time, 25 times and record your results.

17. Based on your results, what is the probability that any two coins will show heads?

18. Based on your results, what is the probability that the first and third coins show tails?

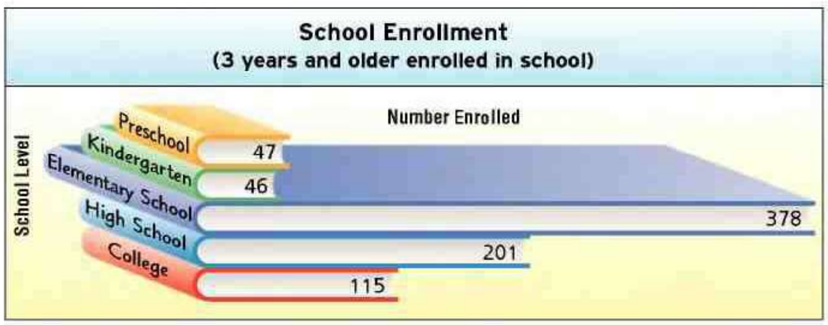
For Exercises 19–21, roll two dice 50 times and record the sums.

19. Based on your results, what is the probability that the sum is 8?

20. Based on your results, what is the probability that the sum is 7, or the sum is greater than 5?

21. If you roll the dice 25 more times, which sum would you expect to see about 10% of the time?

CITY PLANNING **For Exercises 22–24, use the following information.**
The Lewiston City Council sent surveys to randomly selected households to determine current and future enrollment for the local school district. The results of the survey are shown in the table.



22. Find the experimental probability distribution for the number of people enrolled at each level.

23. Based on the survey, what is the probability that a student chosen at random is in elementary school or high school?

24. Suppose the school district is expecting school enrollment to increase by 1800 over the next 5 years due to new buildings in the area. Of the new enrollment, how many will most likely be in kindergarten?

RESTAURANTS **For Exercises 25–27, use the following information.**
A family restaurant gives children a free toy with each children's meal. There are eight different toys that are randomly given. There is an equally likely chance of getting each toy each time.

25. What objects could be used to perform a simulation of this situation?

26. Conduct a simulation until you have one of each toy. Record your results.

27. Based on your results, how many meals must be purchased so that you get all 8 toys?

More About. . .

Animals

Labrador retrievers are the most popular breed of dog in the United States.

Source: American Kennel Club

ANIMALS **For Exercises 28–31, use the following information.**
Refer to Example 4 on page 784. Suppose Ali's dog is expecting a litter of 5 puppies.

28. List the possible outcomes of the genders of the puppies.

29. Perform a simulation and list your results in a table.

30. Based on your results, what is the probability that there will be 3 females and two males in the litter?

31. What is the experimental probability of the litter having at least three male puppies?

32. **CRITICAL THINKING** The captain of a football team believes that the coin the referee uses for the opening coin toss gives an advantage to one team. The referee has players toss the coin 50 times each and record their results. Based on the results, do you think the coin is fair? Explain your reasoning.

Player	1	2	3	4	5	6
Heads	38	31	29	27	26	30
Tails	12	19	21	23	24	20

33. *WRITING IN MATH* Answer the question that was posed at the beginning of the lesson.

How can probability simulations be used in health care?

Include the following in your answer:
- a few sentences explaining experimental probability, and
- an explanation of why an experimental probability of 75% found in 400 trials is more reliable than an experimental probability of 75% found in 50 trials.

Standardized Test Practice

34. Ramón tossed two coins and rolled a die. What is the probability that he tossed two tails and rolled a 3?

Ⓐ $\frac{1}{4}$ Ⓑ $\frac{1}{6}$ Ⓒ $\frac{5}{12}$ Ⓓ $\frac{1}{24}$

35. If a coin is tossed three times, what is the probability that the results will be heads exactly one time?

Ⓐ $\frac{2}{3}$ Ⓑ $\frac{3}{8}$ Ⓒ $\frac{1}{5}$ Ⓓ $\frac{1}{8}$

Graphing Calculator

SIMULATION For Exercises 36–38, use the following information.

When you are performing an experiment that involves a large number of trials that cannot be simulated using an object like a coin or a spinner, you can use the random number generator function on a graphing calculator. The TI-83 Plus program at the right will perform T trials by generating random numbers between 1 and P, the number of possible outcomes.

```
PROGRAM: SIMULATE
:Disp "ENTER THE NUMBER"
:Disp "OF POSSIBLE"
:Disp "OUTCOMES"
:Input P
:Disp "ENTER THE NUMBER"
:Disp "OF TRIALS"
:Input T
:For(N, 1, T)
:randInt(1, P)→S
:Disp S
:Pause
```

36. Run the program to simulate 50 trials of an event that has 15 outcomes. Record your results.

37. What is the experimental probability of displaying the number 10?

38. Repeat the experiment several times. Find the experimental probability of displaying the number 10. Has the probability changed from the probability found in Exercise 37? Explain why or why not.

ENTERTAINMENT For Exercises 39–41, use the following information and the graphing calculator program above.

A CD changer contains 5 CDs with 14 songs each. When "Random" is selected, each CD is equally likely to be chosen as each song.

39. Use the program SIMULATE to perform a simulation of randomly playing 40 songs from the 5 CDs. (*Hint*: Number the songs sequentially from 1, CD 1 track 1, to 70, CD 5 track 14.)

40. Do the experimental probabilities for your simulation support the statement that each CD is equally likely to be chosen? Explain.

41. Based on your results, what is the probability that the first three songs played are on the third disc?

CONTENTS

Maintain Your Skills

Mixed Review

For Exercises 42–44, use the probability distribution for the random variable X, the number of computers per household. *(Lesson 14-4)*

42. Show that the probability distribution is valid.

43. If a household is chosen at random, what is the probability that it has at least 2 computers?

44. Determine the probability of randomly selecting a household with no more than one computer.

Computers per Household	
X = Number of Computers	$P(X)$
0	0.579
1	0.276
2	0.107
3+	0.038

Source: U.S. Dept. of Commerce

For Exercises 45–47, use the following information.
A jar contains 18 nickels, 25 dimes, and 12 quarters. Three coins are randomly selected. Find each probability. *(Lesson 14-3)*

45. picking three dimes, replacing each after it is drawn

46. a nickel, then a quarter, then a dime without replacing the coins

47. 2 dimes and a quarter, without replacing the coins, if order does not matter

Solve each equation. *(Lesson 12-9)*

48. $\frac{2a-3}{a-3} - 2 = \frac{12}{a+3}$ **49.** $\frac{r^2}{r-7} + \frac{50}{7-r} = 14$ **50.** $\frac{x-2}{x} - \frac{x-3}{x-6} = \frac{1}{x}$

51. $\frac{2x-3}{7} - \frac{x}{2} = \frac{x+3}{14}$ **52.** $\frac{5n}{n+1} + \frac{1}{n} = 5$ **53.** $\frac{a+2}{a-2} - \frac{2}{a+2} = \frac{-7}{3}$

54. CONSTRUCTION To paint his house, Lonnie needs to purchase an extension ladder that reaches at least 24 feet off the ground. Ladder manufacturers recommend the angle formed by the ladder and the ground be no more than 75°. What is the shortest ladder he could buy to reach 24 feet safely? *(Lesson 11-7)*

Determine whether the following side measures would form a right triangle. *(Lesson 11-4)*

55. 5, 7, 9 **56.** $3\sqrt{34}$, 9, 15 **57.** 36, 86.4, 93.6

Solve each equation. Check your solutions. *(Lesson 9-6)*

58. $(x-6)^2 = 4$ **59.** $x^2 + 121 = 22x$ **60.** $4x^2 + 12x + 9 = 0$

61. $25x^2 + 20x = -4$ **62.** $49x^2 - 84x + 36 = 0$ **63.** $180x - 100 = 81x^2$

WebQuest Internet Project

America Counts!

It is time to complete your project. Use the information and data you have gathered about populations to prepare a brochure or Web page. Be sure to identify the state you have chosen for this project. Include graphs, tables, and/or calculations in the presentation.

www.algebra1.com/webquest

Chapter 14

Study Guide and Review

Vocabulary and Concept Check

combination (p. 762)
complements (p. 771)
compound event (p. 769)
dependent events (p. 770)
edge (p. 759)
empirical study (p. 782)
event (p. 754)
experimental probability (p. 782)
factorial (p. 755)
finite graph (p. 759)
Fundamental Counting Principle (p. 755)
inclusive (p. 771)
independent events (p. 769)
mutually exclusive (p. 771)
network (p. 759)
node (p. 759)
permutation (p. 760)
probability distribution (p. 777)
probability histogram (p. 778)
random variable (p. 777)
relative frequency (p. 782)
sample space (p. 754)
simple event (p. 769)
simulation (p. 783)
theoretical probability (p. 782)
traceable (p. 759)
tree diagram (p. 754)

Choose the word or term that best completes each sentence.

1. The arrangement or listing in which order is important is called a (*combination*, *permutation*).
2. The notation 10! refers to a (*prime factor*, *factorial*).
3. Rolling one die and then another die are (*dependent*, *independent*) events.
4. The sum of probabilities of complements equals (*0*, *1*).
5. Randomly drawing a coin from a bag is a dependent event if the coins (*are*, *are not*) replaced.
6. Events that cannot occur at the same time are (*inclusive*, *mutually exclusive*).
7. The sum of the probabilities in a probability distribution equals (*0*, *1*).
8. (*Experimental*, *Theoretical*) probabilities are precise and predictable.

Lesson-by-Lesson Review

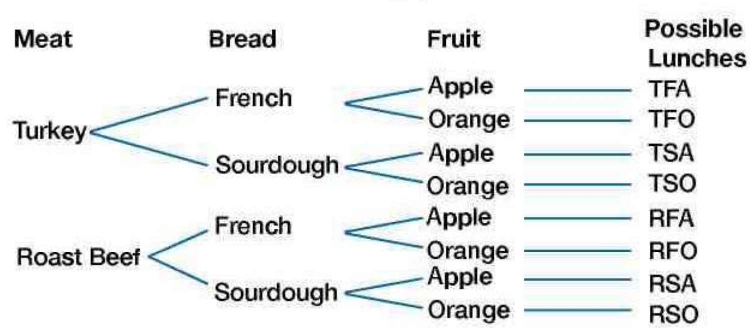
14-1 Counting Outcomes

See pages 754–758.

Concept Summary

- Use a tree diagram to make a list of possible outcomes.
- If an event M can occur m ways and is followed by an event N that can occur n ways, the event M followed by event N can occur $m \cdot n$ ways.

Example When Jerri packs her lunch, she can choose to make a turkey or roast beef sandwich on French or sourdough bread. She also can pack an apple or an orange. Draw a tree diagram to show the number of different ways Jerri can select these items.



There are 8 different ways for Jerri to select these items.

Exercises **Determine the number of outcomes for each event.**
See Examples 1–3 on pages 754 and 755.

9. Samantha wants to watch 3 videos one rainy afternoon. She has a choice of 3 comedies, 4 dramas, and 3 musicals.

10. Marquis buys 4 books, one from each category. He can choose from 12 mystery, 8 science fiction, 10 classics, and 5 biographies.

11. The Jackson Jackals and the Westfield Tigers are going to play a best three-out-of-five games baseball tournament.

14-2 Permutations and Combinations

See pages 760–767.

Concept Summary

- In a permutation, the order of objects is important. $_nP_r = \frac{n!}{(n-r)!}$
- In a combination, the order of objects is not important. $_nC_r = \frac{n!}{(n-r)!r!}$

Examples

1 **Find $_{12}C_8$.**

$$_{12}C_8 = \frac{12!}{(12-8)!8!} = \frac{12!}{4!8!} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4!} = 495$$

2 **Find $_9P_4$.**

$$_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 3024$$

Exercises **Evaluate each expression.** *See Examples 1, 2, and 4 on pages 760–762.*

12. $_4P_2$
13. $_8C_3$
14. $_4C_4$
15. $(_7C_1)(_6C_3)$
16. $(_7P_3)(_7P_2)$
17. $(_3C_2)(_4P_1)$

14-3 Probability of Compound Events

See pages 769–776.

Concept Summary

- For independent events, use $P(A \text{ and } B) = P(A) \cdot P(B)$.
- For dependent events, use $P(A \text{ and } B) = P(A) \cdot P(B \text{ following } A)$.
- For mutually exclusive events, use $P(A \text{ or } B) = P(A) + P(B)$.
- For inclusive events, use $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Example

A box contains 8 red chips, 6 blue chips, and 12 white chips. Three chips are randomly drawn from the box and not replaced. Find *P*(red, white, blue).

First chip: $P(\text{red}) = \frac{8}{26}$ ← number of red chips / ← total number of chips

Second chip: $P(\text{white}) = \frac{12}{25}$ ← number of white chips / ← number of chips remaining

Third chip: $P(\text{blue}) = \frac{6}{24}$ ← number of blue chips / ← number of chips remaining

$$P(\text{red, white, blue}) = \underbrace{P(\text{red})} \cdot \underbrace{P(\text{white})} \cdot \underbrace{P(\text{blue})}$$
$$= \frac{8}{26} \cdot \frac{12}{25} \cdot \frac{6}{24}$$
$$= \frac{576}{15{,}600} \text{ or } \frac{7}{650}$$

Exercises **A bag of colored paper clips contains 30 red clips, 22 blue clips, and 22 green clips. Find each probability if three clips are drawn randomly from the bag and are not replaced.** *See Example 2 on page 770.*

18. P(blue, red, green) **19.** P(red, red, blue) **20.** P(red, green, not blue)

One card is randomly drawn from a standard deck of 52 cards. Find each probability. *See Examples 3 and 4 on pages 771 and 772.*

21. P(diamond or club) **22.** P(heart or red) **23.** P(10 or spade)

14-4 Probability Distributions

See pages 777–781.

Concept Summary

Probability distributions have the following properties.

- For each value of X, $0 \leq P(X) \leq 1$.
- The sum of the probabilities of each value of X is 1.

Example **A local cable provider asked its subscribers how many televisions they had in their homes. The results of their survey are shown in the probability distribution.**

a. Show that the probability distribution is valid.

For each value of X, the probability is greater than or equal to 0 and less than or equal to 1.

$0.18 + 0.36 + 0.34 + 0.08 + 0.04 = 1$, so the probabilities add up to 1.

b. If a household is selected at random, what is the probability that it has fewer than 4 televisions?

$$P(X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$$
$$= 0.18 + 0.36 + 0.34$$
$$= 0.88$$

Televisions per Household

X = Number of Televisions	Probability
1	0.18
2	0.36
3	0.34
4	0.08
5+	0.04

Exercises **The table shows the probability distribution for the number of extracurricular activities in which students at Boardwalk High School participate.** *See Example 2 on page 778.*

24. Show that the probability distribution is valid.

25. If a student is chosen at random, what is the probability that the student participates in 1 to 3 activities?

26. Make a probability histogram of the data.

Extracurricular Activities

X = Number of Activities	Probability
0	0.04
1	0.12
2	0.37
3	0.30
4+	0.17

Chapter 14 For More ...
• Extra Practice, see pages 851–852.
• Mixed Problem Solving, see page 866.

14-5 Probability Simulations

See pages 782–788.

Concept Summary

- Theoretical probability describes expected outcomes, while experimental probabilities describe tested outcomes.
- Simulations are used to perform experiments that would be difficult or impossible to perform in real life.

Example **A group of 3 coins are tossed.**

a. Find the theoretical probability that there will be 2 heads and 1 tail.

Each coin toss can be heads or tails, so there are $2 \cdot 2 \cdot 2$ or 8 possible outcomes. The number of combinations of 2 heads and one tail is $_2C_1$ or $\frac{2!}{1!1!}$ or 2. So, the theoretical probability is $\frac{2}{8}$ or $\frac{1}{4}$.

b. The results of a simulation in which three coins are tossed ten times are shown in the table. What is the experimental probability that there will be 1 head and 2 tails?

Of the 10 trials, 3 resulted in 1 head and 2 tails, so the experimental probability is $\frac{3}{10}$ or 30%.

Outcomes	Frequency
3 heads, 0 tails	1
2 heads, 1 tail	4
1 head, 2 tails	3
0 heads, 3 tails	2

c. Compare the theoretical probability of 2 heads and 1 tail and the experimental probability of 2 heads and 1 tail.

The theoretical probability is $\frac{1}{4}$ or 25%, while the experimental probability is $\frac{3}{10}$ or 30%. The probabilities are close.

Exercises **While studying flower colors in biology class, students are given the Punnett square at the right. The Punnett square shows that red parent plant flowers (Rr) produce red flowers (RR and Rr) and pink flowers (rr).**

See Examples 1, 3, and 4 on pages 782 and 784.

	R	r
R	RR	Rr
r	Rr	rr

27. If 5 flowers are produced, find the theoretical probability that there will be 4 red flowers and 1 pink flower.

28. Describe items that the students could use to simulate the colors of 5 flowers.

29. The results of a simulation of flowers are shown in the table. What is the experimental probability that there will be 3 red flowers and 2 pink flowers?

Outcomes	Frequency
5 red, 0 pink	15
4 red, 1 pink	30
3 red, 2 pink	23
2 red, 3 pink	7
1 red, 4 pink	4
0 red, 5 pink	1

CONTENTS

Chapter 14 Practice Test

Vocabulary and Concepts

1. Seven students lining up to buy tickets for a school play is an example of a (*permutation*, *combination*).
2. Rolling a die and recording the result 25 times would be used to find (*theoretical*, *experimental*) probability.
3. A (*random variable*, *probability distribution*) is the numerical outcome of an event.

Skills and Applications

There are two roads from Ashville to Bakersville, four roads from Bakersville to Clifton, and two roads from Clifton to Derry.

4. Draw a tree diagram showing the possible routes from Ashville to Derry.
5. How many different routes are there from Ashville to Derry?

Determine whether each situation involves a *permutation* or a *combination*. Then determine the number of possible arrangements.

6. Six students in a class meet in a room that has nine chairs.
7. The top four finishers in a race with ten participants.
8. A class has 15 girls and 19 boys. A committee is formed with two girls and two boys, each with a separate responsibility.

A bag contains 4 red, 6 blue, 4 yellow, and 2 green marbles. Once a marble is selected, it is not replaced. Find each probability.

9. P(blue, green)
10. P(yellow, yellow)
11. P(red, blue, yellow)
12. P(blue, red, not green)

The spinner is spun, and a die is rolled. Find each probability.

13. P(yellow, 4)
14. P(red, even)
15. P(purple or white, not prime)
16. P(green, even or less than 5)

During a magic trick, a magician randomly selects a card from a standard deck of 52 cards. Without replacing it, the magician has a member of the audience randomly select a card. Find each probability.

17. P(club, heart)
18. P(black 7, diamond)
19. P(queen or red, jack of spades)
20. P(black 10, ace or heart)

The table shows the number of ways four coins can land heads up when they are tossed at the same time.

Four Coins Tossed	
Number of Heads	Possible Outcomes
0	1
1	4
2	6
3	4
4	1

21. Set up a probability distribution of the possible outcomes.
22. Find the probability that there will be no heads.
23. Find the probability that there will be at least two heads.
24. Find the probability that there will be two tails.
25. **STANDARDIZED TEST PRACTICE** Two numbers a and b can be arranged in two different orders, a, b and b, a. In how many ways can three numbers be arranged?

   (A) 3 (B) 4 (C) 5 (D) 6

Chapter 14 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. If the average of a and b is 20, and the average of a, b, and c is 25, then what is the value of c? (Prerequisite Skill)

   (A) 10 (B) 15
   (C) 25 (D) 35

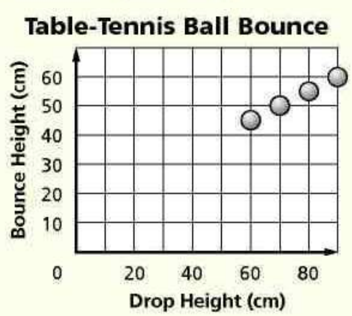
2. The volume of a cube is 27 cubic inches. Its total surface area, in square inches, is (Lesson 3-8)

   (A) 9. (B) $6\sqrt{3}$.
   (C) $18\sqrt{3}$. (D) 54.

3. A truck travels 50 miles from Oakton to Newton in exactly 1 hour. When the truck is halfway between Oakton and Newton, a car leaves Oakton and travels at 60 miles per hour. How many miles has the car traveled when the truck reaches Newton? (Lesson 3-8)

   (A) 25 (B) 30
   (C) 50 (D) 60

4. Which equation would best represent the graphed data? (Lesson 5-7)



   (A) $y = \frac{1}{2}x + 15$ (B) $y = 2x + 15$
   (C) $y = 2x$ (D) $y = \frac{1}{2}x$

5. If a child is equally likely to be born a boy or a girl, what is the probability that a family of 3 children will contain exactly one boy? (Lesson 7-5)

   (A) $\frac{1}{8}$ (B) $\frac{1}{4}$
   (C) $\frac{3}{8}$ (D) $\frac{1}{2}$

6. What is the value of 5^{-2}? (Lesson 8-2)

   (A) -25 (B) $-\frac{1}{25}$
   (C) $\frac{1}{25}$ (D) $-\sqrt{5}$

7. What are the solutions of $x^2 + x = 20$? (Lesson 9-4)

   (A) $-4, 5$ (B) $-2, 10$
   (C) $2, 10$ (D) $4, -5$

8. Two airplanes are flying at the same altitude. One plane is two miles west and two miles north of an airport. The other plane is seven miles west and eight miles north of the same airport. How many miles apart are the airplanes? (Lesson 11-4)

   (A) 2.8 (B) 7.8
   (C) 10.6 (D) 11.0

9. A certain password consists of three characters, and each character is a letter of the alphabet. Each letter can be used more than once. How many different passwords are possible? (Lesson 14-1)

   (A) 78 (B) 2600
   (C) 15,600 (D) 17,576

The Princeton Review Test-Taking Tip

If you are allowed to write in your test booklet, underline key words, do calculations, sketch diagrams, cross out answer choices as you eliminate them, and mark any questions that you skip. But do not make any marks on the *answer sheet* except your answers.

Aligned and verified by The Princeton Review

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

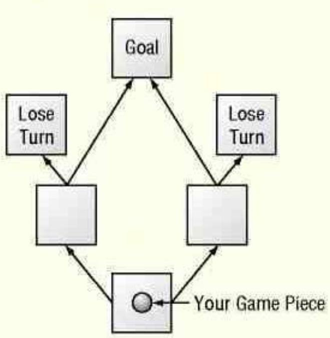
10. What are the coordinates of the point of intersection of the lines represented by the equations $x + 4y = 0$ and $2x - 3y = 11$? (Lesson 7-2)

11. Is $4\left(x - \frac{1}{2}\right)^2 - 1 = 4x^2 - 4x$ true for *all* values of x, *some* values of x, or *no* values of x? (Lesson 8-8)

12. Triangle ABC has sides of length $a = 5$, $b = 7$, and $c = \sqrt{74}$. What is the measure, in degrees, of the angle opposite side c? (Lesson 11-4)

13. All seven-digit telephone numbers in a town begin with the same three digits. Of the last four digits in any given phone number, neither the first nor the last digit can be 0. How many telephone numbers are available in this town? (Lesson 14-2)

14. In the board game shown below, you move your game piece along the arrows from square to square. To determine which direction to move your game piece, you roll a number cube with sides numbered 1, 2, 3, 4, 5, and 6. If you roll 1 or 2, you move your game piece one space to the left. If you roll 3, 4, 5, or 6, you move your game piece one square to the right. What is the probability that you will reach the goal within two turns? (Lesson 14-3)



Part 3 Quantitative Comparison

Compare the quantity in Column A and the Quantity in Column B. Then determine whether:

Ⓐ **the quantity in Column A is greater,**

Ⓑ **the quantity in Column B is greater,**

Ⓒ **the two quantities are equal, or**

Ⓓ **the relationship cannot be determined from the information given.**

	Column A	Column B
15.	the value of x in $3x + 15 > 45$	the value of y in $-2y + 3 > -17$

(Lesson 6-3)

16.	${}_{12}P_4$	${}_{10}C_6$

(Lesson 14-2)

17. A bag contains 5 red marbles, 7 blue marbles, and 2 green marbles. A marble is randomly drawn, not replaced, then another marble is randomly drawn.

P(blue, green)	P(red, red)

(Lesson 14-3)

Part 4 Open Ended

Record your answers on a sheet of paper. Show your work.

18. At WackyWorld Pizza, the Random Special is a random selection of two different toppings on a large cheese pizza. The available toppings are pepperoni, sausage, onion, mushrooms, and green peppers. (Lessons 14-2 and 14-3)

a. How many different Random Specials are possible? Show how you found your answer.

b. If you order the Random Special, what is the probability that it will have mushrooms?