Chapter 9

Factoring

What You'll Learn

- **Lesson 9-1** Find the prime factorizations of integers and monomials.
- **Lesson 9-1** Find the greatest common factors (GCF) for sets of integers and monomials.
- **Lessons 9-2 through 9-6** Factor polynomials.
- **Lessons 9-2 through 9-6** Use the Zero Product Property to solve equations.

Key Vocabulary

- factored form (p. 475)
- factoring by grouping (p. 482)
- prime polynomial (p. 497)
- difference of squares (p. 501)
- perfect square trinomials (p. 508)

Why It's Important

The factoring of polynomials can be used to solve a variety of real-world problems and lays the foundation for the further study of polynomial equations. Factoring is used to solve problems involving vertical motion. For example, the height h in feet of a dolphin that jumps out of the water traveling at 20 feet per second is modeled by a polynomial equation. Factoring can be used to determine how long the dolphin is in the air. *You will learn how to solve polynomial equations in Lesson 9-2.*

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 9.

For Lessons 9-2 through 9-6 — Distributive Property

Rewrite each expression using the Distributive Property. Then simplify.
(For review, see Lesson 1-5.)

1. $3(4 - x)$ **2.** $a(a + 5)$ **3.** $-7(n^2 - 3n + 1)$ **4.** $6y(-3y - 5y^2 + y^3)$

For Lessons 9-3 and 9-4 — Multiplying Binomials

Find each product. *(For review, see Lesson 8-7.)*

5. $(x + 4)(x + 7)$ **6.** $(3n - 4)(n + 5)$ **7.** $(6a - 2b)(9a + b)$ **8.** $(-x - 8y)(2x - 12y)$

For Lessons 9-5 and 9-6 — Special Products

Find each product. *(For review, see Lesson 8-8.)*

9. $(y + 9)^2$ **10.** $(3a - 2)^2$ **11.** $(n - 5)(n + 5)$ **12.** $(6p + 7q)(6p - 7q)$

For Lesson 9-6 — Square Roots

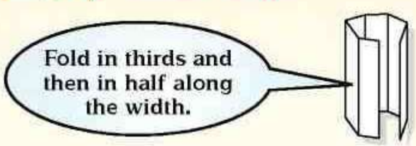
Find each square root. *(For review, see Lesson 2-7.)*

13. $\sqrt{121}$ **14.** $\sqrt{0.0064}$ **15.** $\sqrt{\frac{25}{36}}$ **16.** $\sqrt{\frac{8}{98}}$

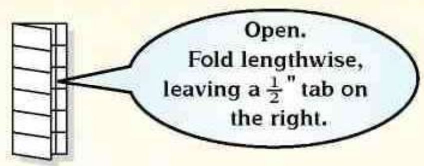
FOLDABLES™ Study Organizer

Make this Foldable to help you organize your notes on factoring. Begin with a sheet of plain $8\frac{1}{2}$" by 11" paper.

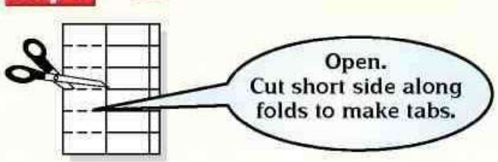
Step 1 Fold in Sixths



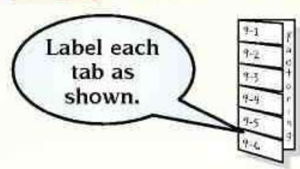
Step 2 Fold Again



Step 3 Cut



Step 4 Label



Reading and Writing As you read and study the chapter, write notes and examples for each lesson under its tab.

9-1 Factors and Greatest Common Factors

What You'll Learn

- Find prime factorizations of integers and monomials.
- Find the greatest common factors of integers and monomials.

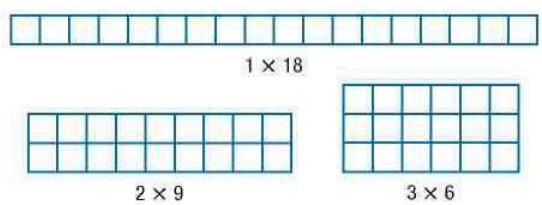
Vocabulary

- prime number
- composite number
- prime factorization
- factored form
- greatest common factor (GCF)

How are prime numbers related to the search for extraterrestrial life?

In the search for extraterrestrial life, scientists listen to radio signals coming from faraway galaxies. How can they be sure that a particular radio signal was deliberately sent by intelligent beings instead of coming from some natural phenomenon? What if that signal began with a series of beeps in a pattern comprised of the first 30 prime numbers ("beep-beep," "beep-beep-beep," and so on)?

PRIME FACTORIZATION Recall that when two or more numbers are multiplied, each number is a *factor* of the product. Some numbers, like 18, can be expressed as the product of different pairs of whole numbers. This can be shown geometrically. Consider all of the possible rectangles with whole number dimensions that have areas of 18 square units.



The number 18 has 6 factors, 1, 2, 3, 6, 9, and 18. Whole numbers greater than 1 can be classified by their number of factors.

Key Concept — Prime and Composite Numbers

Words	Examples
A whole number, greater than 1, whose only factors are 1 and itself, is called a **prime number**.	2, 3, 5, 7, 11, 13, 17, 19
A whole number, greater than 1, that has more than two factors is called a **composite number**.	4, 6, 8, 9, 10, 12, 14, 15, 16, 18

0 and 1 are neither prime nor composite.

Example 1 Classify Numbers as Prime or Composite

Study Tip

Listing Factors
Notice that in Example 1, 6 is listed as a factor of 36 only once.

Factor each number. Then classify each number as *prime* or *composite*.

a. 36

To find the factors of 36, list all pairs of whole numbers whose product is 36.

1×36 $\quad$ 2×18 $\quad$ 3×12 $\quad$ 4×9 $\quad$ 6×6

Therefore, the factors of 36, in increasing order, are 1, 2, 3, 4, 6, 9, 12, 18, and 36. Since 36 has more than two factors, it is a composite number.

b. 23

The only whole numbers that can be multiplied together to get 23 are 1 and 23. Therefore, the factors of 23 are 1 and 23. Since the only factors of 23 are 1 and itself, 23 is a prime number.

Study Tip

Prime Numbers
Before deciding that a number is prime, try dividing it by all of the prime numbers that are less than the square root of that number.

When a whole number is expressed as the product of factors that are all prime numbers, the expression is called the **prime factorization** of the number.

Example 2 Prime Factorization of a Positive Integer

Find the prime factorization of 90.

Method 1

$90 = 2 \cdot 45$ The least prime factor of 90 is 2.

$= 2 \cdot 3 \cdot 15$ The least prime factor of 45 is 3.

$= 2 \cdot 3 \cdot 3 \cdot 5$ The least prime factor of 15 is 3.

All of the factors in the last row are prime. Thus, the prime factorization of 90 is $2 \cdot 3 \cdot 3 \cdot 5$.

Method 2

Use a factor tree.

```
    90
   /  \
  9  ·  10      90 = 9 · 10
 / \    / \
3 · 3 · 2 · 5   9 = 3 · 3 and 10 = 2 · 5
```

All of the factors in the last branch of the factor tree are prime. Thus, the prime factorization of 90 is $2 \cdot 3 \cdot 3 \cdot 5$ or $2 \cdot 3^2 \cdot 5$.

Usually the factors are ordered from the least prime factor to the greatest.

Study Tip

Unique Factorization Theorem
The prime factorization of every number is unique except for the order in which the factors are written.

A negative integer is factored completely when it is expressed as the product of -1 and prime numbers.

Example 3 Prime Factorization of a Negative Integer

Find the prime factorization of -140.

$-140 = -1 \cdot 140$ Express -140 as -1 times 140.

$= -1 \cdot 2 \cdot 70$ $140 = 2 \cdot 70$

$= -1 \cdot 2 \cdot 7 \cdot 10$ $70 = 7 \cdot 10$

$= -1 \cdot 2 \cdot 7 \cdot 2 \cdot 5$ $10 = 2 \cdot 5$

Thus, the prime factorization of -140 is $-1 \cdot 2 \cdot 2 \cdot 5 \cdot 7$ or $-1 \cdot 2^2 \cdot 5 \cdot 7$.

A monomial is in **factored form** when it is expressed as the product of prime numbers and variables and no variable has an exponent greater than 1.

Example 4 Prime Factorization of a Monomial

Factor each monomial completely.

a. $12a^2b^3$

$12a^2b^3 = 2 \cdot 6 \cdot a \cdot a \cdot b \cdot b \cdot b$ $12 = 2 \cdot 6$, $a^2 = a \cdot a$, and $b^3 = b \cdot b \cdot b$

$= 2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot b \cdot b \cdot b$ $6 = 2 \cdot 3$

Thus, $12a^2b^3$ in factored form is $2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot b \cdot b \cdot b$.

b. $-66pq^2$

$-66pq^2 = -1 \cdot 66 \cdot p \cdot q \cdot q$ Express -66 as -1 times 66.

$= -1 \cdot 2 \cdot 33 \cdot p \cdot q \cdot q$ $66 = 2 \cdot 33$

$= -1 \cdot 2 \cdot 3 \cdot 11 \cdot p \cdot q \cdot q$ $33 = 3 \cdot 11$

Thus, $-66pq^2$ in factored form is $-1 \cdot 2 \cdot 3 \cdot 11 \cdot p \cdot q \cdot q$.

GREATEST COMMON FACTOR Two or more numbers may have some common prime factors. Consider the prime factorization of 48 and 60.

$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$ Factor each number.

$60 = 2 \cdot 2 \cdot 3 \cdot 5$ Circle the common prime factors.

The integers 48 and 60 have two 2s and one 3 as common prime factors. The product of these common prime factors, $2 \cdot 2 \cdot 3$ or 12, is called the **greatest common factor (GCF)** of 48 and 60. The GCF is the greatest number that is a factor of both original numbers.

Key Concept — Greatest Common Factor (GCF)

- The GCF of two or more integers is the product of the prime factors common to the integers.
- The GCF of two or more monomials is the product of their common factors when each monomial is in factored form.
- If two or more integers or monomials have a GCF of 1, then the integers or monomials are said to be *relatively prime*.

Study Tip

Alternative Method

You can also find the greatest common factor by listing the factors of each number and finding which of the common factors is the greatest. Consider Example 5a.

15: 1, 3, 5, 15

16: 1, 2, 4, 8, 16

The only common factor, and therefore, the greatest common factor, is 1.

Example 5 GCF of a Set of Monomials

Find the GCF of each set of monomials.

a. 15 and 16

$15 = 3 \cdot 5$ Factor each number.

$16 = 2 \cdot 2 \cdot 2 \cdot 2$ Circle the common prime factors, if any.

There are no common prime factors, so the GCF of 15 and 16 is 1. This means that 15 and 16 are relatively prime.

b. $36x^2y$ **and** $54xy^2z$

$36x^2y = 2 \cdot 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot y$ Factor each number.

$54xy^2z = 2 \cdot 3 \cdot 3 \cdot 3 \cdot x \cdot y \cdot y \cdot z$ Circle the common prime factors.

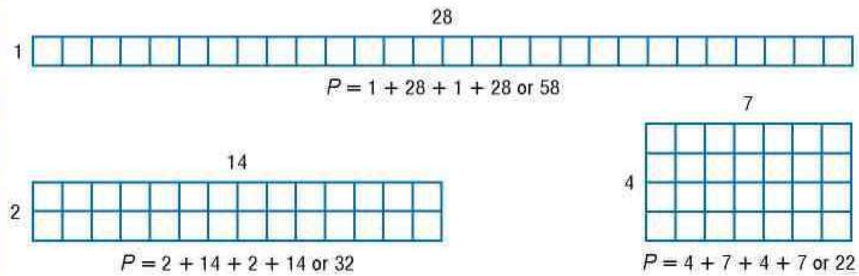
The GCF of $36x^2y$ and $54xy^2z$ is $2 \cdot 3 \cdot 3 \cdot x \cdot y$ or $18xy$.

Example 6 Use Factors

GEOMETRY The area of a rectangle is 28 square inches. If the length and width are both whole numbers, what is the maximum perimeter of the rectangle?

Find the factors of 28, and draw rectangles with each length and width. Then find each perimeter.

The factors of 28 are 1, 2, 4, 7, 14, and 28.



The greatest perimeter is 58 inches. The rectangle with this perimeter has a length of 28 inches and a width of 1 inch.

Check for Understanding

Concept Check

1. **Determine** whether the following statement is *true* or *false*. If false, provide a counterexample.
   All prime numbers are odd.
2. **Explain** what it means for two numbers to be relatively prime.
3. **OPEN ENDED** Name two monomials whose GCF is $5x^2$.

Guided Practice **Find the factors of each number. Then classify each number as *prime* or *composite*.**

4. 8 5. 17 6. 112

Find the prime factorization of each integer.

7. 45 8. -32 9. -150

Factor each monomial completely.

10. $4p^2$ 11. $39b^3c^2$ 12. $-100x^3yz^2$

Find the GCF of each set of monomials.

13. 10, 15 14. $18xy$, $36y^2$ 15. 54, 63, 180
16. $25n$, $21m$ 17. $12a^2b$, $90a^2b^2c$ 18. $15r^2$, $35s^2$, $70rs$

Application

19. **GARDENING** Ashley is planting 120 tomato plants in her garden. In what ways can she arrange them so that she has the same number of plants in each row, at least 5 rows of plants, and at least 5 plants in each row?

Practice and Apply

Find the factors of each number. Then classify each number as *prime* or *composite*.

20. 19 21. 25 22. 80 23. 61
24. 91 25. 119 26. 126 27. 304

Homework Help

For Exercises	See Examples
20–27, 62, 65, 66	1
32–39	2, 3
40–47	4
48–61, 63, 64	5
28–31, 67	6

Extra Practice

See page 839.

GEOMETRY For Exercises 28 and 29, consider a rectangle whose area is 96 square millimeters and whose length and width are both whole numbers.

28. What is the minimum perimeter of the rectangle? Explain your reasoning.

29. What is the maximum perimeter of the rectangle? Explain your reasoning.

COOKIES For Exercises 30 and 31, use the following information.
A bakery packages cookies in two sizes of boxes, one with 18 cookies and the other with 24 cookies. A small number of cookies are to be wrapped in cellophane before they are placed in a box. To save money, the bakery will use the same size cellophane packages for each box.

30. How many cookies should the bakery place in each cellophane package to maximize the number of cookies in each package?

31. How many cellophane packages will go in each size box?

Find the prime factorization of each integer.

32. 39 **33.** -98 **34.** 117 **35.** 102

36. -115 **37.** 180 **38.** 360 **39.** -462

Factor each monomial completely.

40. $66d^4$ **41.** $85x^2y^2$ **42.** $49a^3b^2$ **43.** $50gh$

44. $128pq^2$ **45.** $243n^3m$ **46.** $-183xyz^3$ **47.** $-169a^2bc^2$

Find the GCF of each set of monomials.

48. 27, 72 **49.** 18, 35 **50.** 32, 48

51. 84, 70 **52.** 16, 20, 64 **53.** 42, 63, 105

54. $15a$, $28b^2$ **55.** $24d^2$, $30c^2d$ **56.** $20gh$, $36g^2h^2$

57. $21p^2q$, $32r^2t$ **58.** $18x$, $30xy$, $54y$ **59.** $28a^2$, $63a^3b^2$, $91b^3$

60. $14m^2n^2$, $18mn$, $2m^2n^3$ **61.** $80a^2b$, $96a^2b^3$, $128a^2b^2$

62. **NUMBER THEORY** *Twin primes* are two consecutive odd numbers that are prime. The first pair of twin primes is 3 and 5. List the next five pairs of twin primes.

More About. . .

Marching Bands

Drum Corps International (DCI) is a nonprofit youth organization serving junior drum and bugle corps around the world. Members of these marching bands range from 14 to 21 years of age.

Source: www.dci.org

MARCHING BANDS For Exercises 63 and 64, use the following information.
Central High's marching band has 75 members, and the band from Northeast High has 90 members. During the halftime show, the bands plan to march into the stadium from opposite ends using formations with the same number of rows.

63. If the bands want to match up in the center of the field, what is the maximum number of rows?

64. How many band members will be in each row after the bands are combined?

NUMBER THEORY For Exercises 65 and 66, use the following information.
One way of generating prime numbers is to use the formula $2^p - 1$, where p is a prime number. Primes found using this method are called *Mersenne primes*. For example, when $p = 2$, $2^2 - 1 = 3$. The first Mersenne prime is 3.

65. Find the next two Mersenne primes.

66. Will this formula generate all possible prime numbers? Explain your reasoning.

Online Research **Data Update** What is the greatest known prime number? Visit www.algebra1.com/data_update to learn more.

67. GEOMETRY The area of a triangle is 20 square centimeters. What are possible whole-number dimensions for the base and height of the triangle?

WebQuest

Finding the GCF of distances will help you make a scale model of the solar system. Visit www.algebra1.com/webquest to continue work on your WebQuest project.

68. CRITICAL THINKING Suppose 6 is a factor of ab, where a and b are natural numbers. Make a valid argument to explain why each assertion is true or provide a counterexample to show that an assertion is false.

a. 6 must be a factor of a *or* of b.

b. 3 must be a factor of a *or* of b.

c. 3 must be a factor of a *and* of b.

69. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are prime numbers related to the search for extraterrestrial life?

Include the following in your answer:

- a list of the first 30 prime numbers and an explanation of how you found them, and
- an explanation of why a signal of this kind might indicate that an extraterrestrial message is to follow.

Standardized Test Practice

70. Miko claims that there are at least four ways to design a 120-square-foot rectangular space that can be tiled with 1-foot by 1-foot tiles. Which statement best describes this claim?

(A) Her claim is false because 120 is a prime number.

(B) Her claim is false because 120 is not a perfect square.

(C) Her claim is true because 240 is a multiple of 120.

(D) Her claim is true because 120 has at least eight factors.

71. Suppose Ψ_x is defined as the largest prime factor of x. For which of the following values of x would Ψ_x have the greatest value?

(A) 53 (B) 74 (C) 99 (D) 117

Maintain Your Skills

Mixed Review

Find each product. *(Lessons 8-7 and 8-8)*

72. $(2x - 1)^2$	**73.** $(3a + 5)(3a - 5)$	**74.** $(7p^2 + 4)(7p^2 + 4)$
75. $(6r + 7)(2r - 5)$	**76.** $(10h + k)(2h + 5k)$	**77.** $(b + 4)(b^2 + 3b - 18)$

Find the value of r so that the line that passes through the given points has the given slope. *(Lesson 5-1)*

78. $(1, 2), (-2, r), m = 3$ **79.** $(-5, 9), (r, 6), m = -\frac{3}{5}$

80. RETAIL SALES A department store buys clothing at wholesale prices and then marks the clothing up 25% to sell at retail price to customers. If the retail price of a jacket is \$79, what was the wholesale price? *(Lesson 3-7)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL Use the Distributive Property to rewrite each expression.
*(To review the **Distributive Property**, see Lesson 1-5.)*

81. $5(2x + 8)$	**82.** $a(3a + 1)$	**83.** $2g(3g - 4)$
84. $-4y(3y - 6)$	**85.** $7b + 7c$	**86.** $2x + 3x$

Algebra Activity

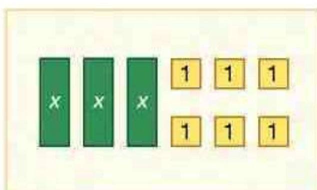
A Preview of Lesson 9-2

Factoring Using the Distributive Property

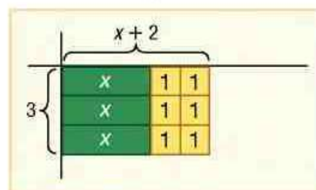
Sometimes you know the product of binomials and are asked to find the factors. This is called factoring. You can use algebra tiles and a product mat to factor binomials.

Activity 1 Use algebra tiles to factor $3x + 6$.

Step 1 Model the polynomial $3x + 6$.



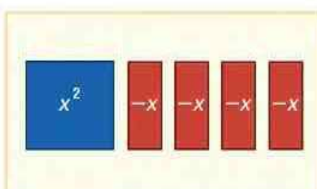
Step 2 Arrange the tiles into a rectangle. The total area of the rectangle represents the product, and its length and width represent the factors.



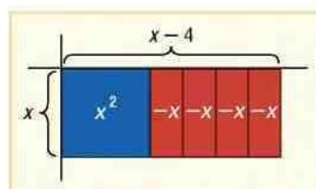
The rectangle has a width of 3 and a length of $x + 2$. So, $3x + 6 = 3(x + 2)$.

Activity 2 Use algebra tiles to factor $x^2 - 4x$.

Step 1 Model the polynomial $x^2 - 4x$.



Step 2 Arrange the tiles into a rectangle.



The rectangle has a width of x and a length of $x - 4$. So, $x^2 - 4x = x(x - 4)$.

Model and Analyze

Use algebra tiles to factor each binomial.

1. $2x + 10$ **2.** $6x - 8$ **3.** $5x^2 + 2x$ **4.** $9 - 3x$

Tell whether each binomial can be factored. Justify your answer with a drawing.

5. $4x - 10$ **6.** $3x - 7$ **7.** $x^2 + 2x$ **8.** $2x^2 + 3$

9. MAKE A CONJECTURE Write a paragraph that explains how you can use algebra tiles to determine whether a binomial can be factored. Include an example of one binomial that can be factored and one that cannot.

9-2 Factoring Using the Distributive Property

What You'll Learn

- Factor polynomials by using the Distributive Property.
- Solve quadratic equations of the form $ax^2 + bx = 0$.

Vocabulary

- factoring
- factoring by grouping

How can you determine how long a baseball will remain in the air?

Nolan Ryan, the greatest strike-out pitcher in the history of baseball, had a fastball clocked at 98 miles per hour or about 151 feet per second. If he threw a ball directly upward with the same velocity, the height h of the ball in feet above the point at which he released it could be modeled by the formula $h = 151t - 16t^2$, where t is the time in seconds. You can use factoring and the Zero Product Property to determine how long the ball would remain in the air before returning to his glove.

Study Tip

Look Back
To review the **Distributive Property**, see Lesson 1-5.

FACTOR BY USING THE DISTRIBUTIVE PROPERTY In Chapter 8, you used the Distributive Property to multiply a polynomial by a monomial.

$$\begin{aligned} 2a(6a + 8) &= 2a(6a) + 2a(8) \\ &= 12a^2 + 16a \end{aligned}$$

You can reverse this process to express a polynomial as the product of a monomial factor and a polynomial factor.

$$\begin{aligned} 12a^2 + 16a &= 2a(6a) + 2a(8) \\ &= 2a(6a + 8) \end{aligned}$$

Thus, a *factored form* of $12a^2 + 16a$ is $2a(6a + 8)$.

Factoring a polynomial means to find its *completely* factored form. The expression $2a(6a + 8)$ is not completely factored since $6a + 8$ can be factored as $2(3a + 4)$.

Example 1 Use the Distributive Property

Use the Distributive Property to factor each polynomial.

a. $12a^2 + 16a$

First, find the GCF of $12a^2$ and $16a$.

$12a^2 = 2 \cdot 2 \cdot 3 \cdot a \cdot a$ Factor each number.

$16a = 2 \cdot 2 \cdot 2 \cdot 2 \cdot a$ Circle the common prime factors.

GCF: $2 \cdot 2 \cdot a$ or $4a$

Write each term as the product of the GCF and its remaining factors. Then use the Distributive Property to factor out the GCF.

$$\begin{aligned} 12a^2 + 16a &= 4a(3 \cdot a) + 4a(2 \cdot 2) && \text{Rewrite each term using the GCF.} \\ &= 4a(3a) + 4a(4) && \text{Simplify remaining factors.} \\ &= 4a(3a + 4) && \text{Distributive Property} \end{aligned}$$

Thus, the completely factored form of $12a^2 + 16a$ is $4a(3a + 4)$.

b. $\mathbf{18cd^2 + 12c^2d + 9cd}$

$18cd^2 = 2 \cdot 3 \cdot 3 \cdot c \cdot d \cdot d$ Factor each number.

$12c^2d = 2 \cdot 2 \cdot 3 \cdot c \cdot c \cdot d$ Circle the common prime factors.

$9cd = 3 \cdot 3 \cdot c \cdot d$

GCF: $3 \cdot c \cdot d$ or $3cd$

$18cd^2 + 12c^2d + 9cd = 3cd(6d) + 3cd(4c) + 3cd(3)$ Rewrite each term using the GCF.

$= 3cd(6d + 4c + 3)$ Distributive Property

The Distributive Property can also be used to factor some polynomials having four or more terms. This method is called **factoring by grouping** because pairs of terms are grouped together and factored. The Distributive Property is then applied a second time to factor a common binomial factor.

Example 2 Use Grouping

Factor $4ab + 8b + 3a + 6$.

$4ab + 8b + 3a + 6$

$= (4ab + 8b) + (3a + 6)$ Group terms with common factors.

$= 4b(a + 2) + 3(a + 2)$ Factor the GCF from each grouping.

$= (a + 2)(4b + 3)$ Distributive Property

CHECK Use the FOIL method.

$$(a + 2)(4b + 3) = \overset{F}{(a)(4b)} + \overset{O}{(a)(3)} + \overset{I}{(2)(4b)} + \overset{L}{(2)(3)}$$

$$= 4ab + 3a + 8b + 6 \checkmark$$

Study Tip

Factoring by Grouping

Sometimes you can group terms in more than one way when factoring a polynomial. For example, the polynomial in Example 2 could have been factored in the following way.

$4ab + 8b + 3a + 6$

$= (4ab + 3a) + (8b + 6)$

$= a(4b + 3) + 2(4b + 3)$

$= (4b + 3)(a + 2)$

Notice that this result is the same as in Example 2.

Recognizing binomials that are additive inverses is often helpful when factoring by grouping. For example, $7 - y$ and $y - 7$ are additive inverses because their sum is 0. By rewriting $7 - y$ in the factored form $-1(y - 7)$, factoring by grouping is made possible in the following example.

Example 3 Use the Additive Inverse Property

Factor $35x - 5xy + 3y - 21$.

$35x - 5xy + 3y - 21 = (35x - 5xy) + (3y - 21)$ Group terms with common factors.

$= 5x(7 - y) + 3(y - 7)$ Factor the GCF from each grouping.

$= 5x(-1)(y - 7) + 3(y - 7)$ $7 - y = -1(y - 7)$

$= -5x(y - 7) + 3(y - 7)$ $5x(-1) = -5x$

$= (y - 7)(-5x + 3)$ Distributive Property

Study Tip

Factoring Trinomials

Since the order in which factors are multiplied does not affect the product, $(-5x + 3)(y - 7)$ is also a correct factoring of $35x - 5xy + 3y - 21$.

Concept Summary — Factoring by Grouping

- **Words** A polynomial can be factored by grouping if all of the following situations exist.
  - There are four or more terms.
  - Terms with common factors can be grouped together.
  - The two common factors are identical or are additive inverses of each other.
- **Symbols** $ax + bx + ay + by = x(a + b) + y(a + b)$
  $= (a + b)(x + y)$

SOLVE EQUATIONS BY FACTORING

Some equations can be solved by factoring. Consider the following products.

$$6(0) = 0 \qquad 0(-3) = 0 \qquad (5 - 5)(0) = 0 \qquad -2(-3 + 3) = 0$$

Notice that in each case, *at least one* of the factors is zero. These examples illustrate the **Zero Product Property**.

Key Concept — Zero Product Property

- **Words** If the product of two factors is 0, then at least one of the factors must be 0.
- **Symbols** For any real numbers a and b, if $ab = 0$, then either $a = 0$, $b = 0$, or both a and b equal zero.

Example 4 Solve an Equation in Factored Form

Solve $(d - 5)(3d + 4) = 0$. Then check the solutions.

If $(d - 5)(3d + 4) = 0$, then according to the Zero Product Property either $d - 5 = 0$ or $3d + 4 = 0$.

$(d - 5)(3d + 4) = 0$	Original equation
$d - 5 = 0 \quad \text{or} \quad 3d + 4 = 0$	Set each factor equal to zero.
$d = 5 \qquad\qquad 3d = -4$	Solve each equation.
$d = -\frac{4}{3}$	

The solution set is $\left\{5, -\frac{4}{3}\right\}$.

CHECK Substitute 5 and $-\frac{4}{3}$ for d in the original equation.

$$(d - 5)(3d + 4) = 0 \qquad\qquad (d - 5)(3d + 4) = 0$$

$$(5 - 5)[3(5) + 4] \stackrel{?}{=} 0 \qquad\qquad \left(-\frac{4}{3} - 5\right)\left[3\left(-\frac{4}{3}\right) + 4\right] \stackrel{?}{=} 0$$

$$(0)(19) \stackrel{?}{=} 0 \qquad\qquad \left(-\frac{19}{3}\right)(0) \stackrel{?}{=} 0$$

$$0 = 0 \ \checkmark \qquad\qquad 0 = 0 \ \checkmark$$

If an equation can be written in the form $ab = 0$, then the Zero Product Property can be applied to solve that equation.

Study Tip

Common Misconception

You may be tempted to try to solve the equation in Example 5 by dividing each side of the equation by x. Remember, however, that x is an *unknown* quantity. If you divide by x, you may actually be dividing by zero, which is undefined.

Example 5 Solve an Equation by Factoring

Solve $x^2 = 7x$. Then check the solutions.

Write the equation so that it is of the form $ab = 0$.

$x^2 = 7x$	Original equation
$x^2 - 7x = 0$	Subtract $7x$ from each side.
$x(x - 7) = 0$	Factor the GCF of x^2 and $-7x$, which is x.
$x = 0 \quad \text{or} \quad x - 7 = 0$	Zero Product Property
$x = 7$	Solve each equation.

The solution set is $\{0, 7\}$. Check by substituting 0 and 7 for x in the original equation.

Check for Understanding

Concept Check

1. **Write** $4x^2 + 12x$ as a product of factors in three different ways. Then decide which of the three is the completely factored form. Explain your reasoning.
2. **OPEN ENDED** Give an example of the type of equation that can be solved by using the Zero Product Property.
3. **Explain** why $(x - 2)(x + 4) = 0$ cannot be solved by dividing each side by $x - 2$.

Guided Practice

Factor each polynomial.

4. $9x^2 + 36x$
5. $16xz - 40xz^2$
6. $24m^2np^2 + 36m^2n^2p$
7. $2a^3b^2 + 8ab + 16a^2b^3$
8. $5y^2 - 15y + 4y - 12$
9. $5c - 10c^2 + 2d - 4cd$

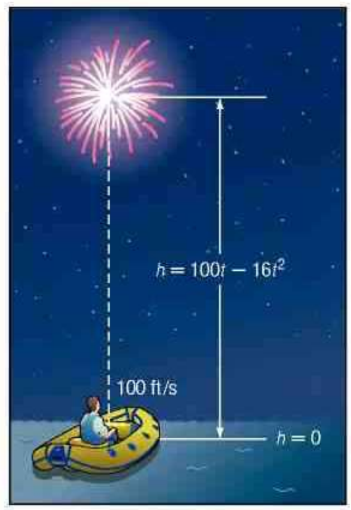
Solve each equation. Check your solutions.

10. $h(h + 5) = 0$
11. $(n - 4)(n + 2) = 0$
12. $5m = 3m^2$

Application

PHYSICAL SCIENCE **For Exercises 13–15, use the information below and in the graphic.**

A flare is launched from a life raft. The height h of the flare in feet above the sea is modeled by the formula $h = 100t - 16t^2$, where t is the time in seconds after the flare is launched.



13. At what height is the flare when it returns to the sea?
14. Let $h = 0$ in the equation $h = 100t - 16t^2$ and solve for t.
15. How many seconds will it take for the flare to return to the sea? Explain your reasoning.

Practice and Apply

Homework Help

For Exercises	See Examples
16–29, 40–47	1
30–39	2, 3
48–61	4, 5

Extra Practice

See page 840.

Factor each polynomial.

16. $5x + 30y$
17. $16a + 4b$
18. $a^5b - a$
19. $x^3y^2 + x$
20. $21cd - 3d$
21. $14gh - 18h$
22. $15a^2y - 30ay$
23. $8bc^2 + 24bc$
24. $12x^2y^2z + 40xy^3z^2$
25. $18a^2bc^2 - 48abc^3$
26. $a + a^2b^2 + a^3b^3$
27. $15x^2y^2 + 25xy + x$
28. $12ax^3 + 20bx^2 + 32cx$
29. $3p^3q - 9pq^2 + 36pq$
30. $x^2 + 2x + 3x + 6$
31. $x^2 + 5x + 7x + 35$
32. $4x^2 + 14x + 6x + 21$
33. $12y^2 + 9y + 8y + 6$
34. $6a^2 - 15a - 8a + 20$
35. $18x^2 - 30x - 3x + 5$
36. $4ax + 3ay + 4bx + 3by$
37. $2my + 7x + 7m + 2xy$
38. $8ax - 6x - 12a + 9$
39. $10x^2 - 14xy - 15x + 21y$

GEOMETRY **For Exercises 40 and 41, use the following information.**

A quadrilateral has 4 sides and 2 diagonals. A pentagon has 5 sides and 5 diagonals. You can use $\frac{1}{2}n^2 - \frac{3}{2}n$ to find the number of diagonals in a polygon with n sides.

40. Write this expression in factored form.
41. Find the number of diagonals in a decagon (10-sided polygon).

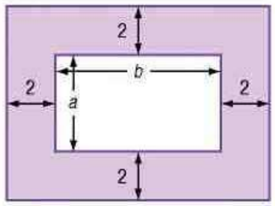
SOFTBALL **For Exercises 42 and 43, use the following information.**
Albertina is scheduling the games for a softball league. To find the number of games she needs to schedule, she uses the equation $g = \frac{1}{2}n^2 - \frac{1}{2}n$, where g represents the number of games needed for each team to play each other team exactly once and n represents the number of teams.

42. Write this equation in factored form.

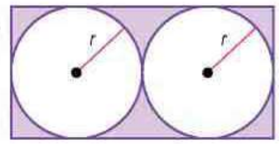
43. How many games are needed for 7 teams to play each other exactly 3 times?

GEOMETRY **Write an expression in factored form for the area of each shaded region.**

44.



45.



GEOMETRY **Find an expression for the area of a square with the given perimeter.**

46. $P = 12x + 20y$ in.

47. $P = 36a - 16b$ cm

Solve each equation. Check your solutions.

48. $x(x - 24) = 0$

49. $a(a + 16) = 0$

50. $(q + 4)(3q - 15) = 0$

51. $(3y + 9)(y - 7) = 0$

52. $(2b - 3)(3b - 8) = 0$

53. $(4n + 5)(3n - 7) = 0$

54. $3z^2 + 12z = 0$

55. $7d^2 - 35d = 0$

56. $2x^2 = 5x$

57. $7x^2 = 6x$

58. $6x^2 = -4x$

59. $20x^2 = -15x$

60. **MARINE BIOLOGY** In a pool at a water park, a dolphin jumps out of the water traveling at 20 feet per second. Its height h, in feet, above the water after t seconds is given by the formula $h = 20t - 16t^2$. How long is the dolphin in the air before returning to the water?

Career Choices

Marine Biologist

Marine biologists study factors that affect organisms living in and near the ocean.

Online Research
For information about a career as a marine biologist, visit: www.algebra1.com/careers

Source: National Sea Grant Library

61. **BASEBALL** Malik popped a ball straight up with an initial upward velocity of 45 feet per second. The height h, in feet, of the ball above the ground is modeled by the equation $h = 2 + 48t - 16t^2$. How long was the ball in the air if the catcher catches the ball when it is 2 feet above the ground?

62. **CRITICAL THINKING** Factor $a^{x+y} + a^xb^y - a^yb^x - b^{x+y}$.

63. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you determine how long a baseball will remain in the air?

Include the following in your answer:

- an explanation of how to use factoring and the Zero Product Property to find how long the ball would be in the air, and
- an interpretation of each solution in the context of the problem.

Standardized Test Practice

64. The total number of feet in x yards, y feet, and z inches is

(A) $3x + y + \frac{z}{12}$.
(B) $12(x + y + z)$.
(C) $x = 3y + 36z$.
(D) $\frac{x}{36} + \frac{y}{12} + z$.

65. **QUANTITATIVE COMPARISON** **Compare the quantity in Column A and the quantity in Column B. Then determine whether:**

(A) **the quantity in Column A is greater,**
(B) **the quantity in Column B is greater,**
(C) **the two quantities are equal, or**
(D) **the relationship cannot be determined from the information given.**

Column A	Column B
the negative solution of $(a - 2)(a + 5) = 0$	the negative solution of $(b + 6)(b - 1) = 0$

Maintain Your Skills

Mixed Review

Factor each number. Then classify each number as *prime* or *composite*. *(Lesson 9-1)*

66. 123
67. 300
68. 67

Find each product. *(Lesson 8-8)*

69. $(4s^3 + 3)^2$
70. $(2p + 5q)(2p - 5q)$
71. $(3k + 8)(3k + 8)$

Simplify. Assume that no denominator is equal to zero. *(Lesson 8-2)*

72. $\frac{s^4}{s^{-7}}$
73. $\frac{18x^3y^{-1}}{12x^2y^4}$
74. $\frac{34p^7q^2r^{-5}}{17(p^3qr^{-1})^2}$

75. **FINANCE** Michael uses at most 60% of his annual FlynnCo stock dividend to purchase more shares of FlynnCo stock. If his dividend last year was $885 and FlynnCo stock is selling for $14 per share, what is the greatest number of shares that he can purchase? *(Lesson 6-2)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each product.**
*(To review **multiplying polynomials**, see Lesson 8-7.)*

76. $(n + 8)(n + 3)$
77. $(x - 4)(x - 5)$
78. $(b - 10)(b + 7)$
79. $(3a + 1)(6a - 4)$
80. $(5p - 2)(9p - 3)$
81. $(2y - 5)(4y + 3)$

Practice Quiz 1

Lessons 9-1 and 9-2

1. Find the factors of 225. Then classify the number as *prime* or *composite*. *(Lesson 9-1)*
2. Find the prime factorization of -320. *(Lesson 9-1)*
3. Factor $78a^2bc^3$ completely. *(Lesson 9-1)*
4. Find the GCF of $54x^3$, $42x^2y$, and $30xy^2$. *(Lesson 9-1)*

Factor each polynomial. *(Lesson 9-2)*

5. $4xy^2 - xy$
6. $32a^2b + 40b^3 - 8a^2b^2$
7. $6py + 16p - 15y - 40$

Solve each equation. Check your solutions. *(Lesson 9-2)*

8. $(8n + 5)(n - 4) = 0$
9. $9x^2 - 27x = 0$
10. $10x^2 = -3x$

Algebra Activity

A Preview of Lesson 9-3

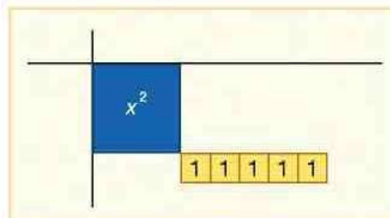
Factoring Trinomials

You can use algebra tiles to factor trinomials. If a polynomial represents the area of a rectangle formed by algebra tiles, then the rectangle's length and width are *factors* of the area.

Activity 1 Use algebra tiles to factor $x^2 + 6x + 5$.

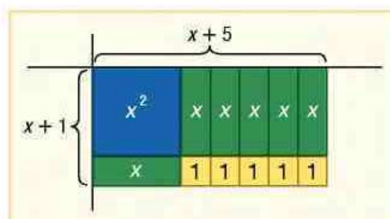
Step 1 Model the polynomial $x^2 + 6x + 5$.

Step 2 Place the x^2 tile at the corner of the product mat. Arrange the 1 tiles into a rectangular array. Because 5 is prime, the 5 tiles can be arranged in a rectangle in one way, a 1-by-5 rectangle.



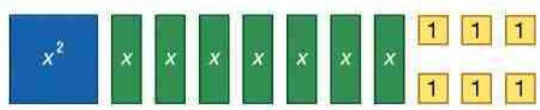
Step 3 Complete the rectangle with the x tiles.

The rectangle has a width of $x + 1$ and a length of $x + 5$. Therefore, $x^2 + 6x + 5 = (x + 1)(x + 5)$.

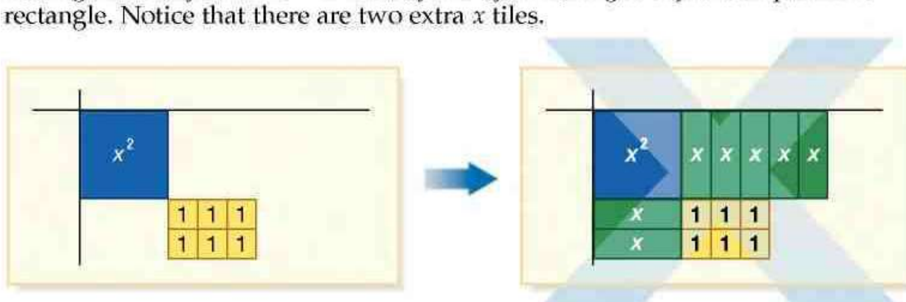


Activity 2 Use algebra tiles to factor $x^2 + 7x + 6$.

Step 1 Model the polynomial $x^2 + 7x + 6$.



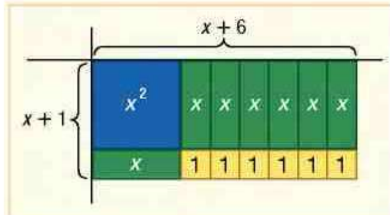
Step 2 Place the x^2 tile at the corner of the product mat. Arrange the 1 tiles into a rectangular array. Since $6 = 2 \times 3$, try a 2-by-3 rectangle. Try to complete the rectangle. Notice that there are two extra x tiles.



(continued on the next page)

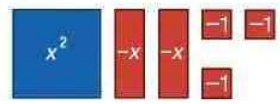
Step 3 Arrange the 1 tiles into a 1-by-6 rectangular array. This time you can complete the rectangle with the x tiles.

The rectangle has a width of $x + 1$ and a length of $x + 6$. Therefore, $x^2 + 7x + 6 = (x + 1)(x + 6)$.

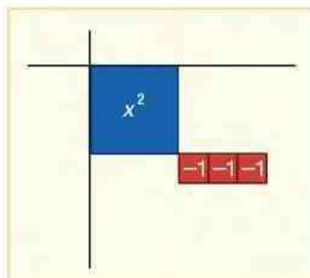


Activity 3 Use algebra tiles to factor $x^2 - 2x - 3$.

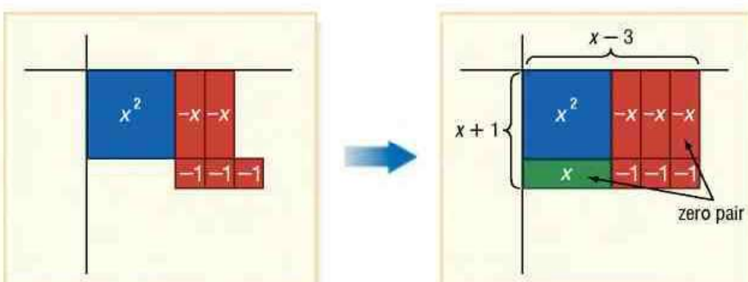
Step 1 Model the polynomial $x^2 - 2x - 3$.



Step 2 Place the x^2 tile at the corner of the product mat. Arrange the 1 tiles into a 1-by-3 rectangular array as shown.



Step 3 Place the x tile as shown. Recall that you can add zero-pairs without changing the value of the polynomial. In this case, add a zero pair of x tiles.



The rectangle has a width of $x + 1$ and a length of $x - 3$. Therefore, $x^2 - 2x - 3 = (x + 1)(x - 3)$.

Model

Use algebra tiles to factor each trinomial.

1. $x^2 + 4x + 3$ **2.** $x^2 + 5x + 4$ **3.** $x^2 - x - 6$ **4.** $x^2 - 3x + 2$

5. $x^2 + 7x + 12$ **6.** $x^2 - 4x + 4$ **7.** $x^2 - x - 2$ **8.** $x^2 - 6x + 8$

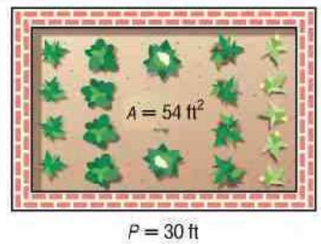
9-3 Factoring Trinomials: $x^2 + bx + c$

What You'll Learn

- Factor trinomials of the form $x^2 + bx + c$.
- Solve equations of the form $x^2 + bx + c = 0$.

How can factoring be used to find the dimensions of a garden?

Tamika has enough bricks to make a 30-foot border around the rectangular vegetable garden she is planting. The booklet she got from the nursery says that the plants will need a space of 54 square feet to grow. What should the dimensions of her garden be? To solve this problem, you need to find two numbers whose product is 54 and whose sum is 15, half the perimeter of the garden.



FACTOR $x^2 + bx + c$ In Lesson 9-1, you learned that when two numbers are multiplied, each number is a factor of the product. Similarly, when two binomials are multiplied, each binomial is a factor of the product.

To factor some trinomials, you will use the pattern for multiplying two binomials. Study the following example.

$$\begin{aligned}
(x+2)(x+3) &= \overset{F}{(x \cdot x)} + \overset{O}{(x \cdot 3)} + \overset{I}{(x \cdot 2)} + \overset{L}{(2 \cdot 3)} && \text{Use the FOIL method.}\\
&= x^2 + 3x + 2x + 6 && \text{Simplify.}\\
&= x^2 + (3 + 2)x + 6 && \text{Distributive Property}\\
&= x^2 + 5x + 6 && \text{Simplify.}
\end{aligned}$$

Observe the following pattern in this multiplication.

$$\begin{aligned}
(x+2)(x+3) &= x^2 + (3+2)x + (2 \cdot 3)\\
(x+m)(x+n) &= x^2 + (n+m)x + mn\\
&= x^2 + \underbrace{(m+n)}x + \underbrace{mn}\\
& \quad\; x^2 + \quad bx \quad + \; c && b = m + n \text{ and } c = mn
\end{aligned}$$

Notice that the coefficient of the middle term is the sum of m and n and the last term is the product of m and n. This pattern can be used to factor quadratic trinomials of the form $x^2 + bx + c$.

Study Tip

Reading Math
A *quadratic trinomial* is a trinomial of degree 2. This means that the greatest exponent of the variable is 2.

Key Concept — Factoring $x^2 + bx + c$

- **Words** To factor quadratic trinomials of the form $x^2 + bx + c$, find two integers, m and n, whose sum is equal to b and whose product is equal to c. Then write $x^2 + bx + c$ using the pattern $(x + m)(x + n)$.
- **Symbols** $x^2 + bx + c = (x + m)(x + n)$ when $m + n = b$ and $mn = c$.
- **Example** $x^2 + 5x + 6 = (x + 2)(x + 3)$, since $2 + 3 = 5$ and $2 \cdot 3 = 6$.

To determine m and n, find the factors of c and use a guess-and-check strategy to find which pair of factors has a sum of b.

Example 1 b and c Are Positive

Factor $x^2 + 6x + 8$.

In this trinomial, $b = 6$ and $c = 8$. You need to find two numbers whose sum is 6 and whose product is 8. Make an organized list of the factors of 8, and look for the pair of factors whose sum is 6.

Factors of 8	Sum of Factors	
1, 8	9	
2, 4	6	The correct factors are 2 and 4.

$$x^2 + 6x + 8 = (x + m)(x + n) \quad \text{Write the pattern.}$$
$$= (x + 2)(x + 4) \quad m = 2 \text{ and } n = 4$$

CHECK You can check this result by multiplying the two factors.

$$(x + 2)(x + 4) = \overset{F}{x^2} + \overset{O}{4x} + \overset{I}{2x} + \overset{L}{8} \quad \text{FOIL method}$$
$$= x^2 + 6x + 8 \checkmark \quad \text{Simplify.}$$

When factoring a trinomial where b is negative and c is positive, you can use what you know about the product of binomials to help narrow the list of possible factors.

Example 2 b Is Negative and c Is Positive

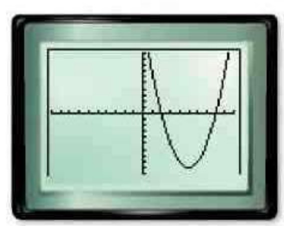
Factor $x^2 - 10x + 16$.

In this trinomial, $b = -10$ and $c = 16$. This means that $m + n$ is negative and mn is positive. So m and n must both be negative. Therefore, make a list of the negative factors of 16, and look for the pair of factors whose sum is -10.

Factors of 16	Sum of Factors	
$-1, -16$	-17	
$-2, -8$	-10	The correct factors are -2 and -8.
$-4, -4$	-8	

$$x^2 - 10x + 16 = (x + m)(x + n) \quad \text{Write the pattern.}$$
$$= (x - 2)(x - 8) \quad m = -2 \text{ and } n = -8$$

CHECK You can check this result by using a graphing calculator. Graph $y = x^2 - 10x + 16$ and $y = (x - 2)(x - 8)$ on the same screen. Since only one graph appears, the two graphs must coincide. Therefore, the trinomial has been factored correctly. ✓



[−10, 10] scl: 1 by [−10, 10] scl: 1

Study Tip

Testing Factors
Once you find the correct factors, there is no need to test any other factors. Therefore, it is not necessary to test -4 and -4 in Example 2.

You will find that keeping an organized list of the factors you have tested is particularly important when factoring a trinomial like $x^2 + x - 12$, where the value of c is negative.

Example 3 b Is Positive and c Is Negative

Factor $x^2 + x - 12$.

In this trinomial, $b = 1$ and $c = -12$. This means that $m + n$ is positive and mn is negative. So either m or n is negative, but not both. Therefore, make a list of the factors of -12, where one factor of each pair is negative. Look for the pair of factors whose sum is 1.

Factors of -12	Sum of Factors
1, -12	-11
-1, 12	11
2, -6	-4
-2, 6	4
3, -4	-1
-3, 4	1

The correct factors are -3 and -4.

$$\begin{aligned} x^2 + x - 12 &= (x + m)(x + n) && \text{Write the pattern.} \\ &= (x - 3)(x + 4) && m = -3 \text{ and } n = 4 \end{aligned}$$

Study Tip

Alternate Method

You can use the opposite of FOIL to factor trinomials. For instance, consider Example 3.

$x^2 + x - 12$

$(x + \square)(x + \square)$

Try factor pairs of -12 until the sum of the products of the Inner and Outer terms is x.

Example 4 b Is Negative and c Is Negative

Factor $x^2 - 7x - 18$.

Since $b = -7$ and $c = -18$, $m + n$ is negative and mn is negative. So either m or n is negative, but not both.

Factors of -18	Sum of Factors
1, -18	-17
-1, 18	17
2, -9	-7

The correct factors are 2 and -9.

$$\begin{aligned} x^2 - 7x - 18 &= (x + m)(x + n) && \text{Write the pattern.} \\ &= (x + 2)(x - 9) && m = 2 \text{ and } n = -9 \end{aligned}$$

SOLVE EQUATIONS BY FACTORING Some equations of the form $x^2 + bx + c = 0$ can be solved by factoring and then using the Zero Product Property.

Example 5 Solve an Equation by Factoring

Solve $x^2 + 5x = 6$. Check your solutions.

$$\begin{aligned} x^2 + 5x &= 6 && \text{Original equation} \\ x^2 + 5x - 6 &= 0 && \text{Rewrite the equation so that one side equals 0.} \\ (x - 1)(x + 6) &= 0 && \text{Factor.} \end{aligned}$$

$$\begin{aligned} x - 1 = 0 \quad &\text{or} \quad x + 6 = 0 && \text{Zero Product Property} \\ x = 1 \quad & \qquad\quad x = -6 && \text{Solve each equation.} \end{aligned}$$

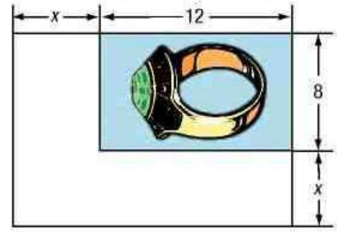
The solution set is $\{1, -6\}$.

CHECK Substitute 1 and -6 for x in the original equation.

$$\begin{aligned} x^2 + 5x &= 6 \\ (1)^2 + 5(1) &\stackrel{?}{=} 6 \\ 6 &= 6 \ \checkmark \end{aligned} \qquad \begin{aligned} x^2 + 5x &= 6 \\ (-6)^2 + 5(-6) &\stackrel{?}{=} 6 \\ 6 &= 6 \ \checkmark \end{aligned}$$

Example 6 Solve a Real-World Problem by Factoring

YEARBOOK DESIGN **A sponsor for the school yearbook has asked that the length and width of a photo in their ad be increased by the same amount in order to double the area of the photo. If the photo was originally 12 centimeters wide by 8 centimeters long, what should the new dimensions of the enlarged photo be?**



Explore Begin by making a diagram like the one shown above, labeling the appropriate dimensions.

Plan Let x = the amount added to each dimension of the photo.

$$\underbrace{\text{The new length}}_{x + 12} \quad \underbrace{\text{times}}_{\cdot} \quad \underbrace{\text{the new width}}_{x + 8} \quad \underbrace{\text{equals}}_{=} \quad \underbrace{\text{the new area.}}_{2\underbrace{(8)(12)}_{\text{old area}}}$$

Solve

$(x + 12)(x + 8) = 2(8)(12)$	Write the equation.
$x^2 + 20x + 96 = 192$	Multiply.
$x^2 + 20x - 96 = 0$	Subtract 192 from each side.
$(x + 24)(x - 4) = 0$	Factor.
$x + 24 = 0$ or $x - 4 = 0$	Zero Product Property
$x = -24$ $\qquad$ $x = 4$	Solve each equation.

Examine The solution set is $\{-24, 4\}$. Only 4 is a valid solution, since dimensions cannot be negative. Thus, the new length of the photo should be $4 + 12$ or 16 centimeters, and the new width should be $4 + 8$ or 12 centimeters.

Check for Understanding

Concept Check

1. **Explain** why, when factoring $x^2 + 6x + 9$, it is not necessary to check the sum of the factor pairs -1 and -9 or -3 and -3.
2. **OPEN ENDED** Give an example of an equation that can be solved using the factoring techniques presented in this lesson. Then, solve your equation.
3. **FIND THE ERROR** Peter and Aleta are solving $x^2 + 2x = 15$.

Peter	Aleta
$x^2 + 2x = 15$	$x^2 + 2x = 15$
$x(x + 2) = 15$	$x^2 + 2x - 15 = 0$
$x = 15$ or $x + 2 = 15$	$(x - 3)(x + 5) = 0$
$x = 13$	$x - 3 = 0$ or $x + 5 = 0$
	$x = 3$ $\qquad$ $x = -5$

Who is correct? Explain your reasoning.

Guided Practice **Factor each trinomial.**

4. $x^2 + 11x + 24$
5. $c^2 - 3c + 2$
6. $n^2 + 13n - 48$
7. $p^2 - 2p - 35$
8. $72 + 27a + a^2$
9. $x^2 - 4xy + 3y^2$

Solve each equation. Check your solutions.

10. $n^2 + 7n + 6 = 0$
11. $a^2 + 5a - 36 = 0$
12. $p^2 - 19p - 42 = 0$
13. $y^2 + 9 = -10y$
14. $9x + x^2 = 22$
15. $d^2 - 3d = 70$

Application

16. **NUMBER THEORY** Find two consecutive integers whose product is 156.

Practice and Apply

Homework Help

For Exercises	See Examples
17–36	1–4
37–53	5
54–56, 61, 62	6

Extra Practice

See page 840.

Factor each trinomial.

17. $a^2 + 8a + 15$
18. $x^2 + 12x + 27$
19. $c^2 + 12c + 35$
20. $y^2 + 13y + 30$
21. $m^2 - 22m + 21$
22. $d^2 - 7d + 10$
23. $p^2 - 17p + 72$
24. $g^2 - 19g + 60$
25. $x^2 + 6x - 7$
26. $b^2 + b - 20$
27. $h^2 + 3h - 40$
28. $n^2 + 3n - 54$
29. $y^2 - y - 42$
30. $z^2 - 18z - 40$
31. $-72 + 6w + w^2$
32. $-30 + 13x + x^2$
33. $a^2 + 5ab + 4b^2$
34. $x^2 - 13xy + 36y^2$

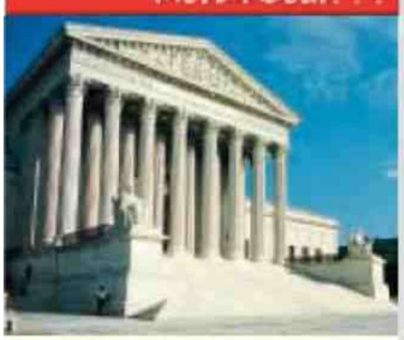
GEOMETRY **Find an expression for the perimeter of a rectangle with the given area.**

35. area $= x^2 + 24x - 81$
36. area $= x^2 + 13x - 90$

Solve each equation. Check your solutions.

37. $x^2 + 16x + 28 = 0$
38. $b^2 + 20b + 36 = 0$
39. $y^2 + 4y - 12 = 0$
40. $d^2 + 2d - 8 = 0$
41. $a^2 - 3a - 28 = 0$
42. $g^2 - 4g - 45 = 0$
43. $m^2 - 19m + 48 = 0$
44. $n^2 - 22n + 72 = 0$
45. $z^2 = 18 - 7z$
46. $h^2 + 15 = -16h$
47. $24 + k^2 = 10k$
48. $x^2 - 20 = x$
49. $c^2 - 50 = -23c$
50. $y^2 - 29y = -54$
51. $14p + p^2 = 51$
52. $x^2 - 2x - 6 = 74$
53. $x^2 - x + 56 = 17x$

More About. . .



Supreme Court

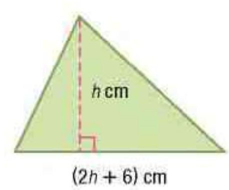
The "Conference handshake" has been a tradition since the late 19th century.

Source: www.supremecourtus.gov

54. **SUPREME COURT** When the Justices of the Supreme Court assemble to go on the Bench each day, each Justice shakes hands with each of the other Justices for a total of 36 handshakes. The total number of handshakes h possible for n people is given by $h = \frac{n^2 - n}{2}$. Write and solve an equation to determine the number of Justices on the Supreme Court.

55. **NUMBER THEORY** Find two consecutive even integers whose product is 168.

56. **GEOMETRY** The triangle has an area of 40 square centimeters. Find the height h of the triangle.



CRITICAL THINKING **Find all values of k so that each trinomial can be factored using integers.**

57. $x^2 + kx - 19$
58. $x^2 + kx + 14$
59. $x^2 - 8x + k, k > 0$
60. $x^2 - 5x + k, k > 0$

RUGBY **For Exercises 61 and 62, use the following information.**
The length of a Rugby League field is 52 meters longer than its width w.

61. Write an expression for the area of the field.
62. The area of a Rugby League field is 8160 square meters. Find the dimensions of the field.

63. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can factoring be used to find the dimensions of a garden?

Include the following in your answer:

- a description of how you would find the dimensions of the garden, and
- an explanation of how the process you used is related to the process used to factor trinomials of the form $x^2 + bx + c$.

Standardized Test Practice

64. Which is the factored form of $x^2 - 17x + 42$?

Ⓐ $(x - 1)(y - 42)$ Ⓑ $(x - 2)(x - 21)$

Ⓒ $(x - 3)(x - 14)$ Ⓓ $(x - 6)(x - 7)$

65. **GRID IN** What is the positive solution of $p^2 - 13p - 30 = 0$?

Graphing Calculator

Use a graphing calculator to determine whether each factorization is correct. Write *yes* or *no*. If no, state the correct factorization.

66. $x^2 - 14x + 48 = (x + 6)(x + 8)$ **67.** $x^2 - 16x - 105 = (x + 5)(x - 21)$

68. $x^2 + 25x + 66 = (x + 33)(x + 2)$ **69.** $x^2 + 11x - 210 = (x + 10)(x - 21)$

Maintain Your Skills

Mixed Review

Solve each equation. Check your solutions. *(Lesson 9-2)*

70. $(x + 3)(2x - 5) = 0$ **71.** $b(7b - 4) = 0$ **72.** $5y^2 = -9y$

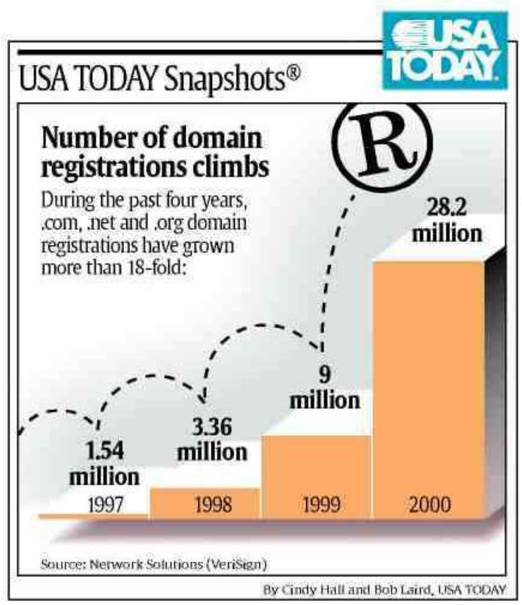
Find the GCF of each set of monomials. *(Lesson 9-1)*

73. 24, 36, 72 **74.** $9p^2q^5, 21p^3q^3$ **75.** $30x^4y^5, 20x^2y^7, 75x^3y^4$

INTERNET **For Exercises 76 and 77, use the graph at the right.** *(Lessons 3-7 and 8-3)*

76. Find the percent increase in the number of domain registrations from 1997 to 2000.

77. Use your answer from Exercise 76 to verify the claim that registrations grew more than 18-fold from 1997 to 2000 is correct.



Getting Ready for the Next Lesson

PREREQUISITE SKILL **Factor each polynomial.**

*(To review **factoring by grouping**, see Lesson 9-2.)*

78. $3y^2 + 2y + 9y + 6$ **79.** $3a^2 + 2a + 12a + 8$ **80.** $4x^2 + 3x + 8x + 6$

81. $2p^2 - 6p + 7p - 21$ **82.** $3b^2 + 7b - 12b - 28$ **83.** $4g^2 - 2g - 6g + 3$

9-4 Factoring Trinomials: $ax^2 + bx + c$

What You'll Learn

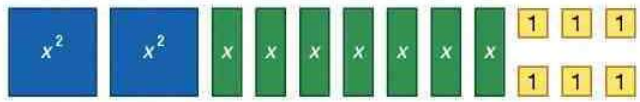
- Factor trinomials of the form $ax^2 + bx + c$.
- Solve equations of the form $ax^2 + bx + c = 0$.

Vocabulary

- prime polynomial

How can algebra tiles be used to factor $2x^2 + 7x + 6$?

The factors of $2x^2 + 7x + 6$ are the dimensions of the rectangle formed by the algebra tiles shown below.



The process you use to form the rectangle is the same mental process you can use to factor this trinomial algebraically.

FACTOR $ax^2 + bx + c$ For trinomials of the form $x^2 + bx + c$, the coefficient of x^2 is 1. To factor trinomials of this form, you find the factors of c whose sum is b. We can modify this approach to factor trinomials whose leading coefficient is not 1.

Study Tip

Look Back
To review **factoring by grouping**, see Lesson 9-2.

$$(2x + 5)(3x + 1) = \overset{F}{6x^2} + \overset{O}{2x} + \overset{I}{15x} + \overset{L}{5}$$ Use the FOIL method.

$$2 \cdot 15 = 30$$

$$6 \cdot 5 = 30$$

Observe the following pattern in this product.

$$6x^2 + 2x + 15x + 5 \qquad ax^2 + mx + nx + c$$
$$6x^2 + 17x + 5 \qquad ax^2 + bx + c$$
$$2 + 15 = 17 \text{ and } 2 \cdot 15 = 6 \cdot 5 \qquad m + n = b \text{ and } mn = ac$$

You can use this pattern and the method of factoring by grouping to factor $6x^2 + 17x + 5$. Find two numbers, m and n, whose product is $6 \cdot 5$ or 30 and whose sum is 17.

Factors of 30	Sum of Factors
1, 30	31
2, 15	17

The correct factors are 2 and 15.

$$\begin{aligned} 6x^2 + 17x + 5 &= 6x^2 + mx + nx + 5 && \text{Write the pattern.} \\ &= 6x^2 + 2x + 15x + 5 && m = 2 \text{ and } n = 15 \\ &= (6x^2 + 2x) + (15x + 5) && \text{Group terms with common factors.} \\ &= 2x(3x + 1) + 5(3x + 1) && \text{Factor the GCF from each grouping.} \\ &= (3x + 1)(2x + 5) && 3x + 1 \text{ is the common factor.} \end{aligned}$$

Therefore, $6x^2 + 17x + 5 = (3x + 1)(2x + 5)$.

Example 1 Factor $ax^2 + bx + c$

a. Factor $7x^2 + 22x + 3$.

In this trinomial, $a = 7$, $b = 22$ and $c = 3$. You need to find two numbers whose sum is 22 and whose product is $7 \cdot 3$ or 21. Make an organized list of the factors of 21 and look for the pair of factors whose sum is 22.

Factors of 21	Sum of Factors
1, 21	22

The correct factors are 1 and 21.

$$
\begin{aligned}
7x^2 + 22x + 3 &= 7x^2 + mx + nx + 3 && \text{Write the pattern.}\\
&= 7x^2 + 1x + 21x + 3 && m = 1 \text{ and } n = 21\\
&= (7x^2 + 1x) + (21x + 3) && \text{Group terms with common factors.}\\
&= x(7x + 1) + 3(7x + 1) && \text{Factor the GCF from each grouping.}\\
&= (7x + 1)(x + 3) && \text{Distributive Property}
\end{aligned}
$$

CHECK You can check this result by multiplying the two factors.

$$
\begin{aligned}
(7x + 1)(x + 3) &= \overset{F}{7x^2} + \overset{O}{21x} + \overset{I}{x} + \overset{L}{3} && \text{FOIL method}\\
&= 7x^2 + 22x + 3 \checkmark && \text{Simplify.}
\end{aligned}
$$

b. Factor $10x^2 - 43x + 28$.

In this trinomial, $a = 10$, $b = -43$ and $c = 28$. Since b is negative, $m + n$ is negative. Since c is positive, mn is positive. So m and n must both be negative. Therefore, make a list of the negative factors of $10 \cdot 28$ or 280, and look for the pair of factors whose sum is -43.

Factors of 280	Sum of Factors
−1, −280	−281
−2, −140	−142
−4, −70	−74
−5, −56	−61
−7, −40	−47
−8, −35	−43

The correct factors are −8 and −35.

Study Tip

Finding Factors

Factor pairs in an organized list so you do not miss any possible pairs of factors.

$$
\begin{aligned}
&10x^2 - 43x + 28\\
&= 10x^2 + mx + nx + 28 && \text{Write the pattern.}\\
&= 10x^2 + (-8)x + (-35)x + 28 && m = -8 \text{ and } n = -35\\
&= (10x^2 - 8x) + (-35x + 28) && \text{Group terms with common factors.}\\
&= 2x(5x - 4) + 7(-5x + 4) && \text{Factor the GCF from each grouping.}\\
&= 2x(5x - 4) + 7(-1)(5x - 4) && -5x + 4 = (-1)(5x - 4)\\
&= 2x(5x - 4) + (-7)(5x - 4) && 7(-1) = -7\\
&= (5x - 4)(2x - 7) && \text{Distributive Property}
\end{aligned}
$$

Sometimes the terms of a trinomial will contain a common factor. In these cases, first use the Distributive Property to factor out the common factor. Then factor the trinomial.

Example 2 Factor When a, b, and c Have a Common Factor

Factor $3x^2 + 24x + 45$.

Notice that the GCF of the terms $3x^2$, $24x$, and 45 is 3. When the GCF of the terms of a trinomial is an integer other than 1, you should first factor out this GCF.

$3x^2 + 24x + 45 = 3(x^2 + 8x + 15)$ Distributive Property

Now factor $x^2 + 8x + 15$. Since the lead coefficient is 1, find two factors of 15 whose sum is 8.

Factors of 15	Sum of Factors
1, 15	16
3, 5	8

The correct factors are 2 and 15.

So, $x^2 + 8x + 15 = (x + 3)(x + 5)$. Thus, the complete factorization of $3x^2 + 24x + 45$ is $3(x + 3)(x + 5)$.

Study Tip

Factoring Completely
Always check for a GCF first before trying to factor a trinomial.

A polynomial that cannot be written as a product of two polynomials with integral coefficients is called a **prime polynomial**.

Example 3 Determine Whether a Polynomial Is Prime

Factor $2x^2 + 5x - 2$.

In this trinomial, $a = 2$, $b = 5$ and $c = -2$. Since b is positive, $m + n$ is positive. Since c is negative, mn is negative. So either m or n is negative, but not both. Therefore, make a list of the factors of $2 \cdot -2$ or -4, where one factor in each pair is negative. Look for a pair of factors whose sum is 5.

Factors of −4	Sum of Factors
1, −4	−3
−1, 4	3
−2, 2	0

There are no factors whose sum is 5. Therefore, $2x^2 + 5x - 2$ cannot be factored using integers. Thus, $2x^2 + 5x - 2$ is a prime polynomial.

SOLVE EQUATIONS BY FACTORING Some equations of the form $ax^2 + bx + c = 0$ can be solved by factoring and then using the Zero Product Property.

Example 4 Solve Equations by Factoring

Solve $8a^2 - 9a - 5 = 4 - 3a$. Check your solutions.

$8a^2 - 9a - 5 = 4 - 3a$ Original equation

$8a^2 - 6a - 9 = 0$ Rewrite so that one side equals 0.

$(4a + 3)(2a - 3) = 0$ Factor the left side.

$4a + 3 = 0$ or $2a - 3 = 0$ Zero Product Property

$4a = -3$ $\quad 2a = 3$ Solve each equation.

$a = -\frac{3}{4}$ $\quad a = \frac{3}{2}$

The solution set is $\left\{-\frac{3}{4}, \frac{3}{2}\right\}$.

CHECK Check each solution in the original equation.

$$8a^2 - 9a - 5 = 4 - 3a \qquad 8a^2 - 9a - 5 = 4 - 3a$$

$$8\left(-\frac{3}{4}\right)^2 - 9\left(-\frac{3}{4}\right) - 5 \stackrel{?}{=} 4 - 3\left(-\frac{3}{4}\right) \qquad 8\left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right) - 5 \stackrel{?}{=} 4 - 3\left(\frac{3}{2}\right)$$

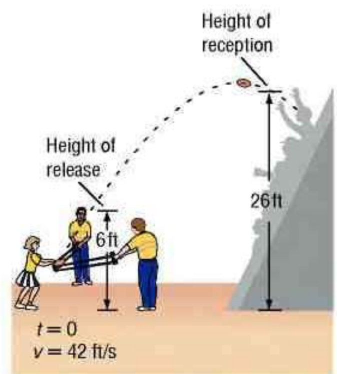
$$\frac{9}{2} + \frac{27}{4} - 5 \stackrel{?}{=} 4 + \frac{9}{4} \qquad 18 - \frac{27}{2} - 5 \stackrel{?}{=} 4 - \frac{9}{2}$$

$$\frac{25}{4} = \frac{25}{4} \ \checkmark \qquad -\frac{1}{2} = -\frac{1}{2} \ \checkmark$$

A *model for the vertical motion of a projected object* is given by the equation $h = -16t^2 + vt + s$, where h is the height in feet, t is the time in seconds, v is the initial upward velocity in feet per second, and s is the starting height of the object in feet.

Example 5 Solve Real-World Problems by Factoring

PEP RALLY **At a pep rally, small foam footballs are launched by cheerleaders using a sling-shot. How long is a football in the air if a student in the stands catches it on its way down 26 feet above the gym floor?**



Use the model for vertical motion.

$h = -16t^2 + vt + s$	Vertical motion model
$26 = -16t^2 + 42t + 6$	$h = 26$, $v = 42$, $s = 6$
$0 = -16t^2 + 42t - 20$	Subtract 26 from each side.
$0 = -2(8t^2 - 21t + 10)$	Factor out -2.
$0 = 8t^2 - 21t + 10$	Divide each side by -2.
$0 = (8t - 5)(t - 2)$	Factor $8t^2 - 21t + 10$.
$8t - 5 = 0$ or $t - 2 = 0$	Zero Product Property
$8t = 5$ $\quad t = 2$	Solve each equation.
$t = \frac{5}{8}$	

The solutions are $\frac{5}{8}$ second and 2 seconds. The first time represents how long it takes the football to reach a height of 26 feet on its way up. The later time represents how long it takes the ball to reach a height of 26 feet again on its way down. Thus, the football will be in the air for 2 seconds before the student catches it.

Study Tip

Factoring When a Is Negative

When factoring a trinomial of the form $ax^2 + bx + c$, where a is negative, it is helpful to factor out a negative monomial.

Check for Understanding

Concept Check

1. **Explain** how to determine which values should be chosen for m and n when factoring a polynomial of the form $ax^2 + bx + c$.
2. **OPEN ENDED** Write a trinomial that can be factored using a pair of numbers whose sum is 9 and whose product is 14.
3. **FIND THE ERROR** Dasan and Craig are factoring $2x^2 + 11x + 18$.

Dasan

Factors of 18	Sum
1, 18	19
3, 6	9
9, 2	11

$2x^2 + 11x + 18$
$= 2(x^2 + 11x + 18)$
$= 2(x + 9)(x + 2)$

Craig

Factors of 36	Sum
1, 36	37
2, 18	20
3, 12	15
4, 9	13
6, 6	12

$2x^2 + 11x + 18$ is prime.

Who is correct? Explain your reasoning.

Guided Practice **Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*.**

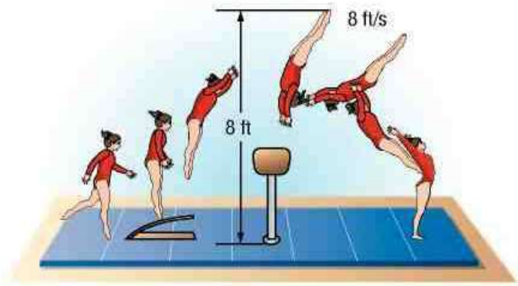
4. $3a^2 + 8a + 4$
5. $2a^2 - 11a + 7$
6. $2p^2 + 14p + 24$
7. $2x^2 + 13x + 20$
8. $6x^2 + 15x - 9$
9. $4n^2 - 4n - 35$

Solve each equation. Check your solutions.

10. $3x^2 + 11x + 6 = 0$ **11.** $10p^2 - 19p + 7 = 0$ **12.** $6n^2 + 7n = 20$

Application

13. GYMNASTICS When a gymnast making a vault leaves the horse, her feet are 8 feet above the ground traveling with an initial upward velocity of 8 feet per second. Use the model for vertical motion to find the time t in seconds it takes for the gymnast's feet to reach the mat. (*Hint*: Let $h = 0$, the height of the mat.)



Practice and Apply

Homework Help

For Exercises	See Examples
14–31	1–3
35–48	4
49–52	5

Extra Practice

See page 840.

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*.

14. $2x^2 + 7x + 5$ **15.** $3x^2 + 5x + 2$ **16.** $6p^2 + 5p - 6$

17. $5d^2 + 6d - 8$ **18.** $8k^2 - 19k + 9$ **19.** $9g^2 - 12g + 4$

20. $2a^2 - 9a - 18$ **21.** $2x^2 - 3x - 20$ **22.** $5c^2 - 17c + 14$

23. $3p^2 - 25p + 16$ **24.** $8y^2 - 6y - 9$ **25.** $10n^2 - 11n - 6$

26. $15z^2 + 17z - 18$ **27.** $14x^2 + 13x - 12$ **28.** $6r^2 - 14r - 12$

29. $30x^2 - 25x - 30$ **30.** $9x^2 + 30xy + 25y^2$ **31.** $36a^2 + 9ab - 10b^2$

CRITICAL THINKING **Find all values of k so that each trinomial can be factored as two binomials using integers.**

32. $2x^2 + kx + 12$ **33.** $2x^2 + kx + 15$ **34.** $2x^2 + 12x + k, k > 0$

Solve each equation. Check your solutions.

35. $5x^2 + 27x + 10 = 0$ **36.** $3x^2 - 5x - 12 = 0$ **37.** $24x^2 - 11x - 3 = 3x$

38. $17x^2 - 11x + 2 = 2x^2$ **39.** $14n^2 = 25n + 25$ **40.** $12a^2 - 13a = 35$

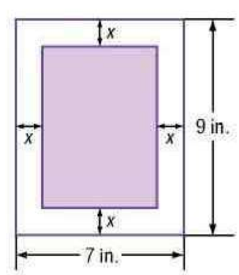
41. $6x^2 - 14x = 12$ **42.** $21x^2 - 6 = 15x$ **43.** $24x^2 - 30x + 8 = -2x$

44. $24x^2 - 46x = 18$ **45.** $\frac{x^2}{12} - \frac{2x}{3} - 4 = 0$ **46.** $t^2 - \frac{t}{6} = \frac{35}{6}$

47. $(3y + 2)(y + 3) = y + 14$ **48.** $(4a - 1)(a - 2) = 7a - 5$

GEOMETRY **For Exercises 49 and 50, use the following information.**

A rectangle with an area of 35 square inches is formed by cutting off strips of equal width from a rectangular piece of paper.



49. Find the width of each strip.

50. Find the dimensions of the new rectangle.

51. CLIFF DIVING Suppose a diver leaps from the edge of a cliff 80 feet above the ocean with an initial upward velocity of 8 feet per second. How long it will take the diver to enter the water below?

More About. . .

Cliff Diving

In Acapulco, Mexico, divers leap from La Quebrada, the "Break in the Rocks," diving headfirst into the Pacific Ocean 105 feet below.

Source: acapulco-travel.web.com.mx

52. **CLIMBING** Damaris launches a grappling hook from a height of 6 feet with an initial upward velocity of 56 feet per second. The hook just misses the stone ledge of a building she wants to scale. As it falls, the hook anchors on the ledge, which is 30 feet above the ground. How long was the hook in the air?

53. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can algebra tiles be used to factor $2x^2 + 7x + 6$?

Include the following in your answer:
- the dimensions of the rectangle formed, and
- an explanation, using words and drawings, of how this geometric guess-and-check process of factoring is similar to the algebraic process described on page 495.

Standardized Test Practice

54. What are the solutions of $2p^2 - p - 3 = 0$?

Ⓐ $-\frac{2}{3}$ and 1 Ⓑ $\frac{2}{3}$ and -1 Ⓒ $-\frac{3}{2}$ and 1 Ⓓ $\frac{3}{2}$ and -1

55. Suppose a person standing atop a building 398 feet tall throws a ball upward. If the person releases the ball 4 feet above the top of the building, the ball's height h, in feet, after t seconds is given by the equation $h = -16t^2 + 48t + 402$. After how many seconds will the ball be 338 feet from the ground?

Ⓐ 3.5 Ⓑ 4 Ⓒ 4.5 Ⓓ 5

Maintain Your Skills

Mixed Review

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lesson 9-3)*

56. $a^2 - 4a - 21$ 57. $t^2 + 2t + 2$ 58. $d^2 + 15d + 44$

Solve each equation. Check your solutions. *(Lesson 9-2)*

59. $(y - 4)(5y + 7) = 0$ 60. $(2k + 9)(3k + 2) = 0$ 61. $12u = u^2$

62. **BUSINESS** Jake's Garage charges \$83 for a two-hour repair job and \$185 for a five-hour repair job. Write a linear equation that Jake can use to bill customers for repair jobs of any length of time. *(Lesson 5-3)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the principal square root of each number.**
(To review ***square roots****, see Lesson 2-7.)*

63. 16 64. 49 65. 36 66. 25
67. 100 68. 121 69. 169 70. 225

Practice Quiz 2

Lessons 9-3 and 9-4

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lessons 9-3 and 9-4)*

1. $x^2 - 14x - 72$ 2. $8p^2 - 6p - 35$ 3. $16a^2 - 24a + 5$
4. $n^2 - 17n + 52$ 5. $24c^2 + 62c + 18$ 6. $3y^2 + 33y + 54$

Solve each equation. Check your solutions. *(Lessons 9-3 and 9-4)*

7. $b^2 + 14b - 32 = 0$ 8. $x^2 + 45 = 18x$
9. $12y^2 - 7y - 12 = 0$ 10. $6a^2 = 25a - 14$

9-5 Factoring Differences of Squares

What You'll Learn

- Factor binomials that are the differences of squares.
- Solve equations involving the differences of squares.

How can you determine a basketball player's hang time?

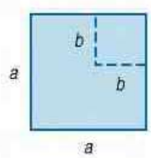
A basketball player's *hang time* is the length of time he is in the air after jumping. Given the maximum height h a player can jump, you can determine his hang time t in seconds by solving $4t^2 - h = 0$. If h is a perfect square, this equation can be solved by factoring using the pattern for the difference of squares.

FACTOR $a^2 - b^2$ A geometric model can be used to factor the difference of squares.

Algebra Activity

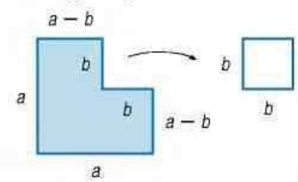
Difference of Squares

Step 1 Use a straightedge to draw two squares similar to those shown below. Choose any measures for a and b.



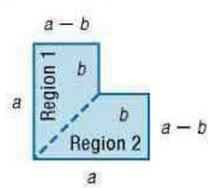
Notice that the area of the large square is a^2, and the area of the small square is b^2.

Step 2 Cut the small square from the large square.

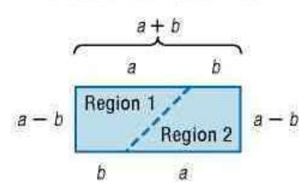


The area of the remaining irregular region is $a^2 - b^2$.

Step 3 Cut the irregular region into two congruent pieces as shown below.



Step 4 Rearrange the two congruent regions to form a rectangle with length $a + b$ and width $a - b$.



Make a Conjecture

1. Write an expression representing the area of the rectangle.
2. Explain why $a^2 - b^2 = (a + b)(a - b)$.

Study Tip

Look Back
To review the **product of a sum and a difference**, see Lesson 8-8.

Key Concept — Difference of Squares

- **Symbols** $a^2 - b^2 = (a + b)(a - b)$ or $(a - b)(a + b)$
- **Example** $x^2 - 9 = (x + 3)(x - 3)$ or $(x - 3)(x + 3)$

We can use this pattern to factor binomials that can be written in the form $a^2 - b^2$.

Example 1 Factor the Difference of Squares

Factor each binomial.

a. $n^2 - 25$

$n^2 - 25 = n^2 - 5^2$ — Write in the form $a^2 - b^2$.

$= (n + 5)(n - 5)$ — Factor the difference of squares.

b. $36x^2 - 49y^2$

$36x^2 - 49y^2 = (6x)^2 - (7y)^2$ — $36x^2 = 6x \cdot 6x$ and $49y^2 = 7y \cdot 7y$

$= (6x + 7y)(6x - 7y)$ — Factor the difference of squares.

If the terms of a binomial have a common factor, the GCF should be factored out first before trying to apply any other factoring technique.

Example 2 Factor Out a Common Factor

Factor $48a^3 - 12a$.

$48a^3 - 12a = 12a(4a^2 - 1)$ — The GCF of $48a^3$ and $-12a$ is $12a$.

$= 12a[(2a)^2 - 1^2]$ — $4a^2 = 2a \cdot 2a$ and $1 = 1 \cdot 1$

$= 12a(2a + 1)(2a - 1)$ — Factor the difference of squares.

Occasionally, the difference of squares pattern needs to be applied more than once to factor a polynomial completely.

Example 3 Apply a Factoring Technique More Than Once

Factor $2x^4 - 162$.

$2x^4 - 162 = 2(x^4 - 81)$ — The GCF of $2x^4$ and -162 is 2.

$= 2[(x^2)^2 - 9^2]$ — $x^4 = x^2 \cdot x^2$ and $81 = 9 \cdot 9$

$= 2(x^2 + 9)(x^2 - 9)$ — Factor the difference of squares.

$= 2(x^2 + 9)(x^2 - 3^2)$ — $x^2 = x \cdot x$ and $9 = 3 \cdot 3$

$= 2(x^2 + 9)(x + 3)(x - 3)$ — Factor the difference of squares.

Study Tip

Common Misconception

Remember that the sum of two squares, like $x^2 + 9$, is not factorable using the difference of squares pattern. $x^2 + 9$ is a prime polynomial.

Example 4 Apply Several Different Factoring Techniques

Factor $5x^3 + 15x^2 - 5x - 15$.

$5x^3 + 15x^2 - 5x - 15$ — Original polynomial

$= 5(x^3 + 3x^2 - x - 3)$ — Factor out the GCF.

$= 5[(x^3 - x) + (3x^2 - 3)]$ — Group terms with common factors.

$= 5[x(x^2 - 1) + 3(x^2 - 1)]$ — Factor each grouping.

$= 5(x^2 - 1)(x + 3)$ — $x^2 - 1$ is the common factor.

$= 5(x + 1)(x - 1)(x + 3)$ — Factor the difference of squares, $x^2 - 1$, into $(x + 1)(x - 1)$.

SOLVE EQUATIONS BY FACTORING

You can apply the Zero Product Property to an equation that is written as the product of any number of factors set equal to 0.

Example 5 Solve Equations by Factoring

Study Tip

Alternative Method

The fraction could also be cleared from the equation in Example 5a by multiplying each side of the equation by 16.

$$p^2 - \frac{9}{16} = 0$$
$$16p^2 - 9 = 0$$
$$(4p + 3)(4p - 3) = 0$$
$$4p + 3 = 0 \text{ or } 4p - 3 = 0$$
$$p = -\frac{3}{4} \qquad p = \frac{3}{4}$$

Solve each equation by factoring. Check your solutions.

a. $p^2 - \frac{9}{16} = 0$

$p^2 - \frac{9}{16} = 0$ — Original equation

$p^2 - \left(\frac{3}{4}\right)^2 = 0$ — $p^2 = p \cdot p$ and $\frac{9}{16} = \frac{3}{4} \cdot \frac{3}{4}$

$\left(p + \frac{3}{4}\right)\left(p - \frac{3}{4}\right) = 0$ — Factor the difference of squares.

$p + \frac{3}{4} = 0$ or $p - \frac{3}{4} = 0$ — Zero Product Property

$p = -\frac{3}{4}$ $\qquad p = \frac{3}{4}$ — Solve each equation.

The solution set is $\left\{-\frac{3}{4}, \frac{3}{4}\right\}$. Check each solution in the original equation.

b. $18x^3 = 50x$

$18x^3 = 50x$ — Original equation

$18x^3 - 50x = 0$ — Subtract $50x$ from each side.

$2x(9x^2 - 25) = 0$ — The GCF of $18x^3$ and $-50x$ is $2x$.

$2x(3x + 5)(3x - 5) = 0$ — $9x^2 = 3x \cdot 3x$ and $25 = 5 \cdot 5$

Applying the Zero Product Property, set each factor equal to 0 and solve the resulting three equations.

$2x = 0$ or $3x + 5 = 0$ or $3x - 5 = 0$

$x = 0$ $\qquad 3x = -5$ $\qquad 3x = 5$

$\qquad x = -\frac{5}{3}$ $\qquad x = \frac{5}{3}$

The solution set is $\left\{-\frac{5}{3}, 0, \frac{5}{3}\right\}$. Check each solution in the original equation.

Standardized Test Practice

Example 6 Use Differences of Two Squares

The Princeton Review

Test-Taking Tip

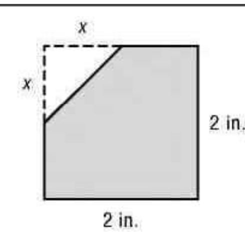
Look to see if the area of an oddly-shaped figure can be found by subtracting the areas of more familiar shapes, such as triangles, rectangles, or circles.

Extended-Response Test Item

A corner is cut off a 2-inch by 2-inch square piece of paper. The cut is x inches from a corner as shown.

a. Write an equation in terms of x that represents the area A of the paper after the corner is removed.

b. What value of x will result in an area that is $\frac{7}{9}$ the area of the original square piece of paper? Show how you arrived at your answer.



Read the Test Item

A is the area of the square minus the area of the triangular corner to be removed.

(continued on the next page)

Solve the Test Item

a. The area of the square is $2 \cdot 2$ or 4 square inches, and the area of the triangle is $\frac{1}{2} \cdot x \cdot x$ or $\frac{1}{2}x^2$ square inches. Thus, $A = 4 - \frac{1}{2}x^2$.

b. Find x so that A is $\frac{7}{9}$ the area of the original square piece of paper, A_o.

$A = \frac{7}{9}A_o$	Translate the verbal statement.
$4 - \frac{1}{2}x^2 = \frac{7}{9}(4)$	$A = 4 - \frac{1}{2}x^2$ and A_o is 4.
$4 - \frac{1}{2}x^2 = \frac{28}{9}$	Simplify.
$4 - \frac{1}{2}x^2 - \frac{28}{9} = 0$	Subtract $\frac{28}{9}$ from each side.
$\frac{8}{9} - \frac{1}{2}x^2 = 0$	Simplify.
$16 - 9x^2 = 0$	Multiply each side by 18 to remove fractions.
$(4 + 3x)(4 - 3x) = 0$	Factor the difference of squares.
$4 + 3x = 0$ or $4 - 3x = 0$	Zero Product Property
$x = -\frac{4}{3}$ $\quad$ $x = \frac{4}{3}$	Solve each equation.

Since length cannot be negative, the only reasonable solution is $\frac{4}{3}$.

Check for Understanding

Concept Check

1. **Describe** a binomial that is the difference of two squares.

2. **OPEN ENDED** Write a binomial that is the difference of two squares. Then factor your binomial.

3. **Determine** whether the difference of squares pattern can be used to factor $3n^2 - 48$. Explain your reasoning.

4. **FIND THE ERROR** Manuel and Jessica are factoring $64x^2 + 16y^2$.

Manuel

$64x^2 + 16y^2$
$= 16(4x^2 + y^2)$

Jessica

$64x^2 + 16y^2$
$= 16(4x^2 + y^2)$
$= 16(2x + y)(2x - y)$

Who is correct? Explain your reasoning.

Guided Practice

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.

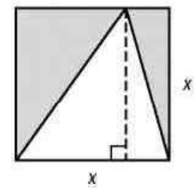
5. $n^2 - 81$
6. $4 - 9a^2$
7. $2x^5 - 98x^3$
8. $32x^4 - 2y^4$
9. $4t^2 - 27$
10. $x^3 - 3x^2 - 9x + 27$

Solve each equation by factoring. Check your solutions.

11. $4y^2 = 25$
12. $17 - 68k^2 = 0$
13. $x^2 - \frac{1}{36} = 0$
14. $121a = 49a^3$

Standardized Test Practice

15. **OPEN ENDED** The area of the shaded part of the square at the right is 72 square inches. Find the dimensions of the square.



Practice and Apply

Homework Help

For Exercises	See Examples
16–33	1–4
34–45	5
47–50	6

Extra Practice
See page 841.

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.

16. $x^2 - 49$
17. $n^2 - 36$
18. $81 + 16k^2$
19. $25 - 4p^2$
20. $-16 + 49h^2$
21. $-9r^2 + 121$
22. $100c^2 - d^2$
23. $9x^2 - 10y^2$
24. $144a^2 - 49b^2$
25. $169y^2 - 36z^2$
26. $8d^2 - 18$
27. $3x^2 - 75$
28. $8z^2 - 64$
29. $4g^2 - 50$
30. $18a^4 - 72a^2$
31. $20x^3 - 45xy^2$
32. $n^3 + 5n^2 - 4n - 20$
33. $(a + b)^2 - c^2$

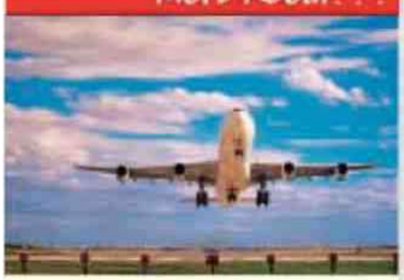
Solve each equation by factoring. Check your solutions.

34. $25x^2 = 36$
35. $9y^2 = 64$
36. $12 - 27n^2 = 0$
37. $50 - 8a^2 = 0$
38. $w^2 - \frac{4}{49} = 0$
39. $\frac{81}{100} - p^2 = 0$
40. $36 - \frac{1}{9}r^2 = 0$
41. $\frac{1}{4}x^2 - 25 = 0$
42. $12d^3 - 147d = 0$
43. $18n^3 - 50n = 0$
44. $x^3 - 4x = 12 - 3x^2$
45. $36x - 16x^3 = 9x^2 - 4x^4$

46. **CRITICAL THINKING** Show that $a^2 - b^2 = (a + b)(a - b)$ algebraically. (*Hint*: Rewrite $a^2 - b^2$ as $a^2 + 0ab - b^2$.)

47. **BOATING** The United States Coast Guard's License Exam includes questions dealing with the breaking strength of a line. The basic breaking strength b in pounds for a natural fiber line is determined by the formula $900c^2 = b$, where c is the circumference of the line in inches. What circumference of natural line would have 3600 pounds of breaking strength?

More About. . .



Aerodynamics

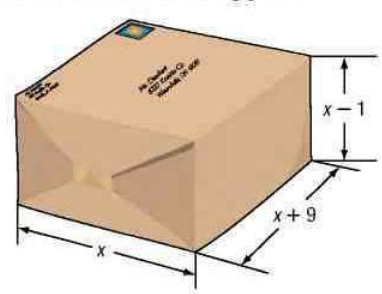
Lift works on the principle that as the speed of a gas increases, the pressure decreases. As the velocity of the air passing over a curved wing increases, the pressure above the wing decreases, lift is created, and the wing rises.

Source: www.glenn.com

48. **AERODYNAMICS** The formula for the pressure difference P above and below a wing is described by the formula $P = \frac{1}{2}dv_1^2 - \frac{1}{2}dv_2^2$, where d is the density of the air, v_1 is the velocity of the air passing above, and v_2 is the velocity of the air passing below. Write this formula in factored form.

49. **LAW ENFORCEMENT** If a car skids on dry concrete, police can use the formula $\frac{1}{24}s^2 = d$ to approximate the speed s of a vehicle in miles per hour given the length d of the skid marks in feet. If the length of skid marks on dry concrete are 54 feet long, how fast was the car traveling when the brakes were applied?

50. **PACKAGING** The width of a box is 9 inches more than its length. The height of the box is 1 inch less than its length. If the box has a volume of 72 cubic inches, what are the dimensions of the box?



51. **CRITICAL THINKING** The following statements appear to prove that 2 is equal to 1. Find the flaw in this "proof."

Suppose a and b are real numbers such that $a = b, a \neq 0, b \neq 0$.

(1)	$a = b$	Given.
(2)	$a^2 = ab$	Multiply each side by a.
(3)	$a^2 - b^2 = ab - b^2$	Subtract b^2 from each side.
(4)	$(a - b)(a + b) = b(a - b)$	Factor.
(5)	$a + b = b$	Divide each side by $a - b$.
(6)	$a + a = a$	Substitution Property; $a = b$
(7)	$2a = a$	Combine like terms.
(8)	$2 = 1$	Divide each side by a.

52. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you determine a basketball player's hang time?

Include the following in your answer:

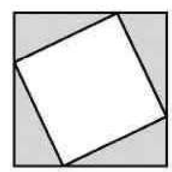
- a maximum height that is a perfect square and that would be considered a reasonable distance for a student athlete to jump, and
- a description of how to find the hang time for this maximum height.

Standardized Test Practice

53. What is the factored form of $25b^2 - 1$?

(A) $(5b - 1)(5b + 1)$ (B) $(5b + 1)(5b + 1)$
(C) $(5b - 1)(5b - 1)$ (D) $(25b + 1)(b - 1)$

54. **GRID IN** In the figure, the area between the two squares is 17 square inches. The sum of the perimeters of the two squares is 68 inches. How many inches long is a side of the larger square?



Maintain Your Skills

Mixed Review

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lesson 9-4)*

55. $2n^2 + 5n + 7$ 56. $6x^2 - 11x + 4$ 57. $21p^2 + 29p - 10$

Solve each equation. Check your solutions. *(Lesson 9-3)*

58. $y^2 + 18y + 32 = 0$ 59. $k^2 - 8k = -15$ 60. $b^2 - 8 = 2b$

61. **STATISTICS** Amy's scores on the first three of four 100-point biology tests were 88, 90, and 91. To get a B+ in the class, her average must be between 88 and 92, inclusive, on all tests. What score must she receive on the fourth test to get a B+ in biology? *(Lesson 6-4)*

Solve each inequality, check your solution, and graph it on a number line. *(Lesson 6-1)*

62. $6 \leq 3d - 12$ 63. $-5 + 10r > 2$ 64. $13x - 3 < 23$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each product.** *(To review **special products**, see Lesson 8-8.)*

65. $(x + 1)(x + 1)$ 66. $(x - 6)(x - 6)$ 67. $(x + 8)^2$
68. $(3x - 4)(3x - 4)$ 69. $(5x - 2)^2$ 70. $(7x + 3)^2$

CONTENTS

Reading Mathematics

The Language of Mathematics

Mathematics is a language all its own. As with any language you learn, you must read slowly and carefully, translating small portions of it at a time. Then you must reread the entire passage to make complete sense of what you read.

In mathematics, concepts are often written in a compact form by using symbols. Break down the symbols and try to translate each piece before putting them back together. Read the following sentence.

$$a^2 + 2ab + b^2 = (a + b)^2$$

The trinomial a squared plus twice the product of a and b plus b squared equals the square of the binomial a plus b.

Below is a list of the concepts involved in that single sentence.

- The letters a and b are variables and can be replaced by monomials like 2 or $3x$ or by polynomials like $x + 3$.
- The square of the binomial $a + b$ means $(a + b)(a + b)$. So, $a^2 + 2ab + b^2$ can be written as the product of two identical factors, $a + b$ and $a + b$.

Now put these concepts together. The algebraic statement $a^2 + 2ab + b^2 = (a + b)^2$ means that any trinomial that can be written in the form $a^2 + 2ab + b^2$ can be factored as the square of a binomial using the pattern $(a + b)^2$.

When reading a lesson in your book, use these steps.

- Read the "What You'll Learn" statements to understand what concepts are being presented.
- Skim to get a general idea of the content.
- Take note of any new terms in the lesson by looking for highlighted words.
- Go back and reread in order to understand all of the ideas presented.
- Study all of the examples.
- Pay special attention to the explanations for each step in each example.
- Read any study tips presented in the margins of the lesson.

Reading to Learn

Turn to page 508 and skim Lesson 9-6.

1. List three main ideas from Lesson 9-6. Use phrases instead of whole sentences.
2. What factoring techniques should be tried when factoring a trinomial?
3. What should you always check for first when trying to factor any polynomial?
4. Translate the symbolic representation of the Square Root Property presented on page 511 and explain why it can be applied to problems like $(a + 4)^2 = 49$ in Example 4a.

9-6 Perfect Squares and Factoring

What You'll Learn

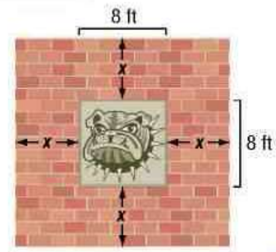
- Factor perfect square trinomials.
- Solve equations involving perfect squares.

Vocabulary

- perfect square trinomials

How can factoring be used to design a pavilion?

The senior class has decided to build an outdoor pavilion. It will have an 8-foot by 8-foot portrayal of the school's mascot in the center. The class is selling bricks with students' names on them to finance the project. If they sell enough bricks to cover 80 square feet and want to arrange the bricks around the art, how wide should the border of bricks be? To solve this problem, you would need to solve the equation $(8 + 2x)^2 = 144$.



Study Tip

Look Back
To review the **square of a sum or difference**, see Lesson 8-8.

FACTOR PERFECT SQUARE TRINOMIALS Numbers like 144, 16, and 49 are perfect squares, since each can be expressed as the square of an integer.

$$144 = 12 \cdot 12 \text{ or } 12^2 \qquad 16 = 4 \cdot 4 \text{ or } 4^2 \qquad 49 = 7 \cdot 7 \text{ or } 7^2$$

Products of the form $(a + b)^2$ and $(a - b)^2$, such as $(8 + 2x)^2$, are also perfect squares. Recall that these are special products that follow specific patterns.

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2\end{aligned} \qquad \begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - ab - ab + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

These patterns can help you factor **perfect square trinomials**, trinomials that are the square of a binomial.

Squaring a Binomial	Factoring a Perfect Square
$(x + 7)^2 = x^2 + 2(x)(7) + 7^2$ $= x^2 + 14x + 49$	$x^2 + 14x + 49 = x^2 + 2(x)(7) + 7^2$ $= (x + 7)^2$
$(3x - 4)^2 = (3x)^2 - 2(3x)(4) + 4^2$ $= 9x^2 - 24x + 16$	$9x^2 - 24x + 16 = (3x)^2 - 2(3x)(4) + 4^2$ $= (3x - 4)^2$

For a trinomial to be factorable as a perfect square, three conditions must be satisfied as illustrated in the example below.

$$4x^2 + 20x + 25$$

1. The first term must be a perfect square. $4x^2 = (2x)^2$

2. The last term must be a perfect square. $25 = 5^2$

3. The middle term must be twice the product of the square roots of the first and last terms. $2(2x)(5) = 20x$

CONTENTS

Key Concept — *Factoring Perfect Square Trinomials*

- **Words** If a trinomial can be written in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$, then it can be factored as $(a + b)^2$ or as $(a - b)^2$, respectively.
- **Symbols** $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$
- **Example** $4x^2 - 20x + 25 = (2x)^2 - 2(2x)(5) + (5)^2$ or $(2x - 5)^2$

Example 1 Factor Perfect Square Trinomials

Determine whether each trinomial is a perfect square trinomial. If so, factor it.

a. $16x^2 + 32x + 64$

1. Is the first term a perfect square? Yes, $16x^2 = (4x)^2$.
2. Is the last term a perfect square? Yes, $64 = 8^2$.
3. Is the middle term equal to $2(4x)(8)$? No, $32x \neq 2(4x)(8)$.

$16x^2 + 32x + 64$ is not a perfect square trinomial.

b. $9y^2 - 12y + 4$

1. Is the first term a perfect square? Yes, $9y^2 = (3y)^2$.
2. Is the last term a perfect square? Yes, $4 = 2^2$.
3. Is the middle term equal to $2(3y)(2)$? Yes, $12y = 2(3y)(2)$.

$9y^2 - 12y + 4$ is a perfect square trinomial.

$9y^2 - 12y + 4 = (3y)^2 - 2(3y)(2) + 2^2$ Write as $a^2 - 2ab + b^2$.
$= (3y - 2)^2$ Factor using the pattern.

In this chapter, you have learned to factor different types of polynomials. The Concept Summary lists these methods and can help you decide when to use a specific method.

Concept Summary — *Factoring Polynomials*

Number of Terms	Factoring Technique		Example
2 or more	greatest common factor		$3x^3 + 6x^2 - 15x = 3x(x^2 + 2x - 5)$
2	difference of squares	$a^2 - b^2 = (a + b)(a - b)$	$4x^2 - 25 = (2x + 5)(2x - 5)$
3	perfect square trinomial	$a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$	$x^2 + 6x + 9 = (x + 3)^2$ $4x^2 - 4x + 1 = (2x - 1)^2$
	$x^2 + bx + c$	$x^2 + bx + c = (x + m)(x + n)$, when $m + n = b$ and $mn = c$.	$x^2 - 9x + 20 = (x - 5)(x - 4)$
	$ax^2 + bx + c$	$ax^2 + bx + c = ax^2 + mx + nx + c$, when $m + n = b$ and $mn = ac$. Then use factoring by grouping.	$6x^2 - x - 2 = 6x^2 + 3x - 4x - 2$ $= 3x(2x + 1) - 2(2x + 1)$ $= (2x + 1)(3x - 2)$
4 or more	factoring by grouping	$ax + bx + ay + by$ $= x(a + b) + y(a + b)$ $= (a + b)(x + y)$	$3xy - 6y + 5x - 10$ $= (3xy - 6y) + (5x - 10)$ $= 3y(x - 2) + 5(x - 2)$ $= (x - 2)(3y + 5)$

When there is a GCF other than 1, it is usually easier to factor it out first. Then, check the appropriate factoring methods in the order shown in the table. Continue factoring until you have written the polynomial as the product of a monomial and/or prime polynomial factors.

Example 2 Factor Completely

Study Tip

Alternative Method
Note that $4x^2 - 36$ could first be factored as $(2x + 6)(2x - 6)$. Then the common factor 2 would need to be factored out of each expression.

Factor each polynomial.

a. $4x^2 - 36$

First check for a GCF. Then, since the polynomial has two terms, check for the difference of squares.

$$\begin{aligned} 4x^2 - 36 &= 4(x^2 - 9) && \text{4 is the GCF.} \\ &= 4(x^2 - 3^2) && x^2 = x \cdot x \text{ and } 9 = 3 \cdot 3 \\ &= 4(x + 3)(x - 3) && \text{Factor the difference of squares.} \end{aligned}$$

b. $25x^2 + 5x - 6$

This polynomial has three terms that have a GCF of 1. While the first term is a perfect square, $25x^2 = (5x)^2$, the last term is not. Therefore, this is not a perfect square trinomial.

This trinomial is of the form $ax^2 + bx + c$. Are there two numbers m and n whose product is $25 \cdot -6$ or -150 and whose sum is 5? Yes, the product of 15 and -10 is -150 and their sum is 5.

$$\begin{aligned} &25x^2 + 5x - 6 \\ &= 25x^2 + mx + nx - 6 && \text{Write the pattern.} \\ &= 25x^2 + 15x - 10x - 6 && m = 15 \text{ and } n = -10 \\ &= (25x^2 + 15x) + (-10x - 6) && \text{Group terms with common factors.} \\ &= 5x(5x + 3) - 2(5x + 3) && \text{Factor out the GCF from each grouping.} \\ &= (5x + 3)(5x - 2) && 5x + 3 \text{ is the common factor.} \end{aligned}$$

SOLVE EQUATIONS WITH PERFECT SQUARES When solving equations involving repeated factors, it is only necessary to set one of the repeated factors equal to zero.

Example 3 Solve Equations with Repeated Factors

Solve $x^2 - x + \frac{1}{4} = 0$.

$$\begin{aligned} x^2 - x + \frac{1}{4} &= 0 && \text{Original equation} \\ x^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 &= 0 && \text{Recognize } x^2 - x + \tfrac{1}{4} \text{ as a perfect square trinomial.} \\ \left(x - \frac{1}{2}\right)^2 &= 0 && \text{Factor the perfect square trinomial.} \\ x - \frac{1}{2} &= 0 && \text{Set repeated factor equal to zero.} \\ x &= \frac{1}{2} && \text{Solve for } x. \end{aligned}$$

Thus, the solution set is $\left\{\frac{1}{2}\right\}$. Check this solution in the original equation.

You have solved equations like $x^2 - 36 = 0$ by using factoring. You can also use the definition of square root to solve this equation.

Study Tip

Reading Math
$\pm\sqrt{36}$ is read as *plus or minus the square root of 36.*

$x^2 - 36 = 0$	Original equation
$x^2 = 36$	Add 36 to each side.
$x = \pm\sqrt{36}$	Take the square root of each side.

Remember that there are two square roots of 36, namely 6 and -6. Therefore, the solution set is $\{-6, 6\}$. This is sometimes expressed more compactly as $\{\pm 6\}$. This and other examples suggest the following property.

Key Concept — *Square Root Property*

- **Symbols** For any number $n > 0$, if $x^2 = n$, then $x = \pm\sqrt{n}$.
- **Example** $x^2 = 9$
  $x = \pm\sqrt{9}$ or ± 3

Example 4 Use the Square Root Property to Solve Equations

Solve each equation. Check your solutions.

a. $(a + 4)^2 = 49$

$(a + 4)^2 = 49$	Original equation
$a + 4 = \pm\sqrt{49}$	Square Root Property
$a + 4 = \pm 7$	$49 = 7 \cdot 7$
$a = -4 \pm 7$	Subtract 4 from each side.
$a = -4 + 7$ or $a = -4 - 7$	Separate into two equations.
$= 3$ $\quad$ $= -11$	Simplify.

The solution set is $\{-11, 3\}$. Check each solution in the original equation.

b. $y^2 - 4y + 4 = 25$

$y^2 - 4y + 4 = 25$	Original equation
$(y)^2 - 2(y)(2) + 2^2 = 25$	Recognize perfect square trinomial.
$(y - 2)^2 = 25$	Factor perfect square trinomial.
$y - 2 = \pm\sqrt{25}$	Square Root Property
$y - 2 = \pm 5$	$25 = 5 \cdot 5$
$y = 2 \pm 5$	Add 2 to each side.
$y = 2 + 5$ or $y = 2 - 5$	Separate into two equations.
$= 7$ $\quad$ $= -3$	Simplify.

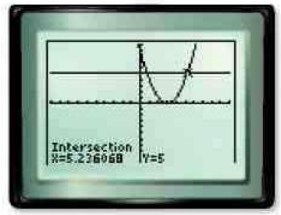
The solution set is $\{-3, 7\}$. Check each solution in the original equation.

c. $(x - 3)^2 = 5$

$(x - 3)^2 = 5$	Original equation
$x - 3 = \pm\sqrt{5}$	Square Root Property
$x = 3 \pm \sqrt{5}$	Add 3 to each side.

Since 5 is not a perfect square, the solution set is $\{3 \pm \sqrt{5}\}$. Using a calculator, the approximate solutions are $3 + \sqrt{5}$ or about 5.24 and $3 - \sqrt{5}$ or about 0.76.

CHECK You can check your answer using a graphing calculator. Graph $y = (x - 3)^2$ and $y = 5$. Using the INTERSECT feature of your graphing calculator, find where $(x - 3)^2 = 5$. The check of 5.24 as one of the approximate solutions is shown at the right.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Check for Understanding

Concept Check

1. **Explain** how to determine whether a trinomial is a perfect square trinomial.
2. **Determine** whether the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.
   $a^2 - 2ab - b^2 = (a - b)^2, b \neq 0$
3. **OPEN ENDED** Write a polynomial that requires at least two different factoring techniques to factor it completely.

Guided Practice

Determine whether each trinomial is a perfect square trinomial. If so, factor it.

4. $y^2 + 8y + 16$
5. $9x^2 - 30x + 10$

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.

6. $2x^2 + 18$
7. $c^2 - 5c + 6$
8. $5a^3 - 80a$
9. $8x^2 - 18x - 35$
10. $9g^2 + 12g - 4$
11. $3m^3 + 2m^2n - 12m - 8n$

Solve each equation. Check your solutions.

12. $4y^2 + 24y + 36 = 0$
13. $3n^2 = 48$
14. $a^2 - 6a + 9 = 16$
15. $(m - 5)^2 = 13$

Application

16. **HISTORY** Galileo demonstrated that objects of different weights fall at the same velocity by dropping two objects of different weights from the top of the Leaning Tower of Pisa. A model for the height h in feet of an object dropped from an initial height h_o in feet is $h = 16t^2 + h_o$, where t is the time in seconds after the object is dropped. Use this model to determine approximately how long it took for the objects to hit the ground if Galileo dropped them from a height of 180 feet.

Practice and Apply

Homework Help

For Exercises	See Examples
17–24	1
25–42	2
43–59	3, 4

Extra Practice

See page 841.

Determine whether each trinomial is a perfect square trinomial. If so, factor it.

17. $x^2 + 9x + 81$
18. $a^2 - 24a + 144$
19. $4y^2 - 44y + 121$
20. $2c^2 + 10c + 25$
21. $9n^2 + 49 + 42n$
22. $25a^2 - 120ab + 144b^2$

23. **GEOMETRY** The area of a circle is $(16x^2 + 80x + 100)\pi$ square inches. What is the diameter of the circle?

24. **GEOMETRY** The area of a square is $81 - 90x + 25x^2$ square meters. If x is a positive integer, what is the least possible perimeter measure for the square?

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.

25. $4k^2 - 100$

26. $9x^2 - 3x - 20$

27. $x^2 + 6x - 9$

28. $50g^2 + 40g + 8$

29. $9t^3 + 66t^2 - 48t$

30. $4a^2 - 36b^2$

31. $20n^2 + 34n + 6$

32. $5y^2 - 90$

33. $24x^3 - 78x^2 + 45x$

34. $18y^2 - 48y + 32$

35. $90g - 27g^2 - 75$

36. $45c^2 - 32cd$

37. $4a^3 + 3a^2b^2 + 8a + 6b^2$

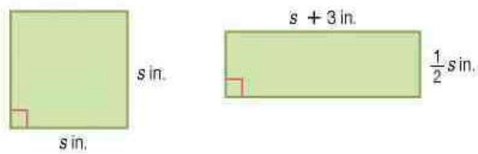
38. $5a^2 + 7a + 6b^2 - 4b$

39. $x^2y^2 - y^2 - z^2 + x^2z^2$

40. $4m^4n + 6m^3n - 16m^2n^2 - 24mn^2$

41. GEOMETRY The volume of a rectangular prism is $x^3y - 63y^2 + 7x^2 - 9xy^3$ cubic meters. Find the dimensions of the prism if they can be represented by binomials with integral coefficients.

42. GEOMETRY If the area of the square shown below is $16x^2 - 56x + 49$ square inches, what is the area of the rectangle in terms of x?



More About. . .

Free-Fall Ride

Some amusement park free-fall rides can seat 4 passengers across per coach and reach speeds of up to 62 miles per hour.

Source: www.pgathrills.com

Solve each equation. Check your solutions.

43. $3x^2 + 24x + 48 = 0$

44. $7r^2 = 70r - 175$

45. $49a^2 + 16 = 56a$

46. $18y^2 + 24y + 8 = 0$

47. $y^2 - \frac{2}{3}y + \frac{1}{9} = 0$

48. $a^2 + \frac{4}{5}a + \frac{4}{25} = 0$

49. $z^2 + 2z + 1 = 16$

50. $x^2 + 10x + 25 = 81$

51. $(y - 8)^2 = 7$

52. $(w + 3)^2 = 2$

53. $p^2 + 2p + 1 = 6$

54. $x^2 - 12x + 36 = 11$

FORESTRY For Exercises 55 and 56, use the following information.
Lumber companies need to be able to estimate the number of board feet that a given log will yield. One of the most commonly used formulas for estimating board feet is the *Doyle Log Rule*, $B = \frac{L}{16}(D^2 - 8D + 16)$, where B is the number of board feet, D is the diameter in inches, and L is the length of the log in feet.

55. Write this formula in factored form.

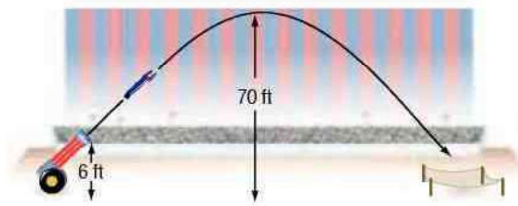
56. For logs that are 16 feet long, what diameter will yield approximately 256 board feet?

FREE-FALL RIDE For Exercises 57 and 58, use the following information.
The height h in feet of a car above the exit ramp of an amusement park's free-fall ride can be modeled by $h = -16t^2 + s$, where t is the time in seconds after the car drops and s is the starting height of the car in feet.

57. How high above the car's exit ramp should the ride's designer start the drop in order for riders to experience free fall for at least 3 seconds?

58. Approximately how long will riders be in free fall if their starting height is 160 feet above the exit ramp?

59. HUMAN CANNONBALL A circus acrobat is shot out of a cannon with an initial upward velocity of 64 feet per second. If the acrobat leaves the cannon 6 feet above the ground, will he reach a height of 70 feet? If so, how long will it take him to reach that height? Use the model for vertical motion.



CRITICAL THINKING **Determine all values of k that make each of the following a perfect square trinomial.**

60. $x^2 + kx + 64$ **61.** $4x^2 + kx + 1$ **62.** $25x^2 + kx + 49$

63. $x^2 + 8x + k$ **64.** $x^2 - 18x + k$ **65.** $x^2 + 20x + k$

66. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can factoring be used to design a pavilion?

Include the following in your answer:

- an explanation of how the equation $(8 + 2x)^2 = 144$ models the given situation, and
- an explanation of how to solve this equation, listing any properties used, and an interpretation of its solutions.

Standardized Test Practice

67. During an experiment, a ball is dropped off a bridge from a height of 205 feet. The formula $205 = 16t^2$ can be used to approximate the amount of time, in seconds, it takes for the ball to reach the surface of the water of the river below the bridge. Find the time it takes the ball to reach the water to the nearest tenth of a second.

Ⓐ 2.3 s Ⓑ 3.4 s Ⓒ 3.6 s Ⓓ 12.8 s

68. If $\sqrt{a^2 - 2ab + b^2} = a - b$, then which of the following statements best describes the relationship between a and b?

Ⓐ $a < b$ Ⓑ $a \leq b$ Ⓒ $a > b$ Ⓓ $a \geq b$

Maintain Your Skills

Mixed Review **Solve each equation. Check your solutions.** *(Lessons 9-4 and 9-5)*

69. $s^2 = 25$ **70.** $9x^2 - 16 = 0$ **71.** $49m^2 = 81$

72. $8k^2 + 22k - 6 = 0$ **73.** $12w^2 + 23w = -5$ **74.** $6z^2 + 7 = 17z$

Write the slope-intercept form of an equation that passes through the given point and is perpendicular to the graph of each equation. *(Lesson 5-6)*

75. $(1, 4), y = 2x - 1$ **76.** $(-4, 7), y = -\frac{2}{3}x + 7$

77. NATIONAL LANDMARKS At the Royal Gorge in Colorado, an inclined railway takes visitors down to the Arkansas River. Suppose the slope is 50% or $\frac{1}{2}$ and the vertical drop is 1015 feet. What is the horizontal change of the railway? *(Lesson 5-1)*

Find the next three terms of each arithmetic sequence. *(Lesson 4-7)*

78. 17, 13, 9, 5, … **79.** $-5, -4.5, -4, -3.5, \ldots$ **80.** 45, 54, 63, 72, …

Chapter 9

Study Guide and Review

Vocabulary and Concept Check

composite number (p. 474)
factored form (p. 475)
factoring (p. 481)
factoring by grouping (p. 482)
greatest common factor (GCF) (p. 476)
perfect square trinomials (p. 508)
prime factorization (p. 475)
prime number (p. 474)
prime polynomial (p. 497)
Square Root Property (p. 511)
Zero Product Property (p. 483)

State whether each sentence is *true* or *false*. If false, replace the underlined word or number to make a true sentence.

1. The number 27 is an example of a prime number.
2. $2x$ is the greatest common factor (GCF) of $12x^2$ and $14xy$.
3. 66 is an example of a perfect square.
4. 61 is a factor of 183.
5. The prime factorization for 48 is $3 \cdot 4^2$.
6. $x^2 - 25$ is an example of a perfect square trinomial.
7. The number 35 is an example of a composite number.
8. $x^2 - 3x - 70$ is an example of a prime polynomial.
9. Expressions with four or more unlike terms can be factored by grouping.
10. $(b - 7)(b + 7)$ is the factorization of a difference of squares.

Lesson-by-Lesson Review

9-1 Factors and Greatest Common Factors

See pages 474–479.

Concept Summary

- Prime number: whole number greater than 1 with exactly two factors
- Composite number: whole number greater than 1 with more than two factors
- The greatest common factor (GCF) of two or more monomials is the product of their common factors.

Example **Find the GCF of $15x^2y$ and $45xy^2$.**

$15x^2y = 3 \cdot 5 \cdot x \cdot x \cdot y$ Factor each number.

$45xy^2 = 3 \cdot 3 \cdot 5 \cdot x \cdot y \cdot y$ Circle the common prime factors.

The GCF is $3 \cdot 5 \cdot x \cdot y$ or $15xy$.

Exercises **Find the prime factorization of each integer.**
See Examples 2 and 3 on page 475.

11. 28 **12.** 33 **13.** 150
14. 301 **15.** -83 **16.** -378

Find the GCF of each set of monomials. *See Example 5 on page 476.*

17. 35, 30 **18.** 12, 18, 40 **19.** $12ab$, $4a^2b^2$
20. $16mrt$, $30m^2r$ **21.** $20n^2$, $25np^5$ **22.** $60x^2y^2$, $35xz^3$

9-2 Factoring Using the Distributive Property

See pages 481–486.

Concept Summary

- Find the greatest common factor and then use the Distributive Property.
- With four or more terms, try factoring by grouping.

  Factoring by Grouping: $ax + bx + ay + by = x(a + b) + y(a + b) = (a + b)(x + y)$
- Factoring can be used to solve some equations.

  Zero Product Property: For any real numbers a and b, if $ab = 0$, then either $a = 0$, $b = 0$, or both a and b equal zero.

Example **Factor $2x^2 - 3xz - 2xy + 3yz$.**

$$\begin{aligned} 2x^2 - 3xz - 2xy + 3yz &= (2x^2 - 3xz) + (-2xy + 3yz) && \text{Group terms with common factors.} \\ &= x(2x - 3z) - y(2x - 3z) && \text{Factor out the GCF from each grouping.} \\ &= (x - y)(2x - 3z) && \text{Factor out the common factor } 2x - 3z. \end{aligned}$$

Exercises **Factor each polynomial.** *See Examples 1 and 2 on pages 481 and 482.*

23. $13x + 26y$
24. $24a^2b^2 - 18ab$
25. $26ab + 18ac + 32a^2$
26. $a^2 - 4ac + ab - 4bc$
27. $4rs + 12ps + 2mr + 6mp$
28. $24am - 9an + 40bm - 15bn$

Solve each equation. Check your solutions. *See Examples 2 and 5 on pages 482 and 483.*

29. $x(2x - 5) = 0$
30. $(3n + 8)(2n - 6) = 0$
31. $4x^2 = -7x$

9-3 Factoring Trinomials: $x^2 + bx + c$

See pages 489–494.

Concept Summary

- Factoring $x^2 + bx + c$: Find m and n whose sum is b and whose product is c. Then write $x^2 + bx + c$ as $(x + m)(x + n)$.

Example **Solve $a^2 - 3a - 4 = 0$. Then check the solutions.**

$a^2 - 3a - 4 = 0$ Original equation

$(a + 1)(a - 4) = 0$ Factor.

$a + 1 = 0$ or $a - 4 = 0$ Zero Product Property

$a = -1$ $\quad a = 4$ Solve each equation.

The solution set is $\{-1, 4\}$.

Exercises **Factor each trinomial.** *See Examples 1–4 on pages 490 and 491.*

32. $y^2 + 7y + 12$
33. $x^2 - 9x - 36$
34. $b^2 + 5b - 6$
35. $18 - 9r + r^2$
36. $a^2 + 6ax - 40x^2$
37. $m^2 - 4mn - 32n^2$

Solve each equation. Check your solutions. *See Example 5 on page 491.*

38. $y^2 + 13y + 40 = 0$
39. $x^2 - 5x - 66 = 0$
40. $m^2 - m - 12 = 0$

9-4 Factoring Trinomials: $ax^2 + bx + c$

See pages 495–500.

Concept Summary

- Factoring $ax^2 + bx + c$: Find m and n whose product is ac and whose sum is b. Then, write as $ax^2 + mx + nx + c$ and use factoring by grouping.

Example **Factor $12x^2 + 22x - 14$.**

First, factor out the GCF, 2: $12x^2 + 22x - 14 = 2(6x^2 + 11x - 7)$. In the new trinomial, $a = 6$, $b = 11$ and $c = -7$. Since b is positive, $m + n$ is positive. Since c is negative, mn is negative. So either m or n is negative, but not both. Therefore, make a list of the factors of $6(-7)$ or -42, where one factor in each pair is negative. Look for a pair of factors whose sum is 11.

Factors of −42	Sum of Factors
−1, 42	41
1, −42	−41
−2, 21	19
2, −21	−19
−3, 14	11

The correct factors are −3 and 14.

$$
\begin{aligned}
6x^2 + 11x - 7 &= 6x^2 + mx + nx - 7 && \text{Write the pattern.}\\
&= 6x^2 - 3x + 14x - 7 && m = -3 \text{ and } n = 14\\
&= (6x^2 - 3x) + (14x - 7) && \text{Group terms with common factors.}\\
&= 3x(2x - 1) + 7(2x - 1) && \text{Factor the GCF from each grouping.}\\
&= (2x - 1)(3x + 7) && 2x - 1 \text{ is the common factor.}
\end{aligned}
$$

Thus, the complete factorization of $12x^2 + 22x - 14$ is $2(2x - 1)(3x + 7)$.

Exercises **Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*.** *See Examples 1–3 on pages 496 and 497.*

41. $2a^2 - 9a + 3$ **42.** $2m^2 + 13m - 24$ **43.** $25r^2 + 20r + 4$

44. $6z^2 + 7z + 3$ **45.** $12b^2 + 17b + 6$ **46.** $3n^2 - 6n - 45$

Solve each equation. Check your solutions. *See Example 4 on page 497.*

47. $2r^2 - 3r - 20 = 0$ **48.** $3a^2 - 13a + 14 = 0$ **49.** $40x^2 + 2x = 24$

9-5 Factoring Differences of Squares

See pages 501–506.

Concept Summary

- Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$ or $(a - b)(a + b)$
- Sometimes it may be necessary to use more than one factoring technique or to apply a factoring technique more than once.

Example **Factor $3x^3 - 75x$.**

$$
\begin{aligned}
3x^3 - 75x &= 3x(x^2 - 25) && \text{The GCF of } 3x^3 \text{ and } 75x \text{ is } 3x.\\
&= 3x(x + 5)(x - 5) && \text{Factor the difference of squares.}
\end{aligned}
$$

Chapter 9 For More ...
• Extra Practice, see pages 839–841.
• Mixed Problem Solving, see page 861.

Exercises **Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.** *See Examples 1–4 on page 502.*

50. $2y^3 - 128y$ **51.** $9b^2 - 20$ **52.** $\frac{1}{4}n^2 - \frac{9}{16}r^2$

Solve each equation by factoring. Check your solutions. *See Example 5 on page 503.*

53. $b^2 - 16 = 0$ **54.** $25 - 9y^2 = 0$ **55.** $16a^2 - 81 = 0$

9-6 Perfect Squares and Factoring

See pages 508–514.

Concept Summary

- If a trinomial can be written in the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$, then it can be factored as $(a + b)^2$ or as $(a - b)^2$, respectively.
- For a trinomial to be factorable as a perfect square, the first term must be a perfect square, the middle term must be twice the product of the square roots of the first and last terms, and the last term must be a perfect square.
- Square Root Property: For any number $n > 0$, if $x^2 = n$, then $x = \pm\sqrt{n}$.

Examples

1 **Determine whether $9x^2 + 24xy + 16y^2$ is a perfect square trinomial. If so, factor it.**

❶ Is the first term a perfect square? Yes, $9x^2 = (3x)^2$.

❷ Is the last term a perfect square? Yes, $16y^2 = (4y)^2$.

❸ Is the middle term equal to $2(3x)(4y)$? Yes, $24xy = 2(3x)(4y)$.

$9x^2 + 24xy + 16y^2 = (3x)^2 + 2(3x)(4y) + (4y)^2$ Write as $a^2 + 2ab + b^2$.

$= (3x + 4y)^2$ Factor using the pattern.

2 **Solve $(x - 4)^2 = 121$.**

$(x - 4)^2 = 121$ Original equation

$x - 4 = \pm\sqrt{121}$ Square Root Property

$x - 4 = \pm 11$ $121 = 11 \cdot 11$

$x = 4 \pm 11$ Add 4 to each side.

$x = 4 + 11$ or $x = 4 - 11$ Separate into two equations.

$= 15$ $= -7$ The solution set is $\{-7, 15\}$.

Exercises **Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*.** *See Example 2 on page 510.*

56. $a^2 + 18a + 81$ **57.** $9k^2 - 12k + 4$

58. $4 - 28r + 49r^2$ **59.** $32n^2 - 80n + 50$

Solve each equation. Check your solutions. *See Examples 3 and 4 on pages 510 and 511.*

60. $6b^3 - 24b^2 + 24b = 0$ **61.** $49m^2 - 126m + 81 = 0$

62. $(c - 9)^2 = 144$ **63.** $144b^2 = 36$

Chapter 9 Practice Test

Vocabulary and Concepts

1. **Give an example** of a prime number and explain why it is prime.
2. **Write** a polynomial that is the difference of two squares. Then factor your polynomial.
3. **Describe** the first step in factoring any polynomial.

Skills and Applications

Find the prime factorization of each integer.

4. 63
5. 81
6. -210

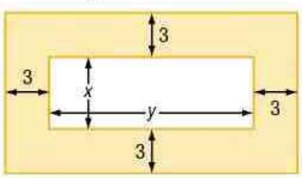
Find the GCF of the given monomials.

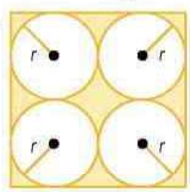
7. 48, 64
8. 28, 75
9. $18a^2b^2$, $28a^3b^2$

Factor each polynomial, if possible. If the polynomial cannot be factored using integers, write *prime*.

10. $25y^2 - 49w^2$
11. $t^2 - 16t + 64$
12. $x^2 + 14x + 24$
13. $28m^2 + 18m$
14. $a^2 - 11ab + 18b^2$
15. $12x^2 + 23x - 24$
16. $2h^2 - 3h - 18$
17. $6x^3 + 15x^2 - 9x$
18. $64p^2 - 63p + 16$
19. $2d^2 + d - 1$
20. $36a^2b^3 - 45ab^4$
21. $36m^2 + 60mn + 25n^2$
22. $a^2 - 4$
23. $4my - 20m + 3py - 15p$
24. $15a^2b + 5a^2 - 10a$
25. $6y^2 - 5y - 6$
26. $4s^2 - 100t^2$
27. $x^3 - 4x^2 - 9x + 36$

Write an expression in factored form for the area of each shaded region.

28. 

29. 

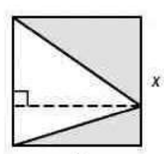
Solve each equation. Check your solutions.

30. $(4x - 3)(3x + 2) = 0$
31. $18s^2 + 72s = 0$
32. $4x^2 = 36$
33. $t^2 + 25 = 10t$
34. $a^2 - 9a - 52 = 0$
35. $x^3 - 5x^2 - 66x = 0$
36. $2x^2 = 9x + 5$
37. $3b^2 + 6 = 11b$

38. **GEOMETRY** A rectangle is 4 inches wide by 7 inches long. When the length and width are increased by the same amount, the area is increased by 26 square inches. What are the dimensions of the new rectangle?

39. **CONSTRUCTION** A rectangular lawn is 24 feet wide by 32 feet long. A sidewalk will be built along the inside edges of all four sides. The remaining lawn will have an area of 425 square feet. How wide will the walk be?

40. **STANDARDIZED TEST PRACTICE** The area of the shaded part of the square shown at the right is 98 square meters. Find the dimensions of the square.



Chapter 9 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

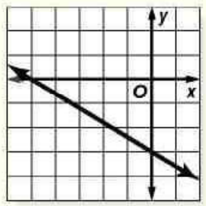
1. Which equation best describes the function graphed below? (Lesson 5-3)

Ⓐ $y = -\frac{3}{5}x - 3$

Ⓑ $y = \frac{3}{5}x - 3$

Ⓒ $y = -\frac{5}{3}x - 3$

Ⓓ $y = \frac{5}{3}x - 3$



2. The school band sold tickets to their spring concert every day at lunch for one week. Before they sold any tickets, they had $80 in their account. At the end of each day, they recorded the total number of tickets sold and the total amount of money in the band's account.

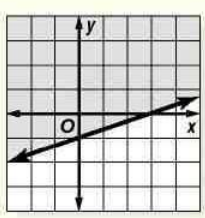
Day	Total Number of Tickets Sold *t*	Total Amount in Account *a*
Monday	12	$176
Tuesday	18	$224
Wednesday	24	$272
Thursday	30	$320
Friday	36	$368

Which equation describes the relationship between the total number of tickets sold t and the amount of money in the band's account a? (Lesson 5-4)

Ⓐ $a = \frac{1}{8}t + 80$ Ⓑ $a = \frac{t + 80}{6}$

Ⓒ $a = 6t + 8$ Ⓓ $a = 8t + 80$

3. Which inequality represents the shaded portion of the graph? (Lesson 6-6)



Ⓐ $y \geq \frac{1}{3}x - 1$

Ⓑ $y \leq \frac{1}{3}x - 1$

Ⓒ $y \leq 3x + 1$

Ⓓ $y \geq 3x - 1$

4. Today, the refreshment stand at the high school football game sold twice as many bags of popcorn as were sold last Friday. The total sold both days was 258 bags. Which system of equations will determine the number of bags sold today n and the number of bags sold last Friday f? (Lesson 7-2)

Ⓐ $n = f - 258$, $f = 2n$ Ⓑ $n = f - 258$, $n = 2f$

Ⓒ $n + f = 258$, $f = 2n$ Ⓓ $n + f = 258$, $n = 2f$

5. Express 5.387×10^{-3} in standard notation. (Lesson 8-3)

Ⓐ 0.0005387 Ⓑ 0.005387

Ⓒ 538.7 Ⓓ 5387

6. The quotient $\frac{16x^8}{8x^4}$, $x \neq 0$, is (Lesson 9-1)

Ⓐ $2x^2$. Ⓑ $8x^2$. Ⓒ $2x^4$. Ⓓ $8x^4$.

7. What are the solutions of the equation $3x^2 - 48 = 0$? (Lesson 9-1)

Ⓐ 4, −4 Ⓑ $4, \frac{1}{3}$

Ⓒ 16, −16 Ⓓ $16, \frac{1}{3}$

8. What are the solutions of the equation $x^2 - 3x + 8 = 6x - 6$? (Lesson 9-4)

Ⓐ 2, −7 Ⓑ −2, −4

Ⓒ 2, 4 Ⓓ 2, 7

9. The area of a rectangle is $12x^2 - 21x - 6$. The width is $3x - 6$. What is the length? (Lesson 9-5)

Ⓐ $4x - 1$ Ⓑ $4x + 1$

Ⓒ $9x + 1$ Ⓓ $12x - 18$

The Princeton Review Test-Taking Tip

Questions 7 and 9 When answering a multiple-choice question, first find an answer on your own. Then, compare your answer to the answer choices given in the item. If your answer does not match any of the answer choices, check your calculations.

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

10. Find all values of x that make the equation $6|x - 2| = 18$ true. (Lesson 6-5)

11. Graph the inequality $x + y \leq 3$. (Lesson 6-6)

12. A movie theater charges $7.50 for each adult ticket and $4 for each child ticket. If the theater sold a total of 145 tickets for a total of $790, how many adult tickets were sold? (Lesson 7-2)

13. Solve the following system of equations.

$3x + y = 8$
$4x - 2y = 14$ (Lesson 7-3)

14. Write $(x + t)x + (x + t)y$ as the product of two factors. (Lesson 9-3)

15. The product of two consecutive odd integers is 195. Find the integers. (Lesson 9-4)

16. Solve $2x^2 + 5x - 12 = 0$ by factoring. (Lesson 9-5)

17. Factor $2x^2 + 7x + 3$. (Lesson 9-5)

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- **Ⓐ the quantity in Column A is greater,**
- **Ⓑ the quantity in Column B is greater,**
- **Ⓒ the two quantities are equal, or**
- **Ⓓ the relationship cannot be determined from the information given.**

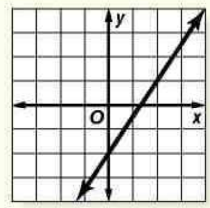
	Column A	Column B
18.	$\|x\| - \|y\|$ if $x = -15$ and $y = -7$	$\|x - y\|$ if $x = -15$ and $y = -7$

(Lesson 2-2)

	Column A	Column B
19.	the value of x in $\frac{2}{3}x - 27 = 39$	the value of y in $\frac{3}{4}y - 55 = 20$

(Lesson 3-4)

20.



20.	the x-intercept of the line whose graph is shown	the x-intercept of the line that is perpendicular to the line graphed above and passes through $(6, -4)$

(Lesson 5-6)

21.	the y value of the solution of $3x - y = 5$ and $x - 3y = -6$	the b value of the solution of $2a - 3b = -3$ and $a + 4b = 24$

(Lesson 7-2)

22.	the GCF of $2x^3$, $6x^2$, and $8x$	the GCF of $18x^3$, $14x^2$, and $4x$

(Lesson 9-1)

Part 4 Open Ended

Record your answers on a sheet of paper. Show your work.

23. Madison is building a fenced, rectangular dog pen. The width of the pen will be 3 yards less than the length. The total area enclosed is 28 square yards. (Lesson 9-4)

a. Using L to represent the length of the pen, write an expression showing the area of the pen in terms of its length.

b. What is the length of the pen?

c. How many yards of fencing will Madison need to enclose the pen completely?

www.algebra1.com/standardized_test

CONTENTS

Chapter 10 Quadratic and Exponential Functions

What You'll Learn

- **Lesson 10-1** Graph quadratic functions.
- **Lessons 10-2 through 10-4** Solve quadratic equations.
- **Lesson 10-5** Graph exponential functions.
- **Lesson 10-6** Solve problems involving exponential growth and exponential decay.
- **Lesson 10-7** Recognize and extend geometric sequences.

Why It's Important

Quadratic functions and equations are used to solve problems about fireworks, to simulate the flight of golf balls in computer games, to describe arches, to determine hang time in football, and to help with water management. Exponential functions are used to describe changes in population, to solve compound interest problems, and to determine concentration of chemicals in a body of water after a spill. Exponential decay is one type of exponential function. Carbon dating uses exponential decay to determine the age of fossils and dinosaurs. *You will learn about carbon dating in Lesson 10-6.*

Key Vocabulary

- **parabola** (p. 524)
- **completing the square** (p. 539)
- **Quadratic Formula** (p. 546)
- **exponential function** (p. 554)
- **geometric sequence** (p. 567)

Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 10.

For Lesson 10-1 — Graph Functions

Use a table of values to graph each equation. *(For review, see Lesson 5-3.)*

1. $y = x + 5$ **2.** $y = 2x - 3$ **3.** $y = 0.5x + 1$ **4.** $y = -3x - 2$

5. $2x - 3y = 12$ **6.** $5y = 10 + 2x$ **7.** $x + 2y = -6$ **8.** $3x = -2y + 9$

For Lesson 10-3 — Perfect Square Trinomials

Determine whether each trinomial is a perfect square trinomial. If so, factor it. *(For review, see Lesson 9-6.)*

9. $t^2 + 12t + 36$ **10.** $a^2 - 14a + 49$ **11.** $m^2 + 18m - 81$ **12.** $y^2 + 8y + 12$

13. $9b^2 - 6b + 1$ **14.** $6x^2 + 4x + 1$ **15.** $4p^2 + 12p + 9$ **16.** $16s^2 - 24s + 9$

For Lesson 10-7 — Arithmetic Sequences

Find the next three terms of each arithmetic sequence. *(For review, see Lesson 4-7.)*

17. $5, 9, 13, 17, \ldots$

18. $12, 5, -2, -9, \ldots$

19. $-4, -1, 2, 5, \ldots$

20. $24, 32, 40, 48, \ldots$

21. $-1, -6, -11, -16, \ldots$

22. $-27, -20, -13, -6, \ldots$

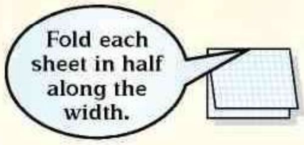
23. $5.3, 6.0, 6.7, 7.4, \ldots$

24. $9.1, 8.8, 8.5, 8.2, \ldots$

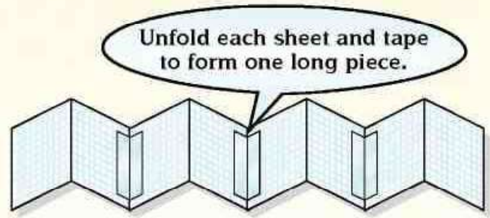
FOLDABLES™ Study Organizer

Make this Foldable to help you organize information on quadratic and exponential functions. Begin with four sheets of grid paper.

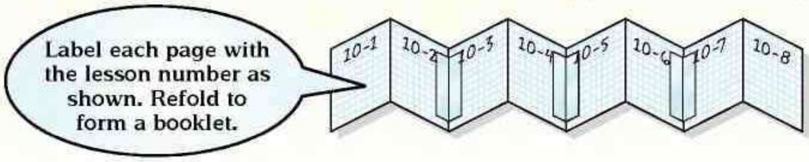
Step 1 Fold in Half



Step 2 Tape



Step 3 Label



Reading and Writing As you read and study the chapter, write notes and examples for each lesson on each page of the journal.

10-1 Graphing Quadratic Functions

What You'll Learn

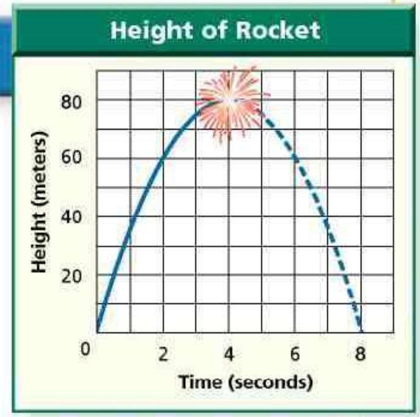
- Graph quadratic functions.
- Find the equation of the axis of symmetry and the coordinates of the vertex of a parabola.

Vocabulary

- quadratic function
- parabola
- minimum
- maximum
- vertex
- symmetry
- axis of symmetry

How can you coordinate a fireworks display with recorded music?

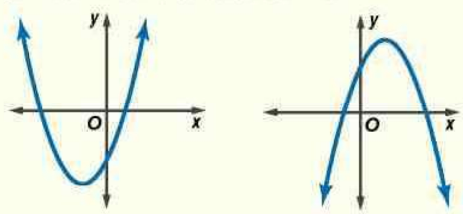
The Sky Concert in Peoria, Illinois, is a 4th of July fireworks display set to music. If a rocket (firework) is launched with an initial velocity of 39.2 meters per second at a height of 1.6 meters above the ground, the equation $h = -4.9t^2 + 39.2t + 1.6$ represents the rocket's height h in meters after t seconds. The rocket will explode at approximately the highest point.



GRAPH QUADRATIC FUNCTIONS The function describing the height of the rocket is an example of a quadratic function. A **quadratic function** can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$. This form of the quadratic function is called the *standard form*. Notice that this polynomial has degree 2 and the exponents are positive. The graph of a quadratic function is called a **parabola**.

Key Concept — Quadratic Function

- **Words** A quadratic function can be described by an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$.
- **Models**

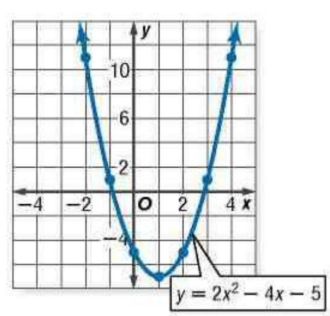


Example 1 Graph Opens Upward

Use a table of values to graph $y = 2x^2 - 4x - 5$.

Graph these ordered pairs and connect them with a smooth curve.

x	y
−2	11
−1	1
0	−5
1	−7
2	−5
3	1
4	11



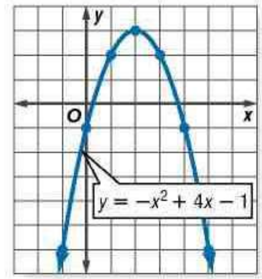
Consider the standard form $y = ax^2 + bx + c$. Notice that the value of a in Example 1 is positive and the curve opens upward. The lowest point, or **minimum**, of the graph is located at $(1, -7)$.

Example 2 Graph Opens Downward

Use a table of values to graph $y = -x^2 + 4x - 1$.

Graph these ordered pairs and connect them with a smooth curve.

x	y
−1	−6
0	−1
1	2
2	3
3	2
4	−1
5	−6



Study Tip

Reading Math
The plural of vertex is vertices. In math, vertex has several meanings. For example, there are the vertex of an angle, the vertices of a polygon, and the vertex of a parabola.

Notice that the value of a in Example 2 is negative and the curve opens downward. The highest point, or **maximum**, of the graph is located at $(2, 3)$. The maximum or minimum point of a parabola is called the **vertex**.

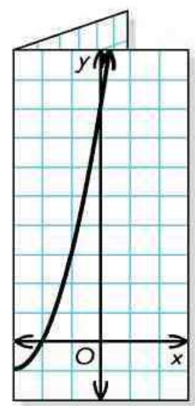
SYMMETRY AND VERTICES Parabolas possess a geometric property called **symmetry**. Symmetrical figures are those in which the figure can be folded in half so that each half matches the other exactly.

Algebra Activity

Symmetry of Parabolas

Model

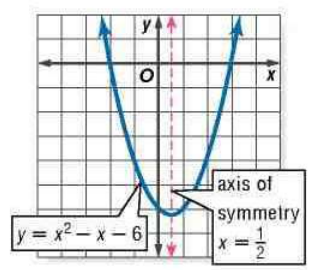
- Graph $y = x^2 + 6x + 8$ on grid paper.
- Hold your paper up to the light and fold the parabola in half so that the two sides match exactly.
- Unfold the paper.



Make a Conjecture

1. What is the vertex of the parabola?
2. Write an equation of the fold line.
3. Which point on the parabola lies on the fold line?
4. Write a few sentences to describe the symmetry of a parabola based on your findings in this activity.

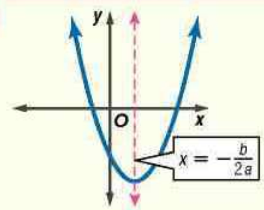
The fold line in the activity above is called the **axis of symmetry** for the parabola. Each point on the parabola that is on one side of the axis of symmetry has a corresponding point on the parabola on the other side of the axis. The vertex is the only point on the parabola that is on the axis of symmetry.



In the graph of $y = x^2 - x - 6$, the axis of symmetry is $x = \frac{1}{2}$. The vertex is $\left(\frac{1}{2}, -6\frac{1}{4}\right)$. Notice the relationship between the values a and b and the equation of the axis of symmetry.

Key Concept — Equation of the Axis of Symmetry of a Parabola

- **Words** The equation of the axis of symmetry for the graph of $y = ax^2 + bx + c$, where $a \neq 0$, is $x = -\frac{b}{2a}$.
- **Model**



You can determine information about a parabola from its equation.

Example 3 Vertex and Axis of Symmetry

Consider the graph of $y = -3x^2 - 6x + 4$.

a. Write the equation of the axis of symmetry.

In $y = -3x^2 - 6x + 4$, $a = -3$ and $b = -6$.

$x = -\frac{b}{2a}$ Equation for the axis of symmetry of a parabola

$x = -\frac{-6}{2(-3)}$ or -1 $a = -3$ and $b = -6$

The equation of the axis of symmetry is $x = -1$.

b. Find the coordinates of the vertex.

Since the equation of the axis of symmetry is $x = -1$ and the vertex lies on the axis, the x-coordinate for the vertex is -1.

$y = -3x^2 - 6x + 4$ Original equation

$y = -3(-1)^2 - 6(-1) + 4$ $x = -1$

$y = -3 + 6 + 4$ Simplify.

$y = 7$ Add.

The vertex is at $(-1, 7)$.

Study Tip

Coordinates of Vertex

Notice that you can find the x-coordinate by knowing the axis of symmetry. However, to find the y-coordinate, you must substitute the value of x into the quadratic equation.

c. Identify the vertex as a maximum or minimum.

Since the coefficient of the x^2 term is negative, the parabola opens downward and the vertex is a maximum point.

d. Graph the function.

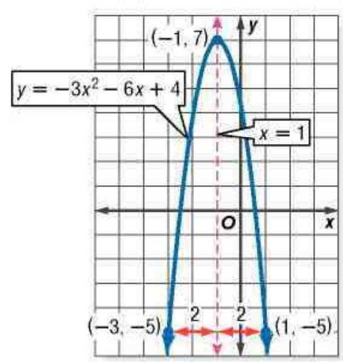
You can use the symmetry of the parabola to help you draw its graph. On a coordinate plane, graph the vertex and the axis of symmetry. Choose a value for x other than -1. For example, choose 1 and find the y-coordinate that satisfies the equation.

$y = -3x^2 - 6x + 4$ Original equation

$y = -3(1)^2 - 6(1) + 4$ Let $x = 1$.

$y = -5$ Simplify.

Graph $(1, -5)$. Since the graph is symmetrical about its axis of symmetry $x = -1$, you can find another point on the other side of the axis of symmetry. The point at $(1, -5)$ is 2 units to the right of the axis. Go 2 units to the left of the axis and plot the point $(-3, -5)$. Repeat this for several other points. Then sketch the parabola.



CHECK Does $(-3, -5)$ satisfy the equation?

$y = -3x^2 - 6x + 4$	Original equation
$-5 \stackrel{?}{=} -3(-3)^2 - 6(-3) + 4$	$y = -5$ and $x = -3$
$-5 = -5$ ✓	Simplify.

The ordered pair $(-3, -5)$ satisfies $y = -3x^2 - 4x + 5$, and the point is on the graph.

Standardized Test Practice

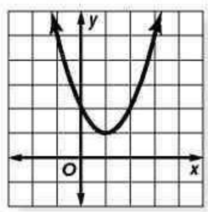
A B C D

Example 4 Match Equations and Graphs

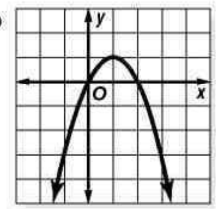
Multiple-Choice Test Item

Which is the graph of $y + 1 = (x + 1)^2$?

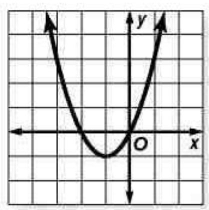
(A)



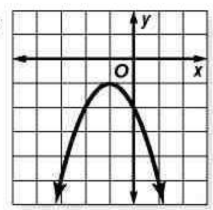
(B)



(C)



(D)



Read the Test Item

You are given a quadratic function, and you are asked to choose the graph that corresponds to it.

Solve the Test Item

First write the equation in standard form.

$y + 1 = (x + 1)^2$	Original equation
$y + 1 = x^2 + 2x + 1$	$(x + 1)^2 = x^2 + 2x + 1$
$y + 1 - 1 = x^2 + 2x + 1 - 1$	Subtract 1 from each side.
$y = x^2 + 2x$	Simplify.

Then find the axis of symmetry of the graph of $y = x^2 + 2x$.

$x = -\frac{b}{2a}$	Equation for the axis of symmetry
$x = -\frac{2}{2(1)}$ or -1	$a = 1$ and $b = 2$

The axis of symmetry is $x = -1$. Look at the graphs. Since only choices C and D have this as their axis of symmetry, you can eliminate choices A and B. Since the coefficient of the x^2 term is positive, the graph opens upward. Eliminate choice D. The answer is C.

The Princeton Review

Test-Taking Tip

Sometimes you can answer a question by eliminating the incorrect choices. For example, in this test question, choices A and B are eliminated because their axes of symmetry are *not* $x = -1$.

Check for Understanding

Concept Check

1. **Compare and contrast** a parabola with a maximum and a parabola with a minimum.

2. **OPEN ENDED** Draw two different parabolas with a vertex of $(2, -1)$.

3. **Explain** how the axis of symmetry can help you graph a quadratic function.

Guided Practice

Use a table of values to graph each function.

4. $y = x^2 - 5$
5. $y = -x^2 + 4x + 5$

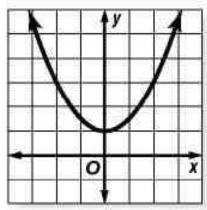
Write the equation of the axis of symmetry, and find the coordinates of the vertex of the graph of each function. Identify the vertex as a maximum or minimum. Then graph the function.

6. $y = x^2 + 4x - 9$
7. $y = -x^2 + 5x + 6$
8. $y = -(x - 2)^2 + 1$

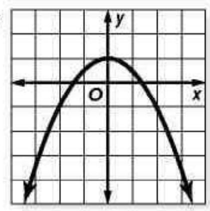
Standardized Test Practice

9. Which is the graph of $y = -\frac{1}{2}x^2 + 1$?

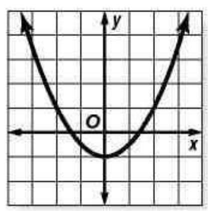
(A)



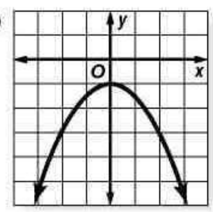
(B)



(C)



(D)



Practice and Apply

Homework Help

For Exercises	See Examples
10–15	1, 2
16–49	3
52, 53	4

Extra Practice

See page 841.

Use a table of values to graph each function.

10. $y = x^2 - 3$
11. $y = -x^2 + 7$
12. $y = x^2 - 2x - 8$
13. $y = x^2 - 4x + 3$
14. $y = -3x^2 - 6x + 4$
15. $y = -3x^2 + 6x + 1$

16. What is the equation of the axis of symmetry of the graph of $y = -3x^2 + 2x - 5$?

17. Find the equation of the axis of symmetry of the graph of $y = 4x^2 - 5x + 16$.

Write the equation of the axis of symmetry, and find the coordinates of the vertex of the graph of each function. Identify the vertex as a maximum or minimum. Then graph the function.

18. $y = 4x^2$
19. $y = -2x^2$
20. $y = x^2 + 2$
21. $y = -x^2 + 5$
22. $y = -x^2 + 2x + 3$
23. $y = -x^2 - 6x + 15$
24. $y = x^2 - 14x + 13$
25. $y = x^2 + 2x + 18$
26. $y = 2x^2 + 12x - 11$
27. $y = 3x^2 - 6x + 4$
28. $y = 5 + 16x - 2x^2$
29. $y = 9 - 8x + 2x^2$

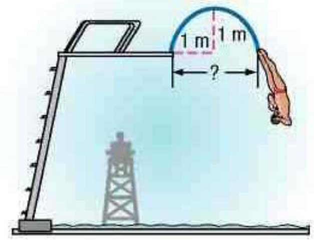
30. $y = 3(x + 1)^2 - 20$ 31. $y = -2(x - 4)^2 - 3$ 32. $y + 2 = x^2 - 10x + 25$

33. $y + 1 = 3x^2 + 12x + 12$ 34. $y - 5 = \frac{1}{3}(x + 2)^2$ 35. $y + 1 = \frac{2}{3}(x + 1)^2$

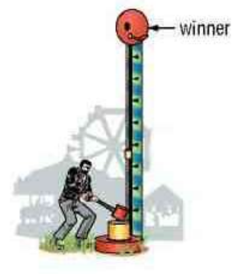
36. The vertex of a parabola is at $(-4, -3)$. If one x-intercept is -11, what is the other x-intercept?

37. What is the equation of the axis of symmetry of a parabola if its x-intercepts are -6 and 4?

38. **SPORTS** A diver follows a path that is in the shape of a parabola. Suppose the diver's foot reaches 1 meter above the height of the diving board at the maximum height of the dive. At that time, the diver's foot is also 1 meter horizontally from the edge of the diving board. What is the distance of the diver's foot from the diving board as the diver descends past the diving board? Explain.



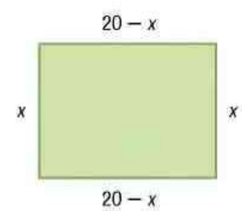
ENTERTAINMENT For Exercises 39 and 40, use the following information.
A carnival game involves striking a lever that forces a weight up a tube. If the weight reaches 20 feet to ring the bell, the contestant wins a prize. The equation $h = -16t^2 + 32t + 3$ gives the height of the weight if the initial velocity is 32 feet per second.



39. Find the maximum height of the weight.

40. Will a prize be won?

PETS For Exercises 41–43, use the following information.
Miriam has 40 meters of fencing to build a pen for her dog.

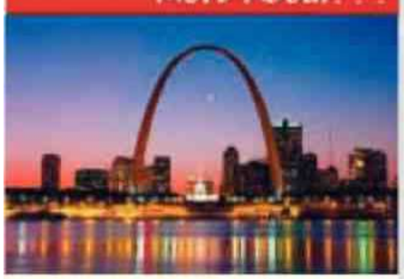


41. Use the diagram at the right to write an equation for the area A of the pen.

42. What value of x will result in the greatest area?

43. What is the greatest possible area of the pen?

More About. . .



Architecture

The Gateway Arch is part of a tribute to Thomas Jefferson, the Louisiana Purchase, and the pioneers who settled the West. Each year about 2.5 million people visit the arch.

Source: *World Book Encyclopedia*

ARCHITECTURE For Exercises 44–46, use the following information.
The shape of the Gateway Arch in St. Louis, Missouri, is a *catenary* curve. It resembles a parabola with the equation $h = -0.00635x^2 + 4.0005x - 0.07875$, where h is the height in feet and x is the distance from one base in feet.

44. What is the equation of the axis of symmetry?

45. What is the distance from one end of the arch to the other?

46. What is the maximum height of the arch?

BRIDES For Exercises 47–49, use the following information.
The equation $a = 0.003x^2 - 0.115x + 21.3$ models the average ages of women when they first married since the year 1940. In this equation, a represents the average age and x represents the years since 1940.

47. Use what you know about parabolas and their minimum values to estimate the year in which the average age of brides was the youngest.

48. Estimate the average age of the brides during that year.

49. Use a graphing calculator to check your estimates.

50. **CRITICAL THINKING** Write a quadratic equation that represents a graph with an axis of symmetry with equation $x = -\frac{3}{8}$.

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

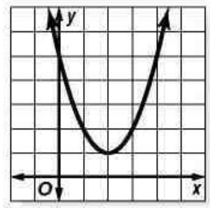
How can you coordinate a fireworks display with recorded music?

Include the following in your answer:

- an explanation of how to determine when the rocket will explode, and
- an explanation of how to determine the height of the rocket when it explodes.

Standardized Test Practice

52. Which equation corresponds to the graph at the right?



Ⓐ $y = x^2 - 4x + 5$

Ⓑ $y = -x^2 + 4x + 5$

Ⓒ $y = x^2 - 4x - 5$

Ⓓ $y = -x^2 + 4x - 5$

53. Which equation does *not* represent a quadratic function?

Ⓐ $y = (x + 3)^2$ Ⓑ $y = 3x^2$ Ⓒ $y = 6x^2 - 1$ Ⓓ $y = x + 5$

Graphing Calculator

MAXIMUM OR MINIMUM **Graph each function. Determine whether the vertex is a maximum or a minimum and give the ordered pair for the vertex.**

54. $y = x^2 - 10x + 25$ 55. $y = -x^2 + 4x + 3$ 56. $y = -2x^2 - 8x - 1$

57. $y = 2x^2 - 40x + 214$ 58. $y = 0.25x^2 - 4x - 2$ 59. $y = -0.5x^2 - 2x + 3$

Maintain Your Skills

Mixed Review

Factor each polynomial, if possible. If the polynomial cannot be factored, write *prime*. *(Lessons 9-5 and 9-6)*

60. $x^2 + 6x - 9$ 61. $a^2 + 22a + 121$ 62. $4m^2 - 4m + 1$

63. $4q^2 - 9$ 64. $2a^2 - 25$ 65. $1 - 16g^2$

Find each sum or difference. *(Lesson 8-5)*

66. $(13x + 9y) + 11y$ 67. $(7p^2 - p - 7) - (p^2 + 11)$

68. **RECREATION** At a recreation and sports facility, 3 members and 3 nonmembers pay a total of \$180 to take an aerobics class. A group of 5 members and 3 nonmembers pay \$210 to take the same class. How much does it cost members and nonmembers to take an aerobics class? *(Lesson 7-3)*

Solve each inequality. Then check your solution. *(Lesson 6-2)*

69. $12b > -144$ 70. $-5w > -125$ 71. $\frac{3r}{4} \leq \frac{2}{3}$

Write an equation of the line that passes through each point with the given slope. *(Lesson 5-4)*

72. $(2, 13), m = 4$ 73. $(-2, -7), m = 0$ 74. $(-4, 6), m = \frac{3}{2}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the x-intercept of the graph of each equation.** *(To review finding **x-intercepts**, see Lesson 4-5.)*

75. $3x + 4y = 24$ 76. $2x - 5y = 14$ 77. $-2x - 4y = 7$

78. $7y + 6x = 42$ 79. $2y - 4x = 10$ 80. $3x - 7y + 9 = 0$

Graphing Calculator Investigation

A Follow-Up of Lesson 10-1

Families of Quadratic Graphs

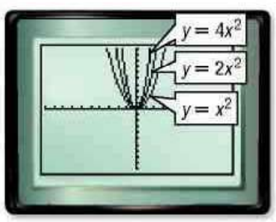
Recall that a *family of graphs* is a group of graphs that have at least one characteristic in common. On page 278, families of linear graphs were introduced. Families of quadratic graphs often fall into two categories—those that have the same vertex and those that have the same shape.

In each of the following families, the parent function is $y = x^2$. Graphing calculators make it easy to study the characteristics of these families of parabolas.

Graph each group of equations on the same screen. Use the standard viewing window. Compare and contrast the graphs.

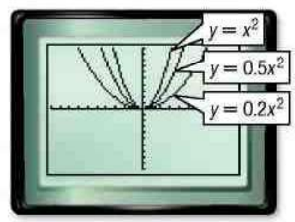
KEYSTROKES: *Review graphing equations on pages 224 and 225.*

a. $y = x^2, y = 2x^2, y = 4x^2$



Each graph opens upward and has its vertex at the origin. The graphs of $y = 2x^2$ and $y = 4x^2$ are narrower than the graph of $y = x^2$.

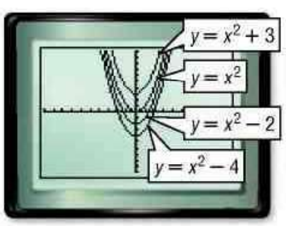
b. $y = x^2, y = 0.5x^2, y = 0.2x^2$



Each graph opens upward and has its vertex at the origin. The graphs of $y = 0.5x^2$ and $y = 0.2x^2$ are wider than the graph of $y = x^2$.

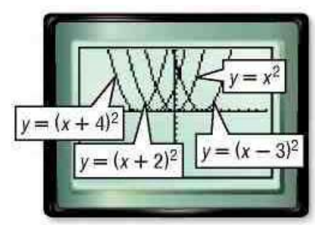
How does the value of a in $y = ax^2$ affect the shape of the graph?

c. $y = x^2, y = x^2 + 3, y = x^2 - 2, y = x^2 - 4$



Each graph opens upward and has the same shape as $y = x^2$. However, each parabola has a different vertex, located along the y-axis. *How does the value of the constant affect the position of the graph?*

d. $y = x^2, y = (x - 3)^2, y = (x + 2)^2, y = (x + 4)^2$



Each graph opens upward and has the same shape as $y = x^2$. However, each parabola has a different vertex located along the x-axis. *How is the location of the vertex related to the equation of the graph?*

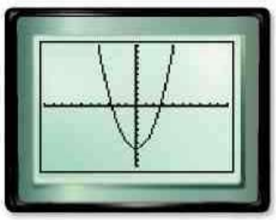
www.algebra1.com/other_calculator_keystrokes

Graphing Calculator Investigation

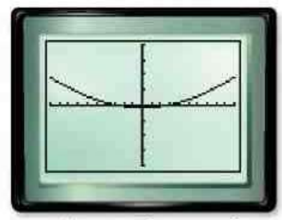
When analyzing or comparing the shapes of various graphs on different screens, it is important to compare the graphs using the same window with the same scale factors. Suppose you graph the same equation using a different window for each. How will the appearance of the graph change?

Graph $y = x^2 - 7$ in each viewing window. What conclusions can you draw about the appearance of a graph in the window used?

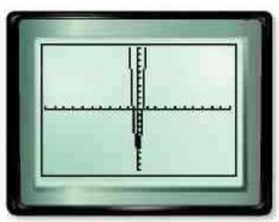
a. standard viewing window



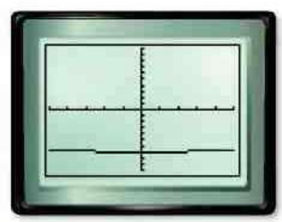
b. $[-10, 10]$ scl: 1 by $[-200, 200]$ scl: 50



c. $[-50, 50]$ scl: 5 by $[-10, 10]$ scl: 1



d. $[-0.5, 0.5]$ scl: 0.1 by $[-10, 10]$ scl: 1



The window greatly affects the appearance of the parabola. Without knowing the window, graph **b** might be of the family $y = ax^2$, where $0 < a < 1$. Graph **c** looks like a member of $y = ax^2 - 7$, where $a > 1$. Graph **d** looks more like a line. However, all are graphs of the same equation.

Exercises

Graph each family of equations on the same screen. Compare and contrast the graphs.

1. $y = -x^2$
$y = -3x^2$
$y = -6x^2$

2. $y = -x^2$
$y = -0.6x^2$
$y = -0.4x^2$

3. $y = -x^2$
$y = -(x + 5)^2$
$y = -(x - 4)^2$

4. $y = -x^2$
$y = -x^2 + 7$
$y = -x^2 - 5$

Use the families of graphs on page 531 and Exercises 1–4 above to predict the appearance of the graph of each equation. Then draw the graph.

5. $y = -0.1x^2$ **6.** $y = (x + 1)^2$ **7.** $y = 4x^2$ **8.** $y = x^2 - 6$

Describe how each change in $y = x^2$ would affect the graph of $y = x^2$. Be sure to consider all values of a, h, and k.

9. $y = ax^2$ **10.** $y = (x + h)^2$ **11.** $y = x^2 + k$ **12.** $y = (x + h)^2 + k$

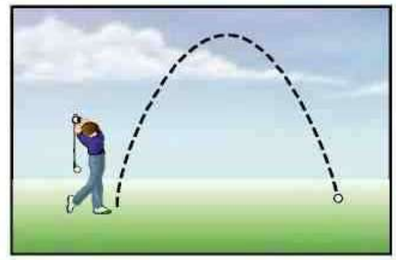
10-2 Solving Quadratic Equations by Graphing

What You'll Learn

- Solve quadratic equations by graphing.
- Estimate solutions of quadratic equations by graphing.

How can quadratic equations be used in computer simulations?

A golf ball follows a path much like a parabola. Because of this property, quadratic functions can be used to simulate parts of a computer golf game. One of the x-intercepts of the quadratic function represents the location where the ball will hit the ground.



Vocabulary

- quadratic equation
- roots
- zeros

SOLVE BY GRAPHING Recall that a quadratic function has standard form $f(x) = ax^2 + bx + c$. In a **quadratic equation**, the value of the related quadratic function is 0. So for the quadratic equation $0 = x^2 - 2x - 3$, the related quadratic function is $f(x) = x^2 - 2x - 3$. You have used factoring to solve equations like $x^2 - 2x - 3 = 0$. You can also use graphing to determine the solutions of equations like this.

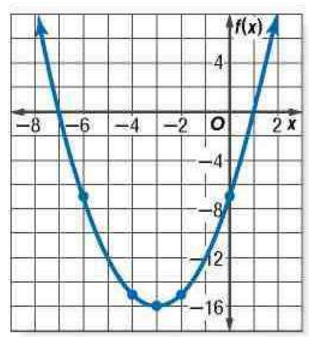
The solutions of a quadratic equation are called the **roots** of the equation. The roots of a quadratic equation can be found by finding the x-intercepts or **zeros** of the related quadratic function.

Example 1 Two Roots

Solve $x^2 + 6x - 7 = 0$ by graphing.

Graph the related function $f(x) = x^2 + 6x - 7$. The equation of the axis of symmetry is $x = -\frac{6}{2(1)}$ or $x = -3$. When x equals -3, $f(x)$ equals $(-3)^2 + 6(-3) - 7$ or -16. So, the coordinates of the vertex are $(-3, -16)$. Make a table of values to find other points to sketch the graph.

x	$f(x)$
−8	9
−6	−7
−4	−15
−3	−16
−2	−15
0	−7
2	9



To solve $x^2 + 6x - 7 = 0$, you need to know where the value of $f(x)$ is 0. This occurs at the x-intercepts. The x-intercepts of the parabola appear to be -7 and 1.

(continued on the next page)

CHECK Solve by factoring.

$x^2 + 6x - 7 = 0$		Original equation
$(x + 7)(x - 1) = 0$		Factor.
$x + 7 = 0$	or $x - 1 = 0$	Zero Product Property
$x = -7$	$x = 1$ ✓	Solve for x.

The solutions of the equation are -7 and 1.

Quadratic equations always have two roots. However, these roots are not always two distinct numbers. Sometimes the two roots are the same number.

Example 2 A Double Root

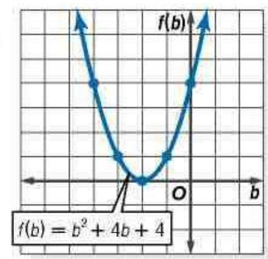
Solve $b^2 + 4b = -4$ by graphing.

First rewrite the equation so one side is equal to zero.

$b^2 + 4b = -4$	Original equation
$b^2 + 4b + 4 = -4 + 4$	Add 4 to each side.
$b^2 + 4b + 4 = 0$	Simplify.

Graph the related function $f(b) = b^2 + 4b + 4$.

b	$f(b)$
-4	4
-3	1
-2	0
-1	1
0	4



Notice that the vertex of the parabola is the b-intercept. Thus, one solution is -2. What is the other solution?

Try solving the equation by factoring.

$b^2 + 4b + 4 = 0$		Original equation
$(b + 2)(b + 2) = 0$		Factor.
$b + 2 = 0$	or $b + 2 = 0$	Zero Product Property
$b = -2$	$b = -2$	Solve for b.

There are two identical factors for the quadratic function, so there is only one root, called a double root. The solution is -2.

Study Tip

Common Misconception

Although solutions found by graphing may appear to be exact, you cannot be sure that they are exact. Solutions need to be verified by substituting into the equation and checking, or by using the algebraic methods that you will learn in this chapter.

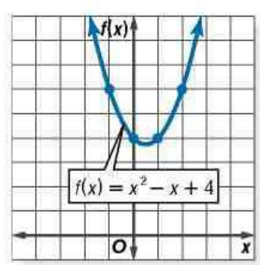
Thus far, you have seen that quadratic equations can have two real roots or one double real root. Can a quadratic equation have no real roots?

Example 3 No Real Roots

Solve $x^2 - x + 4 = 0$ by graphing.

Graph the related function $f(x) = x^2 - x + 4$.

x	$f(x)$
-1	6
0	4
1	4
2	6



The graph has no x-intercept. Thus, there are no real number solutions for this equation.

The symbol ∅, indicating an empty set, is often used to represent no real solutions.

ESTIMATE SOLUTIONS In Examples 1 and 2, the roots of the equation were integers. Usually the roots of a quadratic equation are not integers. In these cases, use estimation to approximate the roots of the equation.

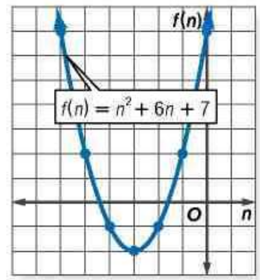
Example 4 Rational Roots

Solve $n^2 + 6n + 7 = 0$ by graphing. If integral roots cannot be found, estimate the roots by stating the consecutive integers between which the roots lie.

Graph the related function $f(n) = n^2 + 6n + 7$.

n	$f(n)$
−6	7
−5	2
−4	−1
−3	−2
−2	−1
−1	2
0	7

Notice that the value of the function changes from negative to positive between the n values of −5 and −4 and between −2 and −1.



The n-intercepts of the graph are between −5 and −4 and between −2 and −1. So, one root is between −5 and −4, and the other root is between −2 and −1.

More About. . .

Football

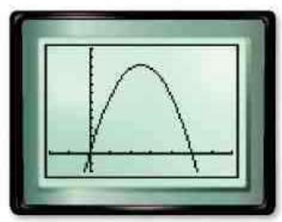
On September 21, 1969, Steve O'Neal set a National Football League record by punting the ball 98 yards.

Source: *The Guinness Book of Records*

Example 5 Estimate Solutions to Solve a Problem

FOOTBALL **When a football player punts a football, he hopes for a long "hang time." Hang time is the total amount of time the ball stays in the air. A time longer than 4.5 seconds is considered good. If a punter kicks the ball with an upward velocity of 80 feet per second and his foot meets the ball 2 feet off the ground, the function $y = -16t^2 + 80t + 2$ represents the height of the ball y in feet after t seconds. What is the hang time of the ball?**

You need to find the solution of the equation $0 = -16t^2 + 80t + 2$. Use a graphing calculator to graph the related function $y = -16t^2 + 80t + 2$. The x-intercept is about 5. Therefore, the hang time is about 5 seconds.



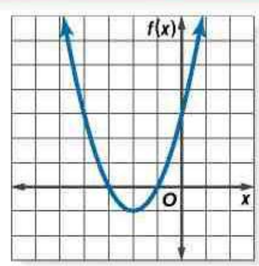
[−2, 7] scl: 1 by [−20, 120] scl: 10

Since 5 seconds is greater than 4.5 seconds, this kick would be considered to have good hang time.

Check for Understanding

Concept Check

1. **State** the real roots of the quadratic equation whose related function is graphed at the right.



2. **Write** the related quadratic function for the equation $7x^2 + 2x = 8$.

3. **OPEN ENDED** Draw a graph to show a counterexample to the following statement.
All quadratic equations have two different solutions.

Guided Practice

Solve each equation by graphing.

4. $x^2 - 7x + 6 = 0$ 5. $a^2 - 10a + 25 = 0$ 6. $c^2 + 3 = 0$

Solve each equation by graphing. If integral roots cannot be found, estimate the roots by stating the consecutive integers between which the roots lie.

7. $t^2 + 9t + 5 = 0$ 8. $x^2 - 16 = 0$ 9. $w^2 - 3w = 5$

Application

10. **NUMBER THEORY** Two numbers have a sum of 4 and a product of -12. Use a quadratic equation to determine the two numbers.

Practice and Apply

Homework Help

For Exercises	See Examples
11–20	1–3
21–34	4
35–46	5

Extra Practice

See page 842.

Solve each equation by graphing.

11. $c^2 - 5c - 24 = 0$ 12. $5n^2 + 2n + 6 = 0$ 13. $x^2 + 6x + 9 = 0$
14. $b^2 - 12b + 36 = 0$ 15. $x^2 + 2x + 5 = 0$ 16. $r^2 + 4r - 12 = 0$

17. The roots of a quadratic equation are -2 and -6. The minimum point of the graph of its related function is at $(-4, -2)$. Sketch the graph of the function.

18. The roots of a quadratic equation are -6 and 0. The maximum point of the graph of its related function is at $(-3, 4)$. Sketch the graph of the function.

19. **NUMBER THEORY** The sum of two numbers is 9, and their product is 20. Use a quadratic equation to determine the two numbers.

20. **NUMBER THEORY** Use a quadratic equation to find two numbers whose sum is 5 and whose product is -24.

More About. . .

Design

The Winter Palace and the rest of the State Hermitage Museum in St. Petersburg, Russia, house 322 art galleries with about three million pieces of art.

Source: *The Guinness Book of Records*

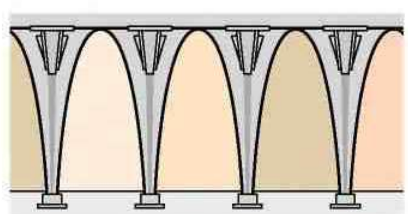
Solve each equation by graphing. If integral roots cannot be found, estimate the roots by stating the consecutive integers between which the roots lie.

21. $a^2 - 12 = 0$ 22. $n^2 - 7 = 0$ 23. $2c^2 + 20c + 32 = 0$
24. $3s^2 + 9s - 12 = 0$ 25. $x^2 + 6x + 6 = 0$ 26. $y^2 - 4y + 1 = 0$
27. $a^2 - 8a = 4$ 28. $x^2 + 6x = -7$ 29. $m^2 - 10m = -21$
30. $p^2 + 16 = 8p$ 31. $12n^2 - 26n = 30$ 32. $4x^2 - 35 = -4x$

33. One root of a quadratic equation is between -4 and -3, and the other root is between 1 and 2. The maximum point of the graph of the related function is at $(-1, 6)$. Sketch the graph of the function.

34. One root of a quadratic equation is between -1 and 0, and the other root is between 6 and 7. The minimum point of the graph of the related function is at $(3, -5)$. Sketch the graph of the function.

DESIGN **For Exercises 35–39, use the following information.**
An art gallery has walls that are sculptured with arches that can be represented by the quadratic function $f(x) = -x^2 - 4x + 12$, where x is in feet. The wall space under each arch is to be painted a different color from the arch itself.



35. Graph the quadratic function and determine its x-intercepts.

36. What is the length of the segment along the floor of each arch?

37. What is the height of the arch?

38. The formula $A = \frac{2}{3}bh$ can be used to estimate the area under a parabola. In this formula, A represents area, b represents the length of the base, and h represents the height. Calculate the area that needs to be painted.

39. How much would the paint for the walls under 12 arches cost if the paint is \$27 per gallon, the painter applies 2 coats, and the manufacturer states that each gallon will cover 200 square feet? (*Hint*: Remember that you cannot buy part of a gallon.)

WebQuest

The graph of the surface areas of the planets can be modeled by a quadratic equation. Visit www.algebra1.com/webquest to continue work on your WebQuest project.

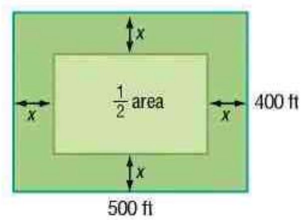
40. **COMPUTER GAMES** Suppose the function $-0.005d^2 + 0.22d = h$ is used to simulate the path of a football at the kickoff of a computer football game. In this equation, h is the height of the ball and d is the horizontal distance in yards. What is the horizontal distance the ball will travel before it hits the ground?

HIKING **For Exercises 41 and 42, use the following information.**
Monya and Kishi are hiking in the mountains and stop for lunch on a ledge 1000 feet above the valley below. Kishi decides to climb to another ledge 20 feet above Monya. Monya throws an apple up to Kishi, but Kishi misses it. The equation $h = -16t^2 + 30t + 1000$ represents the height in feet of the apple t seconds after it was thrown.

41. How long did it take for the apple to reach the ground?

42. If it takes 3 seconds to react, will the girls have time to call down and warn any hikers below? Assume that sound travels about 1000 feet per second. Explain.

WORK **For Exercises 43–46, use the following information.**
Kirk and Montega have accepted a job mowing the soccer playing fields. They must mow an area 500 feet long and 400 feet wide. They agree that each will mow half the area. They decide that Kirk will mow around the edge in a path of equal width until half the area is left.



43. What is the area each person will mow?

44. Write a quadratic equation that could be used to find the width x that Kirk should mow.

45. What width should Kirk mow?

46. The mower can mow a path 5 feet wide. To the nearest whole number, how many times should Kirk go around the field?

47. **CRITICAL THINKING** Where does the graph of $f(x) = \frac{x^3 + 2x^2 - 3x}{x + 5}$ intersect the x-axis?

48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

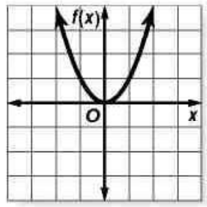
How can quadratic equations be used in computer simulations?

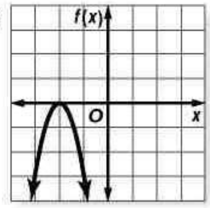
Include the following in your answer:

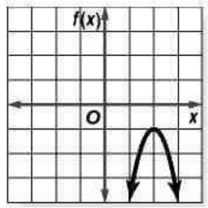
- the meaning of the two roots of a simulation equation for a computer golf game, and
- the approximate location at which the ball will hit the ground if the equation of the path of the ball is $y = -0.0015x^2 + 0.3x$, where y and x are in yards.

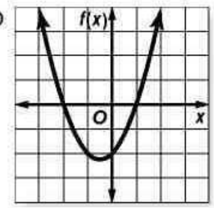
Standardized Test Practice

49. Which graph represents a function whose corresponding quadratic equation has no solutions?

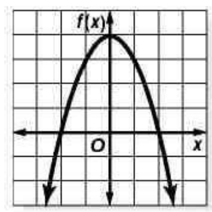
Ⓐ


Ⓑ


Ⓒ


Ⓓ


50. What are the root(s) of the quadratic equation whose related function is graphed at the right?



Ⓐ $-2, 2$ Ⓑ 0

Ⓒ 4 Ⓓ $0, 4$

Graphing Calculator

CUBIC EQUATIONS An equation of the form $ax^3 + bx^2 + cx + d = 0$ is called a **cubic equation**. You can use a graphing calculator to graph and solve cubic equations.

Use the graph of the related function of each cubic equation to estimate the roots of the equation.

51. $x^3 - x^2 - 4x + 4 = 0$ 52. $2x^3 - 11x^2 + 13x - 4 = 0$

Maintain Your Skills

Mixed Review

Write the equation of the axis of symmetry, and find the coordinates of the vertex of the graph of each function. Identify the vertex as a maximum or minimum. Then graph the function. *(Lesson 10-1)*

53. $y = x^2 + 6x + 9$ 54. $y = -x^2 + 4x - 3$ 55. $y = 0.5x^2 - 6x + 5$

Solve each equation. Check your solutions. *(Lesson 9-6)*

56. $m^2 - 24m = -144$ 57. $7r^2 = 70r - 175$ 58. $4d^2 + 9 = -12d$

Simplify. Assume that no denominator is equal to zero. *(Lesson 8-2)*

59. $\frac{10m^4}{30m}$ 60. $\frac{22a^2b^5c^7}{-11abc^2}$ 61. $\frac{-9m^3n^5}{27m^{-2}n^5y^{-4}}$

62. **SHIPPING** An empty book crate weighs 30 pounds. The weight of a book is 1.5 pounds. For shipping, the crate must weigh at least 55 pounds and no more than 60 pounds. What is the acceptable number of books that can be packed in the crate? *(Lesson 6-4)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Determine whether each trinomial is a perfect square trinomial. If so, factor it.** *(To review **perfect square trinomials**, see Lesson 9-6.)*

63. $a^2 + 14 + 49$ 64. $m^2 - 10m + 25$ 65. $t^2 + 16t - 64$

66. $4y^2 + 12y + 9$ 67. $9d^2 - 12d - 4$ 68. $25x^2 - 10x + 1$

10-3 Solving Quadratic Equations by Completing the Square

What You'll Learn

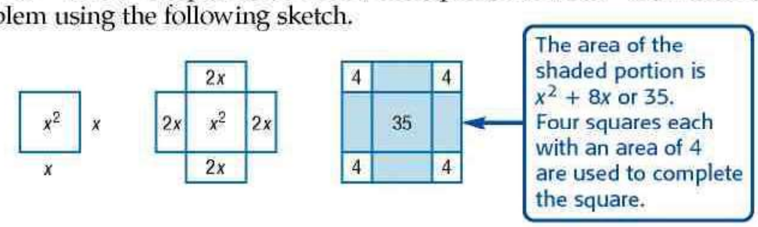
- Solve quadratic equations by finding the square root.
- Solve quadratic equations by completing the square.

Vocabulary

- completing the square

How did ancient mathematicians use squares to solve algebraic equations?

Al-Khwarizmi, born in Baghdad in 780, is considered to be one of the foremost mathematicians of all time. He wrote some of the oldest works on arithmetic and algebra. He wrote algebra in sentences instead of using equations, and he explained the work with geometric sketches. Al-Khwarizmi would have described $x^2 + 8x = 35$ as "A square and 8 roots are equal to 35 units." He would solve the problem using the following sketch.



To solve problems this way today, you might use algebra tiles or a method called completing the square.

FIND THE SQUARE ROOT Some equations can be solved by taking the square root of each side.

Example 1 Irrational Roots

Solve $x^2 - 10x + 25 = 7$ by taking the square root of each side. Round to the nearest tenth if necessary.

$x^2 - 10x + 25 = 7$	Original equation
$(x - 5)^2 = 7$	$x^2 - 10x + 25$ is a perfect square trinomial.
$\sqrt{(x - 5)^2} = \sqrt{7}$	Take the square root of each side.
$\lvert x - 5 \rvert = \sqrt{7}$	Simplify.
$x - 5 = \pm\sqrt{7}$	Definition of absolute value
$x - 5 + 5 = \pm\sqrt{7} + 5$	Add 5 to each side.
$x = 5 \pm \sqrt{7}$	Simplify.

Use a calculator to evaluate each value of x.

$x = 5 + \sqrt{7}$ or $x = 5 - \sqrt{7}$

≈ 7.6 $\qquad\qquad$ ≈ 2.4

The solution set is $\{2.4, 7.6\}$.

COMPLETE THE SQUARE To use the method shown in Example 1, the quadratic expression on one side of the equation must be a perfect square. However, few quadratic expressions are perfect squares. To make any quadratic expression a perfect square, a method called **completing the square** may be used.

Study Tip

Look Back
To review **perfect square trinomials**, see Lesson 9-6.

Consider the pattern for squaring a binomial such as $x + 6$.

$$(x + 6)^2 = x^2 + 2(6)(x) + 6^2$$
$$= x^2 + \underbrace{12x}_{\left(\frac{12}{2}\right)^2} + \underbrace{36}_{6^2}$$

$\left(\frac{12}{2}\right)^2 \rightarrow 6^2$ Notice that one half of 12 is 6 and 6^2 is 36.

Key Concept — Completing the Square

To complete the square for a quadratic expression of the form $x^2 + bx$, you can follow the steps below.

Step 1 Find $\frac{1}{2}$ of b, the coefficient of x.

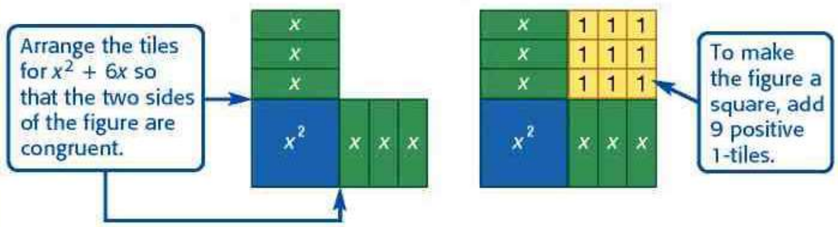
Step 2 Square the result of Step 1.

Step 3 Add the result of Step 2 to $x^2 + bx$, the original expression.

Example 2 Complete the Square

Find the value of c that makes $x^2 + 6x + c$ a perfect square.

Method 1 Use algebra tiles.



$x^2 + 6x + 9$ is a perfect square.

Method 2 Complete the square.

Step 1 Find $\frac{1}{2}$ of 6. $\quad \frac{6}{2} = 3$

Step 2 Square the result of Step 1. $\quad 3^2 = 9$

Step 3 Add the result of Step 2 to $x^2 + 6x$. $\quad x^2 + 6x + 9$

Thus, $c = 9$. Notice that $x^2 + 6x + 9 = (x + 3)^2$.

Example 3 Solve an Equation by Completing the Square

Solve $a^2 - 14a + 3 = -10$ by completing the square.

Step 1 Isolate the a^2 and a terms.

$a^2 - 14a + 3 = -10$ Original equation

$a^2 - 14a + 3 - 3 = -10 - 3$ Subtract 3 from each side.

$a^2 - 14a = -13$ Simplify.

Step 2 Complete the square and solve.

$a^2 - 14a + 49 = -13 + 49$ Since $\left(\frac{-14}{2}\right)^2 = 49$, add 49 to each side.

$(a - 7)^2 = 36$ Factor $a^2 - 14a + 49$.

$a - 7 = \pm 6$ Take the square root of each side.

$a - 7 + 7 = \pm 6 + 7$ Add 7 to each side.

$a = 7 \pm 6$ Simplify.

$$a = 7 + 6 \quad \text{or} \quad a = 7 - 6$$
$$= 13 \qquad\qquad = 1$$

CHECK Substitute each value for a in the original equation.

$$\begin{aligned} a^2 - 14a + 3 &= -10 \\ 1^2 - 14(1) + 3 &\stackrel{?}{=} -10 \\ 1 - 14 + 3 &\stackrel{?}{=} -10 \\ -10 &= -10 \checkmark \end{aligned} \qquad \begin{aligned} a^2 - 14a + 3 &= -10 \\ 13^2 - 14(13) + 3 &\stackrel{?}{=} -10 \\ 169 - 182 + 3 &\stackrel{?}{=} -10 \\ -10 &= -10 \checkmark \end{aligned}$$

The solution set is $\{1, 13\}$.

This method of completing the square cannot be used unless the coefficient of the first term is 1. To solve a quadratic equation in which the leading coefficient is not 1, first divide each term by the coefficient. Then follow the steps for completing the square.

Example 4 Solve a Quadratic Equation in Which $a \neq 1$

ENTERTAINMENT **The path of debris from a firework display on a windy evening can be modeled by a quadratic function. A function for the path of the fireworks when the wind is about 15 miles per hour is $h = -0.04x^2 + 2x + 8$, where h is the height and x is the horizontal distance in feet. How far away from the launch site will the debris land?**

Explore You know the function that relates the horizontal and vertical distances. You want to know how far away from the launch site the debris will land.

Plan The debris will hit the ground when $h = 0$. Use completing the square to solve $-0.04x^2 + 2x + 8 = 0$.

Solve

$-0.04x^2 + 2x + 8 = 0$	Equation for where debris will land
$\frac{-0.04x^2 + 2x + 8}{-0.04} = \frac{0}{-0.04}$	Divide each side by -0.04.
$x^2 - 50x - 200 = 0$	Simplify.
$x^2 - 50x - 200 + 200 = 0 + 200$	Add 200 to each side.
$x^2 - 50x = 200$	Simplify.
$x^2 - 50x + 625 = 200 + 625$	Since $\left(\frac{50}{2}\right)^2 = 625$, add 625 to each side.
$x^2 - 50x + 625 = 825$	Simplify.
$(x - 25)^2 = 825$	Factor $x^2 - 50x + 625$.
$x - 25 = \pm\sqrt{825}$	Take the square root of each side.
$x - 25 + 25 = \pm\sqrt{825} + 25$	Add 25 to each side.
$x = 25 \pm \sqrt{825}$	Simplify.

Use a calculator to evaluate each value of x.

$$x = 25 + \sqrt{825} \quad \text{or} \quad x = 25 - \sqrt{825}$$
$$\approx 53.7 \qquad\qquad \approx -3.7$$

Examine Since you are looking for a distance, ignore the negative number. The debris will land about 53.7 feet from the launch site.

More About. . .

Entertainment

One of the exploded fireworks for the Lake Toya Festival in Japan on July 15, 1988, broke a world record. The diameter of the burst was 3937 feet.

Source: *The Guinness Book of Records*

Check for Understanding

Concept Check

1. **OPEN ENDED** Make a square using one or more of each of the following types of tiles.
   - x^2 tile
   - x tile
   - 1 tile

   Write an expression for the area of your square.

2. **Explain** why completing the square to solve $x^2 - 5x - 7 = 0$ is a better strategy than graphing the related function or factoring.

3. **Describe** the first step needed to solve $5x^2 + 12x = 15$ by completing the square.

Guided Practice

Solve each equation by taking the square root of each side. Round to the nearest tenth if necessary.

4. $b^2 - 6b + 9 = 25$
5. $m^2 + 14m + 49 = 20$

Find the value of c that makes each trinomial a perfect square.

6. $a^2 - 12a + c$
7. $t^2 + 5t + c$

Solve each equation by completing the square. Round to the nearest tenth if necessary.

8. $c^2 - 6c = 7$
9. $x^2 + 7x = -12$
10. $v^2 + 14v - 9 = 6$
11. $r^2 - 4r = 2$
12. $a^2 - 24a + 9 = 0$
13. $2p^2 - 5p + 8 = 7$

Application

14. **GEOMETRY** The area of a square can be doubled by increasing the length by 6 inches and the width by 4 inches. What is the length of the side of the square?

Practice and Apply

Homework Help

For Exercises	See Examples
15–20	1
21–28	2
29–52	3, 4

Extra Practice

See page 842.

Solve each equation by taking the square root of each side. Round to the nearest tenth if necessary.

15. $b^2 - 4b + 4 = 16$
16. $t^2 + 2t + 1 = 25$
17. $g^2 - 8g + 16 = 2$
18. $y^2 - 12y + 36 = 5$
19. $w^2 + 16w + 64 = 18$
20. $a^2 + 18a + 81 = 90$

Find the value of c that makes each trinomial a perfect square.

21. $s^2 - 16s + c$
22. $y^2 - 10y + c$
23. $w^2 + 22w + c$
24. $a^2 + 34a + c$
25. $p^2 - 7p + c$
26. $k^2 + 11k + c$

27. Find all values of c that make $x^2 + cx + 81$ a perfect square.

28. Find all values of c that make $x^2 + cx + 144$ a perfect square.

Solve each equation by completing the square. Round to the nearest tenth if necessary.

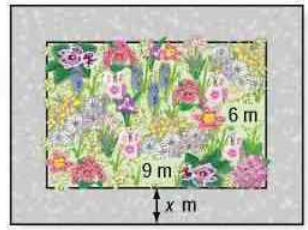
29. $s^2 - 4s - 12 = 0$
30. $d^2 + 3d - 10 = 0$
31. $y^2 - 19y + 4 = 70$
32. $d^2 + 20d + 11 = 200$
33. $a^2 - 5a = -4$
34. $p^2 - 4p = 21$
35. $x^2 + 4x + 3 = 0$
36. $d^2 - 8d + 7 = 0$
37. $s^2 - 10s = 23$
38. $m^2 - 8m = 4$
39. $9r^2 + 49 = 42r$
40. $4h^2 + 25 = 20h$
41. $0.3t^2 + 0.1t = 0.2$
42. $0.4v^2 + 2.5 = 2v$
43. $5x^2 + 10x - 7 = 0$
44. $9w^2 - 12w - 1 = 0$
45. $\frac{1}{2}d^2 - \frac{5}{4}d - 3 = 0$
46. $\frac{1}{3}f^2 - \frac{7}{6}f + \frac{1}{2} = 0$

Solve each equation for x by completing the square.

47. $x^2 + 4x + c = 0$

48. $x^2 - 6x + c = 0$

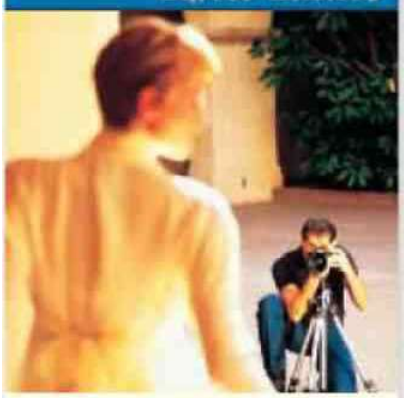
49. PARK PLANNING A plan for a park has a rectangular plot of wild flowers that is 9 meters long by 6 meters wide. A pathway of constant width goes around the plot of wild flowers. If the area of the path is equal to the area of the plot of wild flowers, what is the width of the path?



50. EATING HABITS In the early 1900s, the average American ate 300 pounds of bread and cereal every year. By the 1960s, Americans were eating half that amount. However, eating cereal and bread is on the rise again. The consumption of these types of foods can be modeled by the function $y = 0.059x^2 - 7.423x + 362.1$, where y represents the bread and cereal consumption in pounds and x represents the number of years since 1900. If this trend continues, in what future year will the average American consume 300 pounds of bread and cereal?

Online Research **Data Update** What are the eating habits of Americans? Visit www.algebra1.com/data_update to learn more.

Career Choices



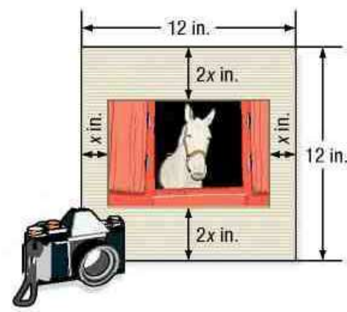
Photographer

Photographers must consider lighting, lens setting, and composition to create the best photograph.

Online Research For information about a career as a photographer, visit: www.algebra1.com/careers

51. CRITICAL THINKING Describe the solution of $x^2 + 4x + 12 = 0$. Explain your reasoning.

52. PHOTOGRAPHY Emilio is placing a photograph behind a 12-inch-by-12-inch piece of matting. The photograph is to be positioned so that the matting is twice as wide at the top and bottom as it is at the sides. If the area of the photograph is to be 54 square inches, what are the dimensions?



53. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How did ancient mathematicians use squares to solve algebraic equations?

Include the following in your answer:

- an explanation of Al-Khwarizmi's drawings for $x^2 + 8x = 35$, and
- a step-by-step algebraic solution with justification for each step of the equation.

Standardized Test Practice

54. Determine which trinomial is *not* a perfect square trinomial.

(A) $a^2 - 26a + 169$
(B) $a^2 + 32a + 256$
(C) $a^2 + 30a - 225$
(D) $a^2 - 44a + 484$

55. Which equation is equivalent to $x^2 + 5x = 14$?

(A) $\left(x + \frac{5}{2}\right)^2 = \frac{81}{4}$
(B) $\left(x - \frac{5}{2}\right)^2 = \frac{45}{4}$
(C) $\left(x + \frac{5}{2}\right)^2 = -\frac{5}{4}$
(D) $\left(x - \frac{5}{2}\right)^2 = -\frac{5}{4}$

Maintain Your Skills

Mixed Review

Solve each equation by graphing. *(Lesson 10-2)*

56. $x^2 + 7x + 12 = 0$ **57.** $x^2 - 16 = 0$ **58.** $x^2 - 2x + 6 = 0$

Use a table of values to graph each equation. *(Lesson 10-1)*

59. $y = 4x^2 + 16$ **60.** $y = x^2 - 3x - 10$ **61.** $y = -x^2 + 3x - 4$

Find each GCF of the given monomials. *(Lesson 9-1)*

62. $14a^2b^3, 20a^3b^2c, 35ab^3c^2$ **63.** $32m^2n^3, 8m^2n, 56m^3n^2$

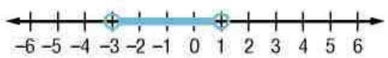
Use substitution to solve each system of equations. If the system does not have exactly one solution, state whether it has no solution or infinitely many solutions. *(Lesson 7-2)*

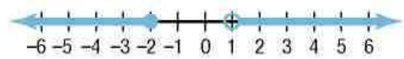
64. $y = 2x$
$x + y = 9$

65. $x = y + 3$
$2x - 3y = 5$

66. $x - 2y = 3$
$3x + y = 23$

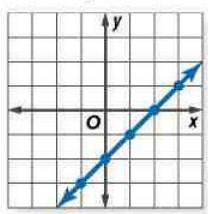
Write a compound inequality for each graph. *(Lesson 6-4)*

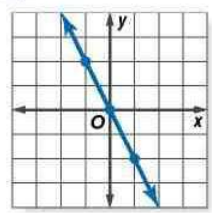
67. 

68. 

69. Write the slope-intercept form of an equation that passes through (8, −2) and is perpendicular to the graph of $5x - 3y = 7$. *(Lesson 5-6)*

Write an equation for each relation. *(Lesson 4-8)*

70. 

71. 

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate $\sqrt{b^2 - 4ac}$ for each set of values. Round to the nearest tenth if necessary.** *(To review **finding square roots**, see Lesson 2-7.)*

72. $a = 1, b = -2, c = -15$ **73.** $a = 2, b = 7, c = 3$

74. $a = 1, b = 5, c = -2$ **75.** $a = -2, b = 7, c = 5$

Practice Quiz 1

Lessons 10-1 through 10-3

Write the equation of the axis of symmetry and find the coordinates of the vertex of the graph of each function. Identify the vertex as a maximum or minimum. Then graph the function. *(Lesson 10-1)*

1. $y = x^2 - x - 6$ **2.** $y = 2x^2 + 3$ **3.** $y = -3x^2 - 6x + 5$

Solve each equation by graphing. If integral roots cannot be found, estimate the roots by stating the consecutive integers between which the roots lie. *(Lesson 10-2)*

4. $x^2 + 6x + 10 = 0$ **5.** $x^2 - 2x - 1 = 0$ **6.** $x^2 - 5x - 6 = 0$

Solve each equation by completing the square. Round to the nearest tenth if necessary. *(Lesson 10-3)*

7. $s^2 + 8s = -15$ **8.** $a^2 - 10a = -24$ **9.** $y^2 - 14y + 49 = 5$ **10.** $2b^2 - b - 7 = 14$

Graphing Calculator Investigation

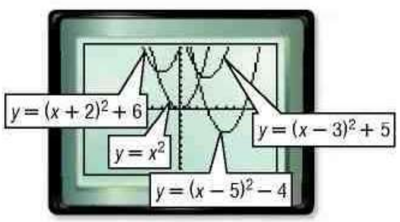
A Follow-Up of Lesson 10-3

Graphing Quadratic Functions in Vertex Form

Quadratic functions written in the form $y = a(x - h)^2 + k$ are said to be in **vertex form**.

Graph each group of equations on the same screen. Use the standard viewing window. Compare and contrast the graphs.

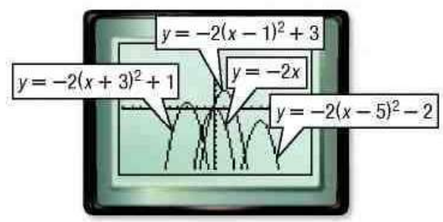
a. $y = x^2$
$y = (x - 3)^2 + 5$
$y = (x + 2)^2 + 6$
$y = (x - 5)^2 - 4$



Each graph opens upward and has the same shape. However, the vertices are different.

Equation	Vertex
$y = x^2$	(0, 0)
$y = (x - 3)^2 + 5$	(3, 5)
$y = (x + 2)^2 + 6$	(−2, 6)
$y = (x - 5)^2 - 4$	(5, −4)

b. $y = -2x^2$
$y = -2(x - 1)^2 + 3$
$y = -2(x + 3)^2 + 1$
$y = -2(x - 5)^2 - 2$



Each graph opens downward and has the same shape. However, the vertices are different.

Equation	Vertex
$y = -2x^2$	(0, 0)
$y = -2(x - 1)^2 + 3$	(1, 3)
$y = -2(x + 3)^2 + 1$	(−3, 1)
$y = -2(x - 5)^2 - 2$	(5, −2)

Exercises

1. Study the relationship between the equations in vertex form and their vertices. What is the vertex of the graph of $y = a(x - h)^2 + k$?

2. Completing the square can be used to change a quadratic equation to vertex form. Copy and complete the steps needed to rewrite $y = x^2 - 2x - 3$ in vertex form.

$y = x^2 - 2x - 3$
$y = (x^2 - 2x + \underline{\ ?\ }) - 3 - \underline{\ ?\ }$
$y = (x - \underline{\ ?\ })^2 - \underline{\ ?\ }$

Complete the square to rewrite each quadratic equation in vertex form. Then determine the vertex of the graph of the equation and sketch the graph.

3. $y = x^2 + 2x - 7$ **4.** $y = x^2 - 4x + 8$ **5.** $y = x^2 + 6x - 1$

www.algebra1.com/other_calculator_keystrokes

10-4 Solving Quadratic Equations by Using the Quadratic Formula

What You'll Learn

- Solve quadratic equations by using the Quadratic Formula.
- Use the discriminant to determine the number of solutions for a quadratic equation.

Vocabulary

- Quadratic Formula
- discriminant

How can the Quadratic Formula be used to solve problems involving population trends?

In the past few decades, there has been a dramatic increase in the percent of people living in the United States who were born in other countries. This trend can be modeled by the quadratic function $P = 0.006t^2 - 0.080t + 5.281$, where P is the percent born outside the United States and t is the number of years since 1960.

To predict when 15% of the population will be people who were born outside of the U.S., you can solve the equation $15 = 0.006t^2 - 0.080t + 5.281$. This equation would be impossible or difficult to solve using factoring, graphing, or completing the square.

QUADRATIC FORMULA You can solve the standard form of the quadratic equation $ax^2 + bx + c = 0$ for x. The result is called the **Quadratic Formula**.

Key Concept — The Quadratic Formula

The solutions of a quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can solve quadratic equations by factoring, graphing, completing the square, or using the Quadratic Formula.

Example 1 Integral Roots

Use two methods to solve $x^2 - 2x - 24 = 0$.

Method 1 Factoring

$x^2 - 2x - 24 = 0$ Original equation

$(x + 4)(x - 6) = 0$ Factor $x^2 - 2x - 24$.

$x + 4 = 0$ or $x - 6 = 0$ Zero Product Property

$x = -4$ $\quad$ $x = 6$ Solve for x.

Study Tip

Look Back

To review **solving equations by factoring**, see Chapter 9.

Method 2 Quadratic Formula

For this equation, $a = 1$, $b = -2$, and $c = -24$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic Formula}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-24)}}{2(1)} \quad a = 1, b = -2, \text{ and } c = -24$$

$$= \frac{2 \pm \sqrt{4 + 96}}{2} \quad \text{Multiply.}$$

$$= \frac{2 \pm \sqrt{100}}{2} \quad \text{Add.}$$

$$= \frac{2 \pm 10}{2} \quad \text{Simplify.}$$

$$x = \frac{2 - 10}{2} \quad \text{or} \quad x = \frac{2 + 10}{2}$$

$$= -4 \qquad\qquad = 6$$

The solution set is $\{-4, 6\}$.

Example 2 Irrational Roots

Solve $24x^2 - 14x = 6$ by using the Quadratic Formula. Round to the nearest tenth if necessary.

Step 1 Rewrite the equation in standard form.

$$24x^2 - 14x = 6 \quad \text{Original equation}$$

$$24x^2 - 14x - 6 = 6 - 6 \quad \text{Subtract 6 from each side.}$$

$$24x^2 - 14x - 6 = 0 \quad \text{Simplify.}$$

Step 2 Apply the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic Formula}$$

$$= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(24)(-6)}}{2(24)} \quad a = 24, b = -14, \text{ and } c = -6$$

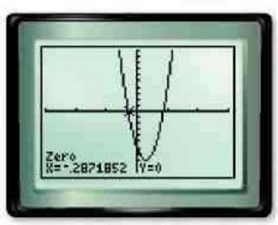
$$= \frac{14 \pm \sqrt{196 + 576}}{48} \quad \text{Multiply.}$$

$$= \frac{14 \pm \sqrt{772}}{48} \quad \text{Add.}$$

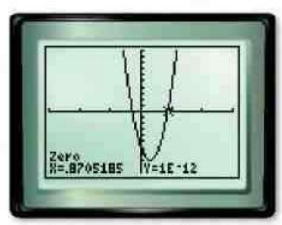
$$x = \frac{14 - \sqrt{772}}{48} \quad \text{or} \quad x = \frac{14 + \sqrt{772}}{48}$$

$$\approx -0.3 \qquad\qquad \approx 0.9$$

Check the solutions by using the CALC menu on a graphing calculator to determine the zeros of the related quadratic function.



[−3, 3] scl: 1 by [−10, 10] scl: 1



[−3, 3] scl: 1 by [−10, 10] scl: 1

The approximate solution set is $\{-0.3, 0.9\}$.

You have studied four methods for solving quadratic equations. The table summarizes these methods.

Concept Summary *Solving Quadratic Equations*

Method	Can Be Used	Comments
graphing	always	Not always exact; use only when an approximate solution is sufficient.
factoring	sometimes	Use if constant term is 0 or factors are easily determined.
completing the square	always	Useful for equations of the form $x^2 + bx + c = 0$, where b is an even number.
Quadratic Formula	always	Other methods may be easier to use in some cases, but this method always gives accurate solutions.

Example 3 Use the Quadratic Formula to Solve a Problem

SPACE TRAVEL The height H of an object t seconds after it is propelled upward with an initial velocity v is represented by $H = -\frac{1}{2}gt^2 + vt + h$, where g is the gravitational pull and h is the initial height. Suppose an astronaut on the Moon throws a baseball upward with an initial velocity of 10 meters per second, letting go of the ball 2 meters above the ground. Use the information at the left to find how much longer the ball will stay in the air than a similarly-thrown baseball on Earth.

In order to find when the ball hits the ground, you must find when $H = 0$. Write two equations to represent the situation on the Moon and on Earth.

Baseball Thrown on the Moon

$$H = -\frac{1}{2}gt^2 + vt + h$$
$$0 = -\frac{1}{2}(1.6)t^2 + 10t + 2$$
$$0 = -0.8t^2 + 10t + 2$$

Baseball Thrown on Earth

$$H = -\frac{1}{2}gt^2 + vt + h$$
$$0 = -\frac{1}{2}(9.8)t^2 + 10t + 2$$
$$0 = -4.9t^2 + 10t + 2$$

These equations cannot be factored, and completing the square would involve a lot of computation. To find accurate solutions, use the Quadratic Formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-10 \pm \sqrt{10^2 - 4(-0.8)(2)}}{2(-0.8)}$$
$$= \frac{-10 \pm \sqrt{106.4}}{-1.6}$$
$$t \approx 12.7 \quad \text{or} \quad t \approx -0.2$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-10 \pm \sqrt{10^2 - 4(-4.9)(2)}}{2(-4.9)}$$
$$= \frac{-10 \pm \sqrt{139.2}}{-9.8}$$
$$t \approx 2.2 \quad \text{or} \quad t \approx -0.2$$

Since a negative number of seconds is not reasonable, use the positive solutions. Therefore, the baseball will stay in the air about 12.7 seconds on the Moon and about 2.2 seconds on Earth. The baseball will stay in the air about 12.7 − 2.2 or 10.5 seconds longer on the Moon.

More About. . .

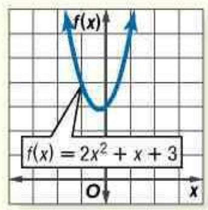
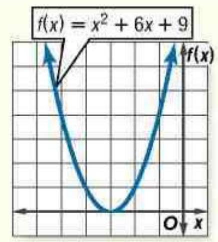
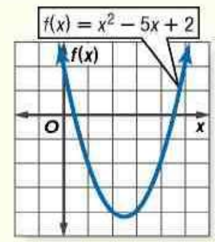
Space Travel

Astronauts have found walking on the Moon to be very different from walking on Earth because the gravitational pull of the moon is only 1.6 meters per second squared. The gravitational pull on Earth is 9.8 meters per second squared.

Source: *World Book Encyclopedia*

THE DISCRIMINANT In the Quadratic Formula, the expression under the radical sign, $b^2 - 4ac$, is called the **discriminant**. The value of the discriminant can be used to determine the number of real roots for a quadratic equation.

Key Concept — Using the Discriminant

Discriminant	negative	zero	positive
Example	$2x^2 + x + 3 = 0$ $x = \frac{-1 \pm \sqrt{1^2 - 4(2)(3)}}{2(2)}$ $= \frac{-1 \pm \sqrt{-23}}{4}$ There are no real roots since no real number can be the square root of a negative number.	$x^2 + 6x + 9 = 0$ $x = \frac{-6 \pm \sqrt{6^2 - 4(1)(9)}}{2(1)}$ $= \frac{-6 \pm \sqrt{0}}{2}$ $= \frac{-6}{2}$ or -3 There is a double root, -3.	$x^2 - 5x + 2 = 0$ $x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)}$ $= \frac{5 \pm \sqrt{17}}{2}$ There are two roots, $\frac{5 + \sqrt{17}}{2}$ and $\frac{5 - \sqrt{17}}{2}$.
Graph of Related Function	 The graph does not cross the x-axis.	 The graph touches the x-axis in one place.	 The graph crosses the x-axis twice.
Number of Real Roots	0	1	2

Example 4 Use the Discriminant

State the value of the discriminant for each equation. Then determine the number of real roots of the equation.

a. $2x^2 + 10x + 11 = 0$

$b^2 - 4ac = 10^2 - 4(2)(11)$ $a = 2$, $b = 10$, and $c = 11$

$= 12$ Simplify.

Since the discriminant is positive, the equation has two real roots.

b. $4t^2 - 20t + 25 = 0$

$b^2 - 4ac = (-20)^2 - 4(4)(25)$ $a = 4$, $b = -20$, and $c = 25$

$= 0$ Simplify.

Since the discriminant is 0, the equation has one real root.

c. $3m^2 + 4m = -2$

Step 1 Rewrite the equation in standard form.

$3m^2 + 4m = -2$ Original equation

$3m^2 + 4m + 2 = -2 + 2$ Add 2 to each side.

$3m^2 + 4m + 2 = 0$ Simplify.

Step 2 Find the discriminant.

$b^2 - 4ac = 4^2 - 4(3)(2)$ $a = 3$, $b = 4$, and $c = 2$

$= -8$ Simplify.

Since the discriminant is negative, the equation has no real roots.

Check for Understanding

Concept Check

1. **Describe** three different ways to solve $x^2 - 2x - 15 = 0$. Which method do you prefer and why?
2. **OPEN ENDED** Write a quadratic equation with no real solutions.
3. **FIND THE ERROR** Lakeisha and Juanita are determining the number of solutions of $5y^2 - 3y = 2$.

Lakeisha

$5y^2 - 3y = 2$

$b^2 - 4ac = (-3)^2 - 4(5)(2)$

$= -31$

Since the discriminant is negative, there are no real solutions.

Juanita

$5y^2 - 3y = 2$

$5y^2 - 3y - 2 = 0$

$b^2 - 4ac = (-3)^2 - 4(5)(-2)$

$= 49$

Since the discriminant is positive, there are two real roots.

Who is correct? Explain your reasoning.

Guided Practice

Solve each equation by using the Quadratic Formula. Round to the nearest tenth if necessary.

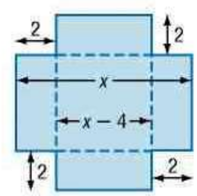
4. $x^2 + 7x + 6 = 0$
5. $t^2 + 11t = 12$
6. $r^2 + 10r + 12 = 0$
7. $3v^2 + 5v + 11 = 0$
8. $4x^2 + 2x = 17$
9. $w^2 + \frac{2}{25} = \frac{3}{5}w$

State the value of the discriminant for each equation. Then determine the number of real roots of the equation.

10. $m^2 + 5m - 6 = 0$
11. $s^2 + 8s + 16 = 0$
12. $2z^2 + z = -50$

Application

13. **MANUFACTURING** A pan is to be formed by cutting 2-centimeter-by-2-centimeter squares from each corner of a square piece of sheet metal and then folding the sides. If the volume of the pan is to be 441 square centimeters, what should the dimensions of the original piece of sheet metal be?



Practice and Apply

Homework Help

For Exercises	See Examples
14–37	1, 2
38–45	4
46–53	3

Extra Practice

See page 842.

Solve each equation by using the Quadratic Formula. Round to the nearest tenth if necessary.

14. $x^2 + 3x - 18 = 0$
15. $v^2 + 12v + 20 = 0$
16. $3t^2 - 7t - 20 = 0$
17. $5y^2 - y - 4 = 0$
18. $x^2 - 25 = 0$
19. $r^2 + 25 = 0$
20. $2x^2 + 98 = 28x$
21. $4s^2 + 100 = 40s$
22. $2r^2 + r - 14 = 0$
23. $2n^2 - 7n - 3 = 0$
24. $5v^2 - 7v = 1$
25. $11z^2 - z = 3$
26. $2w^2 = -(7w + 3)$
27. $2(12g^2 - g) = 15$
28. $1.34d^2 - 1.1d = -1.02$
29. $-2x^2 + 0.7x = -0.3$
30. $2y^2 - \frac{5}{4}y = \frac{1}{2}$
31. $\frac{1}{2}v^2 - v = \frac{3}{4}$

32. **GEOMETRY** The perimeter of a rectangle is 60 inches. Find the dimensions of the rectangle if its area is 221 square inches.

33. **GEOMETRY** Rectangle $ABCD$ has a perimeter of 42 centimeters. What are the dimensions of the rectangle if its area is 80 square centimeters?

34. **NUMBER THEORY** Find two consecutive odd integers whose product is 255.

35. **NUMBER THEORY** The sum of the squares of two consecutive odd numbers is 130. What are the numbers?

36. Without graphing, determine the x-intercepts of the graph of $f(x) = 4x^2 - 9x + 4$.

37. Without graphing, determine the x-intercepts of the graph of $f(x) = 13x^2 - 16x - 4$.

State the value of the discriminant for each equation. Then determine the number of real roots of the equation.

38. $x^2 + 3x - 4 = 0$

39. $y^2 + 3y + 1 = 0$

40. $4p^2 + 10p = -6.25$

41. $1.5m^2 + m = -3.5$

42. $2r^2 = \frac{1}{2}r - \frac{2}{3}$

43. $\frac{4}{3}n^2 + 4n = -3$

44. Without graphing, determine the number of x-intercepts of the graph of $f(x) = 7x^2 - 3x - 1$.

45. Without graphing, determine the number of x-intercepts of the graph of $f(x) = x^2 + 4x + 7$.

More About. . .

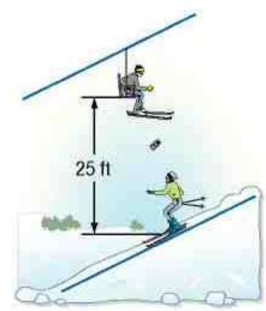
Recreation

Downhill skiing is the most popular type of snow skiing. Skilled skiers can obtain speeds of about 60 miles per hour as they race down mountain slopes.

Source: *World Book Encyclopedia*

RECREATION **For Exercises 46 and 47, use the following information.**

As Darius is skiing down a ski slope, Jorge is on the chair lift on the same slope. The chair lift has stopped. Darius stops directly below Jorge and attempts to toss a disposable camera up to him. If the camera is thrown with an initial velocity of 35 feet per second, the equation for the height of the camera is $h = -16t^2 + 35t + 5$, where h represents the height in feet and t represents the time in seconds.

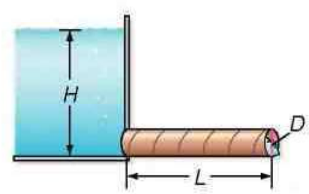


46. If the chair lift is 25 feet above the ground, will Jorge have 0, 1, or 2 chances to catch the camera?

47. If Jorge is unable to catch the camera, when will it hit the ground?

48. **PHYSICAL SCIENCE** A projectile is shot vertically up in the air from ground level. Its distance s, in feet, after t seconds is given by $s = 96t - 16t^2$. Find the values of t when s is 96 feet.

49. **WATER MANAGEMENT** Cox's formula for measuring velocity of water draining from a reservoir through a horizontal pipe is $4v^2 + 5v - 2 = \frac{1200HD}{L}$, where v represents the velocity of the water in feet per second, H represents the height of the reservoir in feet, D represents the diameter of the pipe in inches, and L represents the length of the pipe in feet. How fast is water flowing through a pipe 20 feet long with a diameter of 6 inches that is draining a swimming pool with a depth of 10 feet?



50. **CRITICAL THINKING** If the graph of $f(x) = ax^2 + 10x + 3$ intersects the x-axis in two places, what must be true about the value of a?

CANCER STATISTICS For Exercises 51–53, use the following information. A decrease in smoking in the United States has resulted in lower death rates caused by cancer. In 1965, 42% of adults smoked, compared with less than 25% in 1995. The number of deaths per 100,000 people y can be approximated by $y = -0.048x^2 + 1.87x + 154$, where x represents the number of years after 1970.

51. Use the Quadratic Formula to solve for x when $y = 150$.

52. In what year would you expect the death rate from cancer to be 150 per 100,000?

53. According to the quadratic function, when will the death rate from cancer be 0 per 100,000? Do you think that the prediction is valid? Why or why not?

54. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can the Quadratic Formula be used to solve problems involving population trends?

Include the following in your answer:

- a step-by-step solution of $15 = 0.0055t^2 - 0.0796t + 5.2810$ with justification of each step, and
- an explanation for why the Quadratic Formula is the best way to solve this equation.

Standardized Test Practice

55. Determine the number of solutions of $x^2 - 5x + 8 = 0$.

(A) 0 (B) 1 (C) 2 (D) infinitely many

56. Which expression represents the solutions of $2x^2 + 5x + 1 = 0$?

(A) $\frac{5 \pm \sqrt{17}}{4}$ (B) $\frac{5 \pm \sqrt{33}}{4}$ (C) $\frac{-5 \pm \sqrt{17}}{4}$ (D) $\frac{-5 \pm \sqrt{33}}{4}$

Maintain Your Skills

Mixed Review

Solve each equation by completing the square. Round to the nearest tenth if necessary. *(Lesson 10-3)*

57. $x^2 - 8x = -7$ **58.** $a^2 + 2a + 5 = 20$ **59.** $n^2 - 12n = 5$

Solve each equation by graphing. If integral roots cannot be found, estimate the roots by stating the consecutive integers between which the roots lie. *(Lesson 10-2)*

60. $x^2 - x = 6$ **61.** $2x^2 + x = 2$ **62.** $-x^2 + 3x + 6 = 0$

Factor each polynomial. *(Lesson 9-2)*

63. $15xy^3 + y^4$ **64.** $2ax + 6xc + ba + 3bc$

65. SCIENCE The mass of a proton is 0.0000000000000000000001672 milligram. Write this number in scientific notation. *(Lesson 8-3)*

Graph each system of inequalities. *(Lesson 7-5)*

66. $x \leq 2$, $y + 4 \geq 5$ **67.** $x + y > 2$, $x - y \leq 2$ **68.** $y > x$, $y \leq x + 4$

Solve each inequality. Then check your solution. *(Lesson 6-3)*

69. $2m + 7 > 17$ **70.** $-2 - 3x \geq 2$ **71.** $-20 \geq 8 + 7k$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Evaluate $c(a^x)$ for each of the given values.
*(To review **evaluating expressions with exponents**, see Lesson 1-1.)*

72. $a = 2, c = 1, x = 4$ **73.** $a = 7, c = 3, x = 2$ **74.** $a = 5, c = 2, x = 3$

Graphing Calculator Investigation

A Follow-Up of Lesson 10-4

Solving Quadratic-Linear Systems

Since you can graph multiple functions on a graphing calculator, it is a useful tool when finding the intersection points or solutions of a system of equations in which one equation is quadratic and one is linear.

Solve the following quadratic-linear system of equations.

$y + 1 = x$

$y = -x^2 + 2x + 5$

Step 1 *Solve each equation for y.*

- $y + 1 = x$
  $y = x - 1$
- $y = -x^2 + 2x + 5$

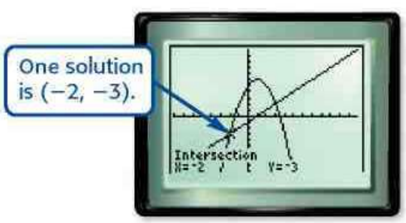
Step 2 *Graph the equations on the same screen.*

- Enter $y = x - 1$ as Y1.
- Enter $y = -x^2 + 2x + 5$ as Y2.
- Graph both in the standard viewing window.

Step 3 *Approximate the intersection point.*

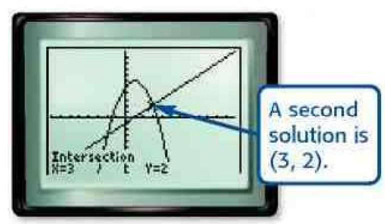
- Use the intersect option on the CALC menu to approximate the first intersection point.

KEYSTROKES: [2nd] [CALC] 5 [ENTER] [ENTER] [ENTER]



Step 4 *Approximate the other intersection point.*

- Use the TRACE feature with the right and left arrow keys to move the cursor near the other intersection point.
- Use the **intersect** option on the CALC menu to approximate the other intersection point.



Thus, the solutions of the quadratic-linear system are $(-2, -3)$ and $(3, 2)$.

Exercises

Use the intersect feature to solve each quadratic-linear system of equations. State any decimal solutions to the nearest tenth.

1. $y = -2(2x + 3)$
$y = x^2 + 2x + 3$

2. $y - 5 = 0$
$y = -x^2$

3. $1.8x + y = 3.6$
$y = x^2 - 3x - 1$

4. $y = -1.4x - 2.88$
$y = x^2 + 0.4x - 3.14$

5. $y = x^2 - 3.5x + 2.2$
$y = 2x - 5.3625$

6. $y = 0.35x - 1.648$
$y = -0.2x^2 + 0.28x + 1.01$

www.algebra1.com/other_calculator_keystrokes

10-5 Exponential Functions

What You'll Learn

- Graph exponential functions.
- Identify data that displays exponential behavior.

Vocabulary

- exponential function

How can exponential functions be used in art?

Earnest "Mooney" Warther was a whittler and a carver. For one of his most unusual carvings, Mooney carved a large pair of pliers in a tree.

From this original carving, he carved another pair of pliers in each handle of the original. Then he carved another pair of pliers in each of those handles. He continued this pattern to create the original pliers and 8 more layers of pliers. Even more amazing is the fact that all of the pliers work.

The number of pliers on each level is given in the table below.

Level	Number of Pliers	Power of 2
Original	1	2^0
First	1(2) = 2	2^1
Second	2(2) = 4	2^2
Third	2(2)(2) = 8	2^3
Fourth	2(2)(2)(2) = 16	2^4
Fifth	2(2)(2)(2)(2) = 32	2^5
Sixth	2(2)(2)(2)(2)(2) = 64	2^6
Seventh	2(2)(2)(2)(2)(2)(2) = 128	2^7
Eighth	2(2)(2)(2)(2)(2)(2)(2) = 256	2^8

GRAPH EXPONENTIAL FUNCTIONS Study the Power of 2 column. Notice that the exponent number matches the level. So we can write an equation to describe y, the number of pliers for any given level x as $y = 2^x$. This type of function, in which the variable is the exponent, is called an **exponential function**.

Key Concept *Exponential Function*

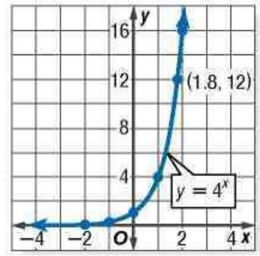
An exponential function is a function that can be described by an equation of the form $y = a^x$, where $a > 0$ and $a \neq 1$.

As with other functions, you can use ordered pairs to graph an exponential function.

Example 1 Graph an Exponential Function with $a > 1$

a. Graph $y = 4^x$. State the y-intercept.

x	4^x	y
-2	4^{-2}	$\frac{1}{16}$
-1	4^{-1}	$\frac{1}{4}$
0	4^0	1
1	4^1	4
2	4^2	16
3	4^3	64



Graph the ordered pairs and connect the points with a smooth curve. The y-intercept is 1. *Notice that the y values change little for small values of x, but they increase quickly as the values of x become greater.*

b. Use the graph to determine the approximate value of $4^{1.8}$.

The graph represents all real values of x and their corresponding values of y for $y = 4^x$. So, the value of y is about 12 when $x = 1.8$. Use a calculator to confirm this value.

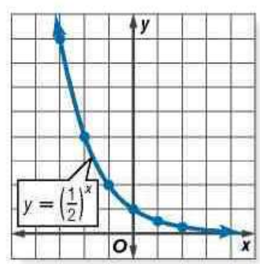
$4^{1.8} \approx 12.12573252$

The graphs of functions of the form $y = a^x$, where $a > 1$, all have the same shape as the graph in Example 1, rising faster and faster as you move from left to right.

Example 2 Graph Exponential Functions with $0 < a < 1$

a. Graph $y = \left(\frac{1}{2}\right)^x$. State the y-intercept.

x	$\left(\frac{1}{2}\right)^x$	y
-3	$\left(\frac{1}{2}\right)^{-3}$	8
-2	$\left(\frac{1}{2}\right)^{-2}$	4
-1	$\left(\frac{1}{2}\right)^{-1}$	2
0	$\left(\frac{1}{2}\right)^0$	1
1	$\left(\frac{1}{2}\right)^1$	$\frac{1}{2}$
2	$\left(\frac{1}{2}\right)^2$	$\frac{1}{4}$



Graph the ordered pairs and connect the points with a smooth curve. The y-intercept is 1. *Notice that the y values decrease less rapidly as x increases.*

b. Use the graph to determine the approximate value of $\left(\frac{1}{2}\right)^{-2.5}$.

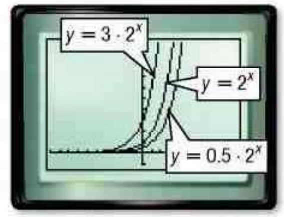
The value of y is about $5\frac{1}{2}$ when $x = -2.5$. Use a calculator to confirm this value.

$\left(\frac{1}{2}\right)^{-2.5} \approx 5.656854249$

Graphing Calculator Investigation

Transformations of Exponential Functions

You can use a graphing calculator to study families of graphs of exponential functions. For example, the graph at the right shows the graphs of $y = 2^x$, $y = 3 \cdot 2^x$, and $y = 0.5 \cdot 2^x$. Notice that the y-intercept of $y = 2^x$ is 1, the y-intercept of $y = 3 \cdot 2^x$ is 3, and the y-intercept of $y = 0.5 \cdot 2^x$ is 0.5. The graph of $y = 3 \cdot 2^x$ is steeper than the graph of $y = 2^x$. The graph of $y = 0.5 \cdot 2^x$ is not as steep as the graph of $y = 2^x$.



[−10, 10] scl: 1 by [−1, 10] scl: 1

Think and Discuss

Graph each family of equations on the same screen. Compare and contrast the graphs.

1. $y = 2^x$, $y = 2^x + 3$, $y = 2^x - 4$

2. $y = 2^x$, $y = 2^{x+5}$, $y = 2^{x-4}$

3. $y = 2^x$, $y = 3^x$, $y = 5^x$

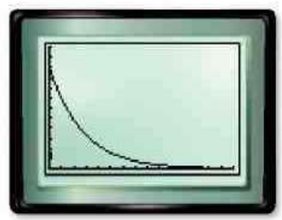
4. $y = 3 \cdot 2^x$, $y = 3(2^x - 1)$, $y = 3(2^x + 1)$

Example 3 Use Exponential Functions to Solve Problems

MOTION PICTURES **Movies tend to have their best ticket sales the first weekend after their release. The sales then follow a decreasing exponential function each successive weekend after the opening. The function $E = 49.9 \cdot 0.692^w$ models the earnings of a popular movie. In this equation, E represents earnings in millions of dollars and w represents the weekend number.**

a. Graph the function. What values of E and w are meaningful in the context of the problem?

Use a graphing calculator to graph the function. Only values where $E \leq 49.9$ and $w > 0$ are meaningful in the context of the problem.



[0, 15] scl: 1 by [0, 60] scl: 5

b. How much did the movie make on the first weekend?

$E = 49.9 \cdot 0.692^w$ Original equation

$E = 49.9 \cdot 0.692^1$ $w = 1$

$E = 34.5308$ Use a calculator.

On the first weekend, the movie grossed about $34.53 million.

c. How much did it make on the fifth weekend?

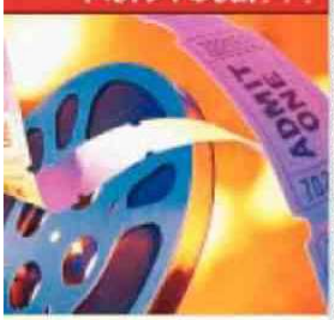
$E = 49.9 \cdot 0.692^w$ Original equation

$E = 49.9 \cdot 0.692^5$ $w = 5$

$E \approx 7.918282973$ Use a calculator.

On the fifth weekend, the movie grossed about $7.92 million.

More About...



Motion Pictures

The first successful photographs of motion were made in 1877. Today, the motion picture industry is big business, with the highest-grossing movie making $600,800,000.

Source: *World Book Encyclopedia*

IDENTIFY EXPONENTIAL BEHAVIOR How do you know if a set of data is exponential? One method is to observe the shape of the graph. But the graph of an exponential function may resemble part of the graph of a quadratic function. Another way is to use the problem-solving strategy *look for a pattern* with the data.

Example 4 Identify Exponential Behavior

Determine whether each set of data displays exponential behavior.

a.

x	0	10	20	30	40	50
y	80	40	20	10	5	2.5

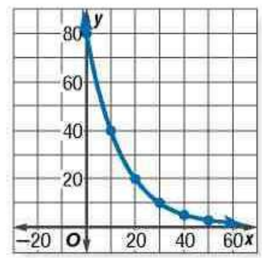
Method 1 Look for a Pattern

The domain values are at regular intervals of 10. Let's see if there is a common factor among the range values.

$$80 \xrightarrow{\times \frac{1}{2}} 40 \xrightarrow{\times \frac{1}{2}} 20 \xrightarrow{\times \frac{1}{2}} 10 \xrightarrow{\times \frac{1}{2}} 5 \xrightarrow{\times \frac{1}{2}} 2.5$$

Since the domain values are at regular intervals and the range values have a common factor, the data are probably exponential. The equation for the data may involve $\left(\frac{1}{2}\right)^x$.

Method 2 Graph the Data



The graph shows a rapidly decreasing value of y as x increases. This is a characteristic of exponential behavior.

b.

x	0	10	20	30	40	50
y	15	21	27	33	39	45

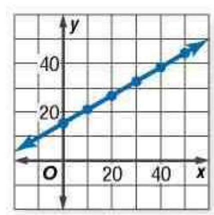
Method 1 Look for a Pattern

The domain values are at regular intervals of 10. The range values have a common difference 6.

$$15 \xrightarrow{+6} 21 \xrightarrow{+6} 27 \xrightarrow{+6} 33 \xrightarrow{+6} 39 \xrightarrow{+6} 45$$

The data do not display exponential behavior, but rather linear behavior.

Method 2 Graph the Data



This is a graph of a line, not an exponential function.

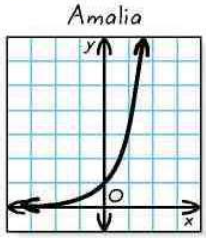
Check for Understanding

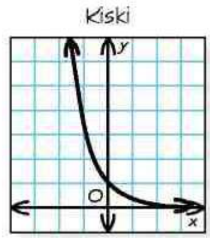
Concept Check

1. **Determine** whether the graph of $y = a^x$, where $a > 0$ and $a \neq 1$, *sometimes*, *always*, or *never* has an x-intercept.

2. **OPEN ENDED** Write an exponential function and graph the function. Describe the graph.

3. **FIND THE ERROR** Amalia and Kiski are graphing $y = \left(\frac{1}{3}\right)^x$.





Who is correct? Explain your reasoning.

Guided Practice

Graph each function. State the y-intercept. Then use the graph to determine the approximate value of the given expression. Use a calculator to confirm the value.

4. $y = 3^x$; $3^{1.2}$
5. $y = \left(\frac{1}{4}\right)^x$; $\left(\frac{1}{4}\right)^{1.7}$
6. $y = 9^x$; $9^{0.8}$

Graph each function. State the y-intercept.

7. $y = 2 \cdot 3^x$
8. $y = 4(5^x - 10)$

Determine whether the data in each table display exponential behavior. Explain why or why not.

9.

x	0	1	2	3	4	5
y	1	6	36	216	1296	7776

10.

x	4	6	8	10	12	14
y	5	9	13	17	21	25

Application

FOLKLORE For Exercises 11 and 12, use the following information.
A wise man asked his ruler to provide rice for feeding his people. Rather than receiving a constant daily supply of rice, the wise man asked the ruler to give him 2 grains of rice for the first square on a chessboard, 4 grains for the second, 8 grains for the third, 16 for the fourth, and so on doubling the amount of rice with each square of the board.

11. How many grains of rice will the wise man receive for the last (64th) square on the chessboard?
12. If one pound of rice has approximately 24,000 grains, how many tons of rice will the wise man receive on the last day? (*Hint*: one ton = 2000 pounds)

Practice and Apply

Homework Help

For Exercises	See Examples
13–26	1, 2
27–32	4
33–41	3

Extra Practice
See page 843.

Graph each function. State the y-intercept. Then use the graph to determine the approximate value of the given expression. Use a calculator to confirm the value.

13. $y = 5^x$; $5^{1.1}$
14. $y = 10^x$; $10^{0.3}$
15. $y = \left(\frac{1}{10}\right)^x$; $\left(\frac{1}{10}\right)^{-1.3}$
16. $y = \left(\frac{1}{5}\right)^x$; $\left(\frac{1}{5}\right)^{0.5}$
17. $y = 6^x$; $6^{0.3}$
18. $y = 8^x$; $8^{0.8}$

Graph each function. State the y-intercept.

19. $y = 5(2^x)$
20. $y = 3(5^x)$
21. $y = 3^x - 7$
22. $y = 2^x + 4$
23. $y = 2(3^x) - 1$
24. $y = 5(2^x) + 4$
25. $y = 2(3^x + 1)$
26. $y = 3(2^x - 5)$

Determine whether the data in each table display exponential behavior. Explain why or why not.

27.

x	−2	−1	0	1
y	−5	−2	1	4

28.

x	0	1	2	3
y	1	0.5	0.25	0.125

29.

x	10	20	30	40
y	16	12	9	6.75

30.

x	−1	0	1	2
y	−0.5	1.0	−2.0	4.0

31.

x	3	6	9	12
y	5	5	5	5

32.

x	5	3	1	−1
y	32	16	8	4

BUSINESS **For Exercises 33–35, use the following information.**
The amount of money spent at West Outlet Mall in Midtown continues to increase. The total $T(x)$ in millions of dollars can be estimated by the function $T(x) = 12(1.12)^x$, where x is the number of years after it opened in 1995.

33. According to the function, find the amount of sales for the mall in the years 2005, 2006, and 2007.

34. Graph the function and name the y-intercept.

35. What does the y-intercept represent in this problem?

36. **BIOLOGY** Mitosis is a process of cell reproduction in which one cell divides into two identical cells. *E. coli* is a fast-growing bacterium that is often responsible for food poisoning in uncooked meat. It can reproduce itself in 15 minutes. If you begin with 100 *E. coli* bacteria, how many will there be in one hour?

More About. . .

Training

The first Boston Marathon was held in 1896. The distance of this race was based on the Greek legend that Pheidippides ran 24.8 miles from Marathon to Athens to bring the news of victory over the Persian army.

Source: www.bostonmarathon.org

TOURNAMENTS **For Exercises 37–39, use the following information.**
In a regional quiz bowl competition, three schools compete and the winner advances to the next round. Therefore, after each round, only $\frac{1}{3}$ of the schools remain in the competition for the next round. Suppose 729 schools start the competition.

37. Write an exponential function to describe the number of schools remaining after x rounds.

38. How many schools are left after 3 rounds?

39. How many rounds will it take to declare a champion?

TRAINING **For Exercises 40 and 41, use the following information.**
A runner is training for a marathon, running a total of 20 miles per week on a regular basis. She plans to increase the distance $D(x)$ in miles according to the function $D(x) = 20(1.1)^x$, where x represents the number of weeks of training.

40. Copy and complete the table showing the number of miles she plans to run.

Week	Distance (miles)
1	
2	
3	
4	

41. The runner's goal is to work up to 50 miles per week. What is the first week that the total will be 50 miles or more?

CRITICAL THINKING **Describe the graph of each equation as a transformation of the graph of $y = 5^x$.**

42. $y = \left(\frac{1}{5}\right)^x$ **43.** $y = 5^x + 2$ **44.** $y = 5^x - 4$

45. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can exponential functions be used in art?

Include the following in your answer:
- the exponential function representing the pliers,
- an explanation of which x and y values are meaningful, and
- the graph of this function.

Standardized Test Practice

46. Which function is an exponential function?

Ⓐ $f(x) = x^2$ Ⓑ $f(x) = 6^x$
Ⓒ $f(x) = x^5$ Ⓓ $f(x) = x^3 + 2x^2 - x + 5$

47. Compare the graphs of $y = 2^x$ and $y = 6^x$.

Ⓐ The graph of $y = 6^x$ steeper than the graph of $y = 2^x$.
Ⓑ The graph of $y = 2^x$ steeper than the graph of $y = 6^x$.
Ⓒ The graph of $y = 6^x$ is the graph of $y = 2^x$ translated 4 units up.
Ⓓ The graph of $y = 6^x$ is the graph of $y = 2^x$ translated 3 units up.

Maintain Your Skills

Mixed Review

Solve each equation by using the Quadratic Formula. Round to the nearest tenth if necessary. *(Lesson 10-4)*

48. $x^2 - 9x - 36 = 0$ 49. $2t^2 + 3t - 1 = 0$ 50. $5y^2 + 3 = y$

Solve each equation by completing the square. Round to the nearest tenth if necessary. *(Lesson 10-3)*

51. $x^2 - 7x = -10$ 52. $a^2 - 12a = 3$ 53. $t^2 + 6t + 3 = 0$

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lesson 9-3)*

54. $m^2 - 14m + 40$ 55. $t^2 - 2t + 35$ 56. $z^2 - 5z - 24$

57. Three times one number equals twice a second number. Twice the first number is 3 more than the second number. Find the numbers. *(Lesson 7-4)*

Solve each inequality. *(Lesson 6-1)*

58. $x + 7 > 2$ 59. $10 \geq x + 8$ 60. $y - 7 < -12$

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Evaluate $p(1 + r)^t$ for each of the given values.**
*(To review **evaluating expressions with exponents**, see Lesson 1-1.)*

61. $p = 5, r = \frac{1}{2}, t = 2$ 62. $p = 300, r = \frac{1}{4}, t = 3$
63. $p = 100, r = 0.2, t = 2$ 64. $p = 6, r = 0.5, t = 3$

Practice Quiz 2

Lessons 10-4 and 10-5

Solve each equation by using the Quadratic Formula. Round to the nearest tenth if necessary. *(Lesson 10-4)*

1. $x^2 + 2x = 35$ 2. $2n^2 - 3n + 5 = 0$ 3. $2v^2 - 4v = 1$

Graph each function. State the y-intercept. *(Lesson 10-5)*

4. $y = 0.5(4^x)$ 5. $y = 5^x - 4$

10-6 Growth and Decay

What You'll Learn

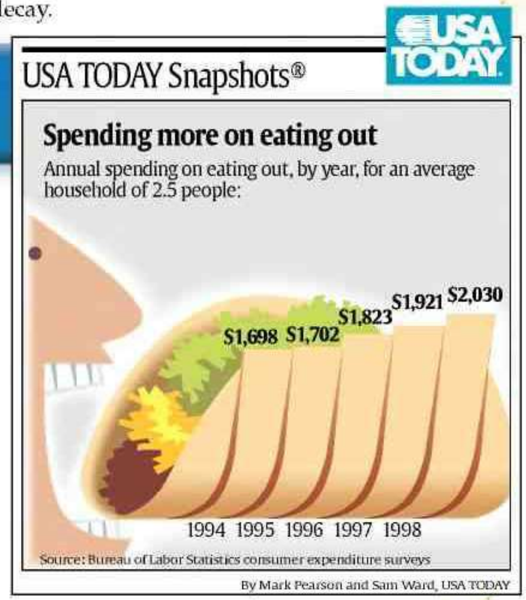
- Solve problems involving exponential growth.
- Solve problems involving exponential decay.

Vocabulary

- exponential growth
- compound interest
- exponential decay

How can exponential growth be used to predict future sales?

The graph shows that the average household in the United States has increased its spending for restaurant meals. In fact, the amount grew at an annual rate of about 4.6% between 1994 and 1998. Let y represent the average amount spent on restaurant meals, and let t represent the number of years since 1994. Then the average amount spent on restaurant meals can be modeled by $y = 1698(1 + 0.046)^t$ or $y = 1698(1.046)^t$.



EXPONENTIAL GROWTH The equation for the average amount spent on restaurant meals is in the form $y = C(1 + r)^t$. This is the general equation for **exponential growth** in which the initial amount C increases by the same percent over a given period of time.

Key Concept — *General Equation for Exponential Growth*

The general equation for exponential growth is $y = C(1 + r)^t$ where y represents the final amount, C represents the initial amount, r represents the rate of change expressed as a decimal, and t represents time.

Example 1 *Exponential Growth*

SPORTS In 1971, there were 294,105 females participating in high school sports. Since then, that number has increased an average of 8.5% per year.

a. Write an equation to represent the number of females participating in high school sports since 1971.

$y = C(1 + r)^t$	General equation for exponential growth
$y = 294{,}105(1 + 0.085)^t$	$C = 294{,}105$ and $r = 8.5\%$ or 0.085
$y = 294{,}105(1.085)^t$	Simplify.

An equation to represent the number of females participating in high school sports is $y = 294{,}105(1.085)^t$, where y represents the number of female athletes and t represents the number of years since 1971.

(continued on the next page)

b. According to the equation, how many females participated in high school sports in the year 2001?

$y = 294{,}105(1.085)^t$ Equation for females participating in sports

$y = 294{,}105(1.085)^{30}$ $t = 2001 - 1971$ or 30

$y \approx 3{,}399{,}340$ In 2001, about 3,399,340 females participated.

One special application of exponential growth is **compound interest**. The equation for compound interest is $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A represents the amount of the investment, P is the principal (initial amount of the investment), r represents the annual rate of interest expressed as a decimal, n represents the number of times that the interest is compounded each year, and t represents the number of years that the money is invested.

Example 2 Compound Interest

HISTORY **Use the information at the left. If the money the Native Americans received for Manhattan had been invested at 6% per year compounded semiannually, how much money would there be in the year 2026?**

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compound interest equation

$A = 24\left(1 + \frac{0.06}{2}\right)^{2(400)}$ $P = 24$, $r = 6\%$ or 0.06, $n = 2$, and $t = 400$

$A = 24(1.03)^{800}$ Simplify.

$A \approx 4.47 \times 10^{11}$ There would be about $447,000,000,000.

More About. . .

History

In 1626, Peter Minuit, governor of the colony of New Netherland, bought the island of Manhattan from the Native Americans for beads, cloth, and trinkets worth 60 Dutch guilders ($24).

Source: *World Book Encyclopedia*

EXPONENTIAL DECAY A variation of the growth equation can be used as the general equation for **exponential decay**. In exponential decay, the original amount decreases by the same percent over a period of time.

Key Concept *General Equation for Exponential Decay*

The general equation for exponential decay is $y = C(1 - r)^t$ where y represents the final amount, C represents the initial amount, r represents the rate of decay expressed as a decimal, and t represents time.

Example 3 Exponential Decay

ENERGY **In 1950, the use of coal by residential and commercial users was 114.6 million tons. Many businesses now use cleaner sources of energy. As a result, the use of coal has decreased by 6.6% per year.**

a. Write an equation to represent the use of coal since 1950.

$y = C(1 - r)^t$ General equation for exponential decay

$y = 114.6(1 - 0.066)^t$ $C = 114.6$ and $r = 6.6\%$ or 0.066

$y = 114.6(0.934)^t$ Simplify.

An equation to represent the use of coal is $y = 114.6(0.934)^t$, where y represents tons of coal used annually and t represents the number of years since 1950.

b. Estimate the estimated amount of coal that will be used in 2015.

$y = 114.6(0.934)^t$ Equation for coal use

$y = 114.6(0.934)^{65}$ $t = 2015 - 1950$ or 65

$y \approx 1.35$ The amount of coal should be about 1.35 million tons.

Sometimes items decrease in value or *depreciate*. For example, most cars and office equipment depreciate as they get older. You can use the exponential decay formula to determine the value of an item at a given time.

Example 4 Depreciation

FARMING A farmer buys a tractor for $50,000. If the tractor depreciates 10% per year, find the value of the tractor in 7 years.

$y = C(1 - r)^t$	General equation for exponential decay
$y = 50{,}000(1 - 0.10)^7$	$C = 50{,}000$, $r = 10\%$ or 0.10, and $t = 7$
$y = 50{,}000(0.90)^7$	Simplify.
$y \approx 23{,}914.85$	Use a calculator.

The tractor will be worth about $23,914.85 or less than half its original value.

Check for Understanding

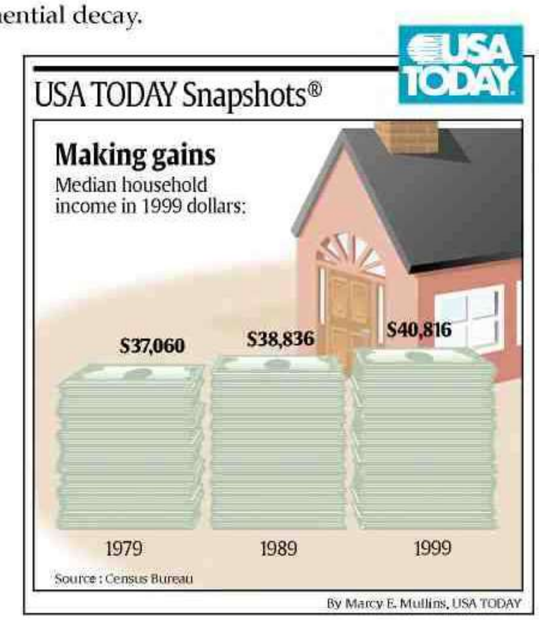
Concept Check

1. **Explain** the difference between *exponential growth* and *exponential decay*.
2. **OPEN ENDED** Write a compound interest problem that could be solved by the equation $A = 500\left(1 + \frac{0.07}{4}\right)^{4(6)}$.
3. **Draw** a graph representing exponential decay.

Guided Practice

INCOME For Exercises 4 and 5, use the graph at the right and the following information.
The median household income in the United States increased an average of 0.5% each year between 1979 and 1999. Assume this pattern continues.

4. Write an equation for the median household income for t years after 1979.
5. Predict the median household income in 2009.



Applications

6. **INVESTMENTS** Determine the amount of an investment if $400 is invested at an interest rate of 7.25% compounded quarterly for 7 years.
7. **POPULATION** In 1995, the population of West Virginia reached 1,821,000, its highest in the 20th century. For the next 5 years, its population decreased 0.2% each year. If this trend continues, predict the population of West Virginia in 2010.
8. **TRANSPORTATION** A car sells for $16,000. If the rate of depreciation is 18%, find the value of the car after 8 years.

Practice and Apply

TECHNOLOGY For Exercises 9 and 10, use the following information.
Computer use around the world has risen 19% annually since 1980.

9. If 18.9 million computers were in use in 1980, write an equation for the number of computers in use for t years after 1980.
10. Predict the number of computers in 2015.

Homework Help

For Exercises	See Examples
9–13, 18 21, 22	1
14, 15	2
16, 17, 25–28	3
19, 20	4

Extra Practice

See page 843.

WEIGHT TRAINING **For Exercises 11 and 12, use the following information.**
The use of free weights for fitness has increased in popularity. In 1997, there were 43.2 million people who used free weights.

11. Assuming the use of free weights increases 6% annually, write an equation for the number of people using free weights t years from 1997.

12. Predict the number of people using free weights in 2007.

13. POPULATION The population of Mexico has been increasing at an annual rate of 1.7%. If the population of Mexico was 100,350,000 in the year 2000, predict its population in 2012.

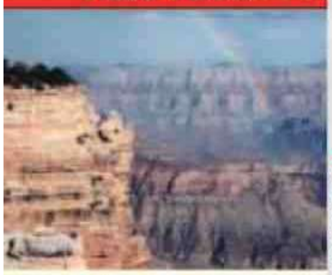
14. INVESTMENTS Determine the amount of an investment if $500 is invested at an interest rate of 5.75% compounded monthly for 25 years.

15. INVESTMENTS Determine the amount of an investment if $250 is invested at an interest rate of 10.3% compounded quarterly for 40 years.

16. POPULATION The country of Latvia has been experiencing a 1.1% annual decrease in population. In 2000, its population was 2,405,000. If the trend continues, predict Latvia's population in 2015.

17. MUSIC In 1994, the sales of music cassettes reached its peak at $2,976,400,000. Since then, cassette sales have been declining. If the annual percent of decrease in sales is 18.6%, predict the sales of cassettes in the year 2009.

More About. . .

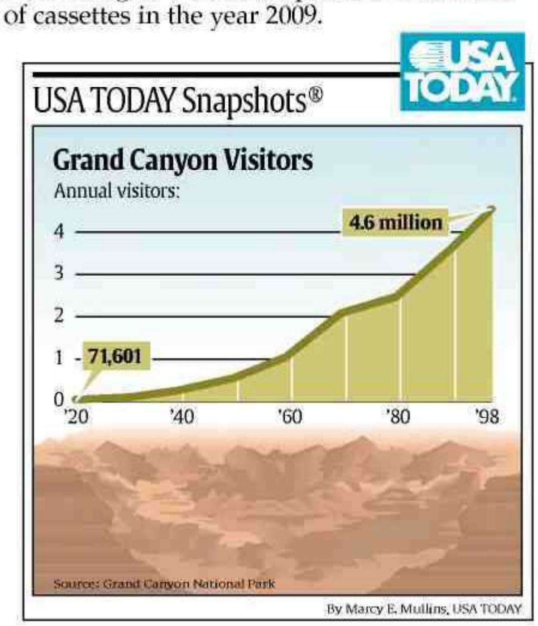


Grand Canyon

The Grand Canyon National Park covers 1,218,375 acres. It has 38 hiking trails, which cover about 400 miles.

Source: *World Book Encyclopedia*

18. GRAND CANYON The increase in the number of visitors to the Grand Canyon National Park is similar to an exponential function. If the average visitation has increased 5.63% annually since 1920, use the graph to predict the number of visitors to the park in 2020.



19. BUSINESS A piece of office equipment valued at $25,000 depreciates at a steady rate of 10% per year. What is the value of the equipment in 8 years?

20. TRANSPORTATION A new car costs $23,000. It is expected to depreciate 12% each year. Find the value of the car in 5 years.

POPULATION **For Exercises 21 and 22, use the following information.**
Since birth rates are going down and people are living longer, the percent of the population that is 65 years old or older continues to rise. The percent of the U.S. population P that is at least 65 years old can be approximated by the equation $P = 3.86(1.013)^t$, where t represents the number of years since 1900.

21. What percent of the population will be 65 years of age or older in the year 2010?

22. Predict the year in which people ages 65 or older will represent 20% of the population if this trend continues. (*Hint*: Make a table.)

CRITICAL THINKING **Each equation represents an exponential rate of change if t is time in years. Determine whether each equation represents growth or decay. Give the annual rate of change as a percent.**

23. $y = 500(1.026^t)$

24. $y = 500(0.761^t)$

ARCHAEOLOGY **For Exercises 25–28, use the following information.**
The *half-life* of a radioactive element is the time that it takes for one-half a quantity of the element to decay. Carbon-14 is found in all living organisms and has a half-life of 5730 years. Archaeologists use this fact to estimate the age of fossils. Consider an organism with an original Carbon-14 content of 256 grams. The number of grams remaining in the organism's fossil after t years is $256(0.5)^{\frac{t}{5730}}$.

25. If the organism died 5730 years ago, what is the amount of Carbon-14 today?
26. If the organism died 1000 years ago, what is the amount of Carbon-14 today?
27. If the organism died 10,000 years ago, what is the amount of Carbon-14 today?
28. If the fossil has 32 grams of Carbon-14 remaining, how long ago did it live? (*Hint*: Make a table.)

29. **RESEARCH** Find the enrollment of your school district each year for the last decade. Find the rate of change from one year to the next. Then, determine the average annual rate of change for those years. Use this information to estimate the enrollment for your school district in ten years.

30. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can exponential growth be used to predict future sales?

Include the following in your answer:
- an explanation of the equation $y = 1698(1 + 0.046)^t$, and
- an estimate of the average family's spending for restaurant meals in 2010.

Standardized Test Practice

31. Which equation represents exponential growth?
Ⓐ $y = 50x^3$ Ⓑ $y = 30x^2 + 10$
Ⓒ $y = 35(1.05^x)$ Ⓓ $y = 80(0.92^x)$

32. Lorena is investing a \$5000 inheritance from her aunt in a certificate of deposit that matures in 4 years. The interest rate is 8.25% compounded quarterly. What is the balance of the account after 4 years?
Ⓐ \$5412.50 Ⓑ \$6865.65 Ⓒ \$6908.92 Ⓓ \$6931.53

Maintain Your Skills

Mixed Review

Graph each function. State the y-intercept. *(Lesson 10-5)*

33. $y = \left(\frac{1}{8}\right)^x$ 34. $y = 2^x - 5$ 35. $y = 4(3^x - 6)$

Solve each equation by using the Quadratic Formula. Round to the nearest tenth if necessary. *(Lesson 10-4)*

36. $m^2 - 9m - 10 = 0$ 37. $2t^2 - 4t = 3$ 38. $7x^2 + 3x + 1 = 0$

Simplify. *(Lesson 8-1)*

39. $m^7(m^3b^2)$ 40. $-3(ax^3y)^2$ 41. $(0.3x^3y^2)^2$

Solve each open sentence. *(Lesson 6-5)*

42. $|7x + 2| = -2$ 43. $|3 - 3x| = 0$ 44. $|t + 4| \geq 3$

45. **SKIING** A course for cross-country skiing is regulated so that the slope of any hill cannot be greater than 0.33. A hill rises 60 meters over a horizontal distance of 250 meters. Does the hill meet the requirements? *(Lesson 5-1)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find the next three terms in each arithmetic sequence.**
*(To review **arithmetic sequences**, see Lesson 4-7.)*

46. 8, 11, 14, 17, … 47. 7, 4, 1, −2, … 48. 1.5, 2.6, 3.7, 4.8, …

Reading Mathematics

Growth and Decay Formulas

Growth and decay problems may be confusing, unless you read them in a simplified, generalized form. The growth and decay formulas that you used in Lesson 10-6 are based on the idea that an initial amount is multiplied by a rate raised to a power of time, which is equivalent to a final amount. If you remember the following formula, all other formulas will be easier to remember.

$$\textit{final amount} = \textit{initial amount} \cdot \textit{rate}^{\textit{time}}$$

Below, we will review the general equation for exponential growth to see how it is related to the generalized formula above.

$$\overbrace{y}^{\text{The final amount}} \quad \overbrace{=}^{\text{equals}} \quad \overbrace{C}^{\text{an initial amount}} \quad \overbrace{\cdot}^{\text{times}} \quad \overbrace{(1 + r)^t}^{\text{the quantity one plus a rate raised to the power of time.}}$$

The only difference from the generalized formula is that rate equals $1 + r$. *Why?*

One represents 100%. If you multiply C by 100%, the final amount is the same as the initial amount. We *add* 1 to the rate r so that the final amount is the initial amount plus the increase.

You can break each growth and decay formula into the following pieces:

- final amount,
- initial amount,
- rate, and
- time.

Reading to Learn

1. Write the general equation for exponential decay. Discuss how it is related to the generalized formula. Why is the rate equal to $1 - r$?

2. Write the formula for compound interest. How is it related to the generalized formula? Why does the rate equal $\left(1 + \frac{r}{n}\right)$? Why does the time equal nt?

3. Suppose that \$2500 is invested at an annual rate of 6%. If the interest is compounded quarterly, find the value of the account after 5 years.

  a. Copy the problem and underline all important numerical data.

  b. Choose the appropriate formula and solve the problem.

4. Angela bought a car for \$18,500. If the rate of depreciation is 11%, find the value of the car in 4 years.

  a. Copy the problem and underline all important numerical data.

  b. Choose the appropriate formula and solve the problem.

5. The population of Centerville is increasing at an average annual rate of 3.5%. If its current population is 12,500, predict its population in 5 years.

  a. Copy the problem and underline all important numerical data.

  b. Choose the appropriate formula and solve the problem.

CONTENTS

10-7 Geometric Sequences

What You'll Learn

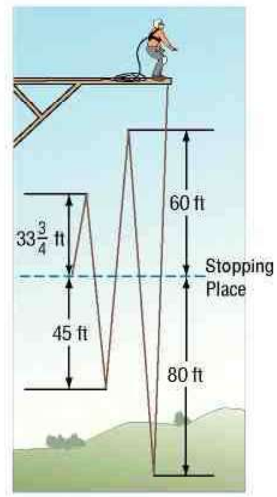
- Recognize and extend geometric sequences.
- Find geometric means.

Vocabulary

- geometric sequence
- common ratio
- geometric means

How can a geometric sequence be used to describe a bungee jump?

A thrill ride is set up with a bungee rope that will stretch when a person jumps from the platform. The ride continues as the person bounces back and forth closer to the stopping place of the rope. Each bounce is only $\frac{3}{4}$ as far from the stopping length as the preceding bounce. If the initial drop is 80 feet past the stopping length of the rope, the following table gives the distance of the first four bounces.



Bounce	Distance (ft)
1	80
2	$\frac{3}{4} \cdot 80$ or 60
3	$\frac{3}{4} \cdot 60$ or 45
4	$\frac{3}{4} \cdot 45$ or $33\frac{3}{4}$

GEOMETRIC SEQUENCES The distance of each bounce is found by multiplying the previous term by $\frac{3}{4}$. The successive distances of the bounces is an example of a **geometric sequence**. The number by which each term is multiplied is called the **common ratio**.

Key Concept — *Geometric Sequence*

- **Words** A geometric sequence is a sequence in which each term after the nonzero first term is found by multiplying the previous term by a constant called the common ratio r, where $r \neq 0, 1$.
- **Symbols** $a, ar, (ar)r$ or $ar^2, (ar^2)r$ or $ar^3, \ldots \quad (a \neq 0; r \neq 0, 1)$
- **Examples** 1, 3, 9, 27, 81, ...

Example 1 Recognize Geometric Sequences

Study Tip

Look Back
To review **arithmetic sequences**, see Lesson 4-7.

Determine whether each sequence is geometric.

a. 0, 5, 10, 15, 20, ...

Determine the pattern.

$$0 \underbrace{\quad}_{+5} 5 \underbrace{\quad}_{+5} 10 \underbrace{\quad}_{+5} 15 \underbrace{\quad}_{+5} 20$$

In this sequence, each term is found by adding 5 to the previous term. This sequence is arithmetic, *not* geometric.

b. 1, 5, 25, 125, 625

$1 \xrightarrow{\times 5} 5 \xrightarrow{\times 5} 25 \xrightarrow{\times 5} 125 \xrightarrow{\times 5} 625$

In this sequence, each term is found by multiplying the previous term times 5. This sequence is geometric.

The common ratio of a geometric sequence can be found by dividing any term by the preceding term.

Example 2 Continue Geometric Sequences

Find the next three terms in each geometric sequence.

a. 4, −8, 16, …

$\frac{-8}{4} = -2$ Divide the second term by the first.

The common factor is -2. Use this information to find the next three terms.

$4, -8, 16, \xrightarrow{\times (-2)} -32 \xrightarrow{\times (-2)} 64 \xrightarrow{\times (-2)} -128$

The next three terms are -32, 64, and -128.

b. 60, 72, 86.4, …

$\frac{72}{60} = 1.2$ Divide the second term by the first.

The common factor is 1.2. Use this information to find the next three terms.

$60, 72, 86.4, \xrightarrow{\times 1.2} 103.68 \xrightarrow{\times 1.2} 124.416 \xrightarrow{\times 1.2} 149.2992$

The next three terms are 103.68, 124.416, and 149.2992.

More About. . .

Geography

Madagascar is a country just east of mainland Africa. It consists of the third largest island in the world and many tiny islands. In 2000, the population of Madagascar was about 15,500,000.

Source: *World Book Encyclopedia*

Example 3 Use Geometric Sequences to Solve a Problem

GEOGRAPHY Madagascar's population has been increasing at an average annual rate of 3%. Use the information at the left to determine the population of Madagascar in 2001, 2002, and 2003.

The population is a geometric sequence in which the first term is 15,500,000 and the common ratio is 1.03.

Year	Population
2000	15,500,000
2001	15,500,000(1.03) or 15,965,000
2002	15,965,000(1.03) or 16,443,950
2003	16,443,950(1.03) or about 16,937,269

The population of Madagascar in the years 2001, 2002, and 2003 will be 15,965,000, 16,443,950, and about 16,937,269, respectively.

As with arithmetic sequences, you can name the terms of a geometric sequence using a_1, a_2, a_3, and so on. Then the nth term is represented as a_n. Each term of a geometric sequence can also be represented using r and its previous term. A third way to represent each term is by using r and the first term a_1.

Sequence	number	2	6	18	54	...	
	symbols	a_1	a_2	a_3	a_4	...	a_n
Expressed in Terms of r and Previous Term	number	2	2(3)	6(3)	18(3)	...	
	symbols	a_1	$a_1 \cdot r$	$a_2 \cdot r$	$a_3 \cdot r$	...	$a_{n-1} \cdot r$
Expressed in Terms of r and First Term	number	2 or $2(3^0)$	2(3) or $2(3^1)$	2(9) or $2(3^2)$	2(27) or $2(3^3)$	...	
	symbols	$a_1 \cdot r^0$	$a_1 \cdot r^1$	$a_1 \cdot r^2$	$a_1 \cdot r^3$	...	$a_1 \cdot r^{n-1}$

The three values in the last column of the table all describe the nth term of a geometric sequence.

Study Tip

Recursive Formulas

When the nth term of a sequence is expressed in terms of the previous term, as in $a_n = a_{n-1} \cdot n$, the formula is called a *recursive formula*.

Key Concept Formula for the nth Term of a Geometric Sequence

The nth term a_n of a geometric sequence with the first term a_1 and common ratio r is given by $a_n = a_1 \cdot r^{n-1}$.

Example 4 nth Term of a Geometric Sequence

Find the sixth term of a geometric sequence in which $a_1 = 3$ and $r = -5$.

$a_n = a_1 \cdot r^{n-1}$	Formula for the nth term of a geometric sequence
$a_6 = 3 \cdot (-5)^{6-1}$	$n = 6$, $a_1 = 3$, and $r = -5$
$a_6 = 3 \cdot (-5)^5$	$6 - 1 = 5$
$a_6 = 3 \cdot (-3125)$	$(-5)^5 = -3125$
$a_6 = -9375$	$3 \cdot (-3125) = -9375$

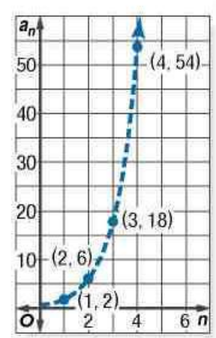
The sixth term of the geometric sequence is -9375.

Geometric sequences are related to exponential functions.

Algebra Activity

Graphs of Geometric Sequences

You can graph a geometric sequence by graphing the coordinates (n, a_n). For example, consider the sequence 2, 6, 18, 54, ... To graph this sequence, graph the points at (1, 2), (2, 6), (3, 18), and (4, 54). Use a dashed curve to connect the points.



Model

Graph each geometric sequence. Name each common ratio.

1. 1, 2, 4, 8, 16, ...
2. 1, −2, 4, −8, 16, ...
3. 81, 27, 9, 3, 1, ...
4. −81, 27, −9, 3, −1, ...
5. 0.2, 1, 5, 25, 125, ...
6. −0.2, 1, −5, 25, −125, ...

Analyze

7. Which graphs appear to be similar to an exponential function?

8. Compare and contrast the graphs of geometric sequences with $r > 0$ and $r < 0$.

9. Compare the formula for an exponential function $y = c(a^x)$ to the value of the nth term of a geometric sequence.

GEOMETRIC MEANS Missing term(s) between two nonconsecutive terms in a geometric sequence are called **geometric means**. In the sequence 100, 20, 4, …, the geometric mean between 100 and 4 is 20. You can use the formula for the nth term of a geometric sequence to find a geometric mean.

Example 5 Find Geometric Means

Find the geometric mean in the sequence 2, ____, 18.

In the sequence, $a_1 = 2$ and $a_3 = 18$. To find a_2, you must first find r.

$a_n = a_1 \cdot r^{n-1}$	Formula for the nth term of a geometric sequence
$a_3 = a_1 \cdot r^{3-1}$	$n = 3$
$18 = 2 \cdot r^2$	$a_3 = 18$ and $a_1 = 2$
$\frac{18}{2} = \frac{2r^2}{2}$	Divide each side by 2.
$9 = r^2$	Simplify.
$\pm 3 = r$	Take the square root of each side.

If $r = 3$, the geometric mean is 2(3) or 6. If $r = -3$, the geometric mean is $2(-3)$ or -6. Therefore, the geometric mean is 6 or -6.

Check for Understanding

Concept Check

1. **Compare and contrast** an arithmetic sequence and a geometric sequence.
2. **Explain** why the definition of a geometric sequence restricts the values of the common ratio to numbers other than 0 and 1.
3. **OPEN ENDED** Give an example of a sequence that is neither arithmetic nor geometric.

Guided Practice

Determine whether each sequence is geometric.

4. 5, 15, 45, 135, …
5. 56, -28, 14, -7, …
6. 25, 20, 15, 10, …

Find the next three terms in each geometric sequence.

7. 5, 20, 80, 320, …
8. 176, -88, 44, -22, …
9. -8, 12, -18, 27, …

Find the nth term of each geometric sequence.

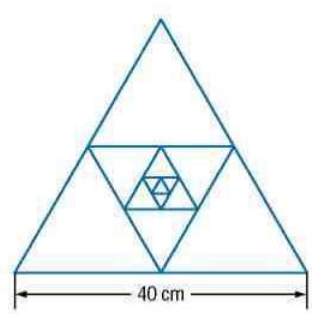
10. $a_1 = 3, n = 5, r = 4$
11. $a_1 = -1, n = 6, r = 2$
12. $a_1 = 4, n = 7, r = -3$

Find the geometric means in each sequence.

13. 7, ____, 28
14. 48, ____, 3
15. -4, ____, -100

Application

16. **GEOMETRY** Consider the inscribed equilateral triangles at the right. The perimeter of each triangle is one-half of the perimeter of the next larger triangle. What is the perimeter of the smallest triangle?



Practice and Apply

Homework Help

For Exercises	See Examples
17–24	1
25–34	2
35–42	4
43–54	5
55–62	3

Extra Practice

See page 843.

Determine whether each sequence is geometric.

17. 2, 6, 18, 54, …
18. 7, 17, 27, 37, …
19. −19, −16, −13, −10, …
20. 640, 160, 40, 10, …
21. 36, 25, 16, 9, …
22. −567, −189, −63, −21, …
23. 20, −90, 405, −1822.5, …
24. −50, 110, −242, 532.4, …

Find the next three terms in each geometric sequence.

25. 1, −4, 16, −64, …
26. −1, −6, −36, −216, …
27. 1024, 512, 256, 128, …
28. 224, 112, 56, 28, …
29. −80, 20, −5, 1.25, …
30. 10,000, −200, 4, −0.08, …
31. $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \ldots$
32. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$

33. GEOMETRY A rectangle is 6 inches by 8 inches. The rectangle is cut in half, and one half is discarded. The remaining rectangle is cut in half, and one half is discarded. This is repeated twice. List the areas of the five rectangles formed.

34. GEOMETRY To bisect an angle means to cut it into two angles with the same measure. Suppose a 160° angle is bisected. Then one of the new angles is bisected. This is repeated twice. List the measures of the four sizes of angles.

Find the *n*th term of each geometric sequence.

35. $a_1 = 5, n = 7, r = 2$
36. $a_1 = 4, n = 5, r = 3$
37. $a_1 = -2, n = 4, r = -5$
38. $a_1 = 3, n = 6, r = -4$
39. $a_1 = -8, n = 3, r = 6$
40. $a_1 = -10, n = 8, r = 2$
41. $a_1 = 300, n = 10, r = 0.5$
42. $a_1 = 14, n = 6, r = 1.5$

Find the geometric means in each sequence.

43. 5, ___, 20
44. 6, ___, 54
45. −9, ___, −225
46. −5, ___, −80
47. 128, ___, 8
48. 180, ___, 5
49. −2, ___, −98
50. −6, ___, −384
51. 7, ___, 1.75
52. 3, ___, 0.75
53. $\frac{3}{5}$, ___, $\frac{3}{20}$
54. $\frac{2}{5}$, ___, $\frac{2}{45}$

55. A ball is thrown vertically. It is allowed to return to the ground and rebound without interference. If each rebound is 60% of the previous height, give the heights of the three rebounds after the initial rebound of 10 meters.

QUIZ GAMES **For Exercises 56 and 57, use the following information.**
Radio station WXYZ has a special game for its listeners. A trivia question is asked, and the player scores 10 points for the first correct answer. Every correct answer after that doubles the player's score.

56. List the scores after each of the first 6 correct answers.

57. Suppose the player needs to answer the question worth more than a million points to win the grand prize of a car. How many questions must be answered correctly in order to earn the car?

POLLUTION **For Exercise 58–60, use the following information.**
A lake was closed because of an accidental pesticide spill. The concentration of the pesticide after the spill was 848 parts per million. Each day the water is tested, and the amount of pesticide is found to be about 75% of what was there the day before.

58. List the level of pesticides in the water during the first week.

59. If a safe level of pesticides is considered to be 12 parts per million or less, when will the lake be considered safe?

60. Do you think the lake will ever be completely free of the pesticide? Explain.

More About. . .

Pollution

On March 23, 1989, 250,000 barrels of oil were spilled affecting 1300 miles of Alaskan coastline. This was the largest oil spill in the United States.

Source: www.oilspill.state.ak.us

CRITICAL THINKING **For Exercises 61 and 62, suppose a sequence is geometric.**

61. If each term of the sequence is multiplied by the same nonzero real number, is the new sequence *always*, *sometimes*, or *never* a geometric sequence?

62. If the same nonzero number is added to each term of the sequence, is the new sequence *always*, *sometimes*, or *never* a geometric sequence?

63. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can a geometric sequence be used to describe a bungee jump?

Include the following in your answer:

- an explanation of how to determine the tenth term in the sequence, and
- the number of rebounds the first time the distance from the stopping place is less than one foot, which would trigger the end of the ride.

Standardized Test Practice

64. Which number is next in the geometric sequence 40, 100, 250, 625, … ?

(A) 900 (B) 1250 (C) 1562.5 (D) 1875

65. **GRID IN** Find the next term in the following geometric sequence.
343, 49, 7, 1, …

Extending the Lesson

For Exercises 66–68, consider the nth term of the sequence 2, 1, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$.

66. As n approaches infinity, what value will the nth term approach?

67. In mathematics, a **limit** is a number that something approaches, but never reaches. What would you consider the limit of the values of the sequence?

68. If n approaches infinity, how is the nth term of a geometric sequences where $0 < r < 1$ different than the nth term of a geometric sequences where $r > 1$?

Maintain Your Skills

Mixed Review

69. **INVESTMENTS** Determine the value of an investment if $1500 is invested at an interest rate of 6.5% compounded monthly for 3 years. *(Lesson 10-6)*

Determine whether the data in each table display exponential behavior. Explain why or why not. *(Lesson 10-5)*

70.

x	3	5	7	9
y	10	12	14	16

71.

x	2	5	8	11
y	0.5	1.5	4.5	13.5

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lesson 9-4)*

72. $7a^2 + 22a + 3$ **73.** $2x^2 - 5x - 12$ **74.** $3c^2 - 3c - 5$

WebQuest Internet Project

Pluto Is Falling from Status as a Distant Planet

It is time to complete your project. Use the information and data you have gathered about the solar system to prepare a brochure, poster, or Web page. Be sure to include the three graphs, tables, diagrams, or calculations in the presentation.

www.algebra1.com/webquest

Investigating Rates of Change

Collect the Data

- The Richter scale is used to measure the force of an earthquake. The table below shows the increase in magnitude for the values on the Richter scale.

Richter Number (x)	Increase in Magnitude (y)	Rate of Change (slope)
1	1	—
2	10	9
3	100	
4	1000	
5	10,000	
6	100,000	
7	1,000,000	

Source: *The New York Public Library Science Desk Reference*

- On grid paper, plot the ordered pairs (Richter number, increase in magnitude).
- Copy the table for the Richter scale and fill in the rate of change from one value to the next. For example, the rate of change for (1, 1) and (2, 10) is $\frac{10-1}{2-1}$ or 9.

Analyze the Data

1. Describe the graph you made of the Richter scale data.

2. Is the rate of change between any two points the same?

Make a Conjecture

3. Can the data be represented by a linear equation? Why or why not?

4. Describe the pattern shown in the rates of change in Column 3.

Extend the Investigation

5. Use a graphing calculator or graphing software to find a regression equation for the Richter scale data. (*Hint*: If you are using the TI-83 Plus, use ExpReg.)

6. Graph the following set of data that shows the amount of energy released for each Richter scale value. Describe the graph. Fill in the third column and describe the rates of change. Find a regression equation for this set of data.

Richter Number (x)	Energy Released (y)	Rate of Change (slope)
1	0.00017 metric ton	
2	0.006 metric ton	
3	0.179 metric ton	
4	5 metric tons	
5	179 metric tons	
6	5643 metric tons	
7	179,100 metric tons	

Source: *The New York Public Library Science Desk Reference*

Chapter 10 Study Guide and Review

Vocabulary and Concept Check

axis of symmetry (p. 525)
common ratio (p. 567)
completing the square (p. 539)
compound interest (p. 562)
discriminant (p. 548)
exponential decay (p. 562)
exponential function (p. 554)
exponential growth (p. 561)
geometric means (p. 570)
geometric sequence (p. 567)
maximum (p. 525)
minimum (p. 525)
parabola (p. 524)
quadratic equation (p. 533)
Quadratic Formula (p. 546)
quadratic function (p. 524)
roots (p. 533)
symmetry (p. 525)
vertex (p. 525)
zeros (p. 533)

Choose the letter of the term that best matches each equation or phrase.

1. $y = C(1 + r)^t$
2. $f(x) = ax^2 + bx + c$
3. a geometric property of parabolas
4. $x = -\frac{b}{2a}$
5. $y = a^x$
6. maximum or minimum point of a parabola
7. $y = C(1 - r)^t$
8. solutions of a quadratic equation
9. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
10. the graph of a quadratic function

a. equation of axis of symmetry
b. exponential decay equation
c. exponential function
d. exponential growth equation
e. parabola
f. Quadratic Formula
g. quadratic function
h. roots
i. symmetry
j. vertex

Lesson-by-Lesson Review

10-1 Graphing Quadratic Functions

See pages 524–530.

Concept Summary

- The standard form of a quadratic function is $y = ax^2 + bx + c$.
- Complete a table of values to graph a quadratic function.
- The equation of the axis of symmetry for the graph of $y = ax^2 + bx + c$, where $a \neq 0$, is $x = -\frac{b}{2a}$.
- The vertex of a parabola is on the axis of symmetry.

Example **Consider the graph of $y = x^2 - 8x + 12$.**

a. Write the equation of the axis of symmetry.

In the equation $y = x^2 - 8x + 12$, $a = 1$ and $b = -8$. Substitute these values into the equation of the axis of symmetry.

$x = -\frac{b}{2a}$ Equation of the axis of symmetry

$= -\frac{-8}{2(1)}$ or 4 $a = 1$ and $b = -8$

The equation of the axis of symmetry is $x = 4$.

www.algebra1.com/vocabulary_review

CONTENTS

b. Find the coordinates of the vertex of the graph.

The x-coordinate of the vertex is 4.

$y = x^2 - 8x + 12$ Original equation

$y = (4)^2 - 8(4) + 12$ $x = 4$

$y = 16 - 32 + 12$ Simplify.

$y = -4$ The coordinates of the vertex are $(4, -4)$.

Exercises **Write the equation of the axis of symmetry, and find the coordinates of the vertex of the graph of each function. Identify the vertex as a maximum or minimum. Then graph the function.** *See Example 3 on pages 526 and 527.*

11. $y = x^2 + 2x$ **12.** $y = -3x^2 + 4$ **13.** $y = x^2 - 3x - 4$
14. $y = 3x^2 + 6x - 17$ **15.** $y = -2x^2 + 1$ **16.** $y = -x^2 - 3x$

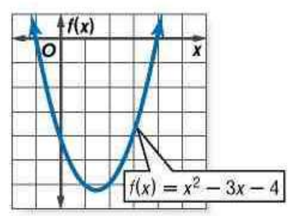
10-2 Solving Quadratic Equations by Graphing

See pages 533–538.

Concept Summary

- The roots of a quadratic equation are the x-intercepts of the related quadratic function.

Example **Solve $x^2 - 3x - 4 = 0$ by graphing.**



Graph the related function $f(x) = x^2 - 3x - 4$.

The x-intercepts are -1 and 4. Therefore, the solutions are -1, and 4.

Exercises **Solve each equation by graphing. If integral roots cannot be found, estimate the roots by stating the consecutive integers between which the roots lie.** *See Examples 1–4 on pages 533–535.*

17. $x^2 - x - 12 = 0$ **18.** $x^2 + 6x + 9 = 0$ **19.** $x^2 + 4x - 3 = 0$
20. $2x^2 - 5x + 4 = 0$ **21.** $x^2 - 10x = -21$ **22.** $6x^2 - 13x = 15$

10-3 Solving Quadratic Equations by Completing the Square

See pages 539–544.

Concept Summary

- Complete the square to make a quadratic expression a perfect square.
- Use the following steps to complete the square of $x^2 + bx$.

Step 1 Find $\frac{1}{2}$ of b, the coefficient of x.

Step 2 Square the result of Step 1.

Step 3 Add the result of Step 2 to $x^2 + bx$, the original expression.

Example **Solve $y^2 + 6y + 2 = 0$ by completing the square. Round to the nearest tenth if necessary.**

$y^2 + 6y + 2 = 0$	Original equation
$y^2 + 6y + 2 - 2 = 0 - 2$	Subtract 2 from each side.
$y^2 + 6y = -2$	Simplify.
$y^2 + 6y + 9 = -2 + 9$	Since $\left(\frac{6}{2}\right)^2 = 9$, add 9 to each side.
$(y + 3)^2 = 7$	Factor $y^2 + 6y + 9$.
$y + 3 = \pm\sqrt{7}$	Take the square root of each side.
$y + 3 - 3 = \pm\sqrt{7} - 3$	Subtract 3 from each side.
$y = -3 \pm \sqrt{7}$	Simplify.

Use a calculator to evaluate each value of y.

$y = -3 + \sqrt{7}$ or $y = -3 - \sqrt{7}$

$y \approx -0.4$ $\quad$ $y \approx -5.6$

The solution set is $\{-5.6, -0.4\}$.

Exercises **Solve each equation by completing the square. Round to the nearest tenth if necessary.** *See Example 3 on pages 540 and 541.*

23. $-3x^2 + 4 = 0$ **24.** $x^2 - 16x + 32 = 0$ **25.** $m^2 - 7m = 5$

26. $4a^2 + 16a + 15 = 0$ **27.** $\frac{1}{2}y^2 + 2y - 1 = 0$ **28.** $n^2 - 3n + \frac{5}{4} = 0$

10-4 Solving Quadratic Equations by Using the Quadratic Formula

See pages 546–552.

Concept Summary

- The solutions of a quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example **Solve $2x^2 + 7x - 15 = 0$ by using the Quadratic Formula.**

For this equation, $a = 2$, $b = 7$, and $c = -15$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Quadratic Formula
$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-15)}}{2(2)}$	$a = 2$, $b = 7$, and $c = -15$
$x = \frac{-7 \pm \sqrt{169}}{4}$	Simplify.

$x = \frac{-7 + 13}{4}$ or $x = \frac{-7 - 13}{4}$

$x = 1\frac{1}{2}$ $\quad$ $x = -5$ $\quad$ The solution set is $\left\{-5, 1\frac{1}{2}\right\}$.

Exercises **Solve each equation by using the Quadratic Formula. Round to the nearest tenth if necessary.** *See Examples 1 and 2 on pages 546 and 547.*

29. $x^2 - 8x = 20$ **30.** $r^2 + 10r + 9 = 0$ **31.** $4p^2 + 4p = 15$
32. $2y^2 + 3 = -8y$ **33.** $2d^2 + 8d + 3 = 3$ **34.** $21a^2 + 5a - 7 = 0$

10-5 Exponential Functions

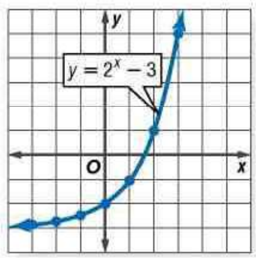
See pages 554–560.

Concept Summary

- An exponential function is a function that can be described by the equation of the form $y = a^x$, where $a > 0$ and $a \neq 1$.

Example **Graph $y = 2^x - 3$. State the y-intercept.**

x	y
−3	−2.875
−2	−2.75
−1	−2.5
0	−2
1	−1
2	1
3	5



Graph the ordered pairs and connect the points with a smooth curve. The y-intercept is -2.

Exercises **Graph each function. State the y-intercept.**
See Examples 1 and 2 on page 555.

35. $y = 3^x + 6$ **36.** $y = 3^{x+2}$ **37.** $y = 2\left(\frac{1}{2}\right)^x$

10-6 Growth and Decay

See pages 561–565.

Concept Summary

- Exponential Growth: $y = C(1 + r)^t$, where y represents the final amount, C represents the initial amount, r represents the rate of change expressed as a decimal, and t represents time.
- Compound Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A represents the amount of the investment, P represents the principal, r represents the annual rate of interest expressed as a decimal, n represents the number of times that the interest is compounded each year, and t represents the number of years that the money is invested.
- Exponential Decay: $y = C(1 - r)^t$, where y represents the final amount, C represents the initial amount, r represents the rate of decay expressed as a decimal, and t represents time.

Chapter 10 For More ...
• Extra Practice, see pages 841–843.
• Mixed Problem Solving, see page 862.

Example **Find the final amount of an investment if \$1500 is invested at an interest rate of 7.5% compounded quarterly for 10 years.**

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compound interest equation

$A = 1500\left(1 + \frac{0.075}{4}\right)^{4 \cdot 10}$ $P = 1500$, $r = 7.5\%$ or 0.075, $n = 4$, and $t = 10$

$A \approx 3153.52$ Simplify.

The final amount in the account is about \$3153.52.

Exercises **Determine the final amount for each investment.**
See Example 2 on page 562.

	Principal	Annual Interest Rate	Time	Type of Compounding
38.	\$2000	8%	8 years	quarterly
39.	\$5500	5.25%	15 years	monthly
40.	\$15,000	7.5%	25 years	monthly
41.	\$500	9.75%	40 years	daily

10-7 Geometric Sequences

See pages 567–572.

Concept Summary

- A geometric sequence is a sequence in which each term after the nonzero first term is found by multiplying the previous term by a constant called the common ratio r, where $r \neq 0$ or 1.
- The nth term a_n of a geometric sequence with the first term a_1, and a common ratio r is given by $a_n = a_1 \cdot r^{n-1}$.

Example **Find the next three terms in the geometric sequence 7.5, 15, 30, … .**

$\frac{15}{7.5} = 2$ Divide the second term by the first.

The common ratio is 2. Find the next three terms.

7.5, 15, 30, 60, 120, 240

×2 ×2 ×2

The next three terms are 60, 120, and 240.

Exercises **Find the nth term of each geometric sequence.**
See Example 4 on page 569.

42. $a_1 = 2, n = 5, r = 2$ **43.** $a_1 = 7, n = 4, r = \frac{2}{3}$ **44.** $a_1 = 243, n = 5, r = -\frac{1}{3}$

Find the geometric means in each sequence. See Example 5 on page 570.

45. 5, ____, 20 **46.** −12, ____, −48 **47.** 1, ____, $\frac{1}{4}$

Chapter 10 Practice Test

Vocabulary and Concepts

Choose the letter of the term that matches each formula.

1. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
2. $y = C(1 + r)^t$
3. $y = C(1 - r)^t$

a. exponential decay equation
b. exponential growth equation
c. Quadratic Formula

Skills and Applications

Write the equation of the axis of symmetry, and find the coordinates of the vertex of the graph of each function. Identify the vertex as a maximum or minimum. Then graph the function.

4. $y = x^2 - 4x + 13$
5. $y = -3x^2 - 6x + 4$
6. $y = 2x^2 + 3$
7. $y = -1(x - 2)^2 + 1$

Solve each equation by graphing. If integral roots cannot be found, estimate the roots by stating the consecutive integers between which the roots lie.

8. $x^2 - 2x + 2 = 0$
9. $x^2 + 6x = -7$
10. $x^2 + 24x + 144 = 0$
11. $2x^2 - 8x = 42$

Solve each equation. Round to the nearest tenth if necessary.

12. $x^2 + 7x + 6 = 0$
13. $2x^2 - 5x - 12 = 0$
14. $6n^2 + 7n = 20$
15. $3k^2 + 2k = 5$
16. $y^2 - \frac{3}{5}y + \frac{2}{25} = 0$
17. $-3x^2 + 5 = 14x$
18. $z^2 - 13z = 32$
19. $3x^2 + 4a = 8$
20. $7m^2 = m + 5$

Graph each function. State the *y*-intercept.

21. $y = \left(\frac{1}{2}\right)^x$
22. $y = 4 \cdot 2^x$
23. $y = \left(\frac{1}{3}\right)^x - 3$

Find the *n*th term of each geometric sequence.

24. $a_1 = 12, n = 6, r = 2$
25. $a_1 = 20, n = 4, r = 3$

Find the geometric means in each sequence.

26. 7, ____, 63
27. $-\frac{1}{3}$, ____, -12

28. **CARS** Ley needs to replace her car. If she leases a car, she will pay $410 a month for 2 years and then has the option to buy the car for $14,458. The current price of the car is $17,369. If the car depreciates at 16% per year, how will the depreciated price compare with the buyout price of the lease?

29. **FINANCE** Find the total amount after $1500 is invested for 10 years at a rate of 6%, compounded quarterly.

30. **STANDARDIZED TEST PRACTICE** Which value is the next value in the pattern $-4, 12, -36, 108, \ldots$?

(A) -324 (B) 324 (C) -432 (D) 432

Chapter 10 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

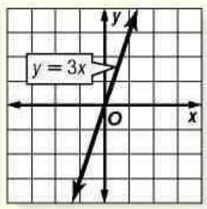
1. The graph of $y = 3x$ is shown. If the line is translated 2 units down, which equation will describe the new line? (Lesson 4-2)

Ⓐ $y = -6x$

Ⓑ $y = 3x - 2$

Ⓒ $y = 3x + 2$

Ⓓ $y = 3(x - 2)$



2. Suppose a varies directly as b, and $a = 21$ when $b = 6$. Find a when $b = 28$. (Lesson 5-2)

Ⓐ 4.5 Ⓑ 8

Ⓒ 98 Ⓓ 126

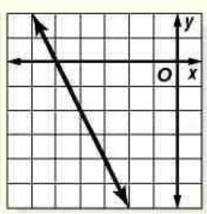
3. Which equation is represented by the graph? (Lesson 5-5)

Ⓐ $y = -2x - 10$

Ⓑ $y = -2x - 5$

Ⓒ $y = 2x + 10$

Ⓓ $y = 2x - 5$



4. At a farm market, apples cost 20¢ each and grapefruit cost 25¢ each. A shopper bought twice as many apples as grapefruit and spent a total of $1.95. How many apples did he buy? (Lesson 7-2)

Ⓐ 3 Ⓑ 4 Ⓒ 5 Ⓓ 6

5. A rectangle has a length of $2x + 3$ and a width of $2x - 6$. Which expression describes the area of the rectangle? (Lesson 8-7)

Ⓐ $4x - 3$ Ⓑ $4x^2 - 18$

Ⓒ $4x^2 - 6x - 18$ Ⓓ $4x^2 + 18x - 18$

6. The solution set for the equation $x^2 + x - 12 = 0$ is (Lesson 9-3)

Ⓐ $\{-4, -3\}$. Ⓑ $\{-4, 3\}$.

Ⓒ $\{4, -3\}$. Ⓓ $\{4, 3\}$.

7. Which equation best represents the data in the table? (Lesson 10-1)

x	y
−3	0
−1	8
0	9
2	5
3	0
4	−7

Ⓐ $y = -x^2 + 3$ Ⓑ $y = -x^2 + 9$

Ⓒ $y = x^2 - 3$ Ⓓ $y = x^2 + 9$

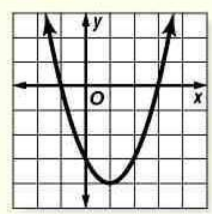
8. Which equation best represents the parabola graphed below? (Lesson 10-1)

Ⓐ $y = x^2 - 2x - 4$

Ⓑ $y = x^2 - 2x - 3$

Ⓒ $y = x^2 + 2x - 3$

Ⓓ $y = x^2 + 2x + 3$



9. At which points does the graph of $f(x) = 2x^2 + 8x + 6$ intersect the x-axis? (Lesson 10-2)

Ⓐ $(-3, 0)$ and $(-2, 0)$

Ⓑ $(-3, 0)$ and $(-1, 0)$

Ⓒ $(1, 0)$ and $(3, 0)$

Ⓓ $(2, 0)$ and $(3, 0)$

The Princeton Review **Test-Taking Tip**

Questions 1 and 7 Sketching the graph of a function or a transformation may help you see which answer choice is correct.

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

10. Monica earned \$18.50, \$23.00, and \$15.00 mowing lawns for 3 consecutive weeks. She wanted to earn an average of at least \$18 per week. What is the minimum she should earn during the 4th week to meet her goal? (Lesson 3-4)

11. Write an equation in slope-intercept form of the line that is perpendicular to the line represented by $8x - 4y + 9 = 0$ and passes through the point at $(2, 3)$. (Lesson 5-6)

12. If $5a + 4b = 25$ and $3a - 8b = 41$, solve for a and b. (Lesson 7-4)

13. Complete the square of $x^2 + 4x - 5$ by finding numbers h and k such that $x^2 + 4x - 5 = (x + h)^2 + k$. (Lesson 10-2)

14. At how many points does the graph of $y = 6x^2 + 11x + 4$ intersect the x-axis? (Lesson 10-3)

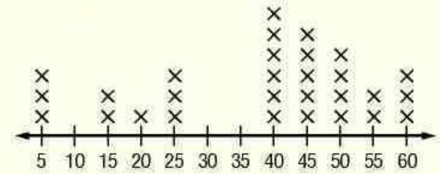
15. The length and width of a rectangle that measures 8 inches by 6 inches are both increased by the same amount. The area of the larger rectangle is twice the area of the original rectangle. How much was added to each dimension of the original rectangle? Round to the nearest hundredth of an inch. (Lesson 10-4)

Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

16.



Column A	Column B
the mean of the data in the line plot	the median of the data in the line plot

(Lesson 2-5)

17.

Column A	Column B
the solution of $-6p = -12$	the solution of $10q = 5$

(Lesson 3-3)

18.

Column A	Column B
5.3×10^3	53,000

(Lesson 8-3)

19.

Column A	Column B
the 14th term of $-2, -4, -8, \ldots$	the 14th term of $2, -4, 8, \ldots$

(Lesson 10-7)

Part 4 Open Ended

Record your answers on a sheet of paper. Show your work.

20. Analyze the graph of $y = -4x^2 + 8x - \frac{15}{4}$. (Lessons 10-1, 10-3)

a. Show that the equation $-4x^2 + 8x - \frac{15}{4} = -4(x - 1)^2 + \frac{1}{4}$ is always true by expanding the right side.

b. Find the equation of the axis of symmetry of the graph of $y = -4x^2 + 8x - \frac{15}{4}$.

c. Does the parabola open upward or downward? Explain how you determined this.

d. Find the values of x, if any, where the graph crosses the x-axis. Write as rational numbers.

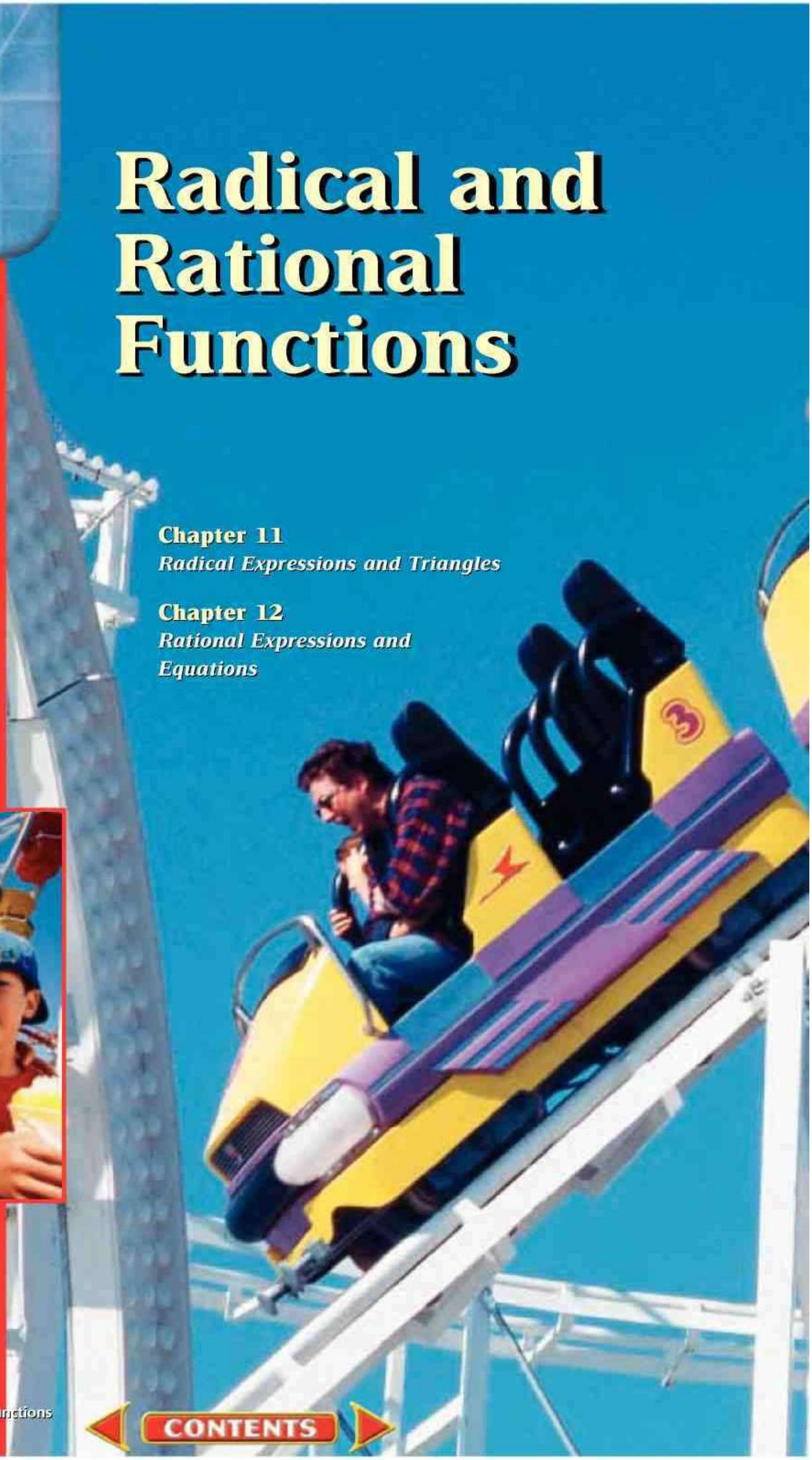
e. Find the coordinates of the maximum or minimum point on this parabola.

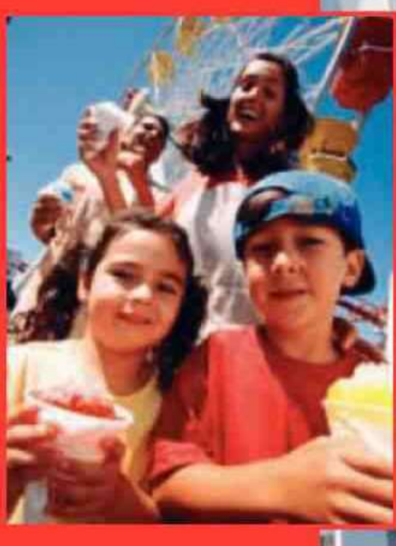
f. Sketch the graph of the equation. Label the maximum or minimum point and the roots.

UNIT 4

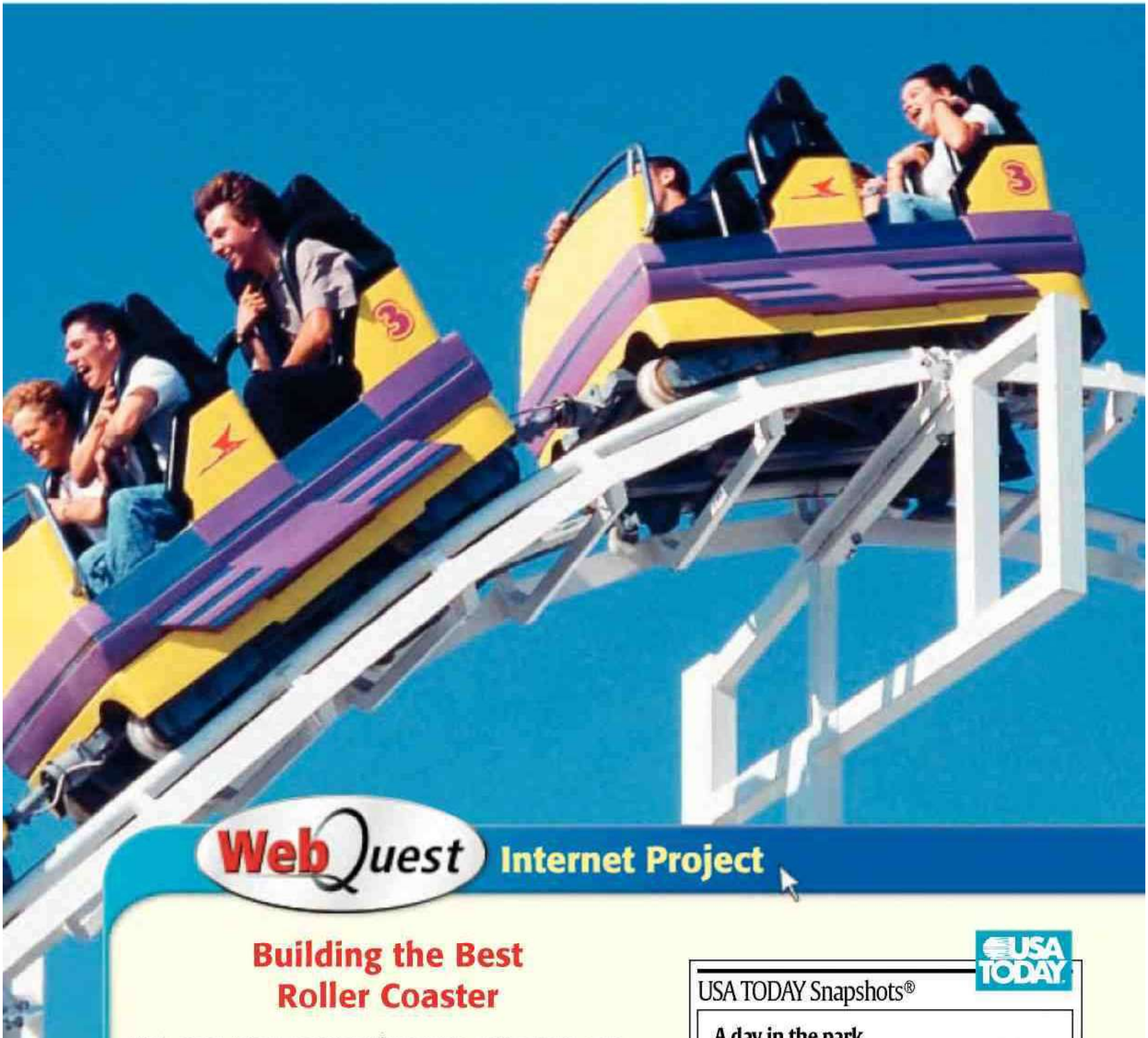
Radical and Rational Functions

Nonlinear functions such as radical and rational functions can be used to model real-world situations such as the speed of a roller coaster. In this unit, you will learn about radical and rational functions.





CONTENTS



WebQuest Internet Project

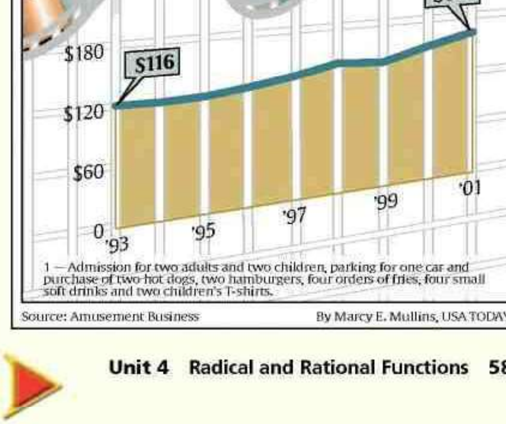
Building the Best Roller Coaster

Each year, amusement park owners compete to earn part of the billions of dollars Americans spend at amusement parks. Often the parks draw customers with new taller and faster roller coasters. In this project, you will explore how radical and rational functions are related to buying and building a new roller coaster.

Log on to www.algebra1.com/webquest. Begin your WebQuest by reading the Task.

Then continue working on your WebQuest as you study Unit 4.

Lesson	11-1	12-2
Page	590	652



Chapter 11 Radical Expressions and Triangles

What You'll Learn

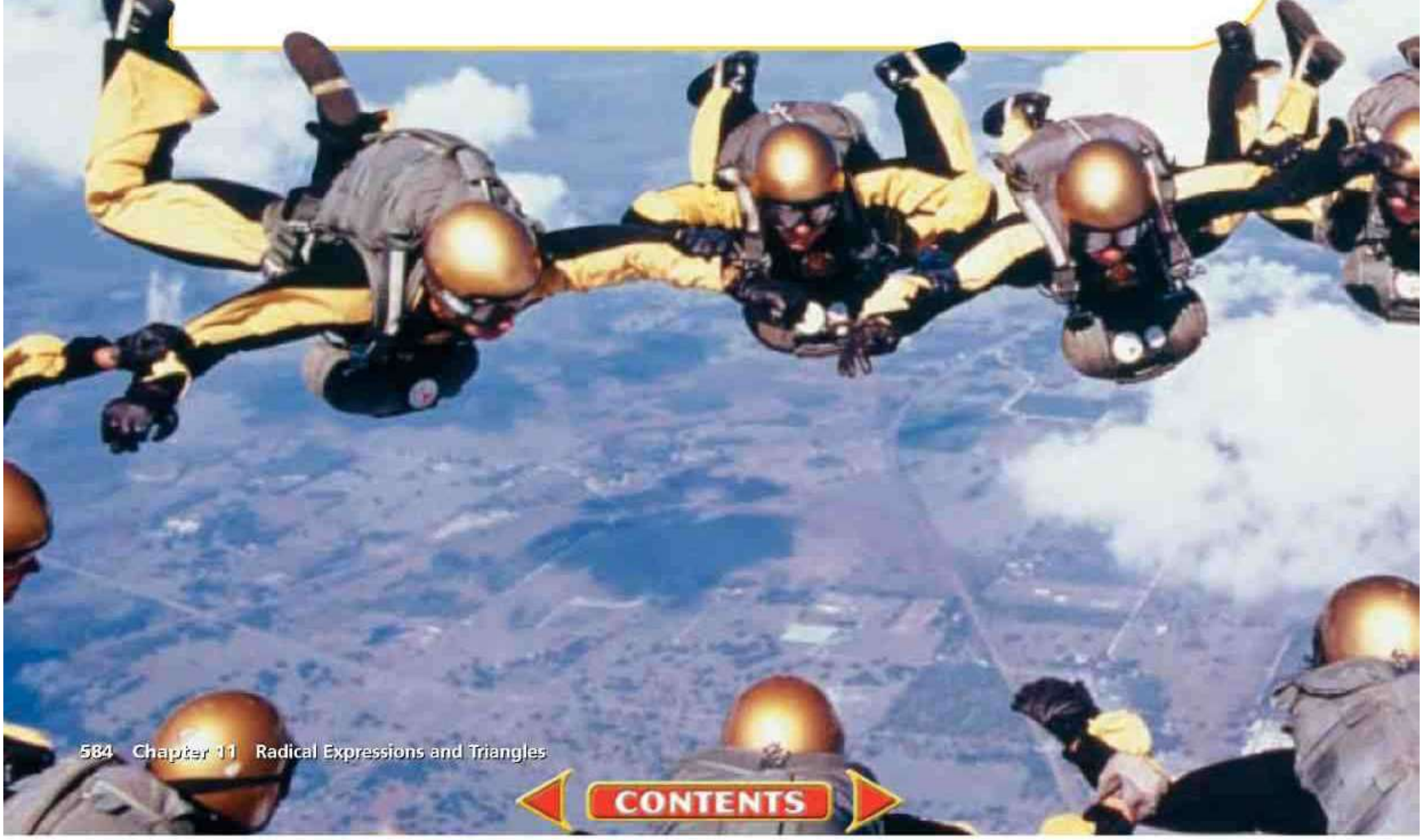
- **Lessons 11-1 and 11-2** Simplify and perform operations with radical expressions.
- **Lesson 11-3** Solve radical equations.
- **Lessons 11-4 and 11-5** Use the Pythagorean Theorem and Distance Formula.
- **Lessons 11-6 and 11-7** Use similar triangles and trigonometric ratios.

Key Vocabulary

- radical expression (p. 586)
- radical equation (p. 598)
- Pythagorean Theorem (p. 605)
- Distance Formula (p. 611)
- trigonometric ratios (p. 623)

Why It's Important

Physics problems are among the many applications of radical equations. Formulas that contain the value for the acceleration due to gravity, such as free-fall times, escape velocities, and the speeds of roller coasters, can all be written as radical equations. *You will learn how to calculate the time it takes for a skydiver to fall a given distance in Lesson 11-3.*



Getting Started

Prerequisite Skills To be successful in this chapter, you'll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 11.

For Lessons 11-1 and 11-4 — Find Square Roots

Find each square root. If necessary, round to the nearest hundredth.
(For review, see Lesson 2-7.)

1. $\sqrt{25}$ **2.** $\sqrt{80}$ **3.** $\sqrt{56}$ **4.** $\sqrt{324}$

For Lesson 11-2 — Combine Like Terms

Simplify each expression. *(For review, see Lesson 1-6.)*

5. $3a + 7b - 2a$ **6.** $14x - 6y + 2y$

7. $(10c - 5d) + (6c + 5d)$ **8.** $(21m + 15n) - (9n - 4m)$

For Lesson 11-3 — Solve Quadratic Equations

Solve each equation. *(For review, see Lesson 9-3.)*

9. $x(x - 5) = 0$ **10.** $x^2 + 10x + 24 = 0$

11. $x^2 - 6x - 27 = 0$ **12.** $2x^2 + x + 1 = 2$

For Lesson 11-6 — Proportions

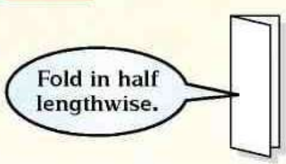
Use cross products to determine whether each pair of ratios forms a proportion. Write *yes* or *no*. *(For review, see Lesson 3-6.)*

13. $\frac{2}{3}, \frac{8}{12}$ **14.** $\frac{4}{5}, \frac{16}{25}$ **15.** $\frac{8}{10}, \frac{12}{16}$ **16.** $\frac{6}{30}, \frac{3}{15}$

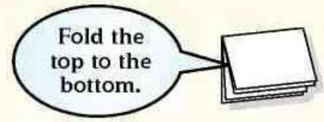
FOLDABLES™ Study Organizer

Make this Foldable to help you organize information about radical expressions and equations. Begin with a sheet of plain $8\frac{1}{2}$" by 11" paper.

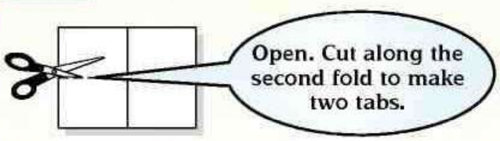
Step 1 Fold in Half



Step 2 Fold Again



Step 3 Cut



Step 4 Label

Reading and Writing As you read and study the chapter, write notes and examples for each lesson under each tab.

11-1 Simplifying Radical Expressions

What You'll Learn

- Simplify radical expressions using the Product Property of Square Roots.
- Simplify radical expressions using the Quotient Property of Square Roots.

Vocabulary

- radical expression
- radicand
- rationalizing the denominator
- conjugate

How are radical expressions used in space exploration?

A spacecraft leaving Earth must have a velocity of at least 11.2 kilometers per second (25,000 miles per hour) to enter into orbit. This velocity is called the *escape velocity*. The escape velocity of an object is given by the radical expression $\sqrt{\frac{2GM}{R}}$, where G is the gravitational constant, M is the mass of the planet or star, and R is the radius of the planet or star. Once values are substituted for the variables, the formula can be simplified.

PRODUCT PROPERTY OF SQUARE ROOTS A **radical expression** is an expression that contains a square root. A **radicand**, the expression under the radical sign, is in simplest form if it contains no perfect square factors other than 1. The following property can be used to simplify square roots.

Key Concept — Product Property of Square Roots

- **Words** For any numbers a and b, where $a \geq 0$ and $b \geq 0$, the square root of a product is equal to the product of each square root.
- **Symbols** $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$
- **Example** $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$

The Product Property of Square Roots and prime factorization can be used to simplify radical expressions in which the radicand is not a perfect square.

Example 1 Simplify Square Roots

Simplify.

a. $\sqrt{12}$

$\sqrt{12} = \sqrt{2 \cdot 2 \cdot 3}$ Prime factorization of 12

$= \sqrt{2^2} \cdot \sqrt{3}$ Product Property of Square Roots

$= 2\sqrt{3}$ Simplify.

b. $\sqrt{90}$

$\sqrt{90} = \sqrt{2 \cdot 3 \cdot 3 \cdot 5}$ Prime factorization of 90

$= \sqrt{3^2} \cdot \sqrt{2 \cdot 5}$ Product Property of Square Roots

$= 3\sqrt{10}$ Simplify.

Study Tip

Reading Math
$2\sqrt{3}$ is read *two times the square root of 3* or *two radical three.*

The Product Property can also be used to multiply square roots.

Study Tip

Alternative Method
To find $\sqrt{3} \cdot \sqrt{15}$, you could multiply first and then use the prime factorization.

$$\sqrt{3} \cdot \sqrt{15} = \sqrt{45} = \sqrt{3^2} \cdot \sqrt{5} = 3\sqrt{5}$$

Example 2 Multiply Square Roots

Find $\sqrt{3} \cdot \sqrt{15}$.

$\sqrt{3} \cdot \sqrt{15} = \sqrt{3} \cdot \sqrt{3} \cdot \sqrt{5}$ Product Property of Square Roots

$= \sqrt{3^2} \cdot \sqrt{5}$ Product Property

$= 3\sqrt{5}$ Simplify.

When finding the principal square root of an expression containing variables, be sure that the result is not negative. Consider the expression $\sqrt{x^2}$. It may seem that $\sqrt{x^2} = x$. Let's look at $x = -2$.

$\sqrt{x^2} \stackrel{?}{=} x$

$\sqrt{(-2)^2} \stackrel{?}{=} -2$ Replace x with -2.

$\sqrt{4} \stackrel{?}{=} -2$ $(-2)^2 = 4$

$2 \neq -2$ $\sqrt{4} = 2$

For radical expressions where the exponent of the variable inside the radical is *even* and the resulting simplified exponent is odd, you must use absolute value to ensure nonnegative results.

$\sqrt{x^2} = |x|$ $\sqrt{x^3} = x\sqrt{x}$ $\sqrt{x^4} = x^2$ $\sqrt{x^5} = x^2\sqrt{x}$ $\sqrt{x^6} = |x^3|$

Example 3 Simplify a Square Root with Variables

Simplify $\sqrt{40x^4y^5z^3}$.

$\sqrt{40x^4y^5z^3} = \sqrt{2^3 \cdot 5 \cdot x^4 \cdot y^5 \cdot z^3}$ Prime factorization

$= \sqrt{2^2} \cdot \sqrt{2} \cdot \sqrt{5} \cdot \sqrt{x^4} \cdot \sqrt{y^4} \cdot \sqrt{y} \cdot \sqrt{z^2} \cdot \sqrt{z}$ Product Property

$= 2 \cdot \sqrt{2} \cdot \sqrt{5} \cdot x^2 \cdot y^2 \cdot \sqrt{y} \cdot |z| \cdot \sqrt{z}$ Simplify.

$= 2x^2y^2|z|\sqrt{10yz}$ The absolute value of z ensures a nonnegative result.

QUOTIENT PROPERTY OF SQUARE ROOTS You can divide square roots and simplify radical expressions that involve division by using the Quotient Property of Square Roots.

Key Concept — Quotient Property of Square Roots

- **Words** For any numbers a and b, where $a \geq 0$ and $b > 0$, the square root of a quotient is equal to the quotient of each square root.
- **Symbols** $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$
- **Example** $\sqrt{\dfrac{49}{4}} = \dfrac{\sqrt{49}}{\sqrt{4}}$

Study Tip

Look Back
To review the **Quadratic Formula**, see Lesson 10-4.

You can use the Quotient Property of Square Roots to derive the Quadratic Formula by solving the quadratic equation $ax^2 + bx + c = 0$.

$ax^2 + bx + c = 0$ Original equation

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ Divide each side by a, $a \neq 0$.

(continued on the next page)

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$ Subtract $\frac{c}{a}$ from each side.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$ Complete the square; $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$ Factor $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$.

$$\left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$ Take the square root of each side.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$ Remove the absolute value symbols and insert $\pm$.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$ Quotient Property of Square Roots

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$ $\sqrt{4a^2} = 2a$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Subtract $\frac{b}{2a}$ from each side.

Study Tip

Plus or Minus Symbol

The $\pm$ symbol is used with the radical expression since both square roots lead to solutions.

Thus, we have derived the Quadratic Formula.

A fraction containing radicals is in simplest form if no prime factors appear under the radical sign with an exponent greater than 1 and if no radicals are left in the denominator. **Rationalizing the denominator** of a radical expression is a method used to eliminate radicals from the denominator of a fraction.

Example 4 Rationalizing the Denominator

Simplify.

a. $\frac{\sqrt{10}}{\sqrt{3}}$

$$\frac{\sqrt{10}}{\sqrt{3}} = \frac{\sqrt{10}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ Multiply by $\frac{\sqrt{3}}{\sqrt{3}}$.

$$= \frac{\sqrt{30}}{3}$$ Product Property of Square Roots

b. $\frac{\sqrt{7x}}{\sqrt{8}}$

$$\frac{\sqrt{7x}}{\sqrt{8}} = \frac{\sqrt{7x}}{\sqrt{2 \cdot 2 \cdot 2}}$$ Prime factorization

$$= \frac{\sqrt{7x}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$ Multiply by $\frac{\sqrt{2}}{\sqrt{2}}$.

$$= \frac{\sqrt{14x}}{4}$$ Product Property of Square Roots

c. $\frac{\sqrt{2}}{\sqrt{6}}$

$$\frac{\sqrt{2}}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}$$ Multiply by $\frac{\sqrt{6}}{\sqrt{6}}$.

$$= \frac{\sqrt{12}}{6}$$ Product Property of Square Roots

$$= \frac{\sqrt{2 \cdot 2 \cdot 3}}{6}$$ Prime factorization

$$= \frac{2\sqrt{3}}{6}$$ $\sqrt{2^2} = 2$

$$= \frac{\sqrt{3}}{3}$$ Divide the numerator and denominator by 2.

Binomials of the form $p\sqrt{q} + r\sqrt{s}$ and $p\sqrt{q} - r\sqrt{s}$ are called **conjugates**. For example, $3 + \sqrt{2}$ and $3 - \sqrt{2}$ are conjugates. Conjugates are useful when simplifying radical expressions because if p, q, r, and s are rational numbers, their product is always a rational number with no radicals. Use the pattern $(a - b)(a + b) = a^2 - b^2$ to find their product.

$$(3 + \sqrt{2})(3 - \sqrt{2}) = 3^2 - (\sqrt{2})^2 \quad a = 3, b = \sqrt{2}$$
$$= 9 - 2 \text{ or } 7 \quad (\sqrt{2})^2 = \sqrt{2} \cdot \sqrt{2} \text{ or } 2$$

Example 5 Use Conjugates to Rationalize a Denominator

Simplify $\frac{2}{6 - \sqrt{3}}$.

$$\frac{2}{6-\sqrt{3}} = \frac{2}{6-\sqrt{3}} \cdot \frac{6+\sqrt{3}}{6+\sqrt{3}} \quad \frac{6+\sqrt{3}}{6+\sqrt{3}} = 1$$

$$= \frac{2(6+\sqrt{3})}{6^2 - (\sqrt{3})^2} \quad (a-b)(a+b) = a^2 - b^2$$

$$= \frac{12 + 2\sqrt{3}}{36 - 3} \quad (\sqrt{3})^2 = 3$$

$$= \frac{12 + 2\sqrt{3}}{33} \quad \text{Simplify.}$$

When simplifying radical expressions, check the following conditions to determine if the expression is in simplest form.

Concept Summary — Simplest Radical Form

A radical expression is in simplest form when the following three conditions have been met.

1. No radicands have perfect square factors other than 1.
2. No radicands contain fractions.
3. No radicals appear in the denominator of a fraction.

Check for Understanding

Concept Check

1. **Explain** why absolute value is not necessary for $\sqrt{x^4} = x^2$.
2. **Show** that $\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}$ for $a > 0$.
3. **OPEN ENDED** Give an example of a binomial in the form $a\sqrt{b} + c\sqrt{d}$ and its conjugate. Then find their product.

Guided Practice

Simplify.

4. $\sqrt{20}$
5. $\sqrt{2} \cdot \sqrt{8}$
6. $3\sqrt{10} \cdot 4\sqrt{10}$
7. $\sqrt{54a^2b^2}$
8. $\sqrt{60x^5y^6}$
9. $\frac{4}{\sqrt{6}}$
10. $\sqrt{\frac{3}{10}}$
11. $\frac{8}{3 - \sqrt{2}}$
12. $\frac{2\sqrt{5}}{-4 + \sqrt{8}}$

Applications

13. **GEOMETRY** A square has sides each measuring $2\sqrt{7}$ feet. Determine the area of the square.

14. **PHYSICS** The period of a pendulum is the time required for it to make one complete swing back and forth. The formula of the period P of a pendulum is $P = 2\pi\sqrt{\frac{\ell}{32}}$, where ℓ is the length of the pendulum in feet. If a pendulum in a clock tower is 8 feet long, find the period. Use 3.14 for π.

Practice and Apply

Homework Help

For Exercises	See Examples
15–18, 41, 44–46	1
19–22, 39, 40, 48, 49	2
23–26	3
27–32, 42, 43, 47	4
33–38	5

Extra Practice

See page 844.

Simplify.

15. $\sqrt{18}$
16. $\sqrt{24}$
17. $\sqrt{80}$
18. $\sqrt{75}$
19. $\sqrt{5} \cdot \sqrt{6}$
20. $\sqrt{3} \cdot \sqrt{8}$
21. $7\sqrt{30} \cdot 2\sqrt{6}$
22. $2\sqrt{3} \cdot 5\sqrt{27}$
23. $\sqrt{40a^4}$
24. $\sqrt{50m^3n^5}$
25. $\sqrt{147x^6y^7}$
26. $\sqrt{72x^3y^4z^5}$
27. $\sqrt{\frac{2}{7}} \cdot \sqrt{\frac{7}{3}}$
28. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{6}{4}}$
29. $\sqrt{\frac{t}{8}}$
30. $\sqrt{\frac{27}{p^2}}$
31. $\sqrt{\frac{5c^5}{4d^5}}$
32. $\frac{\sqrt{9x^5y}}{\sqrt{12x^2y^6}}$
33. $\frac{18}{6 - \sqrt{2}}$
34. $\frac{2\sqrt{5}}{-4 + \sqrt{8}}$
35. $\frac{10}{\sqrt{7} + \sqrt{2}}$
36. $\frac{2}{\sqrt{3} + \sqrt{6}}$
37. $\frac{4}{4 - 3\sqrt{3}}$
38. $\frac{3\sqrt{7}}{5\sqrt{3} + 3\sqrt{5}}$

39. **GEOMETRY** A rectangle has width $3\sqrt{5}$ centimeters and length $4\sqrt{10}$ centimeters. Find the area of the rectangle.

40. **GEOMETRY** A rectangle has length $\sqrt{\frac{a}{8}}$ meters and width $\sqrt{\frac{a}{2}}$ meters. What is the area of the rectangle?

41. **GEOMETRY** The formula for the area A of a square with side length s is $A = s^2$. Solve this equation for s, and find the side length of a square having an area of 72 square inches.

PHYSICS For Exercises 42 and 43, use the following information.

The formula for the kinetic energy of a moving object is $E = \frac{1}{2}mv^2$, where E is the kinetic energy in joules, m is the mass in kilograms, and v is the velocity in meters per second.

42. Solve the equation for v.

43. Find the velocity of an object whose mass is 0.6 kilogram and whose kinetic energy is 54 joules.

44. **SPACE EXPLORATION** Refer to the application at the beginning of the lesson. Find the escape velocity for the Moon in kilometers per second if $G = \frac{6.7 \times 10^{-20} \text{ km}}{s^2 \text{ kg}}$, $M = 7.4 \times 10^{22}$ kg, and $R = 1.7 \times 10^3$ km. How does this compare to the escape velocity for Earth?

WebQuest

The speed of a roller coaster can be determined by evaluating a radical expression. Visit www.algebra1.com/webquest to continue work on your WebQuest project.

Career Choices

Insurance Investigator

Insurance investigators decide whether claims are covered by the customer's policy, assess the amount of loss, and investigate the circumstances of a claim.

Online Research For more information about a career as an insurance investigator, visit: www.algebra1.com/careers

Source: U.S. Department of Labor

INVESTIGATION **For Exercises 45–47, use the following information.**
Police officers can use the formula $s = \sqrt{30fd}$ to determine the speed s that a car was traveling in miles per hour by measuring the distance d in feet of its skid marks. In this formula, f is the coefficient of friction for the type and condition of the road.

45. Write a simplified expression for the speed if $f = 0.6$ for a wet asphalt road.

46. What is a simplified expression for the speed if $f = 0.8$ for a dry asphalt road?

47. An officer measures skid marks that are 110 feet long. Determine the speed of the car for both wet road conditions and for dry road conditions.

GEOMETRY **For Exercises 48 and 49, use the following information.**
Hero's Formula can be used to calculate the area A of a triangle given the three side lengths a, b, and c.

$$A = \sqrt{s(s - a)(s - b)(s - c)}, \text{ where } s = \frac{1}{2}(a + b + c)$$

48. Find the value of s if the side lengths of a triangle are 13, 10, and 7 feet.

49. Determine the area of the triangle.

50. **CRITICAL THINKING** Simplify $\dfrac{1}{a - 1 + \sqrt{a}}$.

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are radical expressions used in space exploration?

Include the following in your answer:

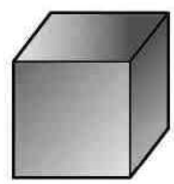
- an explanation of how you could determine the escape velocity for a planet and why you would need this information before you landed on the planet, and
- a comparison of the escape velocity for two astronomical bodies with the same mass, but different radii.

Standardized Test Practice

52. If the cube has a surface area of $96a^2$, what is its volume?

Ⓐ $32a^3$ Ⓑ $48a^3$

Ⓒ $64a^3$ Ⓓ $96a^3$



Surface area of a cube $= 6s^2$

53. If $x = 81b^2$ and $b > 0$, then $\sqrt{x} =$

Ⓐ $-9b$. Ⓑ $9b$.

Ⓒ $3b\sqrt{27}$. Ⓓ $27b\sqrt{3}$.

Graphing Calculator

WEATHER **For Exercises 54 and 55, use the following information.**
The formula $y = 91.4 - (91.4 - t)[0.478 + 0.301(\sqrt{x} - 0.02)]$ can be used to find the windchill factor. In this formula, y represents the windchill factor, t represents the air temperature in degrees Fahrenheit, and x represents the wind speed in miles per hour. Suppose the air temperature is 12°.

54. Use a graphing calculator to find the wind speed to the nearest mile per hour if it feels like −9° with the windchill factor.

55. What does it feel like to the nearest degree if the wind speed is 4 miles per hour?

Extending the Lesson Radical expressions can be represented with fractional exponents. For example, $x^{\frac{1}{2}} = \sqrt{x}$. Using the properties of exponents, simplify each expression.

56. $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$

57. $\left(x^{\frac{1}{2}}\right)^4$

58. $\dfrac{x^{\frac{5}{2}}}{x}$

59. Simplify the expression $\dfrac{\sqrt{a}}{a\sqrt[3]{a}}$.

60. Solve the equation $|y^3| = \dfrac{1}{3\sqrt{3}}$ for y.

61. Write $\left(s^2 t^{\frac{1}{2}}\right)^8 \sqrt{s^5 t^4}$ in simplest form.

Maintain Your Skills

Mixed Review Find the next three terms in each geometric sequence. *(Lesson 10-7)*

62. 2, 6, 18, 54

63. 1, −2, 4, −8

64. 384, 192, 96, 48

65. $\frac{1}{9}, \frac{2}{3}, 4, 24$

66. $3, \frac{3}{4}, \frac{3}{16}, \frac{3}{64}$

67. 50, 10, 2, 0.4

68. BIOLOGY A certain type of bacteria, if left alone, doubles its number every 2 hours. If there are 1000 bacteria at a certain point in time, how many bacteria will there be 24 hours later? *(Lesson 10-6)*

69. PHYSICS According to Newton's Law of Cooling, the difference between the temperature of an object and its surroundings decreases in time exponentially. Suppose a cup of coffee is 95°C and it is in a room that is 20°C. The cooling of the coffee can be modeled by the equation $y = 75(0.875)^t$, where y is the temperature difference and t is the time in minutes. Find the temperature of the coffee after 15 minutes. *(Lesson 10-6)*

Factor each trinomial, if possible. If the trinomial cannot be factored using integers, write *prime*. *(Lesson 9-4)*

70. $6x^2 + 7x - 5$

71. $35x^2 - 43x + 12$

72. $5x^2 + 3x + 31$

73. $3x^2 - 6x - 105$

74. $4x^2 - 12x + 15$

75. $8x^2 - 10x + 3$

Find the solution set for each equation, given the replacement set. *(Lesson 4-4)*

76. $y = 3x + 2$; {(1, 5), (2, 6), (−2, 2), (−4, −10)}

77. $5x + 2y = 10$; {(3, 5), (2, 0), (4, 2), (1, 2.5)}

78. $3a + 2b = 11$; {(−3, 10), (4, 1), (2, 2.5), (3, −2)}

79. $5 - \frac{3}{2}x = 2y$; $\left\{(0, 1), (8, 2), \left(4, -\frac{1}{2}\right), (2, 1)\right\}$

Solve each equation. Then check your solution. *(Lesson 3-3)*

80. $40 = -5d$

81. $20.4 = 3.4y$

82. $\dfrac{h}{-11} = -25$

83. $-65 = \dfrac{r}{29}$

Getting Ready for the Next Lesson **PREREQUISITE SKILL** **Find each product.**
*(To review **multiplying binomials**, see Lesson 8-7.)*

84. $(x - 3)(x + 2)$

85. $(a + 2)(a + 5)$

86. $(2t + 1)(t - 6)$

87. $(4x - 3)(x + 1)$

88. $(5x + 3y)(3x - y)$

89. $(3a - 2b)(4a + 7b)$

11-2 Operations with Radical Expressions

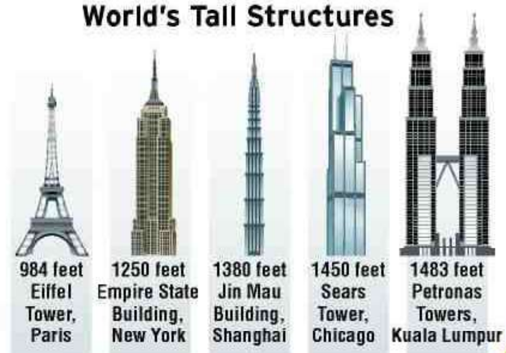
What You'll Learn

- Add and subtract radical expressions.
- Multiply radical expressions.

How can you use radical expressions to determine how far a person can see?

The formula $d = \sqrt{\frac{3h}{2}}$ represents the distance d in miles that a person h feet high can see. To determine how much farther a person can see from atop the Sears Tower than from atop the Empire State Building, we can substitute the heights of both buildings into the equation.

World's Tall Structures



ADD AND SUBTRACT RADICAL EXPRESSIONS Radical expressions in which the radicands are alike can be added or subtracted in the same way that monomials are added or subtracted.

Monomials	**Radical Expressions**
$2x + 7x = (2 + 7)x$	$2\sqrt{11} + 7\sqrt{11} = (2 + 7)\sqrt{11}$
$= 9x$	$= 9\sqrt{11}$
$15y - 3y = (15 - 3)y$	$15\sqrt{2} - 3\sqrt{2} = (15 - 3)\sqrt{2}$
$= 12y$	$= 12\sqrt{2}$

Notice that the Distributive Property was used to simplify each radical expression.

Example 1 Expressions with Like Radicands

Simplify each expression.

a. $\mathbf{4\sqrt{3} + 6\sqrt{3} - 5\sqrt{3}}$

$$4\sqrt{3} + 6\sqrt{3} - 5\sqrt{3} = (4 + 6 - 5)\sqrt{3} \quad \text{Distributive Property}$$
$$= 5\sqrt{3} \quad \text{Simplify.}$$

b. $\mathbf{12\sqrt{5} + 3\sqrt{7} + 6\sqrt{7} - 8\sqrt{5}}$

$$12\sqrt{5} + 3\sqrt{7} + 6\sqrt{7} - 8\sqrt{5} = 12\sqrt{5} - 8\sqrt{5} + 3\sqrt{7} + 6\sqrt{7} \quad \text{Commutative Property}$$
$$= (12 - 8)\sqrt{5} + (3 + 6)\sqrt{7} \quad \text{Distributive Property}$$
$$= 4\sqrt{5} + 9\sqrt{7} \quad \text{Simplify.}$$

In Example 1b, $4\sqrt{5} + 9\sqrt{7}$ cannot be simplified further because the radicands are different. There are no common factors, and each radicand is in simplest form. If the radicals in a radical expression are not in simplest form, simplify them first.

CONTENTS

Example 2 Expressions with Unlike Radicands

Simplify $2\sqrt{20} + 3\sqrt{45} + \sqrt{180}$.

$$\begin{aligned}2\sqrt{20} + 3\sqrt{45} + \sqrt{180} &= 2\sqrt{2^2 \cdot 5} + 3\sqrt{3^2 \cdot 5} + \sqrt{6^2 \cdot 5}\\ &= 2(\sqrt{2^2} \cdot \sqrt{5}) + 3(\sqrt{3^2} \cdot \sqrt{5}) + \sqrt{6^2} \cdot \sqrt{5}\\ &= 2(2\sqrt{5}) + 3(3\sqrt{5}) + 6\sqrt{5}\\ &= 4\sqrt{5} + 9\sqrt{5} + 6\sqrt{5}\\ &= 19\sqrt{5}\end{aligned}$$

The simplified form is $19\sqrt{5}$.

You can use a calculator to verify that a simplified radical expression is equivalent to the original expression. Consider Example 2. First, find a decimal approximation for the original expression.

KEYSTROKES: 2 [2nd] [√] 20 [)] [+] 3 [2nd] [√] 45 [)] [+] [2nd] [√] 180 [)] [ENTER] 42.48529157

Next, find a decimal approximation for the simplified expression.

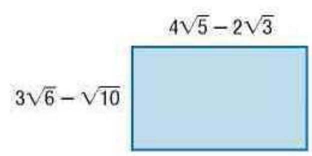
KEYSTROKES: 19 [2nd] [√] 5 [ENTER] 42.48529157

Since the approximations are equal, the expressions are equivalent.

MULTIPLY RADICAL EXPRESSIONS Multiplying two radical expressions with different radicands is similar to multiplying binomials.

Example 3 Multiply Radical Expressions

Find the area of the rectangle in simplest form.



To find the area of the rectangle multiply the measures of the length and width.

$(4\sqrt{5} - 2\sqrt{3})(3\sqrt{6} - \sqrt{10})$

$$\begin{aligned}&= \overbrace{(4\sqrt{5})(3\sqrt{6})}^{\text{First terms}} + \overbrace{(4\sqrt{5})(-\sqrt{10})}^{\text{Outer terms}} + \overbrace{(-2\sqrt{3})(3\sqrt{6})}^{\text{Inner terms}} + \overbrace{(-2\sqrt{3})(-\sqrt{10})}^{\text{Last terms}}\\ &= 12\sqrt{30} - 4\sqrt{50} - 6\sqrt{18} + 2\sqrt{30} && \text{Multiply.}\\ &= 12\sqrt{30} - 4\sqrt{5^2 \cdot 2} - 6\sqrt{3^2 \cdot 2} + 2\sqrt{30} && \text{Prime factorization}\\ &= 12\sqrt{30} - 20\sqrt{2} - 18\sqrt{2} + 2\sqrt{30} && \text{Simplify.}\\ &= 14\sqrt{30} - 38\sqrt{2} && \text{Combine like terms.}\end{aligned}$$

The area of the rectangle is $14\sqrt{30} - 38\sqrt{2}$ square units.

Study Tip

Look Back
To review the **FOIL method**, see Lesson 8-7.

Check for Understanding

Concept Check

1. **Explain** why you should simplify each radical in a radical expression before adding or subtracting.
2. **Explain** how you use the Distributive Property to simplify like radicands that are added or subtracted.
3. **OPEN ENDED** Choose values for x and y. Then find $(\sqrt{x} + \sqrt{y})^2$.

Guided Practice

Simplify each expression.

4. $4\sqrt{3} + 7\sqrt{3}$
5. $2\sqrt{6} - 7\sqrt{6}$
6. $5\sqrt{5} - 3\sqrt{20}$
7. $2\sqrt{3} + \sqrt{12}$
8. $3\sqrt{5} + 5\sqrt{6} + 3\sqrt{20}$
9. $8\sqrt{3} + \sqrt{3} + \sqrt{9}$

Find each product.

10. $\sqrt{2}(\sqrt{8} + 4\sqrt{3})$
11. $(4 + \sqrt{5})(3 + \sqrt{5})$

Applications

12. **GEOMETRY** Find the perimeter and the area of a square whose sides measure $4 + 3\sqrt{6}$ feet.
13. **ELECTRICITY** The voltage V required for a circuit is given by $V = \sqrt{PR}$, where P is the power in watts and R is the resistance in ohms. How many more volts are needed to light a 100-watt bulb than a 75-watt bulb if the resistance for both is 110 ohms?

Practice and Apply

Homework Help

For Exercises	See Examples
14–21	1
22–29	2
30–48	3

Extra Practice

See page 844.

Simplify each expression.

14. $8\sqrt{5} + 3\sqrt{5}$
15. $3\sqrt{6} + 10\sqrt{6}$
16. $2\sqrt{15} - 6\sqrt{15} - 3\sqrt{15}$
17. $5\sqrt{19} + 6\sqrt{19} - 11\sqrt{19}$
18. $16\sqrt{x} + 2\sqrt{x}$
19. $3\sqrt{5b} - 4\sqrt{5b} + 11\sqrt{5b}$
20. $8\sqrt{3} - 2\sqrt{2} + 3\sqrt{2} + 5\sqrt{3}$
21. $4\sqrt{6} + \sqrt{17} - 6\sqrt{2} + 4\sqrt{17}$
22. $\sqrt{18} + \sqrt{12} + \sqrt{8}$
23. $\sqrt{6} + 2\sqrt{3} + \sqrt{12}$
24. $3\sqrt{7} - 2\sqrt{28}$
25. $2\sqrt{50} - 3\sqrt{32}$
26. $\sqrt{2} + \sqrt{\frac{1}{2}}$
27. $\sqrt{10} - \sqrt{\frac{2}{5}}$
28. $3\sqrt{3} - \sqrt{45} + 3\sqrt{\frac{1}{3}}$
29. $6\sqrt{\frac{7}{4}} + 3\sqrt{28} - 10\sqrt{\frac{1}{7}}$

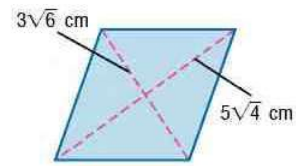
Find each product.

30. $\sqrt{6}(\sqrt{3} + 5\sqrt{2})$
31. $\sqrt{5}(2\sqrt{10} + 3\sqrt{2})$
32. $(3 + \sqrt{5})(3 - \sqrt{5})$
33. $(7 - \sqrt{10})^2$
34. $(\sqrt{6} + \sqrt{8})(\sqrt{24} + \sqrt{2})$
35. $(\sqrt{5} - \sqrt{2})(\sqrt{14} + \sqrt{35})$
36. $(2\sqrt{10} + 3\sqrt{15})(3\sqrt{3} - 2\sqrt{2})$
37. $(5\sqrt{2} + 3\sqrt{5})(2\sqrt{10} - 3)$

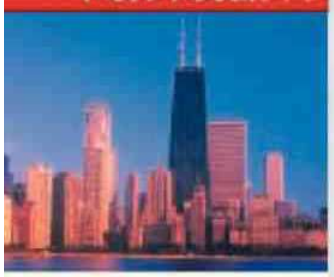
38. **GEOMETRY** Find the perimeter of a rectangle whose length is $8\sqrt{7} + 4\sqrt{5}$ inches and whose width is $2\sqrt{7} - 3\sqrt{5}$ inches.

39. **GEOMETRY** The perimeter of a rectangle is $2\sqrt{3} + 4\sqrt{11} + 6$ centimeters, and its length is $2\sqrt{11} + 1$ centimeters. Find the width.

40. **GEOMETRY** A formula for the area A of a rhombus can be found using the formula $A = \frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of the diagonals of the rhombus. What is the area of the rhombus at the right?



More About...



Distance

The Sears Tower was the tallest building in the world from 1974 to 1996. You can see four states from its roof: Michigan, Indiana, Illinois, and Wisconsin.

Source: www.the-skydeck.com

DISTANCE **For Exercises 41 and 42, refer to the application at the beginning of the lesson.**

41. How much farther can a person see from atop the Sears Tower than from atop the Empire State Building?

42. A person atop the Empire State Building can see approximately 4.57 miles farther than a person atop the Texas Commerce Tower in Houston. Explain how you could find the height of the Texas Commerce Tower.

Online Research **Data Update** What are the tallest buildings and towers in the world today? Visit www.algebra1.com/data_update to learn more.

ENGINEERING **For Exercises 43 and 44, use the following information.**

The equation $r = \sqrt{\frac{F}{5\pi}}$ relates the radius r of a drainpipe in inches to the flow rate F of water passing through it in gallons per minute.

43. Find the radius of a pipe that can carry 500 gallons of water per minute. Round to the nearest whole number.

44. An engineer determines that a drainpipe must be able to carry 1000 gallons of water per minute and instructs the builder to use an 8-inch radius pipe. Can the builder use two 4-inch radius pipes instead? Justify your answer.

MOTION **For Exercises 45–47, use the following information.**

The velocity of an object dropped from a certain height can be found using the formula $v = \sqrt{2gd}$, where v is the velocity in feet per second, g is the acceleration due to gravity, and d is the distance in feet the object drops.

45. Find the speed of an object that has fallen 25 feet and the speed of an object that has fallen 100 feet. Use 32 feet per second squared for g.

46. When you increased the distance by 4 times, what happened to the velocity?

47. **MAKE A CONJECTURE** Estimate the velocity of an object that has fallen 225 feet. Then use the formula to verify your answer.

48. **WATER SUPPLY** The relationship between a city's size and its capacity to supply water to its citizens can be described by the expression $1020\sqrt{P}(1 - 0.01\sqrt{P})$, where P is the population in thousands and the result is the number of gallons per minute required. If a city has a population of 55,000 people, how many gallons per minute must the city's pumping station be able to supply?

49. **CRITICAL THINKING** Find a counterexample to disprove the following statement.

For any numbers a and b, where $a > 0$ and $b > 0$, $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

50. **CRITICAL THINKING** Under what conditions is $(\sqrt{a+b})^2 = (\sqrt{a})^2 + (\sqrt{b})^2$ true?

51. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How can you use radical expressions to determine how far a person can see?

Include the following in your answer:

- an explanation of how this information could help determine how far apart lifeguard towers should be on a beach, and
- an example of a real-life situation where a lookout position is placed at a high point above the ground.

Standardized Test Practice

52. Find the difference of $9\sqrt{7}$ and $2\sqrt{28}$.

(A) $\sqrt{7}$ (B) $4\sqrt{7}$
(C) $5\sqrt{7}$ (D) $7\sqrt{7}$

53. Simplify $\sqrt{3}(4 + \sqrt{12})^2$.

(A) $4\sqrt{3} + 6$ (B) $28\sqrt{3}$
(C) $28 + 16\sqrt{3}$ (D) $48 + 28\sqrt{3}$

Maintain Your Skills

Mixed Review

Simplify. *(Lesson 11-1)*

54. $\sqrt{40}$ 55. $\sqrt{128}$ 56. $-\sqrt{196x^2y^3}$

57. $\frac{\sqrt{50}}{\sqrt{8}}$ 58. $\sqrt{\frac{225c^4d}{18c^2}}$ 59. $\sqrt{\frac{63a}{128a^3b^2}}$

Find the *n*th term of each geometric sequence. *(Lesson 10-7)*

60. $a_1 = 4, n = 6, r = 4$ 61. $a_1 = -7, n = 4, r = 9$ 62. $a_1 = 2, n = 8, r = -0.8$

Solve each equation by factoring. Check your solutions. *(Lesson 9-5)*

63. $81 = 49y^2$ 64. $q^2 - \frac{36}{121} = 0$

65. $48n^3 - 75n = 0$ 66. $5x^3 - 80x = 240 - 15x^2$

Solve each inequality. Then check your solution. *(Lesson 6-2)*

67. $8n \geq 5$ 68. $\frac{w}{9} < 14$ 69. $\frac{7k}{2} > \frac{21}{10}$

70. **PROBABILITY** A student rolls a die three times. What is the probability that each roll is a 1? *(Lesson 2-6)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Find each product.** *(To review **special products**, see Lesson 8-8.)*

71. $(x - 2)^2$ 72. $(x + 5)^2$ 73. $(x + 6)^2$

74. $(3x - 1)^2$ 75. $(2x - 3)^2$ 76. $(4x + 7)^2$

11-3 Radical Equations

What You'll Learn

- Solve radical equations.
- Solve radical equations with extraneous solutions.

Vocabulary

- radical equation
- extraneous solution

How are radical equations used to find free-fall times?

Skydivers fall 1050 to 1480 feet every 5 seconds, reaching speeds of 120 to 150 miles per hour at *terminal velocity*. It is the highest speed they can reach and occurs when the air resistance equals the force of gravity. With no air resistance, the time t in seconds that it takes an object to fall h feet can be determined by the equation $t = \frac{\sqrt{h}}{4}$. How would you find the value of h if you are given the value of t?

RADICAL EQUATIONS Equations like $t = \frac{\sqrt{h}}{4}$ that contain radicals with variables in the radicand are called **radical equations**. To solve these equations, first isolate the radical on one side of the equation. Then square each side of the equation to eliminate the radical.

Example 1 Radical Equation with a Variable

FREE-FALL HEIGHT Two objects are dropped simultaneously. The first object reaches the ground in 2.5 seconds, and the second object reaches the ground 1.5 seconds later. From what heights were the two objects dropped?

Find the height of the first object. Replace t with 2.5 seconds.

$t = \frac{\sqrt{h}}{4}$	Original equation
$2.5 = \frac{\sqrt{h}}{4}$	Replace t with 2.5.
$10 = \sqrt{h}$	Multiply each side by 4.
$10^2 = (\sqrt{h})^2$	Square each side.
$100 = h$	Simplify.

CHECK

$t = \frac{\sqrt{h}}{4}$	Original equation
$t \stackrel{?}{=} \frac{\sqrt{100}}{4}$	$h = 100$
$t \stackrel{?}{=} \frac{10}{4}$	$\sqrt{100} = 10$
$t = 2.5$	Simplify.

The first object was dropped from 100 feet.

The time it took the second object to fall was 2.5 + 1.5 seconds or 4 seconds.

$$t = \frac{\sqrt{h}}{4}$$ Original equation

$$4 = \frac{\sqrt{h}}{4}$$ Replace t with 4.

$$16 = \sqrt{h}$$ Multiply each side by 4.

$$16^2 = (\sqrt{h})^2$$ Square each side.

$$256 = h$$ Simplify.

The second object was dropped from 256 feet. *Check this solution.*

Example 2 Radical Equation with an Expression

Solve $\sqrt{x + 1} + 7 = 10$.

$$\sqrt{x+1} + 7 = 10$$ Original equation

$$\sqrt{x+1} = 3$$ Subtract 7 from each side.

$$(\sqrt{x+1})^2 = 3^2$$ Square each side.

$$x + 1 = 9$$ $(\sqrt{x+1})^2 = x + 1$

$$x = 8$$ Subtract 1 from each side.

The solution is 8. *Check this result.*

EXTRANEOUS SOLUTIONS Squaring each side of an equation sometimes produces extraneous solutions. An **extraneous solution** is a solution derived from an equation that is not a solution of the original equation. Therefore, you must check all solutions in the original equation when you solve radical equations.

Example 3 Variable on Each Side

Solve $\sqrt{x + 2} = x - 4$.

$$\sqrt{x+2} = x - 4$$ Original equation

$$(\sqrt{x+2})^2 = (x-4)^2$$ Square each side.

$$x + 2 = x^2 - 8x + 16$$ Simplify.

$$0 = x^2 - 9x + 14$$ Subtract x and 2 from each side.

$$0 = (x-7)(x-2)$$ Factor.

$x - 7 = 0$ or $x - 2 = 0$ Zero Product Property

$x = 7$ $\quad$ $x = 2$ Solve.

CHECK

$$\sqrt{x+2} = x - 4 \qquad \sqrt{x+2} = x - 4$$

$$\sqrt{7+2} \stackrel{?}{=} 7 - 4 \quad x = 7 \qquad \sqrt{2+2} \stackrel{?}{=} 2 - 4 \quad x = 2$$

$$\sqrt{9} \stackrel{?}{=} 3 \qquad \sqrt{4} \stackrel{?}{=} -2$$

$$3 = 3 \ \checkmark \qquad 2 \neq -2 \ \times$$

Since 2 does not satisfy the original equation, 7 is the only solution.

Study Tip

Look Back
To review **Zero Product Property**, see Lesson 9-2.

Graphing Calculator Investigation

Solving Radical Equations

You can use a TI-83 Plus graphing calculator to solve radical equations such as $\sqrt{3x - 5} = x - 5$. Clear the Y= list. Enter the left side of the equation as Y1 = $\sqrt{3x - 5}$. Enter the right side of the equation as Y2 = $x - 5$. Press $\boxed{\text{GRAPH}}$.

Think and Discuss

1. Sketch what is shown on the screen.
2. Use the intersect feature on the CALC menu, to find the point of intersection.
3. Solve the radical equation algebraically. How does your solution compare to the solution from the graph?

Check for Understanding

Concept Check

1. **Describe** the steps needed to solve a radical equation.
2. **Explain** why it is necessary to check for extraneous solutions in radical equations.
3. **OPEN ENDED** Give an example of a radical equation. Then solve the equation for the variable.
4. **FIND THE ERROR** Alex and Victor are solving $-\sqrt{x - 5} = -2$.

Alex	Victor
$-\sqrt{x-5} = -2$	$-\sqrt{x-5} = -2$
$(-\sqrt{x-5})^2 = (-2)^2$	$(-\sqrt{x-5})^2 = (-2)^2$
$x - 5 = 4$	$-(x - 5) = 4$
$x = 9$	$-x + 5 = 4$
	$x = 1$

Who is correct? Explain your reasoning.

Guided Practice

Solve each equation. Check your solution.

5. $\sqrt{x} = 5$
6. $\sqrt{2b} = -8$
7. $\sqrt{7x} = 7$
8. $\sqrt{-3a} = 6$
9. $\sqrt{8s} + 1 = 5$
10. $\sqrt{7x + 18} = 9$
11. $\sqrt{5x + 1} + 2 = 6$
12. $\sqrt{3x - 5} = x - 5$
13. $4 + \sqrt{x - 2} = x$

Application

OCEANS **For Exercises 14–16, use the following information.**
Tsunamis, or large tidal waves, are generated by undersea earthquakes in the Pacific Ocean. The speed of the tsunami in meters per second is $s = 3.1\sqrt{d}$, where d is the depth of the ocean in meters.

14. Find the speed of the tsunami if the depth of the water is 10 meters.
15. Find the depth of the water if a tsunami's speed is 240 meters per second.
16. A tsunami may begin as a 2-foot high wave traveling 450–500 miles per hour. It can approach a coastline as a 50-foot wave. How much speed does the wave lose if it travels from a depth of 10,000 meters to a depth of 20 meters?

Practice and Apply

Homework Help

For Exercises	See Examples
17–34	1, 2
35–47	3
48–59	1–3

Extra Practice

See page 844.

Solve each equation. Check your solution.

17. $\sqrt{a} = 10$
18. $\sqrt{-k} = 4$
19. $5\sqrt{2} = \sqrt{x}$
20. $3\sqrt{7} = \sqrt{-y}$
21. $3\sqrt{4a} - 2 = 10$
22. $3 + 5\sqrt{n} = 18$
23. $\sqrt{x + 3} = -5$
24. $\sqrt{x - 5} = 2\sqrt{6}$
25. $\sqrt{3x + 12} = 3\sqrt{3}$
26. $\sqrt{2c - 4} = 8$
27. $\sqrt{4b + 1} - 3 = 0$
28. $\sqrt{3r - 5} + 7 = 3$
29. $\sqrt{\frac{4x}{5}} - 9 = 3$
30. $5\sqrt{\frac{4t}{3}} - 2 = 0$
31. $\sqrt{x^2 + 9x + 14} = x + 4$
32. $y + 2 = \sqrt{y^2 + 5y + 4}$

33. The square root of the sum of a number and 7 is 8. Find the number.

34. The square root of the quotient of a number and 6 is 9. Find the number.

Solve each equation. Check your solution.

35. $x = \sqrt{6 - x}$
36. $x = \sqrt{x + 20}$
37. $\sqrt{5x - 6} = x$
38. $\sqrt{28 - 3x} = x$
39. $\sqrt{x + 1} = x - 1$
40. $\sqrt{1 - 2b} = 1 + b$
41. $4 + \sqrt{m - 2} = m$
42. $\sqrt{3d - 8} = d - 2$
43. $x + \sqrt{6 - x} = 4$
44. $\sqrt{6 - 3x} = x + 16$
45. $\sqrt{2r^2 - 121} = r$
46. $\sqrt{5p^2 - 7} = 2p$

47. State whether the following equation is *sometimes*, *always*, or *never* true.
$\sqrt{(x - 5)^2} = x - 5$

AVIATION **For Exercises 48 and 49, use the following information.**
The formula $L = \sqrt{kP}$ represents the relationship between a plane's length L and the pounds P its wings can lift, where k is a constant of proportionality calculated for a plane.

48. The length of the Douglas D-558-II, called the Skyrocket, was approximately 42 feet, and its constant of proportionality was $k = 0.1669$. Calculate the maximum takeoff weight of the Skyrocket.

49. A Boeing 747 is 232 feet long and has a takeoff weight of 870,000 pounds. Determine the value of k for this plane.

More About...

Aviation

Piloted by A. Scott Crossfield on November 20, 1953, the Douglas D-558-2 Skyrocket became the first aircraft to fly faster than Mach 2, twice the speed of sound.

Source: National Air and Space Museum

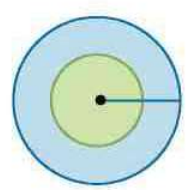
GEOMETRY **For Exercises 50–53, use the figure below. The area A of a circle is equal to πr^2 where r is the radius of the circle.**

50. Write an equation for r in terms of A.

51. The area of the larger circle is 96π square meters. Find the radius.

52. The area of the smaller circle is 48π square meters. Find the radius.

53. If the area of a circle is doubled, what is the change in the radius?



PHYSICAL SCIENCE **For Exercises 54–56, use the following information.**

The formula $P = 2\pi\sqrt{\frac{\ell}{32}}$ gives the period of a pendulum of length ℓ feet. The period P is the number of seconds it takes for the pendulum to swing back and forth once.

54. Suppose we want a pendulum to complete three periods in 2 seconds. How long should the pendulum be?

55. Two clocks have pendulums of different lengths. The first clock requires 1 second for its pendulum to complete one period. The second clock requires 2 seconds for its pendulum to complete one period. How much longer is one pendulum than the other?

56. Repeat Exercise 55 if the pendulum periods are t and $2t$ seconds.

More About. . .

Physical Science

The Foucault pendulum was invented in 1851. It demonstrates the rotation of Earth. The pendulum appears to change its path during the day, but it moves in a straight line. The path under the pendulum changes because Earth is rotating beneath it.

Source: California Academy of Sciences

SOUND **For Exercises 57–59, use the following information.**

The speed of sound V near Earth's surface can be found using the equation $V = 20\sqrt{t + 273}$, where t is the surface temperature in degrees Celsius.

57. Find the temperature if the speed of sound V is 356 meters per second.

58. The speed of sound at Earth's surface is often given at 340 meters per second, but that is only accurate at a certain temperature. On what temperature is this figure based?

59. What is the speed of sound when the surface temperature is below 0°C?

60. **CRITICAL THINKING** Solve $\sqrt{h + 9} - \sqrt{h} = \sqrt{3}$.

61. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are radical equations used to find free-fall times?

Include the following in your answer:

- the time it would take a skydiver to fall 10,000 feet if he falls 1200 feet every 5 seconds and the time using the equation $t = \frac{\sqrt{h}}{4}$, with an explanation of why the two methods find different times, and
- ways that a skydiver can increase or decrease his speed.

Standardized Test Practice

QUANTITATIVE COMPARISON **In Exercises 62 and 63, compare the quantity in Column A and the quantity in Column B. Then determine whether:**

Ⓐ **the quantity in Column A is greater,**

Ⓑ **the quantity in Column B is greater,**

Ⓒ **both quantities are equal, or**

Ⓓ **the relationship cannot be determined from the given information.**

	Column A	Column B
62.	the solution of $\sqrt{x} + 3 = 6$	the solution of $\sqrt{y + 3} = 6$
63.	$(\sqrt{a} - 1)^2$	$\sqrt{(a - 1)^2}$

Graphing Calculator

RADICAL EQUATIONS Use a graphing calculator to solve each radical equation. Round to the nearest hundredth.

64. $3 + \sqrt{2x} = 7$
65. $\sqrt{3x - 8} = 5$
66. $\sqrt{x + 6} - 4 = x$
67. $\sqrt{4x + 5} = x - 7$
68. $x + \sqrt{7 - x} = 4$
69. $\sqrt{3x - 9} = 2x + 6$

Maintain Your Skills

Mixed Review

Simplify each expression. *(Lesson 11-2)*

70. $5\sqrt{6} + 12\sqrt{6}$
71. $\sqrt{12} + 6\sqrt{27}$
72. $\sqrt{18} + 5\sqrt{2} - 3\sqrt{32}$

Simplify. *(Lesson 11-1)*

73. $\sqrt{192}$
74. $\sqrt{6} \cdot \sqrt{10}$
75. $\dfrac{21}{\sqrt{10} + \sqrt{3}}$

Determine whether each trinomial is a perfect square trinomial. If so, factor it. *(Lesson 9-6)*

76. $d^2 + 50d + 225$
77. $4n^2 - 28n + 49$
78. $16b^2 - 56bc + 49c^2$

Find each product. *(Lesson 8-7)*

79. $(r + 3)(r - 4)$
80. $(3z + 7)(2z + 10)$
81. $(2p + 5)(3p^2 - 4p + 9)$

82. **PHYSICAL SCIENCE** A European-made hot tub is advertised to have a temperature of 35°C to 40°C, inclusive. What is the temperature range for the hot tub in degrees Fahrenheit? Use $F = \frac{9}{5}C + 32$. *(Lesson 6-4)*

Write each equation in standard form. *(Lesson 5-5)*

83. $y = 2x + \frac{3}{7}$
84. $y - 3 = -2(x - 6)$
85. $y + 2 = 7.5(x - 3)$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Evaluate $\sqrt{a^2 + b^2}$ for each value of a and b.
*(To review **evaluating expressions**, see Lesson 1-2.)*

86. $a = 3, b = 4$
87. $a = 24, b = 7$
88. $a = 1, b = 1$
89. $a = 8, b = 12$

Practice Quiz 1

Lessons 11-1 through 11-3

Simplify. *(Lesson 11-1)*

1. $\sqrt{48}$
2. $\sqrt{3} \cdot \sqrt{6}$
3. $\dfrac{3}{2 + \sqrt{10}}$

Simplify. *(Lesson 11-2)*

4. $6\sqrt{5} + 3\sqrt{11} + 5\sqrt{5}$
5. $2\sqrt{3} + 9\sqrt{12}$
6. $(3 - \sqrt{6})^2$

7. **GEOMETRY** Find the area of a square whose side measure is $2 + \sqrt{7}$ centimeters. *(Lesson 11-2)*

Solve each equation. Check your solution. *(Lesson 11-3)*

8. $\sqrt{15 - x} = 4$
9. $\sqrt{3x^2 - 32} = x$
10. $\sqrt{2x - 1} = 2x - 7$

Graphing Calculator Investigation

A Follow-Up of Lesson 11-3

Graphs of Radical Equations

In order for a square root to be a real number, the radicand cannot be negative. When graphing a radical equation, determine when the radicand would be negative and exclude those values from the domain.

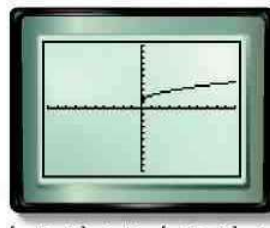
Example 1

Graph $y = \sqrt{x}$. State the domain of the graph.

Enter the equation in the Y= list.

KEYSTROKES: [Y=] [2nd] [$\sqrt{\ }$] [X,T,θ,n] [ )] [GRAPH]

From the graph, you can see that the domain of x is $\{x \mid x \geq 0\}$.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Example 2

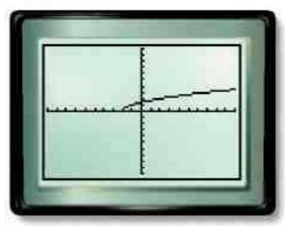
Graph $y = \sqrt{x + 4}$. State the domain of the graph.

Enter the equation in the Y= list.

KEYSTROKES: [Y=] [2nd] [$\sqrt{\ }$] [X,T,θ,n] [+] 4 [ )] [GRAPH]

The value of the radicand will be positive when $x + 4 \geq 0$, or when $x \geq -4$. So the domain of x is $\{x \mid x \geq -4\}$.

This graph looks like the graph of $y = \sqrt{x}$ shifted left 4 units.



[−10, 10] scl: 1 by [−10, 10] scl: 1

Exercises

Graph each equation and sketch the graph on your paper. State the domain of the graph. Then describe how the graph differs from the parent function $y = \sqrt{x}$.

1. $y = \sqrt{x} + 1$ **2.** $y = \sqrt{x} - 3$ **3.** $y = \sqrt{x + 2}$

4. $y = \sqrt{x - 5}$ **5.** $y = \sqrt{-x}$ **6.** $y = \sqrt{3x}$

7. $y = -\sqrt{x}$ **8.** $y = \sqrt{1 - x} + 6$ **9.** $y = \sqrt{2x + 5} - 4$

10. Is the graph of $x = y^2$ a function? Explain your reasoning.

11. Does the equation $x^2 + y^2 = 1$ determine y as a function of x? Explain.

12. Graph $y = |x| \pm \sqrt{1 - x^2}$ in the window defined by $[-2, 2]$ scl: 1 by $[-2, 2]$ scl: 1. Describe the graph.

www.algebra1.com/other_calculator_keystrokes

11-4 The Pythagorean Theorem

What You'll Learn

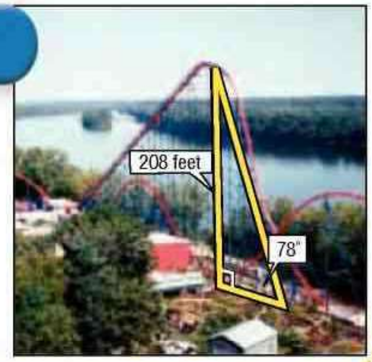
- Solve problems by using the Pythagorean Theorem.
- Determine whether a triangle is a right triangle.

Vocabulary

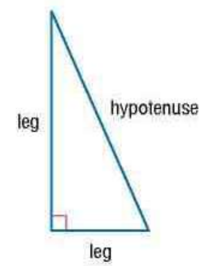
- hypotenuse
- legs
- Pythagorean triple
- corollary

How is the Pythagorean Theorem used in roller coaster design?

The roller coaster *Superman: Ride of Steel* in Agawam, Massachusetts, is one of the world's tallest roller coasters at 208 feet. It also boasts one of the world's steepest drops, measured at 78 degrees, and it reaches a maximum speed of 77 miles per hour. You can use the Pythagorean Theorem to estimate the length of the first hill.



THE PYTHAGOREAN THEOREM In a right triangle, the side opposite the right angle is called the **hypotenuse**. This side is always the longest side of a right triangle. The other two sides are called the **legs** of the triangle.



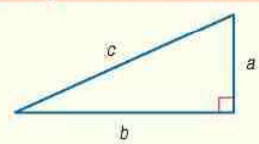
To find the length of any side of a right triangle when the lengths of the other two are known, you can use a formula developed by the Greek mathematician Pythagoras.

Study Tip

Triangles
Sides of a triangle are represented by lowercase letters a, b, and c.

Key Concept — The Pythagorean Theorem

- **Words** If a and b are the measures of the legs of a right triangle and c is the measure of the hypotenuse, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.



- **Symbols** $c^2 = a^2 + b^2$

Example 1 Find the Length of the Hypotenuse

Find the length of the hypotenuse of a right triangle if $a = 8$ and $b = 15$.

$c^2 = a^2 + b^2$ Pythagorean Theorem

$c^2 = 8^2 + 15^2$ $a = 8$ and $b = 15$

$c^2 = 289$ Simplify.

$c = \pm\sqrt{289}$ Take the square root of each side.

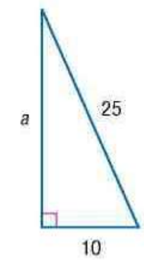
$c = \pm 17$ Disregard -17. Why?

The length of the hypotenuse is 17 units.

Example 2 Find the Length of a Side

Find the length of the missing side.

In the triangle, $c = 25$ and $b = 10$ units.



$c^2 = a^2 + b^2$	Pythagorean Theorem
$25^2 = a^2 + 10^2$	$b = 10$ and $c = 25$
$625 = a^2 + 100$	Evaluate squares.
$525 = a^2$	Subtract 100 from each side.
$\pm\sqrt{525} = a$	Use a calculator to evaluate $\sqrt{525}$.
$22.91 \approx a$	Use the positive value.

To the nearest hundredth, the length of the leg is 22.91 units.

Whole numbers that satisfy the Pythagorean Theorem are called **Pythagorean triples**. Multiples of Pythagorean triples also satisfy the Pythagorean Theorem. Some common triples are (3, 4, 5), (5, 12, 13), (8, 15, 17), and (7, 24, 25).

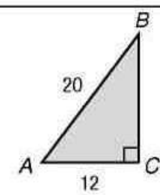
Standardized Test Practice

Example 3 Pythagorean Triples

Multiple-Choice Test Item

What is the area of triangle *ABC*?

(A) 96 units^2 (B) 120 units^2
(C) 160 units^2 (D) 196 units^2



Read the Test Item

The area of a triangle is $A = \frac{1}{2}bh$. In a right triangle, the legs form the base and height of the triangle. Use the measures of the hypotenuse and the base to find the height of the triangle.

Solve the Test Item

Step 1 Check to see if the measurements of this triangle are a multiple of a common Pythagorean triple. The hypotenuse is $4 \cdot 5$ units, and the leg is $4 \cdot 3$ units. This triangle is a multiple of a (3, 4, 5) triangle.

$4 \cdot 3 = 12$

$4 \cdot 4 = 16$

$4 \cdot 5 = 20$

The height of the triangle is 16 units.

Step 2 Find the area of the triangle.

$A = \frac{1}{2}bh$	Area of a triangle
$A = \frac{1}{2} \cdot 12 \cdot 16$	$b = 12$ and $h = 16$
$A = 96$	Simplify.

The area of the triangle is 96 square units. Choice A is correct.

The Princeton Review

Test-Taking Tip

Memorize the common Pythagorean triples and check for multiples such as (6, 8, 10). This will save you time when evaluating square roots.

RIGHT TRIANGLES A statement that can be easily proved using a theorem is often called a **corollary**. The following corollary, based on the Pythagorean Theorem, can be used to determine whether a triangle is a right triangle.

Key Concept *Corollary to the Pythagorean Theorem*

If a and b are measures of the shorter sides of a triangle, c is the measure of the longest side, and $c^2 = a^2 + b^2$, then the triangle is a right triangle. If $c^2 \neq a^2 + b^2$, then the triangle is not a right triangle.

Example 4 Check for Right Triangles

Determine whether the following side measures form right triangles.

a. 20, 21, 29

Since the measure of the longest side is 29, let $c = 29$, $a = 20$, and $b = 21$. Then determine whether $c^2 = a^2 + b^2$.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$29^2 \stackrel{?}{=} 20^2 + 21^2$	$a = 20$, $b = 21$, and $c = 29$
$841 \stackrel{?}{=} 400 + 441$	Multiply.
$841 = 841$	Add.

Since $c^2 = a^2 + b^2$, the triangle is a right triangle.

b. 8, 10, 12

Since the measure of the longest side is 12, let $c = 12$, $a = 8$, and $b = 10$. Then determine whether $c^2 = a^2 + b^2$.

$c^2 = a^2 + b^2$	Pythagorean Theorem
$12^2 \stackrel{?}{=} 8^2 + 10^2$	$a = 8$, $b = 10$, and $c = 12$
$144 \stackrel{?}{=} 64 + 100$	Multiply.
$144 \neq 164$	Add.

Since $c^2 \neq a^2 + b^2$, the triangle is not a right triangle.

Check for Understanding

Concept Check

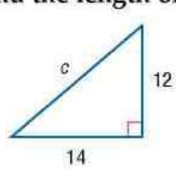
1. **OPEN ENDED** Draw a right triangle and label each side and angle. Be sure to indicate the right angle.

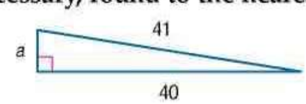
2. **Explain** how you can determine which angle is the right angle of a right triangle if you are given the lengths of the three sides.

3. **Write** an equation you could use to find the length of the diagonal d of a square with side length s.

Guided Practice

Find the length of each missing side. If necessary, round to the nearest hundredth.

4. 

5. 

If c is the measure of the hypotenuse of a right triangle, find each missing measure. If necessary, round to the nearest hundredth.

6. $a = 10, b = 24, c = ?$

7. $a = 11, c = 61, b = ?$

8. $b = 13, c = \sqrt{233}, a = ?$

9. $a = 7, b = 4, c = ?$

Determine whether the following side measures form right triangles. Justify your answer.

10. 4, 6, 9

11. 16, 30, 34

Standardized Test Practice

12. In right triangle XYZ, the length of $\overline{YZ}$ is 6, and the length of the hypotenuse is 8. Find the area of the triangle.

Ⓐ $6\sqrt{7}$ units^2 Ⓑ 30 units^2 Ⓒ 40 units^2 Ⓓ 48 units^2

Practice and Apply

Homework Help

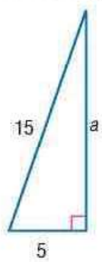
For Exercises	See Examples
13–30	1, 2
31–36	4
37–40	3

Extra Practice

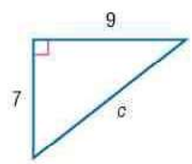
See page 845.

Find the length of each missing side. If necessary, round to the nearest hundredth.

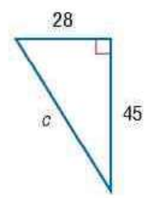
13.



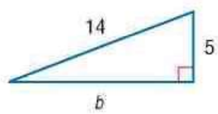
14.



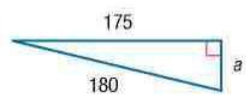
15.



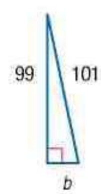
16.



17.



18.



If c is the measure of the hypotenuse of a right triangle, find each missing measure. If necessary, round to the nearest hundredth.

19. $a = 16, b = 63, c = ?$

20. $a = 16, c = 34, b = ?$

21. $b = 3, a = \sqrt{112}, c = ?$

22. $a = \sqrt{15}, b = \sqrt{10}, c = ?$

23. $c = 14, a = 9, b = ?$

24. $a = 6, b = 3, c = ?$

25. $b = \sqrt{77}, c = 12, a = ?$

26. $a = 4, b = \sqrt{11}, c = ?$

27. $a = \sqrt{225}, b = \sqrt{28}, c = ?$

28. $a = \sqrt{31}, c = \sqrt{155}, b = ?$

29. $a = 8x, b = 15x, c = ?$

30. $b = 3x, c = 7x, a = ?$

Determine whether the following side measures form right triangles. Justify your answer.

31. 30, 40, 50

32. 6, 12, 18

33. 24, 30, 36

34. 45, 60, 75

35. 15, $\sqrt{31}$, 16

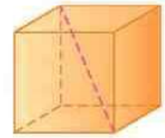
36. 4, 7, $\sqrt{65}$

Use an equation to solve each problem. If necessary, round to the nearest hundredth.

37. Find the length of a diagonal of a square if its area is 162 square feet.

38. A right triangle has one leg that is 5 centimeters longer than the other leg. The hypotenuse is 25 centimeters long. Find the length of each leg of the triangle.

39. Find the length of the diagonal of the cube if each side of the cube is 4 inches long.



40. The ratio of the length of the hypotenuse to the length of the *shorter* leg in a right triangle is 8:5. The hypotenuse measures 144 meters. Find the length of the *longer* leg.

More About . . .

Roller Coasters

Roller Coaster Records in the U.S.

Fastest: *Millennium Force* (2000), Sandusky, Ohio; 92 mph

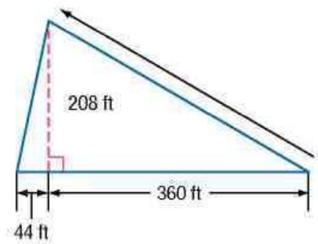
Tallest: *Millenium Force* (2000), Sandusky, Ohio; 310 feet

Longest: *California Screamin'* (2001), Anaheim, California; 6800 feet

Steepest: *Hypersonic XLC* (2001), Doswell, Virginia; 90°

Source: www.coasters2k.com

ROLLER COASTERS **For Exercises 41–43, use the following information and the figure.** Suppose a roller coaster climbs 208 feet higher than its starting point making a horizontal advance of 360 feet. When it comes down, it makes a horizontal advance of 44 feet.



41. How far will it travel to get to the top of the ride?

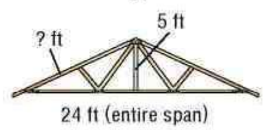
42. How far will it travel on the downhill track?

43. Compare the total horizontal advance, vertical height, and total track length.

44. **RESEARCH** Use the Internet or other reference to find the measurements of your favorite roller coaster or a roller coaster that is at an amusement park close to you. Draw a model of the first drop. Include the height of the hill, length of the vertical drop, and steepness of the hill.

45. **SAILING** A sailboat's mast and boom form a right angle. The sail itself, called a *mainsail*, is in the shape of a right triangle. If the edge of the mainsail that is attached to the mast is 100 feet long and the edge of the mainsail that is attached to the boom is 60 feet long, what is the length of the longest edge of the mainsail?

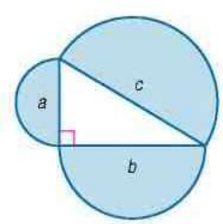
ROOFING **For Exercises 46 and 47, refer to the figures below.**



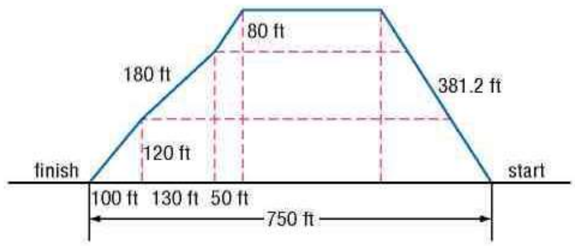
46. Determine the missing length shown in the rafter.

47. If the roof is 30 feet long and it hangs an additional 2 feet over the garage walls, how many square feet of shingles are needed for the entire garage roof?

48. **CRITICAL THINKING** Compare the area of the largest semicircle to the areas of the two smaller semicircles. Justify your reasoning.



49. **CRITICAL THINKING** A model of a part of a roller coaster is shown. Determine the total distance traveled from start to finish and the maximum height reached by the roller coaster.



50. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How is the Pythagorean Theorem used in roller coaster design?

Include the following in your answer:

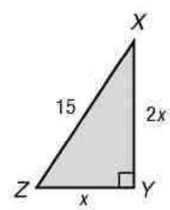
- an explanation of how the height, speed, and steepness of a roller coaster are related, and
- a description of any limitations you can think of in the design of a new roller coaster.

Standardized Test Practice

51. Find the area of $\triangle XYZ$.

Ⓐ $6\sqrt{5}$ units^2 Ⓑ $18\sqrt{5}$ units^2

Ⓒ 45 units^2 Ⓓ 90 units^2



52. Find the perimeter of a square whose diagonal measures 10 centimeters.

Ⓐ $10\sqrt{2}$ cm Ⓑ $20\sqrt{2}$ cm

Ⓒ $25\sqrt{2}$ cm Ⓓ 80 cm

Maintain Your Skills

Mixed Review

Solve each equation. Check your solution. *(Lesson 11-3)*

53. $\sqrt{y} = 12$

54. $3\sqrt{s} = 126$

55. $4\sqrt{2v + 1} - 3 = 17$

Simplify each expression. *(Lesson 11-2)*

56. $\sqrt{72}$

57. $7\sqrt{z} - 10\sqrt{z}$

58. $\sqrt{\frac{3}{7}} + \sqrt{21}$

Simplify. Assume that no denominator is equal to zero. *(Lesson 8-2)*

59. $\frac{5^8}{5^3}$

60. d^{-7}

61. $\frac{-26a^4b^7c^{-5}}{-13a^2b^4c^3}$

62. **AVIATION** Flying with the wind, a plane travels 300 miles in 40 minutes. Flying against the wind, it travels 300 miles in 45 minutes. Find the air speed of the plane. *(Lesson 7-4)*

Getting Ready for the Next Lesson

PREREQUISITE SKILL **Simplify each expression.**

*(To review **simplifying radical expressions**, see Lesson 11-1.)*

63. $\sqrt{(6-3)^2 + (8-4)^2}$

64. $\sqrt{(10-4)^2 + (13-5)^2}$

65. $\sqrt{(5-3)^2 + (2-9)^2}$

66. $\sqrt{(-9-5)^2 + (7-3)^2}$

67. $\sqrt{(-4-5)^2 + (-4-3)^2}$

68. $\sqrt{(20-5)^2 + (-2-6)^2}$

CONTENTS

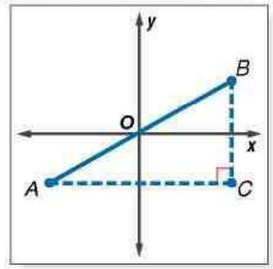
11-5 The Distance Formula

What You'll Learn

- Find the distance between two points on the coordinate plane.
- Find a point that is a given distance from a second point in a plane.

How can the distance between two points be determined?

Consider two points A and B in the coordinate plane. Notice that a right triangle can be formed by drawing lines parallel to the axes through the points at A and B. These lines intersect at C forming a right angle. The hypotenuse of this triangle is the distance between A and B. You can determine the length of the legs of this triangle and use the Pythagorean Theorem to find the distance between the two points. Notice that AC is the difference of the y-coordinates, and BC is the difference of the x-coordinates. So, $(AB)^2 = (AC)^2 + (BC)^2$, and $AB = \sqrt{(AC)^2 + (BC)^2}$.



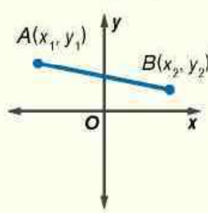
Study Tip

Reading Math
AC is the measure of $\overline{AC}$ and BC is the measure of $\overline{BC}$.

THE DISTANCE FORMULA You can find the distance between any two points in the coordinate plane using a similar process. The result is called the **Distance Formula**.

Key Concept — The Distance Formula

- **Words** The distance d between any two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- **Model**



Example 1 Distance Between Two Points

Find the distance between the points at (2, 3) and (−4, 6).

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\
&= \sqrt{(-4 - 2)^2 + (6 - 3)^2} && (x_1, y_1) = (2, 3) \text{ and } (x_2, y_2) = (-4, 6) \\
&= \sqrt{(-6)^2 + 3^2} && \text{Simplify.} \\
&= \sqrt{45} && \text{Evaluate squares and simplify.} \\
&= 3\sqrt{5} \text{ or about 6.71 units}
\end{aligned}
$$

Example 2 Use the Distance Formula

GOLF **Tracy hits a golf ball that lands 20 feet short and 8 feet to the right of the cup. On her first putt, the ball lands 2 feet to the left and 3 feet beyond the cup. Assuming that the ball traveled in a straight line, how far did the ball travel on her first putt?**

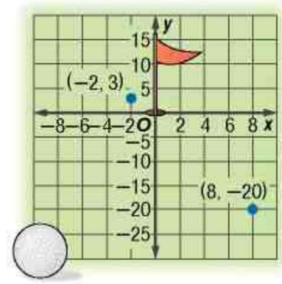
Draw a model of the situation on a coordinate grid. If the cup is at (0, 0), then the location of the ball after the first hit is (8, −20). The location of the ball after the first putt is (−2, 3). Use the Distance Formula.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$= \sqrt{(-2 - 8)^2 + [3 - (-20)]^2}$ $(x_1, y_1) = (8, -20)$, $(x_2, y_2) = (-2, 3)$

$= \sqrt{(-10)^2 + 23^2}$ Simplify.

$= \sqrt{629}$ or about 25 feet



More About. . .

Golf

There are four major tournaments that make up the "grand slam" of golf: Masters, U.S. Open, British Open, and PGA Championship. In 2000, Tiger Woods became the youngest player to win the four major events (called a career grand slam) at age 24.

Source: PGA

FIND COORDINATES Suppose you know the coordinates of a point, one coordinate of another point, and the distance between the two points. You can use the Distance Formula to find the missing coordinate.

Example 3 Find a Missing Coordinate

Find the value of a if the distance between the points at (7, 5) and (a, −3) is 10 units.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Distance Formula

$10 = \sqrt{(a - 7)^2 + (-3 - 5)^2}$ Let $x_2 = a$, $x_1 = 7$, $y_2 = -3$, $y_1 = 5$,

$10 = \sqrt{(a - 7)^2 + (-8)^2}$ and $d = 10$.

$10 = \sqrt{a^2 - 14a + 49 + 64}$ Evaluate squares.

$10 = \sqrt{a^2 - 14a + 113}$ Simplify.

$10^2 = \left(\sqrt{a^2 - 14a + 113}\right)^2$ Square each side.

$100 = a^2 - 14a + 113$ Simplify.

$0 = a^2 - 14a + 13$ Subtract 100 from each side.

$0 = (a - 1)(a - 13)$ Factor.

$a - 1 = 0$ or $a - 13 = 0$ Zero Product Property

$a = 1$ $\quad a = 13$ The value of a is 1 or 13.

Check for Understanding

Concept Check

1. **Explain** why the value calculated under the radical sign in the Distance Formula will never be negative.

2. **OPEN ENDED** Plot two ordered pairs and find the distance between their graphs. Does it matter which ordered pair is first when using the Distance Formula? Explain.

3. **Explain** why there are two values for a in Example 3. Draw a diagram to support your answer.

Guided Practice Find the distance between each pair of points whose coordinates are given. Express in simplest radical form and as decimal approximations rounded to the nearest hundredth if necessary.

4. $(5, -1), (11, 7)$ **5.** $(3, 7), (-2, -5)$

6. $(2, 2), (5, -1)$ **7.** $(-3, -5), (-6, -4)$

Find the possible values of a if the points with the given coordinates are the indicated distance apart.

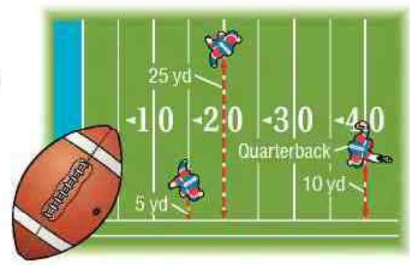
8. $(3, -1), (a, 7); d = 10$ **9.** $(10, a), (1, -6); d = \sqrt{145}$

Applications **10.** **GEOMETRY** An isosceles triangle has two sides of equal length. Determine whether triangle ABC with vertices $A(-3, 4)$, $B(5, 2)$, and $C(-1, -5)$ is an isosceles triangle.

FOOTBALL For Exercises 11 and 12, use the information at the right.

11. A quarterback can throw the football to one of the two receivers. Find the distance from the quarterback to each receiver.

12. What is the distance between the two receivers?



Practice and Apply

Homework Help

For Exercises	See Examples
13–26, 33, 34	1
27–32, 35, 36	3
37–42	2

Extra Practice

See page 845.

Find the distance between each pair of points whose coordinates are given. Express in simplest radical form and as decimal approximations rounded to the nearest hundredth if necessary.

13. $(12, 3), (-8, 3)$ **14.** $(0, 0), (5, 12)$

15. $(6, 8), (3, 4)$ **16.** $(-4, 2), (4, 17)$

17. $(-3, 8), (5, 4)$ **18.** $(9, -2), (3, -6)$

19. $(-8, -4), (-3, -8)$ **20.** $(2, 7), (10, -4)$

21. $(4, 2), \left(6, -\frac{2}{3}\right)$ **22.** $\left(5, \frac{1}{4}\right), (3, 4)$

23. $\left(\frac{4}{5}, -1\right), \left(2, -\frac{1}{2}\right)$ **24.** $\left(3, \frac{3}{7}\right), \left(4, -\frac{2}{7}\right)$

25. $(4\sqrt{5}, 7), (6\sqrt{5}, 1)$ **26.** $(5\sqrt{2}, 8), (7\sqrt{2}, 10)$

Find the possible values of a if the points with the given coordinates are the indicated distance apart.

27. $(4, 7), (a, 3); d = 5$ **28.** $(-4, a), (4, 2); d = 17$

29. $(5, a), (6, 1); d = \sqrt{10}$ **30.** $(a, 5), (-7, 3); d = \sqrt{29}$

31. $(6, -3), (-3, a); d = \sqrt{130}$ **32.** $(20, 5), (a, 9); d = \sqrt{340}$

33. Triangle ABC has vertices at $A(7, -4)$, $B(-1, 2)$, and $C(5, -6)$. Determine whether the triangle has three, two, or no sides that are equal in length.

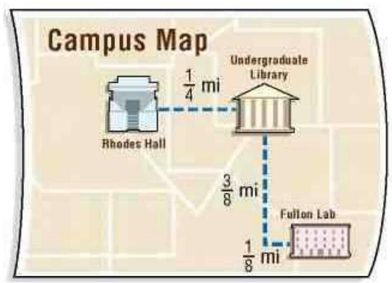
34. If the diagonals of a trapezoid have the same length, then the trapezoid is isosceles. Find the lengths of the diagonals of trapezoid $ABCD$ with vertices $A(-2, 2)$, $B(10, 6)$, $C(9, 8)$, and $D(0, 5)$ to determine if it is isosceles.

35. Triangle LMN has vertices at $L(-4, -3)$, $M(2, 5)$, and $N(-13, 10)$. If the distance from point $P(x, -2)$ to L equals the distance from P to M, what is the value of x?

36. Plot the points $Q(1, 7)$, $R(3, 1)$, $S(9, 3)$, and $T(7, d)$. Find the value of d that makes each side of $QRST$ have the same length.

37. **FREQUENT FLYERS** To determine the mileage between cities for their frequent flyer programs, some airlines superimpose a coordinate grid over the United States. An ordered pair on the grid represents the location of each airport. The units of this grid are approximately equal to 0.316 mile. So, a distance of 3 units on the grid equals an actual distance of 3(0.316) or 0.948 mile. Suppose the locations of two airports are at (132, 428) and (254, 105). Find the actual distance between these airports to the nearest mile.

COLLEGE **For Exercises 38 and 39, use the map of a college campus.**



38. Kelly has her first class in Rhodes Hall and her second class in Fulton Lab. How far does she have to walk between her first and second class?

39. She has 12 minutes between the end of her first class and the start of her second class. If she walks an average of 3 miles per hour, will she make it to her second class on time?

GEOGRAPHY **For Exercises 40–42, use the map at the left that shows part of Minnesota and Wisconsin.**

A coordinate grid has been superimposed on the map with the origin at St. Paul. The grid lines are 20 miles apart. Minneapolis is at $(-7, 3)$.

40. Estimate the coordinates for Duluth, St. Cloud, Eau Claire, and Rochester.

41. Find the distance between the following pairs of cities: Minneapolis and St. Cloud, St. Paul and Rochester, Minneapolis and Eau Claire, and Duluth and St. Cloud.

42. A radio station in St. Paul has a broadcast range of 75 miles. Which cities shown on the map can receive the broadcast?

43. **CRITICAL THINKING** Plot $A(-4, 4)$, $B(-7, -3)$, and $C(4, 0)$, and connect them to form triangle ABC. Demonstrate two different ways to show whether ABC is a right triangle.

44. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can the distance between two points be determined?

Include the following in your answer:

- an explanation how the Distance Formula is derived from the Pythagorean Theorem, and
- an explanation why the Distance Formula is not needed to find the distance between points $P(-24, 18)$ and $Q(-24, 10)$.

Standardized Test Practice

45. Find the distance between points at (6, 11) and $(-2, -4)$.

   (A) 16 units (B) 17 units

   (C) 18 units (D) 19 units

46. Find the perimeter of a square $ABCD$ if two of the vertices are $A(3, 7)$ and $B(-3, 4)$.

   (A) 12 units (B) $12\sqrt{5}$ units

   (C) $9\sqrt{5}$ units (D) 45 units

Maintain Your Skills

Mixed Review

If c is the measure of the hypotenuse of a right triangle, find each missing measure. If necessary, round to the nearest hundredth. *(Lesson 11-4)*

47. $a = 7, b = 24, c = ?$

48. $b = 30, c = 34, a = ?$

49. $a = \sqrt{7}, c = \sqrt{16}, b = ?$

50. $a = \sqrt{13}, b = \sqrt{50}, c = ?$

Solve each equation. Check your solution. *(Lesson 11-3)*

51. $\sqrt{p-2} + 8 = p$

52. $\sqrt{r+5} = r - 1$

53. $\sqrt{5t^2 + 29} = 2t + 3$

COST OF DEVELOPMENT For Exercises 54–56, use the graph that shows the amount of money being spent on worldwide construction. *(Lesson 8-3)*

54. Write the value shown for each continent or region listed in standard notation.

55. Write the value shown for each continent or region in scientific notation.

56. How much more money is being spent in Asia than in Latin America?

Solve each inequality. Then check your solution and graph it on a number line. *(Lesson 6-1)*

57. $8 \le m - 1$

58. $3 > 10 + k$

59. $3x \le 2x - 3$

60. $v - (-4) > 6$

61. $r - 5.2 \ge 3.9$

62. $s + \frac{1}{6} \le \frac{2}{3}$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Solve each proportion. *(To review **proportions**, see Lesson 3-6.)*

63. $\frac{x}{4} = \frac{3}{2}$

64. $\frac{20}{x} = \frac{-5}{2}$

65. $\frac{6}{9} = \frac{8}{x}$

66. $\frac{10}{12} = \frac{x}{18}$

67. $\frac{x+2}{7} = \frac{3}{7}$

68. $\frac{2}{3} = \frac{6}{x+4}$

11-6 Similar Triangles

What You'll Learn

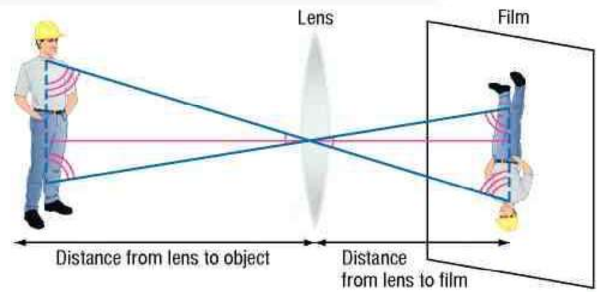
- Determine whether two triangles are similar.
- Find the unknown measures of sides of two similar triangles.

Vocabulary

- similar triangles

How are similar triangles related to photography?

When you take a picture, the image of the object being photographed is projected by the camera lens onto the film. The height of the image on the film can be related to the height of the object using similar triangles.



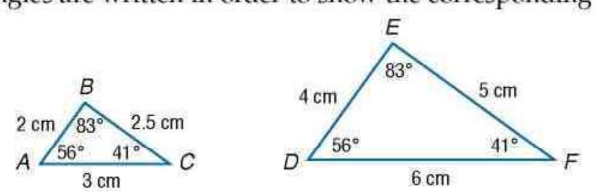
SIMILAR TRIANGLES **Similar triangles** have the same shape, but not necessarily the same size. There are two main tests for similarity.

- If the angles of one triangle and the corresponding angles of a second triangle have equal measures, then the triangles are similar.
- If the measures of the sides of two triangles form equal ratios, or are *proportional*, then the triangles are similar.

Study Tip

Reading Math
The symbol $\sim$ is read *is similar to.*

The triangles below are similar. This is written as $\triangle ABC \sim \triangle DEF$. The vertices of similar triangles are written in order to show the corresponding parts.



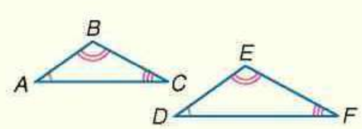
corresponding angles	corresponding sides
$\angle A$ and $\angle D$	$\overline{AB}$ and $\overline{DE} \rightarrow \frac{AB}{DE} = \frac{2}{4} = \frac{1}{2}$
$\angle B$ and $\angle E$	$\overline{BC}$ and $\overline{EF} \rightarrow \frac{BC}{EF} = \frac{2.5}{5} = \frac{1}{2}$
$\angle C$ and $\angle F$	$\overline{AC}$ and $\overline{DF} \rightarrow \frac{AC}{DF} = \frac{3}{6} = \frac{1}{2}$

Study Tip

Reading Math
Arcs are used to show angles that have equal measures.

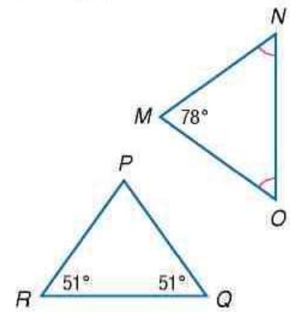
Key Concept — Similar Triangles

- **Words** If two triangles are similar, then the measures of their corresponding sides are proportional, and the measures of their corresponding angles are equal.
- **Symbols** If $\triangle ABC \sim \triangle DEF$, then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.
- **Model**



Example 1 Determine Whether Two Triangles Are Similar

Determine whether the pair of triangles is similar. Justify your answer.



Remember that the sum of the measures of the angles in a triangle is 180°.

The measure of $\angle P$ is $180° - (51° + 51°)$ or $78°$.

In $\triangle MNO$, $\angle N$ and $\angle O$ have the same measure.

Let x = the measure of $\angle N$ and $\angle O$.

$$x + x + 78° = 180°$$
$$2x = 102°$$
$$x = 51°$$

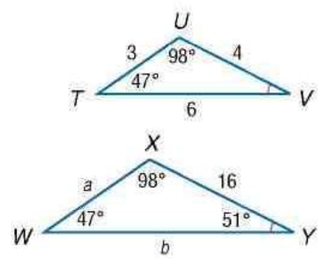
So $\angle N = 51°$ and $\angle O = 51°$. Since the corresponding angles have equal measures, $\triangle MNO \sim \triangle PQR$.

FIND UNKNOWN MEASURES Proportions can be used to find the measures of the sides of similar triangles when some of the measurements are known.

Example 2 Find Missing Measures

Find the missing measures if each pair of triangles below is similar.

a. Since the corresponding angles have equal measures, $\triangle TUV \sim \triangle WXY$. The lengths of the corresponding sides are proportional.



$\frac{WX}{TU} = \frac{XY}{UV}$ Corresponding sides of similar triangles are proportional.

$\frac{a}{3} = \frac{16}{4}$ $WX = a$, $XY = 16$, $TU = 3$, $UV = 4$

$4a = 48$ Find the cross products.

$a = 12$ Divide each side by 4.

$\frac{WY}{TV} = \frac{XY}{UV}$ Corresponding sides of similar triangles are proportional.

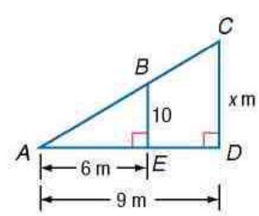
$\frac{b}{6} = \frac{16}{4}$ $WY = b$, $XY = 16$, $TV = 6$, $UV = 4$

$4b = 96$ Find the cross products.

$b = 24$ Divide each side by 4.

The missing measures are 12 and 24.

b. $\triangle ABE \sim \triangle ACD$



$\frac{BE}{CD} = \frac{AE}{AD}$ Corresponding sides of similar triangles are proportional.

$\frac{10}{x} = \frac{6}{9}$ $BE = 10$, $CD = x$, $AE = 6$, $AD = 9$

$90 = 6x$ Find the cross products.

$15 = x$ Divide each side by 6.

The missing measure is 15.

Study Tip

Corresponding Vertices

Always use the corresponding order of the vertices to write proportions for similar triangles.

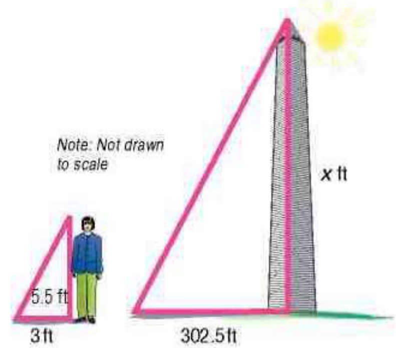
More About...

Washington Monument

The monument has a shape of an Egyptian *obelisk*. A pyramid made of solid aluminum caps the top of the monument.

Example 3 Use Similar Triangles to Solve a Problem

SHADOWS **Jenelle is standing near the Washington Monument in Washington, D.C. The shadow of the monument is 302.5 feet, and Jenelle's shadow is 3 feet. If Jenelle is 5.5 feet tall, how tall is the monument?**



The shadows form similar triangles. Write a proportion that compares the heights of the objects and the lengths of their shadows.

Let x = the height of the monument.

Jenelle's shadow → / monument's shadow → $\dfrac{3}{302.5} = \dfrac{5.5}{x}$ ← Jenelle's height / ← monument's height

$3x = 1663.75$ Cross products

$x \approx 554.6$ feet Divide each side by 3.

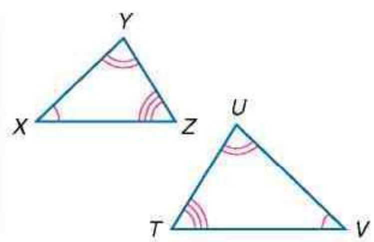
The height of the monument is about 554.6 feet.

Check for Understanding

Concept Check

1. **Explain** how to determine whether two triangles are similar.
2. **OPEN ENDED** Draw a pair of similar triangles. List the corresponding angles and the corresponding sides.
3. **FIND THE ERROR** Russell and Consuela are comparing the similar triangles below to determine their corresponding parts.

Russell	Consuela
$\angle X = \angle T$	$\angle X = \angle V$
$\angle Y = \angle U$	$\angle Y = \angle U$
$\angle Z = \angle V$	$\angle Z = \angle T$
$\triangle XYZ$, $\triangle TUV$	$\triangle XYZ$, $\triangle VUT$

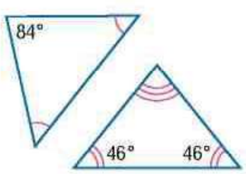


Who is correct? Explain your reasoning.

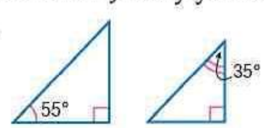
Guided Practice

Determine whether each pair of triangles is similar. Justify your answer.

4.

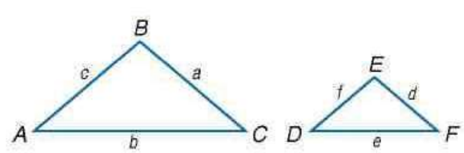


5.



For each set of measures given, find the measures of the missing sides if $\triangle ABC \sim \triangle DEF$.

6. $c = 15, d = 7, e = 9, f = 5$
7. $a = 18, c = 9, e = 10, f = 6$
8. $a = 5, d = 7, f = 6, e = 5$
9. $a = 17, b = 15, c = 10, f = 6$



Application 10. **SHADOWS** If a 25-foot flagpole casts a shadow that is 10 feet long and the nearby school building casts a shadow that is 26 feet long, how high is the building?

Practice and Apply

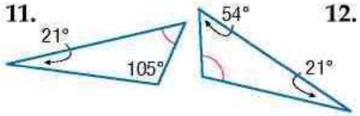
Homework Help

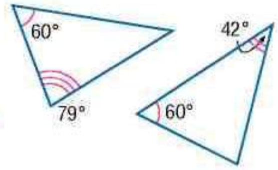
For Exercises	See Examples
11–16	1
17–24	2
25–32	3

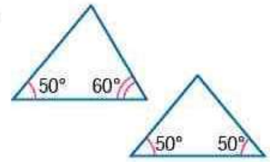
Extra Practice

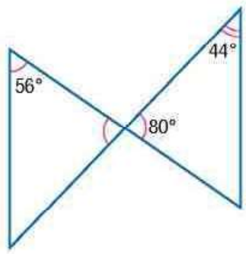
See page 845.

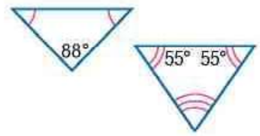
Determine whether each pair of triangles is similar. Justify your answer.

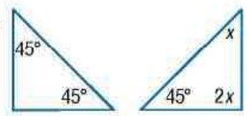
11.


12.


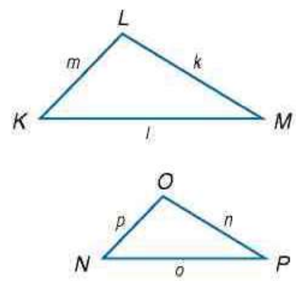
13.


14.


15.


16.


For each set of measures given, find the measures of the missing sides if $\triangle KLM \sim \triangle NOP$.



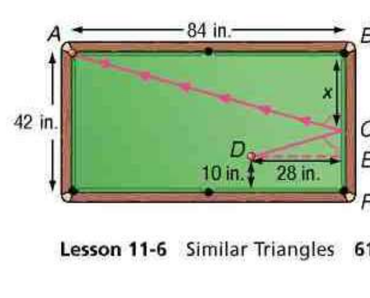
17. $k = 9, n = 6, o = 8, p = 4$
18. $k = 24, \ell = 30, m = 15, n = 16$
19. $m = 11, p = 6, n = 5, o = 4$
20. $k = 16, \ell = 13, m = 12, o = 7$
21. $n = 6, p = 2.5, \ell = 4, m = 1.25$
22. $p = 5, k = 10.5, \ell = 15, m = 7.5$
23. $n = 2.1, \ell = 4.5, p = 3.2, o = 3.4$
24. $m = 5, k = 12.6, o = 8.1, p = 2.5$

25. Determine whether the following statement is *sometimes*, *always*, or *never* true. *If the measures of the sides of a triangle are multiplied by 3, then the measures of the angles of the enlarged triangle will have the same measures as the angles of the original triangle.*

26. **PHOTOGRAPHY** Refer to the diagram of a camera at the beginning of the lesson. Suppose the image of a man who is 2 meters tall is 1.5 centimeters tall on film. If the film is 3 centimeters from the lens of the camera, how far is the man from the camera?

27. **BRIDGES** Truss bridges use triangles in their support beams. Mark plans to make a model of a truss bridge in the scale 1 inch = 12 feet. If the height of the triangles on the actual bridge is 40 feet, what will the height be on the model?

28. **BILLIARDS** Lenno is playing billiards on a table like the one shown at the right. He wants to strike the cue ball at *D*, bank it at *C*, and hit another ball at the mouth of pocket *A*. Use similar triangles to find where Lenno's cue ball should strike the rail.

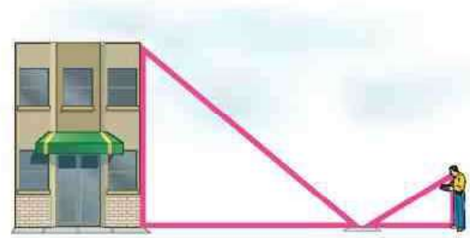


CRAFTS **For Exercises 29 and 30, use the following information.**
Melinda is working on a quilt pattern containing isosceles triangles whose sides measure 2 inches, 2 inches, and 2.5 inches.

29. She has several square pieces of material that measure 4 inches on each side. From each square piece, how many triangles with the required dimensions can she cut?

30. She wants to enlarge the pattern to make similar triangles for the center of the quilt. What is the largest similar triangle she can cut from the square material?

MIRRORS **For Exercises 31 and 32, use the diagram and the following information.**
Viho wanted to measure the height of a nearby building. He placed a mirror on the pavement at point P, 80 feet from the base of the building. He then backed away until he saw an image of the top of the building in the mirror.



31. If Viho is 6 feet tall and he is standing 9 feet from the mirror, how tall is the building?

32. What assumptions did you make in solving the problem?

CRITICAL THINKING **For Exercises 33–35, use the following information.**
The radius of one circle is twice the radius of another.

33. Are the circles similar? Explain your reasoning.

34. What is the ratio of their circumferences? Explain your reasoning.

35. What is the ratio of their areas? Explain your reasoning.

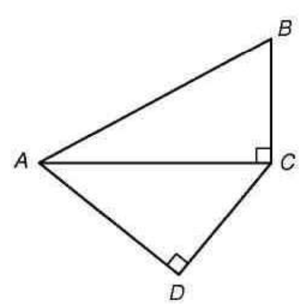
36. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How are similar triangles related to photography?

Include the following in your answer:
- an explanation of the effect of moving a camera with a zoom lens closer to the object being photographed, and
- a description of what you could do to fit the entire image of a large object on the picture.

Standardized Test Practice

For Exercises 37 and 38, use the figure at the right.



37. Which statement is true?

Ⓐ $\triangle ABC \sim \triangle ADC$
Ⓑ $\triangle ABC \sim \triangle ACD$
Ⓒ $\triangle ABC \sim \triangle CAD$
Ⓓ none of the above

38. Which statement is always true?

Ⓐ $AB > DC$ Ⓑ $CB > AD$
Ⓒ $AC > BC$ Ⓓ $AC = AB$

Maintain Your Skills

Mixed Review **Find the distance between each pair of points whose coordinates are given. Express answers in simplest radical form and as decimal approximations rounded to the nearest hundredth if necessary.** *(Lesson 11-5)*

39. $(1, 8), (-2, 4)$
40. $(6, -3), (12, 5)$
41. $(4, 7), (3, 12)$
42. $(1, 5\sqrt{6}), (6, 7\sqrt{6})$

Determine whether the following side measures form right triangles. Justify your answer. *(Lesson 11-4)*

43. 25, 60, 65
44. 20, 25, 35
45. 49, 168, 175
46. 7, 9, 12

Arrange the terms of each polynomial so that the powers of the variable are in descending order. *(Lesson 8-4)*

47. $1 + 3x^2 - 7x$
48. $7 - 4x - 2x^2 + 5x^3$
49. $6x + 3 - 3x^2$
50. $abx^2 - bcx + 34 - x^7$

Use elimination to solve each system of equations. *(Lesson 7-3)*

51. $2x + y = 4$
    $x - y = 5$
52. $3x - 2y = -13$
    $2x - 5y = -5$
53. $0.6m - 0.2n = 0.9$
    $0.3m = 0.45 - 0.1n$
54. $\frac{1}{3}x + \frac{1}{2}y = 8$
    $\frac{1}{2}x - \frac{1}{4}y = 0$

55. **AVIATION** An airplane passing over Sacramento at an elevation of 37,000 feet begins its descent to land at Reno, 140 miles away. If the elevation of Reno is 4500 feet, what should be the approximate slope of descent? (*Hint*: 1 mi = 5280 ft) *(Lesson 5-1)*

Getting Ready for the Next Lesson **PREREQUISITE SKILL** **Evaluate if $a = 6$, $b = -5$, and $c = -1.5$.**
*(To review **evaluating expressions**, see Lesson 1-2.)*

56. $\frac{a}{c}$
57. $\frac{b}{a}$
58. $\frac{a+b}{c}$
59. $\frac{ac}{b}$
60. $\frac{b}{a+c}$
61. $\frac{c}{a+c}$

Practice Quiz 2

Lessons 11-4 through 11-6

If c is the measure of the hypotenuse of a right triangle, find each missing measure. If necessary, round to the nearest hundredth. *(Lesson 11-4)*

1. $a = 14, b = 48, c = ?$
2. $a = 40, c = 41, b = ?$
3. $b = 8, c = \sqrt{84}, a = ?$
4. $a = \sqrt{5}, b = \sqrt{8}, c = ?$

Find the distance between each pair of points whose coordinates are given. *(Lesson 11-5)*

5. $(6, -12), (-3, 3)$
6. $(1, 3), (-5, 11)$
7. $(2, 5), (4, 7)$
8. $(-2, -9), (-5, 4)$

Find the measures of the missing sides if $\triangle BCA \sim \triangle EFD$. *(Lesson 11-6)*

9. $b = 10, d = 7, e = 2, f = 3$
10. $a = 12, c = 9, d = 8, e = 12$

Algebra Activity

A Preview of Lesson 11-7

Investigating Trigonometric Ratios

You can use paper triangles to investigate trigonometric ratios.

Collect the Data

Step 1 Use a ruler and grid paper to draw several right triangles whose legs are in a 7:10 ratio. Include a right triangle with legs 3.5 units and 5 units, a right triangle with legs 7 units and 10 units, another with legs 14 units and 20 units, and several more right triangles similar to these three. Label the vertices of each triangle as *A*, *B*, and *C*, where *C* is at the right angle, *B* is opposite the longest leg, and *A* is opposite the shortest leg.

Step 2 Copy the table below. Complete the first three columns by measuring the hypotenuse (side *AB*) in each right triangle you created and recording its length.

Step 3 Calculate and record the ratios in the middle two columns. Round to the nearest tenth, if necessary.

Step 4 Use a protractor to carefully measure angles *A* and *B* in each right triangle. Record the angle measures in the table.

Side Lengths			Ratios		Angle Measures		
side *BC*	side *AC*	side *AB*	*BC:AC*	*BC:AB*	angle *A*	angle *B*	angle *C*
3.5	5						90°
7	10						90°
14	20						90°
							90°
							90°
							90°

Analyze the Data

1. Examine the measures and ratios in the table. What do you notice? Write a sentence or two to describe any patterns you see.

Make a Conjecture

2. For any right triangle similar to the ones you have drawn here, what will be the value of the ratio of the length of the shortest leg to the length of the longest leg?

3. If you draw a right triangle and calculate the ratio of the length of the shortest leg to the length of the hypotenuse to be approximately 0.573, what will be the measure of the larger acute angle in the right triangle?

11-7 Trigonometric Ratios

What You'll Learn

- Define the sine, cosine, and tangent ratios.
- Use trigonometric ratios to solve right triangles.

Vocabulary

- trigonometric ratios
- sine
- cosine
- tangent
- solve a triangle
- angle of elevation
- angle of depression

How are trigonometric ratios used in surveying?

Surveyors use triangle ratios called trigonometric ratios to determine distances that cannot be measured directly.

- In 1852, British surveyors measured the altitude of the peak of Mt. Everest at 29,002 feet using these trigonometric ratios.
- In 1954, the official height became 29,028 feet, which was also calculated using surveying techniques.
- On November 11, 1999, a team using advanced technology and the Global Positioning System (GPS) satellite measured the mountain at 29,035 feet.

TRIGONOMETRIC RATIOS *Trigonometry* is an area of mathematics that involves angles and triangles. If enough information is known about a right triangle, certain ratios can be used to find the measures of the remaining parts of the triangle. **Trigonometric ratios** are ratios of the measures of two sides of a right triangle. Three common trigonometric ratios are called **sine**, **cosine**, and **tangent**.

Key Concept — Trigonometric Ratios

- **Words**

$$\text{sine of } \angle A = \frac{\text{measure of leg opposite } \angle A}{\text{measure of hypotenuse}}$$

$$\text{cosine of } \angle A = \frac{\text{measure of leg adjacent to } \angle A}{\text{measure of hypotenuse}}$$

$$\text{tangent of } \angle A = \frac{\text{measure of leg opposite } \angle A}{\text{measure of leg adjacent to } \angle A}$$

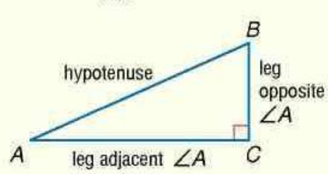
- **Symbols**

$$\sin A = \frac{BC}{AB}$$

$$\cos A = \frac{AC}{AB}$$

$$\tan A = \frac{BC}{AC}$$

- **Model**

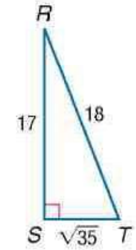


Study Tip

Reading Math
Notice that sine, cosine, and tangent are abbreviated sin, cos, and tan respectively.

Example 1 Sine, Cosine, and Tangent

Find the sine, cosine, and tangent of each acute angle of $\triangle RST$. Round to the nearest ten thousandth.



Write each ratio and substitute the measures. Use a calculator to find each value.

$\sin R = \frac{\text{opposite leg}}{\text{hypotenuse}}$ $\quad \cos R = \frac{\text{adjacent leg}}{\text{hypotenuse}}$ $\quad \tan R = \frac{\text{opposite leg}}{\text{adjacent leg}}$

$= \frac{\sqrt{35}}{18}$ or 0.3287 $\quad = \frac{17}{18}$ or 0.9444 $\quad = \frac{\sqrt{35}}{17}$ or 0.3480

$\sin T = \frac{\text{opposite leg}}{\text{hypotenuse}}$ $\quad \cos T = \frac{\text{adjacent leg}}{\text{hypotenuse}}$ $\quad \tan T = \frac{\text{opposite leg}}{\text{adjacent leg}}$

$= \frac{17}{18}$ or 0.9444 $\quad = \frac{\sqrt{35}}{18}$ or 0.3287 $\quad = \frac{17}{\sqrt{35}}$ or 2.8735

You can use a calculator to find the values of trigonometric functions or to find the measure of an angle. On a graphing calculator, press the trigometric function key, and then enter the value. On a nongraphing scientific calculator, enter the value, and then press the function key. In either case, be sure your calculator is in degree mode. Consider cos 50°.

Graphing Calculator

KEYSTROKES: [COS] 50 [ENTER] .6427876097

Nongraphing Scientific Calculator

KEYSTROKES: 50 [COS] .642787609

Example 2 Find the Sine of an Angle

Find sin 35° to the nearest ten thousandth.

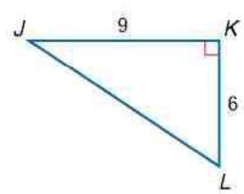
KEYSTROKES: [SIN] 35 [ENTER] .5735764364

Rounded to the nearest ten thousandth, $\sin 35° \approx 0.5736$.

Example 3 Find the Measure of an Angle

Find the measure of $\angle J$ to the nearest degree.

Since the lengths of the opposite and adjacent sides are known, use the tangent ratio.



$\tan J = \frac{\text{opposite leg}}{\text{adjacent leg}}$ Definition of tangent

$= \frac{6}{9}$ $KL = 6$ and $JK = 9$

Now use the [TAN⁻¹] on a calculator to find the measure of the angle whose tangent ratio is $\frac{6}{9}$.

KEYSTROKES: [2nd] [TAN^{-1}] 6 [÷] 9 [ENTER] 33.69006753

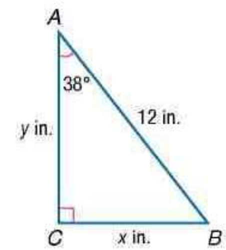
To the nearest degree, the measure of $\angle J$ is 34°.

SOLVE TRIANGLES You can find the missing measures of a right triangle if you know the measure of two sides of a triangle or the measure of one side and one acute angle. Finding all of the measures of the sides and the angles in a right triangle is called **solving the triangle**.

Example 4 Solve a Triangle

Find all of the missing measures in $\triangle ABC$.

You need to find the measures of $\angle B$, $\overline{AC}$, and $\overline{BC}$.



Step 1 Find the measure of $\angle B$. The sum of the measures of the angles in a triangle is 180.

$180° - 90° - 38° = 52°$

The measure of $\angle B$ is 52°.

Step 2 Find the value of x, which is the measure of the side opposite $\angle A$. Use the sine ratio.

$\sin 38° = \frac{x}{12}$ Definition of sine

$0.6157 \approx \frac{x}{12}$ Evaluate sin 38°.

$7.4 \approx x$ Multiply by 12.

$\overline{BC}$ is about 7.4 inches long.

Step 3 Find the value of y, which is the measure of the side adjacent to $\angle A$. Use the cosine ratio.

$\cos 38° = \frac{y}{12}$ Definition of cosine

$0.7880 \approx \frac{y}{12}$ Evaluate cos 38°.

$9.5 \approx y$ Multiply by 12.

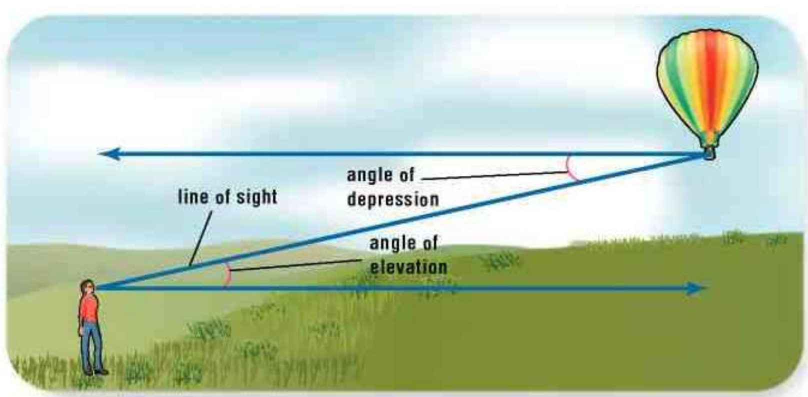
$\overline{AC}$ is about 9.5 inches long.

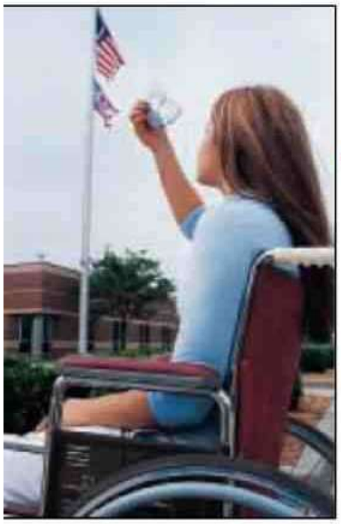
So, the missing measures are 52°, 7.4 in., and 9.5 in.

Study Tip

Verifying Right Triangles
You can use the Pythagorean Theorem to verify that the sides are sides of a right triangle.

Trigonometric ratios are often used to find distances or lengths that cannot be measured directly. In these situations, you will sometimes use an angle of elevation or an angle of depression. An **angle of elevation** is formed by a horizontal line of sight and a line of sight above it. An **angle of depression** is formed by a horizontal line of sight and a line of sight below it.

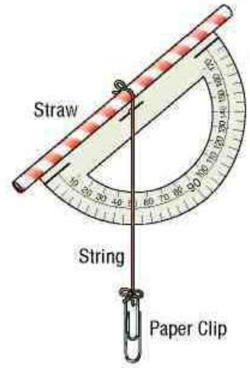




Algebra Activity

Make a Hypsometer

- Tie one end of a piece of string to the middle of a straw. Tie the other end of string to a paper clip.
- Tape a protractor to the side of the straw. Make sure that the string hangs freely to create a vertical or plumb line.
- Find an object outside that is too tall to measure directly, such as a basketball hoop, a flagpole, or the school building.
- Look through the straw to the top of the object you are measuring. Find the angle measure where the string and protractor intersect. Determine the angle of elevation by subtracting this measurement from 90°.
- Measure the distance from your eye level to the ground and from your foot to the base of the object you are measuring.



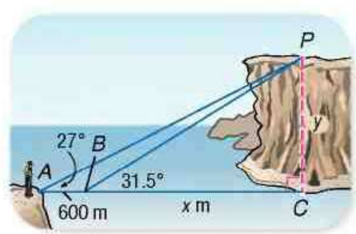
Analyze

1. Make a sketch of your measurements. Use the equation $\tan \text{(angle of elevation)} = \frac{\text{height of object} - x}{\text{distance of object}}$, where x represents distance from the ground to your eye level, to find the height of the object.
2. Why do you have to subtract the angle measurement on the hypsometer from 90° to find the angle of elevation?
3. Compare your answer with someone who measured the same object. Did your heights agree? Why or why not?

Example 5 Angle of Elevation

INDIRECT MEASUREMENT **At point A, Umeko measured the angle of elevation to point P to be 27 degrees. At another point B, which was 600 meters closer to the cliff, Umeko measured the angle of elevation to point P to be 31.5 degrees. Determine the height of the cliff.**

Explore Draw a diagram to model the situation. Two right triangles, $\triangle BPC$ and $\triangle APC$, are formed. You know the angle of elevation for each triangle. To determine the height of the cliff, find the length of $\overline{PC}$, which is shared by both triangles.



Plan Let y represent the distance from the top of the cliff P to its base C. Let x represent BC in the first triangle and let $x + 600$ represent AC.

Solve Write two equations involving the tangent ratio.

$$\tan 31.5° = \frac{y}{x} \quad \text{and} \quad \tan 27° = \frac{y}{600 + x}$$

$$x \tan 31.5° = y \qquad (600 + x)\tan 27° = y$$

Since both expressions are equal to y, use substitution to solve for x.

$x \tan 31.5° = (600 + x) \tan 27°$ Substitute.

$x \tan 31.5° = 600 \tan 27° + x \tan 27°$ Distributive Property

$x \tan 31.5° - x \tan 27° = 600 \tan 27°$ Subtract.

$x(\tan 31.5° - \tan 27°) = 600 \tan 27°$ Isolate x.

$x = \frac{600 \tan 27°}{\quad}$ Divide.

$x \approx 2960$ feet Use a calculator.

Use this value for x and the equation $x \tan 31.5° = y$ to solve for y.

$x \tan 31.5° = y$ Original equation

$2960 \tan 31.5° \approx y$ Replace x with 2960.

$1814 \approx y$ Use a calculator.

The height of the cliff is about 1814 feet.

Examine Examine the solution by finding the angles of elevation.

$\tan B = \frac{y}{x}$	$\tan A = \frac{y}{600 + x}$
$\tan B \stackrel{?}{=} \frac{1814}{2960}$	$\tan A \stackrel{?}{=} \frac{1814}{600 + 2960}$
$B = 31.5°$	$A = 27°$

The solution checks.

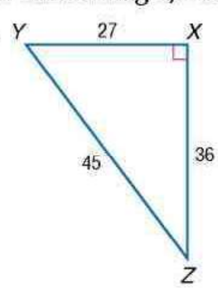
Check for Understanding

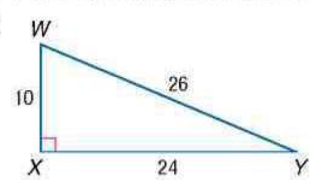
Concept Check

1. **Explain** how to determine which trigonometric ratio to use when solving for an unknown measure of a right triangle.
2. **OPEN ENDED** Draw a right triangle and label the measure of the hypotenuse and the measure of one acute angle. Then solve for the remaining measures.
3. **Compare** the measure of the angle of elevation and the measure of the angle of depression for two objects. What is the relationship between their measures?

Guided Practice

For each triangle, find sin *Y*, cos *Y*, and tan *Y* to the nearest ten thousandth.

4. 

5. 

Use a calculator to find the value of each trigonometric ratio to the nearest ten thousandth.

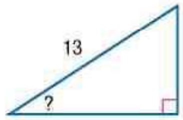
6. $\sin 60°$
7. $\cos 75°$
8. $\tan 10°$

Use a calculator to find the measure of each angle to the nearest degree.

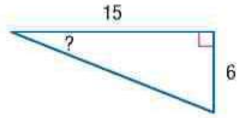
9. $\sin W = 0.9848$
10. $\cos X = 0.6157$
11. $\tan C = 0.3249$

For each triangle, find the measure of the indicated angle to the nearest degree.

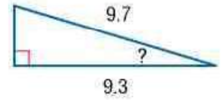
12.



13.

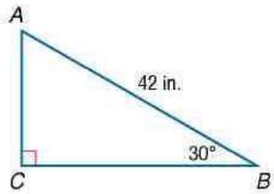


14.

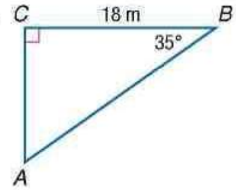


Solve each right triangle. State the side lengths to the nearest tenth and the angle measures to the nearest degree.

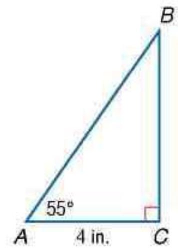
15.



16.



17.



Application

18. **DRIVING** The percent grade of a road is the ratio of how much the road rises or falls in a given horizontal distance. If a road has a vertical rise of 40 feet for every 1000 feet horizontal distance, calculate the percent grade of the road and the angle of elevation the road makes with the horizontal.

Practice and Apply

Homework Help

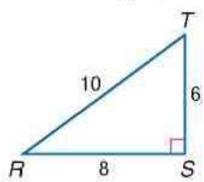
For Exercises	See Examples
19–24	1
25–33	2
34–51	3
52–60	4
61–65	5

Extra Practice

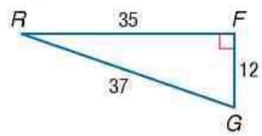
See page 846.

For each triangle, find sin R, cos R, and tan R to the nearest ten thousandth.

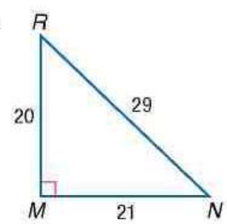
19.



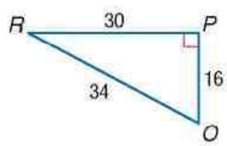
20.



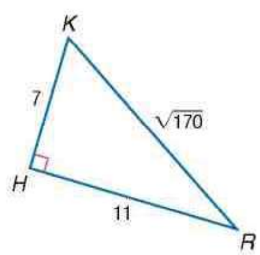
21.



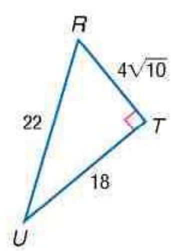
22.



23.



24.



Use a calculator to find the value of each trigonometric ratio to the nearest ten thousandth.

25. sin 30° **26.** sin 80° **27.** cos 45°

28. cos 48° **29.** tan 32° **30.** tan 15°

31. tan 67° **32.** sin 53° **33.** cos 12°

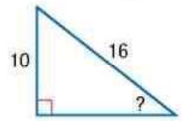
Use a calculator to find the measure of each angle to the nearest degree.

34. cos V = 0.5000 **35.** cos Q = 0.7658 **36.** sin K = 0.9781

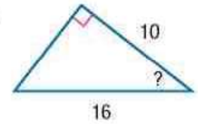
37. sin A = 0.8827 **38.** tan S = 1.2401 **39.** tan H = 0.6473

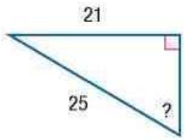
40. sin V = 0.3832 **41.** cos M = 0.9793 **42.** tan L = 3.6541

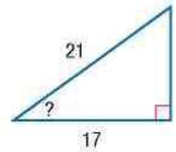
For each triangle, find the measure of the indicated angle to the nearest degree.

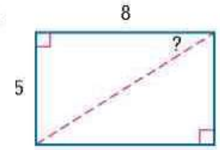
43. 

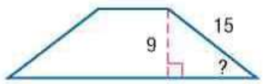
44. 

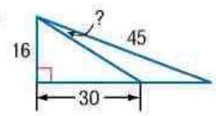
45. 

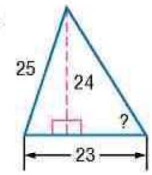
46. 

47. 

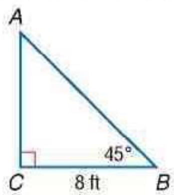
48. 

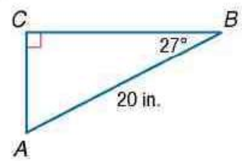
49. 

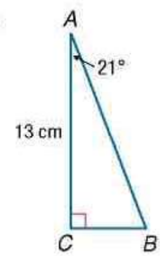
50. 

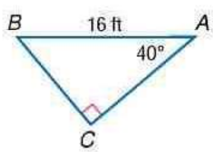
51. 

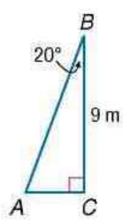
Solve each right triangle. State the side lengths to the nearest tenth and the angle measures to the nearest degree.

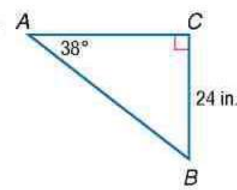
52. 

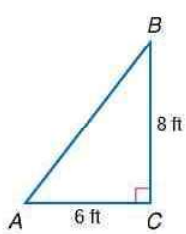
53. 

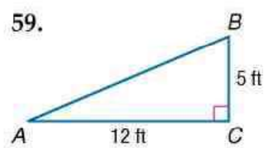
54. 

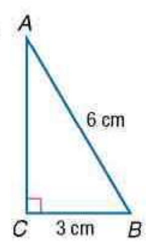
55. 

56. 

57. 

58. 

59. 

60. 

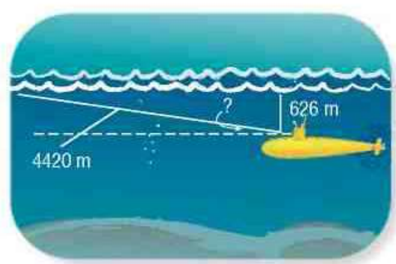
More About. . .

Submarines

In emergency situations, modern submarines are built to allow for a rapid surfacing, a technique called an emergency main ballast blow.

Source: www.howstuffworks.com

SUBMARINES **For Exercises 61 and 62, use the following information.**
A submarine is traveling parallel to the surface of the water 626 meters below the surface. The sub begins a constant ascent to the surface so that it will emerge on the surface after traveling 4420 meters from the point of its initial ascent.



61. What angle of ascent did the submarine make?

62. What horizontal distance did the submarine travel during its ascent?

AVIATION **For Exercises 63 and 64, use the following information.**
Germaine pilots a small plane on weekends. During a recent flight, he determined that he was flying 3000 feet parallel to the ground and that the ground distance to the start of the landing strip was 8000 feet.

63. What is Germaine's angle of depression to the start of the landing strip?

64. What is the distance between the plane in the air and the landing strip on the ground?

65. **FARMING** Leonard and Alecia are building a new feed storage system on their farm. The feed conveyor must be able to reach a range of heights. It has a length of 8 meters, and its angle of elevation can be adjusted from 20° to 5°. Under these conditions, what range of heights is possible for an opening in the building through which feed can pass?

66. **CRITICAL THINKING** An important trigonometric identity is $\sin^2 A + \cos^2 A = 1$. Use the sine and cosine ratios and the Pythagorean Theorem to prove this identity.

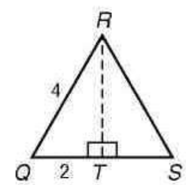
67. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson.

How are trigonometric ratios used in surveying?

Include the following in your answer:
- an explanation of how trigonometric ratios are used to measure the height of a mountain, and
- any additional information you need to know about the point from which you are measuring in order to find the altitude of a mountain.

Standardized Test Practice

For Exercises 68 and 69, use the figure at the right.



68. RT is equal to TS. What is RS?

Ⓐ $2\sqrt{6}$ Ⓑ $2\sqrt{3}$ Ⓒ $4\sqrt{3}$ Ⓓ $2\sqrt{2}$

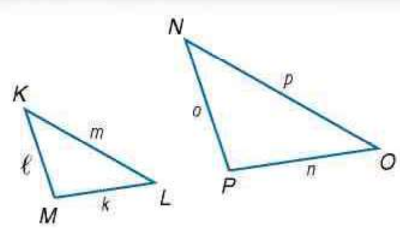
69. What is the measure of $\angle Q$?

Ⓐ 25° Ⓑ 30° Ⓒ 45° Ⓓ 60°

Maintain Your Skills

Mixed Review

For each set of measures given, find the measures of the missing sides if $\triangle KLM \sim \triangle NOP$. *(Lesson 11-6)*



70. $k = 5, \ell = 3, m = 6, n = 10$

71. $\ell = 9, m = 3, n = 12, p = 4.5$

Find the possible values of a if the points with the given coordinates are the indicated distance apart. *(Lesson 11-5)*

72. $(9, 28), (a, -8); d = 39$

73. $(3, a), (10, -1); d = \sqrt{65}$

Find each product. *(Lesson 8-6)*

74. $c^2(c^2 + 3c)$

75. $s(4s^2 - 9s + 12)$

76. $xy^2(2x^2 + 5xy - 7y^2)$

Use substitution to solve each system of equations. *(Lesson 7-2)*

77. $a = 3b + 2$
$4a - 7b = 23$

78. $p + q = 10$
$3p - 2q = -5$

79. $3r + 6s = 0$
$-4r - 10s = -2$

Reading Mathematics

The Language of Mathematics

The language of mathematics is a specific one, but it borrows from everyday language, scientific language, and world languages. To find a word's correct meaning, you will need to be aware of some confusing aspects of language.

Confusing Aspect	Words
Some words are used in English and in mathematics, but have distinct meanings.	factor, **leg**, prime, power, **rationalize**
Some words are used in English and in mathematics, but the mathematical meaning is more precise.	difference, even, **similar**, slope
Some words are used in science and in mathematics, but the meanings are different.	divide, **radical**, solution, variable
Some words are only used in mathematics.	decimal, **hypotenuse**, integer, quotient
Some words have more than one mathematical meaning.	base, **degree**, range, round, square
Sometimes several words come from the same root word.	polygon and polynomial, **radical** and **radicand**
Some mathematical words sound like English words.	**cosine** and cosign, **sine** and sign, sum and some
Some words are often abbreviated, but you must use the whole word when you read them.	**cos** for cosine, **sin** for sine, **tan** for tangent

Words in boldface are in this chapter.

Reading to Learn

1. How do the mathematical meanings of the following words compare to the everyday meanings?

a. factor **b.** leg **c.** rationalize

2. State two mathematical definitions for each word. Give an example for each definition.

a. degree **b.** range **c.** round

3. Each word below is shown with its root word and the root word's meaning. Find three additional words that come from the same root.

a. domain, from the root word *domus*, which means house

b. radical, from the root word *radix*, which means root

c. similar, from the root word *similis*, which means like

Chapter 11 Study Guide and Review

Vocabulary and Concept Check

angle of depression (p. 625)
angle of elevation (p. 625)
conjugate (p. 589)
corollary (p. 607)
cosine (p. 623)
Distance Formula (p. 611)
extraneous solution (p. 599)
hypotenuse (p. 605)
leg (p. 605)
Pythagorean triple (p. 606)
radical equation (p. 598)
radical expression (p. 586)
radicand (p. 586)
rationalizing the denominator (p. 588)
similar triangles (p. 616)
sine (p. 623)
solve a triangle (p. 625)
tangent (p. 623)
trigonometric ratios (p. 623)

State whether each sentence is *true* or *false*. If false, replace the underlined word, number, expression, or equation to make a true sentence.

1. The binomials $-3 + \sqrt{7}$ and $\underline{3 - \sqrt{7}}$ are conjugates.
2. In the expression $-4\sqrt{5}$, the radicand is $\underline{5}$.
3. The sine of an angle is the measure of the opposite leg divided by the measure of the <u>hypotenuse</u>.
4. The <u>longest</u> side of a right triangle is the hypotenuse.
5. After the first step in solving $\sqrt{3x + 19} = x + 3$, you would have $\underline{3x + 19 = x^2 + 9}$.
6. The two sides that form the right angle in a right triangle are called the <u>legs</u> of the triangle.
7. The expression $\underline{\dfrac{2x\sqrt{3x}}{\sqrt{6y}}}$ is in simplest radical form.
8. A triangle with sides having measures of <u>25, 20, and 15</u> is a right triangle.

Lesson-by-Lesson Review

11-1 Simplifying Radical Expressions

See pages 586–592.

Concept Summary

- A radical expression is in simplest form when no radicands have perfect square factors other than 1, no radicands contain fractions, and no radicals appear in the denominator of a fraction.

Example Simplify $\dfrac{3}{5 - \sqrt{2}}$.

$$\frac{3}{5-\sqrt{2}} = \frac{3}{5-\sqrt{2}} \cdot \frac{5+\sqrt{2}}{5+\sqrt{2}} \qquad \text{Multiply by } \frac{5+\sqrt{2}}{5+\sqrt{2}} \text{ to rationalize the denominator.}$$

$$= \frac{3(5) + 3\sqrt{2}}{5^2 - (\sqrt{2})^2} \qquad (a-b)(a+b) = a^2 - b^2$$

$$= \frac{15 + 3\sqrt{2}}{25 - 2} \qquad (\sqrt{2})^2 = 2$$

$$= \frac{15 + 3\sqrt{2}}{23} \qquad \text{Simplify.}$$

www.algebra1.com/vocabulary_review

Exercises **Simplify.** *See Examples 1–5 on pages 586–589.*

9. $\sqrt{\frac{60}{y^2}}$ **10.** $\sqrt{44a^2b^5}$ **11.** $(3 - 2\sqrt{12})^2$

12. $\frac{9}{3 + \sqrt{2}}$ **13.** $\frac{2\sqrt{7}}{3\sqrt{5} + 5\sqrt{3}}$ **14.** $\frac{\sqrt{3a^3b^4}}{\sqrt{8ab^{10}}}$

11-2 Operations with Radical Expressions

See pages 593–597.

Concept Summary

- Radical expressions with like radicands can be added or subtracted.
- Use the FOIL Method to multiply radical expressions.

Examples

1 **Simplify $\sqrt{6} - \sqrt{54} + 3\sqrt{12} + 5\sqrt{3}$.**

$\sqrt{6} - \sqrt{54} + 3\sqrt{12} + 5\sqrt{3}$

$= \sqrt{6} - \sqrt{3^2 \cdot 6} + 3\sqrt{2^2 \cdot 3} + 5\sqrt{3}$ Simplify radicands.

$= \sqrt{6} - (\sqrt{3^2} \cdot \sqrt{6}) + 3(\sqrt{2^2} \cdot \sqrt{3}) + 5\sqrt{3}$ Product Property of Square Roots

$= \sqrt{6} - 3\sqrt{6} + 3(2\sqrt{3}) + 5\sqrt{3}$ Evaluate square roots.

$= \sqrt{6} - 3\sqrt{6} + 6\sqrt{3} + 5\sqrt{3}$ Simplify.

$= -2\sqrt{6} + 11\sqrt{3}$ Add like radicands.

2 **Find $(2\sqrt{3} - \sqrt{5})(\sqrt{10} + 4\sqrt{6})$.**

$(2\sqrt{3} - \sqrt{5})(\sqrt{10} + 4\sqrt{6})$

First terms Outer terms Inner terms Last terms

$= (2\sqrt{3})(\sqrt{10}) + (2\sqrt{3})(4\sqrt{6}) + (-\sqrt{5})(\sqrt{10}) + (-\sqrt{5})(4\sqrt{6})$

$= 2\sqrt{30} + 8\sqrt{18} - \sqrt{50} - 4\sqrt{30}$ Multiply.

$= 2\sqrt{30} + 8\sqrt{3^2 \cdot 2} - \sqrt{5^2 \cdot 2} - 4\sqrt{30}$ Prime factorization

$= 2\sqrt{30} + 24\sqrt{2} - 5\sqrt{2} - 4\sqrt{30}$ Simplify.

$= -2\sqrt{30} + 19\sqrt{2}$ Combine like terms.

Exercises **Simplify each expression.** *See Examples 1 and 2 on pages 593 and 594.*

15. $2\sqrt{3} + 8\sqrt{5} - 3\sqrt{5} + 3\sqrt{3}$ **16.** $2\sqrt{6} - \sqrt{48}$

17. $4\sqrt{27} + 6\sqrt{48}$ **18.** $4\sqrt{7k} - 7\sqrt{7k} + 2\sqrt{7k}$

19. $5\sqrt{18} - 3\sqrt{112} - 3\sqrt{98}$ **20.** $\sqrt{8} + \sqrt{\frac{1}{8}}$

Find each product. *See Example 3 on page 594.*

21. $\sqrt{2}(3 + 3\sqrt{3})$ **22.** $\sqrt{5}(2\sqrt{5} - \sqrt{7})$

23. $(\sqrt{3} - \sqrt{2})(2\sqrt{2} + \sqrt{3})$ **24.** $(6\sqrt{5} + 2)(3\sqrt{2} + \sqrt{5})$

11-3 Radical Expressions

See pages 598–603.

Concept Summary

- Solve radical equations by isolating the radical on one side of the equation. Square each side of the equation to eliminate the radical.

Example **Solve $\sqrt{5-4x}-6=7$.**

$\sqrt{5-4x}-6=7$ Original equation

$\sqrt{5-4x}=13$ Add 6 to each side.

$5-4x=169$ Square each side.

$-4x=164$ Subtract 5 from each side.

$x=-41$ Divide each side by -4.

Exercises **Solve each equation. Check your solution.** *See Examples 2 and 3 on page 599.*

25. $10+2\sqrt{b}=0$ **26.** $\sqrt{a+4}=6$ **27.** $\sqrt{7x-1}=5$

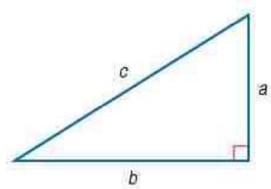
28. $\sqrt{\frac{4a}{3}}-2=0$ **29.** $\sqrt{x+4}=x-8$ **30.** $\sqrt{3x-14}+x=6$

11-4 The Pythagorean Theorem

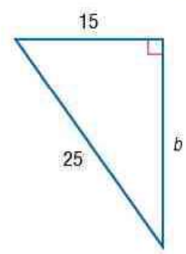
See pages 605–610.

Concept Summary

- If a and b are the measures of the legs of a right triangle and c is the measure of the hypotenuse, then $c^2=a^2+b^2$.
- If a and b are measures of the shorter sides of a triangle, c is the measure of the longest side, and $c^2=a^2+b^2$, then the triangle is a right triangle.



Example **Find the length of the missing side.**



$c^2=a^2+b^2$ Pythagorean Theorem

$25^2=15^2+b^2$ $c=25$ and $a=15$

$625=225+b^2$ Evaluate squares.

$400=b^2$ Subtract 225 from each side.

$20=b$ Take the square root of each side.

Exercises **If c is the measure of the hypotenuse of a right triangle, find each missing measure. If necessary, round answers to the nearest hundredth.**
See Example 2 on page 606.

31. $a=30, b=16, c=?$ **32.** $a=6, b=10, c=?$ **33.** $a=10, c=15, b=?$

34. $b=4, c=56, a=?$ **35.** $a=18, c=30, b=?$ **36.** $a=1.2, b=1.6, c=?$

Determine whether the following side measures form right triangles.
See Example 4 on page 608.

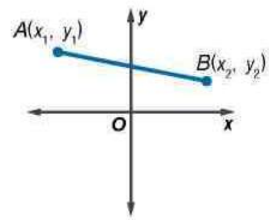
37. 9, 16, 20 **38.** 20, 21, 29 **39.** 9, 40, 41 **40.** 18, $\sqrt{24}$, 30

11-5 The Distance Formula

See pages 611–615.

Concept Summary

- The distance d between any two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Example **Find the distance between the points with coordinates (−5, 1) and (1, 5).**

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	Distance Formula
$= (1 - (-5))^2 + (5 - 1)^2$	$(x_1, y_1) = (-5, 1)$ and $(x_2, y_2) = (1, 5)$
$= \sqrt{6^2 + 4^2}$	Simplify.
$= \sqrt{36 + 16}$	Evaluate squares.
$= \sqrt{52}$ or about 7.21 units	Simplify.

Exercises **Find the distance between each pair of points whose coordinates are given. Express in simplest radical form and as decimal approximations rounded to the nearest hundredth if necessary.** *See Example 1 on page 611.*

41. $(9, -2), (1, 13)$
42. $(4, 2), (7, 9)$
43. $(4, -6), (-2, 7)$
44. $(2\sqrt{5}, 9), (4\sqrt{5}, 3)$
45. $(4, 8), (-7, 12)$
46. $(-2, 6), (5, 11)$

Find the value of a if the points with the given coordinates are the indicated distance apart. *See Example 3 on page 612.*

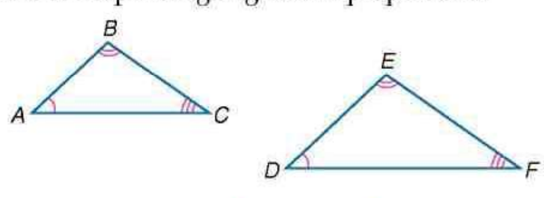
47. $(-3, 2), (1, a)$; $d = 5$
48. $(1, 1), (4, a)$; $d = 5$
49. $(6, -2), (5, a)$; $d = \sqrt{145}$
50. $(5, -2), (a, -3)$; $d = \sqrt{170}$

11-6 Similar Triangles

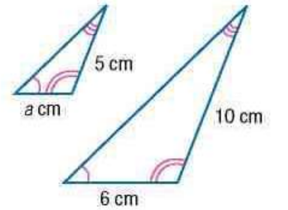
See pages 616–621.

Concept Summary

- Similar triangles have congruent corresponding angles and proportional corresponding sides.
- If $\triangle ABC \sim \triangle DEF$, then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.



Example **Find the measure of side a if the two triangles are similar.**



$\frac{10}{5} = \frac{6}{a}$	Corresponding sides of similar triangles are proportional.
$10a = 30$	Find the cross products.
$a = 3$	Divide each side by 10.

Chapter 11 **For More ...**

- Extra Practice, see pages 844–846.
- Mixed Problem Solving, see page 863.

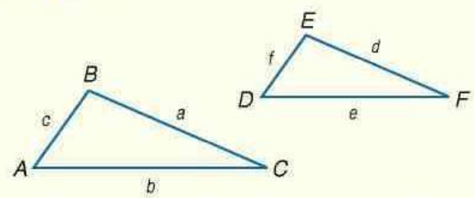
Exercises **For each set of measures given, find the measures of the remaining sides if $\triangle ABC \sim \triangle DEF$.** *See Example 2 on page 617.*

51. $c = 16, b = 12, a = 10, f = 9$

52. $a = 8, c = 10, b = 6, f = 12$

53. $c = 12, f = 9, a = 8, e = 11$

54. $b = 20, d = 7, f = 6, c = 15$



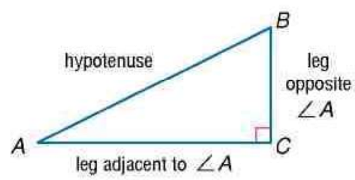
11-7 Trigonometric Ratios

See pages 623–630.

Concept Summary

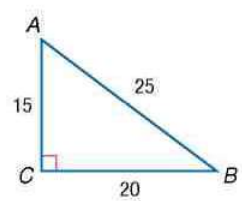
Three common trigonometric ratios are sine, cosine, and tangent.

- $\sin A = \frac{BC}{AB}$
- $\cos A = \frac{AC}{AB}$
- $\tan A = \frac{BC}{AC}$



Example **Find the sine, cosine, and tangent of $\angle A$. Round to the nearest ten thousandth.**

$$\sin A = \frac{\text{opposite leg}}{\text{hypotenuse}}$$
$$= \frac{20}{25} \text{ or } 0.8000$$
$$\cos A = \frac{\text{adjacent leg}}{\text{hypotenuse}}$$
$$= \frac{15}{25} \text{ or } 0.6000$$
$$\tan A = \frac{\text{opposite leg}}{\text{adjacent leg}}$$
$$= \frac{20}{15} \text{ or } 1.3333$$



Exercises **For $\triangle ABC$, find each value of each trigonometric ratio to the nearest ten thousandth.** *See Example 1 on page 624.*

55. $\cos B$

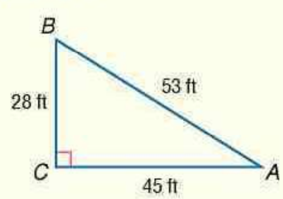
56. $\tan A$

57. $\sin B$

58. $\cos A$

59. $\tan B$

60. $\sin A$



Use a calculator to find the measure of each angle to the nearest degree.
See Example 3 on page 624.

61. $\tan M = 0.8043$ **62.** $\sin T = 0.1212$ **63.** $\cos B = 0.9781$

64. $\cos F = 0.7443$ **65.** $\sin A = 0.4540$ **66.** $\tan Q = 5.9080$

Chapter 11 Practice Test

Vocabulary and Concepts

Match each term and its definition.

1. measure of the opposite side divided by the measure of the hypotenuse
2. measure of the adjacent side divided by the measure of the hypotenuse
3. measure of the opposite side divided by the measure of the adjacent side

a. cosine
b. sine
c. tangent

Skills and Applications

Simplify.

4. $2\sqrt{27} + \sqrt{63} - 4\sqrt{3}$ **5.** $\sqrt{6} + \sqrt{\frac{2}{3}}$ **6.** $\sqrt{112x^4y^6}$

7. $\sqrt{\frac{10}{3}} \cdot \sqrt{\frac{4}{30}}$ **8.** $\sqrt{6}(4 + \sqrt{12})$ **9.** $(1 - \sqrt{3})(3 + \sqrt{2})$

Solve each equation. Check your solution.

10. $\sqrt{10x} = 20$ **11.** $\sqrt{4s} + 1 = 11$ **12.** $\sqrt{4x + 1} = 5$

13. $x = \sqrt{-6x - 8}$ **14.** $x = \sqrt{5x + 14}$ **15.** $\sqrt{4x - 3} = 6 - x$

If c is the measure of the hypotenuse of a right triangle, find each missing measure. If necessary, round to the nearest hundredth.

16. $a = 8, b = 10, c = ?$ **17.** $a = 6\sqrt{2}, c = 12, b = ?$ **18.** $b = 13, c = 17, a = ?$

Find the distance between each pair of points whose coordinates are given. Express in simplest radical form and as decimal approximations rounded to the nearest hundredth if necessary.

19. $(4, 7), (4, -2)$ **20.** $(-1, 1), (1, -5)$ **21.** $(-9, 2), (21, 7)$

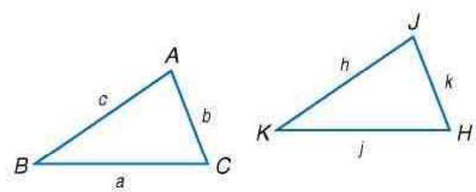
For each set of measures given, find the measures of the missing sides if $\triangle ABC \sim \triangle JKH$.

22. $c = 20, h = 15, k = 16, j = 12$

23. $c = 12, b = 13, a = 6, h = 10$

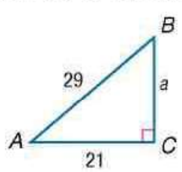
24. $k = 5, c = 6.5, b = 7.5, a = 4.5$

25. $h = 1\frac{1}{2}, c = 4\frac{1}{2}, k = 2\frac{1}{4}, a = 3$

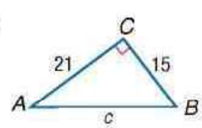


Solve each right triangle. State the side lengths to the nearest tenth and the angle measures to the nearest degree.

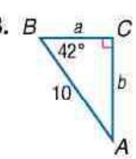
26.



27.



28.

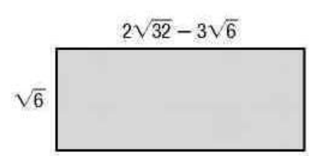


29. SPORTS A hiker leaves her camp in the morning. How far is she from camp after walking 9 miles due west and then 12 miles due north?

30. STANDARDIZED TEST PRACTICE Find the area of the rectangle.

Ⓐ $16\sqrt{2} - 4\sqrt{6}$ units² Ⓑ $16\sqrt{3} - 18$ units²

Ⓒ $32\sqrt{3} - 18$ units² Ⓓ $2\sqrt{32} - 18$ units²



Chapter 11 Standardized Test Practice

Part 1 Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. Which equation describes the data in the table? (Lesson 4-8)

x	-5	-2	1	4
y	11	5	-1	-7

(A) $y = x - 6$ (B) $y = 2x - 1$

(C) $y = 2x + 1$ (D) $y = -2x + 1$

2. The length of a rectangle is 6 feet more than the width. The perimeter is 92 feet. Which system of equations will determine the length in feet ℓ and the width in feet w of the rectangle? (Lesson 7-2)

(A) $w = \ell + 6$, $2\ell + 2w = 92$ (B) $\ell + w = 6$, $\ell w = 92$

(C) $\ell = w + 6$, $2\ell + 2w = 92$ (D) $\ell - w = 6$, $\ell + w = 92$

3. A highway resurfacing project and a bridge repair project will cost \$2,500,000 altogether. The bridge repair project will cost \$200,000 less than twice the cost of the highway resurfacing. How much will the highway resurfacing project cost? (Lesson 7-2)

(A) \$450,000 (B) \$734,000

(C) \$900,000 (D) \$1,600,000

4. If 32,800,000 is expressed in the form 3.28×10^n, what is the value of n? (Lesson 8-3)

(A) 5 (B) 6

(C) 7 (D) 8

5. What are the solutions of the equation $x^2 + 7x - 18 = 0$? (Lesson 9-4)

(A) 2 or -9 (B) -2 or 9

(C) -2 or -9 (D) 2 or 9

6. The function $g = t^2 - t$ represents the total number of games played by t teams in a sports league in which each team plays each of the other teams twice. The Metro League plays a total of 132 games. How many teams are in the league? (Lesson 9-4)

(A) 11 (B) 12

(C) 22 (D) 33

7. One leg of a right triangle is 4 inches longer than the other leg. The hypotenuse is 20 inches long. What is the length of the shorter leg? (Lesson 11-4)

(A) 10 in. (B) 12 in.

(C) 16 in. (D) 18 in.

8. What is the distance from one corner of the garden to the opposite corner? (Lesson 11-4)

(A) 13 yards

(B) 14 yards

(C) 15 yards

(D) 17 yards

9. How many points in the coordinate plane are equidistant from both the x- and y-axes and are 5 units from the origin? (Lesson 11-5)

(A) 0 (B) 1

(C) 2 (D) 4

The Princeton Review **Test-Taking Tip**

Questions 7, 21, and 22 Be sure that you know and understand the Pythagorean Theorem. References to right angles, the diagonal of a rectangle, or the hypotenuse of a triangle indicate that you may need to use the Pythagorean Theorem to find the answer to an item.

Aligned and verified by The Princeton Review

Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

10. A line is parallel to the line represented by the equation $\frac{1}{2}y + \frac{3}{2}x + 4 = 0$. What is the slope of the parallel line? (Lesson 5-6)

11. Graph the solution of the system of linear inequalities $2x - y > 2$ and $3x + 2y < -4$. (Lesson 6-6)

12. The sum of two integers is 66. The second integer is 18 more than half of the first. What are the integers? (Lesson 7-2)

13. The function $h(t) = -16t^2 + v_0t + h_0$ describes the height in feet above the ground $h(t)$ of an object thrown vertically from a height of h_0 feet, with an initial velocity of v_0 feet per second, if there is no air friction and t is the time in seconds that it takes the ball to reach the ground. A ball is thrown upward from a 100-foot tower at a velocity of 60 feet per second. How many seconds will it take for the ball to reach the ground? (Lesson 9-5)

14. Find all values of x that satisfy the equation $x^2 - 8x + 6 = 0$. Approximate irrational numbers to the nearest hundredth. (Lesson 10-4)

15. Simplify the expression $\sqrt[3]{3\sqrt{81}}$. (Lesson 11-1)

16. Simplify the expression $\left(x^{\frac{3}{2}}\right)^{\frac{4}{3}}\left(\frac{\sqrt{x}}{x}\right)$. (Lesson 11-1)

17. The area of a rectangle is 64. The length is $\frac{x^3}{x+1}$, and the width is $\frac{x+1}{x}$. What is x? (Lesson 11-3)

www.algebra1.com/standardized_test

Part 3 Quantative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

- Ⓐ **the quantity in Column A is greater,**
- Ⓑ **the quantity in Column B is greater,**
- Ⓒ **the two quantities are equal, or**
- Ⓓ **the relationship cannot be determined from the information given.**

	Column A	Column B
18.	the value of x in $-13x - 12 = -10x + 3$	the value of y in $12y + 16 = 8y$

(Lesson 3-5)

	Column A	Column B
19.	the slope of $2x - 3y = 10$	the y-intercept of $7x + 4y = 4$

(Lesson 5-3)

	Column A	Column B
20.	the measure of the hypotenuse of a right triangle if the measures of the legs are 10 and 11	the measure of the leg of a right triangle if the measure of the other leg is 13 and the hypotenuse is $\sqrt{390}$

(Lesson 11-4)

Part 4 Open Ended

Record your answers on a sheet of paper. Show your work.

21. Haley hikes 3 miles north, 7 miles east, and then 6 miles north again. (Lesson 11-4)

a. Draw a diagram showing the direction and distance of each segment of Haley's hike. Label Haley's starting point, her ending point, and the distance, in miles, of each segment of her hike.

b. To the nearest tenth of a mile, how far (in a straight line) is Haley from her starting point?

c. How did your diagram help you to find Haley's distance from her starting point?