An integer programming algorithm for constructing maximin distance designs from good lattice point sets

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Outline

1. Introduction
2. Good lattice point sets and the Williams' transformation
3. Integer programming algorithm
4. Results and discussion

Computer models and space-filling designs

- Computer models allow us to simulate complex physical phenomena.

- For example:
  - Car crash simulations (Oyama et al., 2019)
  - Design optimization of combat drones (Siddiqi & Lee, 2019).

- They involve many parameters (factors) and, often, are computationally expensive!

- To overcome this issue, we conduct a computer experiment to build a computationally-cheap surrogate model.

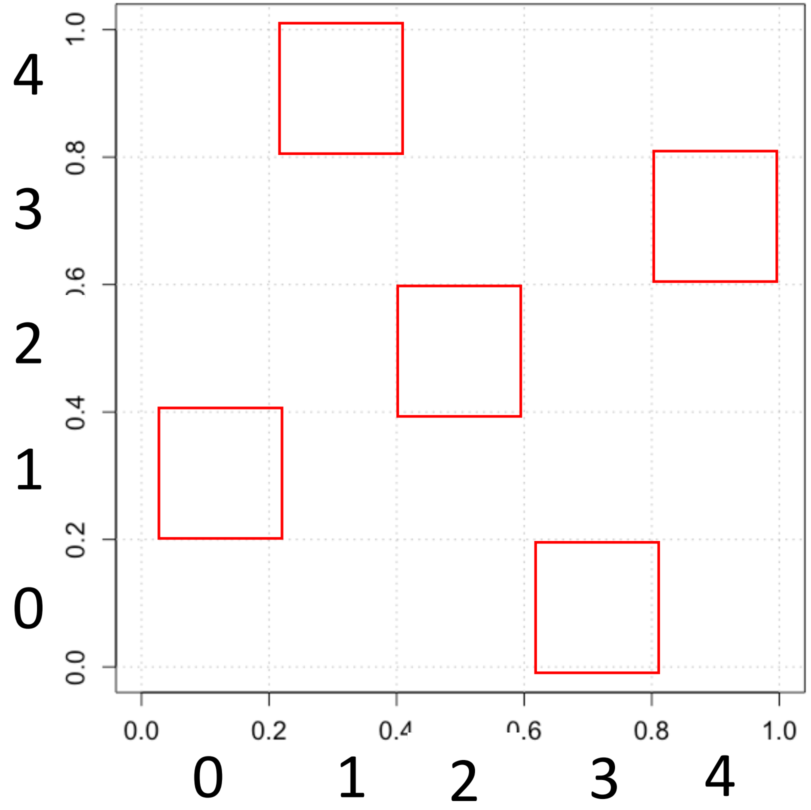
- **Space-filling designs** are attractive for computer experiments because their points (or runs) fill the experimental region uniformly.

Latin hypercube designs (LHDs)

An N-run k-factor LHD is an $N \times k$ matrix whose columns are permutations of the elements in $\{0, \ldots, N-1\}$.

Example 1: 5-run 2-factor LHD

$$\begin{array}{c} \begin{array}{cc} X_1 & X_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 4 \\ 2 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix} \end{array}$$



Latin hypercube designs (LHDs)

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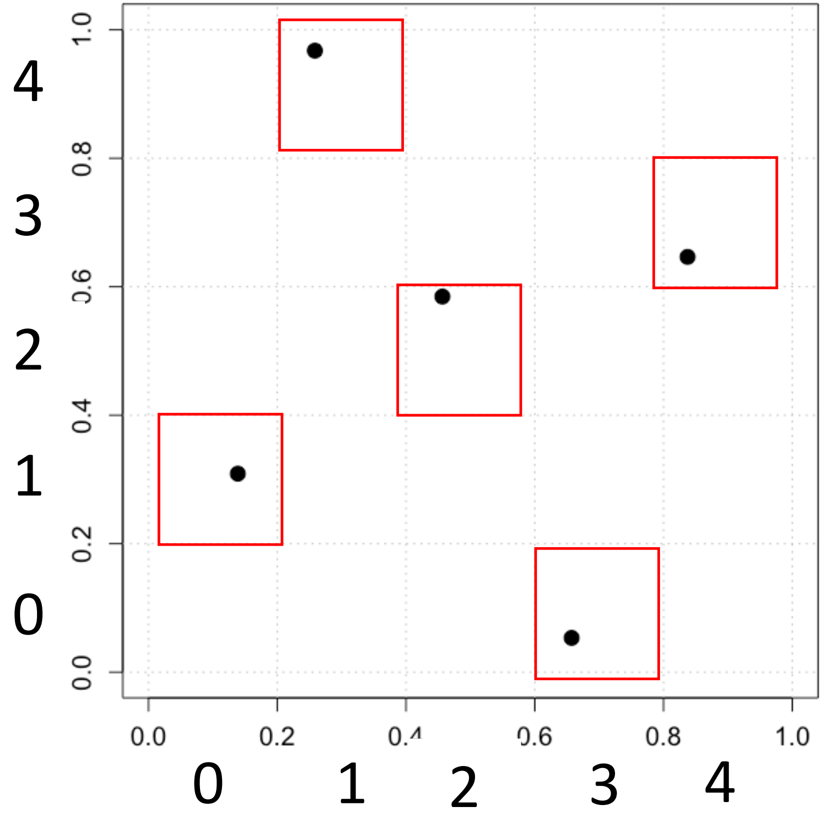


Maximin distance criterion

The maximin distance criterion (Johnson et al., 1990) measures the minimum distance between two rows in an LHD.

Let $\mathbf{x}_i$ be the i-th row of the LHD.

There are two versions:

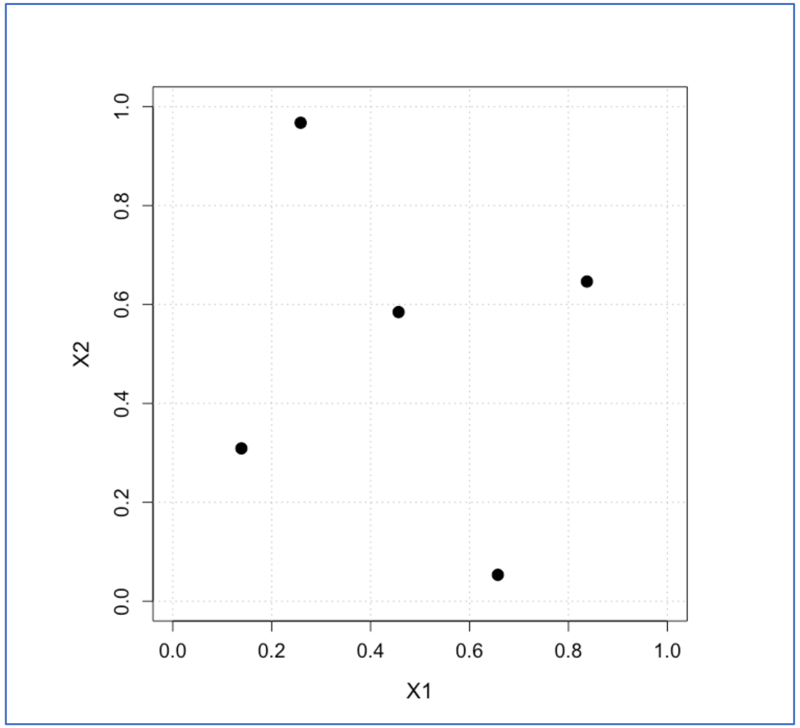
- L_2-distance: $\min\left\{\left\|\mathbf{x}_i - \mathbf{x}_j\right\|_2 \ : \text{for all rows } i \text{ and } j\right\}.$
- L_1-distance: $\min\left\{\left\|\mathbf{x}_i - \mathbf{x}_j\right\|_1 \ : \text{for all rows } i \text{ and } j\right\}.$

Larger values of the maximin distance criterion are preferred.

Example 2: Compare two 5-run 2-factor LHDs in terms of the maximin distance criterion.

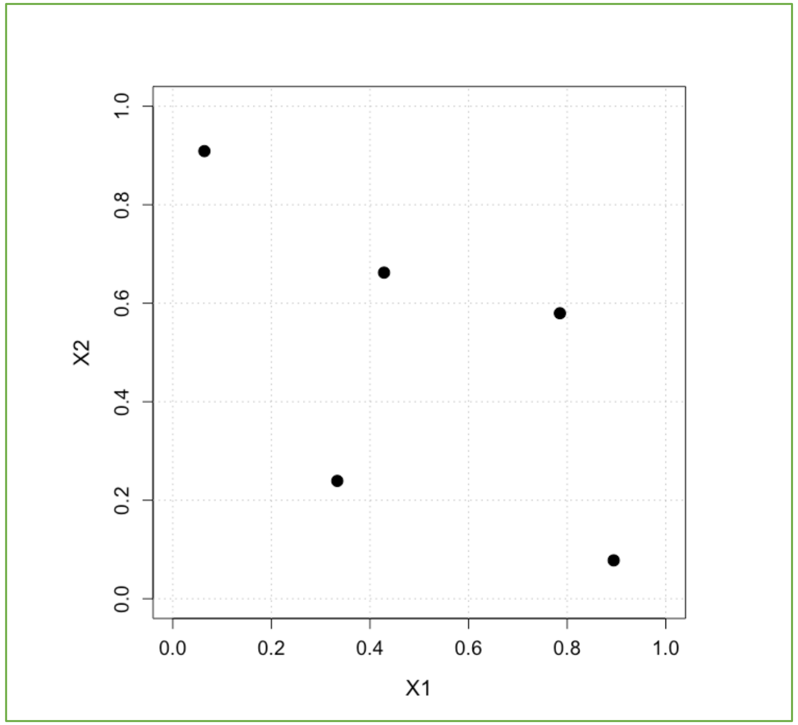
Design 1

$$\begin{pmatrix} 0 & 1 \\ 1 & 4 \\ 2 & 2 \\ 3 & 0 \\ 4 & 3 \end{pmatrix}$$



Design 2

$$\begin{pmatrix} 0 & 4 \\ 1 & 1 \\ 2 & 3 \\ 3 & 2 \\ 4 & 0 \end{pmatrix}$$



Minimum L_1-distance = 3

Minimum L_1-distance = 2

Methods to generate good LHDs

Algorithmic

- Simulated annealing (Morris & Mitchell, 1995; Ba et al., 2015).
- Particle swarm optimization (Chen et al., 2013).
- Iterated local search (Grosso et al., 2009).
- Genetic algorithm (Liefvendahl & Stocki, 2006).

Algebraic

- Good lattice point sets (Zhou & Xu, 2015).
- Williams' transformation (Wang et al., 2018).
- Costas arrays (Xiao & Xu, 2017).
- Orthogonal arrays (Xiao & Xu, 2018).

Among others.

Methods to generate good LHDs

Algorithmic

Limitation:

Computational-performance deteriorates for large number of factors and runs.

Algebraic

Limitation:

Only available for constructing designs of specific sizes.

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Good lattice point (GLP) set

Let $\phi(N)$ be the number of positive integers smaller than and coprime to *N*.

Definition: A GLP set is an $N \times \phi(N)$ matrix whose columns are permutations of the elements in $\{0, \dots, N-1\}$.

Example 3: Construct a GLP set $\mathbf{X}$ with *N* = 7 rows and $\phi(N)$ = 6 columns.

Step 1. Write the positive integers that are smaller and coprime to *N*:
$h_1 = 1,\ h_2 = 2, \dots ,\ h_{\phi(N)} = 6.$

Step 2. Set the elements of $\mathbf{X}$ as $x_{ij} = ih_j \pmod N$ for $i = 1, \dots, N$ and $j = 1, \dots, \phi(N)$.

	h_j values					
i	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1
7	0	0	0	0	0	0

$\mathbf{X}$ is a Latin hypercube design:

- 7 runs and 6 factors
- Minimum L_1-distance = 12

Linear permutations

Zhou and Xu (2015) show that linear permutations of the columns of a GLP set $\mathbf{X}$ may produce a better LHD in terms of the L_1-distance.

Example 3 (cont.): Consider the linear permutation $\mathbf{X} + 4 \pmod 7$.

$$\mathbf{X} = \begin{array}{|cccccc|} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 3 & 1 & 6 & 4 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \qquad \mathbf{X} + 4 \pmod 7 = \begin{array}{|cccccc|} 5 & 6 & 0 & 1 & 2 & 3 \\ 6 & 1 & 3 & 5 & 0 & 2 \\ 0 & 3 & 6 & 2 & 5 & 1 \\ 1 & 5 & 2 & 6 & 3 & 0 \\ 2 & 0 & 5 & 3 & 1 & 6 \\ 3 & 2 & 1 & 0 & 6 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{array}$$

Minimum L_1-distance = 12

Minimum L_1-distance = 13

Williams' transformation

Wang et al. (2018) show that the performance of linearly permuted GLP set can be further improved using the Williams' transformation.

$$W(x) = \begin{cases} 2x & \text{for } 0 \leq x < \dfrac{N}{2} \\ 2(N-x)-1 & \text{for } \dfrac{N}{2} \leq x \leq N \end{cases}.$$

The Williams' transformation is a permutation of $\{0, \dots, N-1\}$.

Example 3 (cont.): Apply the Williams' transformation to each element in $\mathbf{X} + 4 \pmod 7$.

$$\mathbf{X}+4 \pmod 7 = \begin{array}{|cccccc|} \hline 5 & 6 & 0 & 1 & 2 & 3 \\ 6 & 1 & 3 & 5 & 0 & 2 \\ 0 & 3 & 6 & 2 & 5 & 1 \\ 1 & 5 & 2 & 6 & 3 & 0 \\ 2 & 0 & 5 & 3 & 1 & 6 \\ 3 & 2 & 1 & 0 & 6 & 5 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ \hline \end{array}$$

Minimum L_1-distance = 13

$$W(\mathbf{X}+4 \pmod 7) = \begin{array}{|cccccc|} \hline 3 & 1 & 0 & 2 & 4 & 6 \\ 1 & 2 & 6 & 3 & 0 & 4 \\ 0 & 6 & 1 & 4 & 3 & 2 \\ 2 & 3 & 4 & 1 & 6 & 0 \\ 4 & 0 & 3 & 6 & 2 & 1 \\ 6 & 4 & 2 & 0 & 1 & 3 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ \hline \end{array}$$

Minimum L_1-distance = 16

A general construction method

<u>Goal</u>: Construct a good N-run LHD with $\phi(N)$ factors in terms of maximin distance criterion.

GLP set	Linear permutations		Williams' transformation	
	$\mathbf{X} + 0 \pmod{N}$	$\longrightarrow$	$\mathbf{Z}_0 = W(\mathbf{X} + 0) \pmod{N}$	
$\mathbf{X}$	$\mathbf{X} + 1 \pmod{N}$	$\longrightarrow$	$\boxed{\mathbf{Z}_1} = W(\mathbf{X} + 1) \pmod{N}$	**Select best**
	$\vdots$		$\vdots$	
	$\mathbf{X} + N - 1 \pmod{N}$	$\longrightarrow$	$\mathbf{Z}_{N-1} = W(\mathbf{X} + N - 1) \pmod{N}$	

Wang et al. (2018)

Research question

+ GLP sets, linear permutations and the Williams' transformation can generate attractive LHDs (Wang et al., 2018).

- However, the method is limited to LHDs with $\phi(N)$ factors. For N = 30, we can only construct 8-factor LHDs.
- It is unknown how to generate LHDs with more or fewer factors than $\phi(N)$ using the method.

In this talk, we introduce an integer programming algorithm for generating LHDs with flexible numbers of factors.

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Integer programming (IP) algorithm

Integer programming is an optimization method to determine the values of a set of discrete decision variables, so as to maximize or minimize an objective function while satisfying a set of linear constraints (Wolsey, 2020).

Our IP algorithm consists of

- A candidate set of attractive columns.
- A problem formulation for finding maximin distance LHDs.
- The use of state-of-the-art optimization software to solve this problem formulation.

Candidate set: Construction by example

Example 4: Construct a candidate set for LHDs with N = 5 runs.

Step 1. Construct the five $N \times \phi(N)$ designs using the GLP set, linear permutations and the Williams's transformation.

$$\mathbf{Z}_0 = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 4 \\ 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{Z}_1 = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 3 & 0 & 4 & 1 \\ 1 & 4 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \qquad \mathbf{Z}_2 = \begin{pmatrix} 3 & 1 & 0 & 2 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ 2 & 0 & 1 & 3 \\ 4 & 4 & 4 & 4 \end{pmatrix}$$

$$\mathbf{Z}_3 = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 4 & 1 & 2 \\ 2 & 1 & 4 & 0 \\ 4 & 2 & 0 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix} \qquad \mathbf{Z}_4 = \begin{pmatrix} 0 & 2 & 4 & 3 \\ 2 & 3 & 0 & 4 \\ 4 & 0 & 3 & 2 \\ 3 & 4 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Candidate set: Construction by example

Example 4: Construct a candidate set for LHDs with N = 5 runs.

Step 2. Construct an initial candidate set by concatenating the designs.

$$\begin{array}{cccccc} & \mathbf{Z}_0 & \mathbf{Z}_1 & \mathbf{Z}_2 & \mathbf{Z}_3 & \mathbf{Z}_4 \\ \mathbf{C} = & \left(\boxed{\begin{matrix} 2 & 4 & 3 & 1 \\ 4 & 1 & 2 & 3 \\ 3 & 2 & 1 & 4 \\ 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{matrix}} \right. & \boxed{\begin{matrix} 4 & 3 & 1 & 0 \\ 3 & 0 & 4 & 1 \\ 1 & 4 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{matrix}} & \boxed{\begin{matrix} 3 & 1 & 0 & 2 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ 2 & 0 & 1 & 3 \\ 4 & 4 & 4 & 4 \end{matrix}} & \boxed{\begin{matrix} 1 & 0 & 2 & 4 \\ 0 & 4 & 1 & 2 \\ 2 & 1 & 4 & 0 \\ 4 & 2 & 0 & 1 \\ 3 & 3 & 3 & 3 \end{matrix}} & \left. \boxed{\begin{matrix} 0 & 2 & 4 & 3 \\ 2 & 3 & 0 & 4 \\ 4 & 0 & 3 & 2 \\ 3 & 4 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}} \right) \end{array}$$

Candidate set: Construction by example

Example 4: Construct a candidate set for LHDs with N = 5 runs.

Step 2. Construct an initial candidate set by concatenating the designs.

$$\mathbf{C} = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & 1 & 0 & 3 & 1 & 0 & 2 & 1 & 0 & 2 & 4 & 0 & 2 & 4 & 3 \\ 4 & 1 & 2 & 3 & 3 & 0 & 4 & 1 & 1 & 2 & 3 & 0 & 0 & 4 & 1 & 2 & 2 & 3 & 0 & 4 \\ 3 & 2 & 1 & 4 & 1 & 4 & 0 & 3 & 0 & 3 & 2 & 1 & 2 & 1 & 4 & 0 & 4 & 0 & 3 & 2 \\ 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Fully correlated columns: $\begin{pmatrix} 2 \\ 4 \\ 3 \\ 1 \\ 0 \end{pmatrix} = 4 - \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$ (mod 5)

Candidate set: Construction by example

Example 4: Construct a candidate set for LHDs with N = 5 runs.

Step 2. Construct an initial candidate set by concatenating the designs.

$$\mathbf{C} = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & 1 & 0 & 3 & 1 & 0 & 2 & 1 & 0 & 2 & 4 & 0 & 2 & 4 & 3 \\ 4 & 1 & 2 & 3 & 3 & 0 & 4 & 1 & 1 & 2 & 3 & 0 & 0 & 4 & 1 & 2 & 2 & 3 & 0 & 4 \\ 3 & 2 & 1 & 4 & 1 & 4 & 0 & 3 & 0 & 3 & 2 & 1 & 2 & 1 & 4 & 0 & 4 & 0 & 3 & 2 \\ 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Fully correlated columns: $\begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} = 4 - \begin{pmatrix} 0 \\ 3 \\ 2 \\ 1 \\ 4 \end{pmatrix}$ (mod 5)

Candidate set: Construction by example

Example 4: Construct a candidate set for LHDs with N = 5 runs

Step 2. Construct an initial candidate set by concatenating the designs.

$$\mathbf{C} = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & 1 & 0 & 3 & 1 & 0 & 2 & 1 & 0 & 2 & 4 & 0 & 2 & 4 & 3 \\ 4 & 1 & 2 & 3 & 3 & 0 & 4 & 1 & 1 & 2 & 3 & 0 & 0 & 4 & 1 & 2 & 2 & 3 & 0 & 4 \\ 3 & 2 & 1 & 4 & 1 & 4 & 0 & 3 & 0 & 3 & 2 & 1 & 2 & 1 & 4 & 0 & 4 & 0 & 3 & 2 \\ 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 & 1 & 3 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Fully correlated columns: $\begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 0 \end{pmatrix} = 4 - \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{pmatrix}$ (mod 5)

Candidate set: Construction by example

Example 4: Construct a candidate set for LHDs with N = 5 runs.

Step 3. Remove fully correlated columns.

$$\mathbf{C} = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & & 1 & 0 & 2 & 4 & \\ 4 & 1 & 2 & 3 & 3 & 0 & & 0 & 4 & 1 & 2 & \\ 3 & 2 & 1 & 4 & 1 & 4 & & 2 & 1 & 4 & 0 & \\ 1 & 3 & 4 & 2 & 0 & 1 & & 4 & 2 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 2 & 2 & & 3 & 3 & 3 & 3 & \end{pmatrix}$$

Candidate set: Construction by example

Example 4: Construct a candidate set for LHDs with N = 5 runs.

Final candidate set with $\phi(N)N/2$ columns

$$\mathbf{C} = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & 1 & 0 & 2 & 4 \\ 4 & 1 & 2 & 3 & 3 & 0 & 0 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 & 1 & 4 & 2 & 1 & 4 & 0 \\ 1 & 3 & 4 & 2 & 0 & 1 & 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix}$$

Theorem 1. If N is an odd prime, then $\mathbf{C}$ is a maximin L_1-distance LHD.

Problem formulation: Encoding of LHDs

$$\mathbf{C} = \begin{array}{c} \begin{array}{cccccccccc} \color{green}{y_1} & \color{green}{y_2} & \color{green}{y_3} & \color{green}{y_4} & \color{green}{y_5} & \color{green}{y_6} & \color{green}{y_7} & \color{green}{y_8} & \color{green}{y_9} & \color{green}{y_{10}} \end{array} \\ \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & 1 & 0 & 2 & 4 \\ 4 & 1 & 2 & 3 & 3 & 0 & 0 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 & 1 & 4 & 2 & 1 & 4 & 0 \\ 1 & 3 & 4 & 2 & 0 & 1 & 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix} \end{array}$$

The variables y_u are binary:

- $y_u = 1$ if the column is included in the LHD.
- $y_u = 0$ otherwise.

Let k be the number factors in the LHD. We have that

$$\sum_{u=1}^{\phi(N)N/2} y_u = k\,.$$

Calculation of minimum distance

Let c_{iu} denote the element in the *i*-th row and *u*-th column $\mathbf{C}$.

- The L_1-distance between the *i*-th and *j*-th rows in candidate set is

$$\sum_{u=1}^{\phi(N)N/2} \left| c_{iu} - c_{ju} \right|.$$

$$\mathbf{C} = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & 1 & 0 & 2 & 4 \\ 4 & 1 & 2 & 3 & 3 & 0 & 0 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 & 1 & 4 & 2 & 1 & 4 & 0 \\ 1 & 3 & 4 & 2 & 0 & 1 & 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix}$$

- The L_1-distance between *i*-th and *j*-th rows in the LHD is

$$\sum_{u=1}^{\phi(N)N/2} \left| c_{iu} - c_{ju} \right| y_u .$$

$$\begin{array}{c} \begin{matrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{matrix} \\ \mathbf{C} = \begin{pmatrix} 2 & 4 & 3 & 1 & 4 & 3 & 1 & 0 & 2 & 4 \\ 4 & 1 & 2 & 3 & 3 & 0 & 0 & 4 & 1 & 2 \\ 3 & 2 & 1 & 4 & 1 & 4 & 2 & 1 & 4 & 0 \\ 1 & 3 & 4 & 2 & 0 & 1 & 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix} \end{array}$$

The final problem formulation

$$\max_{y_u, t} \quad t$$

Subject to:

(1). $\sum_{u=1}^{\phi(N)N/2} |c_{iu} - c_{ju}| y_u \geq t$
for *all* pairs of rows i and j

(2). $\sum_{u=1}^{\phi(N)N/2} y_u = k$

(3). $t \in \mathbb{N}$

(4). $y_u \in \{0, 1\}$

Solved by optimization solvers: Gurobi, CPLEX or SCIP.

Attractive features:

- Find high-quality designs.
- Provide certificates of optimality.

Theorem 2. If N is even, problem formulation has $\frac{N}{2}\left(\frac{N}{2}-1\right)$ pairs of repeated constraints.

Example 5: Consider the problem formulation with $N = 4$.

$$\max_{y_u, t} \quad t$$

$$\begin{array}{rcl}
\boxed{1y_1 + 1y_2 + 2y_3 + 2y_4 \geq t} \\
1y_1 + 3y_2 + 1y_3 + 3y_4 & \geq & t \\
2y_1 + 2y_2 + 1y_3 + 1y_4 & \geq & t \\
2y_1 + 2y_2 + 1y_3 + 1y_4 & \geq & t \\
3y_1 + 1y_2 + 3y_3 + 2y_4 & \geq & t \\
\boxed{1y_1 + 1y_2 + 2y_3 + 2y_4 \geq t} \\
y_1 + y_2 + y_3 + y_4 & = & k
\end{array}$$

$$t \in \mathbb{N}, y_u \in \{0, 1\}$$

- $\frac{\phi(N)N}{2} = 4$ candidate columns or decision variables.
- $\frac{N}{2}\left(\frac{N}{2}-1\right) = 2$ pairs of repeated constraints.

Theorem 2. If N is even, problem formulation has $\frac{N}{2}\left(\frac{N}{2}-1\right)$ pairs of repeated constraints.

Example 5: Consider the problem formulation with $N = 4$.

$$\max_{y_u, t} \quad t$$

$$\begin{array}{rcl}
1y_1 + 1y_2 + 2y_3 + 2y_4 & \geq & t \\
1y_1 + 3y_2 + 1y_3 + 3y_4 & \geq & t \\
2y_1 + 2y_2 + 1y_3 + 1y_4 & \geq & t \\
3y_1 + 1y_2 + 3y_3 + 2y_4 & \geq & t \\
y_1 + y_2 + y_3 + y_4 & = & k
\end{array}$$

$$t \in \mathbb{N}, y_u \in \{0, 1\}$$

Remove one constraint in each set!

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Numerical comparisons

We obtained design problems with 7 to 30 runs and 4 to 28 factors from Wang et al. (2018).

Construction methods:

- IP algorithm with Gurobi v9 and a maximum search time of 5 min.
- SA: Simulated annealing algorithm with 100 iterations (Ba et al., 2015).
- GA: Genetic algorithm with 100 generations (Liefvendahl and Stocki, 2006)
- WXX: GLP, linear permutations and William's transformation (Wang et al., 2018).
- XX: Costas arrays (Xiao and Xu, 2017).

Results I

		Minimum L_1-distance				
Runs	Factors	IP	SA	GA	WXX	XX
7	6	**16**	15	15	**16**	14
8	4	**11**	**11**	10	10	
9	6	17	**18**	17	16	
10	4	11	11	**12**	11	
11	10	**39**	36	38	**39**	34
12	4	**13**	**13**	**13**	10	
13	12	**54**	52	52	52	48
14	6	**24**	23	**24**	**24**	
15	8	36	35	**37**	36	
16	8	**43**	37	39	36	
17	16	**94**	86	89	**94**	86
18	6	28	28	**30**	28	

Larger the better

Results I

		Minimum L_1-distance				
Runs	Factors	IP	SA	GA	WXX	XX
7	6	**16**	15	15	**16**	14
8	4	**11**	**11**	10	10	
9	6	17	**18**	17	16	
10	4	11	11	**12**	11	
11	10	**39**	36	38	**39**	34
12	4	**13**	**13**	**13**	10	
13	12	**54**	52	52	52	48
14	6	**24**	23	**24**	**24**	
15	8	36	35	**37**	36	
16	8	**43**	37	39	36	
17	16	**94**	86	89	**94**	86
18	6	28	28	**30**	28	

Larger the better

Results II

		Minimum L_1-distance				
Runs	Factors	IP	SA	GA	WXX	XX
19	18	**118**	108	110	115	106
20	8	**47**	43	46	42	
21	12	**77**	73	**77**	76	
22	10	**68**	61	64	**68**	
23	22	**172**	160	161	168	158
24	8	53	50	**54**	36	
25	20	**163**	153	153	162	
26	12	**98**	87	91	**98**	
27	18	**157**	145	147	156	
28	12	**104**	92	97	94	
29	28	270	254	254	**274**	250
30	8	**63**	57	**63**	61	

Larger the better

Results II

		Minimum L_1-distance				
Runs	Factors	IP	SA	GA	WXX	XX
19	18	**118**	108	110	115	106
20	8	**47**	43	46	42	
21	12	**77**	73	**77**	76	
22	10	**68**	61	64	**68**	
23	22	**172**	160	161	168	158
24	8	53	50	**54**	36	
25	20	**163**	153	153	162	
26	12	**98**	87	91	**98**	
27	18	**157**	145	147	156	
28	12	**104**	92	97	94	
29	28	270	254	254	**274**	250
30	8	**63**	57	**63**	61	

Larger the better

Discussion

- Our IP algorithm constructs LHDs that are at least as good as the benchmark methods for 75% of the design problems.
- For larger-sized problems, we propose two modifications to the IP algorithm which allow us to construct LHDs with up to 72 factors and up to 113 runs.
- Use integer programming to construct LHDs that optimize other statistical criteria such as the MaxPro criterion (Joseph et al, 2015).

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An integer programming algorithm for constructing maximin distance designs from good lattice point sets

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Appendix

Using a modified version of the IP algorithm, we constructed LHDs of other practically-relevant sizes.

The modified IP algorithm outperforms the other algorithms in 7 out of 11 instances.

		Minimum L_1-distance		
Runs	Factors	IP	SA	GA
71	7	83	89	**94**
73		87	91	**96**
79		89	95	**100**
83	8	**130**	123	134
89		**133**	127	138
97	9	161	**163**	155
101	10	**203**	191	187
103		**211**	196	190
107		**211**	207	188
109		**212**	206	188
113	11	**244**	242	223

Larger the better