Model Description:
Magnetic Levitation System

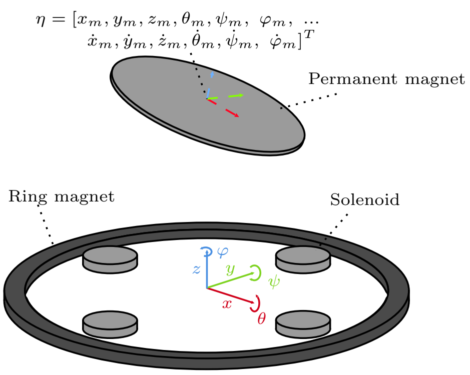

Fig. 1. Illustration of the maglev system.

The maglev system is shown in Figure 1. The base of the system contains a large permanent ring magnet and a set of four electromagnetic solenoids, while the object being lifted is a permanent disk magnet. Hereafter they will be referred to as *the ring magnet*, *the solenoids* and *the levitating magnet*, respectively. The ring magnet and the levitating magnet is oriented such that their positive poles are directed towards each other, achieving magnetic repulsion. The ring magnet provides most of the lift, while the solenoids provide stabilization.

Modeling Assumptions and Simplifications

In modeling, only the magnetic and mechanical properties are considered; it is assumed that other external effects are negligible. Accordingly, the following modeling simplifications are made:

(1) The ring magnet is modeled as bias in the solenoids
(2) The solenoids are modeled as thin wire loops
(3) The levitating magnet is modeled as a solenoid

The first and second simplification are reasonable when the distance between the magnets is sufficiently large. Moreover, a permanent magnet is magnetically equivalent to an electromagnet when only considering the external field, justifying the third simplification.

Physical Laws

The movement is modeled using the *Newton-Euler equations of motion*, where the magnetic field and magnetic force for current carrying wires are modeled using *Biot-Savart's law* and *Laplace Force law*, respectively.

Newton-Euler Equations of Motion

$$\begin{bmatrix} \sum_{i=1}^{n} F_i \\ \sum_{i=1}^{n} \tau_i \end{bmatrix} = \begin{bmatrix} m\mathbf{I}_3 & 0 \\ 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} a \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ \omega \times \mathcal{I}\omega \end{bmatrix} \tag{1}$$

Here, m is mass, a is linear acceleration, F_i is the i'th force acting on an object, $\mathcal{I}$ is an inertia matrix, α is angular acceleration, ω is angular velocity and τ_i is the i'th torque acting on an object.

Biot-Savart's Law

$$\mathbf{B}(p) = \frac{\mu_0}{4\pi} I \int_C \frac{d\ell \times (p - \ell)}{|(p-\ell)|^3} \tag{2}$$

Here, I is current in a wire, $d\ell$ is a differential vector along the line (wire) C, $\mathbf{B}$ is the magnetic field, μ_0 is the permeability of air, p is a point in Cartesian coordinates, and ℓ is point on the line C.

Laplace's Force Law

$$F_b = I \int_C \mathrm{d}\ell \times \mathbf{B} \tag{3}$$

Here, F_b is the magnetic force.

Solutions to Biot-Savarts and Laplace's Force Law

The Biot-savart's law and the Laplace force law include integrals that in general can not be solved analytically. However, González and Cárdenas (2020) present the following solution of the Biot-Savart's law when the current carrying wire *is assumed to be a thin loop*:

$$\begin{aligned}\mathbf{B}(\rho, z) = & \frac{\mu_0 I}{2\pi\sqrt{(\rho' + \rho)^2 + (z - z')^2}} \Bigg\{ \frac{(z - z')}{\rho} \\ & \cdot \left[\frac{\rho'^2 + \rho^2 + (z - z')^2}{(\rho - \rho')^2 + (z - z')^2} \mathrm{E}(k) - \mathrm{K}(k) \right] \boldsymbol{e}_\rho \\ & - \left[\frac{\rho^2 - \rho'^2 + (z - z')^2}{(\rho - \rho')^2 + (z - z')^2} \mathrm{E}(k) - \mathrm{K}(k) \right] \boldsymbol{e}_z \Bigg\}\end{aligned} \tag{4}$$

where

$$k = \frac{4\rho'\rho}{(\rho' + \rho)^2 + (z - z')^2}$$

Here, ρ, ϕ and z are the cylindrical coordinates of the point of interest, and ρ' and z' are the radius and the vertical height of a current loop centered around the z-axis. $K(k)$ and $E(k)$ are the complete elliptical integrals of the first and second kind, respectively.

To approximate the solution of Laplace's force law for a current carrying loop in a magnetic field, Wang and Ren (2014) suggests using discretization to divide the loop into n linear segments. Assuming uniform field strength along each segment, this allows the Laplace force law to be represented as a sum instead of an integral (see (11) and (12)). Wang and Ren (2014) show that this approximation achieve less than 0.07% relative error for $n = 100$.

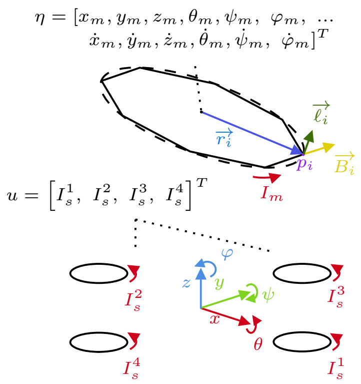

Fig. 2. Illustration of the model of the maglev system

Complete Model Description

The system model is illustrated in Figure 2. The state and input vectors of the system are

$$\begin{aligned} \eta = [x_m, y_m, z_m, \psi_m, \theta_m, \varphi_m, \ldots \\ \dot{x}_m, \dot{y}_m, \dot{z}_m, \dot{\psi}_m, \dot{\theta}_m, \dot{\varphi}_m]^T \quad &\in \mathbb{R}^{12\times 1} \\ u = [I_s^1, I_s^2, I_s^3, I_s^4]^T \quad &\in \mathbb{R}^{4\times 1} \end{aligned} \tag{5}$$

where $[x_m, y_m, x_m]^T$ and $[\psi_m, \theta_m, \varphi_m]^T$ are the Cartesian coordinates and the Euler rotation angles of the center of gravity of the levitating magnet, and I_s^j are the currents in each solenoid.

As suggested by Wang and Ren (2014) the levitating magnet is discretized into n points, p_i, using the discretization

$$p_i = R(\psi_m, \theta_m) \cdot \begin{bmatrix} r_m \cos\left(\frac{i}{n} \cdot 2\pi\right) \\ r_m \sin\left(\frac{i}{n} \cdot 2\pi\right) \\ 0 \end{bmatrix} + [x_m, y_m, z_m]^T,$$
$$i = 1, \ldots, n \tag{6}$$

where r_m is the radius of the levitating magnet, and

$$R(\psi, \theta) = \begin{bmatrix} \cos(\theta) & \sin(\psi)\sin(\theta) & \cos(\psi)\sin(\theta) \\ 0 & \cos(\psi) & -\sin(\psi) \\ -\sin(\theta) & \cos(\theta)\sin(\psi) & \cos(\psi)\cos(\theta) \end{bmatrix} \tag{7}$$

Note that it is assumed that $\varphi_m = 0$ due to symmetries in the system.

Using the simplification for the Biot-Savart's law, the magnetic field is computed at a point $p = [x, y, z]^T$ as

$$\mathbf{B}_x(p) = \mathbf{B}_\rho(\rho, z)\cos\phi \tag{8a}$$
$$\mathbf{B}_y(p) = \mathbf{B}_\rho(\rho, z)\sin\phi \tag{8b}$$
$$\mathbf{B}_z(p) = \mathbf{B}_z(\rho, z) \tag{8c}$$

where the coordinate transformation

$$\rho(p) = \sqrt{x^2 + y^2} \tag{9a}$$
$$\phi(p) = \arctan\frac{y}{x} \tag{9b}$$
$$z(p) = z \tag{9c}$$

is used. Thus, the total magnetic field vector $\overrightarrow{\mathbf{B}}_i(\eta, u)$ produced by the solenoids in a point p_i becomes

$$\overrightarrow{\mathbf{B}}_i(\eta, u) := \sum_{j=1}^{4} \begin{bmatrix} \mathbf{B}_x(p_i - p_s^j) \\ \mathbf{B}_y(p_i - p_s^j) \\ \mathbf{B}_z(p_i - p_s^j) \end{bmatrix} \tag{10}$$

where p_s^j is the Cartesian coordinates of the center of the j'th solenoid. Then, according to the Laplace force law, the total magnetic force on the levitating magnet is approximated by

$$F_b = I_m \sum_{i=1}^{n} \overrightarrow{\ell}_i \times \overrightarrow{\mathbf{B}}_i(\eta, u) \tag{11}$$

where

$$\overrightarrow{\ell}_i = \frac{p_{i+1} - p_{i-1}}{4} \tag{12}$$

Similarly, the magnetic torque is approximated by

$$\tau_b = I_m \sum_{i=1}^{n} \overrightarrow{r}_i \times \overrightarrow{\ell}_i \times \overrightarrow{\mathbf{B}}_i(\eta, u) \tag{13}$$

where

$$\overrightarrow{r}_i = p_i - [x_m, y_m, z_m]^T \tag{14}$$

The moment of inertia of the levitating magnet is

$$\mathcal{I} = diag\left(\left[\frac{1}{4}mr_m^2, \ \frac{1}{4}mr_m^2, \ \frac{1}{2}mr_m^2\right]\right) \tag{15}$$

where m is the mass of the magnet. Assuming full state measurements, the entire system can be compactly represented by

$$\dot{\eta} = A\eta + B\phi(\eta, u) \tag{16a}$$
$$y = C\eta \tag{16b}$$

where

$$A = \begin{bmatrix} O_{6\times 6} & I_6 \\ O_{6\times 6} & O_{6\times 6} \end{bmatrix} \tag{17a}$$
$$B = \begin{bmatrix} O_{6\times 6} \\ I_6 \end{bmatrix} \tag{17b}$$
$$C = I_{12} \tag{17c}$$

and

$$\phi(\eta, u) = \begin{bmatrix} mI_3 & O_{3\times 3} \\ O_{3\times 3} & \mathcal{I} \end{bmatrix}^{-1} \begin{bmatrix} F_b(\eta, u) \\ \tau_b(\eta, u) \end{bmatrix} - \begin{bmatrix} O_{8\times 1} \\ g \\ O_{3\times 1} \end{bmatrix} \tag{18}$$

where I_n and $O_{n\times m}$ are identity and zero matrices, and g is the gravitational constant. Note that the last term in (1) disappears due to symmetries in the system.

REFERENCES

González, M.A. and Cárdenas, D.E. (2020). Analytical expressions for the magnetic field generated by a circular arc filament carrying a direct current. *IEEE Access*, 9, 7483–7495.

Wang, Z.J. and Ren, Y. (2014). Magnetic force and torque calculation between circular coils with nonparallel axes. *IEEE Transactions on Applied Superconductivity*, 24(4), 1–5.