Discrete dynamical systems

Some notes from the homonymous book by J.T.Sandefur

$\mathbb{A}$ $\mathbb{Z}$

1 Introduction

■ **Discrete dynamical system / difference equation** Equations that describe a relationship between one point in time and a previous one

■ **First order d. d. s.** Supposed we have a function $y = f(x)$. A f. o. d. d. s. is a sequence of numbers $A(n)$ for $n = 0, ...$ such that each number after the first one is related to the previous number by the relation

$$A(n+1) = f(A(n))$$

The sequence of numbers given by the relationship $A(n+1) - A(n) = g(A(n))$ is called a first order difference equation. Letting $f(x) = g(x) + x$, the concepts are seen to be equivalent.

■ **Linear** a dynamical system is l. when $f(x)$ is a straight line through the origin ($f(0) = 0$)

$$A(n+1) = 3A(n)$$

■ **Affine** When the function still describes a line, but one that <u>doesn't go through the origin</u>

$$A(n+1) = 2A(n) + 5$$

■ **Nonlinear** When the graph of $f(x)$ is <u>not</u> a straight line

$$A(n+1) = 3A(n)(1 - A(n))$$

■ **Nonautonomous** When the coefficients of $f(A(n))$ depend on n

$$A(n+1) = f(n, A(n))$$

■ **Nonhomogeneous** When the term(s) of the difference equation that do not depend on $A(n)$ depend on n

$$A(n+1) = f(A(n)) + g(n)$$

■ **First order** Because each number $A(n+1)$ depends only on the previous number $A(n) = A(n+1-1)$ (has its value determined by an expression of the sole $A(n)$, except for additional terms)

$$A(n+1) = f(A(n))$$

- **Higher order** A system of the form

$$A(n+m) = f(A(n+m-1), A(n+m-2), ..., A(n))$$

where m is a fixed positive integer is called a higher order d. s. This particular case defined as a m-th order d.s. since each number depends on the previous m ones.

- **Initial values** Values of $A(0)$, $A(1)$, ... necessary for finding the values for all n. For a m-th order d.s. m initial values are needed.

- **D.s. of ≥ 2 equations** if there are more sequences, and each number in each sequence is related to the previous ones in both:

$$A(n+1) = f(A(n), B(n))$$

$$B(n+1) = g(A(n), B(n))$$

The number of equations is equal to the n. of sequences.

- **(!)** if different d.s. represent the same set of equations, they could be said to be the same

- **Equilibrium value** a first order d.s. is given, say $A(n+1) = f(A(n))$. A number a is called an equilibrium value or fixed point for the d.s. if $A(k) = a$ for all values k when the initial value $A(0) = a$.

$$A(k) = a$$

is a constant solution to the d.s.

- **Theorem 1** *The number a is an equilibrium value for the d.s. $A(n+1) = f(A(n))$ if and only if a satisfies the equation $a = f(a)$*

- **Stable e.v. - Attracting f.p.** Suppose a first order d.s. has an e.v. a. It is said to be stable or attracting if there is a number ε, unique to each system, such that, when

$$|A(0) - a| < \varepsilon, \text{ then } \lim_{k\to\infty} A(k) = a$$

- **Unstable e.v. - Repelling f.p.** a is instead said to be unstable or repelling if there is a number ε such that, when

$$0 < |A(0) - a| < \varepsilon, \text{ then } |A(k) - a| > \varepsilon$$

for some, but not necessarily all, values of k.

- **Theorem 2** *The equilibrium value $a = \frac{b}{1-r}$ for the dynamical system*

$$A(n+1) = rA(n) + b, \quad for \;\; r \neq 1$$

is stable if $|r| < 1$, that is, if $-1 < r < 1$, and in fact $\lim_{k\to\infty} A(k) = a$ for any value of $A(0)$. Also, if $|r| > 1$, that is if $r < -1$ or $r > 1$, then a is unstable and $|A(k)|$ goes to infinity for any $A(0) \neq a$. When $r = -1$ it is called a 2-cycle ($A(0) = A(2) = ...$, $A(1) = A(3) = ...$). The equilibrium value in such 2-cycle is neither stable nor unstable. The equilibrium value in this case may be labelled as neutral.