# Walsh Functions and Spaces

## Computational Mathematics and Statistics Seminar
## Multiscale and Computation Seminar

Aleksei G. Sorokin

Illinois Institute of Technology, Department of Applied Mathematics

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# Fourier Series

When $f:[0,1]\to\mathbb{R}$ has an absolutely convergent Fourier series

$$f(x)=\sum_{k\in\mathbb{Z}}\widehat{f}(k)e^{2\pi\mathrm{i}kx}, \qquad \widehat{f}(k)=\int_0^1 f(x)e^{-2\pi\mathrm{i}kx}\mathrm{d}x.$$

When $f^{(\alpha)}$ has an absolutely convergent Fourier series for some $\alpha\in\mathbb{N}_0$ and $f^{(\beta)}$ periodic for all $\beta\in\{1,\dots,\alpha-1\}$

$$f^{(\alpha)}(x)=\sum_{k\in\mathbb{Z}}\widehat{f^{(\alpha)}}(k)e^{2\pi\mathrm{i}kx}, \qquad \widehat{f^{(\alpha)}}(k)=(2\pi\mathrm{i}k)^{\alpha}\widehat{f}(k).$$

# Fourier Spaces

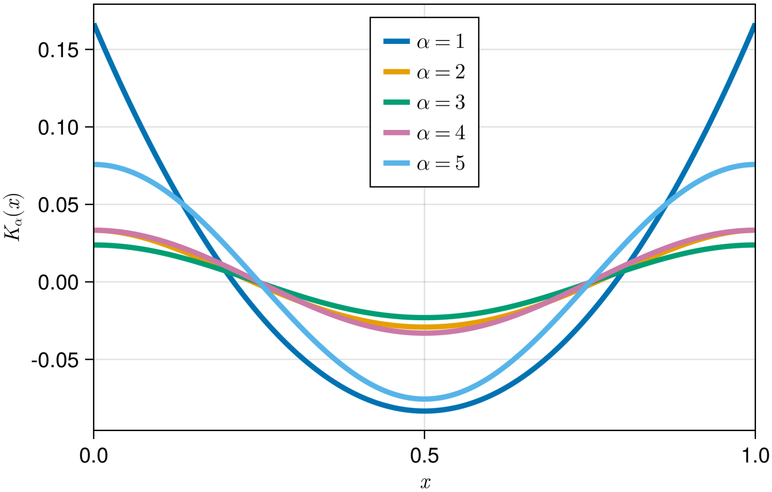
Let $\{x-y\} = (x-y) \mod 1$ and let $B_i$ denote th $i^{\text{th}}$ Bernoulli polynomial.

$$\mathring{K}(x,y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{2\pi \mathrm{i} k(x-y)}}{k^{2\alpha}} = \frac{(-1)^{\alpha+1}(2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(\{x-y\}) = \mathring{K}(\{x-y\}),$$

is the kernel of Sobolev RKHS $\mathring{H}_\alpha$ with $\alpha \in \mathbb{N}$ and

$$\langle f, g\rangle = (-1)^{\alpha}(2\pi)^{-2\alpha} \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x) \mathrm{d}x.$$

$$\mathring{K}(x) = \frac{(-1)^{\alpha+1}(2\pi)^{2\alpha}}{(2\alpha)!} B_{2\alpha}(x)$$



# Digitwise Operations

Prime base $b \geq 2$ expansion of $x \in [0,1)$ is

$$x = .\mathsf{x}_1\mathsf{x}_2\mathsf{x}_3\cdots_b = \sum_{\ell\in\mathbb{N}} \mathsf{x}_\ell b^{-\ell}, \qquad \text{e.g.} \qquad .375 = .011_2,$$

with digitwise addition (digitwise exclusive or in base $b = 2$)

$$x \oplus y := \sum_{\ell\in\mathbb{N}} ((\mathsf{x}_\ell + \mathsf{y}_\ell) \mod b) b^{-\ell}, \qquad \text{e.g.} \qquad .375 \oplus .625 = .011_2 \oplus .101_2 = .110_2 = .75.$$

Similarly for $k \in \mathbb{N}_0$

$$k = \cdots \mathsf{k}_2\mathsf{k}_1\mathsf{k}_0.0_b = \sum_{\ell\in\mathbb{N}_0} \mathsf{k}_\ell b^{\ell}, \qquad \text{e.g.} \qquad 5 = 101_2,$$

$$k \oplus h := \sum_{\ell\in\mathbb{N}_0} ((\mathsf{k}_\ell + \mathsf{h}_\ell) \mod b) b^{\ell}, \qquad \text{e.g.} \qquad 5 \oplus 6 = 101_2 \oplus 110_2 = 011_2 = 3.$$

# Walsh Functions

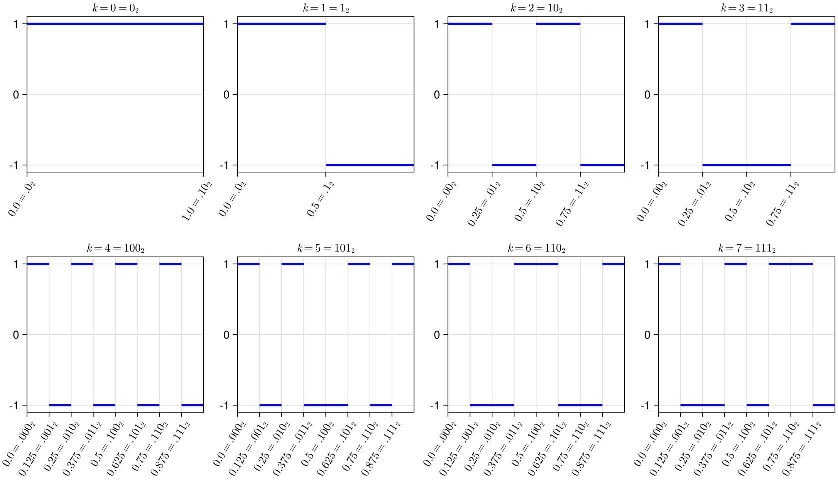
Introduced for base $b = 2$ in [Walsh, 1923] with important results in [Fine, 1949].
Generalized to finite abelian group with a bijection in [Larcher et al., 1996].

For $k \in \mathbb{N}_0$ with $\mathbf{k} = (\mathsf{k}_0, \mathsf{k}_1, \dots)$ and $x \in [0, 1)$ with $\mathbf{x} = (\mathsf{x}_1, \mathsf{x}_2, \dots)$,

$$\mathrm{wal}_k(x) = e^{2\pi \mathrm{i}/b \sum_{\ell=0}^{\infty} \mathsf{k}_\ell \mathsf{x}_{\ell+1}} = e^{2\pi \mathrm{i}/b \mathbf{k}.\mathbf{x}}$$

is a complete orthonormal system in $\mathcal{L}_2([0, 1))$.
Notice similarity to complex exponential basis $\left\{e^{2\pi \mathrm{i} k x} : k \in \mathbb{Z}\right\}$ for Fourier series.

# $b = 2$ Walsh Functions $\mathrm{wal}_k(x) = (-1)^{\sum_{\ell \in \mathbb{N}_0} \mathsf{k}_\ell \mathsf{x}_{\ell+1}}$



# Walsh Function Properties

For any $x, y \in [0,1)$ and $k, h \in \mathbb{N}_0$ and $f \in \mathcal{L}_2([0,1))$

1. $\mathrm{wal}_k(x)\mathrm{wal}_h(x) = \mathrm{wal}_{k\oplus h}(x)$ and $\mathrm{wal}_k(x)\mathrm{wal}_k(y) = \mathrm{wal}_k(x \oplus y)$
2. $$\int_0^1 \mathrm{wal}_k(x)\mathrm{d}x = \begin{cases} 1, & k = 0 \\ 0, & k > 0 \end{cases}$$
3. $$\int_0^1 f(\sigma)\mathrm{d}\sigma = \int_0^1 f(x \oplus \sigma)\mathrm{d}\sigma$$
4. $$\sum_{k=0}^{b^a-1} \mathrm{wal}_k(x) = \begin{cases} b^a, & a < \mathcal{I}(x) - 1 \\ 0, & \text{otherwise} \end{cases}$$

   where $\mathcal{I}(x) = -\lfloor \log_b(x) \rfloor$ is the index first non-zero digit in the base $b$ expansion e.g. with $b = 2$ then $\mathcal{I}(.375) = \mathcal{I}(.011_2) = 2$.

# Weight Function

Write $k \in \mathbb{N}$ as

$$k = \sum_{\ell=1}^{\#k} \mathsf{k}_{a_\ell} b^{a_\ell}$$

with $a_1 > \cdots > a_{\#k} \geq 0$ and $k_{a_\ell} \in \{1, \ldots, b-1\}$.
Weight function for $\alpha \in \mathbb{N}_0$ has

$$\mu_\alpha(k) = \sum_{\ell=1}^{\min(\alpha,\#k)} (a_\ell + 1)$$

with $\mu_0(k) = \mu_\alpha(0) = 0$. $\mu$ sums indices of non-zero digits. For example, with $b = 2$

$$k = 13 = 1101_2 \qquad \text{has} \qquad (a_1, a_2, a_3) = (3, 2, 0)$$

$$\mu_1(k) = (3+1), \quad \mu_2(k) = (3+1)+(2+1), \quad \mu_3(k) = (3+1)+(2+1)+(0+1) = \mu_4(k) = \ldots.$$

# Walsh Series of Smooth Functions

For $\alpha \geq 2$ the Sobolev RKHS $H_\alpha$ with inner product

$$\langle f, g \rangle_\alpha = \sum_{\beta=1}^{\alpha-1} \int_0^1 f^{(\beta)}(x) \mathrm{d}x \int_0^1 g^{(\beta)}(x) \mathrm{d}x + \int_0^1 f^{(\alpha)}(x) g^{(\alpha)}(x) \mathrm{d}x$$

has kernel

$$K_\alpha(x, y) = \sum_{\beta=1}^{\alpha-1} \frac{B_\beta(x) B_\beta(y)}{(\beta!)^2} + \overbrace{(-1)^{\alpha+1} \frac{B_{2\alpha}(\{x-y\})}{(2\alpha)!}}^{\mathring{K}(\{x-y\})}.$$

[Dick, 2008, 2009] show that if $f \in H_\alpha$ then for $\widehat{f}(k) = \int_0^1 f(x) \overline{\mathrm{wal}_k(x)} \mathrm{d}x$ we have

$$\sup_{k \in \mathbb{N}_0} \left| \widehat{f}(k) \right| b^{\mu_\alpha(k)} < \infty \qquad \text{i.e.} \qquad \exists C_{f,\alpha} > 0 \quad \text{s.t.} \quad \left| \widehat{f}(k) \right| \leq \frac{C_{f,\alpha}}{b^{\mu_\alpha(k)}}.$$

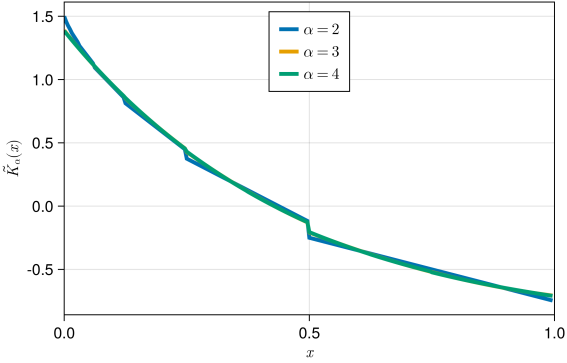
For the $\alpha = 1$ case see [Dick and Pillichshammer, 2005].

# Digitally Shift Invariant Walsh Kernel

$H_\alpha \subset \widetilde{H}_\alpha$ where $\widetilde{H}_\alpha$ is an RKHS with kernel

$$\widetilde{K}_\alpha(x,y) = \sum_{k\in\mathbb{N}} \frac{\mathrm{wal}_k(x \ominus y)}{b^{\mu_\alpha(k)}} = \widetilde{K}_\alpha(x \ominus y).$$

Below $\widetilde{K}_\alpha$ with $b=2$ is shown. Notice discontinuities at $\{2^{-a} : a \in \mathbb{N}\}$ among others.



# Research Connections for Multivariate Functions

**Kernel matrix of pairwise evaluations for kernel interpolation is:**

- circulant with $\mathring{K}(\{\boldsymbol{x}-\boldsymbol{y}\}) := \prod_{j=1}^{d}\left[1+\gamma_j \mathring{K}(\{x_j - y_j\})\right]$ and lattice $\{\boldsymbol{x}_t\}_{t=1}^{N}$.
- block Toepliz with $\widetilde{K}(\boldsymbol{x}\ominus\boldsymbol{y}) := \prod_{j=1}^{d}\left[1+\gamma_j \widetilde{K}(x_j \ominus y_j)\right]$ and digital net $\{\boldsymbol{x}_t\}_{t=1}^{N}$.

**Quasi Monte Carlo Absolute Error**

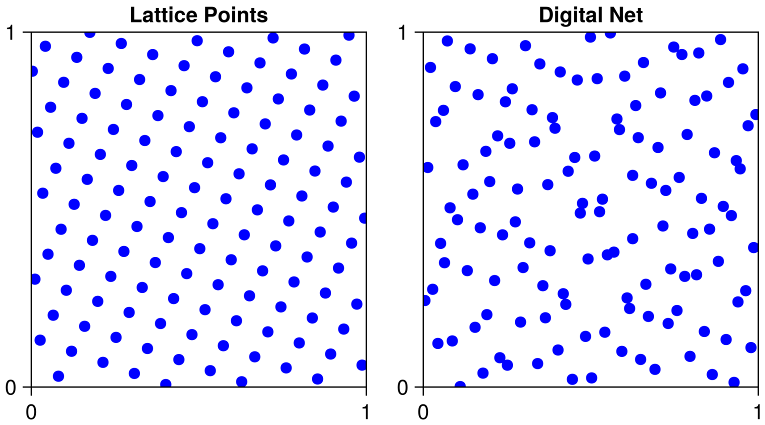
$$\left|\int_{[0,1)^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{t=1}^{N} f(\boldsymbol{x}_t)\right|$$

- is $\mathcal{O}(N^{-\alpha+\delta})$ for $f \in \prod_{j=1}^{d}\left[1+\mathring{H}_\alpha\right]$ using certain lattice sequence $\{\boldsymbol{x}_t\}_{t=1}^{N}$.
- is $\mathcal{O}(N^{-\alpha+\delta})$ for $f \in \prod_{j=1}^{d}[1+H_\alpha]$ using certain digital net $\{\boldsymbol{x}_t\}_{t=1}^{N}$.

where $\delta > 0$ arbitrarily small.

$\star$ Choosing $\{\gamma_j\}_{j=1}^{d}$ judiciously avoids $d$ dependence of the convergence rates.

# Low Discrepancy (Quasi-Random) Point Sets



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