Probabilistic Models for PDEs with Random Coefficients

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Control Extraction Rate to Manage Pressure at Critical Location

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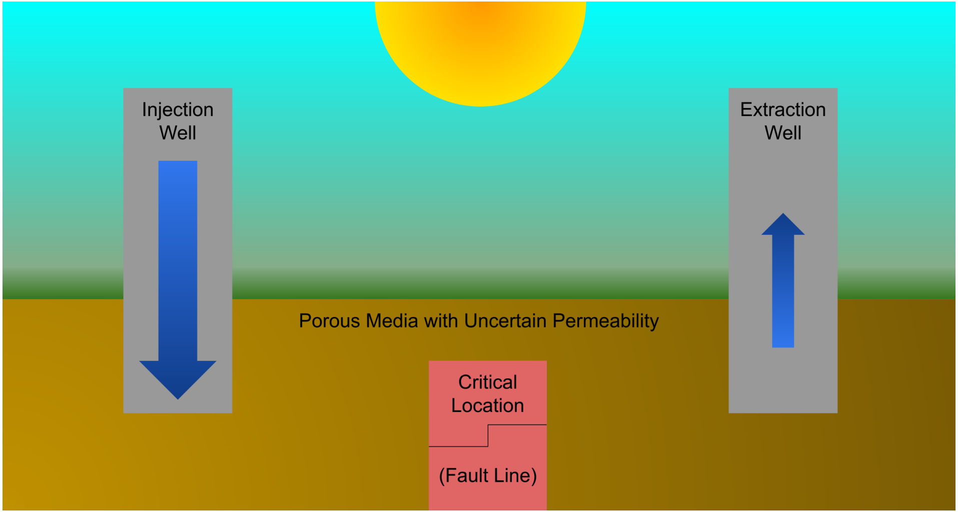

Random Permiability in Porous Media Induces Random Pressure

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Darcy's equation $\nabla.(G(x).\nabla p(E,G,x)) = f(E,x)$ models pressure in porous media

- $G(x)$ a **random** *Gaussian permeability field*, known only by statistical properties
- $p(x)$ the **random** *pressure solution*

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Following Pachalieva et al.[1], we model the *flow rate* as

$$f(x,E) = \begin{cases} I, & x = x_{\text{injection}} \\ -E, & x = x_{\text{extraction}} \\ 0, & \text{else} \end{cases}.$$

- I, the **fixed** *injection rate*
- E, the **variable** *extraction rate*

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Choose Extraction Rate E to Rarely Overpressurizes a Critical Location

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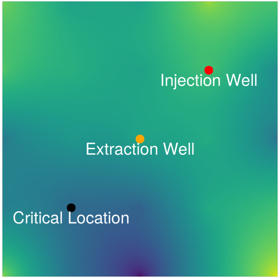
- Given critical location x_{critical} e.g. fault line
- Given *pressure threshold* $\bar{p}$ at the critical location
- $p^c(E, G) := p(E, G, x_{\text{critical}})$, the *critical pressure*

Want **online** choice of smallest E which gives high *confidence*

$$P(p^c(E,G) \leq \bar{p})$$

in keeping low enough pressure at the critical location

Birds-Eye View



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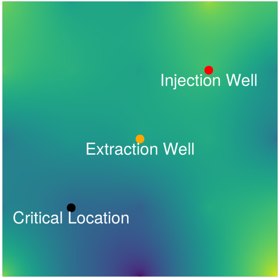

Challenges

- Must approximate $P(p^c(E, G) \leq \bar{p})$ for many different E values
- Approximating $P(p^c(E, G) \leq \bar{p})$ requires many samples of $G(x)$ e.g. over 10,000

Represent $G(x)$ as an Infinite Sum, Approximate $G(x)$ by a Finite Sum

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Karhunen-Loève series expansion of permeability field $G(x)$

$$G(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} Z_j \varphi_j(x)$$

$$Z_1, Z_2, \ldots \overset{\text{IID}}{\sim} \mathcal{N}(0,1)$$

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Since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$, we approximate $G(x)$ by a sum of s terms

$$G(x) \approx \sum_{j=1}^{s} \sqrt{\lambda_j} Z_j \varphi_j(x)$$

$$\boldsymbol{Z} = (Z_1, \ldots, Z_s)^T \sim \mathcal{N}(0, I_s)$$

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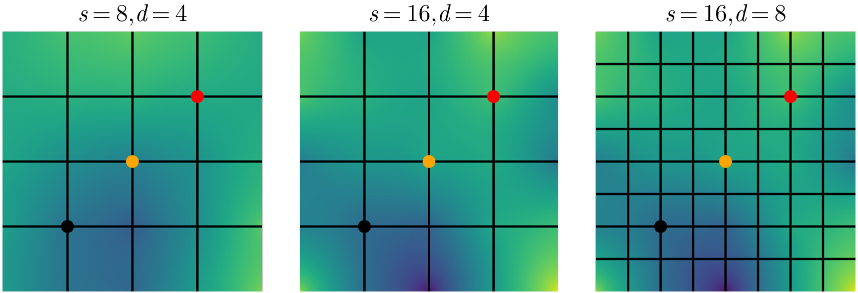

Numerical solution uses DPFEHM: `https://github.com/OrchardLANL/DPFEHM.jl`

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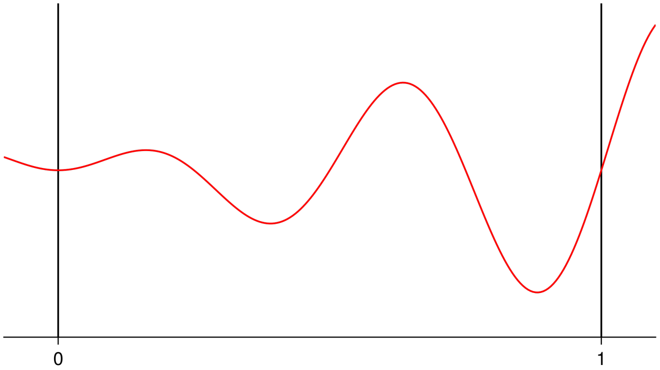

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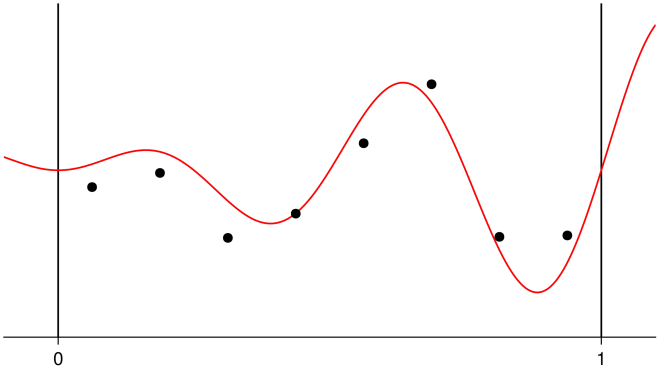

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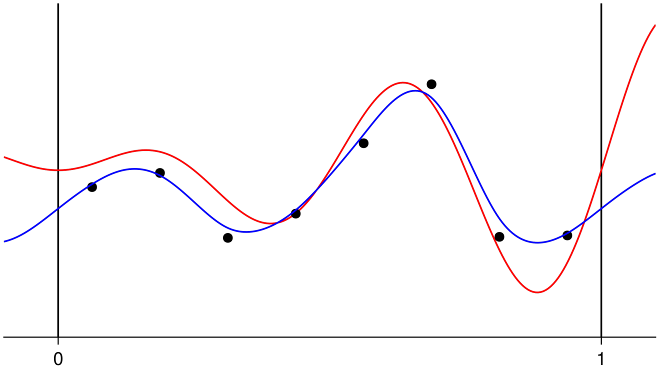

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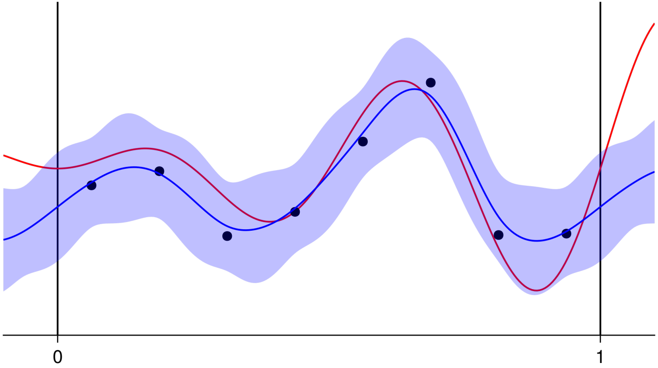

Structuring Nodes and Kernel Enables $\mathcal{O}(n \log n)$ Scaling for GPs

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$\therefore$ Can construct GPs with $\mathcal{O}(n \log n)$ cost when we have control over nodes details

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- Hyperparameter optimization also costs $\mathcal{O}(n \log n)$ for specially structured GPs

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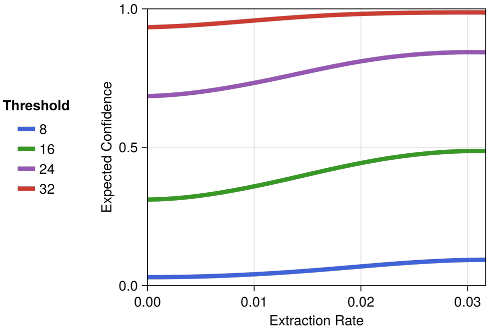

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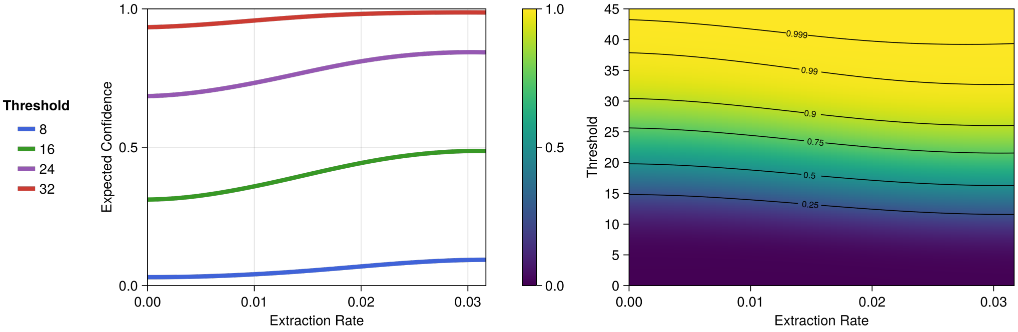

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- **Efficient:** Gaussian process construction in $\mathcal{O}(n \log n)$ instead of $\mathcal{O}(n^3)$ cost
- **Error Aware:** Gaussian process noise calibrated to numerical error
- **Transferable:** Methodology applicable to other PDEs with random coefficients

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- $P(p^c_{s,d}(E,\boldsymbol{Z}) \leq \bar{p}) \neq P(p^c(E,G) \leq \bar{p})$ generally

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$\therefore$ Inexpensive to estimate confidence $P(p^c(E,G) \leq \bar{p})$ online using surrogate

Noisy *Quasi*-Gaussian Process Surrogate

Assume $p^c_{s,d}(t) = p^c(t) + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \zeta_{s,d})$

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$$m_n(t) = K_T(t)\,[K_{TT} + \zeta I_n]^{-1}\,Y$$

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costs $\mathcal{O}(n^3)$ where $K_T(t) = (k(t,t_i))_{i=1}^n, K_{TT} = (k(t_i,t_j))_{i,j=1}^n$.

Quasi-Gaussian Process Regression

Choose special $(t_i)_{i=1}^n$ and k so $[K_{TT} + \zeta I_n]^{-1} Y$ costs $\mathcal{O}(n \log n)$.

Estimating Bound on Standard Deviation of GP Noise

$$\sqrt{\zeta_{s_T,d_T}} = \|p^c - p^c_{s_T,d_T}\| = \sqrt{\mathbb{E}\left[p^c(t) - p^c_{s_T,d_T}(t)\right]^2}, \quad \forall t$$

Rewrite as telescoping sum

$$\sqrt{\zeta_{s_T,d_T}} := \|p^c - p^c_{s_T,d_T}\| \le \sum_{j=T}^{\infty} \|\underbrace{p^c_{s_{j+1},d_j} - p^c_{s_j,d_j}}_{\Delta_{s_{j+1}}}\| + \sum_{j=T}^{\infty} \|\underbrace{p^c_{s_{j+1},d_{j+1}} - p^c_{s_{j+1},d_j}}_{\Delta_{d_{j+1}}}\|.$$

Models $\|\Delta_{s_j}\| = 2^{b_s} s_j^{a_s}$ and $\|\Delta_{d_j}\| = 2^{b_d} d_j^{a_d}$ with $s_j = v_s 2^j$ and $d_j = v_d 2^j$ imply

$$\sqrt{\zeta_{s_T,d_T}} \le \sum_{j=T+1}^{\infty} 2^{b_s} s_j^{a_s} + \sum_{j=T+1}^{\infty} 2^{b_d} d_j^{a_d} = 2^{b_s} v_s^{a_s} \frac{2^{(T+1)a_s}}{1-2^{a_s}} + 2^{b_d} v_d^{a_d} \frac{2^{(T+1)a_d}}{1-2^{a_d}} := \sqrt{\bar{\zeta}_{s_T,d_T}}.$$

Simple Linear Regression Convergence Models