About the joint measurability of observables

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Projector valued measures (PVM)

- Self-adjoint operators as observables:

$$\mathbf{A} \longleftrightarrow \hat{A} = \sum_{i=1}^{n} \alpha_i \left|\alpha_i\right\rangle\!\left\langle\alpha_i\right|$$

- Probabilities for each state $|\psi\rangle$:

$$|\psi\rangle \longrightarrow p(\alpha_i) = \langle\alpha_i|\,\psi\rangle\,\langle\psi|\,\alpha_i\rangle = \mathrm{tr}\left(\left|\alpha_i\right\rangle\!\left\langle\alpha_i\right|\,\left|\psi\right\rangle\!\left\langle\psi\right|\right)$$

- Together, they provide the conditions:

$$\hat{A} \longleftrightarrow \{\left|\alpha_1\right\rangle\!\left\langle\alpha_1\right|, \ldots, \left|\alpha_n\right\rangle\!\left\langle\alpha_n\right|\}$$

$$\left|\alpha_i\right\rangle\!\left\langle\alpha_i\right| \geq 0 \Longleftrightarrow \langle\psi|\,\left|\alpha_i\right\rangle\!\left\langle\alpha_i\right|\,|\psi\rangle \geq 0, \forall\,|\psi\rangle$$

$$\sum_i \left|\alpha_i\right\rangle\!\left\langle\alpha_i\right| = \mathbb{1}, \qquad \sum_i p(\alpha_i) = 1, \forall\,|\psi\rangle$$

Positive operator valued measures (POVM)

- We need $\{E_1, \ldots, E_n\}$ such that :

$$E_i \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{n} E_i = \mathbb{1}$$

- Outcomes optional, $\Omega = \{\omega_1, \ldots, \omega_n\}$,

$$\omega_i \xmapsto{\mathsf{E}} E_i$$

- E.g.

$$\mathsf{E}(\omega_i) = E_i, \qquad \mathsf{E}(\omega_i, \omega_j) = E_i + E_j, \qquad \ldots$$

- **Origin:** Measuring an observable of a system S on a subsystem A *(partial trace, ...)*.

Joint measurability of POVM

$$\{A_1, \dots, A_n\} \qquad \{B_1, \dots, B_m\}$$

- Joint POVM

$$\{R_{ij} \mid i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$$

- We get the former POVM's out of R,

$$A_i = \sum_{j=1}^{m} R_{ij} \qquad B_j = \sum_{i=1}^{n} R_{ij}$$

- In the projective case, $A_i = |\alpha_i\rangle\langle\alpha_i|$, $B_j = |\beta_j\rangle\langle\beta_j|$. Then $\forall i, j$

$$\left[\,|\alpha_i\rangle\langle\alpha_i|\,, |\beta_j\rangle\langle\beta_j|\,\right] = 0 \Rightarrow R_{ij} \stackrel{\text{def}}{=} \overbrace{|\alpha_i\rangle\langle\alpha_i|}^{A_i}\,\underbrace{|\beta_j\rangle\langle\beta_j|}_{B_j} \geq 0$$

Joint measurability of two outcome POVMs

Case for two POVMs

We have $\{P, \mathbb{1} - P\}$ and $\{Q, \mathbb{1} - Q\}$ with *outcome spaces* $\{+, -\}$. There exists an S such that

$$\begin{matrix} \mathbb{1} - P - Q + S \geq 0 \\ P - S \geq 0 \\ Q - S \geq 0 \\ S \geq 0 \end{matrix} \quad \Longleftrightarrow \quad \begin{matrix} R_{++} = S \\ R_{-+} = Q - R_{++} \\ R_{+-} = P - R_{++} \\ R_{--} = \mathbb{1} - P - Q + R_{++} \end{matrix}$$

Joint measurability of two outcome POVMs

Case for two POVMs: Semi-definite program (SDP)

Remember: We have $\{P, \mathbb{1} - P\}$ and $\{Q, \mathbb{1} - Q\}$.

$$\left.\begin{array}{c} \inf \tilde{\lambda} \\ \text{subjected to} \\ \tilde{\lambda}\mathbb{1} - P - Q + S \geq 0 \\ P - S \geq 0 \\ Q - S \geq 0 \\ S \geq 0 \end{array}\right\} \rightarrow \begin{array}{c} \underline{\textit{Semi-definite program}} \\ \\ \lambda \leq 1 \Leftrightarrow \text{Joint measurability} \\ \lambda > 1 \Leftrightarrow \neg\text{Joint measurability} \end{array}$$

Proposition[1]

Let λ and S_0 be the solutions of the SDP above. The parameter $\eta = \max\{0, 1 - \lambda^{-1}\}$ is the least such that the POVMs $(1-\eta)P + \eta E$ and $(1-\eta)Q + \eta E$ are jointly measurable for every E fulfilling $0 \leq E \leq \mathbb{1}$.

[1] M. Wolf, D. Perez-Garcia, C. Fernández . Phys. Rev. Lett., 103:230402, Dec 2009.

Route plan

How do we correct the mistake and provide for a generalization of the parameter?

- Similar but consistent choice of parameters.
- Developement of geometrical and notational framework to circumvent the difficulties of the generalization (*Graphs*).
- Mathematically search for viable properties of these geometrical structures (*subgraphs...*)

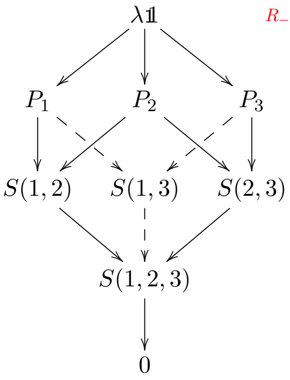
Joint measurability graphs

Joint measurability graph for two POVMs

$$\begin{tikzcd}
\mathbf{0} & & \lambda\mathbb{1} \arrow[dl] \arrow[dr] & & \iff & \lambda\mathbb{1} - P - Q + S \geq 0 \\
\mathbf{1} & P \arrow[dr] & & Q \arrow[dl] & & P - S \geq 0, \quad Q - S \geq 0 \\
\mathbf{2} & & S \arrow[d] & & & S \geq 0 \\
& & 0 & & &
\end{tikzcd}$$

Joint measurability graph for three POVMs

Let us have $\{P_i, \mathbb{1} - P_i\}$ where $i \in \mathcal{N} = \{1, 2, 3\}$.



$$R_{---} = \lambda\mathbb{1} - P_1 - P_2 - P_3 + S(1,2) + S(1,3) + S(2,3) - S(1,2,3) \geq 0$$

$$R_{-+-} = P_2 - S(1,2) - S(2,3) + S(1,2,3) \geq 0$$

$$R_{++-} = S(1,2) - S(1,2,3) \geq 0$$

$$\vdots$$

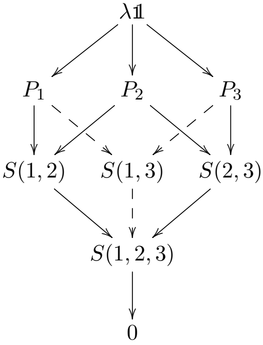
$$R_{+++} = S(1,2,3) \geq 0$$

Joint measurability for three POVMs

Refinement result

Proposition refined

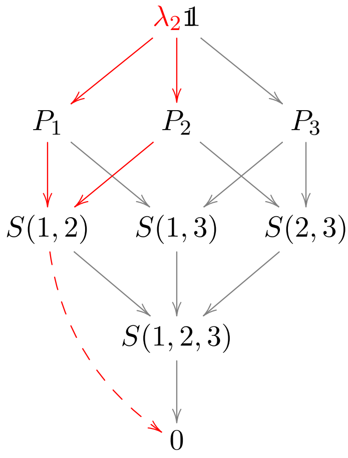
λ is the least number such that $(1-\eta)P_i$ for $i \in \{1, \ldots, n\}$ are jointly measurable. Also if $\lambda \mathbb{1} - P_1 - P_2 - P_3 - \ldots \neq 0$ then there exist operators $0 \leq E \leq \mathbb{1}$ such that $(1-\eta)P_i + \eta E$ are jointly measurable, where $\eta = \max\{0, 1-\lambda^{-1}\}$.



We can make λ in the proposition independent on the choice of P_i, by taking the maximum over all combinations of POVM elements.

Joint measurability for three POVMs

Subgraphs

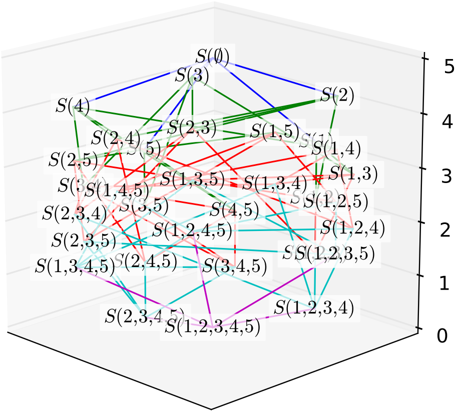


Joint measurability for three POVMs

Parameter tree

$$
\begin{tikzcd}
 & & \lambda\mathbb{1} \arrow[dl] \arrow[d] \arrow[dr] & \\
1 & P_1 \arrow[dd] \arrow[ddr, dashed] & P_2 \arrow[ddl] \arrow[ddr] & P_3 \arrow[ddl, dashed] \arrow[dd] \\
\color{red}{\geq} & & & \\
\color{blue}{\eta_2} & S(1,2) \arrow[ddr] & S(1,3) \arrow[dd, dashed] & S(2,3) \arrow[ddl] \\
\color{red}{\geq} & & & \\
\color{blue}{\eta_3} & & S(1,2,3) \arrow[dd] & \\
\color{red}{\geq} & & & \\
0 & & 0 &
\end{tikzcd}
$$

Thank you very much



for your attention!