CONTINUITY

If $f : D \to \mathbb{R}$ and $c \in D$, then we say that f is *"continuous at c"* if $\lim_{x\to c} f(x) = f(c)$.

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon]$$

Def. Continuous function..
If f is continuous at every point in its domain, we say that f is *continuous*. Notice that now we have $|x - c| < \delta$.

$$(\forall c \in D)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon]$$

Comment. Continuity of Rational functions.

We have shown that if p, q are polynomials, $q(c) \neq 0$ then $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, so all rational functions are continuous wherever their denominator is not zero.

Comment. Continuity of sin and cos functions.

We have shown that $\lim_{x\to c} \sin(x) = \sin(c)$ and $\lim_{x\to c} \cos(x) = \cos(c)$, so sin and cos are continuous functions.

Thm. Continuity of $\sqrt[n]{x}$.

The function $f(x) = \sqrt[n]{x}$ is continuous at any point in its domain (where it makes sense), for $n \in \mathbb{N}$.

Proof.
We want to show that $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$ for all $c > 0$. When n is odd, the proof for $c < 0$ is similar.

$$|f(x) - f(c)| = \left|\sqrt[n]{x} - \sqrt[n]{c}\right| = \frac{|x - c|}{\left|\sum_{k=0}^{n-1} x^{k/n} c^{(n-k)/n}\right|} = \frac{|x - c|}{\sum_{k=0}^{n-1} x^{k/n} c^{(n-k)/n}}$$

Assume $|x - c| < \frac{c}{2}$, so that $-c/2 < x - c < c/2 \Leftrightarrow c/2 < x < 3c/2$.

$$\frac{|x - c|}{\sum_{k=0}^{n-1} x^{k/n} c^{(n-k)/n}} < \frac{|x - c|}{\sum_{k=0}^{n-1} (\frac{c}{2})^{k/n} c^{(n-k)/n}} = \frac{|x - c|}{c \sum_{k=0}^{n-1} 2^{-k/n}} < \frac{|x - c|}{c}$$

So $\delta = \min(\frac{c}{2}, c\varepsilon)$. ■

Corollary. Limit laws.

Note that the **limit laws** immediately tell us that any scalar multiple, a sum, a product, a quotient of two continuous functions is continuous.

Thm. Invariance of domain. (was discussed)

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is an injective continuous map, then $V := f(U)$ is open in $\mathbb{R}^n$ and f is a homeomorphism between U and V.

Thm. Limits of compositions.

Suppose that f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = M$.

If one of the following is true

1. f is continuous at L $(M = f(L))$
2. $\exists e > 0$ such that $g(x) \neq L$ for $0 < |x - c| < e$

then $\lim_{x\to c} f(g(x)) = M$.

Proof.
Fix $\varepsilon > 0$.

(1) Since $\lim_{y\to L} f(y) = M$ we can find a $\hat{\delta} > 0$ such that $0 < |y - L| < \hat{\delta} \Rightarrow |f(y) - M| < \varepsilon$.

(2) Since $\lim_{x\to c} g(x) = L$, $\exists \delta > 0$ such that $0 < |x - c| < \delta \Rightarrow |g(x) - L| < \hat{\delta}$.

We would like to combine these:

$$\begin{aligned} 0 < |x - c| < \delta \Rightarrow |g(x) - L| < \hat{\delta} \\ 0 < |y - L| < \hat{\delta} \Rightarrow |f(y) - M| < \varepsilon \end{aligned}$$

There are two cases.

1. If f is continuous, then (1) becomes $|y - L| < \hat{\delta} \Rightarrow |f(x) - M| < \varepsilon$.
   And so $0 < |x - c| < \delta \Rightarrow |g(x) - L| < \hat{\delta} \Rightarrow |f(g(x)) - M| < \varepsilon$.
2. Since $g(x) \neq L$ when $0 < |x - c| < e$, let $\tilde{\delta} = \min(\delta, e)$.
   So if $0 < |x - c| < \tilde{\delta}$, then $0 < |g(x) - L| < \hat{\delta}$, and consequently $|f(g(x)) - M| < \varepsilon$.

■

Thm.

If f and g are continuous at c, then $f \circ g$ is also continuous at c.

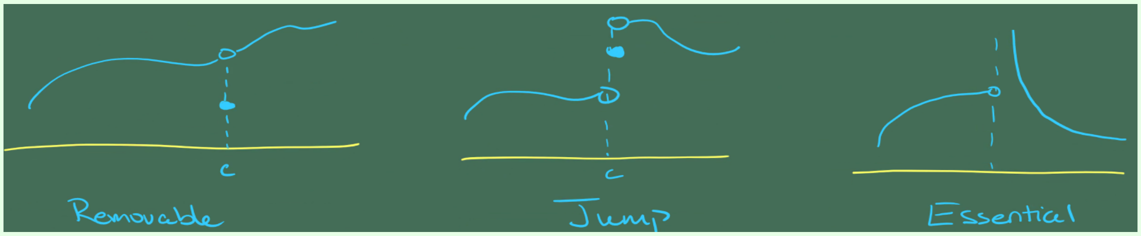
Proof.
We know that $\lim_{x\to c} g(x) = g(c)$ and $\lim_{x\to g(c)} f(x) = f(g(c))$, so $\lim_{x\to c} f(g(x)) = f(g(c))$.

*From this we get the ability to use **substitution** or change the variable.*

TYPES OF DISCONTINUITY

Suppose that f is not continuous at c, and let $L^{\pm} = \lim_{x\to c^{\pm}} f(x)$.

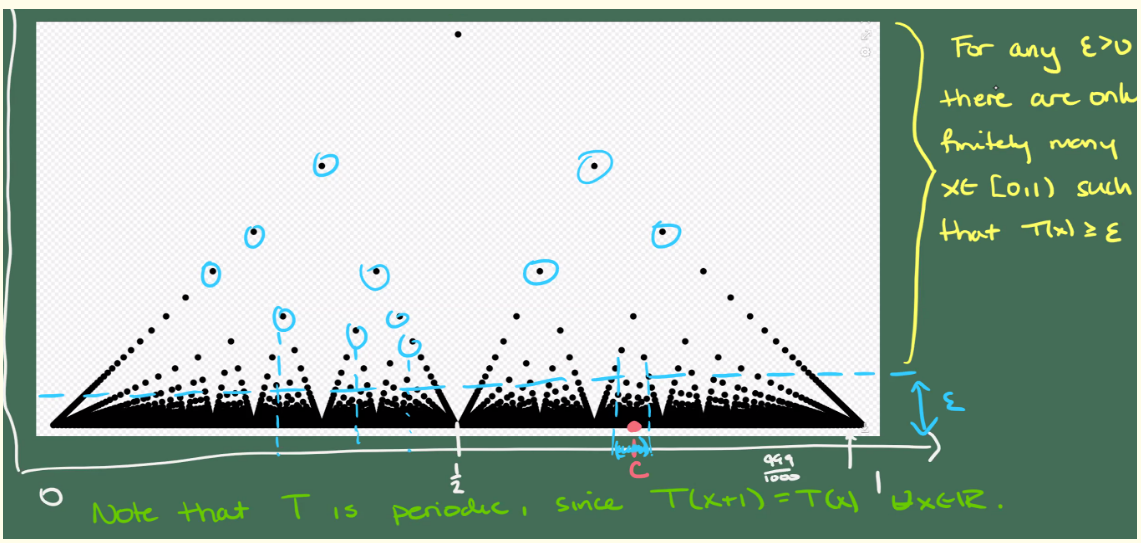
1. We say that c is a *removable discontinuity* if L^+ and L^- exist and $L^+ = L^-$.
2. We say that c is a *jump discontinuity* if L^+ and L^- exist and $L^+ \neq L^-$.
3. We say that c is an *essential discontinuity* if one of L^+ and L^- does not exist.



Def. "Thomae's Function".
We define $T : \mathbb{R} \to \mathbb{R}$ as

$$T(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ and } \gcd(p,q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x = 0 \end{cases}$$

So note that $0 \leq T(x) \leq 1$.
Also, note the T is periodic, since $T(x+1) = T(x)$ for all $x \in \mathbb{R}$.



Note that for any $\varepsilon > 0$ there are only finitely many rationals $x \in [0, 1)$ that $T(x) \geq \varepsilon$.

Thm. "Continuity of Thomae's function"

Thomae's function is continuous at every irrational, and discontinuous at every rational. The discontinuity is removable.

Proof. "Continuity of Thomae's function"
Claim: $\forall c \in \mathbb{R}, \lim_{x \to c} T(x) = 0$. We will prove this on $[0, 1)$.

Let $\varepsilon > 0$ be given and choose the smallest $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ (so $\frac{1}{N-1} > \varepsilon$). Note that if $\varepsilon \geq \frac{1}{2}$ any $\delta > 0$ will work, so we may assume that $0 < \varepsilon < \frac{1}{2}$.

Define $F_c = \left\{\frac{k}{m} : m \in \{2, \dots, N-1\}, k \in \{1, \dots, m-1\}, \gcd(k, m) = 1\right\}$. For every element in F_c, $x \in F_c \Rightarrow T(\frac{k}{m}) = \frac{1}{m} \geq \frac{1}{N-1} > \varepsilon$.

Note that if $x \in F_c$ then writing $x = \frac{k}{m}$ we get $T(x) = T(\frac{k}{m}) = \frac{1}{m} \geq \frac{1}{N-1} > \varepsilon$ and F_c enumerates all numbers whose denominator is at most $N - 1$, so this is precisely the points whose image is greater than ε.

Let $\delta = \min_{x \in F_c \setminus \{c\}} |x - c|$, which exists because $|F_c|$ is finite and is positive since $x \neq c$ for all $x \in F_c \setminus \{c\}$.

Claim: If $0 < |x - c| < \delta$ then $|T(x)| < \varepsilon$.
If $x \in \mathbb{R} \setminus \mathbb{Q}$ then $|T(x)| = |0| = 0 < \varepsilon$.
If $x \in \mathbb{Q}$, note that $x \notin F_c$, so writing $x = \frac{p}{q}$ in lowest terms, $q \geq N$, and so $T(x) = \frac{1}{q} \leq \frac{1}{N} < \varepsilon$. ■

"Dirichlet's function"

The Dirichlet function is continuous nowhere. Limit never exists.

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$