Problem 1.
Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $\lim_{x\to 0} f(x)$ does not exist.

Lemma.
Assume $\lim_{x\to c} f(x)$ exists and $(\forall \delta > 0)(\exists x \in \mathbb{R})[(0 < |x - c| < \delta) \wedge (f(x) = M)]$, then $\lim_{x\to c} f(x) = M$.

Proof.
For the purpose of contradiction, assume that $\lim_{x\to c} f(x) = L \neq M$. Thus, $M - L \neq 0$. Then

$$(\forall \delta > 0)(\exists x \in \mathbb{R})[(0 < |x - c| < \delta) \wedge (|f(x) - L| = |M - L| > 0)].$$

Thus, for $\varepsilon = |M - L| > 0$, we have that $(\forall \delta > 0)(\exists x \in \mathbb{R})[(0 < |x - c| < \delta) \wedge (|f(x) - L| \geq \varepsilon)]$.

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})[0 < |x - c| < \delta \not\Rightarrow |f(x) - L| < \varepsilon]$$

Notice that this is the exact negation of the definition for $\lim_{x\to c} f(x) = L$.

$$\neg(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

Thus, the assumption $\lim_{x\to c} f(x) = L \neq M$ led to a contradiction, and consequently it must be true that $\lim_{x\to c} f(x) = M$. ■

Solution.
We want to prove that $\lim_{x\to 0} f(x)$ does not exist.

For a contradiction, assume that $\lim_{x\to 0} f(x)$ exists.

It will be proven that $(\forall \alpha \in [-1, 1])(\forall \delta > 0)(\exists x \in \mathbb{R})[(0 < |x - 0| < \delta) \wedge (f(x) = \alpha)]$.

Fix some $\alpha_0 \in [-1, 1]$ and $\delta_0 > 0$. Since $(\forall \delta > 0)(\exists n \in \mathbb{N})\left[\frac{1}{n} < \delta\right]$, we can fix $n_0 \in \mathbb{N}$ such that $\delta_0 > \frac{1}{n_0} > \frac{1}{2\pi \cdot n_0} > \frac{1}{2\pi \cdot (n_0+1) + \arcsin(\alpha_0)} > 0$. Then we can take $x_0 = \frac{1}{2\pi \cdot (n_0+1) + \arcsin(\alpha_0)} \in \mathbb{R}$ and we will have $f(x_0) = \sin\left(\frac{1}{x_0}\right) = \sin(2\pi \cdot (n_0 + 1) + \arcsin(\alpha_0)) = \sin(\arcsin(\alpha_0)) = \alpha_0$. Thus, we obtained x_0 such that $(0 < |x_0| < \delta_0) \wedge (f(x_0) = \alpha_0)$.

According to **Lemma**, for every $\alpha \in [-1, 1]$, $\lim_{x\to 0} f(x) = \alpha$. However, a limit can only have a unique value. Thus, the assumption that $\lim_{x\to 0} f(x)$ exists led to a contradiction. ■