FUNCTION

If A and B are sets, a *function* is a unique assignment of every element in A to an element of B. We write $f : A \to B$ to denote a function.

- **Domain** $= A$
- **Codomain** $= B$
- **Range** $= f(A) = \{f(x) : x \in A\} \subseteq B$

Def. Well-defined function.
A map $f : A \to B$ is *well-defined*, if $(\forall x, y \in A)[x = y \Rightarrow f(x) = f(y)]$.

INJECTIVITY

A function $f : A \to B$ is *injective*, if $(\forall x, y \in A)[f(x) = f(y) \Rightarrow x = y]$.

- Every A has unique B.
- Every B comes from unique A or nothing at all.

SURJECTIVITY

A function $f : A \to B$ is *surjective*, if $(\forall y \in B)[\exists x \in A, f(x) = y]$.

- Every element in the codomain is obtainable.

BIJECTIVITY

A function $f : A \to B$ is *bijective*, if it is both *injective* and *surjective*.

- Every element is obtainable in a unique way.
- There is matching between elements of A and B, i.e. function f is *invertible*.

Def. Cardinality.
Cardinality of a set S is the a measure of the "number of elements" in S, denoted $|S|$.

- **Injectivity.** If there is an injection from $S \to T$, then $|S| \leq |T|$.
- **Countability.** A set S is *countable*, if $|S| \leq |\mathbb{N}|$.
- **Bijectivity.** $|S| = |T|$, if there exists a bijection $S \to T$.

Thm. [Cantor–Bernstein-Schröder], a.k.a. the **CBT**

If S, T are two sets with $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$.

- "Left injection" + "Right injection" = "Bijection".

Thm. Transitivity of injectivity.

If $f : B \to C$ and $g : A \to B$ are both injective then their composition $f \circ g$ is also injective.

Proof.
By the definition of injectivity, we have the following.

$$(\forall a_1, a_2)[g(a_1) = g(a_2) \Rightarrow a_1 = a_2]$$

$$(\forall b_1, b_2)[f(b_1) = f(b_2) \Rightarrow b_1 = b_2]$$

Want to show, the following.

$$(\forall a_1, a_2)[f(g(a_1)) = f(g(a_2)) \Rightarrow a_1 = a_2]$$

Assume that $f(g(a_1)) = f(g(a_2))$.

- Since f is injective, $g(a_1) = g(a_2)$.
- Since g is injective, $a_1 = a_2$.

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Problem 1.
If $f : B \to C$ and $g : A \to B$ are such that $f \circ g$ is injective, then g is injective. Also, f does not have to be injective.

Solution.
We have the following.

$$(\forall a, b)[f(g(a)) = f(g(b)) \Rightarrow a = b]$$

Want to show the following.

$$(\forall a, b)[g(a) = g(b) \Rightarrow a = b]$$

Assume $g(a) = g(b)$.

- Since f is a well-defined function, $f(g(a)) = f(g(b))$.
- Since $f \circ g$ is injective, $a = b$.

□

INVERTIBILITY

Inverse of a function $f : A \to B$ is a function $f^{-1} : B \to A$ that satisfies $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

- Function is called *left-invertible*, if $f^{-1} \circ f = \mathrm{id}_A$.
- Function is called *right-invertible*, if $f \circ f^{-1} = \mathrm{id}_B$.

Thm. Injectivity $\Leftrightarrow$ Left-invertibility

Suppose $g : A \to B$ is a function. Then g is injective if and only if there exists a function $h : B \to A$ such that $(h \circ g) = \mathrm{id}_A$.

Proof. ($\Longleftarrow$), (mimics Problem 1.)
We have that $h \circ g = \mathrm{id}_A$, consequently the following is true.

$$(\forall a, b \in A)[h(g(a)) = h(g(b)) \Rightarrow a = b]$$

Want to show the following.

$$(\forall a, b \in A)[g(a) = g(b) \Rightarrow a = b]$$

Find the exact solution in **Problem 1.**

Proof. ($\Longrightarrow$)
The function can be defined explicitly.

1. Fix some element $a_0 \in A$.
2. Define $h : B \to A$ as $\left[h(b) = \begin{cases} a & \text{if } g(a) = b \\ a_0 & \text{otherwise} \end{cases}\right]$.

$$(h \circ g)(a) = h(g(a)) = a, \quad \text{so we have} \quad h \circ g = \mathrm{id}_A$$

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Thm. Surjectivity $\Leftrightarrow$ Right-invertibility

Suppose $g : A \to B$ is a function. Then g is injective if and only if there exists a function $h : B \to A$ such that $(g \circ h) = \mathrm{id}_B$.

Thm. BIJECTIVITY $\Leftrightarrow$ INVERTIBILITY

A function $f : A \to B$ is bijective $\Leftrightarrow$ f has an inverse.

Thm. *A countable union of (disjoint) countable sets is countable.*

For any collection $\{A_i : i \in I, |A_i| \leq |\mathbb{N}|\}$ where $|I| \leq |\mathbb{N}|$, we have $|\cup_{i\in I} A_i| \leq |\mathbb{N}|$.

– ”Countability squared” = ”Countability”

Proof.
We know $\exists f : I \hookrightarrow \mathbb{N}$ which is injective, and $(\forall i \in I)[\exists g_i : A_i \hookrightarrow \mathbb{N}]$ which is injective.

Define a map

$$F : \bigcup_{i\in I} A_i \to \mathbb{N}, \quad a \mapsto 2^{f(n)} \cdot 3^{g_n(a)}, \quad \text{where } a \in A_n.$$

Power of 2 tracks the set. Power of 3 tracks the element within the set. By the *Fundamental Theorem of Arithmetic*, F is truly injective. ∎

Thm. $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

Integers and rationals are countable.

Proof.
Proof if these facts is fairly simply.

1. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \Rightarrow |\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}|$
2. $\mathbb{Z} = \{-1, 0, 1\} \times \mathbb{N} \Rightarrow |\mathbb{Z}| \leq |\mathbb{N}|$, according to the properties of a countable union of countable sets.
3. $\mathbb{Q} = \mathbb{N} \times \mathbb{Z} \Rightarrow |\mathbb{Q}| \leq |\mathbb{N}|$, according to the properties of a countable union of countable sets.

Thus, integers and rationals are truly countable.

Thm. $|\mathbb{R}| > |\mathbb{N}|$

The set of real numbers is not countable.

Proof. ”Gaussian Diagonalization”
This is just an instruction for the proof.

1. Prove that $|(0, 1)| = |\mathbb{R}|$.
2. For the sake of contradiction, assume that $|(0, 1)| = |\mathbb{N}|$.
3. ”Invert” all digits on the diagonal, avoiding assigning the $base - 1$ digit.
4. The obtained number is real and will not match with any of our numbers.
5. Since we claimed to write down all real numbers this is a contradiction. ∎