Problem 2.(i)
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijective function. Show that f is strictly monotone; that is, f is either strictly increasing or strictly decreasing on all $\mathbb{R}$.

Solution.
Note that due to injectivity of f, $(\forall x, y \in \mathbb{R})[f(x) = f(y) \Rightarrow x = y]$. Thus, $f(0) \neq f(1)$. Assume that $f(0) < f(1)$ and it will be proven that f is strictly increasing. The case where $f(0) > f(1)$ and we aim to prove that f is strictly decreasing is analogous. For instance, we could take $g(x) = -f(x)$ with $g(0) < g(1)$ and use that it is strictly increasing to obtain that f is strictly decreasing.

It will be proven that $(\forall x, y, z \in \mathbb{R})[x < y < z \Rightarrow f(x) < f(y) < f(z) \vee f(x) > f(y) > f(z)]$. For a contradiction, assume the opposite and fix $x, y, z \in \mathbb{R}$ such that $x < y < z$ and either $f(x) < f(y) > f(z)$ or $f(x) > f(y) < f(z)$.

If $f(x) < f(y) > f(z)$, take $m = \frac{1}{2}f(y) + \frac{1}{2}\max\{f(x), f(z)\}$. Thus, $m \in (\max\{f(x), f(z)\}, f(y))$. Due to continuity of f, according to the **Intermediate Value Theorem**, exists $x_0 \in [x, y]$ such that $f(x_0) = m$. Similarly, exists $z_0 \in [y, z]$ such that $f(z_0) = m$. Thus, since $f(y) > m$, $x_0 < y < z_0$ and $f(x_0) = f(z_0) = m$. This contradicts injectivity of f, $(\forall x, y \in \mathbb{R})[f(x) = f(y) \Rightarrow x = y]$.

If $f(x) > f(y) < f(z)$, take $m = \frac{1}{2}f(y) + \frac{1}{2}\min\{f(x), f(z)\}$. Thus, $m \in (f(y), \min\{f(x), f(z)\})$. Using the **Intermediate Value Theorem**, we will be able to arrive to a contradiction with injectivity of f in a similar way to the $f(x) < f(y) > f(z)$ case.

Thus, it has been proven that $(\forall x, y, z \in \mathbb{R})[x < y < z \Rightarrow f(x) < f(y) < f(z) \vee f(x) > f(y) > f(z)]$. Consequently,

$$\begin{aligned}
&(\forall x, y, z, t \in \mathbb{R})[x < y < z < t \Rightarrow x < y < z \wedge y < z < t],\\
&(\forall x, y, z, t \in \mathbb{R})[x < y < z < t \Rightarrow (f(x) < f(y) < f(z) \vee f(x) > f(y) > f(z)) \wedge\\
&\qquad\qquad\qquad\qquad\qquad\qquad (f(y) < f(z) < f(t) \vee f(y) > f(z) > f(t))],\\
&(\forall x, y, z, t \in \mathbb{R})[x < y < z < t \Rightarrow f(x) < f(y) < f(z) < f(t) \vee f(x) > f(y) > f(z) > f(t)].
\end{aligned}$$

Now it will be finally proven that $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f(x) < f(y)]$. Fix some $x, y \in \mathbb{R}$ such that $x < y$.

1. If $x = 0 \wedge y = 1$, then, according to the initial assumption $f(0) < f(1)$.
2. If exactly one of x, y belongs to $\{0, 1\}$, we can use the fact that $(\forall x, y, z \in \mathbb{R})[x < y < z \Rightarrow f(x) < f(y) < f(z) \vee f(x) > f(y) > f(z)]$, for numbers in $\{x, y\} \cup \{0, 1\}$, to show that $x < y \Rightarrow f(x) < f(y)$.
3. If none of x, y belong to $\{0, 1\}$, we can use the fact that $(\forall x, y, z, t \in \mathbb{R})[x < y < z < t \Rightarrow f(x) < f(y) < f(z) < f(t) \vee f(x) > f(y) > f(z) > f(t)]$, for numbers in $\{x, y\} \cup \{0, 1\}$, to show that $x < y \Rightarrow f(x) < f(y)$.

Thus, we have finally proven that f is strictly increasing when $f(0) < f(1)$. Consequently, we have proven that f is strictly monotone. ■

Problem 2.(ii)
Suppose that $f : \mathbb{R} \to \mathbb{R}$, and there exists some $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$.

Solution.
We have that $(\forall x, y \in \mathbb{R})[|f(x) - f(y)| \leq K|x - y|]$. WTS that f is uniformly continuous on $\mathbb{R}$.

It will be proven that

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in \mathbb{R})[|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon].$$

Fix some $\epsilon > 0$. Since $K > 0$, we can take $\delta = \frac{\epsilon}{K} > 0$. Since $(\forall x, y \in \mathbb{R})[|f(x) - f(y)| \leq K|x - y|]$,

$$(\forall x, y \in \mathbb{R}) \left[|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq K|x - y| < K\delta = K\frac{\epsilon}{K} = \epsilon \right].$$

Thus, $(\forall x, y \in \mathbb{R})[|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon]$ and f is uniformly continuous. ■