DEDEKIND CUTS and the Construction of $\mathbb{R}$

Suppose we only know about $\mathbb{Q}$ and we want to construct $\mathbb{R}$.

REAL NUMBER as a subset of $\mathbb{Q}$

A *real number* $\underline{r}$ is a subset of $\mathbb{Q}$ such that it is

1. [**Non-trivial**] $\underline{r} \neq \varnothing$
2. [**Proper**] $\underline{r} \neq \mathbb{Q}$
3. [**Closed downwards**] If $y \in \underline{r}$ and $x < y$, then $x \in \underline{r}$.
4. [**No greatest element**] If $y \in \underline{r}$, there exists some $z \in \underline{r}$ with $y < z$.

Thm. "Sum of real numbers"

If $\underline{r}$ and $\underline{s}$ are *real numbers*, then $\underline{r} + \underline{s} = \{x \in \mathbb{Q} : x = a + b, a \in \underline{r}, b \in \underline{s}\}$ is a real number.

Proof.
We shall prove every condition for a real number.

1. **Non-trivial:** Clear. Since both $\underline{r}$ and $\underline{s}$ are non-empty.

$$(\exists a \in \underline{r})(\exists b \in \underline{s})[x = a + b \in \underline{r} + \underline{s}] \implies \underline{r} + \underline{s} \neq \varnothing$$

2. **Proper:** Since $\underline{r}$ and $\underline{s}$ are proper, choose $m \in \mathbb{Q} \setminus \underline{r}$ and $n \in \mathbb{Q} \setminus \underline{s}$. Thus, $m + n \notin \underline{r} + \underline{s}$.
3. **Closed-downwards:** Fix $y \in \underline{r} + \underline{s}$ and let $x < y$. Write $y = a + b$ where $a \in \underline{r}$ and $b \in \underline{s}$. Thus $x < a + b$ or equivalently $x - a < b$. Thus $x - a \in \underline{s}$ and $x = a + (x - a) \in \underline{r} + \underline{s}$.
4. **No greatest element:** Fix $y \in \underline{r} + \underline{s}$ and write $y = a + b$ where $a \in \underline{r}$ and $b \in \underline{s}$. Since $\underline{r}$ and $\underline{s}$ have no greatest elements, $(\exists c \in \underline{r}, d \in \underline{s})[c > a, d > b]$. Thus $y = a + b < c + d$ and since $c + d \in \underline{r} + \underline{s}$, then $\underline{r} + \underline{s}$ has no greatest element.

Def. Comparing $\underline{r}$ and $\underline{s}$.
If $\underline{r}$ and $\underline{s}$ are real numbers, we say

- $\underline{r} \leq \underline{s}$ if $\underline{r} \subseteq \underline{s}$
- $\underline{r} < \underline{s}$ if $\underline{r} \subset \underline{s}$

Def. Negative $\underline{r}$.
If $\underline{r} \in \mathbb{R}$ then

$$-\underline{r} = \{x \in \mathbb{Q} : -x \notin \underline{r} \text{ and } x \neq \min(\mathbb{Q} \setminus \underline{r})\}$$

Def. Absolute value of $\underline{r}$.
If $\underline{r} \in \mathbb{R}$ then

$$|\underline{r}| = \begin{cases} \underline{r} & \text{if } \underline{r} \geq \underline{0} \\ -\underline{r} & \text{if } \underline{r} < \underline{0} \end{cases}$$

Interval

An *interval* I is a subset of $\mathbb{R}$ such that $(\forall a, b \in I)(\forall z \in \mathbb{R})[a < z < b \Rightarrow z \in I]$.

Def. Multiplication of $\underline{r}$ and $\underline{s}$.
If $\underline{r}, \underline{s} \geq \underline{0}$ then

$$\underline{r} \cdot \underline{s} = \underline{0} \cup \{x = a \cdot b \in \mathbb{Q} : (a \in \underline{r}) \wedge (b \in \underline{s}) \wedge (a, b > 0)\}$$

and in general

$$\underline{r} \cdot \underline{s} = \begin{cases} |\underline{r}| \cdot |\underline{s}| & \text{if } \underline{r}, \underline{s} < \underline{0} \text{ or } \underline{r}, \underline{s} > \underline{0} \\ -|\underline{r}| \cdot |\underline{s}| & \text{if } \underline{r}, \underline{s} \text{ have different signs} \\ 0 & \text{if } \underline{r} = \underline{0} \text{ or } \underline{s} = \underline{0} \end{cases}$$

Inequality in $\mathbb{R}$

Let $P = \{x \in \mathbb{R} : x > 0\}$. We say that $x < y$ if $(y - x) = y + (-x) \in P$.

Thm. "Facts about P."

We know the following about P.

1. If $x, y \in P$ then $x + y \in P$ and $x \cdot y \in P$.
2. If $x \in \mathbb{R} \setminus \{0\}$ then either $x \in P$ or $-x \in P$.

Thm. "Facts about inequalities"

Suppose $x, y, u, v \in \mathbb{R}$ and $c > 0$. Then

1. If $x < y$ and $y < u$ then $x < u$.
2. If $x < y$ then $cx < cy$.
3. If $x < y$ and $u < v$ then $x + u < y + v$.

SUPREMUM

Supremum exists only on sets having an upper bound.

- If $S \subseteq \mathbb{R}$ we say that M is an *upper bound* for S if $(\forall x \in S)[x \leq M]$.
- If S has an upper bound we say that S is *bounded from above*.
- If S is bounded from above, its *supremum* is its least upper bound, denoted $\sup(S)$. That is if M is any upper bound then $\sup(S) \leq M$.

There is an 'obvious' analogy for *lower bounds* and *being bounded from below*, in which case the greatest lower bound is the *infimum*, denoted $\inf(S)$.

Thm. COMPLETENESS AXIOM

Every non-empty set which is bounded from above has a supremum.

Proof.
Let $S \subseteq \mathbb{R}$ be non-empty and bounded from above.

Define $\sigma = \bigcup_{\alpha \in S} \alpha$. Want to show, σ is a real number.

1. **Non-empty:** Since $S \neq \varnothing$, there is non-empty $\alpha \in S$, and $\alpha \subseteq \sigma$ so $\sigma \neq \varnothing$.
2. **Proper:** Since S is bounded from above, let μ be an upper bound; that is $(\forall \alpha \in S)[\alpha \leq \mu]$.
   Since $\mu \in \mathbb{R}$, $\exists x \in \mathbb{Q}$ such that $x \notin \mu$, and so $(\forall \alpha \in S)[\alpha \subseteq \mu \Rightarrow x \notin \alpha]$, and so $x \notin \sigma$.
3. **Closed Downwards:** Fix some $y \in \sigma$ and let $x \in \mathbb{Q}$ with $x < y$.
   Since $y \in \sigma$, $\exists \alpha \in S$ such that $y \in \alpha$. (y and x are rationals)
   Since α is closed downwards, $x \in \alpha$ and hence $x \in \sigma$.
4. **No greatest element:** Fix $y \in \sigma$.
   There exists some $\alpha \in S$ with $y \in \alpha$.
   Since α has no greatest element, $(\exists z \in \alpha)[y < z]$, but $z \in \alpha \subseteq \sigma$.
   Thus $(\forall y \in \sigma)(\exists z \in \sigma)[y < z]$.

So σ is a real number. Want to show, it is the *least upper bound.*

1. Note that $(\forall \alpha \in S)[\alpha \subseteq \sigma \Rightarrow \alpha \leq \sigma]$. Thus σ is an upper bound.
2. Suppose μ is some other upper bound. Thus $(\forall \alpha \in S)[\alpha \leq \mu \Rightarrow \alpha \subseteq \mu]$.
   Thus $\sigma = \cup_{\alpha \in S} \alpha \subseteq \mu$, so $\sigma \leq \mu$.

Thus σ is the *least upper bound.* ■

Thm. "Archimedean property"

The naturals are **not** bounded from above.

Corollary.

For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.

Thm. Main *supremum* property.

Suppose that $S \subseteq \mathbb{R}$ and M is an upper bound for S.

$$M = \sup(S) \quad \iff \quad (\forall \varepsilon > 0)(\exists s \in S)\,[(M - \varepsilon) < s \leq M]$$

Proof. ($\implies$)
Assume $M = \sup(S)$. For a contradiction, suppose $(\exists \varepsilon > 0)(\forall s \in S)[s \leq (M - \varepsilon)]$.

Thus $(M - \varepsilon)$ is an upper bound for S, and this brings a contradiction because $\sup(S) = M > (M - \varepsilon)$.

■

Proof. ($\impliedby$)
Assume $(\forall \varepsilon > 0)(\exists s \in S)[(M - \varepsilon) < s \leq M]$. For a contradiction, assume M is not the least upper bound.

Take $\varepsilon = (M - \sup(S)) > 0$. Thus $(\exists s \in S)[(M - \varepsilon) < s \leq M]$. Thus $(\exists s \in S)[sup(S) < s \leq M]$.

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Problem "Sum of suprema"
If $A, B \subset \mathbb{R}$ define their sum the following way.

$$A + B = \{x \in \mathbb{R} : x = a + b \wedge a \in A \wedge b \in B\}$$

Show that $\sup(A + B) = \sup(A) + \sup(B)$.

Proof.
Let $M_A = \sup(A)$, $M_B = \sup(B)$.

1. First, let's show that $M_A + M_B$ is an upper bound for $A + B$.

   If $x \in A + B$, write $x = a + b$, where $a \in A$, $b \in B$. Now $a \leq M_A$, $b \leq M_B$, thus $x = a + b \leq M_A + M_B$. So $M_A + M_B$ is an upper bound, meaning

   $$\sup(A + B) \leq \sup(A) + \sup(B)$$

2. Now we want to show that $M_A + M_B$ is the supremum, using the *main supremum property.*

   Fix $\varepsilon > 0$. We want to find some $x \in A + B$ such that $(M_A + M_B) - \varepsilon < x \leq (M_A + M_B)$.

   According to the *main supremum property* for M_A and M_B,

   $$(\exists a \in A)[M_A - \varepsilon/2 < a \leq M_A],$$

   $$(\exists b \in B)[M_B - \varepsilon/2 < b \leq M_B].$$

   Thus $(M_A + M_B) - \varepsilon < a + b \leq (M_A + M_B)$.

DENSE SETS

A set $S \subseteq \mathbb{R}$ is said to be *dense* if for every open interval $(a, b) \subseteq \mathbb{R}$, we have $(a, b) \cap S \neq \varnothing$.

Thm. "Density of $\mathbb{Q}$ in $\mathbb{R}$"

The rationals are dense among the reals.

Proof.
Fix an open interval (a, b). Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < (b - a)$.

Define $B = \left\{ \frac{m}{N} : m \in \mathbb{Z} \right\}$ for which we claim $B \cap (a, b) \neq \varnothing$.

For a contradiction, assume $B \cap (a, b) = \varnothing$. Let M be the largest integer such that $\frac{M}{N} < a$.

But then $\frac{M+1}{N} > b$. Thus $b - a < \frac{M+1}{N} - \frac{M}{N} = \frac{1}{N} < b - a$.

This is a contradiction, so $B \cap (a, b) \neq \varnothing$. ■