Problem 1.
Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the relation $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is differentiable at 0 with $f'(0) = K$, show that f is in fact differentiable on all of $\mathbb{R}$. Find an expression for f' involving only f and K.

Solution.
We have that $(\forall x, y \in \mathbb{R})[f(x+y) = f(x)f(y)]$ and differentiability at 0, which implies that

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{(0+h) - 0} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = K.$$

It will be proven that f is differentiable on all of $\mathbb{R}$. Fix some $c \in \mathbb{R}$.

Since $(\forall x, y \in \mathbb{R})[f(x+y) = f(x)f(y)]$ and $f'(0) = K$,

$$\begin{aligned} f'(c) &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c+h) - f(c+0)}{h} \\ &= \lim_{h \to 0} \frac{f(c)f(h) - f(c)f(0)}{h} = f(c) \cdot \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f(c) \cdot K. \end{aligned}$$

Thus, for all $c \in \mathbb{R}$, $f'(c)$ exists and it is equal to $f(c) \cdot K$. Consequently, f is differentiable on all of $\mathbb{R}$ and $f'(x) = f(x) \cdot K$ for all $x \in \mathbb{R}$. ■