Problem 3.
Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and bijective, then f^{-1} is also continuous.

Solution.
According to the **Q2.(i)**, f must be strictly monotone. Assume that f is strictly increasing. The case where f is strictly decreasing is completely analogous.

We have that $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f(x) < f(y)]$, the condition of continuity of f, and the condition of bijectivity of f. WTP that f^{-1} is continuous.

It will be proven that f^{-1} is strictly increasing as well. For a contradiction, assume the opposite. Thus, $(\exists f(x), f(y) \in \mathbb{R})[f(x) < f(y) \wedge f^{-1}(f(x)) \geq f^{-1}(f(y))]$. Then, $(\exists x, y \in \mathbb{R})[f(x) < f(y) \wedge x \geq y]$. Fix such x, y. Note that f is injective, and since $f(x) \neq f(y)$ we have $x > y$. Since f is strictly increasing $f(x) > f(y)$, which contradicts $f(x) < f(y)$. Thus, $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f^{-1}(x) < f^{-1}(y)]$.

We want to prove that f^{-1} is continuous or equivalently that

$$(\forall c \in \mathbb{R})(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow |f^{-1}(x) - f^{-1}(c)| < \epsilon].$$

Fix some $c \in \mathbb{R}$ and $\epsilon > 0$. Denote $\hat{c} = f^{-1}(c)$. Since f is strictly increasing $f(\hat{c} - \epsilon) < f(\hat{c}) < f(\hat{c} + \epsilon)$. Take $\delta = \min\{f(\hat{c}) - f(\hat{c} - \epsilon), f(\hat{c} + \epsilon) - f(\hat{c})\}$. Thus,

$$(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow (c - \delta) < x < (c + \delta)],$$

$$(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow (c - (f(\hat{c}) - f(\hat{c} - \epsilon))) < x < (c + (f(\hat{c} + \epsilon) - f(\hat{c})))],$$

$$(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow f(\hat{c} - \epsilon) < x < f(\hat{c} + \epsilon)].$$

Since f^{-1} is strictly increasing,

$$(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow (\hat{c} - \epsilon) < f^{-1}(x) < (\hat{c} + \epsilon)],$$

$$(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow |f^{-1}(x) - \hat{c}| < \epsilon],$$

$$(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow |f^{-1}(x) - f^{-1}(c)| < \epsilon].$$

Thus, the selected δ works and we have successfully proved that f^{-1} is continuous. ■