Problem 2.
Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are such that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, and moreover $f(x) < g(x)$ for all $x \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.

Lemma 1. "A limit of a positive function"
Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $\lim_{x\to c} f(x) = F$ exists and $(\forall x \in \mathbb{R})[f(x) > 0]$, then $F \geq 0$.

Proof.
We have that $(\forall x \in \mathbb{R})[f(x) > 0]$ and the property of the limit F.

$$\lim_{x\to c} f(x) = F \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - F| < \varepsilon]$$

Want to prove that $F \geq 0$.

For a contradiction, assume that $F < 0$.
Since $(\forall x \in \mathbb{R})[f(x) > 0]$ and $(-F) = |F| > 0$, $(\forall x \in \mathbb{R})[|f(x) - F| = |f(x) + (-F)| = f(x) + |F|]$.

Since $F = \lim_{x\to c} f(x)$, we can find $\delta > 0$ such that $(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - F| < \varepsilon = |F|]$.
This leads to the following.

$$(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - F| = f(x) + |F| < |F|]$$

$$(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow f(x) < 0]$$

However, we know that $(\exists x \in \mathbb{R})[0 < |x - c| < \delta]$ and $(\forall x \in \mathbb{R})[f(x) > 0]$. Thus, the assumption that $F < 0$ leads to a contradiction. ■

Solution.
According to the **limit laws**,

$$\lim_{x\to c} g(x) - \lim_{x\to c} f(x) = \lim_{x\to c} g(x) + (-1)\cdot \lim_{x\to c} f(x) = \lim_{x\to c} g(x) + \lim_{x\to c} [(-1)f(x)] = \lim_{x\to c} [g(x) + (-1)\cdot f(x)],$$

$$\lim_{x\to c} g(x) - \lim_{x\to c} f(x) = \lim_{x\to c} [g(x) - f(x)].$$

Since $(\forall x \in \mathbb{R})[f(x) < g(x)]$, it is true that $(\forall x \in \mathbb{R})[(g(x) - f(x)) > 0]$.

Since $(\forall x \in \mathbb{R})[(g(x) - f(x)) > 0]$, according to **Lemma 1**, $\lim_{x\to c}(g(x) - f(x)) \geq 0$.

Thus, we have $\lim_{x\to c} g(x) - \lim_{x\to c} f(x) = \lim_{x\to c}(g(x) - f(x)) \geq 0$ and consequently $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$. ■