W7 Lecture

- $\int$ parameterization
- reciprocal rule

Defn: Suppose $f: I \to \mathbb{R}$ and $c \in I$. We say that f is <u>differentiable at c</u> if

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ exists,}$$

in which case $f'(c)$ is the derivative of f at c.

Example:
$$f'(2) = \lim_{x \to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x-2} = \lim_{x \to 2} \frac{4-x^2}{4x^2(x-2)}$$
$$= \lim_{x \to 2} \frac{(2-x)(2+x)}{4x^2(x-2)} = \lim_{x \to 2} \frac{-(x+2)}{4x^2} = -\frac{4}{4 \cdot 4} = -\frac{1}{4}$$

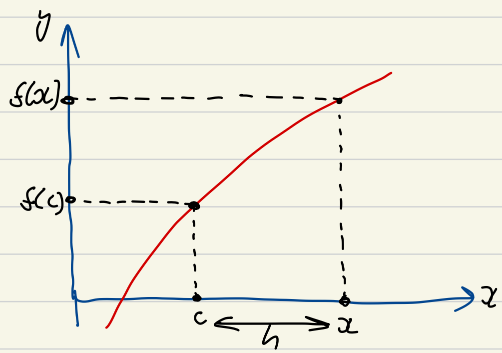
Defn.: If $f: I \to \mathbb{R}$, let $D \subseteq I$ be the points where f is diff. The <u>derivative function</u> is the
$$f': D \to \mathbb{R}$$
$$c \mapsto f'(c)$$

Example: $f: (0, \infty) \to \mathbb{R}$, $x \mapsto 1/x^2$, find f'.

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c} \frac{\frac{1}{x^2} - \frac{1}{c^2}}{x-c} = \lim_{x \to c} \frac{(c-x)(c+x)}{c^2x^2(x-c)}$$
$$= \lim_{x \to c} -\frac{x+c}{c^2x^2} = -\frac{2c}{c^4} = -\frac{2}{c^3}$$

$$\boxed{\text{Tangent line equation} \quad y = f'(x_0)(x-x_0) + f(x_0)}$$

A Different Parameterization



$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Example: $f(x) = \frac{1}{x^2}$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \frac{\frac{1}{(c+h)^2} - \frac{1}{c^2}}{h}$$

$$\lim_{h \to 0} -\frac{(2c+h)\cancel{h}}{(c+h)^2 c^2 \cancel{h}} = \lim_{h \to 0} -\frac{(2c+h)}{(c+h)^2 c^2} = -\frac{2}{c^3}$$

Change of Variables:

$$F(x) = \frac{f(x) - f(c)}{x - c}$$

$$g(x) = c + x$$

$$\lim_{x \to 0} g(x) = c$$

$$\lim_{x \to c} F(c) = f'(c)$$

$$\lim_{x \to 0} F(g(x)) = f'(c)$$

Leibniz Notation: Suppose f is differentiable and $y = f(x)$.

We denote $\boxed{\left.\frac{dy}{dx}\right|_{x=c} = f'(c)}$ "instantaneous rate of change of y with respect to x at x"

$$\boxed{\frac{dy}{dx} = f'(x)}$$

This leads to a derivative operator $\boxed{\frac{d}{dx}}$ ← take the derivative of whatever is to the right of it

$$\frac{d}{dx} f(x) = f'(x)$$

$$\frac{d}{dx} y = \frac{dy}{dx}$$

$$\left.\frac{d}{dx}\right|_{x=2} \frac{1}{x^2} = \frac{1}{4}$$

Higher order derivative

$(f')' = f''$

$f^{(n)} = n^{th}$ derivative

physics notations:

$y = f(x)$

$\dot{y} = f'(x)$

$\ddot{y} = f''(x)$

Prop: If $f : I \to \mathbb{R}$ is a constant function then f is differentiable and $f'(x) = 0$ for all $x \in I$.

Prop: If $f, g : I \to \mathbb{R}$ are differentiable at c, and $\alpha \in \mathbb{R}$

1. $f+g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$
2. αf is differentiable at c and $(\alpha f)'(c) = \alpha f'(c)$

Prf: ① $$(f+g)'(c) = \lim_{x \to c} \frac{(f+g)(c) - (f+g)(x)}{x-c} = \lim \frac{(f(c)-f(x)) + (g(c)-g(x))}{x-c}$$

$$= \lim_{x \to c} \frac{f(c)-f(x)}{x-c} + \lim_{x \to c} \frac{g(c)-g(x)}{x-c} = f'(c) + g'(c)$$

Thm. The function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$ is differentiable for any $x \in \mathbb{R}$ and $f'(x) = n x^{n-1}$.

Prf: Let's compute f'.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x-c} = \lim_{x \to c} \frac{x^n - c^n}{x-c} = \lim_{x \to c} \left[x^{n-1} + c \cdot x^{n-2} + \dots + c^{n-1} \right]$$

$$= c^{n-1} + \dots + c^{n-1} = n c^{n-1}.$$

<u>Theorem</u> If f is differentiable at c, then f is continuous at c.

<u>Proof:</u> We need to show that $\lim\limits_{x \to c} f(x) = f(c)$, and this is equivalent to $\lim\limits_{x \to c} [f(x) - f(c)] = 0$.

$$\left[\lim_{x \to c} f(x) = \lim_{x \to c} [f(x) - f(c) + f(c)] = \overbrace{\lim_{x \to c} [f(x) - f(c)]}^{0} + \overbrace{\lim_{x \to c} f(c)}^{f(c)} = f(c)\right]$$

Thus, $$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} [x - c]$$

$$= \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right) \left[\lim_{x \to c} x - c\right] = f'(c) \cdot 0 = 0 \quad \blacksquare$$

<u>Example:</u> $f(x) = |x|$ at $x = 0$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{|x|}{x} = 1$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

Thus, function $|x|$ is continuous, but non-differentiable at $x = c$.

Defn: A function $f: D \to \mathbb{R}$ is said to be C^1 on D if f is differentiable on D and f' is continuous.

Theorem: If f, g are differentiable at c, then their product is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

$$\frac{d(yz)}{dx} = \frac{dy}{dx} \cdot z + \frac{dz}{dx} \cdot y$$

Prf:

$$(fg)'(c) = \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x-c}$$

$$= \lim_{x \to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c}$$

$$= \lim_{x \to c} \left[f(x) \frac{g(x)-g(c)}{x-c} + g(c) \frac{f(x)-f(c)}{x-c} \right]$$

f diff $\Rightarrow$
$\Rightarrow$ f cont.

f, g are diff.

const

$$= \left[\lim_{x \to c} f(x)\right] \left[\lim_{x \to c} \frac{g(x)-g(c)}{x-c}\right] + g(c) \left[\lim_{x \to c} \frac{f(x)-f(c)}{x-c}\right]$$

$$= f(c) \cdot g'(c) + g(c) \cdot f'(c)$$

Theorem: [Reciprocal Rule] If g is differentiable at c, $g(c) \neq 0$, then $1/g$ is differentiable at c and

$$\left(\frac{1}{g}\right)'(c) = -\frac{g'(c)}{g^2(c)}$$

Pf: $$\left(\frac{1}{g}\right)'(c) = \lim_{x \to c} \frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x - c} = \lim_{x \to c} \frac{g(c) - g(x)}{g(x)g(c)(x-c)}$$

$$= \frac{1}{g(c)} \left[\lim_{x \to c} \frac{1}{g(x)}\right] \left[-\lim_{x \to c} \frac{g(x) - g(c)}{x - c}\right]$$

$$= \frac{1}{g(c)} \cdot \frac{1}{g(c)} \cdot -g'(c) = \boxed{-\frac{g'(c)}{g^2(c)}}$$

Theorem [Quotient rule] If f, g are diff. at c and $g(c) \neq 0$, then f/g is diff. at c and

$$\left(\frac{f}{g}\right)' = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$