

**FUNCTION**

If  $A$  and  $B$  are sets, a *function* is a unique assignment of every element in  $A$  to an element of  $B$ . We write  $f : A \rightarrow B$  to denote a function.

- **Domain** =  $A$
- **Codomain** =  $B$
- **Range** =  $f(A) = \{f(x) : x \in A\} \subseteq B$

**Def.** Well-defined function.

A map  $f : A \rightarrow B$  is *well-defined*, if  $(\forall x, y \in A)[x = y \Rightarrow f(x) = f(y)]$ .

**INJECTIVITY**

A function  $f : A \rightarrow B$  is *injective*, if  $(\forall x, y \in A)[f(x) = f(y) \Rightarrow x = y]$ .

- Every  $A$  has unique  $B$ .
- Every  $B$  comes from unique  $A$  or nothing at all.

**SURJECTIVITY**

A function  $f : A \rightarrow B$  is *surjective*, if  $(\forall y \in B)[\exists x \in A, f(x) = y]$ .

- Every element in the codomain is obtainable.

**BIJECTIVITY**

A function  $f : A \rightarrow B$  is *bijective*, if it is both *injective* and *surjective*.

- Every element is obtainable in a unique way.
- There is matching between elements of  $A$  and  $B$ , i.e. function  $f$  is *invertible*.

**Def.** Cardinality.

*Cardinality* of a set  $S$  is the a measure of the "number of elements" in  $S$ , denoted  $|S|$ .

- **Injectivity.** If there is an injection from  $S \rightarrow T$ , then  $|S| \leq |T|$ .
- **Countability.** A set  $S$  is *countable*, if  $|S| \leq |\mathbb{N}|$ .
- **Bijection.**  $|S| = |T|$ , if there exists a bijection  $S \rightarrow T$ .

**Thm.** [Cantor–Bernstein–Schröder], a.k.a. the **CBT**

If  $S, T$  are two sets with  $|S| \leq |T|$  and  $|T| \leq |S|$ , then  $|S| = |T|$ .

- "Left injection" + "Right injection" = "Bijection".

**Thm.** Transitivity of injectivity.

If  $f : B \rightarrow C$  and  $g : A \rightarrow B$  are both injective then their composition  $f \circ g$  is also injective.

**Proof.**

By the definition of injectivity, we have the following.

$$(\forall a_1, a_2)[g(a_1) = g(a_2) \Rightarrow a_1 = a_2]$$

$$(\forall b_1, b_2)[f(b_1) = f(b_2) \Rightarrow b_1 = b_2]$$

Want to show, the following.

$$(\forall a_1, a_2)[f(g(a_1)) = f(g(a_2)) \Rightarrow a_1 = a_2]$$

Assume that  $f(g(a_1)) = f(g(a_2))$ .

- Since  $f$  is injective,  $g(a_1) = g(a_2)$ .
- Since  $g$  is injective,  $a_1 = a_2$ .

■

**Problem 1.**

If  $f : B \rightarrow C$  and  $g : A \rightarrow B$  are such that  $f \circ g$  is injective, then  $g$  is injective. Also,  $f$  does not have to be injective.

**Solution.**

We have the following.

$$(\forall a, b)[f(g(a)) = f(g(b)) \Rightarrow a = b]$$

Want to show the following.

$$(\forall a, b)[g(a) = g(b) \Rightarrow a = b]$$

Assume  $g(a) = g(b)$ .

- Since  $f$  is a well-defined function,  $f(g(a)) = f(g(b))$ .
- Since  $f \circ g$  is injective,  $a = b$ .

□

**INVERTIBILITY**

*Inverse* of a function  $f : A \rightarrow B$  is a function  $f^{-1} : B \rightarrow A$  that satisfies  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ .

- Function is called *left-invertible*, if  $f^{-1} \circ f = \text{id}_A$ .
- Function is called *right-invertible*, if  $f \circ f^{-1} = \text{id}_B$ .

**Thm.** Injectivity  $\Leftrightarrow$  Left-invertibility

Suppose  $g : A \rightarrow B$  is a function. Then  $g$  is injective if and only if there exists a function  $h : B \rightarrow A$  such that  $(h \circ g) = \text{id}_A$ .

**Proof.** ( $\Leftarrow$ ), (mimics Problem 1.)

We have that  $h \circ g = \text{id}_A$ , consequently the following is true.

$$(\forall a, b \in A)[h(g(a)) = h(g(b)) \Rightarrow a = b]$$

Want to show the following.

$$(\forall a, b \in A)[g(a) = g(b) \Rightarrow a = b]$$

Find the exact solution in **Problem 1**.

**Proof.** ( $\Rightarrow$ )

The function can be defined explicitly.

1. Fix some element  $a_0 \in A$ .

2. Define  $h : B \rightarrow A$  as 
$$h(b) = \begin{cases} a & \text{if } g(a) = b \\ a_0 & \text{otherwise} \end{cases}.$$

$$(h \circ g)(a) = h(g(a)) = a, \quad \text{so we have} \quad h \circ g = \text{id}_A$$

■

**Thm.** Surjectivity  $\Leftrightarrow$  Right-invertibility

Suppose  $g : A \rightarrow B$  is a function. Then  $g$  is surjective if and only if there exists a function  $h : B \rightarrow A$  such that  $(g \circ h) = \text{id}_B$ .

**Thm. BIJECTIVITY  $\Leftrightarrow$  INVERTIBILITY**

A function  $f : A \rightarrow B$  is bijective  $\Leftrightarrow f$  has an inverse.

**Thm.** A countable union of (disjoint) countable sets is countable.

For any collection  $\{A_i : i \in I, |A_i| \leq |\mathbb{N}|\}$  where  $|I| \leq |\mathbb{N}|$ , we have  $|\cup_{i \in I} A_i| \leq |\mathbb{N}|$ .

– "Countability squared" = "Countability"

**Proof.**

We know  $\exists f : I \hookrightarrow \mathbb{N}$  which is injective, and  $(\forall i \in I)[\exists g_i : A_i \hookrightarrow \mathbb{N}]$  which is injective.

Define a map

$$F : \bigcup_{i \in I} A_i \rightarrow \mathbb{N}, \quad a \mapsto 2^{f(i)} \cdot 3^{g_i(a)}, \quad \text{where } a \in A_i.$$

Power of 2 tracks the set. Power of 3 tracks the element within the set. By the *Fundamental Theorem of Arithmetic*,  $F$  is truly injective. ■

**Thm.**  $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

Integers and rationals are countable.

**Proof.**

Proof if these facts is fairly simply.

1.  $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \Rightarrow |\mathbb{N}| \leq |\mathbb{Z}| \leq |\mathbb{Q}|$
2.  $\mathbb{Z} = \{-1, 0, 1\} \times \mathbb{N} \Rightarrow |\mathbb{Z}| \leq |\mathbb{N}|$ , according to the properties of a countable union of countable sets.
3.  $\mathbb{Q} = \mathbb{N} \times \mathbb{Z} \Rightarrow |\mathbb{Q}| \leq |\mathbb{N}|$ , according to the properties of a countable union of countable sets.

Thus, integers and rationals are truly countable.

**Thm.**  $|\mathbb{R}| > |\mathbb{N}|$

The set of real numbers is not countable.

**Proof. "Gaussian Diagonalization"**

This is just an instruction for the proof.

1. Prove that  $|(0, 1)| = |\mathbb{R}|$ .
2. For the sake of contradiction, assume that  $|(0, 1)| = |\mathbb{N}|$ .
3. "Invert" all digits on the diagonal, avoiding assigning the *base* - 1 digit.
4. The obtained number is real and will not match with any of our numbers.
5. Since we claimed to write down all real numbers this is a contradiction. ■