Problem 1.
A *dyadic rational number* is a rational number which, when written in lowest terms, is of the form $p/2^n$ for some $p \in \mathbb{Z}$ and $n \in \mathbb{N}$. Consider the scaled characteristic function of the dyadic rationals:

$$d(x) = \begin{cases} \frac{1}{2^n} & \text{if } x = \frac{p}{2^n},\ p \in \mathbb{Z},\ n \in \mathbb{N},\ \gcd(p, 2^n) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Determine where d is continuous. *Note*: You may freely use the fact that the set of dyadic rational numbers $D = \left\{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ and its complement $\mathbb{R} \setminus D$ are dense in $\mathbb{R}$.

Solution.
For this problem, define $D_0 = \left\{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N}, \gcd(m, 2^n) = 1\right\} = D \setminus \mathbb{Z}$.

$$(\forall x \in \mathbb{R})[d(x) \neq 0 \Leftrightarrow x \in D_0]$$

<u>*Claim 1:*</u> The function d is periodic. For instance, $d(x+1) = d(x)$ for all $x \in \mathbb{R}$.

If $x = \frac{p}{2^n}$, $p \in \mathbb{Z}$, $n \in \mathbb{N}$, $\gcd(p, 2^n) = 1$, then $x + 1 = \frac{p+2^n}{2^n}$, $p + 2^n \in \mathbb{Z}$, $n \in \mathbb{N}$, $\gcd(p + 2^n, 2^n) = 1$, and consequently $d(x) = d(x+1) = \frac{1}{2^n}$. Similarly, we obtain that for such x, $d(x) = d(x-1) \neq 0$.

If $d(x) = 0$, then $d(x+1) = 0$, since otherwise $d(x+1) \neq 0$ and $d(x+1) = d(x) \neq 0$.

Thus, $(\forall x \in \mathbb{R})[d(x) = d(x+1)]$.

<u>*Claim 2:*</u> For every $c \in \mathbb{R}$, $\lim\limits_{x \to c} d(x) = 0$.

Since $(\forall x \in \mathbb{R})[d(x) = d(x+1)]$, it is sufficient to prove that $(\forall c \in [0, 1))\left[\lim\limits_{x \to c} d(x) = 0\right]$.

1. For $c = 0$, we have that $d(0) = 0$ and $(\forall x \in \mathbb{R})[0 \leq d(x) \leq x]$. Since $\lim\limits_{x \to 0} x = \lim\limits_{x \to 0} 0 = 0$, according to the *"Squeeze theorem"*, $\lim\limits_{x \to 0} d(x) = 0$.

2. For $c \in (0, 1)$, fix some $\varepsilon > 0$. Let's prove that $(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |d(x)| < \varepsilon]$. Choose the smallest $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \varepsilon$, so $\frac{1}{2^{N-1}} \geq \varepsilon$. Note that for $\varepsilon > \frac{1}{2}$, any δ would work, so assume $0 < \varepsilon \leq \frac{1}{2}$.

   Take $M = \left\{\frac{m}{2^n} : n \in \{1, 2, \ldots, N-1\}, m \in \{0, 1, \ldots, 2^n\}, \gcd(n, m) = 1\right\}$. Clearly, M is finite.

   If $x \in M$, then $d(x) = \frac{1}{2^n} \geq \frac{1}{2^{N-1}} \geq \varepsilon$.

   If $x \in (0, 1) \setminus M$, then let's show that $d(x) < \varepsilon$. If $x \notin D_0$, $d(x) = 0 < \varepsilon$. If $x \in (D_0 \cap (0, 1)) \setminus M$, then $d(x) = \frac{1}{2^n} \leq \frac{1}{2^N} < \varepsilon$, since this $n \geq N$ because $x \notin M$.

   Define $S_c = \{|x - c| : x \in M \setminus \{c\}\}$. S_c is finite because M is finite. Thus, $\min(S_c)$ exists.

   Take $\delta = \min(\min(S_c), c, 1 - c) > 0$. Thus, $(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow x \notin M \land x \in (0, 1) \Rightarrow d(x) < \varepsilon]$.

   Thus, $(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |d(x) - 0| < \varepsilon]$, meaning that $\lim\limits_{x \to c} d(x) = 0$.

For $c \in D_0$, we have that $\lim\limits_{x \to c} d(x) = 0$ and $d(c) \neq 0$. Thus, d is discontinuous on D_0.

For $c \in \mathbb{R} \setminus D_0$, we have that $\lim\limits_{x \to c} d(x) = d(c) = 0$. Thus, d is continuous on $\mathbb{R} \setminus D_0$.

Thus, d is continuous only on $\mathbb{R} \setminus D_0$. ■