Problem 2.

Commentary: I was not sure how to use the sign-definite requirement, but I did my best.

(a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Suppose there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$, and $g(x) \geq 0$ for all $x \in \mathbb{R}$. If $c \in \mathbb{R}$ and $\lim_{x \to c} g(x) = 0$, show that

$$\lim_{x \to c} f(x)g(x) = 0.$$

(b) Strengthen part (a) by showing that the function f need only be bounded in a deleted open interval of c.

(c) Strengthen part (b) by showing that the function g need only be sign-definite in a deleted open interval of c.

(d) Strengthen part (c) by removing the requirement that the function g is sign-definite in a deleted open interval of c.

Solution. (a)
Want to prove that $\lim_{x \to c} f(x)g(x) = 0$ or $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x)g(x)| < \varepsilon]$.

Fix some $\varepsilon_0 > 0$.
We have $\lim_{x \to c} g(x) = 0$, then $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |g(x)| < \varepsilon]$.
Then, for $\varepsilon = \frac{\varepsilon_0}{M} > 0$, there exists δ_0 such that $(\forall x \in \mathbb{R})[0 < |x - c| < \delta_0 \Rightarrow |g(x)| < \frac{\varepsilon_0}{M}]$.
Since $(\forall x \in \mathbb{R})[|f(x)| \leq M]$, then $(\forall x \in \mathbb{R}) \left[0 < |x - c| < \delta_0 \Rightarrow |f(x)g(x)| = |f(x)||g(x)| < M\dfrac{\varepsilon_0}{M} = \varepsilon_0\right]$.

Thus, we finally obtained that $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x)g(x)| < \varepsilon]$. ■

Solution. (b)
If f is bounded in a deleted open interval of c, then for $\rho > 0$ and $M > 0$, $(\forall x \in (c-\rho, c+\rho))[f(x) \leq M]$.
The previous solution should be modified by taking $\delta = \min(\delta_0, \rho)$. Then, $|g(x)| < \frac{\varepsilon_0}{M}$ and $|f(x)| \leq M$, when $0 < |x - c| < \delta$. ■

Solution. (c)
In fact, the condition that $(\forall x \in \mathbb{R})[g(x) \geq 0]$ is excessive. The case when $(\forall x \in \mathbb{R})[g(x) \leq 0]$ is analogous since $(\forall x \in \mathbb{R})[|-f(x)| = |f(x)|]$. Thus, can get by the fact that g is sign-definite. ■

Solution. (d)
In fact, the condition that g is sign-definite is excessive. We only use the definition of $\lim_{x \to c} g(x) = 0$, where $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |g(x)| < \varepsilon]$, and the fact that $(\forall x \in \mathbb{R})[|f(x)g(x)| = |f(x)||g(x)|]$. Thus, the sign of g is not important for this solution. ■