Problem 2.
Let A be an arbitrary set. The *power set of* A, denoted $\mathcal{P}(A)$, is the set of all subsets of A: $\mathcal{P}(A) = \{S : S \subseteq A\}$. Let P_A denote the collection of all functions from A to the two point set $\{0,1\}$; that is, $P_A = \{f : A \to \{0,1\}\}$. Show that there is a bijection between $\mathcal{P}(A)$ and P_A.

Solution
To prove the existence of a bijection between $\mathcal{P}(A)$ and P_A we shall use *Cantor–Bernstein theorem.*

1. We need to show that $|P_A| \leq |\mathcal{P}(A)|$, meaning there is an injective function from P_A to $\mathcal{P}(A)$. It can be defined explicitly.

$$h : P_A \to \mathcal{P}(A), f \mapsto \{x \in A : f(x) = 1\}$$

   This function is injective because in the case when $h(f_1) = h(f_2)$ that would mean the following.

$$(\forall x \in A)[x \in h(f_1) \Leftrightarrow x \in h(f_2)]$$
$$(\forall x \in A)[f_1(x) = 1 \Leftrightarrow f_2(x) = 1]$$
$$(\forall x \in A)[f_1(x) = f_2(x)]$$
$$f_1 = f_2$$

   Thus h is truly injective.

2. We need to show that $|\mathcal{P}(A)| \leq |P_A|$, meaning there is an injective function from $\mathcal{P}(A)$ to P_A. It can be defined explicitly.

$$k : \mathcal{P}(A) \to P_A, S \mapsto \left[f : A \to \{0,1\}, f(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases} \right]$$

   This function is injective because in the case when $k(S_1) = k(S_2)$ that would mean the following.

$$(\forall x \in A)[k(S_1)(x) = k(S_2)(x)]$$
$$(\forall x \in A)[k(S_1)(x) = 1 \Leftrightarrow k(S_2)(x) = 1]$$
$$(\forall x \in A)[x \in S_1 \Leftrightarrow x \in S_2]$$
$$S_1 = S_2$$

   Thus k is truly injective.

Finally, we have $|P_A| \leq |\mathcal{P}(A)|$ and $|\mathcal{P}(A)| \leq |P_A|$. According to the *Cantor–Bernstein theorem,* $|P_A| = |\mathcal{P}(A)|$. Thus there is a bijection between $\mathcal{P}(A)$ and P_A. ∎