Problem 1.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function, with $c, L \in \mathbb{R}$. Indicate whether each statement below is equivalent to the definition of a limit: $\lim_{x\to c} f(x) = L$. If you believe the statements are equivalent, you need not provide any work. When you believe a statement is not equivalent to the definition of a limit, give an example of an f, c, and L which satisfies one definition but not the other.

Solution.
Original definition: $\lim_{x\to c} f(x) = L \Leftrightarrow (\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$.

(a) $(\exists\varepsilon > 0)(\forall\delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$

This statement is **not** equivalent to the original definition.

For $f(x) = 1, L = 0, c = 0$, the new definition is satisfied. For example, $\varepsilon = 2$ would work, because $(\forall\delta > 0)(\forall x \in \mathbb{R})[0 < |x - 0| < \delta \Rightarrow |1 - 0| < 2]$. However, in terms of the original definition $\lim\limits_{x\to 0} 1 = 1 \neq L = 0$.

(b) $(\forall\varepsilon > 0)(\forall\delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$

This statement is **not** equivalent to the original definition.

For $f(x) = x, L = 0, c = 0$, the new definition is not satisfied. For example $\varepsilon = 1, \delta = 2, x = 1.5$ do not work, because $\neg[0 < |1.5 - 0| < 2 \Rightarrow |1.5 - 0| < 1]$. However, in terms of the original definition $\lim\limits_{x\to 0} x = 0 = L$.

(c) $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| \leq \delta \Rightarrow |f(x) - L| \leq \varepsilon]$

This statement is equivalent to the original definition.

(d) $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in \mathbb{R})[|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$

This statement is **not** equivalent to the original definition.

For $f(x) = \frac{x}{x}, L = 1, c = 0$, the new definition is not satisfied. For example, when $\varepsilon = 1, x = 0$, $(\forall\delta > 0)[|x - c| < \delta]$, but $f(x)$ is not defined for $x = 0$ (division by zero). However, in terms of the original definition $\lim\limits_{x\to 0} \frac{x}{x} = 1 = L$.

(e) $(\exists\varepsilon > 0)(\exists\delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$

This statement is **not** equivalent to the original definition.

For $f(x) = 1, L = 0, c = 0$, the new definition is satisfied. For example, $\varepsilon = 2$, $\delta = 1$ would work, because $(\forall x \in \mathbb{R})[0 < |x - 0| < 1 \Rightarrow |1 - 0| < 2]$. However, in terms of the original definition $\lim\limits_{x\to 0} 1 = 1 \neq L = 0$.

(f) $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$

This statement is equivalent to the original definition.

(g) $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x \in \mathbb{R})[|x - c| > \delta \Rightarrow |f(x) - L| > \varepsilon]$

This statement is **not** equivalent to the original definition.

For $f(x) = |x|, L = 1, c = 0$, the new definition is satisfied. For example, $\delta = \varepsilon + 1$ would work, because $(\forall x \in \mathbb{R})[|x - 0| > \varepsilon + 1 \Rightarrow ||x| - 1| > \varepsilon]$. However, in terms of the original definition $\lim\limits_{x\to 0} |x| = 0 \neq L = 1$.