Problem 2.(a)
By defining $\cos : [0, \pi] \to [-1, 1]$ we get a bijective function, from which a corresponding reciprocal function

$$\sec : [0, \pi] \setminus_{\{\frac{\pi}{2}\}} \to \mathbb{R} \setminus_{(-1,1)}, \qquad \sec(x) = \frac{1}{\cos(x)}$$

is defined. Show that the inverse of secant, $\operatorname{arcsec} : \mathbb{R}\setminus_{(-1,1)} \to [0, \pi]\setminus_{\{\frac{\pi}{2}\}}$ is differentiable on $\mathbb{R}\setminus_{[-1,1]}$ and find its derivative.

Solution. 2.(a)
Take any $x \in \mathbb{R} \setminus (-1, 1)$, then $\operatorname{arcsec}(x) \in [0, \pi] \setminus \{\frac{\pi}{2}\}$, and consequently $\sin(\operatorname{arcsec}(x)) \geq 0$ and $\cos(\operatorname{arcsec}(x)) = \frac{1}{x} \in [-1, 1] \setminus \{0\}$. Then, $\sin(\operatorname{arcsec}(x)) = +\sqrt{1 - \cos^2(\operatorname{arcsec}(x))} = +\sqrt{1 - 1/x^2} \geq 0$.

Also, using the **Reciprocal Rule**, we obtain that for $x \in [0, \pi] \setminus \{\frac{\pi}{2}\}$

$$\sec'(x) = \left(\frac{1}{\cos(x)}\right)' = -\frac{\cos'(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)}.$$

Thus, as $\cos(x) \neq 0$ for $x \in [0, \pi] \setminus \{\frac{\pi}{2}\}$, it can be concluded that sec is a C^1 function (differentiable and has a continuous derivative function) on $[0, \pi/2)$ and $(\pi/2, \pi]$.

Case $\sec : [0, \pi/2) \to [1, \infty)$. Since $\cos : [0, \pi/2) \to (0, 1]$ is bijective and $\sec(x) = 1/\cos(x)$ for all $x \in [0, \pi/2)$, it can be concluded that $\sec : [0, \pi/2) \to [1, \infty)$ is bijective as well.

Case $\sec : (\pi/2, \pi] \to (-\infty, -1]$. Since $\cos : (\pi/2, \pi] \to [-1, 0)$ is bijective and $\sec(x) = 1/\cos(x)$ for all $x \in (\pi/2, \pi]$, it can be concluded that $\sec : (\pi/2, \pi] \to (-\infty, -1]$ is bijective as well.

Now, we can use the **IFT** for $\sec : [0, \pi/2) \to [1, \infty)$ and $\sec : (\pi/2, \pi] \to (-\infty, -1]$ to conclude that for all $x \in \mathbb{R} \setminus [-1, 1]$

$$\operatorname{arcsec}'(x) = \frac{1}{\sec'(\operatorname{arcsec}(x))} = \frac{\cos^2(\operatorname{arcsec}(x))}{\sin(\operatorname{arcsec}(x))} = \frac{1/x^2}{+\sqrt{1 - 1/x^2}} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Notice that now $x \in \mathbb{R} \setminus [-1, 1]$ (was $\mathbb{R} \setminus (-1, 1)$) to keep the denominator $\sin(\operatorname{arcsec}(x)) \neq 0$.

Thus, using the IFT, it was shown that arcsec is differentiable on $\mathbb{R} \setminus [-1, 1]$ and $\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}$. ■

Problem 2.(b)
One may avoid the absolute values that appear in the derivatives of arcsec by changing the definition of these functions. In defining arcsec, we restricted the domain of cos to $[0, \pi]$ in order to ensure injectivity. Alternatively, we could have restricted the domain of cos to $[0, \pi/2] \cup [\pi, 3\pi/2]$, where it is still injective, though not necessarily continuous.

Show that if we define arcsec using this new choice of domain:

$$\operatorname{arcsec} : \mathbb{R} \setminus [-1, 1] \to \left[0, \tfrac{\pi}{2}\right) \cup \left[\pi, \tfrac{3\pi}{2}\right),$$

then the derivative no longer has an absolute value.

Solution. 2.(b)
Notice that now we work with $\sec : [0, \pi/2) \to [1, \infty)$ and $\sec : [\pi, 3\pi/2) \to (-\infty, -1]$ and their inverses.

Case $\sec : [0, \pi/2) \to [1, \infty)$. It perfectly matches **Q2.(a)**. There we proved $\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}$ for $x \in (1, \infty)$.

Case $\sec : [\pi, 3\pi/2) \to (-\infty, -1]$. For any $x \in (-\infty, -1]$, $\operatorname{arcsec}(x) \in [\pi, 3\pi/2)$, and consequently $\sin(\operatorname{arcsec}(x)) \leq 0$ and $\cos(\operatorname{arcsec}(x)) = \frac{1}{x} \in [-1, 0)$. Then, $\sin(\operatorname{arcsec}(x)) = -\sqrt{1 - 1/x^2} \leq 0$. Since the function is still bijective on the given interval and the formula for $\sec'$ stays the same, we can conclude that $\operatorname{arcsec}'(x) = \frac{1/x^2}{-\sqrt{1-1/x^2}} = -\frac{1}{|x|\sqrt{x^2-1}}$ (negative of what it used to be in **Q2.(a)**) for $x \in (-\infty, -1)$.

In this task, the derivative of arcsec no longer has an absolute value since its new formula is conditional

$$\operatorname{arcsec}'(x) = \begin{cases} \frac{1}{|x|\sqrt{x^2-1}} & x \in (1, \infty) \\ -\frac{1}{|x|\sqrt{x^2-1}} & x \in (-\infty, -1) \end{cases}.$$

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