Problem 1.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is $(n+1)$-times differentiable for some positive integer n. If $a, b \in \mathbb{R}$ are such that $f(a) = f(b)$ and $f^{(k)}(a) = 0$ for all $k \in \{1, \ldots, n\}$, show there exists a point $c \in (a, b)$ such that $f^{(n+1)}(c) = 0$.

Solution.
Let's prove by induction that for all $n \in \{0\} \cup \mathbb{N}$ if **(1)** f is $(n+1)$-times differentiable, **(2)** $f(a) = f(b)$, and **(3)** $(\forall k \in \{1, \ldots, n\})[f^{(k)}(a) = 0]$, then $(\exists c \in (a, b))[f^{(n+1)}(c) = 0]$.

For the base case, $n = 0$. We have that **(1)** f is $(0+1)$-times differentiable and **(2)** $f(a) = f(b)$. According to the **MVT**, it is true that $(\exists c \in (a, b)) \left[f^{(1)}(c) = \frac{f(b)-f(a)}{b-a} = 0\right]$, and consequently it was proven that $(\exists c \in (a, b)) \left[f^{(0+1)}(c) = 0\right]$.

Assume that the fact holds for $n = m$. Now, it will be proven for $n = m + 1$.

For the precondition, assume that **(1)** f is $(m+2)$-times differentiable, **(2)** $f(a) = f(b)$, and **(3)** $(\forall k \in \{1, \ldots, m+1\})[f^{(k)}(a) = 0]$. Since **(1)** f is $(m+1)$-times differentiable, **(2)** $f(a) = f(b)$, and **(3)** $(\forall k \in \{1, \ldots, m\})[f^{(k)}(a) = 0]$, according to the induction hypothesis (for $n = m$), we can find c in (a, b) such that $f^{(m+1)}(c) = 0$. Fix such c.

Now, we have that $f^{(m+1)}$ is differentiable, $f^{(m+1)}(a) = 0$, and $f^{(m+1)}(c) = 0$ for some $c \in (a, b)$. According to the **MVT** for $f^{(m+1)}$, it is true that $(\exists d \in (a, c)) \left[f^{(m+1+1)}(d) = \frac{f^{(m+1)}(c)-f^{(m+1)}(a)}{c-a} = 0\right]$. Fix such $d \in (a, c) \subset (a, b)$. Thus, it was proven that $(\exists d \in (a, b)) \left[f^{(m+2)}(d) = 0\right]$.

To sum up, using induction, it was proven that if for some $n \in \{0\} \cup \mathbb{N}$ preconditions **(1)**, **(2)**, and **(3)** hold, then exists $c \in (a, b)$ such that $f^{(n+1)}(c) = 0$. ■