Problem 1.
Let α and β be positive integers, and define the function

$$f_{\alpha,\beta}(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x^\beta}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

(a) Find the values of α and β such that $f_{\alpha,\beta}$ is everywhere continuous.

(b) Find the values of α and β such that $f_{\alpha,\beta}$ is differentiable for $x \neq 0$. Give the formula for $f'_{\alpha,\beta}$ when $x \neq 0$.

(c) Find the values of α and β such that $f_{\alpha,\beta}$ is differentiable at $x = 0$.

(d) Find the values of α and β such that $f'_{\alpha,\beta}$ is continuous at $x = 0$.

Solution. 1.(a)
Fix some $\alpha, \beta \in \mathbb{N}$. It is known that for all $\alpha, \beta \in \mathbb{N}$, x^α, $\sin(x)$, and $\frac{1}{x^\beta}$ are continuous at all $x \neq 0$. Then, $\sin\left(\frac{1}{x^\beta}\right)$ is continuous at all $x \neq 0$ (as a composition of continuous on $\mathbb{R} \setminus \{0\}$ functions), and consequently $x^\alpha \sin\left(\frac{1}{x^\beta}\right)$ is continuous at all $x \neq 0$ (as a product of continuous on $\mathbb{R} \setminus \{0\}$ functions). Thus, $f_{\alpha,\beta}(x)$ is continuous at all $x \in \mathbb{R} \setminus \{0\}$.

Since $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$. We have that $(\forall x \in \mathbb{R} \setminus \{0\}) \left[0 \leq \left|x^\alpha \sin\left(\frac{1}{x^\beta}\right)\right| = |x^\alpha| \cdot \left|\sin\left(\frac{1}{x^\beta}\right)\right| \leq |x^\alpha|\right]$. Thus, as $\lim\limits_{x \to 0} 0 = \lim\limits_{x \to 0} |x^\alpha| = 0$, according to the **Squeeze Theorem**, $\lim\limits_{x \to 0} \left|x^\alpha \sin\left(\frac{1}{x^\beta}\right)\right| = 0$, and $\lim\limits_{x \to 0} x^\alpha \sin\left(\frac{1}{x^\beta}\right) = 0$. Then, as $f_{\alpha,\beta}(x) = x^\alpha \sin\left(\frac{1}{x^\beta}\right)$ for all $x \in \mathbb{R} \setminus \{0\}$, we have that $\lim\limits_{x \to 0} f_{\alpha,\beta}(x) = 0 = f_{\alpha,\beta}(0)$. Thus, it was proven that $f_{\alpha,\beta}$ is continuous at $x = 0$.

To sum up, for every choice of $\alpha, \beta \in \mathbb{N}$, $f_{\alpha,\beta}$ is continuous at all $x \in \mathbb{R}$. ∎

Solution. 1.(b)
Fix some $\alpha, \beta \in \mathbb{N}$.

Since $\sin(x)$ is differentiable on $\mathbb{R}$ and $\frac{1}{x^\beta}$ is differentiable at $x \neq 0$, according to the **Chain Rule**, $\sin\left(\frac{1}{x^\beta}\right)$ is differentiable on $\mathbb{R} \setminus \{0\}$, and for $x \neq 0$

$$\left(\sin\left(\frac{1}{x^\beta}\right)\right)' = \left(\frac{1}{x^\beta}\right)' \cdot \sin'\left(\frac{1}{x^\beta}\right) = \frac{-\beta}{x^{\beta+1}} \cdot \cos\left(\frac{1}{x^\beta}\right) = -\frac{\beta \cos\left(\frac{1}{x^\beta}\right)}{x^{\beta+1}}.$$

Now, according to the **Product Rule**, for $x \neq 0$, $x^\alpha \sin\left(\frac{1}{x^\beta}\right)$ is differentiable and

$$\begin{aligned} f'_{\alpha,\beta}(x) = \left(x^\alpha \sin\left(\frac{1}{x^\beta}\right)\right)' &= (x^\alpha)' \cdot \sin\left(\frac{1}{x^\beta}\right) + x^\alpha \cdot \left(\sin\left(\frac{1}{x^\beta}\right)\right)' \\ &= \alpha x^{\alpha-1} \cdot \sin\left(\frac{1}{x^\beta}\right) - x^\alpha \cdot \frac{\beta \cos\left(\frac{1}{x^\beta}\right)}{x^{\beta+1}} \\ &= \alpha \cdot x^{\alpha-1} \cdot \sin\left(\frac{1}{x^\beta}\right) - \beta \cdot x^{\alpha-\beta-1} \cdot \cos\left(\frac{1}{x^\beta}\right). \end{aligned}$$

Thus, for every choice of $\alpha, \beta \in \mathbb{N}$, $f_{\alpha,\beta}$ is differentiable at all $x \in \mathbb{R} \setminus \{0\}$ and the formula for the derivative function is $f'_{\alpha,\beta}(x) = \alpha \cdot x^{\alpha-1} \cdot \sin\left(\frac{1}{x^\beta}\right) - \beta \cdot x^{\alpha-\beta-1} \cdot \cos\left(\frac{1}{x^\beta}\right)$ at all $x \in \mathbb{R} \setminus \{0\}$. ∎

Solution. 1.(c)
Assume $\alpha = 1$. Fix some $\beta \in \mathbb{N}$. It will be proven that $f_{\alpha,\beta}$ is not differentiable at $x = 0$. For a contradiction assume the opposite, then

$$f'_{\alpha,\beta}(0) = \lim_{x\to 0} \frac{x^\alpha \sin\left(\frac{1}{x^\beta}\right) - f(0)}{x - 0} = \lim_{x\to 0}\left[x^{\alpha-1}\sin\left(\frac{1}{x^\beta}\right)\right] = \lim_{x\to 0}\left[\sin\left(\frac{1}{x^\beta}\right)\right].$$

Notice that for all $\varepsilon > 0$ and for every $z = \sin(\theta) \in [-1, 1]$, exists x in $(0, \varepsilon)$ such that $\sin(\frac{1}{x^\beta}) = z$. This is so, because, if we fix z, θ, and ε, then exists $n \in \mathbb{N}$ such that $(2\pi n + \theta) > \varepsilon^{-\beta} > 0$, which means we can take $x = (2\pi n + \theta)^{-1/\beta} < \varepsilon$, and then $\sin(x^{-\beta}) = \sin(2\pi n + \theta) = \sin(\theta) = z$.

Since $(\forall \epsilon > 0)(\forall z \in [-1, 1])(\exists x \in (0, \epsilon))[\sin(x^{-\beta}) = z]$, we can deduce that $\lim_{x\to 0^+} \sin(x^{-\beta})$ is not defined. Thus, the limit $\lim_{x\to 0}\left[\sin\left(\frac{1}{x^\beta}\right)\right]$ does not exist, as well as the $f'_{\alpha,\beta}(0)$.

Now, assume $\alpha > 1$. Fix some $(\alpha - 1), \beta \in \mathbb{N}$. It will be proven that $f_{\alpha,\beta}$ is differentiable at $x = 0$.

In **Q1.(a)**, using the **Squeeze Theorem**, it was proven that for $a, b \in \mathbb{N}$, $\lim_{x\to 0} x^a \sin(x^{-b}) = 0$. Since now $(\alpha - 1), \beta \in \mathbb{N}$, it can be concluded that $\lim_{x\to 0}[x^{\alpha-1}\sin(x^{-\beta})] = 0$. Then, $f'_{\alpha,\beta}(0)$ exists and

$$f'_{\alpha,\beta}(0) = \lim_{x\to 0} \frac{x^\alpha \sin\left(\frac{1}{x^\beta}\right) - f(0)}{x - 0} = \lim_{x\to 0}\left[x^{\alpha-1}\sin\left(\frac{1}{x^\beta}\right)\right] = 0.$$

To sum up, $f_{\alpha,\beta}$ is differentiable at $x = 0$, iff $\alpha > 1$. ■

Solution. 1.(d)
The function $f'_{\alpha,\beta}$ is continuous at $x = 0$, iff $\lim_{x\to 0} f'_{\alpha,\beta}(x) = f'_{\alpha,\beta}(0) = 0$. Since $f'_{\alpha,\beta}(0)$ exists, $\alpha > 1$.

Then, according to **Q1.(b)**, $f'_{\alpha,\beta}(x) = \left[\alpha \cdot x^{\alpha-1} \cdot \sin\left(x^{-\beta}\right) - \beta \cdot x^{\alpha-\beta-1} \cdot \cos\left(x^{-\beta}\right)\right]$ at all $x \in \mathbb{R} \setminus \{0\}$.

In **Q1.(c)**, it was proven that $\lim_{x\to 0} x^a \sin(x^{-b})$ exists, iff $a > 0$. Since $\cos(\theta) = \sin(\theta + \pi/2)$, in the **Q1.(c)** proof, we can replace $z = \sin(\theta)$ with $z = \sin(\theta + \pi/2)$, to show that $\lim_{x\to 0} x^a \cos(x^{-b})$ exists, iff $a > 0$. The **Q1.(a)** bit is analogous, as it is true that $|\cos(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$. Also, those limits are equal to 0, if they exist.

Thus, $\alpha \cdot x^{\alpha-1} \cdot \sin\left(x^{-\beta}\right)$ is continuous at $x = 0$, iff $\alpha > 1$, and $\beta \cdot x^{\alpha-\beta-1} \cdot \cos\left(x^{-\beta}\right)$ is continuous at $x = 0$, iff $\alpha > \beta + 1$. We already know that $\alpha > 1$. If $1 < \alpha \leq \beta + 1$, $f'_{\alpha,\beta}$ is discontinuous at $x = 0$, since exactly one of its two terms is discontinuous at 0. (For $\lim_{x\to 0} f'_{\alpha,\beta}(x)$ as the sum of limits of its two terms, the first limit is 0, and another one is not defined). Therefore, $f'_{\alpha,\beta}$ as a sum of those terms is continuous at $x = 0$, iff $\alpha > \beta + 1$.

To sum up, $f'_{\alpha,\beta}$ is continuous at $x = 0$, iff $\alpha > \beta + 1$. ■