Problem 1.
Let $D = \left\{x \in \mathbb{Q} : x = \frac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N}\right\}$. Show that D is dense in $\mathbb{R}$; namely that every open interval (a, b) contains at least one element of D.

Lemma. $(\forall n \in \mathbb{N})[2^n > n]$
This fact will be used in the solution of the original problem.

Proof.
This will be a proof by induction.

1. *Base:* For $n = 1$, we have $2^1 > 1$. The lemma holds.
2. *Assumption:* Assume, for $n = k$, $2^k > k$.
3. *Step:* Want to prove the fact for $n = k+1$, namely $2^{k+1} > k+1$, using the assumption.

   According to the assumption and $k \in \mathbb{N}$, we have $2^k > k \geq 1$. Thus, $2^k + 2^k > k + 1$, or equivalently $2^{k+1} > k + 1$.
4. *Conclusion:* The fact holds for $n = 1$. If the fact holds for $n = k$, then it holds for $n = k+1$. According to the *axiom of induction*, the fact holds for all $n \in \mathbb{N}$. ■

Solution.
Fix an open interval (a, b). Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < (b - a)$, as in the proof of $\mathbb{Q}$ being dense in $\mathbb{R}$. Since $(\forall n \in \mathbb{N})[2^n > n]$, we have $\frac{1}{2^N} < \frac{1}{N} < (b - a)$.

Define $D_N = \left\{\frac{m}{2^N} : m \in \mathbb{Z}\right\} \subset D$, for which we claim $D_N \cap (a, b) \neq \varnothing$.

For a contradiction, assume $D_N \cap (a, b) = \varnothing$. Let M be the largest integer such that $\frac{M}{2^N} \leq a$.
But then $\frac{M+1}{2^N} \geq b$. Thus, $(b - a) \leq \frac{M+1}{2^N} - \frac{M}{2^N} = \frac{1}{2^N} < (b - a)$.

This resulted in a contradiction, so $D_N \cap (a, b) \neq \varnothing$. Thus, for every open interval (a, b), $D \cap (a, b) \neq \varnothing$. So, D is dense in $\mathbb{R}$. ■