Def. Absolute value.
The absolute value function is $|\cdot| : \mathbb{R} \to [0, \infty)$,

$$x = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note that if $x, y \in \mathbb{R}$, then the *distance* from x to y is $|x - y|$.
Note that $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$.
And finally, for all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

LIMITS

Goal: Determine the behaviour of a function f near the point c, without ever evaluating $f(c)$.
There is a weird-point graph and Canyon-geologist analogy.

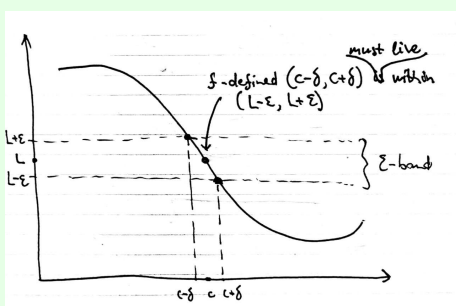
Def. Deleted open interval.
Suppose $f : D \to \mathbb{R}$ is a function, $c \in R$, and f is defined on a deleted open interval around c. (There exists some $\rho > 0$ such that f is defined on $(c - \rho, c) \cup (c, c + \rho)$.)

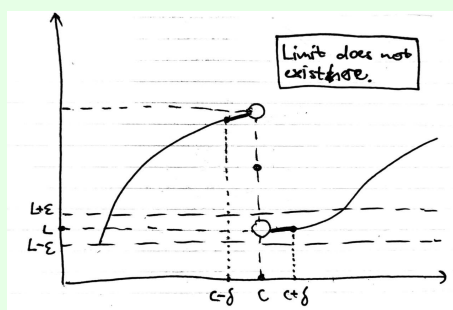
We say that $\lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ satisfying $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

To paraphrase that statement:

$$\lim_{x \to c} f(x) = L \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

1. Given an "error bound" (ε),
2. we can always find some distance (δ) such that
3. if we are no more than that distance away from c
4. then our approximation is within that error bound.





Problem
Show that $\lim_{x \to 4}[2x + 3] = 11$.

Solution.
Let $\varepsilon > 0$ be given, and set $\delta = \frac{\varepsilon}{2}$. Suppose then that $0 < |x - 4| < \delta$, then

$$|[2x + 3] - 11| = |2x - 8| = 2|x - 4| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

Strategy for solving Limits

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

- This (ε) should be arbitrary.
- We get to choose δ, it should dependn on ε.

Goal: Write $|f(x) - L| \leq M \cdot |x - c|$ for some $M > 0$. At this point, take $\delta = \frac{\varepsilon}{M}$

$$|f(x) - L| \leq M|x - c| < M\delta = M \cdot \frac{\varepsilon}{M} = \varepsilon$$

Problem
Show that $\lim_{x \to (-1)}[5 - 4x] = 9$.

Solution. (Rough work)

$$|f(x) - L| = |[5 - 4x] - 9| = |-4x - 4| = 4|x + 1| = M \cdot |x - c|$$

Solution. (Real proof)
Let $\varepsilon > 0$ be given, and set $\delta = \frac{\varepsilon}{M} = \frac{\varepsilon}{4}$. Suppose then that $0 < |x - c| = |x + 1| < \delta$, then

$$|[5 - 4x] - 9| = |-4x - 4| = 4|x + 1| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

$\square$

Problem
Show that $\lim_{x \to 1}[2x^2 + 1] = 3$.

Solution. (Rough work)

$$|f(x) - L| = |[2x^2 + 1] - 3| = |2x^2 - 2| = 2|x^2 - 1| = 2|x - 1||x + 1| \ < M|x - 1|$$

We will have to assume $|x - 1| < 1$, i.e. $\delta \leq 1$.

$$|x - 1| < 1 \Rightarrow |x + 1| \leq |x - 1| + |2| < 3$$

So we set $\delta = \frac{\varepsilon}{6}$.

$$|f(x) - L| = 2|x - 1||x + 1| < 6|x - 1| = M|x - 1|$$

Solution. (Real proof)
Let $\varepsilon > 0$ be given, and set $\delta = \min(1, \frac{\varepsilon}{6})$. Suppose then that $0 < |x - c1| < \delta$. Since $|x - 1| < \delta$ then $|x + 1| < 3$ by (*). Thus,

$$|[2x^2 + 1] - 3| = 2|x - 1||x + 1| < 6|x - 1| < 6\delta \leq 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

$\square$

Problem
Show that $\lim_{x\to 3}\sqrt{x+1}=2$.

ONE-SIDED LIMIT

Suppose that $f:D\to\mathbb{R}$ and $c\in\mathbb{R}$. If f is defined on some interval $(,c+e)$, $e>0$, then we say that $\lim_{x\to c^+} f(x)=L$ if $(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R})[x\in(c-\delta,c)\Rightarrow|f(x)-L|<\varepsilon]$.

Immediately, $\lim_{x\to c} f(x)=L \Leftrightarrow \lim_{x\to c^+} f(x)=L=\lim_{x\to c^-} f(x)$

Problem
Show that $\lim_{x\to 0^+}\frac{x}{|x|}=1$.

Solution.
Let $\varepsilon>0$ be given and set $\delta=\varepsilon$. Note that if $x\in 0,\delta$ then $x>0$ so $|x|=x$, and so

$$|f(x)-L|=\left|\frac{x}{|x|}-1\right|=|1-1|=0<\epsilon$$

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HORIZONTAL ASYMPTOTE

A function f has a horizontal asymptote at L if either $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$.

Def. "Limit to infinity".
Suppose $f:D\to\mathbb{R}$ is defined on (a,∞) for some $a\in\mathbb{R}$. We say that $\lim_{x\to\infty} f(x)=L$ if

$$(\forall\varepsilon>0)(\exists M\in\mathbb{R})(\forall x\in\mathbb{R})[x>M\Rightarrow|f(x)-L|<\varepsilon].$$

For all $\varepsilon>0$, there is some point M after which the function stays entirely in an epsilon band.

Problem
Show $\lim_{x\to\infty}\frac{x^2}{x^2+1}=1$.

Solution. "Rough work"

$$|f(x)-L|=\left|\frac{x^2}{x^2+1}-1\right|=\left|\frac{x^2-x^2-1}{x^2+1}\right|=\left|\frac{-1}{x^2+1}\right|=\frac{1}{x^2+1}$$

We want $\frac{1}{x^2+1}<\varepsilon$ or $x^2+1>\frac{1}{\varepsilon}$, so $x>\sqrt{\frac{1}{\varepsilon}-1}$.

This works so long as $0<\varepsilon<1$.

In the proof, do cases. [If $\varepsilon>0$ take $M=0$.] [If $0<\varepsilon<1$ take $M=\sqrt{1/\varepsilon-1}$.]

Def. $\lim_{x\to\infty} f(x)=\infty$.

$$(\forall N\in\mathbb{R})(\exists M\in\mathbb{R})(\forall x\in\mathbb{R})[x>N\Rightarrow f(x)>M]$$