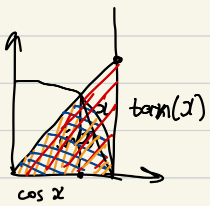
W8 Lecture

- sin', cos'
- chain rule
- IFT

Prop: $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$

Proof:



Thus, $\displaystyle(\forall x\in(0,\tfrac{\pi}{2}))\left[\tfrac{1}{2}\sin(x) \leq \tfrac{1}{2}x \leq \tfrac{1}{2}\tan x = \tfrac{1}{2}\frac{\sin(x)}{\cos(x)}\right]$

$\displaystyle(\forall x\in(0,\tfrac{\pi}{2}))\left[1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos x}\right]$

$\displaystyle(\forall x\in(0,\tfrac{\pi}{2}))\left[1 \geq \frac{\sin x}{x} \geq \cos x\right]$

By the Squeeze Theorem, $\displaystyle 1 = \lim_{x\to 0^+} 1 \geq \lim_{x\to 0^+} \frac{\sin x}{x} \geq \lim_{x\to 0^+} \cos x = 1$, and so $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1.$

For $\displaystyle\lim_{x\to 0}\frac{\cos(x)-1}{x} = \lim_{x\to 0}\frac{\cos x - 1}{x}\cdot\frac{\cos x+1}{\cos x+1} = \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x+1)}$

$\displaystyle = \lim_{x\to 0}\frac{\sin^2 x}{x(\cos x+1)} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x} = 1\cdot 0 = 0$

$\boxed{\textbf{Theorem:}}$ The functions $\sin(x)$ and $\cos(x)$ are differentiable on $\mathbb{R}$, and $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$.

Proof: $\displaystyle\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin x\cdot\cos h + \cos x\cdot\sin h - \sin x}{h}$

$$= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h - 1}{h} + \cos x\cdot\frac{\sin h}{h}\right]$$

$$= \sin x\cdot\left[\lim_{h\to 0}\frac{\cos h-1}{h}\right] + \cos x\cdot\left[\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$= \sin x\cdot 0 + \cos x\cdot 1$$

$$= \cos x \quad \blacksquare$$

Example: $\frac{d}{dx}\tan(x) = \sec^2(x)$

$$f'(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{(\sin(x))'\cos(x) - \sin(x)\cdot(\cos(x))'}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

Theorem: [Chain Rule]

If g is differentiable at c, and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c and

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c)$$

Bad Proof $\lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x-c} = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x-c}$

$$= f'(g(c)) \cdot g'(c)$$

Correct Proof: To apply CoV we need one of

1. $\frac{f(x) - f(g(c))}{x - g(c)}$ is continuous at $g(c)$
2. $g(x) \neq g(c)$ for all x near c

removable discontinuity at $x = g(c)$

Use $F(x) = \begin{cases} \frac{f(x) - f(g(c))}{x - g(c)} & x \neq g(c) \\ f'(g(c)) & x = g(c) \end{cases}$

Claim: $\frac{f(g(x)) - f(g(c))}{x-c} = F(g(x))\,\frac{g(x) - g(c)}{x-c}$ for all x

① $g(x) = g(c)$ LHS $= 0$, RHS $= 0$ ② $g(x) \neq g(c)$ LHS = RHS no Division by Zero

Theorem [Inverse Function Theorem]

Suppose that $f: \mathbb{R} \to \mathbb{R}$ is a C^1 function (differentiable and f' is continuous) and $f'(p) \neq 0$ for some $p \in \mathbb{R}$. There exists an open interval $\mathcal{U}$ containing p and V containing $f(p)$ such that $\hat{f}: \mathcal{U} \to V$, $\hat{f}(x) = f(x)$ is bijective, and $\hat{f}^{-1}$ is C^1 with $(\hat{f}^{-1})'(p) = \dfrac{1}{\hat{f}'(\hat{f}^{-1}(p))} \iff (\hat{f}^{-1})'(q) = \dfrac{1}{\hat{f}'(p)}$ where $q = \hat{f}(p)$.

Proof: Suppose that $f'(p) \neq 0$. Without loss of generality suppose $f'(p) > 0$. Now f' is continuous at p, there is an interval $\mathcal{U}$ containing p such that $f'(x) > 0$ for all $x \in \mathcal{U}$. From the lemma, f' is thus increasing on $\mathcal{U}$ and hence injective. Define $V = f(\mathcal{U})$, so that $\hat{f}: \mathcal{U} \to V$, $x \mapsto f(x)$ is bijective. We know $\hat{f}^{-1}$ exists since $\hat{f}$ is bijective. Moreover, $\hat{f}^{-1}$ is continuous on V (by TT2, Q3).

Let $\varepsilon > 0$ be given. Since f is differentiable at p we know $\displaystyle\lim_{x \to p} \frac{x-p}{f(x)-f(p)} = \frac{1}{f'(p)}$. Thus there exists a $\delta > 0$ such that

if $|x-p| < \delta$ then $\left| \dfrac{x-p}{f(x)-f(p)} - \dfrac{1}{f'(p)} \right| < \varepsilon$.

Since f^{-1} is continuous at $q = f(p)$, there exists a $\hat{\delta} > 0$ such that if $|y-q| < \hat{\delta}$ then $|f^{-1}(y) - f^{-1}(q)| < \delta$. Let $y = f(x)$ then $|x-p| < \delta$, then $\left| \dfrac{x-p}{f(x)-f(p)} - \dfrac{1}{f'(p)} \right| < \varepsilon$, then $\left| \dfrac{f^{-1}(y) - f^{-1}(q)}{y-q} - \dfrac{1}{f'(p)} \right| < \varepsilon$.

So, $$\lim_{y \to q} \frac{f^{-1}(y) - f^{-1}(q)}{y-q} = \frac{1}{f'(p)} = (f^{-1})'(q)$$

Note If f and f^{-1} are differentiable, then

$$\frac{d}{dy} f(f^{-1}(y)) = \frac{d}{dx} y$$

$$f'(f^{-1}(y)) \cdot \left[(f^{-1})'(y)\right] = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

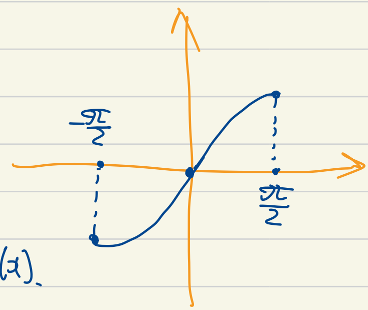
$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Previously, we did not have that f^{-1} is differentiable.

Now we have a prove that if f is a C^1 function and $f(p) \neq 0$ then f^{-1} is differentiable at $q = f(p)$.

Example: Defin $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$, $x \mapsto \sin(x)$ we get

$f'(x) = \cos(x) > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so f is strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $f\left(-\frac{\pi}{2}\right) = -1$, $f\left(\frac{\pi}{2}\right) = 1$ so f is actually bijective.

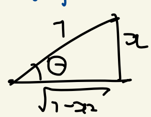


By the Inverse Function Theorem, $f^{-1}: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ exists and is C^1 on $(-1, 1)$. Here $f^{-1}(x) = \arcsin(x)$.
$[\sin^{-1}(x)]$.

By the IFT, $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\arcsin(x))}$

Note that $\sin^2(x) + \cos^2(x) = 1 \Rightarrow \cos(x) = + \sqrt{1 - \sin^2(x)}$

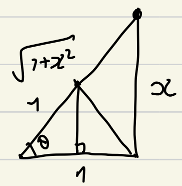
Thus, $\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - x^2}}$.



Example: $\operatorname{atan}(x) : (-\infty, \infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\quad \tan'(x) = \frac{1}{\cos^2(x)}$

By the IFT, $\frac{d}{dx}\operatorname{atan}(x) = \frac{d}{dx}\tan^{-1}(x) = \frac{1}{\tan'(\operatorname{atan}(x))}$

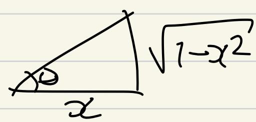
$$\frac{d}{dx}\operatorname{atan}(x) = \cos^2(\operatorname{atan}(x)) = \left[1 \cdot \frac{1}{\sqrt{1+x^2}}\right]^2 = \frac{1}{1+x^2}$$



Example: $\operatorname{acos}(x) : (-1, 1) \longrightarrow (0, \pi)$. $\quad \cos'(x) = -\sin(x)$

By the IFT $\frac{d}{dx}\operatorname{acos}(x) = \frac{d}{dx}\cos^{-1}(x) = \frac{1}{\cos'(\operatorname{acos}(x))}$

$$\frac{d}{dx}\operatorname{acos}(x) = \frac{-1}{\sin(\operatorname{acos}(x))} = -\frac{1}{\sqrt{1-x^2}}$$



$$\boxed{\begin{aligned}
\frac{d}{dx}\operatorname{asin}(x) &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\operatorname{acos}(x) &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}\operatorname{atan}(x) &= \frac{1}{1+x^2}
\end{aligned}}$$

$$\sec : [0,\pi] \setminus \{\tfrac{\pi}{2}\} \longrightarrow \mathbb{R} \setminus (-1,1)$$

$$\cos : [0,\pi] \longrightarrow [-1,1]$$

$$\frac{1}{1/x}$$

Let $\theta = \operatorname{asec}(x) \in [0,\pi] \setminus \frac{\pi}{2} \Rightarrow \sin(\theta) \geq 0,\ \cos\theta \neq 0.$

Also, $\sec'(x) = \left(\frac{1}{\cos(x)}\right)' = -\frac{\cos'(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} =$

$$= \frac{\sqrt{1-\cos^2(x)}}{\cos^2(x)}$$

$$\operatorname{asec}'(x) = \frac{1}{\sec'(\operatorname{asec}(x))} = \frac{\cos^2(\theta)}{\sqrt{1-\cos^2(\theta)}} = \frac{1/x^2}{\sqrt{1-1/x^2}} =$$

$$= \boxed{\frac{1}{|x|\sqrt{x^2-1}}}$$