Problem 2.
Suppose $f : (a,b) \to \mathbb{R}$ is a function, $c \in (a,b)$, and f is differentiable at c with $f'(c) > 0$. Show there exists some $\rho > 0$ such that $(c-\rho, c+\rho) \subseteq (a,b)$ and $f(c-h) < f(c) < f(c+h)$ for all $h \in (0,\rho)$.

Solution.
We have that f is differentiable at $c \in (a,b)$ and $f'(c) > 0$. Thus, $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = K > 0$, and

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall h \in (a-c, b-c)) \left[0 < |h| < \delta \Rightarrow \left|\frac{f(c+h)-f(c)}{h} - K\right| < \epsilon\right].$$

Take $\epsilon = \frac{K}{2} > 0$ and fix a suitable $\delta > 0$. Then, we have

$$(\forall h \in (a-c, b-c)) \left[0 < |h| < \delta \Rightarrow \left|\frac{f(c+h)-f(c)-Kh}{h}\right| < \frac{K}{2}\right],$$
$$(\forall h \in (a-c, b-c)) \left[0 < |h| < \delta \Rightarrow f(c+h) \in \left(f(c) + Kh - \frac{K}{2}|h|, f(c) + Kh + \frac{K}{2}|h|\right)\right].$$

Take $\rho = \min\{\delta, |a-c|, |b-c|\} > 0$. Then, as $a < c < b$, it can be concluded that $(c-\rho) \geq (c-|a-c|) = a$ and $(c+\rho) \leq (c+|b-c|) = b$. Thus, $(c-\rho, c+\rho) \subseteq (a,b)$.
Furthermore, $(\forall h \in (-\rho, 0) \cup (0,\rho))[h \in (a-c, b-c) \wedge 0 < |h| < \delta]$.
Consequently, $(\forall h \in (-\rho, 0) \cup (0,\rho)) \left[f(c+h) \in \left(f(c) + Kh - \frac{K}{2}|h|, f(c) + Kh + \frac{K}{2}|h|\right)\right]$.

Thus, we have $(\forall h \in (0,\rho)) \left[f(c+h) \in \left(f(c) + \frac{K}{2}h, f(c) + \frac{3K}{2}h\right)\right]$, and consequently we obtain that $(\forall h \in (0,\rho))[f(c) < f(c) + \frac{K}{2}h < f(c+h)]$.

Similarly, $(\forall h \in (0,\rho)) \left[f(c+(-h)) \in \left(f(c) + K(-h) - \frac{K}{2}|-h|, f(c) + K(-h) + \frac{K}{2}|-h|\right)\right]$, and then $(\forall h \in (0,\rho)) \left[f(c-h) \in \left(f(c) - \frac{3K}{2}h, f(c) - \frac{K}{2}h\right)\right]$. Thus, $(\forall h \in (0,\rho))[f(c-h) < f(c) - \frac{K}{2}h < f(c)]$.

Now we have finally proven that $(\forall h \in (0,\rho))[f(c-h) < f(c) < f(c+h)]$. ∎