Problem 1.(a)
Let $I \subseteq \mathbb{R}$ be an interval. We say that a function $f : I \to \mathbb{R}$ is *strictly increasing* if whenever $a, b \in I$, then $f(a) < f(b)$. Show that f is injective.

Solution
We shall start with two equivalent definitions of a *strictly increasing* function.

$$(\forall a, b \in I)[a < b \Rightarrow f(a) < f(b)]$$

$$(\forall a, b \in I)[a > b \Rightarrow f(a) > f(b)]$$

These inequalities combined give us the following.

$$(\forall a, b \in I)[a \neq b \Rightarrow f(a) \neq f(b)]$$

Next we apply the fact that $[P \Rightarrow Q] \Leftrightarrow [\neg Q \Rightarrow \neg P]$.

$$(\forall a, b \in I)[\neg(f(a) \neq f(b)) \Rightarrow \neg(a \neq b)]$$

$$(\forall a, b \in I)[f(a) = f(b) \Rightarrow a = b]$$

Thus we have derived the exact definition of *injectivity* for f. ■

Problem 1.(b)
Suppose $f : I \to \mathbb{R}$ is an invertible, strictly increasing function. Show that f^{-1} is also strictly increasing.

Solution
We shall start with the definition of a *strictly increasing* function.

$$(\forall a, b \in I)[a < b \Rightarrow f(a) < f(b)]$$

$$(\forall a, b \in I)[\neg(f(a) < f(b)) \Rightarrow \neg(a < b)]$$
$$(\forall a, b \in I)[f(a) \geq f(b) \Rightarrow a \geq b]$$

Since f is invertible it is *bijective*, meaning it is both injective and surjective.
From *injectivity* we get the following by definition.

$$(\forall a, b \in I)[a = b \Leftrightarrow f(a) = f(b)]$$

Combining the facts that f is both strictly increasing and injective, we obtain the following.

$$(\forall a, b \in I)[(f(a) \geq f(b) \Rightarrow a \geq b) \wedge (a = b \Leftrightarrow f(a) = f(b))]$$

$$(\forall a, b \in I)[f(a) > f(b) \Rightarrow a > b]$$
$$(\forall a, b \in I)[f(a) < f(b) \Rightarrow a < b]$$

Since f is bijective we can replace a with $f^{-1}(x)$ and b with $f^{-1}(y)$ where $x, y \in \mathbb{R}$.

$$(\forall f^{-1}(x), f^{-1}(y) \in I)[f(f^{-1}(x)) < f(f^{-1}(y)) \Rightarrow f^{-1}(x) < f^{-1}(y)]$$

$$(\forall x, y \in \mathbb{R})[x < y \Rightarrow f^{-1}(x) < f^{-1}(y)]$$

Thus we have derived the exact definition of *being strictly increasing* for f^{-1}. ∎

Problem 1.(c)
If n is a positive integer, show that $f : [0, \infty) \to \mathbb{R}, x \mapsto x^n$ is an injective function.

Solution
We shall start by proving $(\forall a, b \in \mathbb{R})[0 \le a < b \Rightarrow a^n < b^n]$ for any $n \in \mathbb{N}$.

Let $a, b \in \mathbb{R}$ satisfy $0 \le a < b$, then we have the following.

1. $b - a > 0$ since $a < b$
2. $\sum_{i=0}^{n-1} b^i a^{(n-1)-i} > 0$ since $b > a \ge 0$, meaning that $b^{n-1} > 0$ and all other terms are at least zero.

Thus the product of $b - a$ and the above-mentioned sum is greater than zero.

$$(b-a) \cdot \sum_{i=0}^{n-1} b^i a^{(n-1)-i} > 0$$

$$b \cdot \sum_{i=0}^{n-1} b^i a^{(n-1)-i} - a \cdot \sum_{i=0}^{n-1} b^i a^{(n-1)-i} > 0$$

$$\left(\sum_{i=0}^{n} b^i a^{n-i} - a^n \right) - \left(\sum_{i=0}^{n} b^i a^{n-i} - b^n \right) > 0$$

$$b^n - a^n > 0$$

We have finally proven that $0 \le a < b \Rightarrow a^n < b^n$, meaning that $f(x) = x^n$ is a strictly increasing function.

$$(\forall a, b \in [0, \infty))[a < b \Rightarrow f(a) < f(b)]$$

Similarly to **1.(a)** we show the following.

$$(\forall a, b \in [0, \infty))[a \ne b \Rightarrow f(a) \ne f(b)]$$

$$(\forall a, b \in [0, \infty))[\neg(f(a) \ne f(b)) \Rightarrow \neg(a \ne b)]$$

$$(\forall a, b \in [0, \infty))[f(a) = f(b) \Rightarrow a = b]$$

Thus we have derived the exact definition of *injectivity* for f. ■