W6 Lecture

- uniform continuity
- IVT
- EVT

General Continuity: "Can draw with a single stroke"

$$(\forall c\in D)(\forall \varepsilon>0)(\exists \delta>0)(\forall x\in D)\big[|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon\big]$$

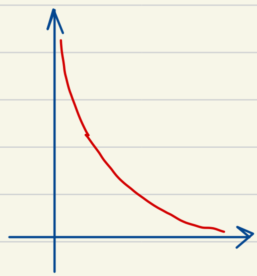
Uniform Continuity: "Limits the speed of growth, kinda"

$$(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\in D)\big[|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon\big]$$

Examples:

$f:(0,\infty)\to\mathbb{R}$, $x\mapsto x^2$	NOT u. CTS
$g:[0,10]\to\mathbb{R}$, $x\mapsto x^2$	u. CTS
$h:(0,\infty)\to\mathbb{R}$, $x\mapsto\sqrt{x}$	u. CTS

Problem 1°: Show $f:(0,\infty)\to\mathbb{R}$, $x\mapsto 1/x$ is <u>not</u> u. cts.



Soln: WTS $\exists\, \varepsilon>0, \forall\, \delta>0, \exists\, x,y\in(0,\infty)$
$|x-y|<\delta$ and $|f(x)-f(y)|\geq\varepsilon$

Set $\varepsilon=1$, and let $\delta>0$ be given.

Case 1°: If $\delta>1$, let $x=\frac{1}{10}$, $y=\frac{1}{100}$ $\Rightarrow |x-y|<\delta$ and $\left|\frac{1}{x}-\frac{1}{y}\right|\geq 1$

Case 2°: If $\delta\leq 1$, let $x=\delta$, $y=\frac{\delta}{2}$ $\Rightarrow |x-y|<\delta$ and $\left|\frac{1}{x}-\frac{1}{y}\right|\geq 1$ $\blacksquare$

Problem 2° Show that $f : [a,\infty) \to \mathbb{R}$, $x \mapsto 1/x$ is uniformly continuous for any $a > 0$.

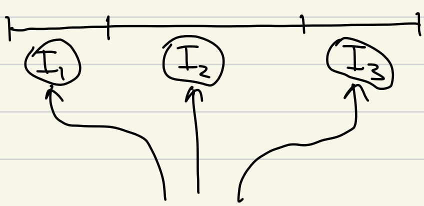
Soln: Rough work: $|f(x)-f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x-y}{xy}\right| = \frac{|x-y|}{xy} \leq \frac{|x-y|}{a^2}$ $(x, y \geq a)$

Thus, choose $\delta = a^2\varepsilon$.

Proof: Let $\varepsilon > 0$ be given and set $\delta = a^2\varepsilon$.

Thus, if $|x-y| < \delta$ then $|f(x)-f(y)| = \frac{|x-y|}{xy} \leq \frac{|x-y|}{a^2} < \frac{a^2\varepsilon}{a^2} = \varepsilon$

Exercise: If $I_n = [a_n, b_n]$, $n = 1, \ldots, m$, are closed intervals such that $b_n = a_{n+1}$ for all $n = 1, \ldots, m-1$, and f is uniformly continuous on each I_n, then f is uniformly cts on $\bigcup_{n=1}^{m} I_n = [a_1, b_m]$.



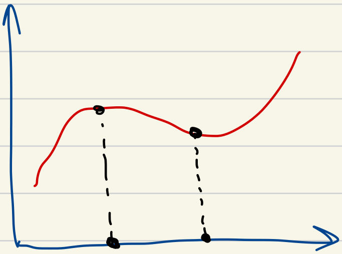
If f is uni. cts on each, the uni. cts on the whole interval.

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in I_n)$$
$$[|x-y| < \delta \implies |f(x)-f(y)| < \varepsilon]$$

Sdn: Let $\varepsilon > 0$ be given.
Find δ_n such that
$(\forall x, y \in I_n)[|x-y| < \delta_n \implies |f(x)-f(y)| < \frac{\varepsilon}{2}]$
Find $\delta_0 = \frac{1}{2}\min\{|a_n - b_n| : n = 1, \ldots n\}$.

Take $\delta = \min(\delta_0, \delta_1, \ldots, \delta_m)$
Fix $x, y \in \bigcup_{i=1} I_i$ Assume $x \leq y$. For the sake of uniformity
If $(\exists i \in \{1, \ldots n\})[x, y \in I_i]$
If $(\exists i \in \{1, \ldots m\})[x \in I_i, y \in I_{i+1}]$
$|x-y| < \delta_n \implies |x-a_i| + |y-a_i| \leq 2\delta_n$
$\implies |f(x)-f(a_i)| + |f(a_i)-f(y)| \geq |f(x)-f(y)|$
$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} >$

Thm. If $f:[a,b]\to\mathbb{R}$ is continuous, then it is uniformly continuous.



Comment: $D=[a,b]$ is both closed and bounded

1). **Closed:** $f:(0,1)\to\mathbb{R}$, $x\mapsto 1/x$ is not U.CTS

2). **Bounded:** $g:[0,\infty)\to\mathbb{R}$, $x\mapsto x^2$ is not U.CTS

Proof: Fix some $\varepsilon>0$ and define

$$C(\varepsilon)=\{c\in[a,b]:(\exists\,\delta>0)(\forall x,y\in[a,c])\,[|x-y|<\delta\Rightarrow|f(x)-f(y)|<\varepsilon]\}$$

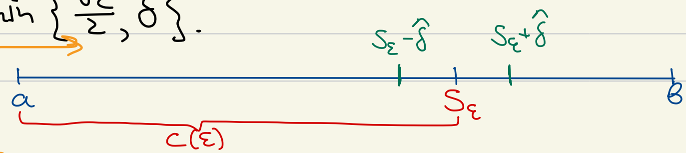
This is the set of points in $[a,b]$ where f is uniformly continuous for this choice of ε.

1. Show $\sup C(\varepsilon)=b$, i.e. f is U.CTS for this ε on $[a,b]$.
2. This doesn't depend on ε.

Note that $a\in C(\varepsilon)$, and b is an upper bound.
By the Completeness Axiom, $S_\varepsilon=\sup C(\varepsilon)$ exists.

Claim: $S_\varepsilon=b$.

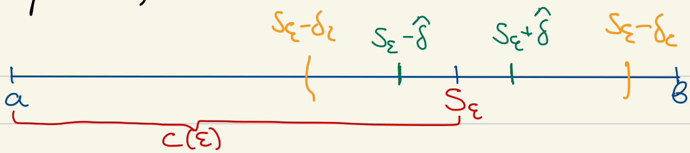
Pf: Certainly $S_\varepsilon\le b$. For a contradiction, assume $S_\varepsilon<b$.
Thus, $S_\varepsilon\in[a,b]$ and so f is continuous at S_ε.
Therefore, choose $\delta_c>0$ such that $|x-S_\varepsilon|<\delta_c\Rightarrow|f(x)-f(S_\varepsilon)|<\frac{\varepsilon}{2}$.
Let δ be the number which works for $C(\varepsilon)$,
and set $\hat{\delta}=\min\{\frac{\delta_c}{2},\delta\}$.



continuity req. at S_ε

to ensure distance between any 2 points is $\le\delta_c$

Condition of continuity at S_ε pushes $C(\varepsilon)$ interval $\Rightarrow$ S_ε is not a supremum, unless



Suppose $|x-y| < \hat{\delta}$

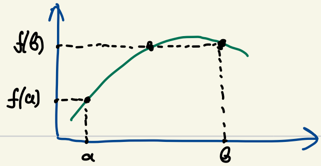
Case 1°: If $x,y \in [a, S_\varepsilon]$ then $|x-y| < \hat{\delta} \leq \delta$
$\Rightarrow |f(x)-f(y)| < \varepsilon.$

Case 2°: If $x \in [a, S_\varepsilon)$ and $y \in (S_\varepsilon-\hat{\delta}, S_\varepsilon+\hat{\delta})$
then $|x-S_\varepsilon| \leq |x-y| + |y-S_\varepsilon| \leq \hat{\delta}+\hat{\delta} \leq \frac{\delta_c}{2}+\frac{\delta_c}{2} = \delta_c.$
Thus, $|f(x)-f(y)| \leq |f(x)-f(S_\varepsilon)| + |f(S_\varepsilon)-f(y)| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$

Case 3°: If $x,y \in (S_\varepsilon-\hat{\delta}, S_\varepsilon+\hat{\delta})$ then $|x-y| \leq |x-S_\varepsilon| + |y-S_\varepsilon|$
$< \hat{\delta}+\hat{\delta} \leq \frac{\delta_c}{2}+\frac{\delta_c}{2} = \delta_c$
so $|f(x)-f(y)| \leq |f(x)-f(S_\varepsilon)| + |f(y)-f(S_\varepsilon)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$

Thus, $\hat{\delta}$ shows ε-uni.cts on $[a, S_\varepsilon+\hat{\delta}]$, this contradicts the fact that S_ε is an upperbound.

Thus, $S_\varepsilon = b$ for every $\varepsilon > 0$.
Convince yourself that $b \in C(\varepsilon)$. $\Rightarrow$ f is u.cts on $[a,b]$ ∎



Theorem [Intermediate Value Theorem]

If $f:[a,b] \to \mathbb{R}$ is continuous and $f(a) < f(b)$, then for every $d \in [f(a), f(b)]$, there exists some $c \in [a,b]$ such that $f(c) = d$.

Theorem [Extreme Value Theorem]

If $f:[a,b] \to \mathbb{R}$ is continuous, there exist $m, M \in [a,b]$, such that for all $x \in [a,b]$, $f(m) \leq f(x) \leq f(M)$.