Problem 2.
Suppose $f : I \to \mathbb{R}$ is a bounded function.

(i) If $g : I \to \mathbb{R}$ is another bounded function such that $f(x) \leq g(x)$ for all $x \in I$, show that $\sup(f) \leq \sup(g)$.

(ii) Show that $\sup(-f) = -\inf(f)$.

(iii) Show that $-\inf(|f|) \leq \sup(f) \leq \sup(|f|)$.

Solution. (i)
Fix bounded functions $f, g : I \to \mathbb{R}$ such that $(\forall x \in I)[f(x) < g(x)]$. Want to show, $\sup(f) \leq \sup(g)$.

For a contradiction, assume $\sup(f) > \sup(g)$. Let $\sup(f) = M$ and $\sup(g) = N$. Write the property of suprema

$$(\forall \varepsilon > 0)(\exists s \in f(I))[M - \varepsilon < s \leq M] \text{ and } (\forall \varepsilon > 0)(\exists s \in g(I))[N - \varepsilon < s \leq N]$$

or equivalently

$$(\forall \varepsilon > 0)(\exists x \in I)[M - \varepsilon < f(x) \leq M] \text{ and } (\forall \varepsilon > 0)(\exists x \in I)[N - \varepsilon < g(x) \leq N].$$

Take $\varepsilon = M - N > 0$ for the property of $\sup(f) = M$, then we can fix $x_0 \in I$ such that

$$N = M - (M - N) = M - \varepsilon < f(x_0) \leq M.$$

Since $(\forall x \in I)[f(x) < g(x)]$, then we have $g(x_0) > f(x_0) > N$. However, since $N = \sup(g)$, the following must be true, $g(x_0) \leq N$. Thus, the assumption $\sup(f) > \sup(g)$ led to a contradiction. ■

Solution. (ii)
Take $\sup(-f) = M$. Want to prove $\inf(f) = (-M)$.

Since $(\forall x \in I)[-f(x) \leq M]$, after multiplying both sides by (-1), we obtain $(\forall x \in I)[f(x) \geq (-M)]$. Thus, $(-M)$ is a lower bound of the set $f(I)$ and the function f.

According to the property of suprema,

$$(\forall \varepsilon > 0)(\exists s \in -f(I))[M - \varepsilon < s \leq M], \text{ or equivalently } (\forall \varepsilon > 0)(\exists x \in I)[M - \varepsilon < -f(x) \leq M]$$

After multiplying all sides of the inequality by (-1), we have obtain the following.

$$(\forall \varepsilon > 0)(\exists x \in I)[(-M) \leq f(x) < (-M) + \varepsilon]$$

Since $(-M)$ is a lower bound of f and the above-mentioned statement is true, $\inf(f) = (-M)$. This is according to the property of infima, analogous to the property of suprema, we discussed in the class.

Thus, $\sup(-f) = -\inf(f) = M$. ■

Solution. (iii)
First of all, it will be shown that if f is bounded, then $|f|$ and $-|f|$ are bounded as well.

The fact that f is bound is equivalent to $(\exists M \in \mathbb{R})(\forall x \in I)[|f(x)| \leq M]$. Let's fix such M that satisfies the previous statement. Then we have $(\forall x \in I)[||f(x)|| = |f(x)| \leq M]$ and $(\forall x \in I)[|-|f(x)|| = |f(x)| \leq M]$. Thus, both $|f|$ and $-|f|$ are bounded.

Those inequalities themselves will be proven separately.

1. Want to prove, $-\inf(|f|) \leq \sup(f)$.

   According to **2.(ii)**, $-\inf(|f|) = \sup(-|f|)$.

   Since f and $-|f|$ are bounded functions and $(\forall x \in I)[-|f(x)| \leq f(x)]$, according to **2.(i)**, we have $-\inf(|f|) = \sup(-|f|) \leq \sup(f)$.

2. Want to prove, $\sup(f) \leq \sup(|f|)$.

   Since f and $|f|$ are bounded functions and $(\forall x \in I)[f(x) \leq |f(x)|]$, according to **2.(i)**, we have $\sup(f) \leq \sup(|f|)$.

Thus, $-\inf(|f|) \leq \sup(f) \leq \sup(|f|)$. ■