Problem 2.
Suppose I is an open interval and $f : I \to \mathbb{R}$ is a function satisfying $|f(x) - f(y)| \leq K|x - y|$ for some $K > 0$ and all $x, y \in I$. Show that f is continuous.

Solution.
It is given that for some K, we have $(\forall x \in I)(\forall y \in I)[|f(x) - f(y)| \leq K|x - y|]$.

Now it shall be proven that $(\forall x \in I)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in I)[|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon]$, which is the definition of f being *continuous* at every point $x \in I$.

Fix some $x \in I$ and $\varepsilon > 0$. Since $K > 0$, we take $\delta = \frac{\varepsilon}{K} > 0$.
Since $(\forall x, y \in I)[|f(x) - f(y)| \leq K|x - y|]$, this leads us to the fact that

$$(\forall y \in I)\left[|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq K|x - y| < K\delta = K\frac{\varepsilon}{K} = \varepsilon\right].$$

Thus, we have proven that $(\forall x \in I)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in I)[|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon]$, meaning that f is *continuous*. ■