Def. Absolute value.

The absolute value function is $|\cdot| : \mathbb{R} \to [0, \infty)$,

$$x = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note that if $x, y \in \mathbb{R}$, then the *distance* from x to y is $|x - y|$.
Note that $|x| < a \Leftrightarrow -a < x < a \Leftrightarrow x \in (-a, a)$.
And finally, for all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

LIMITS

Goal: Determine the behaviour of a function f near the point c, without ever evaluating $f(c)$.
There is a weird-point graph and Canyon-geologist analogy.

Def. Deleted open interval.

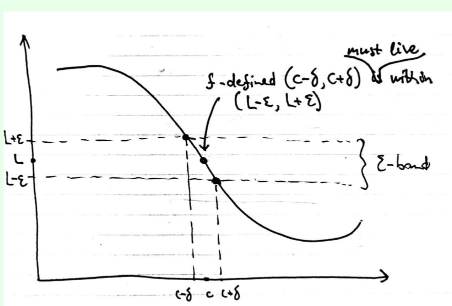
Suppose $f : D \to \mathbb{R}$ is a function, $c \in R$, and f is defined on a deleted open interval around c. (There exists some $\rho > 0$ such that f is defined on $(c - \rho, c) \cup (c, c + \rho)$.)

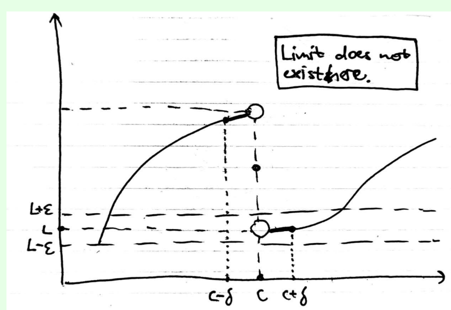
We say that $\lim_{x \to c} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ satisfying $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$.

To paraphrase that statement:

$$\lim_{x \to c} f(x) = L \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

1. Given an "error bound" (ε),
2. we can always find some distance (δ) such that
3. if we are no more than that distance away from c
4. then our approximation is within that error bound.





Problem
Show that $\lim_{x \to 4}[2x + 3] = 11$.

Solution.
Let $\varepsilon > 0$ be given, and set $\delta = \frac{\varepsilon}{2}$. Suppose then that $0 < |x - 4| < \delta$, then

$$|[2x + 3] - 11| = |2x - 8| = 2|x - 4| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

Strategy for solving Limits

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

- This (ε) should be arbitrary.
- We get to choose δ, it should dependn on ε.

Goal: Write $|f(x) - L| \leq M \cdot |x - c|$ for some $M > 0$. At this point, take $\delta = \frac{\varepsilon}{M}$

$$|f(x) - L| \leq M|x - c| < M\delta = M \cdot \frac{\varepsilon}{M} = \varepsilon$$

Problem
Show that $\lim_{x \to (-1)}[5 - 4x] = 9$.

Solution. (Rough work)

$$|f(x) - L| = |[5 - 4x] - 9| = | - 4x - 4| = 4|x + 1| = M \cdot |x - c|$$

Solution. (Real proof)
Let $\varepsilon > 0$ be given, and set $\delta = \frac{\varepsilon}{M} = \frac{\varepsilon}{4}$. Suppose then that $0 < |x - c| = |x + 1| < \delta$, then

$$|[5 - 4x] - 9| = | - 4x - 4| = 4|x + 1| < 4\delta = 4 \cdot \frac{\varepsilon}{4} = \varepsilon$$

$\square$

Problem
Show that $\lim_{x \to 1}[2x^2 + 1] = 3$.

Solution. (Rough work)

$$|f(x) - L| = |[2x^2 + 1] - 3| = |2x^2 - 2| = 2|x^2 - 1| = 2|x - 1||x + 1| \ < M|x - 1|$$

We will have to assume $|x - 1| < 1$, i.e. $\delta \leq 1$.

$$|x - 1| < 1 \Rightarrow |x + 1| \leq |x - 1| + |2| < 3$$

So we set $\delta = \frac{\varepsilon}{6}$.

$$|f(x) - L| = 2|x - 1||x + 1| < 6|x - 1| = M|x - 1|$$

Solution. (Real proof)
Let $\varepsilon > 0$ be given, and set $\delta = \min(1, \frac{\varepsilon}{6})$. Suppose then that $0 < |x - c1| < \delta$. Since $|x - 1| < \delta$ then $|x + 1| < 3$ by (*). Thus,

$$|[2x^2 + 1] - 3| = 2|x - 1||x + 1| < 6|x - 1| < 6\delta \leq 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

$\square$

Problem
Show that $\lim_{x \to 3} \sqrt{x+1} = 2$.

ONE-SIDED LIMIT

Suppose that $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$. If f is defined on some interval $(, c+e)$, $e > 0$, then we say that $\lim_{x \to c^+} f(x) = L$ if $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[x \in (c - \delta, c) \Rightarrow |f(x) - L| < \varepsilon]$.

Immediately, $\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$

Problem
Show that $\lim_{x \to 0^+} \frac{x}{|x|} = 1$.

Solution.
Let $\varepsilon > 0$ be given and set $\delta = \varepsilon$. Note that if $x \in 0, \delta$ then $x > 0$ so $|x| = x$, and so

$$|f(x) - L| = \left| \frac{x}{|x|} - 1 \right| = |1 - 1| = 0 < \epsilon$$

. $\square$

HORIZONTAL ASYMPTOTE

A function f has a horizontal asymptote at L if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

Def. "Limit to infinity".
Suppose $f : D \to \mathbb{R}$ is defined on (a, ∞) for some $a \in \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = L$ if

$$(\forall \varepsilon > 0)(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})[x > M \Rightarrow |f(x) - L| < \varepsilon].$$

For all $\varepsilon > 0$, there is some point M after which the function stays entirely in an epsilon band.

Problem
Show $\lim_{x \to \infty} \frac{x^2}{x^2+1} = 1$.

Solution. "Rough work"

$$|f(x) - L| = \left| \frac{x^2}{x^2+1} - 1 \right| = \left| \frac{x^2 - x^2 - 1}{x^2+1} \right| = \left| \frac{-1}{x^2+1} \right| = \frac{1}{x^2+1}$$

We want $\frac{1}{x^2+1} < \varepsilon$ or $x^2 + 1 > \frac{1}{\varepsilon}$, so $x > \sqrt{\frac{1}{\varepsilon} - 1}$.

This works so long as $0 < \varepsilon < 1$.

In the proof, do cases. [If $\varepsilon > 0$ take $M = 0$.] [If $0 < \varepsilon < 1$ take $M = \sqrt{1/\varepsilon - 1}$.]

Def. $\lim_{x \to \infty} f(x) = \infty$.

$$(\forall N \in \mathbb{R})(\exists M \in \mathbb{R})(\forall x \in \mathbb{R})[x > N \Rightarrow f(x) > M]$$