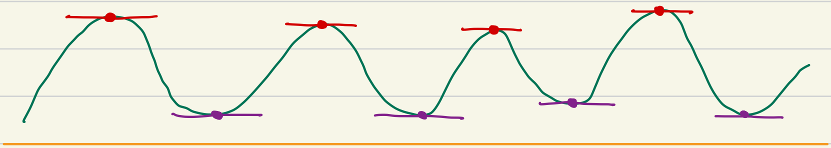
W9 Lecture.

- local min/max
- MVT

Defn: If $f: D \to \mathbb{R}$, a point $c \in D$ is a local max (min) if there exists some $\rho > 0$ such that $f(x) \leq f(c)$ $(f(x) \geq f(c))$ $\forall x \in D \cap (c-\rho, c+\rho)$. We say that c a global max (min) is $f(x) \leq f(c)$ $(f(x) \geq f(c))$ for all $x \in D$.



Defn: If f is differentiable, we say that a point is a critical point of f if $f'(c) = 0$.

Example: $f(x) = \dfrac{x}{1+x^2}$

$$f'(x) = \frac{(1+x^2) - 2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = 0.$$ So the critical points $x = \pm 1$.

Theorem: If $f: (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$ and c is a local max/min of f, then $f'(c) = 0$; that is c is a critical points.

Pf: Without loss of generality, suppose c is a local max. (if not, apply this result to $-f$). Let $\rho > 0$ be such that $f(x) \leq f(c)$ for all $x \in (a,b) \cap (c-\rho, c+\rho)$.

We know $f(x) \leq f(c) \Rightarrow f(x) - f(c) \leq 0$

Case 1°: Suppose $x \in (a,b) \cap (c, c+\rho)$. Then, $\dfrac{f(x)-f(c)}{x-c} \leq 0$.

Thus, $\displaystyle\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = f'(c) \leq 0$

Case 2°: Suppose $x \in (a,b) \cap (c-\rho, c)$. Then, $\dfrac{f(x)-f(c)}{x-c} \geq 0$.

Thus, $\displaystyle\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = f'(c) \geq 0$.

Thus, $0 \leq f'(c) \leq 0$, and $f'(c) = 0$. $\blacksquare$

[If c is a local min, define $g(x) = -f(x)$, so that c is a local max of g. By what we proved, $g'(c) = 0 = -f'(c)$, so $f'(c) = 0$.]

<u>Note:</u> Being a critical point is necessary for local extrema, but not sufficient. There are critical points which are <u>not</u> local extrema. For example, $f(x) = x^3$ has a critical point at $c = 0$, but 0 is not a max/min of x^3.

<u>Thm</u> [Rolle's Theorem] If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

<u>Pf:</u> Without loss of generality, $f(a) = f(b) = 0$. Otherwise, define $\hat{f}(x) = f(x) - f(a)$, then apply the result.

Note that if is constant, then $f'(x) \equiv 0$ for all $x \in (a,b)$, so we are done. By the Extreme Value Theorem, f achieves its max and min on $[a,b]$. If f is not constant, one of these must not be 0. (If min and max are 0, $0 \leq f(x) \leq 0$ for all $x \Rightarrow f(x) = 0$).

Assume the maximum is not zero (otherwise apply the result to $-f$). Let $c \in [a,b]$ so that $f(c)$ is the global max. Note that $c \neq a, b$. Thus, $c \in (a,b)$ and by our proposition, $f'(c) = 0$. ∎

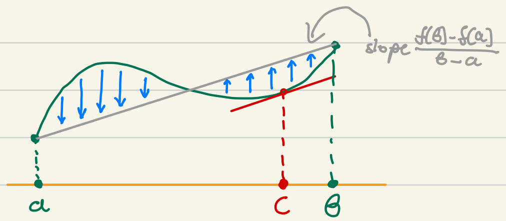
Exercise: Suppose $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f''(x) \neq 0$ $\forall x \in \mathbb{R}$.
Show that f has at most two roots.

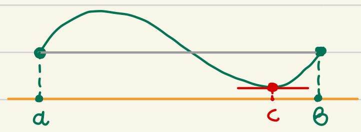
Pf: For the sake of contradiction, suppose f has three roots. Call them $r_1 < r_2 < r_3$. Note that f is continuous on $[r_1, r_2]$ and $[r_2, r_3]$, and differentiable (r_1, r_2) and (r_2, r_3), and $f(r_1) = f(r_2) = f(r_3)$.
By Rolle's Theorem, $\exists$ $c_1 \in (r_1, r_2)$, $c_2 \in (r_1, r_2)$ such that $f'(c_1) = f'(c_2) = 0$. Now $c_1 < c_2$, f' is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2).
So by Rolle's Theorem, $\exists$ $d_1 \in (c_1, c_2)$ such that $f''(d_1) = 0$, a contradiction. Thus, f has at most 2 roots. ■

Note: between two roots of f, lies a root of f'.

Thm [Mean Value Theorem] If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), there exists $c\in(a,b)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ or
$f(b)-f(a) = f'(c)\,(b-a)$.

Idea: "Rotate" the image to get Rolle's Theorem.





Pf: Define $F(x) = \underbrace{f(x)}_{\text{function}} - \dfrac{f(b)-f(a)}{b-a}\cdot\underbrace{(x-a)}_{\text{function}}$.

Note F is continuous on $[a,b]$ and differentiable on (a,b).
Furthermore, $F(a)=f(a)$ and $F(b)=f(a)$.

By Rolle's Theorem, $\exists\, c\in(a,b)$ such that $F'(c)=0$.

$$F'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \implies f'(c) = \frac{f(b)-f(a)}{b-a}$$

Corollary: If $f:\mathbb{R}\to\mathbb{R}$ is differentiable and $f'(x)=0\ \forall x\in\mathbb{R}$, then f is a constant function.

Proof: Fix $x\in\mathbb{R}$, and lets show $f(x)=f(0)$.
Assume $x>0$ with the same proof for $x<0$.
Now f is continuous on $[0,x]$ and differentiable on $(0,x]$ so by the MVT, $\exists\ c\in(0,x)$ such that
$f(x)-f(0) = \underbrace{f'(c)}_{0}\,x = 0$ and so $f(x)=f(0)$. ∎

Corollary: If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable and $f'(x) = g'(x)$ then $\exists\ c \in \mathbb{R}$ such that $f(x) = g(x) + c$.

Corollary: If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ $\forall x \in \mathbb{R}$ then f is strictly increasing.

__Pf:__ We want to show that if $x < y$ then $f(x) < f(y)$. Fix $x, y \in \mathbb{R}$ with $x < y$. Now if is continuous on $[x, y]$ and differentiable on (x, y) thus by the MVT the exists $c \in (x, y)$ such that

$$f(y) - f(x) = \underbrace{f'(c)}_{>0}\ \underbrace{(y - x)}_{>0} > 0 \quad \text{so} \quad f(y) > f(y) \quad \blacksquare$$

Thm [L'Hopital's Rule]: If f, g are differentiable near c and $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then
$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Pf: Let $\delta_1 > 0$ such that f, g are differentiable on $(c-\delta_1, c+\delta_1)$.
Let $\delta_1 > 0$ such that $g'(x) \neq 0$ on $(c-\delta_2, c+\delta_2)$.
$\delta = \min(\delta_1, \delta_2)$

We will prove the theorem as $x \to c^+$. The proof for $x \to c^-$ is nearly identical.
Suppose $h \in (0, \delta)$, define
$$F(x) = \begin{cases} f(x) & x \neq c \\ 0 & x = c \end{cases} \qquad G(x) = \begin{cases} g(x) & x \neq c \\ 0 & x = c \end{cases}$$

Note that F, G are continuous at c, and differentiable on $(c, c+\delta)$.

Define $H(x) = F(x)\overbrace{G(c+h)}^{\text{const}} - G(x)\overbrace{F(c+h)}^{\text{const}}$, which is continuous on $[c, c+h]$ and differentiable on $(c, c+h)$.

Also, $H(c) = \underbrace{H(c)}_{0} G(c+h) - \underbrace{G(h)}_{0} F(c+h) = 0.$
$$H(c+h) = F(c+h)G(c+h) - F(c+h)G(c+h) = 0.$$

So by the MVT, $\exists\, \theta \in (c, c+h)$ such that
$$H(c+h) - H(c) = H'(\theta)h \Rightarrow H'(\theta) = 0$$
$$0 = H'(\theta) = F'(\theta)G(c+h) - G'(\theta)F(c+h) \Rightarrow \frac{F'(\theta)}{G'(\theta)} = \frac{F(c+h)}{G(c+h)} \Rightarrow \frac{f'(\theta)}{g'(\theta)} = \frac{f(c+h)}{g(c+h)}$$

Take the limit $h \to c^+$. Since $c < \theta < c+h$, $\theta \to c^+$ as $h \to 0^+$.

$$\lim_{\theta \to c^+} \frac{f'(\theta)}{g'(\theta)} = \lim_{h \to 0^+} \frac{f(c+h)}{g(c+h)} = \lim_{x \to c^+} \frac{f(x)}{g(x)}.$$

Example: $\displaystyle \lim_{x \to 0} \frac{\tan(4x) - 4x}{16x^3}$

$$\lim_{x \to 0} \frac{\tan(4x) - 4x}{16x^3} \overset{\langle L'H \rangle}{=} \lim_{x \to 0} \frac{(4x)' \cdot \tan'(4x) - 4}{48x^2} = \lim_{x \to 0} \frac{4 \cdot \frac{1}{\cos^2(4x)} - 4}{48x^2}$$

$$= \lim_{x \to 0} \frac{\frac{\sin^2(4x)}{\cos^2(4x)}}{12x^2} = \lim_{x \to 0} \frac{\tan^2(4x)}{12x^2} = \frac{1}{12}\left[\lim_{x \to 0} \frac{\tan(4x)}{x}\right]^2$$

$$\overset{\langle L'H \rangle}{=} \frac{1}{12}\left[\lim_{x \to 0} \frac{4 \cdot \frac{1}{\cos^2(4x)}}{1}\right]^2 = \frac{1}{12}\left[4\right]^2 = \boxed{\frac{4}{3}}$$