Problem 1.(i)
Suppose that $f, g : [0, \infty) \to \mathbb{R}$ are functions satisfying $f(x) \leq g(x)$ for all $x \in [0, \infty)$. If $\lim_{x\to\infty} f(x) = \infty$, show that $\lim_{x\to\infty} g(x) = \infty$.

Solution.
Denote the domain of the functions f, g, as $D = [0, \infty)$.

We have that $(\forall x \in D)[f(x) \leq g(x)]$ and $\lim_{x\to\infty} f(x) = \infty$. WTP $\lim_{x\to\infty} g(x) = \infty$.

The statement that $\lim_{x\to\infty} f(x) = \infty$ is equivalent to $(\forall N \in \mathbb{R})(\exists m \in D)(\forall x \in D)[x > m \Rightarrow f(x) > N]$. Since $(\forall x \in \mathbb{R})[f(x) \leq g(x)]$, we have $(\forall N \in \mathbb{R})(\exists m \in D)(\forall x \in D)[x > m \Rightarrow g(x) \geq f(x) > N]$.

Thus, we have proved $(\forall N \in \mathbb{R})(\exists m \in D)(\forall x \in D)[x > m \Rightarrow g(x) > N]$, which is equivalent to the desired $\lim_{x\to\infty} g(x) = \infty$. ■

Problem 1.(ii)
Suppose that $f : [0, \infty) \to \mathbb{R}$ is strictly increasing and bounded from above. Show that $\lim_{x\to\infty} f(x)$ exists.

Solution.
Denote the domain of the function f, as $D = [0, \infty)$.

We have that $(\forall x, y \in D)[x < y \Rightarrow f(x) < f(y)]$ and $(\exists M \in \mathbb{R})(\forall x \in D)[f(x) \leq M]$. Fix M from the bounded-from-above condition. WTS that $\lim_{x\to\infty} f(x)$ exists.

Take the set $S = f(D)$. S is not empty because $f(0) \in S$. S is bounded from above because $S = \{f(x) : x \in D\}$ and $(\forall x \in D)[f(x) \leq M]$. Thus, according to the **Completeness axiom**, S has a supremum $\sup(S) = L$.

It will be proven that $\lim_{x\to\infty} f(x) = L$ or equivalently that

$$(\forall \epsilon > 0)(\exists m \in D)(\forall x \in D)[x > m \Rightarrow |f(x) - L| < \epsilon].$$

For a contradiction, assume the opposite and fix $\epsilon > 0$ such that

$$(\forall m \in D)(\exists x \in D)[x > m \wedge (f(x) \leq (L - \epsilon) \vee f(x) \geq (L + \epsilon))].$$

Since L is also an upper bound of $f(D)$, $(\forall x \in D)[f(x) \leq L < (L + \epsilon)]$. Thus, we can only have

$$(\forall m \in D)(\exists x \in D)[x > m \wedge f(x) \leq (L - \epsilon)].$$

Assume that there exists $x_0 \in D$ such that $f(x_0) > (L - \epsilon)$. Then we can find $x_1 \in D$ such that $x_1 > x_0$ and $f(x_1) \leq (L - \epsilon) < f(x_0)$. This contradicts the fact that $(\forall x, y \in D)[x < y \Rightarrow f(x) < f(y)]$. Thus, we have $(\forall x \in D)[f(x) \leq (L - \varepsilon)]$. However, this means that $(L - \epsilon)$ is an upper bound of $f(D)$ smaller than the supremum L. Thus, we have arrived at a contradiction and $\lim_{x\to\infty} f(x) = L$. ■