W10 Lecture

- n^{th} order Taylor Polynomial
- some Topology

Defn: A function is C^n on an interval I if it is n-times differentiable and $f^{(n)}$ is continuous on I.

We say that f is C^∞ on I if it is infinitely differentiable.

Goal: Find a function f and a point $\underline{a}$. Now find a polynomial which is a "good" approximation to f at $\underline{a}$.

Define $p_{n,a}(x) = \sum_{k=0}^{n} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

Define $r_{n,a}(x) = f(x) - p_{n,a}(x) \longleftarrow$ this is the error/remainder

- We will say that $p_{n,a}$ is a good linear approximation if its remainder $r_{n,a}$ vanishes faster than linearly near a.

$$\boxed{\lim_{x \to a} \frac{r_{n,a}(x)}{x-a} = 0}$$

- $p_{n,a}$ is a good quadratic approximation near a if

$$\boxed{\lim_{x \to a} \frac{r_{n,a}(x)}{(x-a)^2} = 0}$$

In general, $p_{n,a}$ is a good k^{th}-order approximation if

$$\boxed{\lim_{x \to a} \frac{r_{n,a}(x)}{(x-a)^k} = 0}$$

Claim: If $p_{n,a}$ is a k^{th}-order approximation of f, then

$$r_{n,a}^{(j)}(a) = 0 \text{ for } j = 0, 1, 2, \dots, k \quad \left(\begin{array}{l} f \text{ is } C^k \\ p_{n,a} \text{ is } C^\infty \end{array}\right)$$

Inductively, let's start with $j=0$.

$$r_{n,a}(a) = \lim_{x \to a} r_{n,a}(x) = \lim_{x \to a} \frac{r_{n,a}(x)}{(x-a)^k} \cdot (x-a)^k$$

$$= \left[\lim_{x \to a} \frac{r_{n,a}(x)}{(x-a)^k}\right] \cdot \left[\lim_{x \to a} (x-a)^k\right] = 0$$

Suppose that $r_{n,a}^{(j)} = 0$ for all $j = 0, 1, \dots, m$ $(m < k)$

$$0 = \lim_{x \to a} \frac{r_{n,a}(x)}{(x-a)^k} \overset{\langle L'H \rangle}{=} \lim_{x \to a} \frac{r_{n,a}'(x)}{k(x-a)^{k-1}} \overset{\langle L'H \rangle}{=} \lim_{x \to a} \frac{r_{n,a}''(x)}{k(k-1)(x-a)^{k-2}}$$

$$= \dots \overset{\langle L'H \rangle}{=} \lim_{x \to a} \frac{r_{n,a}^{(m)}(x)}{\frac{k!}{(k-m)!}(x-a)^{k-m}} \overset{\langle L'H \rangle}{=} \lim_{x \to a} \frac{r_{n,a}^{(m+1)}(x)}{\frac{k!}{(k-m-1)!}(x-a)^{k-m-1}}$$

Thus $r_{n,a}^{(m)}(a) = \lim_{x \to a} r_{n,a}^{(m)}(x) = \lim_{x \to a} \frac{r_{n,a}^{(m+1)}}{(x-a)^{k-m-1}} \cdot \lim_{x \to a} (x-a)^{k-m-1}$

$=$

Thus we've shown that $r_{n,a}^{(j)}(a) = 0$, $j = 0, 1, \dots, k$. That is,

$$r_{n,a}^{(j)}(x) = f^{(j)}(x) - p_{n,a}^{(j)}(x)$$

Note that $\frac{d^j}{dx^j}(x-a)^k = \begin{cases} \frac{k!}{(k-j)!}(x-a)^{k-j} & \text{if } j \leq k \\ 0 & \text{if } j > k \end{cases}$

Thus $\frac{d^j}{dx^j} p_{n,a}(x) = \frac{d^j}{dx^j} \sum_{k=0}^{n} c_k (x-a)^k = \sum_{k=0}^{n} c_k \frac{d^j}{dx^j}(x-a)^k$

$$= \sum_{k=j}^{n} \frac{c_k \cdot k!}{(k-j)!}(x-a)^{k-j}$$

Thus $\dfrac{d^j}{dx^j} p_{n,a}(x) = \displaystyle\sum_{k=j}^{n} \frac{c_k \cdot k!}{(k-j)!}(x-a)^{k-j} = \sum_{l=0}^{n-j} \frac{c_{l+j}\,(l+j)!}{l!}(x-a)^l$

Since $r_{n,a}^{(j)}(a) = 0 = f^{(j)}(a) - p_{n,a}^{(j)}(a) = f^{(j)}(a) - \dfrac{c_j \cdot j!}{0!}(x-a)^0$

$$\Rightarrow \boxed{c_j = \frac{f^{(j)}(a)}{j!}}$$

Defn: If f is C^n at the point a, the n^{th} order Taylor polynomial to f at a is

$$\boxed{p_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k}$$

Thm: Suppose that f is C^{n+1} on the interval I, and $a \in I$. If $p_{n,a}$ is the Taylor polynomial for f at a, $r_{n,a} = f - p_{n,a}$, then for all $x \in I$, there exists some c between a and x such that

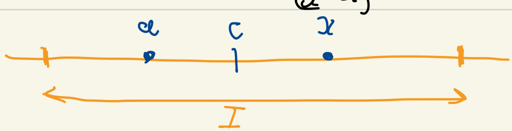
$$r_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Important: Even though f is C^{n+1}, we're only taking the n^{th} order Taylor Polynomial.

Proof: Assume $x > a$, the proof for $x < a$ is almost identical.

Define $g(t) = r_{n,a}(t) - \dfrac{r_{n,a}(x)}{(x-a)^{n+1}}(t-a)^{n+1}$

Note: $g(x) = 0$



<u>Claim:</u> $g^{(k)}(a) = 0$ for all $k = 0, \dots, n$

$$g(a) = r_{n,a}(a) - \underbrace{\frac{r_{n,a}(x)}{(x-a)^{n+1}}(a-a)^{n+1}}_{0} = r_{n,a}(a) = 0$$

Taking j^{th} derivative:

$$g^{(j)}(t) = r_{n,a}^{(j)}(t) - \frac{r_{n,a}(x)}{(x-a)^{n+1}} \frac{(n+1)!}{(n+1-j)!}(t-a)^{n+1-j}$$

$$g^{(j)}(a) = r_{n,a}^{(j)}(a) - \underbrace{\cdots}_{\text{smth}} \underbrace{(a-a)^{n+1-j}}_{0} = 0$$

Thus, $g^{(j)}(a) = 0$ for all $j = 0, \dots, n$.

By A7, Q1, there exists a $c \in (a, x)$ such that

$g^{(n+1)}(c) = 0$.

$$g^{(n+1)}(t) = r_{n,a}^{(n+1)}(t) - \frac{r_{n,a}(x)}{(x-a)^{n+1}}(n+1)!$$

$$= f^{(n+1)}(t) - \frac{r_{n,a}(x)}{(x-a)^{n+1}}(n+1)!$$

Thus $g^{(n+1)}(c) = 0 = f^{(n+1)}(c) - \frac{r_{n,a}(x)}{(x-a)^{n+1}}(n+1)!$

$$\Rightarrow \quad r_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \blacksquare$$

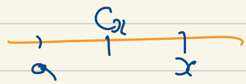
Corollary: If f is C^{n+1} on I, $a \in I$, $p_{n,a}$ is the n^{th} Taylor Polynomial at a, then $p_{n,a}$ is a good n^{th}-order approximation.

Proof: Let $r_{n,a} = f - p_{n,a}$

$$\lim_{x\to a} \frac{r_{n,a}(x)}{(x-a)^n} = \lim_{x\to a} \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)$$ using the theorem

Note that $\lim_{x\to a} c_x = a$ and since $f^{(n+1)}$ is continuous,

$$\lim_{x\to a} f^{(n+1)}(c_x) = f^{(n+1)}(a)$$



$$= \frac{1}{n+1}\left[\lim_{x\to a} f^{(n+1)}(c_x)\right]\left[\lim_{x\to a}(x-a)\right] = 0$$

Example: Let $f(x) = \sin(x)$. Find the n^{th}-order Taylor Polynomial for f at 0.

Soln: We know that $p_{n,0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$

We need to find $f^{(k)}(0)$ in general.

$$\left.\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ f^{(2)}(0) &= -\sin(0) = 0 \\ f^{(3)}(0) &= -\cos(0) = -1 \end{aligned}\right\} \text{then repeat.}$$

$$f^{(2k)}(0) = 0 \text{ for all } k$$

$$f^{(2k+1)}(0) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

Thus, $$p_{2k+1,0}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$p_{1,0}(x) = x$$

$$p_{3,0}(x) = x - \frac{x^3}{3!}$$

$$p_{5,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$p_{7,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$\blacksquare$

Corollary:

Suppose $f : [-1,1] \to \mathbb{R}$ and $p_{n,0}$ is the n^{th} order TP.
$x \mapsto \sin(x)$

We know, $\forall x \in [-1,1]$, $\exists\, c_x$ such that

$$|r_{n,a}(x)| = \left| \frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1} \right| = \frac{\overbrace{|f^{(n+1)}(c_x)|}^{\text{bounded by } 1}}{(n+1)!} \underbrace{|x|^n}_{\text{bounded by } 1}$$

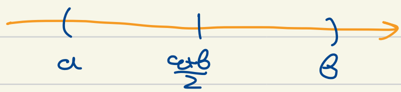
$$\leq \frac{1}{(n+1)!}$$

= Topology =

Defn: If $a, r \in \mathbb{R}$ and $r > 0$ we define the open ball of radius r centered at a to be

$$\boxed{B_r(a) = \{x \in \mathbb{R} : |x-a| < r\}}$$

Note that $|x-a| < r \iff -r < x-a < +r \iff a-r < x < a+r$; that is, $B_r(a) = (a-r, a+r)$.



Also note $(a,b) = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$.

There is a bijective correspondence between open balls and open intervals.

Defn. Let $U \subseteq \mathbb{R}$. A point $a \in \mathbb{R}$ is

1. an interior point if $\exists\, r > 0$ such that $B_r(a) \subseteq U$.
2. a boundary point if $\forall\, r > 0$, $B_r(a) \cap U \neq \emptyset$ and $B_r(a) \cap U^c \neq \emptyset$.

Note: If a is an interior point, $a \in U$.
However, boundary points do not neet to be in U.

Example: The set of interior points of $\mathbb{Q}$ is empty. / $U^{int} = \emptyset$
The boundary points of $\mathbb{Q}$ is $\mathbb{R}$. / $\partial U = \mathbb{R}$

Claim: $(0,1)^{int} = (0,1) \Leftarrow r = \min(x, 1-x)$
$\partial(0,1) = \{0, 1\}$

Defn. A set $U \subseteq \mathbb{R}$ is said to be open if every point in U is an interior point; that is, $U^{int} = U$.
On the other hand, U is closed if U^c is open.

Note1: A set can be neither open nor closed.
For example, $U = (0,1]$, $U^c = (-\infty, 0] \cup (1, \infty)$.

Note2: A set can be both open and closed? Yes!
$\mathbb{R}$ is both open and closed.
① $\mathbb{R}$ is open.
② $\mathbb{R}^c = \emptyset$. Thus, $\mathbb{R}^c$ is open.
$\emptyset$ is also both open and closed.

$\Rightarrow$ only $\emptyset$ and $\mathbb{R}$ are "clopen" in $\mathbb{R}$

Prop.: ① $\mathbb{R}$ and $\emptyset$ are both open.
② If $\{U_i\}_{i \in I}$ is an arbitrary collection of open sets, then $\bigcup_{i \in I} U_i$ is open. I does not have to be countable.

Pick some $x \in \bigcup_{i \in I} U_i$, so $x \in U_{i_0}$ for some $i_0 \in I$.
Now U_{i_0} is open, so $\exists\, r > 0$ such that
$B_r(x) \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i$.

③ If $U_1, \dots, U_n$ is a finite collection of open sets, then $\bigcap_{i=1}^{n} U_i$ is open. If $\bigcap_{i=1}^{n} U_i = \emptyset$ we're done, so assume it's not empty.
Let $x \in \bigcap_{i=1}^{n} U_i$, so $x \in U_i$ for all $i = 1, \dots, n$.
Since each U_i is open, $\exists\, r_i > 0$ such that
$B_{r_i}(x) \subseteq U_i$. Set $r = \min\{r_1, \dots, r_n\}$.
Now $B_r(x) \subseteq B_{r_i}(x) \subseteq U_i$ so $B_r(x) \subseteq \bigcap_{i=1}^{n} U_i$,
so x is an interior point.