Thm. "Limits are unique"

If $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} f(x) = M$ then $L = M$.

Proof.
Convince yourself that if $|a| < \varepsilon$ for all $\varepsilon > 0$, then $a = 0$.

Fix some ε, and choose $\delta_1, \delta_2 > 0$ such that
if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \varepsilon/2$
and $0 < |x - c| < \delta_2$, then $|f(x) - M| < \varepsilon/2$.

Want to show $|L - M| < \varepsilon, \forall \varepsilon > 0$.

$$|L - M| = |L - f(x) + f(x) - M| \leq |L - f(x)| + |f(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, $|L - M| < \varepsilon$ for all $\varepsilon > 0$, hence $L = M$. ■

Thm. "Function does not explode at $x = c$"

Suppose $\lim_{x\to c} f(x)$ exists and $f(c)$ is defined. Then there exists some $\rho > 0$ such that f is bounded on $(c - \rho, c + \rho)$.

Proof.
Set $\varepsilon = 1$, and let δ be such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$. Taking $\rho = \delta$ we have $L - 1 < f(x) < L + 1$ for all $x \in (c - \delta, c + \delta) \setminus \{c\}$.

Set $M = max\{|L - 1|, |L + 1|, f(c)\}$ so that $|f(x)| \leq M$ for all $x \in (c - \delta, c + \delta)$. ■

Locally bounded function

A function is *locally bounded* at $x \in D$ if $\exists e > 0$ such that f is bounded on $(x - e, x + e)$.

Def. Nowhere locally bounded function.

These functions exist. For example, $f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \text{smallest denominator} & \text{if } x \in \mathbb{Q} \end{cases}$

LIMIT LAWS

Suppose f, g are defined in a deleted open interval of a point c, and both $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ exist.

1. $(\forall\alpha \in \mathbb{R})[\lim_{x\to c}[\alpha f(x)] = \alpha L]$
2. $\lim_{x\to c}[f(x) + g(x)] = L + M$
3. $\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot M$
4. $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$

Proof. "Limit sum LAW"
Let $\varepsilon > 0$ be given and pick $\delta_f > 0$ and $\delta_g > 0$ so that

$$0 < |x - c| < \delta_f \Rightarrow |f(x) - L| < \varepsilon/2$$

$$0 < |x - c| < \delta_g \Rightarrow |g(x) - M| < \varepsilon/2$$

Set $\delta = \min\{\delta_f, \delta_g\}$ so that if $0 < |x - c| < \delta$ then both facts about deltas are true. Thus, suppose $0 < |x - c| < \delta$.

$$|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Proof. "Limit product LAW"
Rough work:

$$|f(x)g(x) - LM|$$

$$|f(x)g(x) - g(x)L + g(x)L - ML|$$

$$|g(x)[f(x) - L] + L[g(x) - M]|$$

$$\leq |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M|$$

Real proof:

Since $\lim_{x\to c} g(x)$, we know that g is locally bounded, so fix ρ, N such that $0 < |x - c| < \rho \Rightarrow |g(x)| < N$. Choose $\delta_f, \delta_g > 0$ so that

$$0 < |x - c| < \delta_f \Rightarrow |f(x) - L| < \frac{\varepsilon}{2N}$$

$$0 < |x - c| < \delta_g \Rightarrow |g(x) - M| < \frac{\varepsilon}{2|L|}$$

Set $\delta = \min\{\delta_f, \delta_g, \rho\}$ so that all is true. Hence if $0 < |x - c| < \delta$ then from "rough work"

$$|f(x)g(x) - LM| \leq |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M| < N \cdot \frac{\varepsilon}{2N} + |L| \cdot \frac{\varepsilon}{2|L|} = \varepsilon$$

"Limit of a polynomial"

If p is a polynomial $[p \in \mathbb{R}[x]]$ then $\lim_{x \to c} p(x) = p(c)$, for all $c \in \mathbb{R}$.

Proof.
Write $p(x) = \sum_{k=0}^{n} a_k x^k$, and remember that $\lim_{x \to c} x = c$. Thus,

$$\lim_{x \to c} p(x) = \lim_{x \to c} \sum_{k=0}^{n} a_k x^k = \sum_{k=0}^{n} \lim_{x \to c} \left[a_k x^k\right] = \sum_{k=0}^{n} a_k \left[\lim_{x \to c} x^k\right] = \sum_{k=0}^{n} a_k \left[\lim_{x \to c} x\right]^k = \sum_{k=0}^{n} a_k c^k = p(c)$$

Thm. "Squeeze Theorem"

Suppose $f, g, h : D \to \mathbb{R}$ and $f(x) \leq g(x) \leq h(x)$ for all $x \in D$. If $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} g(x) = L$.

Proof.
Let $\varepsilon > 0$ be given and choose $\delta_f, \delta_h > 0$ such that

$$0 < |x - c| < \delta_f \Rightarrow |f(x) - L| < \varepsilon \Leftrightarrow L - \varepsilon < f(x) < L + \varepsilon$$

$$0 < |x - c| < \delta_h \Rightarrow |h(x) - L| < \varepsilon \Leftrightarrow L - \varepsilon < h(x) < L + \varepsilon$$

Set $\delta = \min\{\delta_f, \delta_h\}$, so if $0 < |x - c| < \delta$ then $L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$, so

$$L - \varepsilon < g(x) < L + \varepsilon \Leftrightarrow |g(x) - L| < \varepsilon$$

■

Problem "Squeeze theorem example 1"
Show that $\lim_{x \to c} |f(x)| = 0 \Leftrightarrow \lim_{x \to c} f(x) = 0$.

Since $0 \leq |f(x)|$ we can use 0 as a lower bound for the squeeze theorem.

Problem "Squeeze theorem example 2"
Show that $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Proof.
Since $|\sin\left(\frac{1}{x}\right)| \leq 1$, we know $0 \leq |x \sin\left(\frac{1}{x}\right)| \leq |x|$.

By the Squeeze theorem, since $\lim_{x \to 0} 0 = \lim_{x \to 0} |x| = 0$, we know $\lim_{x \to 0} \left|x \sin\left(\frac{1}{x}\right)\right| = 0$, and hence we have $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Thm. "Zero limit of sin" that gives us limits of all trig functions.

$\lim_{x\to 0} \sin(x) = 0$

Proof.
Suppose $x \in \left[0, \frac{\pi}{2}\right)$. Thus $0 \le |\sin(x)| \le |x|$ if $x \in \left[0, \frac{\pi}{2}\right)$.

This is also true if $x \in \left(-\frac{\pi}{2}, 0\right]$, so $0 \le |\sin(x)| \le |x|$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. By the Squeeze Theorem, $\lim_{x\to 0} \sin(x) = 0$.

Corollary. "Zero limit of cos"

Note that $\lim_{x\to 0} \sin^2(x) = 0$ by the Limit Laws.

Let $\varepsilon > 0$ be given and choose $\delta_1 > 0$ such that $0 < |x| < \delta_1 \Rightarrow |\sin^2(x)| \le \varepsilon$.

Set $\delta = \min(\delta_1, \pi/4)$, and note that if $0 < |x| < \delta$, then, since $\cos(x) \ge 0$,

$$\cos(x) - 1 = (\cos(x) - 1) \cdot \frac{\cos(x) + 1}{\cos(x) + 1} = \frac{\cos^2(x) - 1}{\cos(x) + 1} = -\frac{\sin^2(x)}{\cos(x) + 1} \le -\sin^2(x)$$

$$|\cos(x) - 1| \le |\sin^2(x)| < \varepsilon$$

Corollary. "Limit of sin and cos"

$\lim_{x\to c} \sin(x) = \sin(c)$ and $\lim_{x\to c} \cos(x) = \cos(c)$.

Proof.
Recall that $\lim_{x\to c} \sin(x) = \lim_{x\to 0} \sin(x + c)$, so

$$\lim_{x\to 0} \sin(x + c) = \lim_{x\to 0} [\sin(x)\cos(c) + \cos(x)\sin(c)] = \cos(c)\left[\lim_{x\to 0} \sin(x)\right] + \sin(c)\left[\lim_{x\to 0} \cos(x)\right]$$

$$\lim_{x\to c} \sin(x) = \lim_{x\to 0} \sin(x + c) = \sin(c)$$