Problem 3.
A function $f : B \to C$ is said to be *courageous* if whenever $g_1, g_2 : A \to B$ are a pair of functions such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. Show that f is courageous if and only if f is injective.

Proof. ($\Longleftarrow$)
We have f as an injective function, meaning $(\forall x, y \in B)[f(x) = f(y) \Rightarrow x = y]$.
Want to prove, $(\forall g_1, g_2 : A \to B)[f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2]$.

Fix some $g_1, g_2 : A \to B$. Assume $f \circ g_1 = f \circ g_2$. Thus, $(\forall a \in A)[f(g_1(a)) = f(g_2(a))]$.
Since $(\forall x, y \in B)[f(x) = f(y) \Rightarrow x = y]$, we obtain $(\forall a \in A)[g_1(a) = g_2(a)]$, and $g_1 = g_2$. ■

Proof. ($\Longrightarrow$)
Fix a function $f : B \to C$. Assume $(\forall g_1, g_2 : A \to B)[f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2]$. Want to prove that f is injective, meaning $(\forall x, y \in B)[f(x) = f(y) \Rightarrow x = y]$.

For a contradiction, assume f is not injective, meaning $(\exists x, y \in B)[f(x) = f(y) \wedge x \neq y]$. Fix $x_0, y_0 \in B$ such that $f(x_0) = f(y_0) \wedge x_0 \neq y_0$.

Take constant functions $g_1 : A \to B, g_1(a) = x_0$, and $g_2 : A \to B, g_2(a) = y_0$.

Since $x_0 \neq y_0$, $(\forall a \in A)[g_1(a) \neq g_2(a)]$ and $g_1 \neq g_2$.
Since $f(x_0) = f(y_0) = c$, $(\forall a \in A)[f(g_1(a)) = f(g_2(a)) = c]$ and $f \circ g_1 = f \circ g_2$.

Thus, $(\exists g_1, g_2 : A \to B)[f \circ g_1 = f \circ g_2 \wedge g_1 \neq g_2]$. This is a contradiction with an original assumption that $(\forall g_1, g_2 : A \to B)[f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2]$. ■