Problem 2.
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $f'(x) \neq 0$ for all $x \in \mathbb{R}$. If

$$\lim_{x\to\pm\infty} f(x) \text{ does not exist,}$$

show that f has exactly one root.

Solution. "1st part"
First of all, let's prove that f has no more than one root. For contradiction, assume f has two or more roots. Denote two of them as a, b such that $a < b$. Then, $a < b$ and $f(a) = f(b) = 0$. According to the **MVT**, it must be true that exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. (Notice that in general, if $f(a) = f(b)$, exists $c \in (a, b)$ such that $f'(c) = 0$.) Since $(\forall x \in \mathbb{R})[f'(x) \neq 0]$ and $f'(c) = 0$, we arrived at a contradiction. Thus, f has one or zero roots.

Now, for contradiction, assume f has no roots. It will be shown that either $(\forall x \in \mathbb{R})[f(x) < 0]$ or $(\forall x \in \mathbb{R})[f(x) > 0]$. Assume that for some $x, y \in \mathbb{R}$, $f(x) < 0$ and $f(y) > 0$. According to the **IVT**, exists $c \in (\min\{x, y\}, \max\{x, y\})$ such that $f(c) = 0$. Consequently, either $(\forall x \in \mathbb{R})[f(x) < 0]$ or $(\forall x \in \mathbb{R})[f(x) > 0]$. Without loss of generality, assume $(\forall x \in \mathbb{R})[f(x) > 0]$. The other case, where $(\forall x \in \mathbb{R})[f(x) < 0]$, is analogous. Simply take $g = -f$, for which $(\forall x \in \mathbb{R})[g(x) > 0]$, and we will arrive at a contradiction for g exactly the same way.

Notice that if f has a local min (max) at some point c, then for that point c, $f'(c) = 0$. Thus, f does not have local min (max) on $\mathbb{R}$. According to the **EVT**, for any closed interval $[a, b] \subset \mathbb{R}$, exist $m, M \in [a, b]$ such that $(\forall x \in [a, b])[f(m) \leq f(x) \leq f(M)]$. If for a fixed closed interval $[a, b]$, exist $c \in (a, b)$ such that $f(c) = f(m)$ (or $f(c) = f(M)$), then c is a local min (max), because for $\rho = \min\{c - a, b - c\}$, $(\forall x \in (c - \rho, c + \rho))[f(c) \leq f(x)]$ (or $(\forall x \in (c - \rho, c + \rho))[f(c) \geq f(x)]$). Thus, $(\forall c \in (a, b))[f(c) \neq f(m) \wedge f(c) \neq f(M)]$ and it must be true that $\{m, M\} = \{a, b\}$.

Thus, it was proven that for any $[a, b] \subset \mathbb{R}$, $(\forall x \in (a, b))[f(x) \in (\min\{f(a), f(b)\}, \max\{f(a), f(b)\})]$. Let's show that f is strictly monotone. Notice that $f(0) \neq f(1)$ because of the **MVT**. Without the loss of generality, assume that $f(0) < f(1)$ and we aim to prove that f is strictly increasing. Otherwise, if $f(0) > f(1)$, take $g(x) = f(1 - x)$, for which $g(0) < g(1)$, and using the first case we will be able to show that g is strictly increasing and $f(x) = g(1 - x)$ is strictly decreasing.

Take any $x \in \mathbb{R} \setminus \{0, 1\}$.

1. If $x < 0$, then since $0 \in (x, 1)$, we have that $f(x) < f(0) < f(1)$.
2. If $0 < x < 1$, then since $x \in (0, 1)$, we have that $f(0) < f(x) < f(1)$.
3. If $1 < x$, then since $1 \in (0, x)$, we have that $f(0) < f(1) < f(x)$.

Thus, $(\forall x \in \mathbb{R})[x \leq 0 \Rightarrow f(x) \leq f(0) \wedge x \geq 0 \Rightarrow f(x) \geq f(0)]$. Take any $a, b \in \mathbb{R}$ such that $a < b$.

1. If $a < b \leq 0$, then since $0 \in (a, 1)$, we have that $f(a) < f(0) < f(1)$. Consequently, since $b \in (a, 1)$, we have that $f(a) < f(b) < f(1)$.
2. If $a \leq 0 \leq b$, then $f(a) \leq f(0)$ and $f(0) \leq f(b)$. Consequently, since $f(a) \neq f(b)$, because of the **MVT**, we have that $f(a) < f(b)$.
3. If $0 \leq a < b$, then, as $f(-1) < f(0)$, since $0 \in (-1, b)$, we have that $f(-1) < f(0) < f(b)$. Consequently, since $a \in (-1, b)$, we have that $f(-1) < f(a) < f(b)$.

Thus, $(\forall a, b \in \mathbb{R})[a < b \Rightarrow f(a) < f(b)]$ and f can be assumed to be strictly increasing. For the case where f is strictly decreasing take strictly increasing $g(x) = f(1 - x)$ and arrive at the contradiction the same way. Notice $g'(x) = -f'(1 - x) \neq 0$ for all $x \in \mathbb{R}$ and $\pm\infty$-limits of g also do not exist.

SOLUTION CONTINUED ON THE NEXT PAGE

Solution. "2nd part"
Notice that f is continuous, strictly increasing, and bounded from below. Let's prove that $\lim_{x\to-\infty} f(x)$ exists to achieve contradiction for the assumption that f has no roots. It will be proven that this limit is equal to $\inf[f(\mathbb{R})] = L$ (image of f, $f(\mathbb{R})$, is non-empty and bounded from below because $(\forall x \in \mathbb{R})[f(x) > 0]$).

It will be proven that $\lim\limits_{x\to-\infty} f(x) = L$ or equivalently that

$$(\forall \epsilon > 0)(\exists m \in \mathbb{R})(\forall x \in \mathbb{R})[x < m \Rightarrow |f(x) - L| < \epsilon].$$

For a contradiction, assume the opposite and fix $\epsilon > 0$ such that

$$(\forall m \in \mathbb{R})(\exists x \in \mathbb{R})[x < m \wedge (f(x) \leq (L - \epsilon) \vee f(x) \geq (L + \epsilon))].$$

Since L is also a lower bound of $f(\mathbb{R})$, $(\forall x \in \mathbb{R})[f(x) \geq L > (L - \epsilon)]$. Thus, we can only have

$$(\forall m \in \mathbb{R})(\exists x \in \mathbb{R})[x < m \wedge f(x) \geq (L + \epsilon)].$$

Assume that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) < (L + \epsilon)$. Then we can find $x_1 \in \mathbb{R}$ such that $x_1 < x_0$ and $f(x_1) \geq (L + \epsilon) > f(x_0)$. This contradicts the fact that $(\forall x, y \in \mathbb{R})[x < y \Rightarrow f(x) < f(y)]$. Thus, we have $(\forall x \in \mathbb{R})[f(x) \geq (L + \varepsilon)]$. However, this means that $(L + \epsilon)$ is a lower bound of $f(\mathbb{R})$ greater than the infimum L. Thus, we have arrived at a contradiction and $\lim_{x\to-\infty} f(x) = L$. Consequently, f must have at least one root.

To sum up, it was proven that f has no more than one root and that f has at least one root. Thus, we have finally proven that f has exactly one root. ■