Lattice based approaches

Learnable representations for languages

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Outline

This lecture

Take basic congruence based approaches and fix them.

- Start by considering a different CF approach
- Move to a mildly context-sensitive formalism to fix some efficiency issues.

Palindrome language

$L = \{\lambda, aa, aba, baab, \dots\}$

Inconvenient truth

$u \equiv_L v$ implies $u = v$

$[u] = \{u\}$

Proof

- Assume $|u| \leq |v|$
- $uu^R \in L$, so $vu^R \in L$
- So $v = uw$, and w is palindrome.
- w' is w with a changed to b and vice versa
- $uw'u^R \in L$, so $uww'u^R \in L$, so $|w| = 0$

Problem 2

CF language

$L = \{a^n b^n | n \geq 0\} \cup \{a^n b^{2n} | n \geq 0\}$

We need non-terminals that generate

- $a^n b^n$
- $a^n b^{2n}$
- ...

But *aabb* is not congruent to *aaabbb*!

Representational primitives

Strings

$[u] = \{v | v \equiv_L u\}$
The smallest possible sets
Gives us a partition of Σ^*

Contexts

$I[l, r] = \{v | lvr \in L\}$
These are the largest possible sets.
Overlap

Representational primitives

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These are the largest possible sets.
Overlap

Later we will look at:

Intersections

Given a set of contexts F
For every subset $C \subseteq F$
$C' = \{w | \forall (l, r) \in C, lwr \in L\}$
$\bigcap_{(l,r) \in C} I[l, r]$

Congruence based models

Primal

A priori rules

$[uv] \rightarrow [u][v]$
$[a] \rightarrow a$

Certain rules

$S \rightarrow [u]$

Defeasible rules

$[u] \rightarrow [v]$ iff $u \equiv_L v$

Outline

Crude approach

Single contexts

$I[l, r] = \{v | lvr \in L\}$

- $L = I[\lambda, \lambda]$

Finite set of contexts F that contains (λ, λ). Each context will be a non-terminal.

$\{a^n b^n | n \geq 0\}$

$I[a, b] = L$
$I[a, \lambda] = \{a^n b^{n+1} | n \geq 0\}$
$I[a, a] = \emptyset$
$I[a, abbb] = \{a\}$

Basic rules

Assume we have MQs

Lexical rules: certain

$[l, r] \to a$
Valid if $lar \in L$

The start symbol: a priori

$[\lambda, \lambda]$

Branching rules: defeasible

$[l_N, r_N] \to [l_P, r_P], [l_Q, r_Q]$
Valid only if $I[l_P, r_P] I[l_Q, r_Q] \subseteq I[l_N, r_N]$

Testing branching rules

Define a set of strings: K. Test whether:

$$(I[l_P, r_P] \cap K)(I[l_Q, r_q] \cap K) \subseteq I[l_N, r_N]$$

- Take every $u, v \in K$ such that $l_P u r_p, l_Q v r_q \in L$
- Test whether $l_N u v r_N \in L$

As K increases, the test gets harder.

Example

$L = \{a^n b^n | n \geq 0\}$

Suppose we have $(\lambda, b), (a, \lambda), (\lambda, \lambda)$.

- Consider $ab = a \circ b$
- $ab \in I[\lambda, \lambda]$
- $a \in I[\lambda, b]$
- $b \in I[a, \lambda]$

Example

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- So maybe we have a rule $[\lambda, \lambda] \rightarrow [\lambda, b], [a, \lambda]$

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- So maybe we have a rule $[\lambda, \lambda] \rightarrow [\lambda, b], [a, \lambda]$
- But consider $aab \circ abb$

Example

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Suppose we have $(\lambda, abb), (a, \lambda), (\lambda, \lambda)$.

- Consider $ab = a \circ b$
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Example

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- Consider $ab = a \circ b$
- $ab \in I[\lambda, \lambda]$
- $a \in I[\lambda, abb]$
- $b \in I[a, \lambda]$
- So maybe we have a rule $[\lambda, \lambda] \rightarrow [\lambda, abb], [a, \lambda]$
- $I[\lambda, abb] = \{a\}$ – so valid.

Not a partition

In congruence class methods, we have a partition, so there is a unique N, $N \rightarrow P, Q$.
Multiple overlapping rules.

- $[\lambda, \lambda] \rightarrow [\lambda, abb], [a, \lambda]$
- $[\lambda, \lambda] \rightarrow [\lambda, abb], [aab, \lambda]$
- $[\lambda, \lambda] \rightarrow [\lambda, a], [aab, \lambda]$

Not a partition

In congruence class methods, we have a partition, so there is a unique N, $N \to P, Q$.
Multiple overlapping rules.

- $[\lambda, \lambda] \to [\lambda, abb], [a, \lambda]$
- $[\lambda, \lambda] \to [\lambda, abb], [aab, \lambda]$
- $[\lambda, \lambda] \to [\lambda, a], [aab, \lambda]$
- Unary rules: $[\lambda, a] \to [\lambda, abb]$

Algorithm idea

- Increase K as much as possible – $Sub(\{w_1, \ldots, w_n\})$.
- Eventually this will remove all incorrect rules
- When we undergenerate, increase F to get more rules
- $F = Con(\{w_1, \ldots, w_n\})$.

Algorithm

Data: Input alphabet Σ

Result: A sequence of CFGs $G_1, G_2, \ldots$

$K \leftarrow \Sigma \cup \{\lambda\}$, $F \leftarrow \{(\lambda, \lambda)\}$, $E = \{\}$;
$G = \texttt{Make}(K, F)$;
repeat
    $E \leftarrow E \cup \{w\}$;
    $K \leftarrow \mathit{Sub}(E)$;
    if *there is some* $w \in E$ *that is not in* $L(G)$ **then**
        $F \leftarrow \mathit{Con}(E)$;
        $G \leftarrow \texttt{Make}(K, F)$;
    end
    else
        $G \leftarrow \texttt{Make}(K,F)$;
    end
    Output G;
until w ;

Algorithm 1: CFG learning algorithm

Example

$L = \{a^n b^n | n \geq 0\}$

Start:

- $F = \{(\lambda, \lambda)\}$
- $K = \{a, b\lambda\}$

Grammar

- $[\lambda, \lambda] \rightarrow \lambda$
- $[\lambda, \lambda] \rightarrow [\lambda, \lambda], [\lambda, \lambda]$

$L(G_0) = \{\lambda\}$

Example

$L = \{a^n b^n | n \geq 0\}$

Step 1: receive example *ab*, not in current L.

- $K = \{\lambda, a, b, ab\}$
- $F = \{(\lambda, \lambda), (a, \lambda), (\lambda, b), (a, b)\}$

Lexical rules

- $[\lambda, \lambda] \rightarrow \lambda$
- $[\lambda, b] \rightarrow a$ and $[a, \lambda] \rightarrow b$

64 possible branching rules
But $[a, b]$ the same as $[\lambda, \lambda]$ so 27

Example

$L = \{a^n b^n | n \geq 0\}$

Step 1: receive example *ab*, not in current L.

- $K = \{\lambda, a, b, ab\}$
- $F = \{(\lambda, \lambda), (a, \lambda), (\lambda, b), (a, b)\}$

- $I[\lambda, \lambda] \cap K = \{\lambda, ab\}$
- $I[a, \lambda] \cap K = \{b\}$
- $I[\lambda, b] \cap K = \{a\}$
- $I[a, b] \cap K = \{\lambda, ab\}$

Example

$L = \{a^n b^n | n \geq 0\}$

Group 1

$[\lambda, \lambda] \to X, Y.$

	$[\lambda, \lambda]$	$[a, \lambda]$	$[\lambda, b]$
$[\lambda, \lambda]$	$ab \circ ab$	$\lambda \circ b$	$\lambda \circ a$
$[a, \lambda]$	$b \circ \lambda$	$b \circ b$	$b \circ a$
$[\lambda, b]$	$a \circ \lambda$	YES?	$a \circ a$

$$(I[\lambda, b] \cap K)(I[a, \lambda] \cap K) = \{a\}\{b\} = \{ab\} \subseteq I[\lambda, \lambda]$$

(Wrong!)

Example

$L = \{a^n b^n | n \geq 0\}$

Group 2

$[a, \lambda] \rightarrow X, Y.$

	$[\lambda, \lambda]$	$[a, \lambda]$	$[\lambda, b]$
$[\lambda, \lambda]$	$\lambda \circ \lambda$	YES?	$\lambda \circ a$
$[a, \lambda]$	$b \circ ab$	$b \circ b$	$b \circ a$
$[\lambda, b]$	$a \circ \lambda$	$a \circ b$	$a \circ a$

Example

$L = \{a^n b^n | n \geq 0\}$

Group 3

$[\lambda, b] \to X, Y.$

	$[\lambda, \lambda]$	$[a, \lambda]$	$[\lambda, b]$
$[\lambda, \lambda]$	$\lambda \circ \lambda$	$\lambda \circ b$	$ab \circ a$
$[a, \lambda]$	$b \circ ab$	$b \circ b$	$b \circ a$
$[\lambda, b]$	YES?	$a \circ b$	$a \circ a$

Example

$L = \{a^n b^n | n \geq 0\}$

Resulting grammar has rules:

- $[\lambda, \lambda] \rightarrow [\lambda, b], [a, \lambda]$
- $[a, \lambda] \rightarrow [\lambda, \lambda], [a, \lambda]$
- $[\lambda, b] \rightarrow [\lambda, b], [\lambda, \lambda]$

These are all incorrect.
Lexical rules:

- $[\lambda, \lambda] \rightarrow \lambda$
- $[\lambda, b] \rightarrow a$ and $[a, \lambda] \rightarrow b$

$L(G_1) = \{\lambda, ab, aabb, aababb \dots\}$

Example

$L = \{a^n b^n | n \geq 0\}$

- Next positive datum: *aabb*
- K includes *aab* and *abb* which knocks out these rules.
- Increase F to include $(\lambda, abb), (aab, \lambda)$

Example

$L = \{a^n b^n | n \geq 0\}$

- Next positive datum: *aabb*
- K includes *aab* and *abb* which knocks out these rules.
- Increase F to include $(\lambda, abb), (aab, \lambda)$
- Correct rules:
  - $[\lambda, \lambda] \to [\lambda, b][aab, \lambda]$
  - $[\lambda, \lambda] \to [\lambda, abb][aab, \lambda]$
  - $[\lambda, \lambda] \to [\lambda, abb][a, \lambda]$
  - $[\lambda, b] \to [\lambda, abb][\lambda, \lambda]$ etc.

Example

$$L_{nd} = \{a^n b^n c^m | n, m \geq 0\} \cup \{a^m b^n c^n | n, m \geq 0\}$$

$L(G, N)$	F_N
λ	$(aaabb, bccc)$
a	$(\lambda, abbccc)$
b	$(aaab, bccc)$
c	$(aaabbc, \lambda)$
c^*	(c, λ)
a^*	(λ, a)
$\{a^n b^n \| n \geq 0\}$	(a, b)
$\{a^n b^{n+1} \| n \geq 0\}$	(aa, b)
$\{b^n c^n \| n \geq 0\}$	(b, c)
$\{b^n c^{n+1} \| n \geq 0\}$	(bb, c)
L_{nd}	(λ, λ)

Classes are too large

Intermediate model

Fix a constant $f = 1, 2, \ldots$

Consider all subsets of F of at most f contexts

Polynomial size $|F|^f$

Very crude: doesn't really solve the problem.

Special case $f = 2$

Palindrome language

- $(\lambda, ba), (\lambda, aa)$ defines $\{a\}$
- $[(\lambda, \lambda)] \to [(\lambda, ba), (\lambda, aa)], [(a, \lambda)]$

$a^n b^n \cup a^n b^{2n}$

- $(\lambda, \lambda), (a, b)$ define $a^n b^n$
- $(\lambda, \lambda), (a, bb)$ define $a^n b^{2n}$

Outline

Introduction

Dual CFG model

- Single context
- Multiple contexts

Lattice approaches

- Linguistic justification
- Theory
- Formal concept analysis

Syntactic Concept Lattice

Background

- We can try all possible generalisations defined by any possible set of features
- use lattice theory
- Mildly context-sensitive approach

Sets of strings and contexts

Ambiguity

Simplified example

Suppose $N \stackrel{*}{\Rightarrow} u$, $M \stackrel{*}{\Rightarrow} v$
And $N \stackrel{*}{\Rightarrow} w$ $M \stackrel{*}{\Rightarrow} w$

Then we might have $C_L(w) = C_L(u) \cup C_L(v)$.

Motivation I

"can" is ambiguous:

- A can of beans
- I can see that

Motivation I

"can" is ambiguous:

- A can of beans
- I can see that

Distribution of "can" will be (roughly) the **union** of the distribution of count nouns and distribution of auxiliary verbs.

Motivation II

"may" is ambiguous:

- I like to go to the beach in may
- I may go to the beach

Motivation II

“may” is ambiguous:

- I like to go to the beach in may
- I may go to the beach

Distribution of auxiliary verbs is (roughly) the **intersection** of the distribution of “may” and “can”.

Rulon Wells

Immediate Constituents, Language 1947

It is easy to define a focus-class embracing a large variety of sequence classes but characterized by only a few environments; it is also easy to define one characterized by a great many environments in which all its members occur but on the other hand poor in the number of diverse sequence-classes that it embraces. What is difficult, but far more important than either of the easy tasks, is to define focus-classes rich both in the number of environments chracterizing them and at the same time in the diversity of sequence classes that they embrace.

- Concepts high up in the lattice have a few contexts, but lots of strings
- Concepts low down have a larger number of contexts, but only a few strings.

Key points

- We are dealing with sets of strings, sets of contexts
- These will overlap in very complex ways
- We may have words whose distributions are combinations of distributions of more primitive elements.
- The appropriate structure is a lattice.

Partially ordered sets

Definition

A set with a relation $\leq$ that is

transitive $X \leq Y \leq Z$ means $X \leq Z$

reflexive $X \leq X$

antisymmetric $X \leq Y$ and $Y \leq X$ means $X = Y$

Examples

- numbers with normal partial order
- sets with $X \leq Y$ iff $X \subseteq Y$
- sets with $X \leq Y$ iff $Y \subseteq X$

Lattice

Not all posets are lattices.
Two operations:

- $\vee$, least upper bound, join
- $\wedge$, greatest lower bound meet

Basic axioms

$$X \wedge X = X \quad (1)$$

$$X \wedge Y = Y \wedge X \quad (2)$$

$$(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z) \quad (3)$$

$$X \vee (X \wedge Y) = X \quad (4)$$

$$(5)$$

Complete lattice has $\top$ and $\bot$, that are identities for the two operations.

General theory

Formal concept analysis:

- Set of strings
- Set of contexts

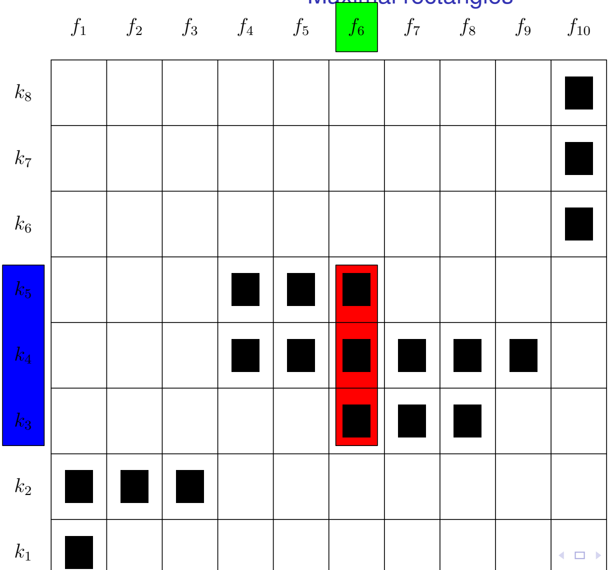
Concepts

Maximal rectangles

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
k_8										■
k_7										■
k_6										■
k_5				■	■	■				
k_4				■	■	■	■	■	■	
k_3						■	■	■		
k_2	■	■	■							
k_1	■									

Concepts

Maximal rectangles



Concepts

Maximal rectangles

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
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k_5				■	■	■				
k_4				■	■	■	■	■	■	
k_3						■	■	■		
k_2	■	■	■							
k_1	■									

Concepts

Maximal rectangles

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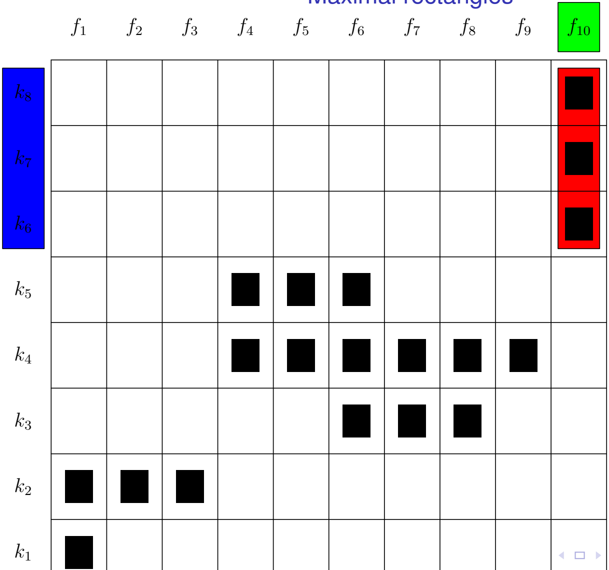
k_3						■	■	■		
k_2	■	■	■							
k_1	■									

Concepts

Maximal rectangles



Concepts

Maximal rectangles

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
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k_4				■	■	■	■	■	■	
k_3						■	■	■		
k_2	■	■	■							
k_1	■									

Concepts

Maximal rectangles

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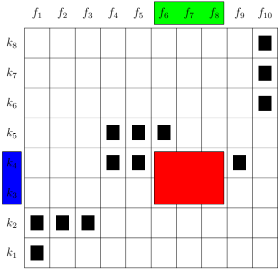
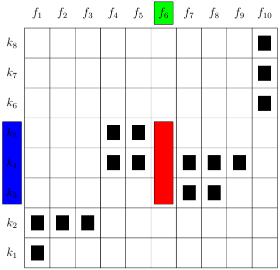
k_2	■	■	■							
k_1	■									

Partial order





Top and bottom

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
k_8										■
k_7										■
k_6										■
k_5				■	■	■				
k_4				■	■	■	■	■	■	
k_3						■	■	■		
k_2	■	■	■							
k_1	■									

Top and bottom

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
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k_5				■	■	■				
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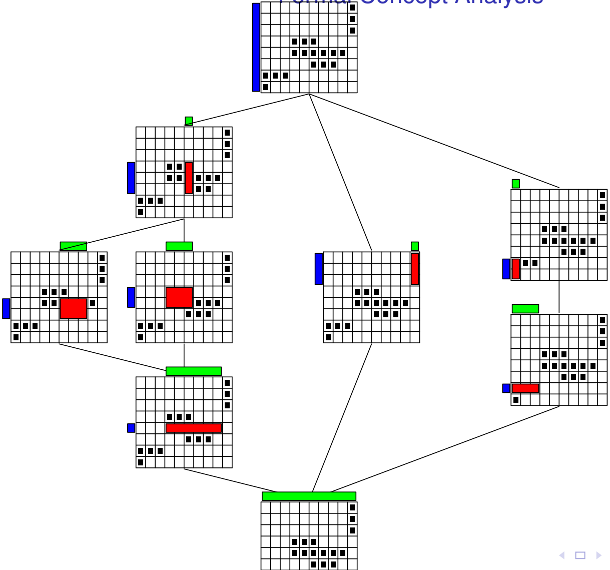
k_3						■	■	■		
k_2	■	■	■							
k_1	■									

Complete Lattice

Formal Concept Analysis



Polar maps

S is a set of strings, and C is a set of contexts.

Polar maps

$S' = \{(l, r) : \forall w \in S \;\; lwr \in L\}$
$C' = \{w : \forall (l, r) \in C \;\; lwr \in L\}$

$L = \{(\lambda, \lambda)\}'$

Concept

A syntactic concept is an ordered pair $\langle S, C \rangle$.
where $C' = S$ and $S' = C$.
Equivalently a maximal pair such that $C \odot S \subseteq L$.

Basic operations

Basic

If $S \subseteq T$, then $S' \supseteq T'$

$S \subseteq S''$

Lemma

$S''' = S', C''' = C'$

Proof: $S \subseteq S''$ so $S' \supseteq S'''$

$(S') \subseteq (S')''$

Closure operator

If $S = S''$ then S is closed.
If $\langle S, C \rangle$ is a concept, then S and C are both closed.
We can consider it as the lattice of closed sets of strings, or the lattice of closed sets of contexts, or both together.

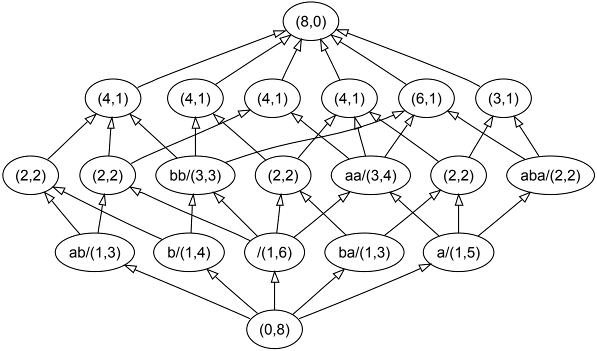
Lattice

Palindrome language over *a*, *b*

	(λ, λ)	(a, λ)	(ab, λ)	(aa, λ)	(b, λ)	(ba, λ)	(λ, b)	(λ, a)
aba	■		■					
bb	■				■		■	
ba		■	■				■	
ab					■	■		■
aa	■	■		■				■
b	■				■	■	■	
a	■	■	■	■				■

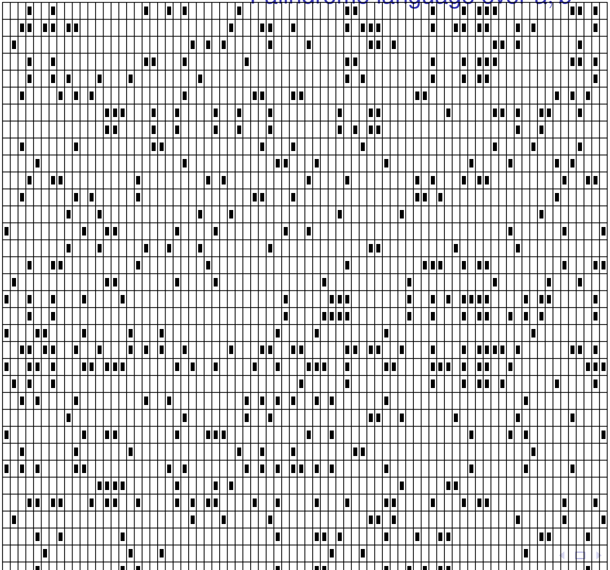
Lattice

Many rectangles

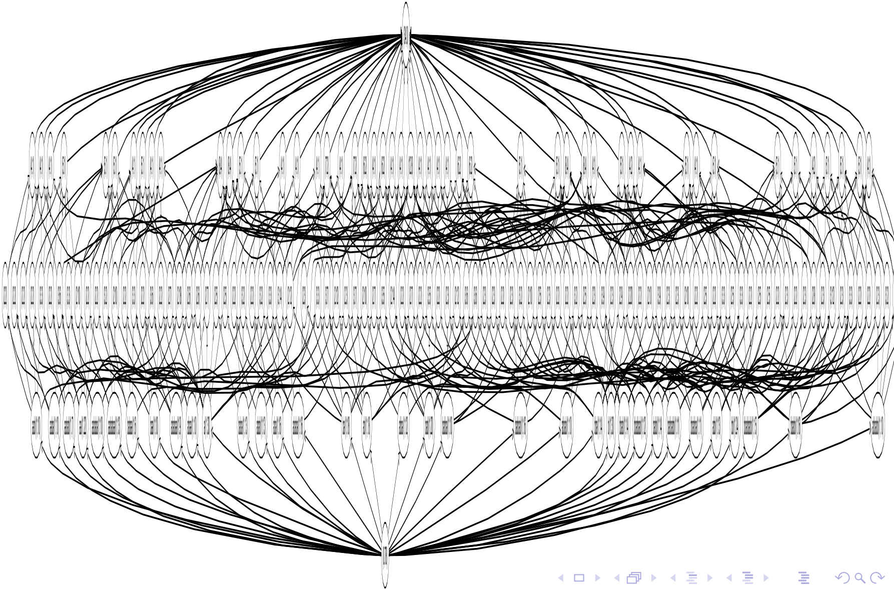


Lattice

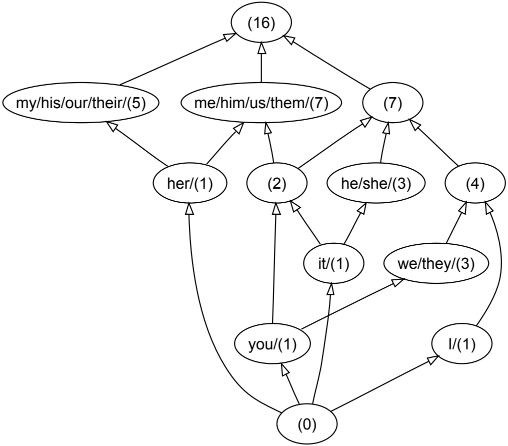
Palindrome language over *a*, *b*



Lattice



Linguistic concepts



Formally

Polar maps

$S' = \{(l, r) \in F : \forall w \in S \;\; lwr \in L\}$
$C' = \{w \in K : \forall (l, r) \in C \;\; lwr \in L\}$

Concept

Ordered pair $\langle S, C \rangle$

- $S \subseteq K$ the set of strings
- $C \subseteq F$ is a set of contexts

$S' = C$ and $C' = S$

$\mathcal{C}(S) = \langle S'', S' \rangle$

Relation to CFGs

Define

Given a CFG G for each non-terminal N

- Yield: $Y(N) = \{w | N \stackrel{*}{\Rightarrow} w\}$
- Contexts: $C(N) = \{(l, r) | S \stackrel{*}{\Rightarrow} lNr\}$.

Clearly $C(N) \odot Y(N) \subseteq L$
Each non-terminal will be a rectangle – but not necessarily maximal.

Technical detail

- These rectangles are "concepts" which form a complete lattice $\mathfrak{B}(K, L, F)$
- We use a concatenation operation $X \circ Y$ and a lower bound $X \wedge Y$.

Concatenation

$\langle S_1, C_1 \rangle \circ \langle S_2, C_2 \rangle = \langle (S_1 S_2)'', (S_1 S_2)''' \rangle$
Given two sets of strings S_x, S_y
Concatenate them $S_x S_y$
Find the shared set of contexts C_{xy}
Result is the concept defined by C'_{xy}

Dyck language

λ, ab, abab, aabb, abaabb ...

	(λ,λ)	(a,λ)	(λ,b)
λ	■		
a			■
b		■	
ab	■		

- $L = \langle\{\lambda, ab\}, (\lambda,\lambda)\rangle$
- $A = \langle\{a\}, (\lambda, b)\rangle$
- $B = \langle\{b\}, (a, \lambda)\rangle$
- $\top$
- $\bot$

Dyck language

- $L = \langle \{\lambda, ab\}, (\lambda, \lambda) \rangle$
- $A = \langle \{a\}, (\lambda, b) \rangle$
- $B = \langle \{b\}, (a, \lambda) \rangle$
- $\top$
- $\bot$

	$\top$	L	A	B	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
L	$\top$	L	A	B	$\bot$
A	$\top$	A	$\top$	L	$\bot$
B	$\top$	B	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Goal

Predict which concept a string is in:

- Define function $\phi : \Sigma^* \rightarrow \mathfrak{B}(K, L, F)$
- A string w is in the language if $\phi(w)$ has the context (λ, λ).
- We want $\phi(w) = \langle S, C \rangle$ to mean that $C_L(w) \cap F = C$.

Recursive definition

- $\phi(a) = \mathcal{C}(a)$ (look it up)
- $\phi(ab) = \phi(a) \circ \phi(b)$
- $\phi(abc) = \phi(ab) \circ \phi(c)$, OR $\phi(a) \circ \phi(bc)$

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Distributional lattice grammars

Derivation: efficient $\mathcal{O}(|w|^3)$ algorithm

Definition

A distributional lattice grammar (DLG) is a tuple $\langle K, D, F \rangle$

- K is a finite subset of strings that includes Σ and λ
- F is a finite set of contexts that includes (λ, λ)
- D is a finite subset of $F \odot KK$

Definition

$\phi : \Sigma^* \to \mathfrak{B}(K, D, F)$.

- for all $a \in \Sigma$, $\phi(a) = \mathcal{C}(a)$
- for all w with $|w| > 1$,

$$\phi(w) = \bigwedge_{u,v \in \Sigma^+ : uv = w} \phi(u) \circ \phi(v)$$

Derivation example

Dyck language

$\{\lambda, ab, aabb, abab, aaababbb \ldots\}$
$F = \{(\lambda, \lambda), (\lambda, b), (a, \lambda)\}$
$K = \{\lambda, a, b, ab\}$

- $\top = \langle K, \emptyset \rangle$
- $\bot = \langle \emptyset, F \rangle$
- $\mathbf{L} = \langle \{\lambda, ab\}, \{(\lambda, \lambda)\} \rangle$
- $A = \langle \{a\}, \{(\lambda, b)\} \rangle$
- $B = \langle \{b\}, \{(a, \lambda)\} \rangle$

Derivation example

Dyck language

$\{\lambda, ab, aabb, abab, aaababbb \dots\}$
$F = \{(\lambda, \lambda), (\lambda, b), (a, \lambda)\}$
$K = \{\lambda, a, b, ab\}$

1. $\phi(a) = A$
2. $\phi(ab) = \phi(a) \circ \phi(b) = \mathbf{L}$, $\phi(aa) = \top$ **...**
3. $\phi(aab) = (\phi(a) \circ \phi(ab)) \wedge (\phi(aa) \circ \phi(b)) = A \wedge \top = A$
4. **...**
5. $\phi(aaababbb) = \phi(a) \circ \phi(aababbb) \wedge \cdots = \mathbf{L}$

Example

$L = \{a^n b^n c^m | n, m \geq 0\} \cup \{a^m b^n c^n | n, m \geq 0\}$

	(λ, λ)	$(aaabb, bccc)$	$(aaabbc, \lambda)$	$(\lambda, abbccc)$	$(aaab, bccc)$	(aa, bbc)	$(aa, bbbc)$	(abb, cc)	$(abbb, cc)$
bcc									■
aab							■		
$bbcc$	■							■	
bc	■							■	
abc	■								
$aabb$	■					■			
ab	■					■			
c	■		■						■
b					■				
a	■			■			■		

Example

$L = \{a^n b^n c^m | n, m \geq 0\} \cup \{a^m b^n c^n | n, m \geq 0\}$

	(aa, bbc)	(λ, λ)	(abb, cc)	$(aaabb, bccc)$	$(aaabbc, \lambda)$	$(\lambda, abbccc)$	$(aaab, bccc)$	$(aa, bbbc)$	$(abbb, cc)$
bcc									■
aab								■	
abc		■							
$aabb$	■	■							
ab	■	■							
λ	■	■	■	■					
$bbcc$		■	■						
bc		■	■						
c		■			■				■
b							■		

Example

$L = \{a^n b^n c^m | n, m \geq 0\} \cup \{a^m b^n c^n | n, m \geq 0\}$

	(aa, bbc)	(λ, λ)	(abb, cc)	(aaabb, bccc)	(aaabbc, λ)	(λ, abbccc)	(aaab, bccc)	(aa, bbbc)	(abbb, cc)
bcc									■
aab								■	
abc		■							
aabb	■	■							
ab	■	■							
λ	■	■	■	■					
bbcc		■	■						
bc		■	■						
c		■			■				■
b							■		

Example

$L = \{a^n b^n c^m | n, m \geq 0\} \cup \{a^m b^n c^n | n, m \geq 0\}$

	(aa, bbc)	(λ, λ)	(abb, cc)	(aaabb, bccc)	(aaabbc, λ)	(λ, abbccc)	(aaab, bccc)	(aa, bbbc)	(abbb, cc)
bcc									■
aab								■	
abc		■							
aabb	■	■							
ab	■	■							
λ	■	■	■	■					
bbcc		■	■						
bc		■	■						
c		■			■				■
b							■		

Learnability I

Search

Language is defined by choice of K and F
How can we find suitable K and F?

Lemma 1

as we increase K the language defined by $\langle K, L, F\rangle$ decreases monotonically
It will always converge to a subset of L in a finite time

Lemma 2

As we increase the set of contexts F the language monotonically increases.
Any sufficiently large set of contexts will do.

Search problem is trivial

Naive Algorithm

Start with $F = \{(\lambda, \lambda)\}, K = \Sigma \cup \{\lambda\}$

- If we see a string that is not in our hypothesis, the hypothesis is too small, and we add contexts to F
- Add strings to K if it will change the lattice at all.

Clark, (CoNLL, 2010)

DLGs can be learnt from positive data and MQs
Polynomial update time

Power of Representation

Language class

Let $\mathcal{L}$ be the set of all languages L such that there is a *finite* set of contexts F s.t. $L = L(\mathfrak{B}(\Sigma^*, L, F))$

Learnable class includes

1. All regular languages
2. Some but not all CFLs (all the examples so far)
3. Some non context free languages

Not in DLG?

$L = \{a^n b | n > 0\} \cup \{a^n c^m | m > n > 0\}$

	(a, λ)	(aa, λ)	(aaa, λ)	(a^4, λ)	(a^5, λ)	(a^6, λ)	(a^7, λ)
c^9	■	■	■	■	■	■	■
c^8	■	■	■	■	■	■	■
c^7	■	■	■	■	■	■	■
c^6	■	■	■	■	■	■	■
c^5	■	■	■	■	■	■	
c^4	■	■	■	■	■		
ccc	■	■	■	■			
cc	■	■	■				
c	■	■					
b	■	■	■	■	■	■	■

Context sensitive example

MIX language variant

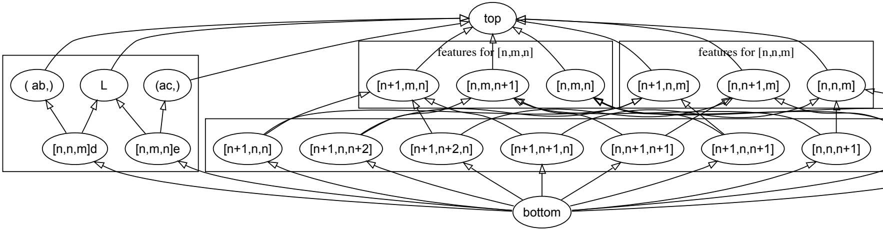
Let $M = \{(a, b, c)^*\}$, we consider the language $L = L_{abc} \cup L_{ab} \cup L_{ac}$ where

- $L_{ab} = \{wd | w \in M, |w|_a = |w|_b\}$,
- $L_{ac} = \{we | w \in M, |w|_a = |w|_c\}$,
- $L_{abc} = \{wf | w \in M, |w|_a = |w|_b = |w|_c\}$.

$F = \{(\lambda, \lambda), (\lambda, d), (\lambda, ad), (\lambda, bd), (\lambda, e), (\lambda, ae), (\lambda, ce), (\lambda, f), (ab, \lambda), (ac, \lambda)\}$

This is a non-context-free language in the learnable class.

Lattice


top
features for [n,m,n]
features for [n,n,m]
(ab,)
L
(ac,)
[n+1,m,n]
[n,m,n+1]
[n,m,n]
[n+1,n,m]
[n,n+1,m]
[n,n,m]
[n,n,m]d
[n,m,n]e
[n+1,n,n]
[n+1,n,n+2]
[n+1,n+2,n]
[n+1,n+1,n]
[n,n+1,n+1]
[n+1,n,n+1]
[n,n,n+1]
bottom

Outline

Syntactic concept lattice

Infinite limit

Let $K \to \Sigma^*$ and $F \to \Sigma^* \times \Sigma^*$

$\mathfrak{B}(K, L, F) \to \mathfrak{B}(L)$

- $\mathfrak{B}(L)$ is the *syntactic concept lattice*
- Same construction as the Universal Automaton for regular languages
- $\mathfrak{B}(L)$ is finite iff L is regular

Basic properties

Partial order

$\langle S_1, C_1 \rangle \leq \langle S_2, C_2 \rangle$ iff $S_1 \subseteq S_2$ iff $C_1 \supseteq C_2$

Lattice

The set of concepts of a language form a complete lattice

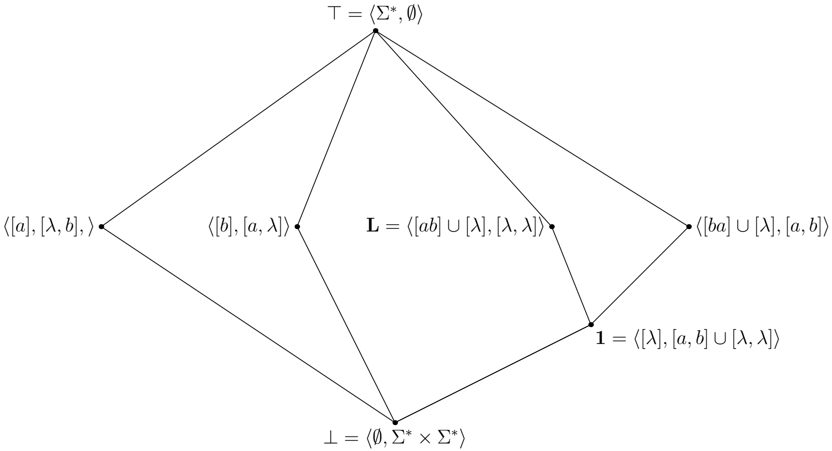
$\langle S_x, C_x \rangle \wedge \langle S_y, C_y \rangle = \langle S_x \cap S_y, (S_x \cap S_y)' \rangle$

Finite iff L is regular

Typical concepts

- Language $\langle L, \{(\lambda, \lambda)\}'' \rangle = \mathcal{C}(L) = \mathcal{C}((\lambda, \lambda))$
- Top $\top = \langle \Sigma^*, \emptyset \rangle$
- Bottom $\perp = \langle \emptyset, \Sigma^* \times \Sigma^* \rangle$
- Unit $\mathbf{1} = \mathcal{C}(\lambda)$

$L = (ab)^*$


$\top = \langle \Sigma^*, \emptyset \rangle$
$\langle [a], [\lambda, b], \rangle$
$\langle [b], [a, \lambda] \rangle$
$\mathbf{L} = \langle [ab] \cup [\lambda], [\lambda, \lambda] \rangle$
$\langle [ba] \cup [\lambda], [a, b] \rangle$
$\mathbf{1} = \langle [\lambda], [a, b] \cup [\lambda, \lambda] \rangle$
$\perp = \langle \emptyset, \Sigma^* \times \Sigma^* \rangle$

Concatenation is a monoid

Associativity

$(X \circ Y) \circ Z = X \circ (Y \circ Z)$

Lattice ordered monoid: monotonicity of concatenation w.r.t. partial order:

$X \leq Y$ then $X \circ Z \leq Y \circ Z$ etc.

Complete residuated lattice

- $X = \langle S_x, C_x \rangle$ and $Y = \langle S_y, C_y \rangle$ are concepts.
- Then define the residual $X/Y = \mathcal{C}(C_x \odot (\lambda, S_y))$
- $Y \setminus X = \mathcal{C}(C_x \odot (S_y, \lambda))$

These are unique, and satisfy the following conditions:

Lemma

$Y \leq X \setminus Z$ iff $X \circ Y \leq Z$ iff $X \leq Z/Y$.

Categorial grammar

Lambek calculus and CG

The Lambek calculus is based entirely on the theory of residuation.

Maximisation

- $Y \circ Z \leq X$

Categorial grammar

Lambek calculus and CG

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- $Y \circ Z \leq X$
- $(X/Z) \circ Z \leq X$

Categorial grammar

Lambek calculus and CG

The Lambek calculus is based entirely on the theory of residuation.

Maximisation

- $Y \circ Z \leq X$
- $(X/Z) \circ Z \leq X$
- $(X/Z) \circ (X/Z) \backslash X \leq X$
- Also have the opposite direction $X/(Z \backslash X) \circ (Z \backslash X) \leq X$

Structural descriptions from the lattice

Congruence based approaches

Get parse trees, but they are not useful.

Admissible structures for a string w

Each span has a concept $\psi[i, j]$

- $\psi[i, j] \geq \mathcal{C}(w[i : j])$
- $\psi[i, j] \geq \bigwedge_k \psi[i, k] \circ \psi[k, j]$
- $\psi[0, l] \leq \mathcal{C}(L)$

Maximal structures

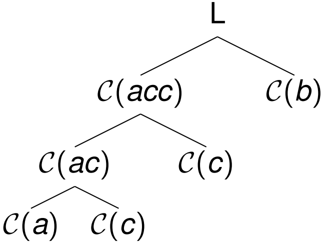
The set of maximal structures under the natural partial order can be viewed as the set of structural descriptions.
Discard $\top$ symbols and construct a graph or DAG.

Simple ambiguous language

Example

Dyck language with ambiguous symbol:
$\{ab, aabb, abab, aaababbb \ldots\}$
Add a symbol c which can be an a or a b
$\{ab, ac, cb, aabb, cccc, abab, aaacabbb \ldots\}$

Consider $accb$ – this could be $(ab)(ab)$ or $(a(ab)b)$.



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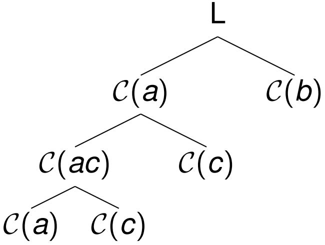

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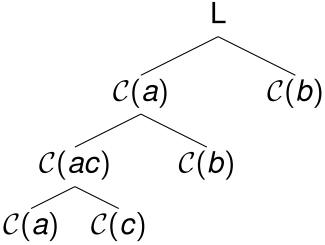

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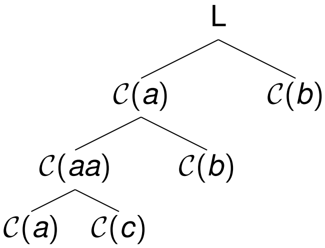

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Add a symbol *c* which can be an *a* or a *b*
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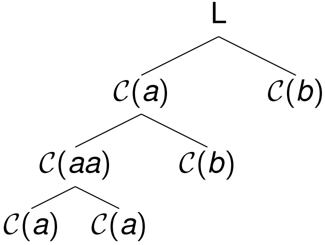

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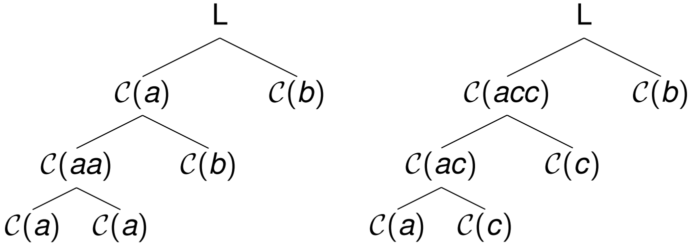

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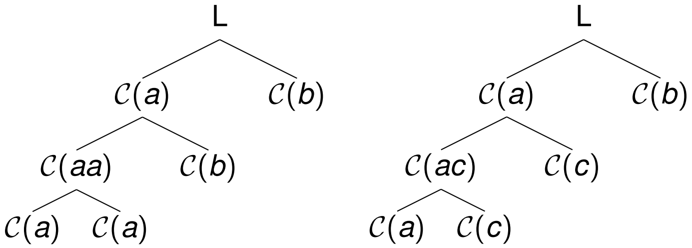

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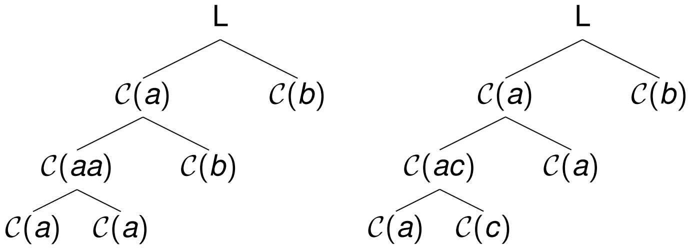

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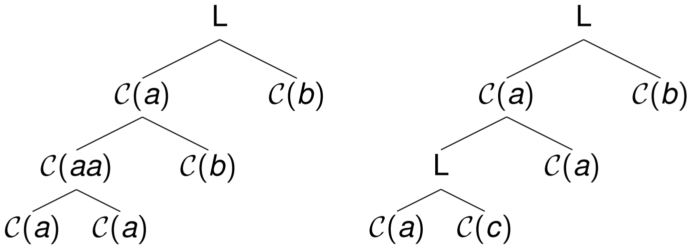

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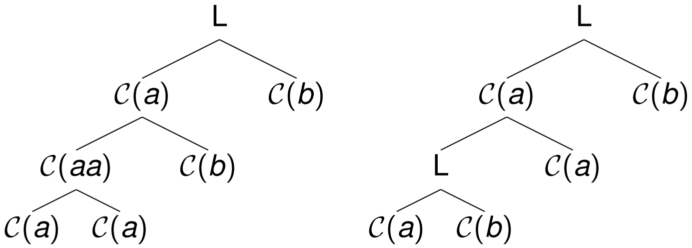

Tension between two notions of language

1950s

Abstraction - Chomsky

- Rich abstract structure that you need to model ambiguity etc.

Requires representations like CFGs, TAGs etc.

Learnability - Shannon

- Observable properties that mean you can learn

n-gram models

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n-gram models

Not incompatible

There is a very rich abstract structure which is also observable and thus learnable.

CFG from the lattice

Alternatively we can stick with a CFG

- Just consider a polynomial number of elements of $\mathfrak{B}(K, L, F)$
- Fix f, and consider only concepts formed from $F^{\leq f}$
- Rules
  - $X \rightarrow YZ$ if $X \geq Y \circ Z$.
  - $X \rightarrow Y$ if $X \geq Y$
  - $S = \mathcal{C}((\lambda, \lambda))$
  - $X \rightarrow a$ if $\mathcal{C}(a) \leq X$

Clark, ICGI, 2010

Polynomial learnability from positive data and MQs, for fixed f

Conclusion

Richly learnable class:

- The class of languages is not obviously wrong.
- The learnability model is still too weak, but we would expect probabilistic results to be obtainable.
- Switch to a more efficient context-sensitive representation in order obtain efficient results.

However, the context-sensitivity may not be strong enough.