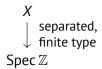
Weil-étale cohomology for n < 0

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 \downarrow separated,
 \downarrow finite type
Spec \mathbb{Z}

$$\zeta_X(s) := \prod_{\substack{X \in X \\ 1 - \#(\mathcal{O}_{X,X}/\mathfrak{m})^{-s}}} \frac{1}{1 - \#(\mathcal{O}_{X,X}/\mathfrak{m})^{-s}}. \quad (\operatorname{Re} s > \dim X)$$

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Conjecture: meromorphic continuation to $s \in \mathbb{C}$.

► Riemann: $\zeta(s) = \prod_{p} \frac{1}{1-p^{-s}} = \zeta_{\text{Spec } \mathbb{Z}}(s)$.

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- ▶ **Dedekind**: $\zeta_F(s) = \zeta_{\text{Spec } \mathcal{O}_F}(s)$ for a number field F/\mathbb{Q} .
- ▶ Hasse-Weil: X/\mathbb{F}_a , then

$$\zeta_X(s)=Z_X(q^{-s}),$$

where

$$Z_X(t) = \exp\left(\sum_{m\geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m\right) \overset{\mathsf{Dwork}}{\in} \mathbb{Q}(t).$$

(Cf. Weil conjectures.)

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- ► **Special value** (leading Taylor coefficient) at *s* = *n*:

$$\zeta_{\chi}^{*}(s) := \lim_{s \to n} (s - n)^{-d_n} \zeta_{\chi}(s).$$

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- ► Special value $\zeta_F^*(0) = -\frac{\#H^1(\operatorname{Spec} \mathcal{O}_F, \mathbb{G}_m)}{\#H^0(\operatorname{Spec} \mathcal{O}_F, \mathbb{G}_m)_{tors}} R_F$, $R_F :=$ Dirichlet regulator $\in \mathbb{R}$.

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- ▶ Formulas for other $n \in \mathbb{Z}$? \mathcal{G} \mathcal{G}

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$$\lambda \colon \mathbb{R} \xrightarrow{\cong} (\underbrace{\det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n))}_{\text{free }\mathbb{Z}\text{-mod of rk }1}) \otimes \mathbb{R}.$$

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From now on fix n < 0

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- ► Calculations: few and hard ®
- ► **Conjecture** (Lichtenbaum): $H^i(X_{\acute{e}t}, \mathbb{Z}^c(n))$ are finitely generated.

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▶ Long exact sequence of $H^{i}_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R}$: need a **regulator**.

► **Kerr–Lewis–Müller-Stach** (2006) \Longrightarrow for $X_{\mathbb{C}}$ is smooth and quasi-projective:

$$Reg \colon R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)) o R\Gamma_{BM}(G_\mathbb{R},X(\mathbb{C}),\mathbb{R}(n))[1].$$

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is a quasi-isomorphism.

► Splitting over \mathbb{R} + Beilinson's conjecture \Longrightarrow l.e.s.

$$\cdots \to H^{i}_{W,c}(X,\mathbb{Z}(n))\otimes \mathbb{R} \to H^{i+1}_{W,c}(X,\mathbb{Z}(n))\otimes \mathbb{R} \to \cdots$$

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meromorphic continuation of $\zeta_X(s)$ around s=n<0, $X_{\mathbb{C}}$ is smooth quasi-projective, Lichtenbaum's and Beilinson's conjectures.

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► Note: this would imply $d_n = \sum_i (-1)^i \dim_{\mathbb{R}} H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), \mathbb{R}(n)).$

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- ► If *X* is proper and regular, then **C**(*X*, *n*) is equivalent to the conjecture of Flach and Morin.
- (Whenever makes sense) compatible with the Tamagawa number conjecture (Bloch-Kato-Fontaine-Perrin-Riou).
- ▶ Well-behaved under decompositions: for $Z \to X \leftarrow U$ holds $\zeta_X(s) = \zeta_Z(s) \cdot \zeta_U(s)$ (obviously), and in fact

$$\mathbf{C}(X,n) \iff \mathbf{C}(Z,n) + \mathbf{C}(U,n).$$

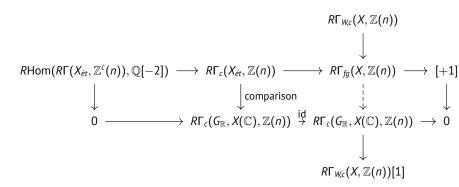
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Consider the étale sheaf $\mathbb{Z}(n) := \bigoplus_{p} \varinjlim_{r} j_{p!} \mu_{p'}^{\otimes n}[-1]$, where $j_p : X[1/p] \hookrightarrow X$.

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- ▶ A regulator for non-smooth $X_{\mathbb{C}}$?
- A less ad-hoc definition of Weil-étale complexes? Morally, there should be a Grothendieck topology behind everything.

Thank you!