Zeta-values from Euler to Weil-étale cohomology

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- Faulhaber's formula: $\sum_{1\leq i\leq n} i^k = \frac{1}{k+1}\sum_{0\leq i\leq k}\binom{k+1}{i} B_i\, n^{k+1-i}$ (**Johann Faulhaber**, *"Academia Algebræ"*, 1631; Bernoulli, 1713).

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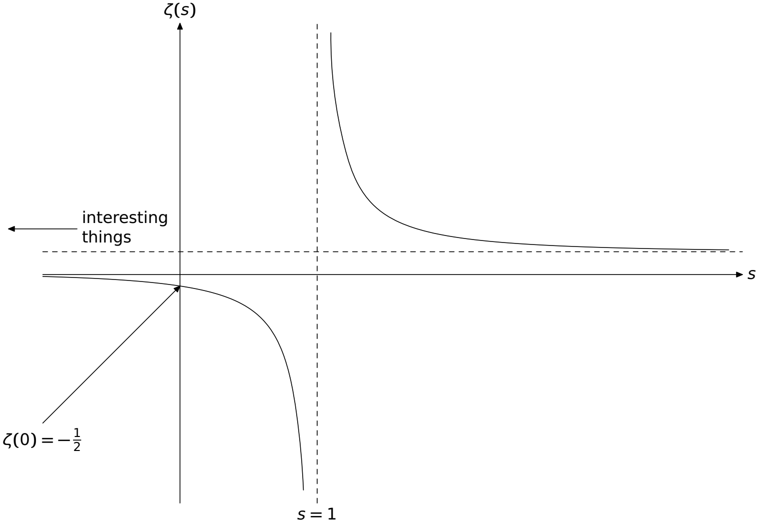
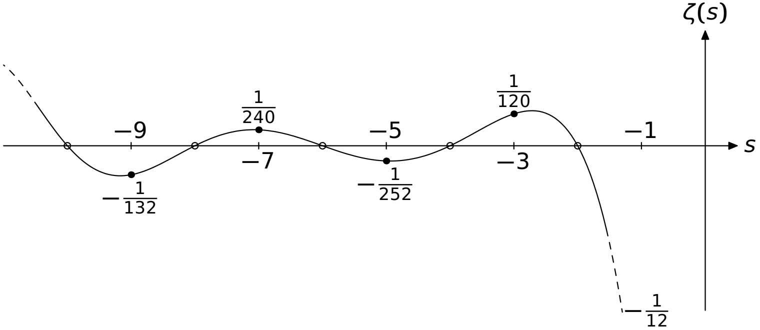
- Euler’s calculation $\zeta(2k) = (-1)^{k+1}\, B_{2k}\, \frac{2^{2k-1}}{(2k)!}\, \pi^{2k}$ becomes

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \text{ for } n = 1, 2, 3, 4, \ldots$$



ζ(s)
interesting
things
s
ζ(0) = −1/2
s = 1


$\zeta(s)$
s
-9
-7
-5
-3
-1
$\frac{1}{240}$
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- Conjecture: $\zeta(2k+1)$ are transcendental, algebraically independent.

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$$\zeta_F(s) := \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_F \\ \mathfrak{a}\neq 0}} \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{a})^s} = \prod_{\substack{\mathfrak{p}\subset\mathcal{O}_F \\ \text{prime}}} \frac{1}{1 - N_{F/\mathbb{Q}}(\mathfrak{p})^{-s}}. \qquad (\operatorname{Re} s > 1)$$

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- Note: $\zeta_{\mathbb{Q}}(s) = \zeta(s)$.

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$$\zeta_F(1-s) = |\Delta_F|^{s-1/2} \left(\cos \tfrac{\pi s}{2}\right)^{r_1+r_2} \left(\sin \tfrac{\pi s}{2}\right)^{r_2} \left(2\,(2\pi)^{-s}\,\Gamma(s)\right)^{d} \zeta_F(s),$$

where
$r_1 :=$ real places, $r_2 :=$ conjugate pairs of complex places;
$d := [F : \mathbb{Q}] = r_1 + 2\,r_2$ and $\Delta_F :=$ discriminant.

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- (Trivial) zeros:

s:	0	-1	-2	-3	-4	-5	$\cdots$
order:	$r_1 + r_2 - 1$	r_2	$r_1 + r_2$	r_2	$r_1 + r_2$	r_2	$\cdots$

Class number formula

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$$\lim_{s\to 1}(s-1)\,\zeta_F(s) = \frac{2^{r_1}\,(2\pi)^{r_2}\,\#\,\mathrm{Cl}(F)}{\#\mu_F \cdot \sqrt{|\Delta_F|}}\,R_F,$$

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- Zero at $s = 0$:

$$\lim_{s \to 0} s^{-(r_1+r_2-1)}\, \zeta_F(s) = -\frac{\#\,\mathrm{Cl}(F)}{\#\mu_F}\, R_F.$$

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- Example: $F = \mathbb{Q}$, $n = 1$, $\mathrm{Sp}_2 = \mathrm{SL}_2$,

$$\chi(\mathrm{SL}_2(\mathbb{Z})) = -\frac{1}{12} = -\frac{B_2}{2} = \zeta(-1)$$

("orbifold Euler characteristic" of $\mathcal{H}/\,\mathrm{SL}_2(\mathbb{Z})$).

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- Special values may be studied via
  K-theory $K_n(X)$ or motivic cohomology $H^i(X, \mathbb{Z}(n))$.

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$$K_0(\mathcal{C}) := \frac{\mathbb{Z}\langle \text{isomorphism classes of objects of } \mathcal{C} \rangle}{[B] = [A] + [C] \text{ for each s.e.s. } 0 \to A \to B \to C \to 0}.$$

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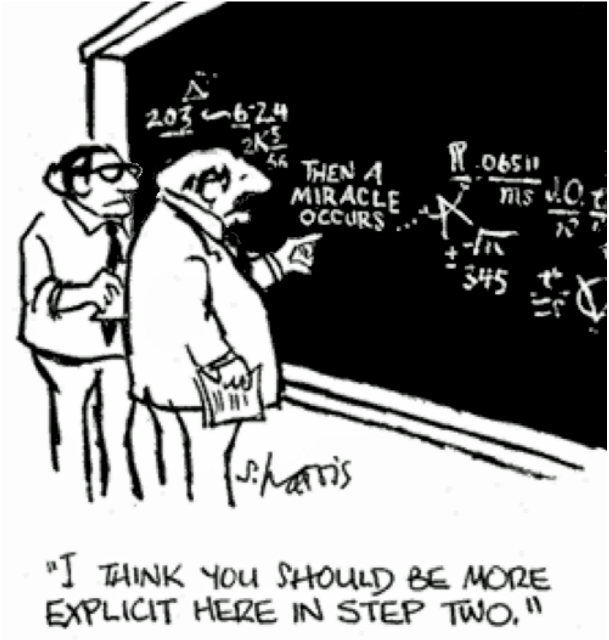
- **Quillen**, 1973:

$$K_0(\mathcal{C}) \cong \pi_1(BQ\mathcal{C}, 0),$$
$$K_n(\mathcal{C}) := \pi_{n+1}(BQ\mathcal{C}, 0);$$

Q — Quillen's "Q-construction",
B — geometric realization of the nerve.



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- Note: $\operatorname{rk} K_{2n+1}(\mathcal{O}_F) =$ order of zero of $\zeta_F(s)$ at $s = -n$.

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- **Elbaz-Vincent**, **Gangl**, **Soulé**, 2002: $K_5(\mathbb{Z}) \cong \mathbb{Z}$.

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- **Lee**, **Szczarba**, 1976: $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$.
- **Rognes**, 2000: $K_4(\mathbb{Z}) = 0$.
- **Elbaz-Vincent**, **Gangl**, **Soulé**, 2002: $K_5(\mathbb{Z}) \cong \mathbb{Z}$.
- Using the Bloch–Kato conjecture (**Voevodsky**, **Rost**, **...**):

n:	2	3	4	5
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/48\mathbb{Z}$	0	$\mathbb{Z}$
n:	6	7	8	9
$K_n(\mathbb{Z})$:	0	$\mathbb{Z}/240\mathbb{Z}$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
n:	10	11	12	13
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/1008\mathbb{Z}$	(0?)	$\mathbb{Z}$
n:	14	15	16	17
$K_n(\mathbb{Z})$:	0	$\mathbb{Z}/480\mathbb{Z}$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
n:	18	19	20	21
$K_n(\mathbb{Z})$:	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/528\mathbb{Z}$	(0?)	$\mathbb{Z}$
n:	22	23	24	25
$K_n(\mathbb{Z})$:	$\mathbb{Z}/691\mathbb{Z}$	$\mathbb{Z}/65\,520\mathbb{Z}$	(0?)	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

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- Example: $\zeta(-11) = -\frac{B_{12}}{12} = \frac{691}{12\cdot 2730} = \frac{691}{2^3\cdot 3^2\cdot 5\cdot 7\cdot 13}$, $\frac{\#K_{22}(\mathbb{Z})}{\#K_{23}(\mathbb{Z})} = \frac{691}{65520} = \frac{691}{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}$.

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- $H^i(X_{\text{ét}}, \mathbb{Z}(n))$ might be better for studying the zeta-values.

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   where $\lambda\colon\mathbb{R}\xrightarrow{\cong}\left(\bigotimes_{i\in\mathbb{Z}}\det_{\mathbb{Z}}H^i_{W,c}(X,\mathbb{Z}(0))^{(-1)^i}\right)\otimes\mathbb{R}$.

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- My thesis, work in progress: $H^i_{W,c}(X, \mathbb{Z}(n))$
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- Construction via the cycle complexes $\mathbb{Z}(n)$, following Flach and Morin (input: conjectures on finite generation and boundedness).

Thank you!