Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

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Outline

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* Definition of Weil-étale complexes

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- **Zeta function** of an **arithmetic scheme**:

$$\begin{array}{c} X \\ {\scriptstyle \text{separated}} \Big\downarrow {\scriptstyle } \\ {\scriptstyle \text{of finite type}} \\ \operatorname{Spec}\mathbb{Z} \end{array} \rightsquigarrow \zeta(X,s) := \prod_{x\in X_0} \frac{1}{1-N(x)^{-s}} \quad (\operatorname{Re} s > \dim X)$$

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- Two quantities for each $n \in \mathbb{Z}$:
  $d_n := \operatorname{ord}_{s=n} \zeta(X,s) :=$ **vanishing order** at $s = n$,
  special value $\zeta^*(X,n) := \lim_{s \to n} (s-n)^{-d_n} \zeta(X,s)$.

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* First results about finite generation of motivic cohomology are from the XIX century :–)
$H^0(\mathrm{Spec}\,\mathcal{O}_F, \mathbb{Z}^c(0))$ is finite;
$H^{-1}(\mathrm{Spec}\,\mathcal{O}_F, \mathbb{Z}^c(0))$ is of rank $r_1 + r_2 - 1$.

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- **Flach** and **Morin**, 2016: $X_{/\mathbb{Z}}$ proper, regular, $n \in \mathbb{Z}$.

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From now on, n is strictly negative!

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* For $X_{/k}$ smooth, RHS $\cong$ Voevodsky's motivic complex (not our case).

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* As always here, $n < 0$.

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  (well-defined up to an iso in $\mathbf{D}(\mathbb{Z})$).
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- $R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1]$.

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- $G_{\mathbb{R}} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $R\Gamma_c(X(\mathbb{C}), (2\pi i)^n A) \Rightarrow$ $G_{\mathbb{R}}$-equivariant cohomology with compact support:

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- Note:

$$H^i_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R}) \cong H^i_c(X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})^{G_{\mathbb{R}}}$$

($H^{>0}(G_{\mathbb{R}}, \ldots)$ is always 2-torsion!)

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- In our case $n < 0$: a morphism in $\mathbf{D}(\mathbb{Z})$

$$Reg\colon R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})[1].$$

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- **Conjecture B**(X, n): the $\mathbb{R}$-dual is a quasi-isomorphism

$$Reg^\vee\colon R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \xrightarrow{\simeq} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})$$

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- Define

$$R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\xrightarrow{\smile\theta}R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1]$$

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is an acyclic complex of finite dimensional vector spaces.

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- **Properties of determinants** $\Rightarrow$ (*) induces a *canonical* iso

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- $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is constructed up to iso in $\mathbf{D}(\mathbb{Z})$, but $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is *canonically* defined and *canonically* seen as a lattice in $\mathbb{R}$.

Main conjecture

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1) the vanishing order is given by

$$\mathrm{ord}_{s=n}\,\zeta(X, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X, \mathbb{Z}(n)).$$

Conjecture $\mathbf{C}(X,n)$

For an arithmetic scheme X and $n < 0$

a) assume $\mathbf{L}^c(X_{\acute{e}t},n)$;
b) assume that $X_{\mathbb{C}}$ is smooth, quasi-projective;
   assume $\mathbf{B}(X,n)$;
c) assume meromorphic continuation near $s=n$ for $\zeta(X,s)$.

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2) the special value is given up to sign by

$$\lambda(\zeta^*(X,n)^{-1})\cdot\mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X,\mathbb{Z}(n)).$$

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- **Flach** and **Morin** (2016):
  their special value conjecture $\iff$
  Tamagawa number conjecture
  (**Bloch, Kato, Fontaine, Perrin-Riou**).

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- Under the assumptions a), b),

$$\begin{aligned}
\mathrm{ord}_{s=n}\,\zeta(X,s) &\overset{?}{=} \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n)) \\
&= \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{R}} H^i_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})^{G_{\mathbb{R}}} \\
&= \chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})).
\end{aligned}$$

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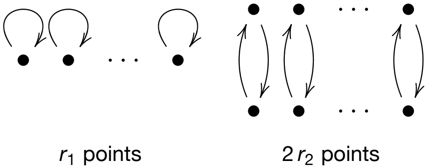

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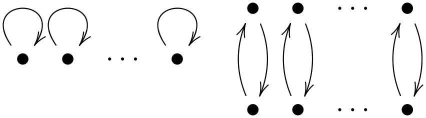

r_1 points $\qquad$ $2\,r_2$ points

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$$V := ((2\pi i)^n\,\mathbb{R})^{\oplus r_1} \oplus ((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2}.$$

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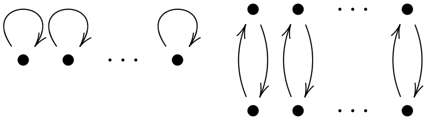

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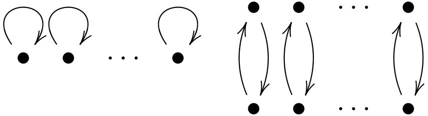

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- $$\chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) = \dim_{\mathbb{R}} V^{G_{\mathbb{R}}} = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}. \end{cases}$$

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- For $n < 0$

$$\operatorname{ord}_{s=n} \zeta(X, s) = \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n \mathbb{R})) = 0.$$

Okay, so how can we obtain new results?

Stability properties

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- **Theorem.** For the conjecture:

$$U \hookrightarrow X \leftarrow Z \rightsquigarrow \text{two out of three } \mathbf{C}(U,n),\ \mathbf{C}(X,n),\ \mathbf{C}(Z,n) \implies \text{the other one},$$
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  additivity of χ for the triangle
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- **Special value and affine space**:
  $$p\colon \mathbb{A}^r_X \to X \Longrightarrow Rp_*\mathbb{Z}^c(n) \simeq \mathbb{Z}^c(n-r)[2r] \Longrightarrow R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)) \simeq R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r].$$

Thank you!