Zeta-values of arithmetic schemes at negative integers and Weil-étale cohomology

Alexey Beshenov
Université de Bordeaux / Universiteit Leiden

Advised by Baptiste Morin and Bas Edixhoven

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Outline

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- Motivic cohomology
- Weil-étale complexes
- Regulator
- Main conjecture
- Stability properties

* Definition of Weil-étale complexes

Motivation

Motivation: zeta values

- **Zeta function** of an **arithmetic scheme**:

$$\begin{array}{r c} & X \\ \text{separated} & \Big\downarrow \\ \text{of finite type} & \\ & \operatorname{Spec}\mathbb{Z} \end{array} \rightsquigarrow \zeta(X,s) := \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}} \quad (\operatorname{Re} s > \dim X)$$

- **Conjecture**: meromorphic continuation to $\mathbb{C}$.
- Two quantities for each $n \in \mathbb{Z}$:
  $d_n := \operatorname{ord}_{s=n} \zeta(X,s) :=$ **vanishing order** at $s = n$,
  special value $\zeta^*(X,n) := \lim_{s \to n} (s-n)^{-d_n} \zeta(X,s)$.

Motivation: cohomological interpretation of zeta values

- May be traced to **Dirichlet** (1839):

$$\zeta_F^*(0) = \zeta^*(\operatorname{Spec} \mathcal{O}_F, 0) = -\frac{h_F}{\# \mu_F} R_F.$$

- Actually,

$$\cdots = -\frac{\# H^0(\operatorname{Spec} \mathcal{O}_F, \mathbb{Z}^c(0))}{\# H^{-1}(\operatorname{Spec} \mathcal{O}_F, \mathbb{Z}^c(0))_{tors}} R_F,$$

where $\mathbb{Z}^c(0) \cong \mathbb{G}_m[1]$ (to be explained in a moment).

* First results about finite generation of motivic cohomology are from the XIX century :–)
$H^0(\operatorname{Spec} \mathcal{O}_F, \mathbb{Z}^c(0))$ is finite;
$H^{-1}(\operatorname{Spec} \mathcal{O}_F, \mathbb{Z}^c(0))$ is of rank $r_1 + r_2 - 1$.

Motivation: Weil-étale cohomology (very brief history)

- **Lichtenbaum**: d_n and $\zeta^*(X, n)$ conjecturally come from "**Weil-étale cohomology**".
- **Lichtenbaum**, 2005: first studies for $X_{/\mathbb{F}_q}$.
- **Geisser**, 2004: further results for $X_{/\mathbb{F}_q}$.
- **Lichtenbaum**, 2009: $X = \operatorname{Spec} \mathcal{O}_F$, $n = 0$.
- **Morin**, 2014: $X_{/\mathbb{Z}}$ proper, regular, $n = 0$.
- **Flach** and **Morin**, 2016: $X_{/\mathbb{Z}}$ proper, regular, $n \in \mathbb{Z}$.

Motivation: my thesis

- **Goal**: construct Weil-étale cohomology following Flach and Morin...
- For *any* arithmetic scheme X (harder).
- For $n < 0$ (easier).

From now on, n is strictly negative!

Motivic cohomology

Motivic cohomology

- **Bloch**, 1986: **cycle complexes** $\mathbb{Z}(n)$, **higher Chow groups** $CH^i(X, p) = H^{2i-p}(X, \mathbb{Z}(i))$.
- **Geisser**, 2010: **dualizing complexes.**

$$(X, n) \rightsquigarrow \mathbb{Z}^c(n), \text{ complex of sheaves on } X_{\acute{e}t}.$$

- $\mathbb{Z}^c(n) = \mathbb{Z}(d-n)[2d], \quad d = \dim X$.

* For $X_{/k}$ smooth, RHS $\cong$ Voevodsky's motivic complex (not our case).

Motivic cohomology: Borel–Moore behavior of $\mathbb{Z}^c(n)$

- **Borel–Moore homology** for locally compact spaces.
- **Localization triangles** for $U \hookrightarrow X \leftarrow Z$:

$$R\Gamma_{BM}(Z, \mathbb{Z}) \to R\Gamma_{BM}(X, \mathbb{Z}) \to R\Gamma_{BM}(U, \mathbb{Z}) \to \cdots [1]$$

- **Verdier**: $R\Gamma_{BM}(X, \mathbb{Z}) = R\Gamma(X, p^! \underline{\mathbb{Z}}) \cong R\mathrm{Hom}(R\Gamma_c(X, \mathbb{Z}), \mathbb{Z})$, where $p\colon X \to *$.
- Dual to cohomology with compact support:

$$R\Gamma_c(U, \mathbb{Z}) \to R\Gamma_c(X, \mathbb{Z}) \to R\Gamma_c(Z, \mathbb{Z}) \to \cdots [1]$$

- **Geisser**, 2010: for schemes,

$$R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)) \to R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)) \to \cdots [1]$$

Motivic cohomology: assumption on finite generation

- **Conjecture $\mathbf{L}^c(X_{ét}, n)$:**
  $H^i(X_{ét}, \mathbb{Z}^c(n))$ are finitely generated for all $i \in \mathbb{Z}$.

* As always here, $n < 0$.

Weil-étale complexes

Weil-étale complexes

Theorem.

- For an arithmetic scheme X and $n < 0$, assume $\mathbf{L}^c(X_{\acute{e}t}, n)$.
- We may construct a perfect complex of abelian groups

$$R\Gamma_{W,c}(X, \mathbb{Z}(n))$$

  (well-defined up to an iso in $\mathbf{D}(\mathbb{Z})$).
- **perfect** = $H^i_{W,c}(X, \mathbb{Z}(n))$ are finitely generated; zero for almost all i.

Weil-étale complexes: splitting

- $R\Gamma_{W,c}(X,\mathbb{Z}(n)) \otimes \mathbb{R} \cong R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1] \oplus R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]$.
- $G_{\mathbb{R}} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $R\Gamma_c(X(\mathbb{C}),(2\pi i)^n\,A) \Rightarrow$ $G_{\mathbb{R}}$-equivariant cohomology with compact support:

$$R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,A) := R\Gamma(G_{\mathbb{R}},R\Gamma_c(X(\mathbb{C}),(2\pi i)^n\,A)).$$

- Grothendieck spectral sequence:

$$E_2^{pq} = H^p(G_{\mathbb{R}},H_c^q(X(\mathbb{C}),(2\pi i)^n\,A)) \Longrightarrow H_c^{p+q}(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,A).$$

- Note:

$$H_c^i(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R}) \cong H_c^i(X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})^{G_{\mathbb{R}}}$$

($H^{>0}(G_{\mathbb{R}},\ldots)$ is always 2-torsion!)

Regulator

Regulator

- A nice construction due to **Kerr, Lewis, and Müller-Stach** (2006).
- Usually:

$$\begin{matrix}\text{motivic cohomology} \\ \text{(higher Chow groups)}\end{matrix} \text{ of } X \xrightarrow{Reg} \text{Deligne (co)homology of } X(\mathbb{C}).$$

- In our case $n < 0$: a morphism in $\mathbf{D}(\mathbb{Z})$

$$Reg\colon R\Gamma(X_{ét}, \mathbb{Z}^c(n)) \to R\Gamma_{BM}(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[1].$$

- **Drawback**: need to assume that $X_{\mathbb{C}}$ is smooth, quasi-projective.
- **Conjecture B**(X, n): the $\mathbb{R}$-dual is a quasi-isomorphism

$$Reg^\vee\colon R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})[-1] \xrightarrow{\simeq} R\mathrm{Hom}(R\Gamma(X_{ét}, \mathbb{Z}^c(n)), \mathbb{R})$$

Morphism $\smile \theta$

- Define

$$R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\xrightarrow{\smile\theta}R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1]$$

$$\begin{array}{c}
R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}\\
\text{splitting}\downarrow\cong\\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})[-1]\ \oplus\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]\\
\twoheaddownarrow\\
R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})[-1]\\
\text{conjecturally }\cong\downarrow Reg^\vee\\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})\\
\rightarrowtail\\
R\mathrm{Hom}(R\Gamma(X_{\acute{e}t},\mathbb{Z}^c(n)),\mathbb{R})\ \oplus\ R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})\\
\text{splitting}\downarrow\cong\\
R\Gamma_{W,c}(X,\mathbb{Z}(n))\otimes\mathbb{R}[1]
\end{array}$$

Trivialization map λ

- Assume $\mathbf{L}^c(X_{\acute{e}t}, n)$ and $\mathbf{B}(X, n)$. Then

$$\cdots \to H^i_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \xrightarrow{\smile\theta} H^{i+1}_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R} \to \cdots \quad (*)$$

  is an acyclic complex of finite dimensional vector spaces.
- **Knudsen, Mumford**, 1976: determinants of complexes.

  $C^\bullet$, perfect object in $\mathbf{D}(\mathbb{Z}) \rightsquigarrow \det_{\mathbb{Z}} C^\bullet$, $\mathbb{Z}$-module of rank 1.
- **Properties of determinants** $\Rightarrow$ (*) induces a *canonical* iso

$$\lambda\colon \mathbb{R} \xrightarrow{\cong} (\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)) \otimes \mathbb{R}.$$

- $R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is constructed up to iso in $\mathbf{D}(\mathbb{Z})$, but $\det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n))$ is *canonically* defined and *canonically* seen as a lattice in $\mathbb{R}$.

Main conjecture

Conjecture $\mathbf{C}(X, n)$

For an arithmetic scheme X and $n < 0$

a) assume $\mathbf{L}^c(X_{ét}, n)$;
b) assume that $X_{\mathbb{C}}$ is smooth, quasi-projective;
   assume $\mathbf{B}(X, n)$;
c) assume meromorphic continuation near $s = n$ for $\zeta(X, s)$.

Then

1) the vanishing order is given by

$$\mathrm{ord}_{s=n}\, \zeta(X, s) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot i \cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X, \mathbb{Z}(n)).$$

2) the special value is given up to sign by

$$\lambda(\zeta^*(X, n)^{-1}) \cdot \mathbb{Z} = \det_{\mathbb{Z}} R\Gamma_{W,c}(X, \mathbb{Z}(n)).$$

Main conjecture: relation to other statements

- For X regular, proper,
  $\mathbf{C}(X,n) \iff$ conjectures of **Flach** and **Morin** (2016).
- **Flach** and **Morin** (2016):
  their special value conjecture $\iff$
  Tamagawa number conjecture
  (**Bloch, Kato, Fontaine, Perrin-Riou**).

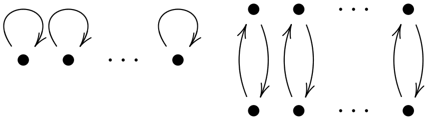
Main conjecture: meaning of the vanishing order

- What is that sum $\sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n))$?
- Under the assumptions a), b),

$$\begin{aligned}\mathrm{ord}_{s=n}\,\zeta(X,s) &\stackrel{?}{=} \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \mathrm{rk}_{\mathbb{Z}}\, H^i_{W,c}(X,\mathbb{Z}(n)) \\ &= \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{R}} H^i_c(X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})^{G_{\mathbb{R}}} \\ &= \chi(R\Gamma_c(G_{\mathbb{R}},X(\mathbb{C}),(2\pi i)^n\,\mathbb{R})).\end{aligned}$$

Toy example: number rings

- Let $X = \operatorname{Spec} \mathcal{O}_F$, $[F : \mathbb{Q}] = r_1 + 2r_2$. The $G_\mathbb{R}$-space $X(\mathbb{C})$:



r_1 points $\qquad$ $2\,r_2$ points

- $R\Gamma_c(X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})$: a single $\mathbb{R}$-vector space

$$V := ((2\pi i)^n\,\mathbb{R})^{\oplus r_1} \oplus ((2\pi i)^n\,\mathbb{R} \oplus (2\pi i)^n\,\mathbb{R})^{\oplus r_2}.$$

- $G_\mathbb{R}$-actions: $z \mapsto \bar{z}$ and $(z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$.
- $$\chi(R\Gamma_c(G_\mathbb{R}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R})) = \dim_\mathbb{R} V^{G_\mathbb{R}} = \begin{cases} r_2, & n \text{ odd}, \\ r_1 + r_2, & n \text{ even}. \end{cases}$$

Dull example: varieties over finite fields

- For $X_{/\mathbb{F}_q}$ we have $X(\mathbb{C}) = \emptyset$.
- **Weil's zeta function**:

$$\zeta(X, s) = \exp\left(\sum_{k\geq 1} \frac{\#X(\mathbb{F}_{q^k})}{k} q^{-ks}\right).$$

- For $n < 0$

$$\mathrm{ord}_{s=n}\, \zeta(X, s) = \chi(R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\, \mathbb{R})) = 0.$$

Okay, so how can we obtain new results?

Stability properties

- For the zeta function:

$$U \hookrightarrow X \leftarrow Z \rightsquigarrow \zeta(X,s) = \zeta(U,s)\cdot\zeta(Z,s),$$
$$\mathbb{A}^r_X := \mathbb{A}^r_{\mathbb{Z}} \times X \rightsquigarrow \zeta(\mathbb{A}^r_X, s) = \zeta(X, s-r).$$

- **Theorem.** For the conjecture:

$$U \hookrightarrow X \leftarrow Z \rightsquigarrow \text{two out of three } \mathbf{C}(U,n),\ \mathbf{C}(X,n),\ \mathbf{C}(Z,n)$$
$$\Longrightarrow \text{the other one},$$
$$\mathbb{A}^r_X := \mathbb{A}^r_{\mathbb{Z}} \times X \rightsquigarrow \mathbf{C}(\mathbb{A}^r_X, n) \iff \mathbf{C}(X, n-r).$$

Stability properties: proof idea

- **Vanishing orders and localization**:
additivity of χ for the triangle
$$R\Gamma_c(G_{\mathbb{R}}, U(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \to R\Gamma_c(G_{\mathbb{R}}, X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \to R\Gamma_c(G_{\mathbb{R}}, Z(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \to \cdots [1]$$
- **Vanishing orders and affine space**:
$G_{\mathbb{R}}$-equivariant quasi-isomorphism
$$R\Gamma_c(\mathbb{C}^r \times X(\mathbb{C}), (2\pi i)^n\,\mathbb{R}) \simeq R\Gamma_c(X(\mathbb{C}), (2\pi i)^{n-r}\,\mathbb{R})[-2r].$$
- **Special value and localization**:
look also at the dual to the Borel–Moore triangle for $\mathbb{Z}^c(n)$:
$$R\mathrm{Hom}(R\Gamma(U_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \to R\mathrm{Hom}(R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \to R\mathrm{Hom}(R\Gamma(Z_{\acute{e}t}, \mathbb{Z}^c(n)), \mathbb{R}) \to \cdots [1]$$
- **Special value and affine space**:
$$p\colon \mathbb{A}^r_X \to X \Longrightarrow Rp_*\mathbb{Z}^c(n) \simeq \mathbb{Z}^c(n-r)[2r] \Longrightarrow R\Gamma(\mathbb{A}^r_{X,\acute{e}t}, \mathbb{Z}^c(n)) \simeq R\Gamma(X_{\acute{e}t}, \mathbb{Z}^c(n-r))[2r].$$

Thank you!