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Consider the TD-SE for two level system:

$$
\begin{aligned}
i\hbar\dot{c}_0 &= E_0 c_0 - i\hbar d_{01} c_1 \\
i\hbar\dot{c}_1 &= -i\hbar d_{10} c_0 + E_1 c_1
\end{aligned}
. \quad (1)
$$

Introduce new variables:

$$a_i = -i\frac{E_i}{\hbar}, \quad (2)$$

$$b = -d_{01}. \quad (3)$$

$$d_{10} = -d_{01}^* \Rightarrow d_{10} = b^*$$

The Eqs. (1) then turn to:

$$
\begin{aligned}
\dot{c}_0 &= a_0 c_0 + b c_1 \\
\dot{c}_1 &= -b^* c_0 + a_1 c_1
\end{aligned}
. \quad (4)
$$

but we use purely real d, so $b^* = b$.

The Liouvillian of this system of first order ODE (ordinary differential equations) can be written as:

$$
\begin{aligned}
iL &= \dot{c}_0\frac{\partial}{\partial c_0} + \dot{c}_1\frac{\partial}{\partial c_1} = (a_0 c_0 + b c_1)\frac{\partial}{\partial c_0} + (-b c_0 + a_1 c_1)\frac{\partial}{\partial c_1} \\
&= a_0 c_0\frac{\partial}{\partial c_0} + a_1 c_1\frac{\partial}{\partial c_1} + b\left(c_1\frac{\partial}{\partial c_0} - c_0\frac{\partial}{\partial c_1}\right) \equiv iL_0 + iL_1 + iL_{01}
\end{aligned}
. \quad (5)
$$

The solution of the system Eq. (4) with initial conditions $\{c_0(0), c_1(0)\}$ for time t is equivalent to propagation of the variables c_0 and c_1 by evolution operator $\exp(iL\cdot t)$ for such time:

$$\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = \exp(iL\cdot t)\begin{pmatrix} c_0(0) \\ c_1(0) \end{pmatrix}. \quad (6)$$

The evolution operator (propagator) $\exp(iL\cdot t)$ can be factorized using the second-order Trotter formula, which for general non-commuting operators A and B can be written as:

$$\exp(A+B) = \exp\left(\frac{A}{2}\right)\exp(B)\exp\left(\frac{A}{2}\right). \quad (7)$$

If the operators A and B commute, the factorization (7) is exact and can be simplified:

$$\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A). \quad (8)$$

Using Eqs. (7) and then (8) and the definition of the sub-Liouvillian operators in Eq. (5), the evolution operator in Eq. (6) can be factorized as:

$$
\begin{aligned}
&\exp(iL\cdot t) \equiv \exp((iL_0 + iL_1 + iL_{01})\cdot t) = \\
&= \exp\left((iL_0 + iL_1)\frac{t}{2}\right)\exp(iL_{01}\cdot t)\exp\left((iL_0 + iL_1)\frac{t}{2}\right) = \\
&= \exp\left(iL_0\frac{t}{2}\right)\exp\left(iL_1\frac{t}{2}\right)\exp(iL_{01}\cdot t)\exp\left(iL_0\frac{t}{2}\right)\exp\left(iL_1\frac{t}{2}\right)
\end{aligned}
. \quad (9)
$$

To obtain explicit solution to the Eq. (6) we need to know the action of operators in the RHS of Eq. (9). It can be shown that the action of operators of type $\exp\left(\alpha x \frac{\partial}{\partial x}\right)$ on variable x is equivalent to multiplication of variable x by the constant $\exp(\alpha)$:

$$\exp\left(\alpha x \frac{\partial}{\partial x}\right) : x \to \exp(\alpha) x \,. \tag{10}$$

Thus,

$$\exp\left(iL_0 \frac{t}{2}\right)\exp\left(iL_1 \frac{t}{2}\right) : \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \to \begin{pmatrix} \exp\left(a_0 \frac{t}{2}\right) c_0 \\ \exp\left(a_1 \frac{t}{2}\right) c_1 \end{pmatrix}, \tag{11}$$

or, equivalently:

$$\begin{pmatrix} \tilde{c}_0\left(\frac{t}{2}\right) \\ \tilde{c}_1\left(\frac{t}{2}\right) \end{pmatrix} = \exp\left(iL_0 \frac{t}{2}\right)\exp\left(iL_1 \frac{t}{2}\right)\begin{pmatrix} c_0(0) \\ c_1(0) \end{pmatrix} = \begin{pmatrix} \exp\left(a_0 \frac{t}{2}\right) c_0 \\ \exp\left(a_1 \frac{t}{2}\right) c_1 \end{pmatrix}, \tag{12}$$

The action of the operator of type $\exp\left(\varphi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)$ is equivalent to rotation of the phase space point (x, y) by the angle φ:

$$\exp\left(\varphi\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right) : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x\cos\varphi + y\sin\varphi \\ -x\sin\varphi + y\cos\varphi \end{pmatrix}. \tag{13}$$

Thus,

$$\begin{gathered} \begin{pmatrix} c_0\left(\frac{t}{2}\right) \\ c_1\left(\frac{t}{2}\right) \end{pmatrix} = \exp(iL_{01} \cdot t)\begin{pmatrix} \tilde{c}_0\left(\frac{t}{2}\right) \\ \tilde{c}_1\left(\frac{t}{2}\right) \end{pmatrix} = \begin{pmatrix} \tilde{c}_0\left(\frac{t}{2}\right)\cos bt + \tilde{c}_1\left(\frac{t}{2}\right)\sin bt \\ -\tilde{c}_0\left(\frac{t}{2}\right)\sin bt + \tilde{c}_1\left(\frac{t}{2}\right)\cos bt \end{pmatrix} = \\ = \begin{pmatrix} \exp\left(a_0 \frac{t}{2}\right) c_0 \cos bt + \exp\left(a_1 \frac{t}{2}\right) c_1 \sin bt \\ -\exp\left(a_0 \frac{t}{2}\right) c_0 \sin bt + \exp\left(a_1 \frac{t}{2}\right) c_1 \cos bt \end{pmatrix} \end{gathered} \tag{14}$$

Finally, applying the operator in Eq. (11) one more time we get the answer:

$$\begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix} = \exp\left(iL_0 \frac{t}{2}\right)\exp\left(iL_1 \frac{t}{2}\right)\begin{pmatrix} c_0\left(\frac{t}{2}\right) \\ c_1\left(\frac{t}{2}\right) \end{pmatrix} =$$

$$= \begin{pmatrix} \exp\left(a_0 \frac{t}{2}\right)\left[\exp\left(a_0 \frac{t}{2}\right)c_0 \cos bt + \exp\left(a_1 \frac{t}{2}\right)c_1 \sin bt\right] \\ \exp\left(a_1 \frac{t}{2}\right)\left[-\exp\left(a_0 \frac{t}{2}\right)c_0 \sin bt + \exp\left(a_1 \frac{t}{2}\right)c_1 \cos bt\right] \end{pmatrix} =. \qquad (15)$$

$$= \begin{pmatrix} \left[\exp(a_0 t)c_0 \cos bt + \exp\left(\frac{a_1 + a_0}{2} t\right)c_1 \sin bt\right] \\ \left[-\exp\left(\frac{a_1 + a_0}{2} t\right)c_0 \sin bt + \exp(a_1 t)c_1 \cos bt\right] \end{pmatrix}$$

Then,

$$|c_0(t)|^2 = \left(\exp(-a_0 t)c_0^* \cos b^* t + \exp\left(-\frac{a_1 + a_0}{2} t\right)c_1^* \sin b^* t\right)\times$$

$$\times\left(\exp(a_0 t)c_0 \cos bt + \exp\left(\frac{a_1 + a_0}{2} t\right)c_1 \sin bt\right) =$$

$$= |c_0|^2 \cos^2 bt + |c_1|^2 \sin^2 bt + \exp\left(\frac{a_1 - a_0}{2} t\right)c_0^* c_1 \cos bt \sin bt \qquad (16)$$

$$+ \exp\left(-\frac{a_1 - a_0}{2} t\right)c_1^* c_0 \cos bt \sin bt =$$

$$= |c_0|^2 \cos^2 bt + |c_1|^2 \sin^2 bt + \sin(2bt) \cdot \mathrm{Re}\left[\exp\left(\frac{a_1 - a_0}{2} t\right)c_0^* c_1\right]$$

In case when we started with pure state 0 (so $c_0 = 1$ and $c_1 = 0$), the Eq. (16) simplifies: $|c_0(t)|^2 = \cos^2 bt$.
Thus, the rate does not depend on the energy difference.