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Metric & Topological Spaces

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Course schedule

Metrics

Definition and examples. Limits and continuity. Open sets and neighbourhoods. Characterizing limits and continuity using neighbourhoods and open sets. [3]

Topology

Definition of a topology. Metric topologies. Further examples. Neighbourhoods, closed sets, convergence and continuity. Hausdorff spaces. Homeomorphisms. Topological and non-topological properties. Completeness. Subspace, quotient and product topologies. [3]

Connectedness

Definition using open sets and integer-valued functions. Examples, including intervals. Components. The continuous image of a connected space is connected. Path-connectedness. Path-connected spaces are connected but not conversely. Connected open sets in Euclidean space are path-connected. [3]

Compactness

Definition using open covers. Examples: finite sets and $[0, 1]$. Closed subsets of compact spaces are compact. Compact subsets of a Hausdorff space must be closed. The compact subsets of the real line. Continuous images of compact sets are compact. Quotient spaces. Continuous real-valued functions on a compact space are bounded and attain their bounds. The product of two compact spaces is compact. The compact subsets of Euclidean space. Sequential compactness. [3]

Appropriate books

W.A. Sutherland *Introduction to Metric and Topological Spaces.* Clarendon 1975 (£21.00 paperback).

A.J. White *Real Analysis: an Introduction.* Addison-Wesley 1968 (out of print)

B. Mendelson *Introduction to Topology.* Dover, 1990 (£5.27 paperback)

Contents

1 Metric spaces

1.1 Introduction

We start by considering Euclidean space $\mathbb{R}^n$, equipped with the standard Euclidean inner product: given vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with coordinates x_i, y_j, respectively, we define

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} x_i \, y_i,$$

which is also referred to as the dot product $\mathbf{x} \cdot \mathbf{y}$.

From this, we can define the *Euclidean norm* on $\mathbb{R}^n$:

$$\|\mathbf{x}\| := (\mathbf{x}, \mathbf{x})^{1/2},$$

which represents the length of the vector $\mathbf{x}$.

This allows us to define a *distance function*:

$$d_2(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

This is an example of a *metric*.

> **Definition.** A *metric space* (X, d) consists of a set X and a function, called the *metric*, $d : X \times X \to \mathbb{R}$ such that for all $P, Q, R \in X$:
>
> (i) $d(P, Q) \geq 0$ with equality if and only if $P = Q$;
> (ii) $d(P, Q) = d(Q, P)$;
> (iii) $d(P, Q) + d(Q, R) \geq d(P, R)$.

Condition (iii) is called the *triangle inequality.* This comes from a simple result in Euclidean space. Any triangle with vertices P, Q and R satisfies the following property:

> the sum of the lengths of two sides of the triangle will be at least the length of the third side.

In other words, travelling along straight line segments from P to Q, and from Q to R, the length of the journey is at least that of travelling directly from P to R.

Proposition 1.1. *The Euclidean distance function d_2 on $\mathbb{R}^n$ is a metric (called the* Euclidean metric*).*

Proof. Conditions (i) and (ii) are obvious in thse case, so we only need to prove (iii). For (iii), we use the Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{n} x_i \, y_i \right)^2 \leq \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{j=1}^{n} y_j^2 \right),$$

or in the inner product notation,

$$(\mathbf{x}, \mathbf{y})^2 \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

To prove the triangle inequality, we take $P = \mathbf{0} \in \mathbb{R}^n$, and Q to have position vector $\mathbf{x}$ with respect to P, and R to have position vector $\mathbf{y}$ with respect to $Q = \mathbf{0}$, so R has position vector $\mathbf{x} + \mathbf{y}$ with respect to P.

Cauchy-Schwarz then gives

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) \\ &= \|\mathbf{x}\|^2 + 2(\mathbf{x},\mathbf{y}) + \|\mathbf{y}\|^2 \\ &\le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\ &= \left(\|\mathbf{x}\| + \|\mathbf{y}\|\right)^2 . \end{aligned}$$

Taking square roots gives

$$d(P,R) = \|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\| = d(P,Q) + d(Q,R). \qquad \square$$

For completeness, we now state and prove Cauchy-Schwarz:

Lemma 1.2 (Cauchy-Schwarz inequality)**.** *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*, we have*

$$(\mathbf{x},\mathbf{y})^2 \le \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 .$$

Proof. The quadratic polynomial in the real variable λ given by

$$(\lambda\mathbf{x}+\mathbf{y}, \lambda\mathbf{x}+\mathbf{y}) = \|\mathbf{x}\|^2 \lambda^2 + 2(\mathbf{x},\mathbf{y})\lambda + \|\mathbf{y}\|^2$$

is positive semi-definite (that is, non-negative for all λ). A quadratic polynomial $a\lambda^2 + b\lambda + c$ is positive semi-definite if and only if $a \ge 0$ and $b^2 \le ac$. This gives us the desired inequality. $\square$

Remarks.

(i) In the Euclidean case, we have equality in the triangle inequality if and only if Q lies on the straight line segment PR.

(ii) This argument just given also proves Cauchy Schwarz for integrals: if f and g are continuous functions on $[0,1]$, then

$$\int (\lambda f + g)^2 \Longrightarrow \left(\int_0^1 fg\right)^2 \le \int_0^1 f^2 \int_0^1 g^2$$

as in the previous argument.

Examples 1.3.

(i) For $X = \mathbb{R}^n$, the functions

$$d_1(\mathbf{x},\mathbf{y}) := \sum_{i=1}^n |x_i - y_i| \qquad \text{or} \qquad d_\infty(\mathbf{x},\mathbf{y}) := \max_i |x_i - y_i| ,$$

are both metrics.

(ii) For any set X, and $\mathbf{x}, \mathbf{y} \in X$, we can define the *discrete metric*

$$d_{\text{disc}}(\mathbf{x},\mathbf{y}) := \begin{cases} 1 & \text{if } \mathbf{x} \ne \mathbf{y}, \\ 0 & \text{if } \mathbf{x} = \mathbf{y}. \end{cases}$$

(iii) If we take $X = C[0,1]$ to be the set of continuous functions on $[0,1]$, then we can define metrics d_1, d_2, d_∞ on X:

$$d_1(f,g) := \int_0^1 |f-g| \, \mathrm{d}x,$$

$$d_2(f,g) := \left(\int_0^1 (f-g)^2 \, \mathrm{d}x \right)^{1/2},$$

$$d_\infty(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)|.$$

For d_2, the triangle inequality follows from Cauchy-Schwarz for integrals and the same argument as in the proof of lemma 1.2.

(iv) *British Rail metric.* Consider $\mathbb{R}^n$ with the Euclidean metric d (in the case $n=2$) and let O denote the origin $\mathbf{0}$. Define a new metric ρ on $\mathbb{R}^n$ by

$$\rho(P,Q) := \begin{cases} d(P,\mathbf{0}) + d(\mathbf{0},Q) & \text{if } P \neq Q, \\ 0 & \text{if } P = Q, \end{cases}$$

that is, all the journeys from P to $Q \neq P$ must go via $\mathbf{0}$. (This is called the British Rail metric because "All rail journeys have to go via London".)

Some metrics satisfy a stronger triangle inequality.

Definition. A metric space (X,d) is *ultra-metric* if d satisfies a stronger condition (iii)$'$: for all $P, Q, R \in X$,

$$d(P,R) \leq \max\left\{d(P,Q), d(Q,R)\right\}$$

Example 1.4. Take $X = \mathbb{Z}$ and p a prime number. The *p-adic metric* is then defined as

$$d(m,n) := \begin{cases} 0 & \text{if } m = n, \\ 1/p^r & \text{if } m \neq n, \text{where } r = \max\{s \in \mathbb{N} \text{ with } p^s \mid (m-n)\}. \end{cases}$$

We claim that this is an ultrametric. For proof, suppose $d(m,n) = 1/p^{r_1}$ and $d(n,q) = 1/p^{r_2}$. Then

$$\left.\begin{matrix} p^{r_1} \mid (m-n) \\ p^{r_2} \mid (n-q) \end{matrix}\right\} \implies p^{\min\{r_1,r_2\}} \mid (m-q).$$

So for some $r \geq \min\{r_1, r_2\}$, we have

$$d(m,q) = \frac{1}{p^r} \leq \frac{1}{p^{\min\{r_1,r_2\}}} = \max\left\{\frac{1}{p^{r_1}}, \frac{1}{p^{r_2}}\right\} = \max\left\{d_p(m,n), d_p(n,q)\right\},$$

as desired.

This can be extended to a p-adic metric on $\mathbb{Q}$. For any $x, y \in \mathbb{Q}$ with $x \neq y$, we can write $x - y = p^r m/n$, $r \in \mathbb{Z}$, with m, n coprime to p. Then we define $d(x,y) = 1/p^r$ as before. Minor modifications to the prior proof will show that this yields $(\mathbb{Q}, d_p)$ as an ultra-metric space.

Definition. We say that two metrics ρ_1 and ρ_2 on a set X are *Lipschitz equivalent* if there are some $0 < \lambda_1 \leq \lambda_2 \in \mathbb{R}$ such that

$$\lambda_1 \rho_1 \leq \rho_2 \leq \lambda_2 \rho_1.$$

Remark. For metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$, we can show that

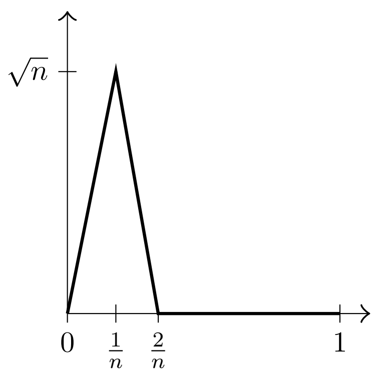
$$d_1 \geq d_2 \geq d_\infty \geq d_2/\sqrt{n} \geq d_1/n,$$

and so these are Lipschitz equivalent.

Of course, not all metrics are Lipschitz equivalent. Consider the following counterexample:

Proposition 1.5. *On $C[0,1]$, the metric d_1 and d_∞ are not Lipschitz equivalent.*

Proof. For $n \geq 2$, let $f_n \in C[0,1]$ be given by



$$f_n(x) = \begin{cases} x/\sqrt{n} & \text{if } 0 \leq x < 1/n, \\ 2\sqrt{n} - x/\sqrt{n} & \text{if } 1/n \leq x < 2/n, \\ 0 & \text{if } 2/n \leq x \leq 1. \end{cases}$$

Then $d_1(f_n, 0)$ is the area of the triangle, while $d_\infty(f_n, 0)$ is $\sqrt{n}$. Thus we have

$$\lim_{n\to\infty} d_1(f_n, 0) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} d_\infty(f_n, 0) = \infty,$$

and so these two metrics cannot possibly be Lipschitz equivalent. $\square$

Exercise 1.6. Continuing the example above, show that $d_2(f_n, 0) = \sqrt{2/3}$ for all n, and so d_2 is not Lipschitz equivalent to d_1 or d_∞ on $C[0,1]$.

1.2 Open balls and open sets

Definition. Let (X, d) be a metric space, $P \in X$ and $\delta > 0$. The *open ball of radius δ about p* is given by

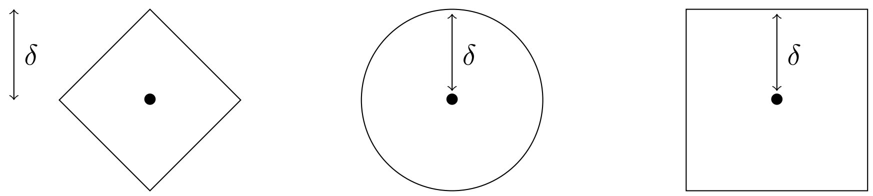
$$B_d(P, \delta) := \{Q \in X : d(P, Q) < \delta\},$$

often also denoted by $B(P, \delta)$ or $B_\delta(P)$.

Examples 1.7.

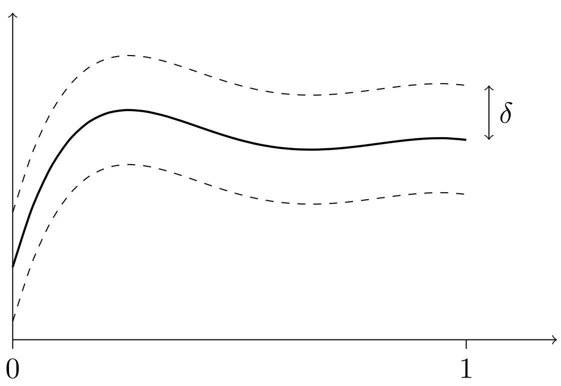
(i) In $(\mathbb{R}, d_1)$, open balls are open intervals of the form $(P - \delta, P + \delta)$.

(ii) In $\mathbb{R}^2$, we obtain different open balls depending on our choice of metric:
- In $(\mathbb{R}^2, d_1)$, we obtain tilted squares or "diamonds".
- In $(\mathbb{R}^2, d_2)$, we obtain open discs of radius δ.
- In $(\mathbb{R}^2, d_\infty)$, open balls are squares.



$(\mathbb{R}_2, d_1)$ $(\mathbb{R}_2, d_2)$ $(\mathbb{R}_2, d_\infty)$

(iii) In $(C[0, 1], d_\infty)$, the open ball of radius δ is the area swept out by translating the image of the function up and down by a distance δ.



(iv) For any set X, in (X, d_{disc}), we have $B(P, \frac{1}{2}) = \{P\}$ for all $P \in X$.

Definition. A subset $U \subset X$ of a metric space (X, d) is called an *open subset* if, for all $P \in U$, there is some $\delta > 0$ such that the open ball $B(P, \delta)$ is contained in U. (Note that the empty set $\emptyset$ is open, as is the whole space X.)

Under this definition, an open subset is a union of (usually infinitely many) open balls.

As the opposite to this definition, a subset $F \subset X$ is a *closed subset* if $X \backslash F$ is open.

Example 1.8. Analogously to open balls, we can define *closed balls*:

$$\overline{B}(P,\delta) := \left\{Q \in X : d(P,Q) \leq \delta\right\},$$

which is the union of the open ball $B(P,\delta)$ and its boundary. The name is appropriate: these are indeed closed.

Consider: if $Q \notin \overline{B}(P,\delta)$, then $d(P,Q) > \delta$, and we can find $\delta' < d(P,Q) - \delta$. Then consider a point $R \in B(Q,\delta')$. Then

$$d(P,R) \geq d(P,Q) - d(R,Q) > d(P,Q) - \delta' > \delta.$$

Thus $B(Q,\delta') \subset X \backslash \overline{B}$. Then $\overline{B}(P,\delta)$ is closed since the complment is open.

Lemma 1.9.

(i) Both X and $\emptyset$ are open subsets of (X,d).
(ii) If $\{U_i : i \in I\}$ are open subsets of (X,d), then so is $\bigcup_{i\in I} U_i$.
(iii) If U_1, U_2 are open subsets, then so is $U_1 \cap U_2$.

Proof. Both (i) and (ii) are easy, and left as exercises.

For (iii): if $P \in U_1 \cap U_2$, then there are open balls $B(P,\delta_1) \subset U_1$ and $B(P,\delta_2) \subset U_2$. Thus, if $\delta = \min\{\delta_1,\delta_2\}$, then $B(P,\delta) \subset U_1 \cap U_2$. $\square$

Definition. If P is a point in (X,d), then an *open neighbourhood N of P* is an open subset $N \ni P$, such as the open balls centred on P.

Example 1.10. Under the British Rail metric ρ on $\mathbb{R}^2$, what are the open neighbourhoods of a point P? Recall that we have a point $0 \in X$,

$$\rho(P,Q) = \begin{cases} d(P,0) + d(0,Q) & \text{if } P \neq Q, \\ 0 & \text{if } P = Q, \end{cases}$$

where d is the Euclidean metric.

Hence, if $P \neq 0$, $\delta < d(P,0)$, then we have $B_\rho(P,\delta) = \{P\}$. If $P = 0$, $B_\rho(P,\delta)$ is an open Euclidean disc of radius δ.

Thus, if $U \subset (\mathbb{R}^2,\rho)$ is open, then either $0 \notin U$ and U is arbitrary, or $U \supseteq B_{\text{eucl}}(0,\delta)$ for some $\delta > 0$ (U contains a Euclidean disc).

1.3 Limits and continuity

Definition. Suppose $x_1, x_2, \dots$ is a sequence of points in a metric space (X, d). We say that x_n *converges to a limit* x (denoted $x_n \to x$) if $d(x_n, x) \to 0$ as $n \to \infty$.

Equivalently, for any $\epsilon > 0$, there is some $N(\epsilon)$ such that $x_n \in B(x, \epsilon)$ for all $n \geq N$. Both of these definitions should be familiar from *Analysis*.

Examples 1.11.

(i) $1 + p + p^2 + \dots + p^{n-1} \to \dfrac{1}{1-p}$ in $(\mathbb{Q}, d_p)$.

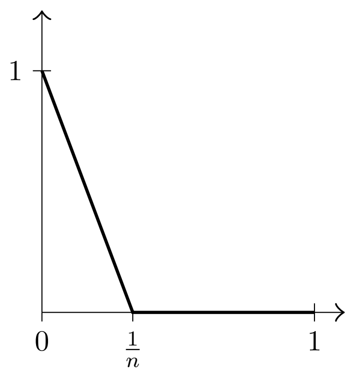
(ii) Consider the sequences of functions f_n defined in the proof of proposition 1.5. We have different limits, depending on our choice of metric. Clearly in (X, d_1), we have $f_n \to 0$, but this is not the case in (X, d_2) or (X, d_∞).

Proposition 1.12. *We have $x_n \to x$ in (X, d) if and only if, for any open neighbourhood $U \ni x$, there is some N such that $x_n \in U$ for all $n \geq N$.*

Proof. The "if" direction is clear by taking $U = B(x, \epsilon)$, for arbitrary ϵ. For the converse, given an open set $U \ni x$, there is some $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Hence there is some N such that $x_n \in B(x, \epsilon) \subset U$ for all $n \geq N$. $\square$

This allows us to rephrase $x_n \to x$ in terms of open subsets in (X, d).

Example 1.13. Take $X = C[0, 1]$, $d = d_1$, and the function g_n given by:



$$g_n(x) = \begin{cases} 1 - nx & \text{if } 0 \leq x < 1/n, \\ 0 & \text{if } 1/n \leq x \leq 1. \end{cases}$$

Then $g_n(0) = 1$ for all n, but $d_1(g_n, 0) \to 0$; that is, $g_n \to 0$ in (X, d_1) as $n \to \infty$.

Definition. We say that a function $f : (X, \rho_1) \to (Y, \rho_2)$ is

- *continuous* at $x \in X$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho_1(x, x') < \delta$ implies $\rho_2(f(x), f(x')) < \epsilon$ for all $x' \in X$.
- *uniformly continuous on* X if, given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho_1(x, x') < \delta$ implies $\rho_2(f(x), f(x')) < \epsilon$ for all $x, x' \in X$.

Remark. We may rephrase this: $f : (X, \rho_1) \to (Y, \rho_2)$ is continuous at $x \in X$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$; that is, $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$.

Lemma 1.14. *If $f : (X, \rho_1) \to (Y, \rho_2)$ is continuous and $x_n \to x$ in (X, ρ_1), then $f(x_n) \to f(x)$ in (Y, ρ_2).*

Proof. Given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho_1(x, x') < \delta$ implies $\rho_2(f(x), f(x')) < \epsilon$. As $x_n \to x$, we know there exists N such that for all $n \geq N$, $\rho_1(x_n, x) < \delta$. Hence, for $n \geq N$, $\rho_2(f(x_n), f(x)) < \epsilon$, and so $f(x_n) \to f(x)$. □

> **Example 1.15.** Consider the identity map $\mathrm{id} : (C[0,1], d_\infty) \to (C[0,1], d_1)$. Since $d_\infty(f, g) < \epsilon$ is equivalent to $\sup_{x\in[0,1]} \left|f(x) - g(x)\right| < \epsilon >$, which implies $d_1(f, g) < \epsilon$, we see that id is continuous.
>
> However, we can use the functions f_n in the proof of proposition 1.5 to see that the identity in the opposite direction, $\mathrm{id} : (C[0,1], d_1) \to (C[0,1], d_\infty)$, is not continuous. We note that $d_1(f_n, 0) \to 0$ but $d_\infty(f_n, 0) \to \infty$ as $n \to \infty$.

Now we wish to express continuity of maps purely in terms of open sets.

Proposition 1.16.

(i) A map $f : (X, \rho_1) \to (Y, \rho_2)$ of a metric space is continuous if and only if, for any open subset $U \subset Y$, the pre-image $f^{-1}(U)$ is open in (X, ρ_1).

(ii) The map f is continuous if and only if, for every closed subset $F \subset Y$, the pre-image $f^{-1}(F)$ is closed in (X, ρ_1).

Proof.

(i) ($\Leftarrow$) Take $U = B_{\rho_2}(f(x), \epsilon)$. If $f^{-1}(U)$ is open, then there exists $\delta > 0$ such that $B_{\rho_1}(x, \delta) \subset f^{-1}(U)$; that is, $\rho_1(x', x) < \delta$ implies $\rho_2(f(x'), f(x)) < \epsilon$.

($\Rightarrow$) If $U \subset Y$ is open, then consider a point $x \in f^{-1}(U)$. Since U is open, we can choose a open ball $B_{\rho_2}(f(x), \epsilon) \subset U$. Since f is continuous at x, there exists $\delta > 0$ such that $B_{\rho_1}(x, \delta) \subset f^{-1}(B_{\rho_2}(f(x), \epsilon)) \subset f^{-1}(U)$. Since this is true for all $x \in f^{-1}(U)$, we have $f^{-1}(U)$ open.

(ii) Note that $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$. So if F is closed, then $Y \backslash F$ is open and hence $f^{-1}(Y \backslash F)$ is open. Then $X \backslash f^{-1}(F)$ is open and $f^{-1}(F)$ is closed.

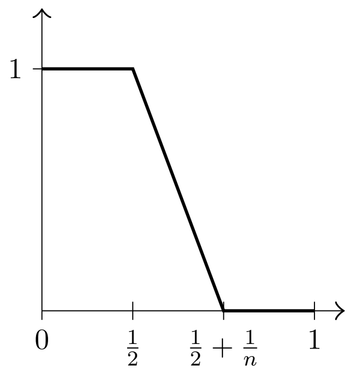
Conversely, if U is open in Y, then $Y \backslash U$ is closed in Y. Then $f^{-1}(Y \backslash U) = X - f^{-1}(U)$ is closed in X, and hence $f^{-1}(U)$ is open in X; that is, f is continuous. □

1.4 Completeness

Definition. A metric space (X, ρ) is called *complete* if, for any sequence $x_1, x_2, \ldots \in X$ such that, given $\epsilon > 0$, there exists N such that for all $m, n \geq N$, $\rho(x_m, x_n) < \epsilon$, we have $x_n \to x$ for some limit point x. That is, every Cauchy sequence in X converges in X.

Recall that $(\mathbb{R}, d_1)$ is complete; this is sometimes referred to as *Cauchy's principle of convergence*. However, neither $(\mathbb{Q}, d_1)$ nor $((0,1) \subset \mathbb{R}, d_{\text{Eucl}})$ are complete.

Example 1.17. Let $X = C[0,1]$ and $\rho = d_1$. This is not complete, for consider f_n as shown:



$$f_n(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2}, \\ 1 - nx & \text{if } \frac{1}{2} \leq x < \frac{1}{2} + \frac{1}{n}, \\ 0 & \text{if } \frac{1}{2} + \frac{1}{n} \leq x \leq 1. \end{cases}$$

Then for $m, n \geq N$, we have $\rho(f_m, f_n) \leq 1/N$.

Now, if $f_n \to f \in C[0,1]$, then $\int_0^1 |f_n - f| \to f$. But

$$\begin{aligned} \int_0^1 |f_n - f| &\geq \int_0^{1/2} \big(|f-1| - |f_n - 1|\big) + \int_{1/2}^1 \big(|f| - |f_n|\big) \\ &\to \int_0^{1/2} |f-1| + \int_{1/2}^1 |f| \geq 0. \end{aligned}$$

Thus we must have

$$\int_0^{1/2} |f-1| = 0 \qquad \text{and} \qquad \int_{1/2}^1 |f| = 0,$$

and our limit is given by

$$f(x) = \begin{cases} 1 & x \leq 1/2, \\ 0 & x > 1/2, \end{cases}$$

but this contradicts $f \in C[0,1]$. Hence this space is not complete.

2 Topological spaces

2.1 Introduction

We've already discussed some of the properties of open subsets of metric spaces. We can abstract these for a definition of a *topological space*.

Definition. A *topological space* (X, τ) consists of a set X and a set (called the *topology*) τ of subsets of X (hence $\tau \subset \wp(X)$, the power set of X). By definition, we call the elements of τ the *open sets*, satisfying the three properties

(i) $X, \emptyset \in \tau$;
(ii) If $U_i \in \tau$ for all $i \in I$, then $\bigcap_{i \in I} U_i \in \tau$;
(iii) If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$ (or similarly for finite intersections).

In this sense, a metric space (X, ρ) gives rise to a topology, which we call the *metric topology*.

Two metrics ρ_1 and ρ_2 on a set X are called (topologically) *equivalent* if the associated topologies are the same.

Exercise 2.1. Show that Lipschitz equivalence implies equivalence.

Example 2.2. The discrete metrix on a set X gives rise to the *discrete topology*, in which *all* subsets are open; that is, $\tau = \wp(X)$.

Examples 2.3 (Non-metric topologies)**.**

(i) Let X be a set with at least two elements, and take $\tau = \{\emptyset, X\}$. This is the *indiscrete topology*.
(ii) Let X be any (infinite) set, and take

$$\tau = \{\emptyset\} \cup \{Y \cup X \text{ such that } X \backslash Y \text{ is finite}\}.$$

Then (X, τ) is a topological space, and τ is the *cofinite topology*.

If X is $\mathbb{R}$ or $\mathbb{C}$, then this is called the *Zariski topology*, and open sets are "complements of zeroes of polynomials". This, and similar Zariski topologies on $\mathbb{R}^n$ and $\mathbb{C}^n$, are very important in Part II *Algebraic Geometry*.
(iii) Let X be any (uncountable) set, such as $\mathbb{R}$ or $\mathbb{C}$, and take

$$\tau = \{\emptyset\} \cup \{Y \cup X \text{ such that } X \backslash Y \text{ is countable}\}.$$

This is the *co-countable topology*.
(iv) Finally, we consider some finite topologies. Take $X = \{a, b\}$. Then there are four distinct topologies:
- Discrete (a metric topology);
- Indiscrete;
- $\{\emptyset, \{a\}, \{a, b\}\}$;
- $\{\emptyset, \{b\}, \{a, b\}\}$.

Remark. A subset $Y \subset X$ of a topological space (X, τ) is called *closed* if $X \backslash Y$ is open, just as we defined in metric spaces.

We can describe a topology on a set X by specifying the *closed sets* in X; these will satisfy

(i) Both $\emptyset$ and X are closed;
(ii) If F_i, for $i \in I$, are closed, then so too is $\bigcap_{i \in I} F_i$.

This description is sometimes more natural, such as with examples (ii) and (iii) above.

Example 2.4. The *half-open interval topology* τ on $\mathbb{R}$ consists of the arbitrary unions of half-open intervals $[a, b)$, for $a < b$ and $a, b \in \mathbb{R}$. Clearly, $\emptyset, R \in \tau$, and τ is closed under unions. To show that this is a topology, we must prove that it is also closed under finite intersections.

Suppose $U_1, U_2 \in \tau$. Then for any $P \in U_1 \cap U_2$, there is a half-open interval $[a, b)$ containing P with $[a, b) \subset U_1 \cap U_2$, and thus $U_1 \cap U_2$ is open.

For consider: since $P \in U_1$, we have $P \in [a_1, b_1) \subset U$. Similarly, since $P \in U_2$, we have $P \in [a_2, b_2) \subset U_2$. Now let $a = \max\{a_1, a_2\}$ and $b = \min\{b_1, b_2\}$, and then $P \in [a, b) \subset U_1 \cap U_2$.

Thus this is indeed a topology.

Definition. If P is a topological space (X, τ), then an *open neighbourhood* of P is any open set $U \in \tau$ such that $P \in U$.

A sequence of points x_n converges to a limit x (written $x_n \to x$) if, for any open neighbourhood $U \ni x$, there is some N such that for all $n \geq N$, $x_n \in U$.

Given topological spaces (X, τ_1) and (Y, τ_2), a map $f : (X, \tau_1) \to (Y, \tau_2)$ is *continuous* if, for any open set $U \subset Y$, the pre-image $f^{-1}(U)$ is open in X.

From this definition, we deduce that f is continuous if and only if, for any closed set $F \subset Y$, the pre-image $f^{-1}(F)$ is closed in X.

Examples 2.5. The identity map $(\mathbb{R}, \tau_{\text{Eucl}}) \to (\mathbb{R}, \text{cofinite topology})$ is continuous, since closed sets in the cofinite topology (that is, finite sets in $\mathbb{R}$), are closed in the Euclidean topology.

However, the identity map $(\mathbb{R}, \tau_{\text{Eucl}}) \to (\mathbb{R}, \text{co-countable topology})$ is not, since $\mathbb{Q} \subset \mathbb{R}$ is closed in the co-countable topology, but not in the Euclidean topology.

Definition. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is a *homeomorphism* if

(i) f is bijective;
(ii) Both f and f^{-1} are continuous.

In this case, the open subsets of X correspond precisely (under the bijection f) with the open subsets of Y. This is an equivalence relation between the two topological spaces: from a topological point of view, the spaces are the "same".

Example 2.6. Let τ_1 and τ_2 denote the Euclidean topology on $\mathbb{R}$ and $(-1,1) \subset \mathbb{R}$, respectively. Both are metric topologies. Now consider the function f, with inverse g, given by

$$\begin{array}{rcrcl} f & : & \mathbb{R} & \longrightarrow & (-1,1) \\ & & x & \longmapsto & x/(|x|+1) \end{array} \qquad \begin{array}{rcrcl} g & : & (-1,1) & \longrightarrow & \mathbb{R} \\ & & y & \longmapsto & y/(|y|-1) \end{array}$$

Here, both f and g are continuous, and so f (and hence g) are homeomorphisms.

This example shows that "completeness" is a property of the metric, and not just a "topological property". Note that $\mathbb{R}$ is complete with the Euclidean topology, while $(-1,1)$ is not. However, it also shows that, under the homeomorphism, we obtain a complete metric ρ on $(-1,1)$ coming from d_{Eucl} on $\mathbb{R}$ which is topologicall equivalent to d_{Eucl} on $(-1,1)$.

Definition. We call a property on topological spaces a *topological property* if, given two homeomorphic spaces (X,τ_1) and (Y,τ_2), one has the property if and only if the other has the property also. That is, (X,τ_1) has the property if and only if (Y,τ_2) has the property.

Let us consider an important example of a topological property:

Definition. A topological space (X,τ) is called *Hausdorff* if, for any $P \neq Q$, $P, Q \in X$, there are disjoint open sets $U \ni P$ and $V \ni Q$; that is, we can separate points by open sets.

Clearly this is a topological property. Let us consider some examples:

Examples 2.7.

(i) $\mathbb{R}$ with the cofinite topology is not Hausdorff, since any two non-empty open sets intersect non-trivially, and the same is true for examples 2.3 (i) to (iii). But any metric space is clearly Hausdorff, which is why the topologies in examples 2.3 are non-metric.

(ii) The half-open interval topology on $\mathbb{R}$ is Hausdorff: if $a < b$, $a, b \in \mathbb{R}$, then we have $[a,b) \cap [b, b+1) = \emptyset$. We will see on the examples sheet that this is *not* a metric topology.

2.2 Closed sets

> **Definition.** For any set $A \subset X$, we say that $x_0 \in X$ is an *accumulation point* for A if any open neighbourhood U of x_0 has $U \cap A \neq \emptyset$. (Sometimes these are called *limit points.*)

Lemma 2.8. *A set is closed if and only if it contains all of its accumulation points; that is, if $x_0 \in X$ is an accumulation point for A, then $x_0 \in A$.*

Proof. ($\Rightarrow$) Suppose A is closed and $x_0 \notin A$. Then take $U = X \backslash A$, an open neighbourhood of x_0. But then $U \cap A = \emptyset$, and so x_0 is not an accumulation point.

($\Leftarrow$) Suppose A is not closed, then $X \backslash A$ is not open. Then there exists $x_0 \in X \backslash A$ such that no open neighbourhood U of x_0 is contained in $X \backslash A$; that is, any open neighbourhood U of x_0 has $U \cap A \neq \emptyset$. Then x_0 is an accumulation point of A. $\square$

Remark. Suppose we have a convergent sequence $x_n \to x \in X$, with $x_n \in A$ for all n. Then any open neighbourhood U of x_0 contains all x_n for $n \gg 0$. Thus x is an accumulation point for A, and if A is closed, we must have $x \in A$.

> **Definition.** Let (X, τ) be a topological space and $P \in X$. If there are nested open neighbourhoods of P, given by $N_1 \supset N_2 \supset N_3 \supset \cdots$, such that for any open neighbourhood U of P, there exists an m such that $N_i \subset U$ for all $i \leq m$, then we say that (X, τ) has *countable bases of neighbourhoods* (or is *first countable*).

> **Example 2.9.** Any metric space is first countable: given $P \in X$, we may take the open balls $B(P, 1/n)$, for $n = 1, 2, 3, \ldots$.

Lemma 2.10. *Suppose (X, τ) is first countable, and we have a subset $A \subset X$. If all convergent subsequences $x_n \to x$, $x_n \in A$ for all n, have their limit $x \in A$, then A is closed.*

Proof. It is sufficient to prove that any accumulation point for A is the limit of some sequence $x_n \in A$, for all n. This gives us A closed.

Suppose x is an accumulation point for A, and let $N_1 \supset N_2 \supset N_3 \cdots$ be a base of open neighbourhoods for x. Then for each i, there exists $x_i \in A \cap N_i$. Hence we construct a sequence x_i, for $i = 1, 2, \ldots$.

For any open neighbourhood U of x, there is an m such that $N_i \subset U$ for all $i \geq m$. Hence $x_i \in U$ for all $i \geq m$, which implies $x_n \to x$. Hence $x \in A$ by the assumption. $\square$

On the first examples sheet, we will see a space which shows that we need the first countable condition here.

2.3 Interiors and closures

Suppose (X, τ) is a topological space.

- If $\{U_i\}_{i\in I}$ are open in (X, τ), then so is $\bigcup_{i\in I} U_i$;
- If $\{F_i\}_{i\in I}$ are closed in (X, τ), then so is $\bigcap_{i\in I} F_i$;

as $X \backslash \bigcap_{i\in I} F_i = \bigcup_{i\in I} U_i \backslash F_i$ is open. This motivates the following definitions:

Definition. Given any subset $A \subset X$, the *interior of* A, denoted $\mathrm{Int}(A) = A^\circ$, is the union of all open subsets contained in A. Then $\mathrm{Int}(A)$ is an open subset, $\mathrm{Int}(A) \subset A$, and in fact, it is the largest open subset contained in A.

The *closure of* A, denoted $\mathrm{Cl}(A)$ or $\overline{A}$, is the intersection of all closed sets which contain A. Then $\mathrm{Cl}(A)$ is a closed subset of X containing A; that is, $A \subset \mathrm{Cl}(A)$. In fact, $\mathrm{Cl}(A)$ is the *smallest* closed subset containing A.

Remarks.

(i) For any set A, we have $\mathrm{Int}(A) \subset A \subset \mathrm{Cl}(A)$.
(ii) Suppose $A, B \subseteq X$ and $A \subseteq B$. Then $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$ and $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$.

Examples 2.11.

(i) Consider $\mathbb{Q} \subset \mathbb{R}$ with the Euclidean sopology. Since any non-empty open subset contains irrationals, we have $\mathrm{Int}(\mathbb{Q}) = \emptyset$. Also $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$.
(ii) Consider $[0, 1]$. It is easy to see that $\mathrm{Int}([0, 1]) = (0, 1)$. We simply remove the limit points. We can easily go the other way: $\mathrm{Cl}((0, 1)) = [0, 1]$.

Definition. The *boundary* or *frontier* of A is $\partial A = \overline{A} \backslash A^\circ$.

We say that a subset $A \subseteq X$ is *dense* if $\mathrm{Cl}(A) = X$.

Proposition 2.12. *For any set $A \subset X$:*

(i) $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) = \mathrm{Int}(\mathrm{Cl}(A))$*;*
(ii) $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))) = \mathrm{Cl}(\mathrm{Int}(A))$.

Proof.

(i) Since $\mathrm{Int}(\mathrm{Cl}(A))$ is open and $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, taking interiors of both sides gives
$$\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))).$$
Now since $\mathrm{Cl}(A)$ is closed and $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(A)$, taking closures and then interiors of both sides gives
$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) \subseteq \mathrm{Int}(\mathrm{Cl}(A)).$$

(ii) Similar argument; see the first examples sheet. □

Thus, if we start from an arbitrary set $A \subset X$ and take successive interiors and closures, then we may achieve *at most seven* distinct sets:
$$A, \mathrm{Int}(A), \mathrm{Cl}(A), \mathrm{Cl}(\mathrm{Int}(A)), \mathrm{Int}(\mathrm{Cl}(A)), \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))), \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))).$$

Indeed, there is a set $A \subset \mathbb{R}$ for which these seven sets are distinct (see the first examples sheet).

2.4 Base of open sets for a topology

> **Definition.** Given a topological space (X, τ), a collection $\mathcal{B}$ of open subsets $\{U_i\}_{i \in I}$ form a *base* or *basis* for the topology if *any* open set is the union of open sets from $\mathcal{B}$.

So we might ask, when does an arbitrary collection of subsets $\{U_i\}_{i \in I}$ form the base of some topology? It forms a base if, for all i, j, the intersection $U_i \cap U_j$ is the union of sets U_k from the collection. If so, then we can define a topology by specifying that any *open subset* is just a union of U_i in the collection.

> **Example 2.13.** The half-open interval topology on $\mathbb{R}$ has a base consisting of intervals $[a, b)$ for $a < b$, $a, b \in \mathbb{R}$.

This motivates the following definition:

> **Definition.** A topological space (X, τ) is called *secound countable* if it has a countable base of open sets.

Clearly, second countable implies first countable. Consider: for any $P \in X$ and an open set $U \ni P$, we have U as a union of bases of open sets. These open sets U_i satisfy $U_i \subset U$ and $P \in U_i$, which we require.

2.5 Subspace topology

> **Definition.** If (X, τ) is a topological space and $Y \subset X$, then the *subspace topology on Y* has open sets
> $$\tau|_Y := \{U \cap Y : U \in \tau\}.$$
> It is easy to see that this is a topology. Moreover, consider the inclusion map $i : Y \hookrightarrow X$, then i is continuous (since $i^{-1}(U) = U \cap Y$), and the subspace topology is the "smallest" topology for which i is continuous.

Proposition 2.14.

(i) If $\mathcal{B}$ is a base for a topology τ, then
$$\mathcal{B}|_Y := \{U \cap Y : U \in \mathcal{B}\}$$
is a base for a subspace topology.

(ii) If (X, ρ) is a metric space, and ρ_1 is the restriction of the metric to Y, then the subspace topology on Y induced from the ρ-metric on X is the same as the ρ_1-metric topology on Y.

Proof.

(i) This is clear: for any open $U = \bigcap_\alpha U_\alpha$, if $U_\alpha \in \mathcal{B}$, then $U \cap Y = \bigcap_\alpha (U_\alpha \cap Y)$.

(ii) The base for the metric topology on X is given by the open balls $B_\rho(x, \delta)$, for $x \in X$. If $x \in Y$, then $B_\rho(x, \delta) \cap Y = B_{\rho_1}(x, \delta) \subset Y$. In general, $B_\rho(x, \delta) \cap Y$ is the union of ρ_1-balls.

Consider: given $y \in B_\rho(x, \delta) \cap Y$, choose δ' such that $B_\rho(y, \delta') \subseteq B_\rho(x, \delta)$. Then $B_{\rho_1}(y, \delta') = B_\rho(y, \delta') \cap Y \subseteq B_\rho(x, \delta) \cap Y$. Then $B_\rho(x, \delta) \cap Y$, a base for the subspace topology, is the union of open ρ_1-balls. $\square$

2.6 Quotient spaces

Definition. Suppose (X, τ) is a topological space and $\sim$ is an equivalence relation on X. Let $Y = X/\sim$ be the quotient set, and $q : Y \to X$ taking $x \mapsto [x]$ be the quotient set.

Then the *quotient topology* on Y is given by

$$\left\{ U \subseteq Y : q^{-1}(U) \in \tau \right\},$$

the subsets of U of the quotient set Y, for which the union of the equivalence classes in X corresponding to points of U is an open subset of X.

Remark. Now q is continuous, and the quotient topology is the "largest" topology on Y for which this is true. It is easy to see that it does form a topology, since τ is a topology on X.

If $f : X \to Z$ is a continuous map of topological spaces such that $x \sim y$ implies $f(x) = f(y)$, then there is a unique factorisation, and $\overline{f}$, defined by $\overline{f}([x]) = f(x)$, is continuous. (As $q^{-1}(\overline{f}^{-1}(U)) = f^{-1}(U)$ is open in X, so $\overline{f}$ is continuous.)

$$\begin{tikzcd}
X \arrow[r, "q"] \arrow[dr, "f"'] & X/\sim \arrow[d, dashed, "\exists!\overline{f}"] \\
& Z
\end{tikzcd}$$

Examples 2.15.

(i) Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Then the map

$$\begin{array}{cccccc} \phi & : & R/\sim & \longrightarrow & \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \\ & & [x] & \longmapsto & e^{2\pi i x} \end{array}$$

is both well-defined and a homeomorphism. (See examples sheet 1, question 15.)

(ii) Define the two-dimensional torus T^2 to be $\mathbb{R}^2/\sim$, where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$. The topological comes from a metric on T^2 (examples sheet 1, question 18) and so is well-defined. (Note that we sometimes denote T^2 as $\mathbb{R}^2/\mathbb{Z}^2$).

Remark. In general, we can get some rather nasty (non-Hausdorff, for example) topologies for an arbitrary equivalence relation.

(iii) *Special case.* If $A \subset X$, then we can define $\sim$ on X by $x \sim y$ if and only if $x = y$ or $x, y \in A$. The quotient space is sometimes written as X/A, in which we scrunch A down to a point. (Note the conflict with example (ii).) Usually we take A to be closed.

For example, if D is the closed unit disc in $\mathbb{C}$, then the boundary is the unit circle C, and D/C is homeomorphic to S^2, the unit sphere. (See examples sheet 2, question 13.)

Lemma 2.16. *Suppose (X, τ) is Hausdorff and $A \subset X$ is closed. Suppose further that for any $x \notin A$, there are open sets U, V with $U \cap V = \emptyset$, $U \supseteq A$ and $V \ni x$. Then X/A is Hausdorff.*

Proof. Given two points $\overline{x} \neq \overline{y}$ in $X \backslash A$, we have two possibilities:

(i) Neither $\overline{x}$ nor $\overline{y}$ correspond to A. In this case, there are unique $x, y \in X$ corresponding to $\overline{x}$ and $\overline{y}$. Thus there are

$$U_x \supset A,\ V_x \ni x \text{ such that } U_x \cap V_x = \emptyset;$$
$$U_y \supset A,\ V_y \ni y \text{ such that } U_y \cap V_y = \emptyset.$$

Since X is Hausdorff, without loss of generality we may assume that $V_x \cap V_y = \emptyset$. Thus the correpsonding open sets $q(V_x)$ and $q(V_y)$ in X/A separate $\overline{x}$ and $\overline{y}$.

(ii) We have $\overline{x} = q(x)$, where $x \in X \backslash A$, and $\overline{y}$ corresponding to A. Then there exist open sets $U \supset A$ and $V \ni x$ such that $U \cap V = \emptyset$. The corresponding open sets $q(U)$, $q(V)$ in X/A separate $\overline{x}$ and $\overline{y}$. $\square$

2.7 Product topologies

> **Definition.** Given topological space (X, τ) and (Y, σ), we define the *product topology* $\tau \times \sigma$ on $X \times Y$ as follows: $W \subseteq X \times Y$ is open if and only if, for all $x, y \in X$, there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $x \in U$, $y \in V$ and $U \times V \subseteq W$.
>
> In other words, $X \times Y$ has a base of open sets of the form $U \times V$, where U is open in X and V is open in Y.

Notice that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, and so this does indeed form the base of a topology. We can extend our definition to a product of countably many spaces:

> **Definition.** If $(X_i, \tau_i)_{i=1}^n$ are topological spaces, then the product topology on $\prod_{i=1}^n X_i$ is defined by having a base of open sets of the form $\prod_{i=1}^n U_i$, where U_i is open in X_i, for $i = 1, \ldots, n$.

Again, this forms a topology.

> **Example 2.17.** Consider $\mathbb{R}$ with the usual topology. The product topology on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is just the usual metric topology. Basic open sets in the product topology include the open rectangles $I_1 \times I_2$ (a product of open intervals I_1 and I_2 in $\mathbb{R}$), and these form a base for the usual topology on $\mathbb{R}^2$.

Lemma 2.18. *Given topological spaces* (X_1, τ_1) *and* (X_2, τ_2), *the projection maps* $\pi_i : (X_1 \times X_2, \tau_1 \times \tau_2) \to (X_i, \tau_i)$ *are continuous. Moreover, given a topological space* (Y, τ) *with continuous map* $f_i : Y \to X_i$, *there is a unique factorisation, and* f *(in the diagram) is continuous.*

$$\begin{tikzcd}
 & & X_1 \\
Y \arrow[urr, bend left, "f_1"] \arrow[r, dashed, "\exists ! f"] \arrow[drr, bend right, "f_2"'] & X_1 \times X_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & X_2
\end{tikzcd}$$

Proof. The first part is easy: we use $\pi_1^{-1}(U) = U \times X_2$ and $\pi_2^{-1}(V) = X_1 \times V$.

For the second part, define $f(y) = (f_1(y), f_2(y)) \in X_1 \times X_2$. The fact that f is unique, making the diagram commute, is obvious. For any basic open set $U \times V \subseteq X_1 \times X_2$, where U is open in X_1 and V is open in X_2, we have $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}$ open in Y.

Further, since any open set W in $X_1 \times X_2$ is the union of basic open sets, $f^{-1}(W)$ is the union of their inverse, and so $f^{-1}(W)$ is open in W. Thus f is continuous. $\square$

3 Connectedness

3.1 Basic notions

Definition. A topological space X is *disconnected* if there are non-empty open subsets U, V with $U \cap V = \emptyset$ and $X = U \cup V$. Otherwise X is called *connected.*

In other words, X is connected if and only if, given open sets U, V with $U \cap V = \emptyset$ and $X = U \cup V$, either $U = \emptyset$ or $V = \emptyset$ (equivalently, either $U = X$ or $V = X$).

Clearly connectedness is a topological property. We can extend this definition to subsets:

Definition. If Y is a subset of the topological space X, then Y is disconnected in the subspace topology if and only if there are U, V open in X such that $U \cap V = \emptyset$, $V \cap Y \neq \emptyset$ but $U \cap V \cap Y = \emptyset$ and $Y \subset U \cup V$. In this case, we say that U and V *disconnect* Y.

Proposition 3.1. *Let X be a topological space. Then the following are equivalent:*

(i) *X is connected;*
(ii) *The only subsets of X that are both open and closed are $\emptyset$ and X;*
(iii) *Every continuous function $f : X \to \mathbb{Z}$ is constant.*

Proof. (i) $\iff$ (ii) is trivial, since for $U \subset X$, we consider that $U \cup (X \backslash U) = X$.

(i) $\implies$ (iii). Suppose there is a non-constant map $f : X \to \mathbb{Z}$, then there exists $m < n$ such that both m, n are in $f(X)$. Then $f^{-1}(\{k : k \leq m\})$ and $f^{-1}(\{k : k > m\})$ are open, non-empty sets disconnecting X; that is, X is *not* connected.

(iii) $\implies$ (i). Suppose that X is not connected; that is, there are non-empty, disjoint open U, V such that $X = U \cup V$. Consider the map $f : X \to \mathbb{Z}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in U, \\ 1 & \text{if } x \in V. \end{cases}$$

This is continuous (even locally constant) function, but not globally constant. □

Proposition 3.2. *A continuous image of a connected space is connected.*

Proof. If $f : X \to Y$ is a surjective continuous map of topological spaces and if U, V disconnect Y, then their pre-images $f^{-1}(U), f^{-1}(V)$ disconnect X. Thus, if X is connected, then so is Y. □

Connectedness in R

Definition. A set $I \subseteq \mathbb{R}$ is called an *interval* if, given $x, z \in I$ with $x \leq z$, we have $y \in I$ for all y such that $x \leq y \leq z$.

We have the cases $\inf I = a \in \mathbb{R}$ and $a \in I$, or $a \notin I$, or $\inf I = -\infty$. Similarly, we have the cases $\sup I = b \in \mathbb{R}$ and $b \in I$, or $b \notin I$, or $\sup I = \infty$. Thus any interval I takes the form $[a, b]$, $[a, b)$, $(a, b]$, $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$ or $(-\infty, \infty)$.

Theorem 3.3

A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Proof. First suppose that $X \subseteq \mathbb{R}$ is not an interval. Then we can find $x < y < z$ with $x, z \in X$ but $y \notin X$. Then $(-\infty, y)$ and (y, ∞) disconnect X.

Now suppose that I is an interval, and $I \subseteq U \cup V$, where U, V are open subsets of $\mathbb{R}$ disconnecting I. Then there exists $u \in U \cap I$, $v \in V \cap I$ where, without loss of generality, $u < v$. Since I is an interval, $[u, v] \subseteq I$. Let $s = \sup[u, v] \cap U$.

If $s \in U$, then $s < v$, so $s \neq v$. As U is open, there is $\delta > 0$ with $(s - \delta, s + \delta) \subset U$. Hence there is $s' \in [u, v] \cap U$ with $s' > s$, contradicting s as an upper bound.

If $s \in V$, then there exists δ such that $(s - \delta, s + \delta) \subset V$. Then $(s - \delta, s + \delta) \cap U = \emptyset$ implies $[u, v] \cap U \subset [u, s - \delta]$, which contradicts s as a *least* upper bound. Thus I is connected. □

Corollary 3.4 (Intermediate value theorem)**.** *Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. If $y \in [f(a), f(b)]$ (or $[f(b)], f(a)]$ as appropriate), then there exists $x \in [a, b]$ such that $f(x) = y$.*

Proof. If not, then $f^{-1}((-\infty, y))$ and $f^{-1}((y, \infty))$ disconnect $[a, b]$. □

Connected subsets and subspaces

Proposition 3.5. *Let $\{Y_\alpha\}_{\alpha \in A}$ be connected subsets of a topological space X, such that $Y_\alpha \cap Y_\beta \neq \emptyset$ for all $\alpha, \beta \in A$. Then their union $Y = \bigcup_{\alpha \in A} Y_\alpha$ is connected.*

Proof. Use proposition 3.1: we wish to prove that any continuous function $f : Y \to \mathbb{Z}$ is constant. Now, the restriction $f_\alpha = f\big|_{Y_\alpha}$ is constant on Y_α, for all α; say, $f_\alpha(x) = n_\alpha$ for $x \in Y_\alpha$. Then for all $\beta \neq \alpha$, there exists $z \in Y_\alpha \cap Y_\beta$, and we have $n_\alpha = f(z) = n_\beta$. Thus f is constant on Y. □

Definition. A *connected component* of a topological space X is a *maximal* conected subset Y; that is, if $Z \subseteq X$ is connected and $Z \supseteq Y$, then $Z = Y$.

Each point $x \in X$ is contained in a unique connected component of X, namely $\bigcup\{Z$ connected subset of X with $x \in Z\}$. This is connected by proposition 3.5, since $x \in Z$ for each Z, and clearly it's maximal.

Example 3.6. Take $X = \{0\} \cup \{1/n : n = 1, 2, 3, \ldots\}$, with the subspace topology. The connected component containing $1/n$ is clearly just $\{1/n\}$, both open and closed in X. The component containing 0 is $\{0\}$, which is closed but not open.

Proposition 3.7. *If Y is a connected subset of a topological space X, then the closure $\overline{Y}$ is connected.*

Proof. Suppose $f : \overline{Y} \to \mathbb{Z}$ is continuous. Then proposition 3.1 tells us that $f|_Y$ is constant; say $f(y) = m$ for all $y \in Y$. Then given any $x \in \overline{Y}$, let $f(x) = n$; then $f^{-1}(n)$ is an open neighbourhood of x in Y; that is, of the form $U \cap \overline{Y}$ with U open in X.

Since $x \in \overline{Y}$, this contains a point of Y (where f takes the value m), so $n = m$, and hence f is constant on $\overline{Y}$. Hence $\overline{Y}$ is connected. □

Remark. Thus connected components of a space are always closed (since they are a *maximal* connected subset), but as in the above example, not necessarily open.

> **Definition.** We call a space X *totally disconnected* if its only connected components are single points; that is, its only connected subsets are single points.

Any discrete topological space is totally disconnected, as was the previous example.

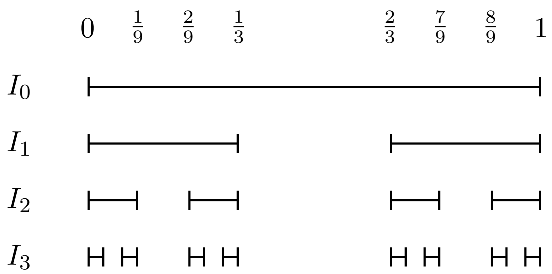
Lemma 3.8. *If X is a topological space and for all $x, y \in X$ with $x \neq y$, X may be disconnected by $U, V \subseteq X$, where $x \in U$ and $y \in V$, then X is totally disconnected.*

Proof. For any subset Y with points $x \neq y$, then for U and V as above, $U \cap Y$ and $V \cap Y$ disconnect Y. □

The irrationals $\mathbb{R} \backslash \mathbb{Q}$ are totally disconnected: use the above lemma, and the fact that the rationals are dense in the irrationals.

The Cantor set

We start with $I_0 = [0, 1]$, and remove $(\frac{1}{3}, \frac{2}{3})$ to obtain $I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. We then remove the middle third from both $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ to obtain I_2. Proceed recursively:



and so on. Then the *Cantor set* C is given by

$$C = \bigcap_{n \geq 0} I_n.$$

We can understand this in terms of ternary (base 3) expansions; that is, expansions of the form $0 \cdot a_1 a_2 a_3 \ldots$, where each $a_i = 0, 1, 2$. Without loss of generality, we impose $1/3 = 0.022\ldots$.

Then I_1 consists of numbers with $a_1 = 0$ or 2. Then I_2 has $a_1 = 0$ or 2, and $a_2 = 0$ or 2, and so on for I_3. Thus C consists of numbers with a ternary expansion where each $a_i = 0$ or 2.

Suppose now we are given two points $x, y \in C$ with $x \neq y$. For some n, the ternary expansions will differ in the nth place (and without loss of generality, they are the same in all previous places). Now $C \subset I_n$, which consists of 2^n closed intervals, one of which contains x, and one of which contains y.

Then we can disconnect I_n by open U, V, where $U \cap V = \emptyset$, $x \in U$ and $y \in V$. Then $C = (U \cap C) \cup (V \cap C)$, and $x \in U \cap C$, $y \in V \cap C$. Then lemma 3.8 tells us that C is totally disconnected.

Note that both C and its complement are uncountable (clear from the ternary expansion).

3.2 Path connectedness

Definition. Let X be a topological space and $x, y \in X$. A *path* from x to y is a continuous function $\phi : [a, b] \to X$ such that $\phi(a) = x$, $\phi(b) = y$) (We sometimes take take $a = 0$, $b = 1$.)

X is *path-connected* if, for all $x, y \in X$, there is a path from x to y.

Proposition 3.9. *The continuous image of a path-connected space is path-connected.*

Proof. Suppose X is path-connected and $f : X \to Y$ is a continuous surjection. Given $y_1, y_2 \in Y$, choose $x_i \in f_i^{-1}(y_i)$, $i = 1, 2$ and a path $\gamma : [a, b] \to X$ from x_1 to x_2. Then there is a path $f \circ \gamma : [a, b] \to Y$ from y_1 to y_2. $\square$

Proposition 3.10. *Path connectedness implies connectedness.*

Proof. Suppose X is a topological space and $X = U \cup V$, with U, V disconnecting X. Suppose for contradiction that X is not path-connected. If it were, then choose $u \in U$, $v \in V$, and there is a continuous function $\phi : [a, b] \to X$ such that $\phi(a) = u$, $\phi(b) = v$. Then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are non-empty open subsets of $[a, b]$ which disconnect $[a, b]$. $\square$

Note that the converse is false: connectedness does not imply path connectedness, as the following example shows:

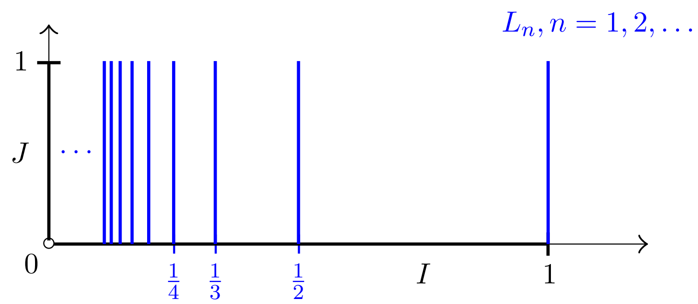
Example 3.11. In $\mathbb{R}^2$, first define

$$I = \{(x, 0) : 0 < x \leq 1\} \qquad \text{and} \qquad J = \{(0, y) : 0 < y \leq 1\}.$$

Then for $n = 1, 2, \ldots$, define

$$L_n = \{(1/n, y) : 0 \leq y \leq 1\}.$$

Now set $X = I \cup J \cup \left(\bigcup_{n \geq 1} L_n\right)$ with the subspace topology.



It is easy to see that X is not path-connected: for any continuous path $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ from $(1, 0)$ to $(0, 1)$, there is some s such that $\gamma_1(s) = 0$ and $\gamma_1(t) > 0$ for all $t < s$, so we must also have $\gamma_2(s) = 0$. This means that γ passes through $(0, 0)$, but this is not in X.

However, such an X is connected. Suppose $f : X \to \mathbb{Z}$ is a continuous function. Then f is constant on J and on $Y = I \cup \left(\bigcup_{n\geq 1} L_n\right)$. However, the points $(1/n, 1/2) \in Y$ have limit $(0, 1/2) \in J$ as $n \to \infty$, and by the continuity of f, the two constants agree. Thus f is constant on X.

Given a topological space X, we can define an equivalence relation $\sim$ on X by $x \sim y$ if and only if there is a path from x to y in X. This is indeed an equivalence relation:

- Reflexivity: $x \sim x$ is trivial.
- Symmetry: if $\phi : [a, b] \to X$ is a path from x to y, then $\psi(t) = \phi(-t)$ gives a path $\psi : [-b, -a] \to X$ from y to x.
- Transitivity: suppose $x \sim y$ and $y \sim z$. Then there are paths $\phi : [a, b] \to X$ and $\psi : [c, d] \to X$ with $\phi(a) = x$, $\phi(b) = y = \psi(c)$ and $\psi(d) = z$. So define a new path $\chi : [a, b + d - c] \to X$ by

$$\chi(t) = \begin{cases} \phi(t) & a \leq t \leq b, \\ \psi(t + c - b) & b \leq t \leq b + d - c. \end{cases}$$

  Then χ is continuous at all points $t \in [a, b + d - c]$ (which is easy to check), and it gives a path from x to z. Thus $x \sim z$.

Definition. The equivalence classes of $\sim$ are called the *path-connected components* of X.

Theorem 3.12

Let X be an open subset of Euclidean space $\mathbb{R}^n$. Then X is connected if and only if X is path connected.

Proof. Path-connectedness implies connectedness by proposition 3.10.

Now the converse: suppose X is connected and $x \in X$. Let U be the equivalence class of x under the equivalence relation $\sim$ defined above.

Now, U is open in X: suppose $y \in U$, whence $x \sim y$. Since X is open, there exists $\delta > 0$ such that $B(y, \delta) \subseteq X$. Then for all $z \in B(y, \delta)$, we have $y \sim z$, by taking the straight line segment. Transitivity implies that $x \sim z$ for all $z \in B(y, \delta)$, and thus $B(y, \delta) \subset U$, and U is open.

Similarly, $X \backslash U$ is open. Suppose $y \in X \backslash U$. Since X is open, there exists $\delta > 0$ such that $B(y, \delta) \subseteq X$. For $z \in B(y, \delta)$, we have $y \sim z$ as above, and then $x \not\sim z$, hence $B(y, \delta) \subsetneq X \backslash U$.

Since X is connected, we must have $X \backslash U = \emptyset$ and $U = X$. Hence X is path-connected, since $x \sim y$ for all $y \in X$. $\square$

3.3 Products of connected spaces

Proposition 3.13. *Let X and Y be topological spaces. If X and Y are path-connected, then so too is $X \times Y$, with the product topology.*

Proof. Given (x_1, y_1) and $(x_2, y_2) \in X \times Y$, we know that there are paths $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to Y$ with $\gamma_1(0) = x_1$, $\gamma_1(1) = x_2$, $\gamma_2(0) = y_1$ and $\gamma_2(1) = y_2$.

Now define a map $\gamma : [0,1] \to X \times Y$ by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. The base for the topology on $X \times Y$ consists of the open sets $U \times V$, with U open in X and V open in Y. So it is sufficient to prove that $\gamma^{-1}(U \times V)$ us open for all such U and V. But $\gamma^{-1}(U \times V) = \gamma^{-1}(U) \cap \gamma^{-1}(V)$ is clearly open. So γ is continuous and defines a path from (x_1, y_1) to (x_2, y_2). □

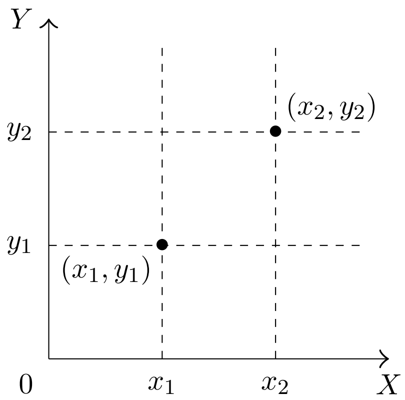
Proposition 3.14. *If X and Y are connected, then so too is $X \times Y$ with the product topology.*

First we make some general comments about product topologies. Given $y \in Y$, the set $X \times \{y\}$ with the subspace topology is homeomorphic to X, using the projection map $\pi_1 : X \times \{y\} \to X$. We already know that this map is a continuous bijection.

However, a base for the topology on $X \times Y$ consists of open sets $U \times V$, with U open in X and V open in Y. This implies that a base for the subspace topology on $X \times \{y\}$ consists of subsets $U \times \{y\}$, for U open subsets of X. Thus, under $\pi_1|_{X \times \{y\}}$, open sets do correspond, and hence $\pi_1 : X \times \{y\} \to X$ is a homeomorphism.

Similarly, for $x \in X$, $\{x\} \times X$ is homeomorphic to Y, and so $X \times \{y\}$ is connected for all $y \in Y$. Thus $\{x\} \times X$ is connected for all $x \in X$.

Proof of proposition 3.14. Given a continuous function $f : X \times Y \to \mathbb{Z}$, it is obvious that f is constant on each slice $\{x\} \times Y$ and $X \times \{y\}$, by connectedness.



Given arbitary points $(x_1, y_1), (x_2, y_2) \in X \times Y$, we deduce that $f(x_1, y_1) = f(x_1, y_2) = f(x_2, y_2)$ (see diagram). Hence f is constant on $X \times Y$ and $X \times Y$ is connected. □

Remark. A similar argument also proves proposition 3.13: there is a path joining (x_1, y_1) to (x_1, y_2) and a path joining (x_1, y_2) to $(x_2, y - 2)$ in $X \times Y$.

4 Compactness

4.1 Basic notions

Definition. Let (X, τ) be a topological space. An *open cover of* X is a collection of open subsets $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ such that $X = \bigcup_{\gamma\in\Gamma} U_\gamma$.

If $Y \subset X$, then an open cover of Y is a collection of open subsets in X $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ such that $Y \subset \bigcup_{\gamma\in\Gamma} U_\gamma$.

Remark. Such an open cover of Y provides a base of open sets $\mathcal{U} = \{U_\gamma \cap Y : \gamma \in \Gamma\}$ for Y with the subspace topology, and conversely.

Definition. A *subcover* of an open cover $\mathcal{U}$ is a subcollection $\mathcal{V} \subseteq \mathcal{U}$ which is still an open cover of Y.

Example 4.1. The intervals $I_n = (-n, n)$, where $n = 1, 2, \ldots$, form an open cover of $\mathbb{R}$, and I_{n^2} is a proper subcover. The intervals $J_n = (n-1, n+1)$, where $n \in \mathbb{Z}$, form an open cover of $\mathbb{R}$ with no proper subcover.

Definition. A topological space (X, τ) is *compact* if every open cover has a finite subcover.

Examples 4.2. In this sense, $\mathbb{R}$ is not compact, as the open covers described above have no finite subcovers. Any finite topological space is compact, as is any set with the indiscrete topology (the only open subsets of X being $\emptyset$ and X), or with the cofinite topology.

With this definition, compactness is a topological property.

Lemma 4.3. *Let (X, τ) be a topological space with $Y \subset X$. Then Y is compact in the subspace topology if and only if every open cover $\{U_\gamma\}$ of Y has a finite subcover.*

Proof. First let $\{U_\gamma : \gamma \in \Gamma\}$ be an open cover of Y, then $Y = \bigcup_{\gamma\in\Gamma}(U_\gamma \cap Y)$, where the $U_\gamma \cap Y$ are open in Y. Since Y is compact, there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $Y = \bigcup_{i=1}^n (U_{\gamma_i} \cap Y)$, and then $\{U_{\gamma_i} : i = 1, \ldots, n\}$ covers Y.

Now the converse. Suppose $Y = \bigcup_{\gamma\in\Gamma} V_\gamma$, where the V_γ are open in Y. Write $V_\gamma = U_\gamma \cap Y$, where the U_γ are open in X and form an open cover of Y. Then there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $Y \subseteq \bigcup_{i=1}^n U_{\gamma_i}$, and hence $Y = \bigcup_{i=1}^n V_{\gamma_i}$. □

Example 4.4. The open interval $(0, 1)$ is not compact: consider the open cover by intervals $(1/n, 1 - 1/n$ for $n = 3, 4, \ldots$.

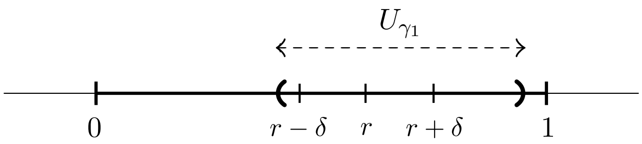
Theorem 4.5: Heine-Borel theorem

The closed interval $[a, b] \subset$ is compact.

Proof. Let $[a, b] \subset \bigcup_{\gamma \in \Gamma} U_\gamma$, for U_γ open in $\mathbb{R}$. Then set

$$K = \left\{x \in [a, b] : [a, x] \text{ is contained in a finite union of the } U_\gamma\right\}.$$

Clearly $a \in K$, so $K \neq \emptyset$. Let $r = \sup K$. Then $r \in [a, b]$, and so $r \in U_{\gamma_1}$ for some $\gamma_1 \in \Gamma$. Since U_{γ_1} is open, there exists $\delta > 0$ such that $[r - \delta, r + \delta] \subseteq U_{\gamma_1}$.



By the definition of r, there exists $c \in [r - \delta, r]$ such that $[a, c]$ is contained in a finite union of the U_γ. Hence, the same is true for $[a, r + \delta] \cap [a, b]$ (we just need to include U_{γ_1} also). But this contradicts r as an upper bound, unless $r = b$, in which case, the above argument tells us that $[a, b]$ is covered by finitely many of the U_γ. (There exists $c \in [b - \delta, b]$ such that $[a, c]$ is covered by finitely many U_γ, and include U_{γ_1} also.)

Thus $[a, b]$ is compact. □

Proposition 4.6. *A continuous image of a compact set is compact.*

Proof. Suppose $f : (X, \tau) \to (Y, \sigma)$ is a continuous map of topological spaces, and $K \subseteq X$ is compact. Then we wish to show that $f(K)$ is compact.

Suppose $f(K) \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$, with U_γ open in Y. Since f is continuous, $K \subset \bigcup_{\gamma \in \Gamma} f^{-1}(U_\gamma)$, and each $f^{-1}(U_\gamma)$ is open in X. Since K is compact, there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $K \subseteq \bigcup_{i=1}^n f^{-1}(U_{\gamma_i})$, and hence $f(K) \subseteq \bigcup_{i=1}^n U_{\gamma_i}$. □

Proposition 4.7. *A closed subset of a compact topological space X is compact.*

Proof. Let X be a compact topological space, and $K \subseteq X$ be closed. If $K = \emptyset$ then this is trivial, so assume not. Suppose $K \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$, where the U_γ are open in X. Then $X = (X \backslash K) \cup \left(\bigcup_{\gamma \in \Gamma} U_\gamma\right)$, where $X \backslash K$ is also open. Since X is compact, there is a finite subcover, and hence there exists $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $X = (X \backslash K) \cup \left(\bigcup_{i=1}^n U_{\gamma_i}\right)$. Thus $K \subseteq \bigcup_{i=1}^n U_{\gamma_i}$. □

Proposition 4.8. *Every compact subset of a Hausdorff topological space is closed.*

Proof. Let X be a Hausdorff topological space, and $K \subseteq X$ be compact. If $K = X$, this is trivial, so suppose not. Then we show that $X \backslash K$ is open.

Given $x \in X \backslash K$, for any $y \in K<$ there are disjoint open sets with $U_y \ni x$ and $V_y \ni y$. Now $\{V_u : y \in K\}$ is an open cover of K. Since K is compact, there exist $y_1, \ldots, y_n \in K$ such that $K \subseteq \bigcup_{i=1}^n V_{y_i}$. Then $U = \bigcap_{i=1}^n U_{y_i}$ is an open neighbourhood of x with $U \cap K = \emptyset$, so $U \subseteq X \backslash K$, as required. □

Corollary 4.9. *A set $X \subset \mathbb{R}$ is compact if and only if it is closed and bounded.*

Proof. If $X \subset \mathbb{R}$ is compact, then proposition 4.8 tells us that X is closed (since $\mathbb{R}$ is Hausdorff). It is also bounded: if not, the open sets $U_m = \{x \in \mathbb{R} : |x| < m\}$ form an open cover of X with no finite subcover, contradiction.

Now the converse. Suppose $X \subset \mathbb{R}$ is closed and bounded. Then there is M such that $|x| \leq M$ for all $x \in X$ and so $X \subset [-M, M]$. Since X is closed, $\mathbb{R} \backslash X$ is open, and so $(\mathbb{R} \backslash X) \cap [-M, M]$ is open in $[-M, M]$, and X is closed in $[-M, M]$. Then by Heine-Borel, X is compact, and so X is compact by proposition 4.7. $\square$

Remark. We'll see later that the product of finitely many compact spaces is compact, and hence $[-M, M]^n$ is a compact subset of $\mathbb{R}^n$. So a set $X \subset \mathbb{R}^n$ is bounded if and only if $\exists\, M$ such that $X \subset [-M, M]^n$. From the proof of corollary 4.9, this extends immediately to show that:

Corollary 4.10. *A subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Examples 4.11.

(i) Combining this with theorem 3.3, the only connected compact subsets of $\mathbb{R}$ are closed intervals $[a, b]$.

(ii) The Cantor set $C \subset [0, 1]$ was defined by $C = \bigcap_{n \geq 0} I_n$, where $[0, 1] = I_0 \supset I_1 \supset I_2 \supset \cdots$, and I_n was the diagonal union of 2^n closed intervals. Thus each I_n is closed, and so C is closed and bounded. Thus C is compact.

We can now combine propositions 4.6, 4.7 and 4.8 into a particular useful result:

Corollary 4.12. *Suppose X is a compact space, Y is a Hausdorff space, and $f : X \to Y$ is a continuous bijection. Then f is a homeomorphism.*

Proof. Let $g : Y \to X$ be the inverse map f^{-1}. We must show that this is continuous. Let $F \subseteq X$ be closed.

Since X is compact, proposition 4.7 tells us that F is compact. Since f is continuous, proposition 4.6 tells us that $g^{-1}(F) = f(F)$ is compact. Finally, Y is Hausdorff, so by proposition 4.8, $g^{-1}(F)$ is closed in Y. Hence g is continuous. $\square$

This result is particularly useful in identifying quotient spaces.

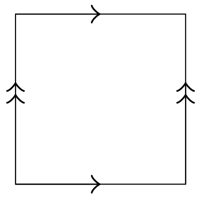
Example 4.13. Define $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$, and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle (with the subspace topology from $\mathbb{C}$). Now consider the map given by

$$\begin{array}{rcrcl} f & : & \mathbb{R} & \longrightarrow & \mathbb{T} \\ & & x & \longmapsto & \exp(2\pi i x) \end{array}$$

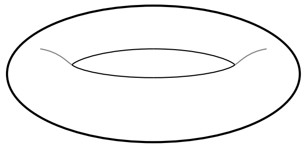
This is continuous and induces a bijection $\overline{f} : \mathbb{R}/\sim\, \to \mathbb{T}$, which, by definition of the quotient topology, is also continuous. (See our remark about quotient topologies on page 19.)

But the quotient map $q : \mathbb{R} \to \mathbb{R}/\sim$ restricts to a continuous surjection $[0, 1] \to \mathbb{R}/\sim$. Since $[0, 1]$ is compact, proposition 4.6 tells us that $\mathbb{R}/\sim$ is compact. Then $\mathbb{T}$ is Hausdorff, and so corollary 4.12 tells us that $\overline{f}$ is a homeomorphism.

In a similar way, we can show that the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic both the product space $S^1 \times S^1$ (where S^1 is the unit circle), and to the embedded torus $X \subset \mathbb{R}^3$, consisting of points $((2+\cos\phi)\cos\theta, (2+\cos\phi)\sin\theta, \sin\phi)$, where $0 \leq \theta, \phi < 2\pi$.



the 2-D torus $\mathbb{R}^2/\mathbb{Z}^2$



the embedded torus $X \subseteq \mathbb{R}^3$

In *Analysis I*, you learnt that continuous real-valued functions on $[a, b] \subseteq \mathbb{R}$ are bounded and attain their bounds. Here, we are using the compactness of $[a, b]$. This statement is still true, for instance, for continuous real-value functions on the Cantor set.

Proposition 4.14. *Continuous real-valued functions on a compact space X are bounded and attain their bounds.*

Proof. Suppose X is compact and $f : X \to \mathbb{R}$ is continuous. Then proposition 4.6 tells us that $f(X)$ is compact, and so corollary 4.10 tells us that $f(X)$ is bounded and closed.

Since $f(X)$ is closed, it contains all of its accumulation points. But $\sup f(X)$ and $\inf f(X)$ (which exist because $f(X)$ is bounded) are accumilation points for $f(X)$, so $\sup f(X), \inf f(X) \in f(X)$, which gives the desire result. $\square$

Theorem 4.15

The product of two compact spaces is compact.

Proof. Suppose X and Y are compact and that $X \times Y = \bigcup_{\gamma \in \Gamma} U_\gamma$, where the U_γ are open in $X \times Y$.

By the definition of the product topology, each U_γ is the union of "basic open sets" of the form $V \times W$, where V is open in X and W is open in Y. Thus

$$X \times Y = \bigcup_{\delta \in \Delta} V_\delta \times W_\delta,$$

with V_δ open in X, W_δ open in Y and $V_\delta \times W_\delta$ a subset of some U_γ.

Now let $x \in X$, and then we have

$$\{x\} \times Y \subseteq \bigcup_{\delta \in \Delta} V_\delta \times W_\delta,$$

such that $x \in V_\delta$.

Now, since Y is compact, there exist $\delta_1, \ldots, \delta_m$ such that $Y = \bigcup_{i=1}^m W_{\delta_i}$.

Then let $V_x = \bigcap_{i=1}^n V_{\delta_i}$ be an open neighbourhood of x such that $V_x \times Y \subseteq \bigcap_{i=1}^n V_{\delta_i} \times W_{\delta_i}$. The V_x obtained in this way form an open cover of X, and so there exist $x_1, \ldots, x_n$ such that $X = \bigcup_{j=1}^n V_{x_j}$.

Now $X \times Y = \bigcup_{j=1}^n V_{x_j} \times Y$ and each $V_{x_j} \times Y$ has a finite cover by $V_\delta \times W_\delta$'s. Thus $X \times Y$ has a finite cover by such sets. Since each $V_\delta \times W_\delta$ is a subset of some U_γ, $X \times Y$ has a finite cover by U_γ's. Thus $X \times Y$ is compact. $\square$

Remarks. Given topological spaces X, Y, Z, the product $X \times Y \times Z$ is homeomorphic to $X \times (Y \times Z)$ under the obvious map (since a base for the topology of $X \times Y \times Z$ consists of subsets $U \times V \times W$, and a base for the topology of $X \times (Y \times Z)$ consists of subsets $U \times (V \times W)$, for U open in X, V open in Y and W open in Z. Hence open sets in $X \times Y \times Z$ correspond to open sets in $X \times (Y \times Z)$.) By induction, the above theorem implies that the product of finitely many compact spaces is compact.

Now applying corollary 4.10, we see that $[-M, M]^n$ is a compact subset of $\mathbb{R}^n$. The proof of the same corollary may be extended to show that $X \subseteq \mathbb{R}^n$ is compact if and only if X is closed and bounded.

Proposition 4.16. *Let X be a compact metric space, Y a metric space and $f : X \to Y$ a continuous map. Then f is* uniformly continuous*; that is, given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$.*

Proof. Since f is continuous, for all $x \in X$, there exists δ_x such that $d_X(x, y) < 2\delta_x$ implies $d_Y(f(x), f(y)) < \epsilon/2$. Now let

$$U_x = \left\{y : d_X(x, y) < \delta_x\right\}.$$

The U_x form an open cover of X, and so there exist $x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^n U_{x_i}$. Let $\delta = \min\{\delta_{x_i}\}$.

Suppose now $d_X(y, z) < \delta$; since the U_{x_i} form a cover, we can find x_i such that $d(y, x_i) < \delta_{x_i}$. Since $d_X(y, z) < \delta < \delta_{x_i}$, we deduce that $d_X(z, x_i) < \delta + \delta_i < 2\delta_i$ (from the triangle inequality). Thus

$$d_Y(f(y), f(z)) < d_Y(f(y), f(x_i)) + d_Y(f(x_i), f(z)) < \epsilon/2 + \epsilon/2 = \epsilon. \qquad \square$$

4.2 Sequential compactness

> **Definition.** A topological space is *sequentially compact* if every sequence in X has a convergent subsequence.

Remark. For general topological spaces, the property of compactness and sequential compactness are independent; neither implies the other.

Proposition 4.17. *Any compact metric space is sequentially compact.*

Notice that this can be reduced to the Bolzano-Weierstrass theorem: namely, that any closed bounded subset of $\mathbb{R}^n$ is sequentially compact.

Proof. Suppose (X, d) is a metric space and $(x_n)_{n=1}^\infty$ is a sequence in X with no convergent subsequence (in particular, there are infinitely many distinct x_n). We claim that for all $x \in X$, there exists $\delta > 0$ such that $d(x, x_n) < \delta$ for at most finitely many n.

If not, then there exists $x \in X$ such that for all $m > 0$ in $\mathbb{N}$, $d(x, x_n) < 1/m$ for infinitely many n, and hence there is a subsequence of (x_n) converging to x, which is a contradiction.

For each x, pick such a $\delta = \delta(x)$ and let $U_x = \{y : d(x, y) < \delta\}$. Each U_x contains x_n for only finitely many n. But $\{U_x : x \in X\}$ is an open cover for X, for which no finite subcover can exist. Hence X is not compact. $\square$

Exercise 4.18. Show directly that if $X \subseteq \mathbb{R}^n$ is sequentially compact, then it is bounded, closed and hence compact. (Bounded, since otherwise we can find (x_n) such that $d(x_n, x_0) > n$, which implies that there is no convergent subsequence.)

More generally, we have:

Theorem 4.19

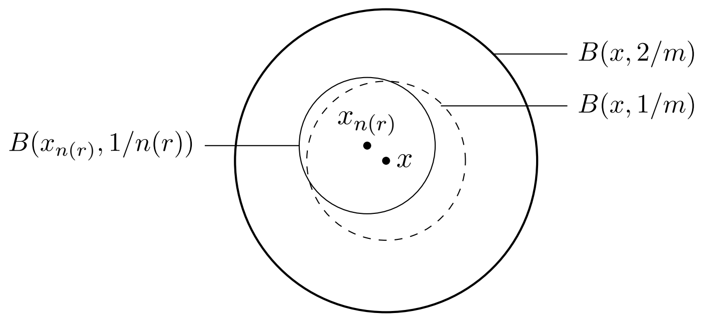
Suppose (X, d) is a sequentially compact metric space. Then

(i) Given any $\epsilon > 0$, there exists $x_1, \dots, x_n$ such that $X = \bigcup_{i=1}^n B(x_i, \epsilon)$;
(ii) Given any open cover $\mathcal{U}$ of X, there exists $\epsilon > 0$ such that for all $x \in X$, $B(x, \epsilon)$ is contained in some element U of $\mathcal{U}$.
(iii) (X, d) is compact.

Proof.

(i) Suppose not. Then by induction, we can construct a sequence (x_n) in X such that $d(x_m, x_n) \geq \epsilon$ for all $m \neq n$. Clearly such a sequence has no convergent subsequence, since no subsequence can satisfy the Cauchy condition. Contradiction.

(ii) Suppose not. Then there exists an open cover $\mathcal{U}$ of X such that for all n, there exists $x_n \in X$ such that $B(x_n, 1/n) \not\subseteq U$, for all $U \in \mathcal{U}$. But (x_n) has a subsequence $(x_{n(r)})$ tending to $x \in X$. So let $x \in U_0$, for some $U_0 \in \mathcal{U}$. Since U_0 is open, there exists $m > 0$ such that $B(x, 2/m) \subseteq U_0$.



Now, there exists N such that $x_{n(r)} \in B(x, 1/m)$ for all $r \geq N$. Additionally, if $n(r) > m$, and $y \in B(x_{n(r)}, 1/n(r))$, then $d(x, y) \leq d(x, x_{n(r)}) + d(x_{n(r)}, y) < 2/m$. So for such $n(r)$, $B(x_{n(r)}, 1/n(r)) \subseteq B(x, 2/m) \subseteq U_0$. Contradiction.

(iii) Let $\mathcal{U}U$ be an open cover of X. Choose $\epsilon > 0$ as in (ii). For this ϵ, using (i), there exists $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n B(x_i, \epsilon)$. For each i, $B(x_i, \epsilon) \subseteq U_i$ for some $U_i \in \mathcal{U}$, by (ii). Thus $X = \bigcup_{i=1}^n U_i$, and X is compact. $\square$

End of notes