EM HW2 B06901147 李宥頡

(1)(a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z} + u$

Sol: 令 $u = XYZ$

$X'YZ = XY'Z - XYZ' + XYZ$

同除 XYZ 得

$\frac{X'}{X} = \frac{Y'}{Y} - \frac{Z'}{Z} + 1 = -\lambda$

$\therefore X' = -\lambda X$

$\frac{Y'}{Y} - \frac{Z'}{Z} + 1 = -\lambda$

$\frac{Y'}{Y} = \frac{Z'}{Z} - \lambda - 1 = -\mu$

$\begin{cases} \therefore Y' + \mu Y = 0 \\ Z' - (\lambda + 1 - \mu) Z = 0 \end{cases}$

$\therefore X' = -\lambda X$

(代 $\alpha = 0$)

$\lambda = 0$, $\lambda = \alpha^2 > 0$ 和 $\lambda = -\alpha^2 < 0$

都為 $X' = -\alpha^2 X$

$X = C_2 e^{\alpha^2 x} + C_3 e^{-\alpha^2 x}$

$= C_2 e^{\lambda x} + C_3 e^{-\lambda x}$

$Y' = -\mu Y$

$\mu = 0$ (代 $\beta = 0$)

$\mu = \beta^2 > 0$ 和 $\mu = -\beta^2 < 0$

都為 $Y = C_5 e^{\beta^2 y} + C_6 e^{-\beta^2 y}$

$Y = C_5 e^{\mu y} + C_6 e^{-\mu y}$

$\therefore Z' = (\lambda + 1 - \mu) Z$

$Z = C_7 e^{\lambda - \mu + 1 z} + C_8 e^{-(\lambda - \mu + 1) z}$

$\therefore u = \left(C_2 e^{\lambda x} + C_3 e^{-\lambda x}\right)\left(C_5 e^{\mu y} + C_6 e^{-\mu y}\right)\left(C_7 e^{(\lambda - \mu + 1) z} + C_8 e^{-(\lambda - \mu + 1) z}\right)$ #

(1)(b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2\frac{\partial u}{\partial t}$ $\quad 0 \le x \le 2 \quad y > 0 \,.\, t > 0 \quad \frac{\partial u}{\partial y} = -u \quad u(0,y,t) = u(2,y,t) = 0 \quad \therefore X(0) = 0 ,\ X(2) = 0$

$XY'T = -XYT \quad \therefore Y' = -Y$

Sol: 令 $u = XYT$

$X''YT = XY'T + 2XYT'$

同除 XYT

$\frac{X''}{X} = \frac{Y'}{Y} + 2\frac{T'}{T} \overset{令}{=} -\lambda$

$\begin{cases} X'' = -\lambda X \\ \frac{Y'}{Y} + 2\frac{T'}{T} = -\lambda \Rightarrow \frac{Y'}{Y} = \frac{-2T'}{T} - \lambda \overset{令}{=} -\mu \end{cases}$

$\Rightarrow \begin{cases} Y' = -\mu Y \\ \frac{2T'}{T} + \lambda = \mu \Rightarrow 2T' + (\lambda - \mu)T = 0 \end{cases}$

Case 1 : $\lambda = 0$

$X'' = 0$

$X = C_1 x + C_2$

代入 b.c $X(0) = 0 = X(2)$

$0 = C_2$

$0 = 2C_1 \quad \therefore C_1 = 0$

$\therefore X = 0 \quad \therefore u = 0$

Case 2 : $\lambda = -\alpha^2 < 0$

$X'' - \alpha^2 X = 0$

$X = C_3 \cosh\alpha x + C_4 \sinh\alpha x$

代入 b.c $X(0) = 0 = X(2) = 0$

$0 = C_3$

$0 = C_4 \sinh\alpha \times 2 \quad \therefore X = 0$

$\therefore C_4 = 0 \quad u = 0$

Case 3 $\lambda = \alpha^2 > 0$

$X'' + \alpha^2 X = 0$

$X = C_5 \cos\alpha x + C_6 \sin\alpha x$

代入 b.c

$0 = C_5$

$0 = C_6 \sin 2\alpha$

$\alpha = \frac{n\pi}{2} ,\ \lambda = \frac{n^2\pi^2}{4} \quad n = 0, 1, 2, \dots$

$\frac{\lambda + 1}{2} = \frac{\frac{n^2\pi^2}{4} + 1}{2} = \frac{n^2\pi^2 + 4}{8}$

$Y' = -\mu Y$

from $\frac{\partial u}{\partial y} = -u$

$\therefore XY'T = -XYT$

$\therefore Y' = -Y$

$\therefore \mu = 1$

$Y' = -Y$

$\frac{dY}{dy} = -Y$

$\frac{dY}{Y} = -dy \quad \therefore \ln Y = -y + C$

$\therefore Y = e^{-y+C} = C' e^{-y}$

$\therefore 2T' + (\lambda - \mu)T = 0$

$\Rightarrow 2T' + (\lambda + 1)T = 0$

for $\lambda > 0$

$T' + \frac{\lambda + 1}{2} T = 0$

$T' = -\frac{\lambda + 1}{2} T \quad \frac{dT}{dt} = -\frac{\lambda + 1}{2} T$

$\frac{dT}{T} = -\frac{\lambda + 1}{2} dt$

$\ln T = -\frac{\lambda + 1}{2} t + C \quad T = C'' e^{-\frac{\lambda + 1}{2} t}$

$\therefore u_n = C_6 \sin\alpha x \cdot C' e^{-y} \cdot C'' e^{-\frac{\lambda + 1}{2} t}$

$= A_n \sin\frac{n\pi}{2} x \cdot e^{-y} e^{-\frac{n^2\pi^2 + 4}{8} t}$

$n = 0$

$\sin\frac{0\pi x}{2} = 0 \quad \therefore u_0 = 0$

$$u = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2} e^{-y} e^{-\frac{n^2\pi^2 + 4}{8} t}$$

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(1)(c) $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$, $0\le r\le 1$, $0\le\theta\le\frac{\pi}{2}$ $u(r,0)=u(r,\frac{\pi}{2})=0$ $u(1,\theta)=f(\theta)$.

$\Theta(0)=\Theta(\frac{\pi}{2})=0$

~~且~~ $\Theta(\theta)=\Theta(\theta+2\pi)$ (沒用)

sol: $u=R\Theta$

$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0 \Rightarrow r^2R''\Theta+rR'\Theta+R\Theta''=0$

同除 $R\Theta$

$\frac{r^2R''}{R}+\frac{rR'}{R}+\frac{\Theta''}{\Theta}=0$ 令 $\frac{r^2R''+rR'}{R}=\frac{-\Theta''}{\Theta}=\lambda$ $\Rightarrow$ $\begin{cases} r^2R''+rR'-\lambda R=0 \\ \Theta''+\lambda\Theta=0 \end{cases}$

Case1 $\lambda=0$

$\Theta''=0$ $\therefore \Theta=C_1\theta+C_2$

$0=C_2$ $0=\frac{\pi}{2}C_1$, $C_1=0$

$\therefore \Theta=0$ $u=0$

Case2 $\lambda>0$

$\lambda=\alpha^2$

$\Theta''+\alpha^2\Theta=0$

$\Theta=C_3\cos\alpha\theta+C_4\sin\alpha\theta$

$0=C_3$

$0=C_4\sin\frac{\pi}{2}\alpha$ $\alpha=2n$

$n=0,1,2,3,4,\dots$

$\lambda_n=4n^2$ $\Theta=C_4\sin 2n\theta$

Case3 $\lambda<0$, $\lambda=-\alpha^2$

$\Theta''-\alpha^2\Theta=0$

$\Theta=C_5\cosh\alpha\theta+C_6\sinh\alpha\theta$

$0=C_5$

$0=C_6\sinh\frac{\pi}{2}\alpha$ $\Theta=0$

$u=0$

$\therefore C_6=0$

$r^2R''+rR'-\lambda R=0$ (Cauchy-Euler)

$m(m-1)+m-4n^2=0$

$m^2=4n^2$ $\therefore m=\pm 2n$

$R=C_1r^{2n}+C_2r^{-2n}$ $n=1,2,3\dots$

$R=C_3+C_4\ln r$ $n=0$

$\because R(0)$ shouldn't be ∞

$\therefore C_2=C_4=0$

$\therefore \begin{cases} R=C_1r^{2n} & n=1,2,3 \\ R=C_3 & n=0 \end{cases}$

$\therefore u_0=0$ ($\because \sin 2n\theta=0$)

$u_n=C_1C_4\sin 2n\theta\cdot r^{2n}=A_nr^{2n}\sin 2n\theta$ $n=1,2,3\dots$

$\therefore u(r,\theta)=\sum_{n=1}^{\infty}A_nr^{2n}\sin 2n\theta$

$u(1,\theta)=f(\theta)$ $\therefore f(\theta)=\sum_{n=1}^{\infty}A_n\sin 2n\theta$ $\frac{n\pi x}{L}$

$\frac{n\pi}{L}=2n$

$A_n=\frac{2}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}f(\theta)\sin 2n\theta\,d\theta$ #

$L=\frac{\pi}{2}$

(1)(d) $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$ $\quad u(1,z)=z \quad 0<z<1 \quad \frac{\partial u}{\partial z}\Big|_{z=0}=0 \quad u(r,1)=0 \quad 0<r<1$

$RZ'(0)=0 \therefore Z'(0)=0 \qquad \therefore Z(1)=0$

Sol:

$u=RZ$

$R''Z+\frac{1}{r}R'Z+RZ''=0$

同除 $R\,Z$

$\frac{R''}{R}+\frac{R'}{rR}+\frac{Z''}{Z}=0$

$\therefore \frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$

$$\begin{cases} rR''+R'+\lambda rR=0 \\ Z''-\lambda Z=0 \end{cases}$$

① case $\lambda=0$

$rR''+R'=0 \quad \therefore R=C_1+C_2\ln r \quad \because \ln(0)\to-\infty \quad \therefore C_2=0$

$\therefore R=C_1$

$Z''=0 \quad Z=C_3+C_4z \quad \because Z(1)=0 \quad \therefore C_3+C_4=0 \quad C_3=-C_4$

$Z'=C_4 \quad Z'(0)=0 \quad \therefore C_4=0 \quad \therefore Z=0 \quad \therefore u=0$

② $\lambda>0,\ \lambda=\alpha^2$

$Z''-\alpha^2Z=0$

$Z=C_1\cosh\alpha z+C_2\sinh\alpha z \qquad Z(1)=0$

$Z'=\alpha C_1\sinh\alpha z+\alpha C_2\cosh\alpha z \qquad Z'(0)=0$

$0=\alpha C_2 \quad \therefore C_2=0$

$\therefore Z=C_1\cosh\alpha z \quad (Z(1)=0)$

$0=C_1\cosh\alpha \qquad \cosh\alpha\neq0$

$\therefore C_1=0$

$\therefore Z=0$

$\therefore u=0$

③ $\lambda<0,\ \lambda=-\alpha^2$

$Z''+\alpha^2Z=0$

$Z=C_1\cos\alpha z+C_2\sin\alpha z$

$Z'=-\alpha C_1\sin\alpha z+\alpha C_2\cos\alpha z$

$Z'(0)=0=\alpha C_2 \quad \therefore C_2=0$

$Z=C_1\cos\alpha z \qquad \cos\alpha=0$

$Z(1)=0 \quad 0=C_1\cos\alpha \qquad \alpha=(n+\frac{1}{2})\pi \quad n=0,1,2\ldots$

$\therefore \lambda=-(n+\frac{1}{2})^2\pi^2$

$\therefore rR''+R'-\alpha^2rR=0$

$R=C_1I_0(\alpha r)+C_2K_0(\alpha r) \quad \because K_0(0)\to\infty \quad \therefore C_2=0$

$R=C_1I_0(\alpha r)$

$\therefore u_n=C_1I_0(\alpha r)\cos\alpha z=A_n\left(I_0(-(n+\frac{1}{2})^2\pi^2 r)\right)\cos\left(-(n+\frac{1}{2})^2\pi^2 z\right))$

$\therefore u=\sum_{n=0}^{\infty}A_nI_0(\alpha_n r)\cos(\alpha_n z)$

$u(1,z)=z=\sum_{n=0}^{\infty}A_nI_0(\alpha_n)\cos(\alpha_n z)$ #

$A_n=\frac{2}{1}\int_0^1 z\cos((n+\frac{1}{2})\pi z)\,dz \qquad =(n+\frac{1}{2})\pi-0=(n+\frac{1}{2})\pi$ #

令 $t=(n+\frac{1}{2})\pi z \quad dt=(n+\frac{1}{2})\pi dz$

$A_n=2\int\frac{t}{(n+\frac{1}{2})\pi}\cos t\frac{dt}{(n+\frac{1}{2})\pi}=\frac{2}{[(n+\frac{1}{2})\pi]^2}\,t\sin t+\cos t=\frac{2}{((n+\frac{1}{2})\pi)^2}\left((n+\frac{1}{2})\pi z\sin(n+\frac{1}{2}\pi)z+\cos(n+\frac{1}{2})\pi z\right)\Big|_0^1$

(2) $\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial t^2} = 0 \quad x>0,\ t>0$

$u(0,t) = \sin t \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = -2\cos t$

$u(x,0) = 0 \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

Sol: $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial^2 u}{\partial t^2}$

$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = L\left\{4\frac{\partial^2 u}{\partial t^2}\right\}$

$a^2 \frac{d^2}{dx^2} L\{u\} = s^2 L\{u\} - su(x,0) - u_t(x,0)$

$\therefore \frac{1}{4}\frac{\partial^2 U}{\partial x^2} - s^2 U = -su(x,0) - u_t(x,0)$

$= 0 - 0 = 0$

$\therefore \frac{1}{4}\frac{\partial^2 U}{\partial x^2} - s^2 U = 0 \quad U_c(x,s) = C_1\cosh 2sx + C_2\sinh 2sx$

$\frac{\partial^2 U}{\partial x^2} - 4s^2 U = 0 \quad U_p = 0$

$\left(\frac{s}{\frac{1}{2}}\right)^2 \quad \therefore U = C_1\cosh 2sx + C_2\sinh 2sx$

$\therefore U(x,s) = C_1\cosh 2sx + C_2\sinh 2sx$

$L\{u(0,t)\} = L\{\sin t\} = U(0,s) = \frac{1}{s^2+1}$

$\therefore C_1 = \frac{1}{s^2+1}$

$L\left\{\left.\frac{\partial u}{\partial x}\right|_{x=0}\right\} = L\{-2\cos t\} = \left.\frac{dU}{dx}\right|_{x=0} = \frac{-2s}{s^2+1}$

$\frac{dU}{dx} = 2sC_1\sinh 2sx + 2sC_2\cosh 2sx$

$\left.\frac{dU}{dx}\right|_{x=0} = 2sC_2 = \frac{-2s}{s^2+1} \quad \therefore C_2 = \frac{-1}{s^2+1}$

$\therefore U = \frac{1}{s^2+1}(\cosh 2sx - \sinh 2sx) = \frac{1}{s^2+1}\exp(-2sx)$

$a = 2x$

$\therefore u = L^{-1}\{U\}$

$\therefore u = \sin(t-2x)\,u(t-2x)$ #

(3) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ $\quad x>0,\ 0<y<2$

$u(0,y)=0$ $\qquad$ Fourier sine transform

$u(x,0)=f(x)$ $\quad u(x,2)=0$

Sol:

$$\mathcal{F}_{s,x\to f}\{u(x,y)\}=\int_0^\infty u(x,y)\sin(2\pi f x)\,dx=U(f,y)$$

$$\therefore\ -4\pi^2f^2U(f,y)+2\pi f\times 0+\frac{\partial^2 U}{\partial y^2}=0$$

$$\frac{\partial^2 U}{\partial y^2}-4\pi^2f^2U=0,\quad U=C_1\cosh 2\pi f y+C_2\sinh 2\pi f y$$

$$\begin{cases} u(x,0)=f(x) & U(f,0)=\mathcal{F}_{s,x\to f}\{f(x)\}=\int_0^\infty f(x)\sin 2\pi f x\,dx \\ u(x,2)=0 & U(f,2)=\mathcal{F}_{s,x\to f}\{0\}=0 \end{cases}$$

$$\therefore U(f,0)=C_1=\int_0^\infty f(x)\sin 2\pi f x\,dx$$

$$U(f,2)=0$$

$$\therefore 0=C_1\cosh 4\pi f+C_2\sinh 4\pi f$$

$$\therefore C_2=-C_1\coth 4\pi f$$

$$\therefore U=C_1(\cosh 2\pi f y-\coth 4\pi f\sinh 2\pi f y)$$

$$\therefore u(x,y)=\mathcal{F}_{s,f\to x}[U]=4\int_0^\infty\int_0^\infty f(x)\sin 2\pi f x\,dx\,(\cosh 2\pi f y-\coth 4\pi f\sinh 2\pi f y)\sin 2\pi f x\,df$$

(4) 1. $\exp(x)$ $\qquad f(x)=\exp(x)$

$$g(x) = f\left(\frac{-1}{2}x+\frac{1}{2}\right)$$

$$g(x)\simeq\sum_{n=0}^{N} C_n P_n(x) \qquad C_n=(n+\tfrac{1}{2})\int_{-1}^{1} g(x)\,P_n(x)\,dx$$

$$C_0 = \frac{1}{2}\int_{-1}^{1} e^{-\frac{1}{2}x+\frac{1}{2}}\,\underset{1}{\overset{\shortparallel}{P_0(x)}}\,dx = e-1$$

$$C_1=\frac{3}{2}\int_{-1}^{1} e^{-\frac{1}{2}x+\frac{1}{2}}\,x\,dx = 3(e-3)$$

$$C_2=\frac{5}{2}\int_{-1}^{1} e^{-\frac{1}{2}x+\frac{1}{2}}\,\frac{1}{2}(3x^2-1)\,dx = 35(e-9.5)$$

$\vdots$

$\therefore\ g(x)=C_0P_0(x)+C_1P_1(x)+\cdots$

$$\therefore\ f(x)=\sum_{n=0}^{N} C_n P_n\left(-2\left(x-\tfrac{1}{2}\right)\right)$$

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(b)