Final B06901147 李宥頡

1. $y'' = y'^2$

Sol: $u = y'$

$u' = u^2 \quad \frac{du}{dx} = u^2 \quad \frac{du}{u^2} = dx \quad -\frac{1}{u} = x + C \quad \frac{1}{u} = -x - C$

$u = -\frac{1}{x+c} = \frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{1}{x+c} \quad dy = -\frac{1}{x+c}dx$

C, C' Const

$y = \int -\frac{1}{x+c}dx = -\ln|x+c| + C'$ #

2.(a) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ $\quad 0<x<5 \quad 0<y<5$

$u(0,y)=0$ $\quad \left.\frac{\partial u}{\partial y}\right|_{y=0}=0$

$u(5,y)=0$

$u(x,0)=\sin\frac{2}{5}\pi x\cos\frac{\pi}{5}x$

Sol: 令 $u=XY$

$X(0)=0$

$\Rightarrow X(5)=0 \quad Y'(0)=0$

$X''Y+XY''=0$

$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda$ $\quad \begin{cases} X''+\lambda X=0 \\ Y''-\lambda Y=0 \end{cases}$

$\lambda=0$

$X''=0$

$X=C_1+C_2x$

By b.c.

$X(0)=0=C_1$

$X(5)=0=5C_2 \quad \therefore C_2=0$

$\therefore X=0$

$u=XY$
$=0$

$\lambda<0$

$\lambda=-\alpha^2, \alpha>0$

$X''-\alpha^2X=0$

$X=d_4\cosh\alpha x+d_5\sinh\alpha x$

$X(0)=0=d_4+0 \quad \therefore d_4=0$

$X(5)=0=d_5\sinh 5\alpha \quad \therefore d_5=0$ $\quad \alpha$ should $\in\mathbb{R}$.

不考慮 $\sinh 5\alpha=0$

$\therefore X=0$

$u=0$

$\lambda>0$

$\lambda=\alpha^2, \alpha>0$

$X''+\alpha^2X=0$

$X=C_1\cos\alpha x+C_2\sin\alpha x$

$X(0)=0=C_1$

$X(5)=0=C_2\sin 5\alpha$ $\quad \therefore \alpha=\frac{n\pi}{5}$

$\therefore X_n(x)=C_2\sin\frac{n\pi}{5}x \quad \lambda=\alpha^2=\frac{n^2\pi^2}{25}$

$Y''(y)-\frac{n^2\pi^2}{25}Y=0$

$Y_n(y)=C_3\cosh\frac{n\pi}{5}y+C_4\sinh\frac{n\pi}{5}y$

$Y'=\frac{n\pi}{5}C_3\sinh\frac{n\pi}{5}y+\frac{n\pi}{5}C_4\cosh\frac{n\pi}{5}y$

$Y'(0)=0=0+\frac{n\pi}{5}C_4 \quad \therefore C_4=0$

$\therefore Y_n=C_3\cosh\frac{n\pi}{5}y$

$\therefore u = XY = C_2 \sin\frac{n\pi}{5}x \, C_3 \cosh\frac{n\pi}{5}y = A_n \sin\frac{n\pi x}{5} \cosh\frac{n\pi y}{5}$

$\therefore u = \sum A_n \sin\frac{n\pi x}{5} \cosh\frac{n\pi y}{5}$ #

where

$u(x,0) = \sum A_n \sin\frac{n\pi}{5}x \qquad A_n = \frac{2}{5}\int_0^5 \sin\frac{2}{5}\pi x \cos\frac{\pi}{5}x \sin\frac{n\pi}{5}x\,dx$ #

$= \sin\frac{2\pi}{5}x \cos\frac{\pi}{5}x$

(b) 令 $u = XYZ$

$X''YZ + XY''Z + XYZ'' = 0$ $\qquad \dfrac{X''}{X} + \dfrac{Y''}{Y} + \dfrac{Z''}{Z} = 0$ $\qquad \dfrac{X''}{X} = \dfrac{-Y''}{Y} - \dfrac{Z''}{Z} = -\lambda$

$\therefore X'' + \lambda X = 0$ $\quad$ 令 $\dfrac{Y''}{Y} = \mu$ $\quad \therefore \dfrac{Z''}{Z} = +\mu + \lambda$

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' + \mu Y = 0 \\ Z'' - (\mu + \lambda) Z = 0 \end{cases}$$

B.C. $X(0) = X(1) = 0$

$Y(0) = Y(1) = 0$

$Z(0) = 0$

$\lambda = 0$ $\quad X'' = 0$

$X = C_1 + C_2 x$

$0 = C_1$

$0 = C_2$

$\therefore u = 0$

$\lambda < 0$

$\lambda = -\alpha^2$

$X'' - \alpha^2 X = 0$

$X = C_1 \cosh \alpha x + C_2 \sinh \alpha x$

$X(0) = C_1 = 0$ $\qquad C_2 = 0$

$X(1) = C_2 \sinh \alpha = 0$

$\therefore u = 0$

$\lambda > 0$

$\lambda = \alpha^2$

$X'' + \alpha^2 X = 0$

$X = C_1 \cos \alpha x + C_2 \sin \alpha x$

$X(0) = 0 = C_1$

$X(1) = 0 = C_2 \sin \alpha$

$\alpha = m\pi$ $\qquad \therefore \lambda = m^2 \pi^2$

$X_m = C_2 \sin m\pi x$

$\mu = 0$

$Y'' = 0$ $\quad Y = C_1 + C_2 y$

$Y(0) = 0 = C_1$

$Y(1) = 0 = C_2$

$Y = 0$ $\therefore u = 0$

$\mu < 0$

$\mu = -\beta^2$

$Y'' - \beta^2 Y = 0$

$Y = C_1 \cosh \beta y + C_2 \sinh \beta y$

$Y(0) = C_1 = 0$ $\qquad C_2 = 0$

$Y(1) = C_2 \sinh \beta = 0$

$\therefore u = 0$

$\mu > 0$

$\mu = +\beta^2$

$Y'' + \beta^2 Y = 0$

$Y = C_1' \cos \beta y + C_2' \sin \beta y$

$Y(0) = C_1' = 0$

$Y(1) = 0 = C_2' \sin \beta$

$\beta = n\pi$ $\quad \mu = n^2 \pi^2$

$Y_n = C_2' \sin n\pi y$

$$Z'' - (\mu+\lambda)Z = 0 \quad \therefore Z'' - (m^2\pi^2 + n^2\pi^2)Z = 0$$

$$Z = C_1 \cosh\sqrt{m^2\pi^2+n^2\pi^2}\,z + C_2 \sinh\sqrt{m^2\pi^2+n^2\pi^2}\,z$$

$$Z(0) = 0 = C_1 \quad \therefore Z = C_2 \sinh\sqrt{m^2\pi^2+n^2\pi^2}\,z$$

$$\therefore u = XYZ \quad u_{mn} = A_{mn}\sin m\pi x \sin n\pi y \sinh\sqrt{m^2\pi^2+n^2\pi^2}\,z$$

$$\therefore u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin m\pi x \sin n\pi y \sinh\sqrt{m^2\pi^2+n^2\pi^2}\,z$$

(c) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + y \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=1}$

Sol: Method 2 for nonhomogeneous PDE

$u(x,y) = v(x,y) + \psi(y) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \quad \therefore \left.\frac{\partial v}{\partial x}\right|_{x=0} = \left.\frac{\partial v}{\partial x}\right|_{x=1} = 0$

$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial y} + \frac{\partial \psi}{\partial y} + y$

$X'(0) = 0$

$X'(1) = 0$

Prob A: $\frac{\partial \psi}{\partial y} + y = 0 \quad \therefore \frac{d\psi}{dy} = -y \quad \psi = -\frac{1}{2}y^2 + C$

Prob B: $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial y}$

$V = XY \quad X''Y = XY' \quad \therefore \frac{X''}{X} = \frac{Y'}{Y} = -\lambda$

$\begin{cases} X'' + \lambda X = 0 \\ Y' + \lambda Y = 0 \end{cases}$

$\lambda = 0 \quad X'' = 0 \quad X = C_1 + C_2 x \quad X' = C_2 = 0$

$Y' = 0 \quad Y = C_3 \quad \therefore V = XY = C_1 C_3 = C_4$

$\lambda > 0 \quad \lambda = \alpha^2$

$X'' + \alpha^2 X = 0$

$X = c_1 \cosh\alpha x + c_2 \sinh\alpha x \quad X' = \alpha C_1 \sinh\alpha x + \alpha C_2 \cosh\alpha x$

$Y = C_3 e^{-\alpha^2 y}$

$X'(0) = 0 = \alpha C_2 \quad \therefore C_2 = 0 \quad \therefore V = 0$

$X'(1) = 0 = \alpha C_1 \sinh\alpha \quad \therefore C_1 = 0$

$\lambda < 0 \quad \lambda = -\alpha^2$

$X'' - \alpha^2 X = 0 \quad X = C_1 \cos\alpha x + C_2 \sin\alpha x \quad X' = -\alpha C_1 \sin\alpha x + \alpha C_2 \cos\alpha x$

$X'(0) = 0 = \alpha C_2 \quad \therefore C_2 = 0$

$Y' - \alpha^2 Y = 0 \quad Y = C_3 e^{\alpha^2 y}$

$X'(1) = 0 = -\alpha C_1 \sin\alpha$

$\therefore \alpha = n\pi \quad \therefore V_n = C_4 + C_1C_3 e^{n^2\pi^2 y} \cos n\pi x$

$$= A_0 + A_n e^{n^2\pi^2 y} \cos n\pi x$$

$$\therefore V = \sum_{n=1}^{\infty} A_0 + A_n e^{n^2\pi^2 y} \cos n\pi x$$

$$\therefore u = \sum_{n=1}^{\infty} A_0 + A_n e^{n^2\pi^2 y} \cos n\pi x + -\frac{1}{2}y^2 + C.$$

$$= A_0' - \frac{1}{2}y^2 + \sum_{n=1}^{\infty} A_n e^{n^2\pi^2 y} \cos n\pi x \quad /\!\!/$$

9, (a) $y(x)=\begin{cases} 1-|x| & -2<x<2 \\ 0 & \text{ow} \end{cases}$

$\cancel{?}$ 3 $g(x)=y\left(\frac{-2-2}{2}x+\frac{-2+2}{2}\right)=y(-2x)=\begin{cases} 1-|-2x| \\ 0 \end{cases}=\begin{cases} 1-|2x| & -1<x<1 \\ 0 & \text{ow} \end{cases}$

$P_0(x)=1 \quad P_1(x)=x \quad P_2(x)=\frac{1}{2}(3x^2-1)$

$g_0=\frac{1}{2}\int_{-1}^{1}1-|2x|\,dx=\frac{1}{2}\times2\int_0^1 1-2x\,dx=\int_0^1 1-2x\,dx=x-x^2\Big|_0^1$

$=1-1=0$

$g_1=\frac{3}{2}\int_{-1}^{1}(1-|2x|)\,x\,dx$

$=\frac{3}{2}\left(\int_0^1 x-2x^2\,dx+\int_{-1}^{0}x+2x^2\,dx\right)$

$=\frac{3}{2}\left(\frac{1}{2}x^2-\frac{2}{3}x^3\Big|_0^1+\frac{1}{2}x^2+\frac{2}{3}x^3\Big|_{-1}^{0}\right)=\frac{3}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{-1}{2}+\frac{2}{3}\right)$

$=0$

$g_2=\frac{5}{2}\int_{-1}^{1}(1-|2x|)\left(\frac{1}{2}(3x^2-1)\right)dx=\frac{5}{4}\left(\int_0^1(1-2x)(3x^2-1)\,dx+\int_{-1}^{0}(1+2x)(3x^2-1)\,dx\right)$

$\int_0^1(1-2x)(3x^2-1)\,dx=\int_0^1-6x^3+3x^2+2x-1\,dx=-\frac{3}{2}x^4+x^3+x^2-x\Big|_0^1=-\frac{1}{2}$

$\int_{-1}^{0}(1+2x)(3x^2-1)\,dx=\int_{-1}^{0}6x^3+3x^2-2x-1\,dx=\frac{3}{2}x^4+x^3-x^2-x\Big|_{-1}^{0}=-\left(\frac{3}{2}-\cancel{1}-\cancel{1}+1\right)$

$=-\frac{1}{2}$

$\therefore g_2=\frac{5}{4}(-1)=-\frac{5}{4}$

$\therefore\ g_2(x) = -\frac{5}{4}x^2$ $\qquad P_2 = \frac{1}{2}(3x^2-1)$

$$f(x) \approx -\frac{5}{4}P_2\left(\frac{2}{-4}(x-0)\right) = -\frac{5}{4}P_2\left(-\frac{1}{2}x\right)$$

$$= -\frac{5}{4}\cdot\frac{1}{2}\left(3\cdot\frac{x^2}{4}-1\right)$$

$$= -\frac{15}{32}x^2+\frac{5}{8} \ \#$$

$C_2 = -\frac{15}{32}$ $\quad C_0 = \frac{5}{8}\ \#$

$C_1 = 0$

(b) $y[n] = (3,1,3,5,3,5)$

S.f: $\vec{a_0} = \frac{\vec{b_0}}{|b_0|} = \frac{1}{\sqrt{6}}(1,1,1,1,1,1)$

$\vec{g_1} = \vec{b_1} - \vec{a_0}\sum_{n=1}^{6} b_1[n]\ a_0[n] = (1,-1,1,-1,1,-1) - \frac{1}{\sqrt{6}}(1,1,1,1,1,1)\cdot 0$

$= (1,-1,1,-1,1,-1)$

$\vec{a_1} = \frac{\vec{g_1}}{|g_1|} = \frac{1}{\sqrt{6}}(1,-1,1,-1,1,-1)$

$\vec{g_2} = \vec{b_2} - \vec{a_0}\sum^{6} b_2[n]\,a_0[n] - \vec{a_1}\sum^{6} b_2[n]\,a_1[n]$

$= (-2,-2,0,0,2,2)$ $\qquad \sum y[n]\,a_0[n] = \frac{1}{\sqrt{6}}\times 15$

$\sum y[n]\,a_1[n] = \frac{1}{\sqrt{6}}(-2)$

$\therefore\ \vec{a_2} = \frac{(-2,-2,0,0,2,2)}{\sqrt{16}}$ $\qquad \sum y[n]\,a_2[n] = \frac{1}{2}\times 4$

$= \left(-\frac{1}{2},-\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}(-1,-1,0,0,1,1)$

$\therefore y[n] = \frac{15}{\sqrt{6}} a_0[n] + \frac{-2}{\sqrt{6}} a_1[n] + 2a_2(n)$

17×6

$1-2+3-4+5-6$
$= 9-11$
$= -2$

$\vec{a_0} = \frac{1}{\sqrt{6}} \vec{b_0}$

$\vec{b_2} - \vec{a_0}\frac{21}{\sqrt{6}} - \vec{a_1}\frac{-2}{\sqrt{6}}$

$\vec{a_1} = \frac{1}{\sqrt{6}} \vec{b_1}$

$\frac{21}{24}$

$\vec{a_2} = \frac{\vec{g_2}}{|\vec{g_2}|} = \frac{\vec{b_2} - \frac{21}{\sqrt{6}}\frac{1}{\sqrt{6}}\vec{b_0} + \frac{2}{6}\vec{b_1}}{\sqrt{\frac{1}{6}}} = \frac{1}{4}\vec{b_2} - \frac{7}{8}\vec{b_0} + \frac{1}{12}\vec{b_1}$

$\therefore \frac{15}{\sqrt{6}} \times \frac{1}{\sqrt{6}}\vec{b_0} + \frac{-2}{\sqrt{6}}\frac{1}{\sqrt{6}}\vec{b_1} + 2\cdot\left(\frac{1}{4}\vec{b_2} - \frac{7}{8}\vec{b_0} + \frac{1}{12}\vec{b_1}\right)$

$= \left(\frac{\cancel{15}^{5}}{\cancel{6}_{2}} - \frac{7}{4}\right)\vec{b_0} + \left(\frac{-2}{6} + \frac{1}{6}\right)\vec{b_1} + \frac{1}{2}\vec{b_2}$

$\therefore d_0 = \frac{3}{4} \quad d_1 = -\frac{1}{6} \quad d_2 = \frac{1}{2}$ #

4.

(a) $F(e^{-2\pi x^2}\cos 6\pi x)$

$$F(e^{-2\pi x^2}) = \int_{-\infty}^{\infty} e^{-2\pi x^2} e^{-j2\pi f x}\,dx = \frac{e^{-\frac{\pi f^2}{2}}}{\sqrt{2}}$$

$$F[e^{-ax^2}] = \int_{-\infty}^{\infty} e^{-ax^2} e^{-j2\pi f x}\,dx$$
$$= \int_{-\infty}^{\infty} e^{-ax^2}(\cos 2\pi f x - i\underbrace{\sin 2\pi f x}_{odd})\,dx$$
$$= \int_{-\infty}^{\infty} e^{-ax^2}\cos 2\pi f x\,dx$$
$$= \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi f^2}{a}}$$

$$F(\cos 6\pi x) = \frac{1}{2}\left[\delta(f-3)+\delta(f+3)\right]$$

$$\therefore\ G(f) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}\pi f^2} * \frac{1}{2}\left[\delta(f-3)+\delta(f+3)\right]$$
$$= \frac{1}{2\sqrt{2}}\left[e^{-\frac{1}{2}\pi(f-3)^2} + e^{-\frac{1}{2}\pi(f+3)^2}\right]$$ #

(b)

$$F(e^{-\pi x^2}) = \sqrt{\frac{\pi}{\pi}}\, e^{-\pi f^2} = e^{-\pi f^2}$$

$$F(e^{-2\pi x^2}) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}\pi f^2}$$

$$F(e^{-3\pi x^2}) = \sqrt{\frac{\pi}{3\pi}}\, e^{-\frac{1}{3}\pi f^2} = \frac{1}{\sqrt{3}} e^{-\frac{1}{3}\pi f^2}$$

$$\frac{3+2+6}{6}$$

$\therefore$ Conv → multiplication

$$所求 = \frac{1}{\sqrt{6}} e^{-\pi f^2(1+\frac{1}{2}+\frac{1}{3})} = \frac{1}{\sqrt{6}} e^{-\frac{11}{6}\pi f^2}$$ #

(c) $\exp(-x^2) * \delta(x+1) * \delta(2x) * \delta(3x)$

$= \exp(-(x+1)^2) * \delta(2x) * \delta(3x)$

$F(\exp(-(x+1)^2)) \times \frac{1}{2} \times \frac{1}{3} \xrightarrow{F^{-1}} \frac{1}{6} e^{-(x+1)^2}$ #

5. Sol: $\Delta x = \frac{1}{2}$

$g(x) = g_{-1}\,\mathrm{sinc}\left(\frac{x}{\Delta x}+1\right) + g_{-\frac{1}{2}}\,\mathrm{sinc}\left(\frac{x}{\Delta x}+\frac{1}{2}\right) + g_0\,\mathrm{sinc}\left(\frac{x}{\Delta x}+0\right) + g_{\frac{1}{2}}\left(\frac{x}{\Delta x}-\frac{1}{2}\right)$

$+ g_1\left(\frac{x}{\Delta x}-1\right)$

$= \mathrm{sinc}(2x+1) + \mathrm{sinc}\left(2x+\frac{1}{2}\right) + 2\,\mathrm{sinc}\,2x + 3\,\mathrm{sinc}\left(2x-\frac{1}{2}\right) + 3\,\mathrm{sinc}(2x-1)$

#

6. $A=\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\end{bmatrix}$

(a) Jordan form

Sol: $\det\begin{pmatrix}1-\lambda&1&0\\0&1-\lambda&0\\0&1&1-\lambda\end{pmatrix}=(1-\lambda)^3=0 \quad \lambda=1,1,1$

$\lambda=1$

$\begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ $\begin{cases}a_1+a_2=a_1\\a_2=a_2\\a_2+a_3=a_3\end{cases} \Rightarrow a_2=0$ 取 $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ✓

$\begin{pmatrix}1&1&0\\0&1&0\\0&1&1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix}$ $\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $a_2=1$ ✗

換取 $\begin{pmatrix}0\\0\\1\end{pmatrix}$ ✗ 換取 $\begin{pmatrix}1\\0\\1\end{pmatrix}$ 取

$\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$ $\begin{matrix}a_2=0\\0=0\\a_2=1\end{matrix}$ ✗ $\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\begin{matrix}a_2=1\\0=0\\a_2=1\end{matrix}$ $\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\therefore A=EDE^{-1}$

$D=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$ $E=\begin{pmatrix}1&0&1\\0&1&0\\1&0&0\end{pmatrix}$ $E^{-1}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&-1\end{pmatrix}$ #

(b) $A^{100}=ED^{100}E^{-1}$

$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}=I+\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ $\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

$\left[I+\overset{B}{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}}\right]^2=I^2+2\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}+B^2$ $B^n=0$ if $n\geq 2$

$=I+2B$

$$(I+B)^{100} = C^{100}_{100} I^{100} + C^{100}_{99} I^{99} B + \underbrace{C^{100}_{98} I^{98} B^{2} \cdots}_{0}$$

$$= 100 I + 99 I B$$

$$= \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix} + \begin{pmatrix} 0 & 99 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 100 & 99 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix}$$

$$\therefore A^{100} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 100 & 99 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 100 & 99 & 100 \\ 0 & 100 & 0 \\ 100 & 99 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 100 & 99 & 0 \\ 0 & 100 & 0 \\ 0 & 99 & 100 \end{pmatrix} ※$$

(c) $\lim_{\alpha \to 0} (\|A\|_\alpha)^\alpha = 5$ ※

$\|A\|_1 = 1 \times 5 = 5$ ※

$\|A\|_2 = \sqrt{1^2+1^2+1^2+1^2+1^2} = \sqrt{5}$ ※

$\|A\|_\infty = \max A[m,n] = 1$ ※

7. SVD $A=\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & 0\end{bmatrix}$ 2×3

Sol. $B=A^HA=\underbrace{\begin{bmatrix}1 & 1\\ -1 & 1\\ 1 & 0\end{bmatrix}}_{3\times 2}\underbrace{\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & 0\end{bmatrix}}_{2\times 3}=\begin{pmatrix}2 & 0 & 1\\ 0 & 2 & -1\\ 1 & -1 & 1\end{pmatrix}$, $C=AA^H=\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}$

$\det\begin{pmatrix}2-\lambda & 0 & 1\\ 0 & 2-\lambda & -1\\ 1 & -1 & 1-\lambda\end{pmatrix}=(1-\lambda)(2-\lambda)^2-[2-\lambda+2-\lambda]=0$ $\lambda=0,2,3$

for $\lambda=3$

$\begin{pmatrix}2 & 0 & 1\\ 0 & 2 & -1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}3a_1\\ 3a_2\\ 3a_3\end{pmatrix}$ $2a_1+a_3=3a_1$ $a_1=a_3$; $2a_2-a_3=3a_2$ $a_2=-a_3$; $a_1-a_2+a_3=3a_3$ $a_1=a_3$ $\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$ 取 $\begin{pmatrix}\frac{1}{\sqrt3}\\ -\frac{1}{\sqrt3}\\ \frac{1}{\sqrt3}\end{pmatrix}$

for $\lambda=2$

$\begin{pmatrix}2 & 0 & 1\\ 0 & 2 & -1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}2a_1\\ 2a_2\\ 2a_3\end{pmatrix}$ $2a_1+a_3=2a_1$ $a_3=0$; $2a_2-a_3=2a_2$; $a_1-a_2+a_3=2a_3$ $a_1=a_2$ 取 $\begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\\ 0\end{pmatrix}$

for $\lambda=0$

$\begin{pmatrix}2 & 0 & 1\\ 0 & 2 & -1\\ 1 & -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ $2a_1+a_3=0$ $2a_1=-a_3$; $2a_2-a_3=0$ $2a_2=a_3$; $a_1-a_2+a_3=0$ $\begin{pmatrix}1\\ -1\\ -2\end{pmatrix}$ 取 $\begin{pmatrix}\frac{1}{\sqrt6}\\ -\frac{1}{\sqrt6}\\ -\frac{2}{\sqrt6}\end{pmatrix}$

$B=VDV^H$ $D=\begin{pmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{pmatrix}$ $V=\begin{pmatrix}\frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6}\\ -\frac{1}{\sqrt3} & \frac{1}{\sqrt2} & -\frac{1}{\sqrt6}\\ \frac{1}{\sqrt3} & 0 & -\frac{2}{\sqrt6}\end{pmatrix}$

$\det\begin{pmatrix}3-\lambda & 0\\ 0 & 2-\lambda\end{pmatrix}=(3-\lambda)(2-\lambda)=0$ $\lambda=2,3$

$\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}2a_1\\ 2a_2\end{pmatrix}$ $3a_1=2a_1$; $2a_2=2a_2$ 取 $\begin{pmatrix}0\\ 1\end{pmatrix}$

$C=\tilde{U}\Omega\tilde{U}^H$

$\Omega=\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}$ $\tilde{U}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}3a_1\\ 3a_2\end{pmatrix}$ $3a_1=3a_1$; $\cancel{3}2a_2=3a_2$ 取 $\begin{pmatrix}1\\ 0\end{pmatrix}$

$a_1-a_2=\frac{1}{\sqrt2}$

$$\therefore S_1 = \tilde{U}^H A V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$\therefore S_1 = S \qquad \tilde{U} = U$$

$A = U S V^H$ where $S = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$ $V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$ #

8.

(a) Entropy of X

sol:

$$\text{Entropy}(X) = -\sum_n P_X(n)\ln[P_X(n)] = -\left[\frac{2}{5}\ln\left(\frac{2}{5}\right) + \left(\frac{1}{5}\ln\frac{1}{5}\right)\times 3\right] = -\frac{3}{5}\ln\frac{1}{5} - \frac{2}{5}\ln\frac{2}{5}$$

$$= \frac{3}{5}\ln 5 + \frac{2}{5}\ln 2 \quad \#$$

$$\text{Entropy}(Y) = -\left(\frac{1}{5}\ln\frac{1}{5}\right)\times 5 = -\ln\frac{1}{5} = \ln 5 \quad \#$$

(b)

$$D_{KL}(X\|Y) = \frac{2}{5}\ln\frac{\frac{2}{5}}{\frac{1}{5}} + \left(\frac{1}{5}\ln\frac{\frac{1}{5}}{\frac{1}{5}}\right)\times 3 = \frac{2}{5}\ln 2 + \frac{1}{5}\ln 1 \times 3$$

$$= \frac{2}{5}\ln 2 \quad \#$$