HW3 B06901147 李育頡

(1)(a) $g(x) = e^{-\frac{\pi}{4}x^2}\cos(4\pi x)$

Sol:

$$G(f) = F(e^{-\frac{\pi}{4}x^2}) * F(\cos 4\pi x)$$

$$F(e^{-\frac{\pi}{4}x^2}) = \int_{-\infty}^{\infty} e^{-\frac{\pi}{4}x^2} e^{-j2\pi f x}\, dx = \int_{-\infty}^{\infty} e^{-(\frac{\pi}{4}x^2 + j2\pi f x)}\, dx = \sqrt{\frac{\pi}{\frac{\pi}{4}}}\, e^{\frac{(j2\pi f)^2}{4\cdot\frac{\pi}{4}}}$$

$$= 2e^{\frac{-4\pi^2 f^2}{\pi}} = 2e^{-4\pi f^2}$$

$$F(\cos 4\pi x) = \frac{1}{2}\left(\delta(f-2) + \delta(f+2)\right)$$

$2\pi k = 4\pi$

$k = 2$

$$\therefore G(f) = \int_{-\infty}^{\infty} 2e^{-4\pi\mu^2} \frac{1}{2}\left(\delta(f-\mu-2) + \delta(f-\mu+2)\right) d\mu$$

$$= \int_{-\infty}^{\infty} e^{-4\pi\mu^2}\delta(f-2-\mu)\, d\mu + \int_{-\infty}^{\infty} e^{-4\pi\mu^2}\delta(f+2-\mu)\, d\mu$$

$$\because \int_{-\infty}^{\infty} f(x)\delta(x-x_0)\, dx = f(x_0)$$

$$\therefore G(f) = e^{-4\pi(f-2)^2} + e^{-4\pi(f+2)^2}$$

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(1)(b) $\begin{cases} g(x)=x & \text{for } 1<x<5 \\ g(x)=0 & \text{otherwise} \end{cases}$

$g(x)=x\,\Pi\left(\frac{x-3}{4}\right)$

$\because F[\Pi(x)]=\mathrm{sinc}(f)$

$\frac{x-3}{4}=\frac{1}{4}(x-3)$

By Time Shifting $F[\Pi(x-3)]=\mathrm{sinc}(f)\cdot e^{-j2\pi f\times 3}=\mathrm{sinc}(f)\,e^{-j6\pi f}$

By Scaling $F(\Pi(\frac{1}{4}(x-3))=\frac{1}{\frac{1}{4}}\left(\mathrm{sinc}\left(\frac{f}{\frac{1}{4}}\right)e^{-j6\pi\frac{f}{\frac{1}{4}}}\right)=4\left(\mathrm{sinc}(4f)e^{-j24\pi f}\right)$

$\because F(x\,g(x))=\frac{j}{2\pi}G'(f)$

$=\frac{j}{2\pi}\frac{d}{df}(4\,\mathrm{sinc}(4f)e^{-j24\pi f})$

$=\frac{j}{2\pi}\left(4\,\frac{d\,\mathrm{sinc}(4f)}{df}\cdot e^{-j24\pi f}+4\,\mathrm{sinc}(4f)\frac{d}{df}e^{-j24\pi f}\right)$

$=\frac{j}{2\pi}\left(\frac{4\cdot 4f\cos 4f-\sin 4f}{(4f)^2}\cdot 4\cdot e^{-j24\pi f}+4\,\mathrm{sinc}(4f)e^{-j24\pi f}\cdot -j24\pi\right)$

$=\frac{j}{2\pi}\left(4e^{-j24\pi f}\left(\frac{16f\cos 4f-\sin 4f}{16f^2}+\mathrm{sinc}4f\cdot -j24\pi\right)\right)$ #

(c) $g(x) = 6+x \quad -6<x<-5$

$g(x)=1 \quad -5<x<5$

$g(x)=6-x \quad 5<x<6$

$$g(x)=(6+x)\Pi\left(\frac{x-(-\frac{11}{2})}{1}\right)+\Pi\left(\frac{x-(0)}{10}\right)+(6-x)\Pi\left(\frac{x-\frac{11}{2}}{1}\right)$$

$$=(6+x)\Pi(x+\tfrac{11}{2})+\Pi(\tfrac{x}{10})+(6-x)\Pi(x-\tfrac{11}{2})$$

$$F(\Pi(x))=\mathrm{sinc}(f)$$

$$F(\Pi(\tfrac{x}{1}))=\mathrm{sinc}(f)$$

$$F(\Pi(x+\tfrac{11}{2}))=e^{-j2\pi f\frac{11}{2}}\mathrm{sinc}f=e^{+j11\pi f}\mathrm{sinc}f$$

$$F(\Pi(x-\tfrac{11}{2}))=e^{-j2\pi f\frac{11}{2}}\mathrm{sinc}f=e^{-j11\pi f}\mathrm{sinc}f$$

$$F(x\Pi(x+\tfrac{11}{2}))=\frac{j}{2\pi}\left(e^{j11\pi f}\mathrm{sinc}f\cdot 11\pi j+e^{j11\pi f}\frac{\cos\pi x-\mathrm{sinc}x}{\pi x}\right)$$

$$F(x\Pi(x-\tfrac{11}{2}))=\frac{j}{2\pi}\left(e^{-j11\pi f}\mathrm{sinc}f\cdot -11\pi j+e^{-j11\pi f}\frac{\cos\pi x-\mathrm{sinc}x}{\pi x}\right)$$

$$\therefore G(f)=6e^{j11\pi f}\mathrm{sinc}f+10\,\mathrm{sinc}\,10f+6e^{-j11\pi f}\mathrm{sinc}f+ \underline{\qquad\qquad}$$

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(2) (a) $\mathrm{sinc}(x) * \mathrm{sinc}(2x) * \mathrm{sinc}(3x) * (\sin x + \cos 10x)$

$F[\mathrm{sinc}(x)] = \Pi(f)$

$F[\mathrm{sinc}(2x)] = \frac{1}{2}\Pi\left(\frac{f}{2}\right)$

$F[\mathrm{sinc}(3x)] = \frac{1}{3}\Pi\left(\frac{f}{3}\right)$

$F[\sin x + \cos 10x) = F[\sin x] + F(\cos 10x)$

$= \frac{-j}{2}\delta\left(f - \frac{1}{2\pi}\right) + \frac{j}{2}\delta\left(f + \frac{1}{2\pi}\right) + \frac{1}{2}\delta\left(f - \frac{10}{2\pi}\right) + \frac{1}{2}\delta\left(f + \frac{10}{2\pi}\right)$

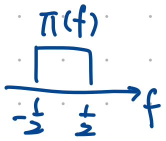
$2\pi k = 1$

$k = \frac{1}{2\pi}$

$2\pi k = 10$

$k = \frac{10}{2\pi}$

$\therefore$ 原式作FT $= \Pi(f) \cdot \frac{1}{2}\Pi(f/2) \cdot \frac{1}{3}\Pi(f/3) \cdot \left(-\frac{j}{2}\delta\left(f - \frac{1}{2\pi}\right) + \frac{j}{2}\delta\left(f + \frac{1}{2\pi}\right) + \frac{1}{2}\delta\left(f - \frac{10}{2\pi}\right) + \frac{1}{2}\delta\left(f + \frac{10}{2\pi}\right)\right)$



$= \frac{1}{6}\Pi(f) \cdot \left(\frac{-j}{2}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right) + \frac{1}{2}\delta\left(f - \frac{10}{2\pi}\right) + \delta\left(f + \frac{10}{2\pi}\right)\right)$

$= \frac{1}{6}\Pi(f) \cdot \frac{-j}{2}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right) = \frac{1}{6}\left(-\frac{j}{2}\left(\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right)\right)$

$\frac{1}{2\pi} \doteqdot \frac{1}{6.28} \doteqdot 0.159 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\frac{10}{2\pi} \doteqdot 1.59 \notin \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\overset{F^{-1}}{\Rightarrow} \frac{1}{6}\sin x$ ※

(b) $\delta(x) * \delta(2x) * \delta(3x) * (\sin x + \cos 10x)$

$= \delta(2x) * \delta(3x) * (\sin x + \cos 10x)$

$F \Rightarrow \frac{1}{2} \times \frac{1}{3} \times F(\sin x + \cos 10x)$

$\Downarrow F^{-1}$

$\frac{1}{6}(\sin x + \cos 10x)$ ※

3. $\int_{-\infty}^{\infty}(x^2+1)\,e^{-\pi x^2}\,dx$

Integration Property:

$$G(0)=\int_{-\infty}^{\infty}g(x)\,dx \qquad g(0)=\int_{-\infty}^{\infty}G(f)\,df$$

Sol: $g(x)=(x^2+1)e^{-\pi x^2}$

$$\frac{4\pi^2+2\pi}{4\pi^2}$$

$$g(0)=1\times 1=1=\int_{-\infty}^{\infty}G(f)\,df$$

$$G(f)=\mathcal{F}[g(x)]=\mathcal{F}[(x^2+1)e^{-\pi x^2}]$$

$$H_0(\sqrt{2\pi}x)=1$$

$$H_1(\sqrt{2\pi}x)=2\sqrt{2\pi}x$$

$$H_2(\sqrt{2\pi}x)=4\cdot 2\pi x^2-2$$

$$=8\pi x^2-2 \qquad -\frac{2}{8\pi}$$

$$\therefore x^2+1=\frac{1}{8\pi}H_2(\sqrt{2\pi}x)+\left(\frac{2}{8\pi}+1\right)H_0(\sqrt{2\pi}x)$$

$$\therefore \mathcal{F}\left((x^2+1)e^{-\pi x^2}\right)=\mathcal{F}\left(\frac{1}{8\pi}H_2(\sqrt{2\pi}x)e^{-\pi x^2}+\left(\frac{2}{8\pi}+1\right)H_0(\sqrt{2\pi}x)\,e^{-\pi x^2}\right)$$

$$=\frac{1}{8\pi}(-j)^2e^{-\pi f^2}H_2(\sqrt{2\pi}f)+\left(\frac{2}{8\pi}+1\right)(-j)^0e^{-\pi f^2}H_0(\sqrt{2\pi}f)$$

$$=-\frac{1}{8\pi}e^{-\pi f^2}\cdot(8\pi f^2-2)+\left(\frac{2}{8\pi}+1\right)e^{-\pi f^2}\times 1$$

$$=-f^2e^{-\pi f^2}+\frac{2}{8\pi}e^{-\pi f^2}+\frac{2}{8\pi}e^{-\pi f^2}+e^{-\pi f^2}=G(f)$$

所求 $=G(0)=\frac{2}{8\pi}+\frac{2}{8\pi}+1=\frac{2}{4\pi}+1=1+\frac{1}{2\pi}$ #

8. $G(f,h) = (\cos(f-h) + \sin(f+h))\exp(j(f^2+h^2)) = F(g(x,y))$

$\mathrm{Re}(g(x,y) + g(-x,-y))$

$$\Rightarrow F(\mathrm{Re}(g(x,y) + g(-x,-y))) = \frac{1}{2}\left(G(f,h) + G^*(-f,-h)\right)$$

$$= \frac{1}{2}\left((\cos(f-h) + \sin(f+h))\exp(j(f^2+h^2)) + \Big(\overset{\cos(f-h)}{\overset{\|}{\cos(-f+h)}} + \overset{-\sin(f+h)}{\overset{\|}{\sin(-f-h)}}\Big)\exp(-j(f^2+h^2)) \right)$$

$$= \frac{1}{2}\left(\cos(f-h)\left(e^{j(f^2+h^2)} + e^{-j(f^2+h^2)}\right) + \sin(f+h)\left(e^{j(f^2+h^2)} - e^{-j(f^2+h^2)}\right) \right)$$

$$= \cos(f-h)\frac{e^{j(f^2+h^2)} + e^{-j(f^2+h^2)}}{2} + \sin(f+h)\frac{e^{j(f^2+h^2)} - e^{-j(f^2+h^2)}}{2j}\, j$$

$$= \cos(f-h)\cdot\cos(f^2+h^2) + j\sin(f+h)\sin(f^2+h^2)$$

$\because \cos x = \frac{e^{ix} + e^{-ix}}{2}$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

5.

$$N\sum_{n=0}^{N-1}|g(n)|^2=\sum_{m=0}^{N-1}|G[m]|^2$$

$$G[m]=\sum_{n=0}^{N-1}g(n)e^{-j\frac{2\pi nm}{N}}$$

$$\because |G[m]|^2=\sum_{n=0}^{N-1}g[n]\sum_{n'=0}^{N-1}g^*[n']e^{j\frac{2\pi(n-n')m}{N}}$$

$$\therefore \sum_{m=0}^{N-1}|G[m]|^2=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}g[n]\sum_{n'=0}^{N-1}g^*[n']e^{j\frac{2\pi(n-n')m}{N}}$$

$$=\sum_{n=0}^{N-1}g[n]\sum_{n'=0}^{N-1}g^*[n']\sum_{m=0}^{N-1}e^{-j\frac{2\pi(n-n')m}{N}}$$

$$\parallel$$

$$\frac{1\left(e^{\frac{-j2\pi(n-n')}{N}\cdot N}-1\right)}{e^{\frac{-j2\pi(n-n')}{N}}-1}=\frac{e^{-j2\pi(n-n')}-1}{e^{\frac{-j2\pi(n-n')}{N}}-1}$$

當 $n\neq n'$ $e^{j2\pi(n-n')}$

$$=\cos(2\pi(n-n'))-i\sin(2\pi(n-n'))$$

$=1$ $\therefore$ 值 $=0$

當 $n=n'$ $\Rightarrow$ 為 $\sum_{m=0}^{N-1}1=N$

$$=\sum_{n=0}^{N-1}g[n]\sum_{n'=0}^{N-1}g^*[n']\,N\,\delta_{nn'}=\sum_{n=0}^{N-1}|g[n]|^2N \text{ ※}$$