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On the Modal Logic of Jeffrey Conditionalization

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In memory of Marcin Mostowski.

Abstract. We continue the investigations initiated in the recent papers (Brown et al. in The modal logic of Bayesian belief revision, 2017; Gyenis in Standard Bayes logic is not finitely axiomatizable, 2018) where Bayes logics have been introduced to study the general laws of Bayesian belief revision. In Bayesian belief revision a Bayesian agent revises (updates) his prior belief by conditionalizing the prior on some evidence using the Bayes rule. In this paper we take the more general Jeffrey formula as a conditioning device and study the corresponding modal logics that we call Jeffrey logics, focusing mainly on the countable case. The containment relations among these modal logics are determined and it is shown that the logic of Bayes and Jeffrey updating are very close. It is shown that the modal logic of belief revision determined by probabilities on a finite or countably infinite set of elementary propositions is *not finitely axiomatizable*. The significance of this result is that it clearly indicates that axiomatic approaches to belief revision might be severely limited.

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1. Background and Overview

This paper continues the investigations initiated in the recent papers [7, 11] where Bayes logics have been introduced to study the modal logical properties

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of statistical inference (Bayesian belief revision) based on Bayes conditionalization.

Suppose $(X, \mathcal{B}, p)$ is a probability space where the probability measure p describes knowledge of statistical information of elements of $\mathcal{B}$. In the terminology of probabilistic belief revision one says that elements in $\mathcal{B}$ stand for the propositions that an agent regards as possible statements about the world, and the probability measure p represents an agent's *prior* degree of beliefs in the truth of these propositions. Belief revision is about to learn new pieces of information: Learning proposition $A \in \mathcal{B}$ to be true, the agent revises his prior p on the basis of this evidence and replaces p with some new probability measure q (often called *posterior*) that can be regarded as the probability measure that the agent *infers* from p on the basis of the information (evidence) that A is true. This transition from p to q is what is called statistical inference. We say in this situation that "q *can* be learned from p"[1] and that "it is *possible* to obtain/learn q from p". This clearly is a modal talk and calls for a logical modeling in terms of concepts of modal logic. Indeed, the core idea of the paper [7] was to look statistical inference as an accessibility relation between probability measures: the probability measure q can be accessed from the probability measure p if for some evidence A we can infer from p to q. (For more motivation on how exactly modal logic come to the picture we refer to the introduction of [7]).

But how do we get q? One possible answer is a fundamental model of statistical inference, the standard Bayes model that relies on Bayes conditionalization of probabilities: given a prior probability measure p and an evidence $A \in \mathcal{B}$ the inferred measure q is defined by conditionalizing p upon A using the Bayes rule:

$$q(H) \doteq p(H \mid A) = \frac{p(H \cap A)}{p(A)} \quad (\forall H \in \mathcal{B}) \tag{1.1}$$

provided $p(A) \neq 0$. When q can be obtained from p using Bayes conditionalization upon some evidence A we say that q is *Bayes accessible* from p. The paper [7] studied the logical aspects of this type of inference from the perspective of modal logic and also hints that a similar analysis could be carried out when Bayes accessibility is replaced by the more general accessibility based on Jeffrey conditionalization.

Indeed, Bayesian belief revision is just a particular type of belief revision: Various rules replacing the Bayes rule have been considered in the context of belief change, and one important particular type is Jeffrey conditionalization (see [9,32]). Jeffrey conditioning is a way of inferring to a new probability q from the prior probability p and from an *uncertain* evidence r_i assigned to a

[1] This terminology is common in the literature of machine learning or artificial intelligence [3, 22], and it might be slightly confusing because one also says the "Agent learns the *evidence*". But the conceptual structure of the situation is clear: The Agent's "learning" q means the Agent *infers* q from some evidence (using conditionalization as inference device, see later).

finite[2] partition $\{E_i\}_{i<n}$ of X ($r_i \geq 0$, $\sum_{i<n} r_i = 1$, $p(E_i) > 0$) by making use of the Jeffrey rule:

$$q(H) \doteq \sum_{i<n} p(H \mid E_i) r_i \tag{1.2}$$

Jeffrey conditioning provides a more general method than Bayesian conditioning: if we assume that an element of the partition becomes certain (i.e. $r_i = 1$ for some index i), then the Jeffrey rule (1.2) reduces to the Bayes rule $q(H) = p(H \mid E_i)$. On this basis the Bayes rule is a special case of the Jeffrey rule. Taking the Jeffrey rule as an inference device gives rise to what we call *Jeffrey accessibility*: we say that q can be Jeffrey accessed from p if q can be obtained from p using (1.2) with some uncertain evidence. The aim of the current paper is to study the modal logical character of Jeffrey accessibility in a similar manner as it had been done in [7, 11] with Bayes accessibility.

For monographic works on Bayesianism we refer to [6, 19, 32]; for papers discussing basic aspects of Bayesianism, including conditionalization, see [14, 16–18, 30, 31]; for a discussion of Jeffrey's conditionalization, see [10].

Two remarks are in order here. First, in the literature of probabilistic updating apart from the Bayes and Jeffrey rules various other rules have been studied to update a prior probability, such as entropy maximalization or minimalization principles among others. Conditionalizing is a concept and technique in probability theory that is much more general than the Bayes rule (1.1) (also called "ratio formula" [25]). Both the Bayes rule and Jeffrey rule are special cases of conditioning with respect to a σ-field, see [4, Chapters 33–34] and [13] for further discussion of the relation of Bayes and Jeffrey rules to the theory of conditionalization via conditional expectation determined by σ-fields. We refer to [9] for a comparison of such methods.

Second, let us note here that there is a huge literature on other types of belief revision as well. Without completeness we mention: the AGM postulates in the seminal work of Alchurrón–Gärdenfors–Makinson [1]; the dynamic epistemic logic [29]; van Benthem's dynamic logic for belief revision [28]; probabilistic logics, e.g. Nilsson [23]; and probabilistic belief logics [2]. Typically, in this literature beliefs are modeled by sets of formulas defined by the syntax of a given logic and axioms about modalities are intended to prescribe how a belief represented by a formula should be modified when new information and evidence are provided. Viewed from the perspective of such theories of belief revision our intention in this paper, following [7], is very different. We do not try to give a plausible set of axioms in some nicely designed logic to capture desired features of (probabilistic) belief revision. On the contrary, we take the model that is actually used in applications of probabilistic learning theory and aim at an in-depth study of this model from a purely logical perspective. Bayesian probabilistic inference is relevant not only for belief change:

[2] Finiteness of the partition does not play a crucial role here, in fact, it turns out from Sect. 3 that from the modal logical point of view allowing infinite (countable) partitions does not make any difference. See the discussion in Sect. 3.

Bayes and Jeffrey conditionalization are the typical and widely applied inference rules also in situations where probability is interpreted not as subjective degree of belief but as representing objective matters of fact. Finding out the logical properties of such types of probabilistic inference has thus a wide interest going way beyond the confines of belief revision.

Below we recall the most important preliminary definitions from [7] and define the central subjects of the present paper. Concerning notions in modal logic we refer to the books Blackburn–Rijke–Venema [5] and Chagrov–Zakharyaschev [8]. We take the standard unimodal language given by the grammar

$$a \mid \perp \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \tag{1.3}$$

defining formulas φ, where a belongs to a nonempty countable set Φ of propositional letters. We use more-or-less standard notation and terminology but to be on the safe side the most basic concepts are recalled in the "Appendix".

Formal background. For a measurable space $\langle X, \mathcal{B}\rangle$ we denote by $M(X, \mathcal{B})$ the set of all probability measures over $\langle X, \mathcal{B}\rangle$. $M(X, \mathcal{B})$ serves as the set of "possible worlds" in the Kripkean terminology and Bayes accessibility relation has been defined in [7] as follows: For $v, w \in M(X, \mathcal{B})$ we say that w is *Bayes accessible* from v if there is an $A \in \mathcal{B}$ such that $w(\cdot) = v(\,\cdot \mid A)$. We denote the Bayes accessibility relation on $M(X, \mathcal{B})$ by $R(X, \mathcal{B})$. [7] introduces the notion of Bayes frames and Bayes logics:

Definition 1.1 (*Bayes frames*). A Bayes frame is a Kripke frame $\langle W, R\rangle$ that is isomorphic, as a directed graph, to $\mathcal{F}(X, \mathcal{B}) = \langle M(X, \mathcal{B}), R(X, \mathcal{B})\rangle$ for a measurable space $\langle X, \mathcal{B}\rangle$.

For convenience, we rely on the convention that elementary events $\{x\}$ for $x \in X$ always belong to the algebra $\mathcal{B}$; the reader can easily convince himself that this convention can be bypassed for the purposes of this paper. As a result, note that if the measurable space $\langle X, \mathcal{B}\rangle$ is finite or countably infinite, then $\mathcal{B}$ must be the powerset algebra $\wp(X)$. Therefore, in the countable case (i.e. when X is countable) instead of writing $\mathcal{F}(X, \wp(X))$, $M(X, \wp(X))$ or $R(X, \wp(X))$ we sometimes simply write $\mathcal{F}(X)$, $M(X)$ or $R(X)$, respectively.

Definition 1.2 (*Bayes logics*). A family of normal modal logics have been defined in [7] based on finite or countable or countably infinite or all Bayes frames as follows.

$$\mathbf{BL}_{<\omega} = \{\phi : \ (\forall n \in \mathbb{N})\ \mathcal{F}(n, \wp(n)) \Vdash \phi\} \tag{1.4}$$

$$\mathbf{BL}_{\omega} = \{\phi : \ \mathcal{F}(\omega, \wp(\omega)) \Vdash \phi\} \tag{1.5}$$

$$\mathbf{BL}_{\leq\omega} = \mathbf{BL}_{<\omega} \cap \mathbf{BL}_{\omega} \tag{1.6}$$

$$\mathbf{BL} = \{\phi : \ (\forall \text{ Bayes frames } \mathcal{F})\ \mathcal{F} \Vdash \phi\} \tag{1.7}$$

We call $\mathbf{BL}_{<\omega}$ (resp. $\mathbf{BL}_{\leq\omega}$) the logic of finite (resp. countable) Bayes frames; however, observe that the set of possible worlds $M(X, \mathcal{B})$ of a Bayes frame $\mathcal{F}(X, \mathcal{B})$ is finite if and only if X is a one-element set, otherwise it is at least of cardinality continuum.

Bayes logics in Definition 1.2 capture the laws of Bayesian learning: $\mathbf{BL}_{<\omega}$ is the set of general laws of Bayesian learning based on all finite Bayes frames, while the *general laws of Bayesian learning* independent of the particular representation $\langle X, \mathcal{B}\rangle$ of the events is then the modal logic $\mathbf{BL}$. The following theorem has been proved in [7].[3]

Theorem 1.3. *The following (non)containments hold.*

- $\mathbf{S4} \subseteq \mathbf{BL} \subseteq \mathbf{BL}_{\omega} = \mathbf{BL}_{\leq\omega} \subsetneq \mathbf{BL}_{<\omega}$,
- $\mathbf{S4.1} \subsetneq \mathbf{BL}_{\omega}$,
- $\mathbf{S4.1} + \mathbf{Grz} \subsetneq \mathbf{BL}_{<\omega}$.

The logic of finite Bayes frames has completely been described in [7] and, in particular, it has been shown that

- $\mathbf{BL}_{<\omega}$ has the finite frame property [7, Proposition 5.8],
- $\mathbf{BL}_{<\omega}$ is *not* finitely axiomatizable [7, Propositions 5.9].

In a similar manner we define Jeffrey accessibility: Given two measures $p, q \in M(X, \mathcal{B})$ we say that q is Jeffrey accessible from p if there is a finite partition $\{E_i\}_{i<n}$ and uncertain evidence r_i assigned to this partition ($r_i \geq 0$, $\sum_{i<n} r_i = 1$, $p(E_i) > 0$) such that Eq. (1.2) holds. Denote the corresponding accessibility relation by $J(X, \mathcal{B})$.

Definition 1.4 (*Jeffrey frames*). A Jeffrey frame is a Kripke frame $\langle W, R\rangle$ that is isomorphic, as a directed graph, to $\mathcal{J}(X, \mathcal{B}) = \langle M(X, \mathcal{B}), J(X, \mathcal{B})\rangle$ for a measurable space $\langle X, \mathcal{B}\rangle$.

A remark similar as above applies here: if the underlying set X of the measurable space $\langle X, \mathcal{B}\rangle$ is countable, then we may write $\mathcal{J}(X)$ and $J(X)$ instead of the longer $\mathcal{J}(X, \wp(X))$ and $J(X, \wp(X))$.

Definition 1.5 (*Jeffrey logics*). A family of normal modal logics is defined for a cardinal κ and $\ltimes \in \{=, <, \leq\}$ as follows.

$$\mathbf{JL}_{\ltimes\kappa} = \{\phi : \text{ (for all } \langle X, \mathcal{B}\rangle \text{ with } |X| \ltimes \kappa) \quad \mathcal{J}(X, \mathcal{B}) \Vdash \phi\} \tag{1.8}$$

$$\mathbf{JL} = \{\phi : \text{ } (\forall \text{ Jeffrey frames } \mathcal{J}) \ \mathcal{J} \Vdash \phi\} \tag{1.9}$$

We call $\mathbf{JL}_{<\omega}$ (resp. $\mathbf{JL}_{\omega}$) the logic of finite (resp. countable) Jeffrey frames or sometimes we use the term "finite (resp. countable) Jeffrey logic".

Jeffrey logics in Definition 1.5 capture the laws of Jeffrey updating: $\mathbf{JL}_{<\omega}$ is the set of general laws of Jeffrey learning based on all finite Jeffrey frames, while the *general laws of Jeffrey learning* independent of the particular representation $\langle X, \mathcal{B}\rangle$ of the events is then the modal logic $\mathbf{JL}$.

From the point of view of applications of probabilistic updating the most important classes of Bayes and Jeffrey frames are the ones determined by measurable spaces $\langle X, \mathcal{B}\rangle$ having a finite or a countably infinite X. Taking the first steps, this paper focuses only on the case with countable X, nevertheless, questions similar to what we ask here could be raised in connection with standard

[3] Some of the basic terminology of modal logic, such as what **S4** is, is recalled in the "Appendix".

Borel spaces, e.g. when $\mathcal{B}$ is the Borel (or Lebesgue) σ-algebra over the unit interval $X = [0, 1]$. It seems that countability of X serves as a dividing line and continuous spaces require different techniques than the ones employed here (cf. [11] where Bayes updating over standard Borel spaces was investigated).

Overview of the paper. Firstly, in Sect. 2 we discuss the connections of Jeffrey logics to a list of modal axioms that are often considered in the literature. In particular, Proposition 2.1 shows that $\mathbf{JL} \vdash \mathbf{S4}$ but $\mathbf{JL} \nvdash \mathbf{M}$ (thus $\mathbf{JL} \nvdash \mathbf{S4.1}$), $\mathbf{JL}_{\leq\omega} \vdash \mathbf{S4.1}$ and $\mathbf{JL}_{\ltimes\kappa} \nvdash \mathbf{Grz}$ for any $\ltimes \in \{=, <, \leq\}$ and $\kappa > 1$. Then, Theorem 2.2 clarifies the containments between the different Jeffrey logics:

$$\mathbf{S4} \subseteq \mathbf{JL} \subseteq \mathbf{JL}_{\omega} = \mathbf{JL}_{\leq\omega} \subseteq \mathbf{JL}_{<\omega} \subseteq \mathbf{JL}_{n+k} \subseteq \mathbf{JL}_{n} \tag{1.10}$$

In Sect. 3 we prove that the logic of Jeffrey updating (in the countable case) coincide with the logic of absolute continuity (see Theorem 3.7). The interesting part is when X is countably infinite: as a side result it turns out that from the modal logical point of view it does not matter whether or not we allow infinite partitions in the Jeffrey formula (1.2). In other words, the general laws that apply to Jeffrey learning are the same in both cases (and coincide with that of absolute continuity).

In Sect. 4 we ask the question "how close Bayes and Jeffrey logics are?" It turns out that finiteness of X serves as a dividing line: there is a proper containment

$$\mathbf{JL}_{n} \subsetneq \mathbf{BL}_{n} \quad \text{and} \quad \mathbf{JL}_{<\omega} \subsetneq \mathbf{BL}_{<\omega} \tag{1.11}$$

The case with countably infinite X, however, seems to show a completely different behavior. Theorem 4.6 disqualifies a large class of normal modal logics $\mathbf{L}$ that can possibly be put in between $\mathbf{JL}_{\omega} \subsetneq \mathbf{L} \subsetneq \mathbf{BL}_{\omega}$. We also show that $\mathbf{JL}_{\omega}$ is indistinguishable from $\mathbf{BL}_{\omega}$ within a large class of modal formulas (Corollary 4.7). It seems that the standard techniques fail to make a distinction between $\mathbf{JL}_{\omega}$ and $\mathbf{BL}_{\omega}$ and thus we conjecture that they are indeed the same. This we articulated in Problem 4.8: Are the logics $\mathbf{JL}_{\omega}$ and $\mathbf{BL}_{\omega}$ the same?

Finally, Sect. 5 deals with finite axiomatizability of the logics $\mathbf{JL}_{<\omega}$ and $\mathbf{JL}_{\omega}$. Theorem 5.8 states that the logic of finite Jeffrey frames $\mathbf{JL}_{<\omega}$ is not finitely axiomatizable, while Theorem 5.16 claims the same non finite axiomatizability result for $\mathbf{JL}_{\omega}$ (moreover countable Jeffrey logics are not axiomatizable by *any* set of formulas using finitely many variables). The situation is thus similar to that of Bayes logics (recall that $\mathbf{BL}_{<\omega}$ is not finitely axiomatizable, see [7, Propositions 5.9]). Such no-go results have a philosophical significance: they tell us that there is no finite set of formulas from which all general laws of Bayesian belief revision and Bayesian learning based on probability spaces with a countable set of propositions can be deduced. Bayesian learning and belief revision based on such simple probability spaces are among the most important instances of probabilistic updatings because they are widely used in applications. If the axiomatic approach to belief revision is not capable to characterize the logic of the simplest, paradigm form of belief revision, then this casts doubt on the general enterprize that aims at axiomatizations of belief revision systems. The cases with finite or countably infinite X require different techniques, therefore this section is divided into two subsections, accordingly.

2. Modal Principles of Jeffrey Updating

In this section we discuss the connections of Jeffrey logics to a list of modal axioms that are often considered in the literature: the T, 4, M and Grz axioms (see "Appendix").

We claim first that each Jeffrey frame is an **S4**-frame, that is, the accessibility relation of the frame is reflexive and transitive. Take any Jeffrey frame $\mathcal{J}(X,\mathcal{B}) = \langle W, R\rangle$. As we mentioned earlier Bayes conditioning is a special case of Jeffrey conditioning. This immediately implies reflexivity of R as for all probability measures $w \in W$ we have $w(\cdot) = w(\,\cdot \mid X)$. As for transitivity, suppose $u, v, w \in W$ with uRv and vRw. Taking into account the Jeffrey formula (1.2), we need two partitions $\{E_i\}$ and $\{F_j\}$ and uncertain evidences r_i and s_j assigned to these partitions such that $v(H) = \sum_i u(H \mid E_i)r_i$ and $w(H) = \sum_j v(H \mid F_j)s_j$. Checking transitivity of R requires some efforts but only basic algebra is involved (such as reordering sums) and thus we skip the lengthy calculations and only hint that one should take the common refinement $\{E_i \cap F_j\}_{i,j}$ of the two partitions with suitable values $t_{i,j}$ calculated from the values r_i and s_j. Consequently $\mathcal{J}(X,\mathcal{B}) \Vdash \mathbf{S4}$ and therefore $\mathbf{S4} \subseteq \mathbf{JL}$.

An **S4**-frame is an **S4.1**-frame if it validates the axiom M that requires the existence of endpoints: the frame $\mathcal{J}(X,\mathcal{B}) = \langle W, R\rangle$ validates M if and only if R has endpoints in the following sense:

$$\forall w \exists u (wRu \ \wedge \ \forall v(uRv \rightarrow u = v)) \tag{2.1}$$

If X is countable, then the Dirac measures $\delta_{\{x\}}$ for $x \in X$ are endpoints: take any $u \in W$ and pick $x \in X$ such that $u(\{x\}) \neq 0$. Then $\delta_{\{x\}} = u(\,\cdot \mid \{x\})$. It follows that $\mathbf{S4.1} \subseteq \mathbf{JL}_{\leq\omega}$. (We will see later on that this containment is proper as $\mathbf{JL}_{\leq\omega}$ is not finitely axiomatizable).

On the other hand, we claim that $M \notin \mathbf{JL}$ and consequently $\mathbf{S4.1} \not\subseteq \mathbf{JL}$. To this end it is enough to give an example for a Jeffrey frame $\mathcal{J}(X,\mathcal{B})$ in which there are paths without endpoints. Consider the frame $\mathcal{J}([0,1],\mathcal{B})$ where $[0,1]$ is the unit interval and $\mathcal{B}$ is the Borel σ-algebra. Then, for the Lebesgue measure w we have

$$\mathcal{J}([0,1],\mathcal{B}) \not\models \exists u(wRu \ \wedge \ \forall v(uRv \rightarrow u = v)) \tag{2.2}$$

We note that none of the logics $\mathbf{JL}_{\ltimes\kappa}$ (for $\ltimes \in \{=,<,\leq\}$ and $\kappa > 1$) contain the Grzegorczyk axiom Grz as a Jeffrey frame $\mathcal{J}$ always contains a complete subgraph of cardinality continuum.

Summing up we get the following proposition.

Proposition 2.1. *The following statements hold:*

- $\mathbf{JL} \vdash \mathbf{S4}$ *but* $\mathbf{JL} \nvdash \mathbf{M}$*, in particular* $\mathbf{JL} \nvdash \mathbf{S4.1}$.
- $\mathbf{JL}_{\leq\omega} \vdash \mathbf{S4.1}$.
- $\mathbf{JL}_{\ltimes\kappa} \nvdash \mathbf{Grz}$ *for any* $\ltimes \in \{=,<,\leq\}$ *and* $\kappa > 1$. □

The containments between different Jeffrey logics are clarified in the next theorem.

Theorem 2.2. *The following containments hold.*

$$\mathbf{S4} \subseteq \mathbf{JL} \subseteq \mathbf{JL}_\omega = \mathbf{JL}_{\leq\omega} \subseteq \mathbf{JL}_{<\omega} \subseteq \mathbf{JL}_{n+k} \subseteq \mathbf{JL}_n \tag{2.3}$$

Proof. From the very definition the following containments are straightforward:

$$\mathbf{JL} \subseteq \mathbf{JL}_{\leq\omega} \subseteq \mathbf{JL}_{<\omega} \subseteq \mathbf{JL}_n \quad \text{and} \quad \mathbf{JL} \subseteq \mathbf{JL}_{\leq\omega} \subseteq \mathbf{JL}_\omega \tag{2.4}$$

Next we show $\mathbf{JL}_m \subseteq \mathbf{JL}_n$ for $m > n$ and $\mathbf{JL}_\omega \subseteq \mathbf{JL}_{<\omega}$. The proof relies on Lemma 2.3. If $\langle X, \mathcal{B}\rangle$ and $\langle Y, \mathcal{S}\rangle$ are measurable spaces, then we say that $\langle X, \mathcal{B}\rangle$ can be embedded into $\langle Y, \mathcal{S}\rangle$ ($\langle X, \mathcal{B}\rangle \hookrightarrow \langle Y, \mathcal{S}\rangle$ in symbols) if there is a surjective measurable function $f : Y \to X$ such that $f^{-1} : \mathcal{B} \to \mathcal{S}$ is a σ-algebra homomorphism.

Lemma 2.3. *If $\langle X, \mathcal{B}\rangle \hookrightarrow \langle Y, \mathcal{S}\rangle$, then $\mathcal{J}(Y, \mathcal{S}) \twoheadrightarrow \mathcal{J}(X, \mathcal{B})$*

Proof. Let $f : Y \to X$ be a surjective measurable function ($f^{-1} : \mathcal{B} \to \mathcal{S}$ is a σ-algebra homomorphism). For a probability measure $p \in M(Y, \mathcal{S})$ let us assign the probability measure $F(p) \in M(X, \mathcal{B})$ defined by the equation

$$F(p)(A) = p\big(f^{-1}(A)\big) \quad (A \in \mathcal{B}) \tag{2.5}$$

Then $F : \mathcal{J}(Y, \mathcal{S}) \twoheadrightarrow \mathcal{J}(X, \mathcal{B})$ is a surjective bounded morphism. □

Now, for $m > n$ we have $\mathcal{J}(m) \twoheadrightarrow \mathcal{J}(n)$ and $\mathcal{J}(\omega) \twoheadrightarrow \mathcal{J}(n)$. Hence, the containments $\mathbf{JL}_m \subseteq \mathbf{JL}_n$ for $m > n$ and $\mathbf{JL}_\omega \subseteq \mathbf{JL}_{<\omega}$ follow. We also obtain $\mathbf{JL}_\omega = \mathbf{JL}_{\leq\omega}$ as $\mathbf{JL}_{\leq\omega} = \mathbf{JL}_\omega \cap \mathbf{JL}_{<\omega}$. □

3. Relation to Absolute Continuity

Considering Eq. (1.2) [or even Eq. (1.1)] it is easy to see that q has value 0 on every element $H \in \mathcal{B}$ which has p-probability zero. The technical expression of this is that q is absolutely continuous with respect to p. Therefore absolute continuity is necessary for Bayes or Jeffrey accessibility. In general, for $p, q \in M(X, \mathcal{B})$ we say that q is absolutely continuous with respect to p ($q \ll p$ in symbols) if $p(A) = 0$ implies $q(A) = 0$ for all $A \in \mathcal{B}$.

Let us now assume that $X = \{x_0, \ldots, x_{n-1}\}$ is finite and take any probability measure $p \in M(X, \wp(X))$. If $q \in M(X, \wp(X))$ is a probability measure such that $q \ll p$, then by taking the partition $E_i = \{x_i\}$ for $i < n$ and the uncertain evidence $r_i = q(E_i)$, we get

$$q(H) = \sum_{i<n} p(H \mid E_i) r_i \tag{3.1}$$

for all $H \subseteq X$. This means that given any prior probability p and an other probability q that is absolutely continuous with respect to p, if the probability space is finite, then q can be obtained from p by the Jeffrey rule. In other words, absolute continuity and Jeffrey accessibility coincide in the finite case. This motivates us to introduce Kripke frames where the accessibility relation is defined by absolute continuity, as follows.

Definition 3.1. For a probability space $\langle X, \mathcal{B}\rangle$ we define the Kripke frame

$$\mathcal{A}(X,\mathcal{B}) = \big\langle M(X,\mathcal{B}),\ \gg \big\rangle \tag{3.2}$$

where $\gg$ stands for absolute continuity: For probability measures $p, q \in M(X,\mathcal{B})$ we write $p \gg q$ (or $q \ll p$) if $p(A) = 0$ implies $q(A) = 0$ for all $A \in \mathcal{B}$.

Definition 3.2 (*Logics of Absolute Continuity*). In a similar manner to Definitions 1.2 and 1.5 we define a family of normal modal logics based on absolute continuity. Let κ be a cardinal and $\ltimes \in \{=, <, \leq\}$.

$$\mathbf{ACL}_{\ltimes\kappa} = \{\phi :\ (\text{for all } \langle X,\mathcal{B}\rangle \text{ with } |X| \ltimes \kappa)\ \ \mathcal{A}(X,\mathcal{B}) \Vdash \phi\} \tag{3.3}$$

$$\mathbf{ACL} = \{\phi :\ (\forall \langle X,\mathcal{B}\rangle)\ \ \mathcal{A}(X,\mathcal{B}) \Vdash \phi\} \tag{3.4}$$

Our observation at the beginning of this section proves the next proposition.

Proposition 3.3. $\mathbf{JL}_n = \mathbf{ACL}_n$ *and* $\mathbf{JL}_{<\omega} = \mathbf{ACL}_{<\omega}$ *for any* $n \in \mathbb{N}$.

Proof. For a finite X a probability measure $q \in M(X, \wp(X))$ can be obtained from $p \in M(X, \wp(X))$ by means of Jeffrey conditionalizing if and only if $p \gg q$. This implies that the frames $\mathcal{A}(X)$ and $\mathcal{J}(X)$ are identical. Consequently $\mathbf{ACL}_n = \Lambda(\mathcal{A}(n)) = \Lambda(\mathcal{J}(n)) = \mathbf{JL}_n$, and $\mathbf{ACL}_{<\omega} = \bigcap_n \mathbf{ACL}_n = \bigcap_n \mathbf{JL}_n = \mathbf{JL}_{<\omega}$. $\square$

What about the countably infinite case? The answer depends on whether or not we allow infinite partitions in the Jeffrey formula (1.2).

If we allow infinite partitions in the Jeffrey formula (1.2) and X is countably infinite, say $X = \mathbb{N}$, then taking the partition $E_i = \{i\}$ and the values $r_i = q(\{i\})$ for $i \in \mathbb{N}$, Jeffrey formula leads to

$$q(H) = \sum_{i\in\mathbb{N}} p(H \mid E_i) r_i \tag{3.5}$$

for all $H \in \wp(\mathbb{N})$, provided $p, q \in M(\mathbb{N}, \wp(\mathbb{N}))$ are such that $p \gg q$. This immediately ensures that q is Jeffrey accessible from p if and only if q is absolutely continuous with respect to p, and in particular $\mathbf{JL}_\omega = \mathbf{ACL}_\omega$.

There are good reasons, however, to keep the partition in the Jeffrey formula finite. The requirement that the uncertain evidence is given by a probability measure on a *proper*, *non-trivial* partition can be important: otherwise, as we have seen it, every probability measure can be obtained from itself as evidence—a triviality. The recent paper [12] argues that even in the finite case (i.e. when X is finite) it makes sense not to consider the trivial partition in the Jeffrey rule [as we did in Eq. (3.1)]. However, by sticking to all proper partitions in Jeffrey accessibility we would lost transitivity[4] which is a well-desired property in the context of learning theory. The natural way to overcome this problem is to not allow *all* proper partitions but rather just the restricted set of *finite* partitions. This way we can keep transitivity and also, as we will shortly see, the infinite Jeffrey logic $\mathbf{JL}_\omega$ will still coincide with $\mathbf{ACL}_\omega$. In other words, from the logical point of view whether or not we allow infinite

[4] The common refinement of two proper partitions can lead to the trivial partition, see the example in Figure 4 in [12].

partitions in the Jeffrey rule (1.2) does not make any difference. The rest of this section is devoted to prove this statement.

Recall that for a countable X, the support of a probability measure $u \in M(X, \wp(X))$ is the set $\mathrm{supp}(u) = \{x \in X : \ u(\{x\}) \neq 0\}$.

Lemma 3.4. *Let p, q, r be probability measures over the measure space $\langle \mathbb{N}, \wp(\mathbb{N})\rangle$ and suppose that both q and r are Jeffrey accessible from p and* $\mathrm{supp}(q) = \mathrm{supp}(r)$. *Then r is Jeffrey accessible from q and vice versa.*

Proof. Let p, q and r be as in the statement. According to Proposition 7.2, as both q and r are Jeffrey accessible from p, we have that the Radon–Nikodym derivatives $\frac{dq}{dp}$ and $\frac{dr}{dp}$ are step functions p-almost everywhere. As $\mathrm{supp}(q) = \mathrm{supp}(r)$, q and r are mutually absolutely continuous. In order to get that r is Jeffrey accessible from q, it is enough (by Proposition 7.2 again) to check that the Radon–Nikodym derivative $\frac{dr}{dq}$ is a step function, q-almost everywhere. But $\frac{dr}{dq} = \frac{dr}{dp} \cdot \frac{dp}{dq}$ except for a q-measure zero set, and it is straightforward that the product of two step functions is a step function. That q is Jeffrey accessible from r is completely similar. $\square$

Proposition 3.5. $\mathbf{JL}_\omega \subseteq \mathbf{ACL}_\omega$.

Proof. It is enough to prove that $\mathcal{A}(\omega) \trianglelefteq \mathcal{J}(\omega)$. Indeed, we claim that whenever $p \in \mathcal{J}(\omega)$ is a faithful measure (meaning that it has full support $\mathrm{supp}(p) = \omega$), then the generated subframe $\mathcal{J}^p$ is isomorphic to $\mathcal{A}(\omega)$. For this we only need that if q and r are Jeffrey accessible from p and $\mathrm{supp}(q) = \mathrm{supp}(r)$, then r is Jeffrey accessible from q and vice versa. This exactly is Lemma 3.4. $\square$

Proposition 3.6. $\mathbf{JL}_\omega \supseteq \mathbf{ACL}_\omega$.

Proof. It is enough to prove that $\biguplus \mathcal{A}(\omega) \twoheadrightarrow \mathcal{J}(\omega)$ for a suitable disjoint union $\biguplus \mathcal{A}(\omega)$. Indeed, recall that $\biguplus \mathcal{A}(\omega) \twoheadrightarrow \mathcal{J}(\omega)$ implies $\Lambda\big(\mathcal{A}(\omega)\big) \subseteq \Lambda\big(\biguplus \mathcal{A}(\omega)\big) \subseteq \Lambda\big(\mathcal{J}(\omega)\big)$, that is, $\mathbf{ACL}_\omega \subseteq \mathbf{JL}_\omega$. The construction is as follows.

For a non-empty subset $X \subseteq \omega$ consider those probability measures in $\mathcal{J}(\omega)$ whose support is X and write

$$S_X = \big\{u \in M(\omega, \wp(\omega)) : \ \mathrm{supp}(u) = X\big\}. \tag{3.6}$$

As the Jeffrey accessibility relation is transitive, S_X can be partitioned into clicks. A click is a maximal subset K of S_X such that any two $u, v \in K$ are mutually Jeffrey accessible. Let K_α^X be an enumeration of the clicks of S_X ($\alpha < \kappa_X$ for some cardinal κ_X depending on X). Note that each K_α^X is either a 1-element set or has continuum many elements depending on whether or not X is a 1-element set.

For each $\alpha < \kappa_\omega$ take a disjoint copy $\mathcal{A}_\alpha(\omega)$ of $\mathcal{A}(\omega)$ and write

$$A_\alpha^X = \big\{u \in \mathcal{A}_\alpha(\omega) : \ \mathrm{supp}(u) = X\big\}. \tag{3.7}$$

Note that each A_α^ω has continuum many elements.

Finally, take arbitrary bijections $F_\alpha : A_\alpha^\omega \to K_\alpha^\omega$ (for $\alpha < \kappa_\omega$) and let $F = \bigcup_{\alpha<\kappa_\omega} F_\alpha$ be the union of these bijections. As the copies A_α^ω are disjoint,

F is a well-defined bijection between $\bigcup_\alpha A_\alpha^\omega$ and S_ω (this latter set is taken as a subset of $\mathcal{J}(\omega)$).

The content of Lemma 3.4 can be interpreted in our context as follows. Take any probability $p \in \mathcal{J}(\omega)$ with $\mathrm{supp}(p) = X$. Suppose $Y \subseteq X$ is a non-empty subset and q, r are measures in $\mathcal{J}(\omega)$ such that $\mathrm{supp}(q) = \mathrm{supp}(r) = Y$ and both q and r can be Jeffrey accessed from p. Then q and r must belong to the same click of S_Y. It follows that F can be extended from $\bigcup_\alpha A_\alpha^\omega$ to the entire $\bigcup_\alpha \mathcal{A}(\omega)_\alpha$ in a homomorphic way. Checking that this extension is indeed a bounded morphism is not hard and left to the reader. $\square$

Summing up, independently of whether or not we allow infinite partitions in the Jeffrey formula (1.2) we obtained the following theorem.

Theorem 3.7. *For all countable cardinals κ and $\ltimes \in \{=, <, \leq\}$ we have*

$$\mathbf{JL}_{\ltimes\kappa} = \mathbf{ACL}_{\ltimes\kappa}. \tag{3.8}$$

Proof. The equations $\mathbf{JL}_n = \mathbf{ACL}_n$ for $n \in \omega$ and $\mathbf{JL}_{<\omega} = \mathbf{ACL}_{<\omega}$ is Proposition 3.3. Combining Propositions 3.5 and 3.6 we get $\mathbf{JL}_\omega = \mathbf{ACL}_\omega$. Finally, $\mathbf{JL}_{\leq\omega} = \mathbf{ACL}_{\leq\omega}$ follows from the previous results and the definition. $\square$

This result enables us to use the frames $\mathcal{A}(n)$ and $\mathcal{A}(\omega)$ instead of the more complex Jeffrey frames $\mathcal{J}(n)$ and $\mathcal{J}(\omega)$.

4. How Close Bayes and Jeffrey Logics Are?

One of the main results in [7] is Theorem 5.2 which relates Bayes logics to the strongest modal companion of Medvedev's logic of finite problems. We start by recalling definitions and theorems from [7,26]. Medvedev's logic of finite problems and its extension to infinite problems by Skvortsov originate in intuitionistic logic. (For an overview we refer to the book [8] and to Shehtman [26]; Medvedev's logic of finite problems is covered in the papers [15,20,21,24, 26,27].)

Definition 4.1 (*Medvedev frames*). A *Medvedev frame* is a frame that is isomorphic (as a directed graph) to $\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq \rangle$ for a non-empty *finite* set X.

For convenience, as a slight abuse of notation, we will call every frame of the form $\langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq \rangle$ (X being finite or infinite) a Medvedev frame and we will use the notation

$$\mathcal{P}_X^0 = \langle \wp(X) \smallsetminus \{\emptyset\}, \supseteq \rangle \tag{4.1}$$

A hierarchy or normal modal logics that correspond to the frames $\mathcal{P}^0_X$ can be given:

$$\mathbf{ML}_n = \{\phi : \ \mathcal{P}^0_n \Vdash \phi\} \tag{4.2}$$

$$\mathbf{ML}_{<\omega} = \{\phi : \ (\forall n \in \mathbb{N})\, \mathcal{P}^0_n \Vdash \phi\} \tag{4.3}$$

$$\mathbf{ML}_\omega = \{\phi : \ \mathcal{P}^0_\omega \Vdash \phi\} \tag{4.4}$$

$$\mathbf{ML}_{\leq\omega} = \mathbf{ML}_{<\omega} \cap \mathbf{ML}_\omega \tag{4.5}$$

$$\mathbf{ML} = \bigcap_\alpha \mathbf{ML}_\alpha \tag{4.6}$$

Observe that for $\alpha < \beta$ we have $\mathcal{P}^0_\alpha \trianglelefteq \mathcal{P}^0_\beta$, consequently $\mathbf{ML}_\beta \subseteq \mathbf{ML}_\alpha$. Since there are countably many modal formulas and proper class many cardinals, there must exists a cardinal α_0 such that the sequence $\mathbf{ML}_\alpha$ stabilizes, i.e. $\mathbf{ML} = \mathbf{ML}_{\alpha_0}$ or equivalently for all $\beta \geq \alpha_0$ we have $\mathbf{ML}_\beta = \mathbf{ML}_{\alpha_0}$. Theorem 5.2 of [7] states the containments below.

$$\begin{array}{ccccccccc}
\mathbf{ML} & \subseteq & \mathbf{ML}_\omega & = & \mathbf{ML}_{\leq\omega} & \subsetneq & \mathbf{ML}_{<\omega} & \subsetneq & \mathbf{ML}_n \\
\rotatebox[origin=c]{90}{$\subsetneq$} & & \parallel & & \parallel & & \parallel & & \parallel \\
\mathbf{BL} & \subsetneq & \mathbf{BL}_\omega & = & \mathbf{BL}_{\leq\omega} & \subsetneq & \mathbf{BL}_{<\omega} & \subsetneq & \mathbf{BL}_n
\end{array} \tag{4.7}$$

A consequence of this result is that when the underlying set X of the measurable space is countable we can use the more easy-to-handle Medvedev frames instead of Bayes frames.

Lemma 4.2. *For a countable X the mapping $f : \mathcal{A}(X) \twoheadrightarrow \mathcal{P}^0(X)$ defined by*

$$f(p) = \mathrm{supp}(p) \tag{4.8}$$

is a surjective bounded morphism.

Proof. Surjectivity of f is straightforward. f is a homomorphism (preserves accessibility) because for $p, q \in M(X, \wp(X))$ we have $p \gg q$ if and only if $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$. To verify the zig-zag property, suppose $\mathrm{supp}(p) \supseteq A$. We need $q \in M(X, \wp(X))$ such that $p \gg q$ and $\mathrm{supp}(q) = A$. Finding such a q is easy, take for example the conditional probability $q(\cdot) = p(\cdot \mid A)$. □

Corollary 4.3. $\mathbf{ACL}_{\ltimes\kappa} \subseteq \mathbf{ML}_{\ltimes\kappa}$ *holds for $\ltimes \in \{=, <, \leq\}$ and κ countable.*

Proof. Immediate from Lemma 4.2. □

Corollary 4.4. $\mathbf{JL}_{\ltimes\kappa} \subseteq \mathbf{BL}_{\ltimes\kappa}$ *holds for $\ltimes \in \{=, <, \leq\}$ and κ countable.*

Proof. Combine Corollary 4.3, Theorem 3.7 and Theorem 5.2. in [7]. □

We note that none of the logics $\mathbf{ACL}_n$ (for $n > 1$) contain the Grzegorczyk axiom Grz as $\mathcal{A}(n)$ always contain a complete subgraph of cardinality continuum. It is easy to check that Medvedev frame over a finite set $\mathcal{P}^0(n)$ validate $\mathbf{Grz}$, that is, $Grz \in \mathbf{BL}_{<\omega}$. Therefore we get

$$\mathbf{JL}_n \subsetneq \mathbf{BL}_n \quad \text{and} \quad \mathbf{JL}_{<\omega} \subsetneq \mathbf{BL}_{<\omega} \tag{4.9}$$

To have all the containments between Bayes and Jeffrey logics the only question remained is whether $\mathbf{JL}_\omega = \mathbf{BL}_\omega$. (By Corollary 4.4 we know that $\mathbf{JL}_\omega \subseteq \mathbf{BL}_\omega$.) The Grzegorczyk axiom does not differentiate between $\mathbf{JL}_\omega$ and

$\mathbf{BL}_\omega$ as none of these logics contain the formula Grz. In fact, we prove that $\mathbf{JL}_\omega$ is indistinguishable from $\mathbf{BL}_\omega$ within a large class of modal formulas. On the other hand the standard techniques (generated subframes, bounded morphisms) to prove the equality of the two logics do not seem to work: none of $\mathcal{A}(\omega) \trianglelefteq \mathcal{P}^0(\omega)$ or $\biguplus \mathcal{P}^0(\omega) \twoheadrightarrow \mathcal{A}(\omega)$ holds.

We call a Kripke frame $\mathcal{F}$ *click free* if there are no two different worlds in $\mathcal{F}$ that are mutually accessible, i.e. the largest click in $\mathcal{F}$ has size at most one. Note that click freeness enables reflexive points.

Lemma 4.5. *Let $\mathcal{F}$ be a click free* **S4***-frame. Then $\mathcal{P}^0(\omega) \twoheadrightarrow \mathcal{F}$ if and only if $\mathcal{A}(\omega) \twoheadrightarrow \mathcal{F}$.*

Proof. ($\Rightarrow$) The claim that $\mathcal{P}^0(\omega) \twoheadrightarrow \mathcal{F}$ implies $\mathcal{A}(\omega) \twoheadrightarrow \mathcal{F}$ is straightforward as any bounded morphism $f : \mathcal{P}^0(\omega) \twoheadrightarrow \mathcal{F}$ can be lifted up to a bounded morphism $f^+ : \mathcal{A}(\omega) \twoheadrightarrow \mathcal{F}$ by letting $f^+(p) \doteq f(\mathrm{supp}(p))$ for $p \in M(\omega, \wp(\omega))$.

($\Leftarrow$) That $\mathcal{A}(\omega) \twoheadrightarrow \mathcal{F}$ implies $\mathcal{P}^0(\omega) \twoheadrightarrow \mathcal{F}$ can be verified by observing that click freeness of $\mathcal{F}$ ensures that all points of a click in $\mathcal{A}(\omega)$ must be mapped to the same point of $\mathcal{F}$. Thus any morphism $f : \mathcal{A}(\omega) \twoheadrightarrow \mathcal{F}$ can be pushed down to a morphism $f^- : \mathcal{P}^0 \twoheadrightarrow \mathcal{F}$ by letting for all $\emptyset \neq X \subseteq \omega$, $f^-(X) \doteq f(p)$ for any $p \in M(\omega, \wp(\omega))$ with $\mathrm{supp}(p) = X$. □

Theorem 4.6. *There is no normal modal logic* **L** *such that*

$$\mathbf{JL}_\omega \subsetneq \mathbf{L} \subsetneq \mathbf{BL}_\omega \tag{4.10}$$

and **L** *is the logic of a click free* **S4***-frame $\mathcal{F}$ with $\mathcal{A}(\omega) \twoheadrightarrow \mathcal{F}$.*

Proof. Immediate from Lemma 4.5. □

The previous theorem tells us that if we would like to distinguish $\mathbf{JL}_\omega$ from $\mathbf{BL}_\omega$, then the standard technique of finding a bounded morphic image of $\mathcal{A}(\omega)$ that does the distinction fails (provided that this bounded morphic image is transitive and click free). We note that every modal formula is validated on a suitable finite, transitive, click free frame, thus Theorem 4.6 gives the impression that the two logics $\mathbf{JL}_\omega$ and $\mathbf{BL}_\omega$ coincide. Applying the same technique the next Corollary tells us that $\mathbf{JL}_\omega$ is indistinguishable from $\mathbf{BL}_\omega$ within a large class of modal formulas called Jankov—de Jongh formulas (cf. Theorem 7.1 in the "Appendix").

Corollary 4.7. *For a transitive, click free frame $\mathcal{F}$ we have*

$$\chi_{\mathcal{F}} \in \mathbf{JL}_\omega \quad \Leftrightarrow \quad \chi_{\mathcal{F}} \in \mathbf{BL}_\omega \tag{4.11}$$

Proof. Combine Lemma 4.5 with Theorem 7.1. □

We end this section by an open problem.

Problem 4.8. *Are the logics $\mathbf{JL}_\omega$ and $\mathbf{BL}_\omega$ the same?*

Part of the question in Problem 4.8 is this: Is there any frame $\mathcal{F}$ such that $\mathcal{A}(\omega) \twoheadrightarrow \mathcal{F}$ but $\mathcal{P}^0(\omega) \not\twoheadrightarrow \mathcal{F}$? Such a frame $\mathcal{F}$ must contain a proper click (must not be click free).

5. Non Finite Axiomatizability

For a natural number l a logic $\mathbf{L}$ is l-axiomatizable if it has an axiomatization using only formulas whose propositional variables are among $p_1, \ldots, p_l$. Every finitely axiomatizable logic is l-axiomatizable for a suitable l: take l to be the maximal number of variables the finitely many axioms in question use.

The main message of this section is that countable Jeffrey logics $\mathbf{JL}_{<\omega}$, $\mathbf{JL}_{\omega}$ and $\mathbf{JL}_{\leq\omega}$ are not finitely axiomatizable. In fact, it turns out from the proof that they cannot even be axiomatized with (possibly infinitely many) formulas using the same finitely many propositional letters. Thus, these logics are not finite schema axiomatizable either. The finite and the countably infinite cases require slightly different techniques, therefore we split the proof into two subsections, accordingly.

Also recall that the logic of countable Jeffrey frames is proved to be equal to that of absolute continuity (see Theorem 3.7). This allows us to use the frames $\mathcal{A}(X)$ of absolute continuity rather than the more complicated Jeffrey frames $\mathcal{J}(X)$. Phrasing it differently: we in fact show that the logics $\mathbf{ACL}_{<\omega}$, $\mathbf{ACL}_{\omega}$ and $\mathbf{ACL}_{\leq\omega}$ are not finitely axiomatizable and then refer to the fact that $\mathbf{JL}_{\ltimes\kappa} = \mathbf{ACL}_{\ltimes\kappa}$ for all countable cardinals κ and $\ltimes \in \{=, <, \leq\}$ (see Theorem 3.7).

5.1. The Finite Case

We aim at proving $\mathbf{ACL}_{<\omega}$ is not finitely axiomatizable. We show first that $\mathbf{ACL}_{<\omega}$ is a logic of *finite* frames, thus it has the finite frame property.

For each $k, n \in \mathbb{N}$ we define the finite frame $\mathcal{A}_k(n)$ as follows. Take the frame $\mathcal{A}(n)$. For each non-singleton set $A \subseteq n$ the frame $\mathcal{A}(n)$ contains a complete subgraph of cardinality continuum (measures p with support $\operatorname{supp}(p) = A$). Replace this infinite complete graph with the complete graph on k vertices and keep everything else fixed. A more precise definition is the following.

Definition 5.1. Let $n, k > 0$ be natural numbers. For each non-singleton set $a \in \wp(n) - \{\emptyset\}$ take new distinct points $[a]_1, \ldots, [a]_k$, and for each singleton $a \in \wp(n)$ take $[a]_1 = \cdots = [a]_k$ to be a single new point. The set of possible worlds of the frame $\mathcal{A}_k(n)$ is the set

$$A_k(n) = \big\{[a]_1, \ldots, [a]_k : \ a \in \wp(n) - \{\emptyset\}\big\} \tag{5.1}$$

For two points $[a]_i, [b]_j \in A_k(n)$ we define the accessibility relation $\to$ as

$$[a]_i \to [b]_j \quad \text{ if and only if } \quad a \supseteq b \tag{5.2}$$

Lemma 5.2. *For all n and k we have $\mathcal{A}(n) \twoheadrightarrow \mathcal{A}_k(n)$.*

Proof. For a measure $p \in M(n)$ the support $\operatorname{supp}(p)$ is a non-empty subset of n, therefore $[\operatorname{supp}(p)]_1$, ..., $[\operatorname{supp}(p)]_k$ are elements of $A_k(n)$. Take any mapping $f : M(n) \to A_k(n)$ such that

$$f(p) \ = \ [\operatorname{supp}(p)]_i \qquad \text{for some } i \in \{1, \ldots, k\} \tag{5.3}$$

and f is a surjection. Such a mapping clearly exists as for each $a \in \wp(n) - \{\emptyset\}$ we have

$$|\{p: \ \mathrm{supp}(p) = a\}| = 2^{\aleph_0} > k \tag{5.4}$$

We claim that f is a surjective bounded morphism:

Homomorphism. Take $p, q \in M(n)$ and suppose $f(p) = [\mathrm{supp}(p)]_i$, $f(q) = [\mathrm{supp}(q)]_j$. Then $p \gg q$ if and only if $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$ if and only if $[\mathrm{supp}(p)]_i \to [\mathrm{supp}(q)]_j$.

Zag property. Assume $f(p) \to [a]_i$ for some $a \in \wp(n) - \{\emptyset\}$. This can be the case if and only if $\mathrm{supp}(p) \supseteq a$. By surjectivity of f there is q such that $f(q) = [a]_i$, whence $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$ which means $p \gg q$. □

Lemma 5.3. *For each modal formula φ there is $k \in \mathbb{N}$ such that $\mathcal{A}(n) \nVdash \varphi$ implies $\mathcal{A}_k(n) \nVdash \varphi$.*

Proof. We prove that if φ uses the propositional letters $p_1, \ldots, p_k$ only, then $\mathcal{A}(n) \nVdash \varphi$ implies $\mathcal{A}_{2^k}(n) \nVdash \varphi$. If $\mathcal{A}(n) \nVdash \varphi$, then there is an evaluation V such that the model $\langle \mathcal{A}(n), V\rangle \nVdash \varphi$. The truth of a formula in a model depends only on the evaluation of the propositional letters the formula uses, therefore we may assume that V is restricted to $p_1, \ldots, p_k$.

For $x \in \mathcal{A}(n)$ we define a 0–1 sequence of length k according to whether $x \in V(p_i)$ holds for $1 \le i \le k$:

$$P_x(i) = \begin{cases} 1 & \text{if } x \in V(p_i) \\ 0 & \text{otherwise.} \end{cases} \qquad (1 \le i \le k) \tag{5.5}$$

As there are 2^k different 0–1 sequences of length k, the number of possible P_x's is at most 2^k.

Take any *surjective* mapping $f : \mathcal{A}(n) \to \mathcal{A}_{2^k}(n)$ such that

$$f(x) = [\mathrm{supp}(x)]_i \qquad \text{for some } i \in \{1, \ldots, k\} \tag{5.6}$$

and for $x, y \in \mathcal{A}(n)$ with $\mathrm{supp}(x) = \mathrm{supp}(y)$ we have

$$P_x = P_y \quad \text{implies} \quad f(x) = f(y) \tag{5.7}$$

Such a mapping f must exist as for each non-singleton $a \in \wp(n) - \{\emptyset\}$ we have 2^k elements $[a]_1, \ldots, [a]_{2^k}$ in $\mathcal{A}_{2^k}(n)$, and this is the number of the possible P_x's. Let us now define the evaluation V' over $\mathcal{A}_{2^k}(n)$ by

$$V'(p_i) = \{f(x): \ x \in V(p_i)\} \tag{5.8}$$

for $1 \le i \le k$. Condition (5.7) ensures that if x and y agree on $p_1, \ldots, p_k$, then so do the images $f(x)$ and $f(y)$. Thus, V' is well-defined. Following the proof of 5.2 one obtains that

$$f : \langle \mathcal{A}(n), V\rangle \twoheadrightarrow \langle \mathcal{A}_{2^k}(n), V'\rangle \tag{5.9}$$

is a surjective bounded morphism. As $\langle \mathcal{A}(n), V\rangle \Vdash \neg\varphi$ we arrive at $\langle \mathcal{A}_{2^k}(n), V'\rangle \Vdash \neg\varphi$. This means $\mathcal{A}_{2^k}(n) \nVdash \varphi$. □

Proposition 5.4. $\mathbf{ACL}_{<\omega} = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \Lambda\,(\mathcal{A}_k(n))$.

Proof. By combining Lemmas 5.2 and 5.3 the equality

$$\bigcap_{n=1}^{\infty} \Lambda\left(\mathcal{A}(n)\right) = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \Lambda\left(\mathcal{A}_k(n)\right) \tag{5.10}$$

follows immediately. The left-hand side of the equation is the definition of $\mathbf{ACL}_{<\omega}$. □

Corollary 5.5. *Finite Jeffrey logic* $\mathbf{JL}_{<\omega}$ *has the finite frame property.*

Proof. $\mathbf{JL}_{<\omega} = \mathbf{ACL}_{<\omega}$ by Theorem 3.7 and thus Proposition 5.4 implies

$$\mathbf{JL}_{<\omega} = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \Lambda\left(\mathcal{A}_k(n)\right) \tag{5.11}$$

As each frame $\mathcal{A}_k(n)$ is finite, the proof is complete. □

Proposition 5.6. *Let* K *be a class of finite, transitive frames, closed under point-generated subframes. For every finite, transitive, point-generated frame* $\mathcal{F}$ *we have*

$$\mathcal{F} \Vdash \Lambda(\mathsf{K}) \ \textit{ if and only if } \ \exists(\mathcal{G} \in \mathsf{K})\, \mathcal{G} \twoheadrightarrow \mathcal{F}. \tag{5.12}$$

Proof. ($\Leftarrow$) If there is $\mathcal{G} \in \mathbf{K}$ such that $\mathcal{G} \twoheadrightarrow \mathcal{F}$, then $\Lambda(\mathsf{K}) \subseteq \Lambda(\mathcal{G}) \subseteq \Lambda(\mathcal{F})$.

($\Rightarrow$) By way of contradiction suppose $\mathcal{G} \not\twoheadrightarrow \mathcal{F}$ for all $\mathcal{G} \in \mathbf{K}$. Then by Theorem 7.1 we have $\mathcal{G} \Vdash \chi(\mathcal{F})$ for all $\mathcal{G} \in \mathsf{K}$, in particular, $\chi(\mathcal{F}) \in \Lambda(\mathsf{K})$. It is straightforward to see that $\mathcal{F} \nVdash \chi(\mathcal{F})$, thus $\mathcal{F} \nVdash \Lambda(\mathsf{K})$. □

Theorem 5.7. $\mathbf{ACL}_{<\omega}$ *is not finitely axiomatizable.*

Proof. A logic $\mathbf{L}$ is not finitely axiomatizable if and only if for any formula $\phi \in \mathbf{L}$ there is a frame $\mathcal{F}_\phi$ such that $\mathcal{F}_\phi \nVdash \mathbf{L}$ but $\mathcal{F}_\phi \Vdash \phi$.

We will use the proof that the modal counterpart of Medvedev's logic of finite problems, $\mathbf{ML}_{<\omega}$, is not finitely axiomatizable. We refer to [26] where it has been proved that for each modal formula $\phi \in \mathbf{ML}_{<\omega}$ there is a finite, transitive, point-generated frame $\mathcal{G}_\phi$ such that $\mathcal{G}_\phi \Vdash \phi$ while $\mathcal{G}_\phi \nVdash \mathbf{ML}_{<\omega}$. The construction therein is such that $\mathcal{G}_\phi$ is click free.

We intend to show that $\mathcal{G}_\phi \nVdash \mathbf{ACL}_{<\omega}$. This is enough because $\mathbf{ACL}_{<\omega} \subset \mathbf{ML}_{<\omega}$. By Proposition 5.4 $\mathbf{ACL}_{<\omega}$ is the logic of the class $\mathsf{K} = \{\mathcal{A}_k(n) : n, k \in \mathbb{N}\}$ of finite, transitive frames, closed under point-generated subframes. Therefore, to show $\mathcal{G}_\phi \nVdash \mathbf{ACL}_{<\omega}$, by Proposition 5.6 it is enough to prove that $\mathcal{G}_\phi$ is not a bounded morphic image of any $\mathcal{A}_k(n)$. Suppose, seeking a contradiction, that there exists a bounded morphism $f : \mathcal{A}_k(n) \twoheadrightarrow \mathcal{G}_\phi$. Then for each $a \in \wp(n) - \{\emptyset\}$ the elements $[a]_1, \ldots, [a]_k$ should be mapped into the same point x_a in $\mathcal{G}_\phi$. This is because the points $[a]_i$ are all accessible from each other, while in $\mathcal{G}_\phi$ there are no non-singleton sets in which points are mutually accessible. It follows that f induces a bounded morphism $f^* : \mathcal{P}^0(n) \to \mathcal{G}_\phi$ from the Medvedev frame $\mathcal{P}^0(n)$ into $\mathcal{G}_\phi$ by letting $f^*(a) = x_a$ for $a \in \wp(n) - \{\emptyset\}$. But this is impossible as $\mathcal{G}_\phi \nVdash \mathbf{ML}_{<\omega}$. □

Theorem 5.8. *Finite Jeffrey logic* $\mathbf{JL}_{<\omega}$ *is not finitely axiomatizable.*

Proof. $\mathbf{JL}_{<\omega} = \mathbf{ACL}_{<\omega}$ by Theorem 3.7 and this latter logic is not finitely axiomatizable by Theorem 5.7. $\square$

5.2. The Countably Infinite Case

To gain non finite axiomatizability results for the countable Jeffrey logic $\mathbf{JL}_\omega$ we follow the method presented in Shehtman [26] and we recall the most important lemmas that we make use of.

Definition 5.9 [26]. For $m > 0$ and $k > 2$ the Chinese lantern is the **S4**-frame $\mathcal{C}(k, m)$ formed by the set

$$\{(i, j) : (1 \le i \le k-2, 0 \le j \le 1) \text{ or } (i = k-1, 1 \le j \le m) \text{ OR } (i = k, j = 0)\} \tag{5.13}$$

with the accessibility relation being an ordering:

$$(i, j) \le (i', j') \text{ iff } (i, j) = (i', j') \text{ OR } i > i' \tag{5.14}$$

$\mathcal{C}(m, k)$ is illustrated on page 373 in [26], however, we will not need any particular information about $\mathcal{C}$ apart from two lemmas that we recall below.

Lemma 5.10 (Lemma 22 in [26]). *Let ϕ be a modal formula using l variables and let $m > 2^l$. Then $\mathcal{C}(k, m) \nVdash \phi$ implies $\mathcal{C}(k, 2^l) \nVdash \phi$.*

Lemma 5.11 (Lemma 24 in [26]). *For any $n > 1$ we have $\mathcal{C}(2^n, 2^n) \Vdash \mathbf{ML}_{<\omega}$.*

Let $\mathcal{F} = \langle W, \le\rangle$ be a finite ordering (partially ordered set) and pick $x \in W$. y is an immediate successor of x if $x < y$ and there is no $x < z < y$. (As usual $<$ means $\le \cap \neq$). The branch index $b_{\mathcal{F}}(x)$ is the cardinality of the set of immediate successors of x, and the depth $d_{\mathcal{F}}(x)$ is the least upper bound of cardinalities of chains in $\mathcal{F}$ whose least element is x. Thus, $d_{\mathcal{F}}(x) = 1$ means that x has no immediate successors. $\mathcal{F}$ is *duplicate-free* if it is finite, generated and $b_{\mathcal{F}}(u) \neq 1$ for any $u \in W$ (cf. Shehtman [26]).

Lemma 5.12. $\mathcal{P}^0(\omega) \not\twoheadrightarrow \mathcal{C}(k, 2^k)$.

Proof. This is essentially Lemma 17 in [26]. Note that $\mathcal{C}(k, m)$ is duplicate-free. The point $u = (k, 0)$ in $\mathcal{C}(k, 2^k)$ has depth $d(u) = k$ and branch index $b(u) = 2^k$. If there were $\mathcal{P}^0(\omega) \twoheadrightarrow \mathcal{C}(k, 2^k)$, then by Lemma 17 in [26] we would have $b(u) < 2^{d(u)}$ which is impossible. $\square$

Theorem 5.13. *Let* $\mathbf{L}$ *be a normal modal logic with* $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. *Suppose that for every $l \ge 1$ and $k > l$ there is $n \ge k$ such that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. Then* $\mathbf{L}$ *is not l-axiomatizable for any number l.*

Proof. By way of contradiction suppose $\mathbf{L}$ is l-axiomatizable, that is, $\mathbf{L} = \mathbf{S4} + \Sigma$ where Σ is a set of formulas that can use only the first l propositional variables. Let $k = 2^l$. By assumption there is $n \ge k$ so that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. That Σ axiomatizes $\mathbf{L}$ means that every formula in $\mathbf{L}$ can be derived (in the normal modal calculus) from a finite set of axioms from Σ. Therefore there is an l-formula $\phi \in \mathbf{L}$ such that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{S4} + \phi$. This implies, by Theorem 7.1(B), that $\mathcal{C}(k, 2^n) \nVdash \phi$. As $n \ge k = 2^l > l$, Lemma 5.10 ensures $\mathcal{C}(k, 2^l) \nVdash \phi$. In particular, $\mathcal{C}(k, 2^l) \nVdash \mathbf{L}$.

On the other hand Lemma 5.10 implies (as $k = 2^l$) that $\mathcal{C}(k, 2^l) \Vdash \mathbf{ML}_{<\omega}$. By assumption $\mathbf{L} \subseteq \mathbf{ML}_{<\omega}$ so it follows that $\mathcal{C}(k, 2^l) \Vdash \mathbf{L}$ which is a contradiction. □

Corollary 5.14. *Let $\mathcal{F}$ be a frame, $\mathbf{L} = \Lambda(\mathcal{F})$ and assume $\mathbf{S4} \subseteq \mathbf{L} \subseteq \mathbf{ML}_{<\omega}$. Suppose for any $k \geq 1$ there is $n \geq k$ such that for all $u \in \mathcal{F}$ we have $\mathcal{F}^u \not\twoheadrightarrow \mathcal{C}(k, 2^n)$. Then $\mathbf{L}$ is not l-axiomatizable for any finite number l.*

Proof. Under the given assumptions Theorem 7.1(A) implies that for all $k \geq 1$ there is $n \geq k$ so that $\chi(\mathcal{C}(k, 2^n)) \in \mathbf{L}$. Then Theorem 5.13 applies. □

Theorem 5.15. $\mathbf{ACL}_\omega$ *is not finitely axiomatizable (in fact, it is not l-axiomatizable for any finite number l).*

Proof. We intend to apply Corollary 5.14. $\mathbf{ACL}_\omega = \Lambda(\mathcal{A}(\omega))$ and the containments $\mathbf{S4} \subseteq \mathbf{ACL}_\omega \subseteq \mathbf{ML}_{<\omega}$ hold (see Theorem 2.2). Write $\mathcal{A} = \mathcal{A}(\omega)$. In order to use Corollary 5.14 we only need to verify that for any $k \geq 1$ there is $n \geq k$ such that for all $u \in \mathcal{A}$ we have $\mathcal{A}^u \not\twoheadrightarrow \mathcal{C}(k, 2^n)$. It is easy to see that for all $u \in \mathcal{A}$, $\mathcal{A}^u$ is isomorphic either to $\mathcal{A}(\omega)$ or to $\mathcal{A}(n)$ depending on whether or not u has an infinite support. Therefore it is enough to check $\mathcal{A} \not\twoheadrightarrow \mathcal{C}(k, 2^k)$. As $\mathcal{C}(k, 2^k)$ is a transitive, click free frame, according to Lemma 4.5 if $\mathcal{A} \twoheadrightarrow \mathcal{C}(k, 2^k)$, then we also have $\mathcal{P}^0(\omega) \twoheadrightarrow \mathcal{C}(k, 2^k)$. But this latter is impossible by Lemma 5.12. □

Theorem 5.16. *Countable Jeffrey logic* $\mathbf{JL}_\omega$ *is not finitely axiomatizable (in fact, it is not l-axiomatizable for any finite number l).*

Proof. $\mathbf{JL}_\omega = \mathbf{ACL}_\omega$ by Theorem 3.7 and this latter logic is not finitely axiomatizable by Theorem 5.15. □

6. Closing Words and Further Research Directions

Our aim was to study the modal logical character of Jeffrey accessibility in a similar manner as it has been done in [7,11] concerning Bayes accessibility. We have seen that the modal logic of Jeffrey learning always extends $\mathbf{S4}$, and extends $\mathbf{S4.1}$ only if the underlying measurable space is countable (see Proposition 2.1). Containments between the different Jeffrey logics were clarified in Theorem 2.2:

$$\mathbf{S4} \subseteq \mathbf{JL} \subseteq \mathbf{JL}_\omega = \mathbf{JL}_{\leq\omega} \subseteq \mathbf{JL}_{<\omega} \subseteq \mathbf{JL}_{n+k} \subseteq \mathbf{JL}_n. \tag{6.1}$$

and the relations of Jeffrey logics to Bayes logics were also drawn in Sect. 4:

$$\mathbf{JL}_n \subsetneq \mathbf{BL}_n \quad \text{and} \quad \mathbf{JL}_{<\omega} \subsetneq \mathbf{BL}_{<\omega} \tag{6.2}$$

Equality between $\mathbf{JL}_\omega$ and $\mathbf{BL}_\omega$ remained open. Theorem 4.6 and Corollary 4.7 hints that they might be equal and in Problem 4.8 we ask whether the logics $\mathbf{JL}_\omega$ and $\mathbf{BL}_\omega$ coincide.

We regard Sect. 5 the main result of the paper. Theorem 5.8 states that the logic of finite Jeffrey frames $\mathbf{JL}_{<\omega}$ is not finitely axiomatizable, while Theorem 5.16 claims the same non finite axiomatizability result for $\mathbf{JL}_\omega$ (moreover countable Jeffrey logics are not axiomatizable by *any* set of formulas using finitely many variables). The picture is thus analogous to that of Bayes logics, see [7, Propositions 5.9]. The significance of these results is that they clearly indicate that axiomatic approaches to belief revision might be severely limited.

It is a longstanding open question whether the strongest modal companion of Medvedev's logic of finite problems $\mathbf{ML}_{<\omega}$ (and thus Bayes logic $\mathbf{BL}_{<\omega}$) is recursively axiomatizable (see [8, Chapter 2]). Since the class of Medvedev frames is a recursive class of finite frames, $\mathbf{BL}_{<\omega}$ is co-recursively enumerable. It follows that if $\mathbf{ML}_{<\omega}$ is recursively axiomatizable, then $\mathbf{BL}_{<\omega}$ is decidable. According to Corollary 5.5 finite Jeffrey logic $\mathbf{JL}_{<\omega}$ has the finite frame property. The proof reveals that $\mathbf{JL}_{<\omega}$ is a logic of a recursive class of finite frames, thus $\mathbf{JL}_{<\omega}$ is co-recursively enumerable, as well. We are not aware any similar result for $\mathbf{JL}_{\omega}$, neither do we know whether Jeffrey logics are recursively axiomatizable. We conjecture that recursive axiomatizability of Jeffrey logics would solve the similar question for Medvedev's logic, thus the problem might be severely hard.

Problem 6.1. *Are any of the logics* $\mathbf{JL}_{<\omega}$ *and* $\mathbf{JL}_{\omega}$ recursively *axiomatizable?*

Finally, we have already mentioned that in the literature of probabilistic updating apart from the Bayes and Jeffrey rules various other rules have been studied to update a prior probability, such as entropy maximalization or minimalization principles, among others. We do believe that a similar analysis should be carried out when Bayes or Jeffrey accessibility is replaced by some other accessibility relation based on these various probability updating principles.

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7. Appendix

ω is the least infinite cardinal (that is, the set of natural numbers). By a frame we always understand a Kripke frame, that is, a structure of the form $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set (of possible worlds) and $R \subseteq W \times W$

a binary relation (accessibility). Kripke models are tuples $\mathfrak{M} = \langle W, R, [\![\cdot]\!]\rangle$ based on frames $\mathcal{F} = \langle W, R\rangle$, and $[\![\cdot]\!] : \Phi \to \wp(W)$ is an evaluation of propositional letters. Truth of a formula φ at world w is defined in the usual way by induction:

- $\mathfrak{M}, w \Vdash p \iff w \in [\![p]\!]$ for propositional letters $p \in \Phi$.
- $\mathfrak{M}, w \Vdash \varphi \wedge \psi \iff \mathfrak{M}, w \Vdash \varphi$ AND $\mathfrak{M}, w \Vdash \psi$.
- $\mathfrak{M}, w \Vdash \neg\varphi \iff \mathfrak{M}, w \nVdash \varphi$.
- $\mathfrak{M}, w \Vdash \Diamond\varphi \iff$ there is v such that wRv and $\mathfrak{M}, v \Vdash \varphi$.

Formula φ is valid over a frame $\mathcal{F}$ ($\mathcal{F} \Vdash \varphi$ in symbols) if and only if it is true at every point in every model based on the frame. For a class C of frames the modal logic of C is the set of all modal formulas that are valid on every frame in C:

$$\Lambda(\mathsf{C}) = \left\{\phi : \ (\forall \mathcal{F} \in \mathsf{C})\ \mathcal{F} \Vdash \phi\right\} \tag{7.1}$$

$\Lambda(\mathsf{C})$ is always a normal modal logic. Let us recall the most standard list of modal axioms (frame properties) that are often considered in the literature (cf. [5,8]). Such axioms are

K $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
T $\Box\phi \to \phi$
4 $\Box\phi \to \Box\Box\phi$
M $\Box\Diamond\phi \to \Diamond\Box\phi$
Grz $\Box(\Box(\phi \to \Box\phi) \to \phi) \to \phi$

Let us recall some of the standard frame properties corresponding to these axioms (cf. [5,8]).

Logic	Axioms	Adequate frames
K	K	All frames
T	K+T	Reflexive frames
4	K+4	Transitive frames
S4	K+T+4	Preorders
S4.1	K+T+4+M	Preorders in which every point sees an endpoint
S4.Grz	K+T+4+Grz	Preorders without infinite chains

For two frames $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{G} = \langle W', R'\rangle$ we write $\mathcal{F} \trianglelefteq \mathcal{G}$ if $\mathcal{F}$ is (isomorphic as a frame to) a generated subframe of $\mathcal{G}$. We recall that if $\mathcal{F} \trianglelefteq \mathcal{G}$, then $\mathcal{G} \Vdash \phi$ implies $\mathcal{F} \Vdash \phi$, whence $\Lambda(\mathcal{G}) \subseteq \Lambda(\mathcal{F})$ (see Theorem 3.14 in [5] where the symbol $\rightarrowtail$ was used instead of $\trianglelefteq$). If $w \in W$, then we write $\mathcal{F}^w$ to denote the subframe of $\mathcal{F}$ generated by w, and we call such subframes point-generated subframes. Further, let $\mathcal{F} \twoheadrightarrow \mathcal{G}$ denote a surjective, bounded morphism (sometimes called p-morphisms). Such morphisms preserve the accessibility relation and have the zig-zag property (see [5]). Recall that if $\mathcal{F} \twoheadrightarrow \mathcal{G}$, then $\mathcal{F} \Vdash \phi$ implies $\mathcal{G} \Vdash \phi$, hence $\Lambda(\mathcal{F}) \subseteq \Lambda(\mathcal{G})$ (see Theorem 3.14 in [5]). We also recall that $(\forall i)\ \mathcal{F}_i \Vdash \phi$ implies $\biguplus \mathcal{F}_i \Vdash \phi$ (for the definition of the disjoint union $\biguplus$ of frames see Definition 3.13 in [5]). In the special case when $\mathcal{F}_i = \mathcal{F}$ it follows that $\Lambda(\mathcal{F}) \subseteq \Lambda(\biguplus \mathcal{F})$ (Theorem 3.14 in [5]).

The next theorem is due to Jankov and de Jongh.

Theorem 7.1 (cf. Proposition 4 in [26]). *Let $\mathcal{F}$ be a generated finite* **S4**-*frame. Then there is a modal formula $\chi(\mathcal{F})$ with the following properties:*

(A) *For any* **S4**-*frame $\mathcal{G}$ we have $\mathcal{G} \not\Vdash \chi(\mathcal{F})$ if and only if $\exists u\, \mathcal{G}^u \twoheadrightarrow \mathcal{F}$.*

(B) *For any logic $\mathbf{L} \supseteq \mathbf{S4}$ we have $\mathbf{L} \subseteq \Lambda(\mathcal{F})$ if and only if $\chi(\mathcal{F}) \notin \mathbf{L}$.*

Let X be countable and consider probability measures $p, q \in M(X, \wp(X))$. Suppose that q is absolutely continuous to p. The Radon–Nikodym derivative $\frac{dq}{dp}$ can be expressed as

$$\frac{dq}{dp}(x) = \begin{cases} \frac{q(\{x\})}{p(\{x\})} & \text{if } p(\{x\}) \neq 0 \\ * & \text{otherwise.} \end{cases}$$

where $*$ denotes any value ($\frac{dq}{dp}$ is determined up to p-measure zero, only). This is because

$$\sum_{x \in A} f(x) q(\{x\}) = \int_A f\ dq = \int_A f \frac{dq}{dp}\ dp = \sum_{x \in A} f(x) \frac{q(\{x\})}{p(\{x\})} p(\{x\})$$

Proposition 7.2. *Let X be a countable set and consider probability measures $p, q \in M(X, \wp(X))$. The following are equivalent*

(a) *q is Jeffrey accessible from p: there is a* finite *partition $\{E_i\}_i$ of X such that $p(E_i) \neq 0$ and*

$$q(H) = \sum_i p(H \mid E_i) q(E_i) \qquad \textit{for all } H \subseteq X$$

(b) *q is absolutely continuous with respect to p and the range of the function $\frac{dq}{dp}$ is finite except for a p-measure zero set.*

Proof. ($\Rightarrow$) Suppose q is Jeffrey accessible from p, that is, there is a finite partition $\{E_i\}_{i<n}$ of X with $p(E_i) \neq 0$ and a probability measure $r : \{E_i\} \to [0, 1]$ such that

$$q(H) = \sum_{i<n} \frac{p(H \cap E_i)}{p(E_i)} r(E_i) \qquad (H \in \mathcal{S}) \tag{7.2}$$

As $p(H) = 0$ implies $q(H) = 0$, q is absolutely continuous with respect to p. Suppose for some $i < n$ there are $a, b \in E_i$ which are not p-measure zero elementary events. Then by (7.2) we have

$$q(\{a\}) = \frac{p(\{a\})}{p(E_i)} r(E_i), \quad q(\{b\}) = \frac{p(\{b\})}{p(E_i)} r(E_i) \quad \Longrightarrow \quad \frac{q(\{a\})}{p(\{a\})} = \frac{q(\{b\})}{p(\{b\})}$$

This shows that $\frac{dq}{dp}$ is constant on each E_i (p-almost everywhere).

($\Leftarrow$) Let $X^+ = \{x \in X :\ p(\{x\}) > 0\}$ and $X^0 = X \setminus X^+$. By $q \ll p$ it follows that the Radon–Nikodym derivative $\frac{dq}{dp}$ exists and by assumption it is a step function on X^+. Define

$$E_y = \left\{ x \in X^+ :\ \frac{q(\{x\})}{p(\{x\})} = y \right\} \qquad \text{for } y \in \mathsf{ran}\left(\frac{dq}{dp}\right)$$

Then $\{E_y\}_{y\in\mathsf{ran}(dq/dp)}$ is a finite partition of X^+ and $\frac{dq}{dp}$ is constant on each block E_y of this partition. Pick an arbitrary E_y and replace it by $E_y \cup X^0$ (we need to have a partition of X such that the blocks of this partition are not p-measure zero, thus we can get rid of X^0 by adding it to any block E_y). For convenience we denote this new block by the same letter E_y. Define a measure r on the σ-subalgebra generated by the partition $\{E_y\}_{y\in\mathsf{ran}(dq/dp)}$ by the equation

$$r(E_y) = y \cdot p(E_y).$$

Note that r defines a probability measure because

$$\begin{aligned}\sum_{y\in\mathsf{ran}(dq/dp)} r(E_y) &= \sum_y y \cdot p(E_y) = \sum_y y \sum_{x\in E_y} p(\{x\}) \\ &= \sum_{x\in X} \frac{dq}{dp}(x) p(\{x\}) = \sum_X q(\{x\}) = 1\end{aligned}$$

We claim that

$$q(H) = \sum_{y\in\mathsf{ran}(dq/dp)} \frac{p(H\cap E_y)}{p(E_y)} r(E_y) \qquad (\text{for } H \in \mathcal{S})$$

For all $x \in X$ there is a unique j such that $x \in E_j$. Therefore

$$\sum_y \frac{p(\{x\}\cap E_y)}{p(E_y)} r(E_y) = \frac{p(\{x\})}{p(E_j)} r(E_j) = q(\{x\})$$

This latter equation holds because $x \in E_j$ iff $\frac{dq}{dp}(x) = j$ iff $\frac{q(\{x\})}{p(\{x\})} = j$, consequently

$$r(E_j) = j \cdot p(E_j) = \frac{q(\{x\})}{p(\{x\})} p(E_j)$$

This completes the proof. □

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