A Primer for Modal Game Theory Logic

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Abstract

A survey of various theories and frameworks to provide a foundation of game theory, which is the intersection decision theory, probabilistic logic, epistemic logic. Assuming knowledge of a first-year course in discrete mathematics covering predicate logic, probability, as well as some familiarity with modal logics. This introduction aims to provide a broad overview and motivate deeper research, by *getting to the fun stuff*. Certain details & definitions are taken for granted, as each of the four topics discussed has its own deep intricacies.

1 Overview & Foundations

The purpose of the paper is to survey and provide a primer for the study of the more complex modal game theory logic. Game theory is a multi-agent expansion of decision theory, however, a game can also consist of multiple steps and incomplete information – which is uncertainty. Given this, game theory often leverages probabilistic logic & epistemic logic, in order to expand upon the tools provided in decision theory. This primer for modal game theory logic doubles as an introduction to decision-making under uncertainty with probabilistic modal logic.

1.1 Modal logic

Modal logic is a family of logics with similar rules. At the basic level, upon propositional logic, there is the addition of two operations, 'necessarily' and 'possibly'. These are informally like the universal quantification operator, $\forall$, that operate over a proposition p, as $\forall x p(x)$. Similarly, in modal logic 'necessarily' is $\Box$, while 'possibly' is $\diamond$. The key difference is there is no predicate variable x.

For example, $\Box p \rightarrow p$, means "if p is necessary, then p". The two modal operations iterate over "worlds", which can be informally thought of as a row of a truth table. $\Box p \rightarrow p$ corresponds to the reflexivity axiom. There are many other potential axioms such as transitive and symmetric, however, they are often debatable, even inside a single modal logic.

Similar to how the universal quantification ($\forall x$) and existential quantification ($\exists x$) operators can interdefined, this also applies to the modal operators, $\diamond p \iff \neg\Box\neg$P. Figure 1.1 shows the potential axioms in modal logics.

1.2 Epistemic logic

Epistemic logics allow one to reason about knowledge in some way. That is, it allows the formal exploration of the implications of epistemic principles. Similarly, Doxastic logic allows one to reason about belief. Epistemic logic contains a knowledge operator and is governed by appropriate axioms. The modal operator is κ, for example, $\kappa_a \phi$ reads "Agent a Knows that κ". In terms of semantics, a Kripe model is used, and "possible worlds" are interpreted as *all worlds the agent considers epistemically possible relative to its current information*. For instance, in a context, reflexivity means the agent can only access worlds with the same information as the current. Here are two examples of

Name	Axiom	Condition on Frames	R is...
(D)	$\Box A \to \Diamond A$	$\exists u wRu$	Serial
(M)	$\Box A \to A$	wRw	Reflexive
(4)	$\Box A \to \Box\Box A$	$(wRv \,\&\, vRu) \Rightarrow wRu$	Transitive
(B)	$A \to \Box\Diamond A$	$wRv \Rightarrow vRw$	Symmetric
(5)	$\Diamond A \to \Box\Diamond A$	$(wRv \,\&\, wRu) \Rightarrow vRu$	Euclidean
(CD)	$\Diamond A \to \Box A$	$(wRv \,\&\, wRu) \Rightarrow v = u$	Functional
$(\Box M)$	$\Box(\Box A \to A)$	$wRv \Rightarrow vRv$	Shift Reflexive
$(C4)$	$\Box\Box A \to \Box A$	$wRv \Rightarrow \exists u(wRu \,\&\, uRv)$	Dense
(C)	$\Diamond\Box A \to \Box\Diamond A$	$wRv \,\&\, wRx \Rightarrow \exists u(vRu \,\&\, xRu)$	Convergent

how to read the notation, what is known is true $\kappa_a\phi \to \phi$, and second, what is known is known to be known $\kappa_a\phi \to \kappa_a\kappa_a\phi$

1.2.1 Formal definition

Model M, (W, R_K, V), represents the epistemic state of an agent. In which, W is the set of scenarios, R_K is a binary relation on W for epistemically accessible, finally, V is a valuation function assigning to each atomic sentence p, which is a subset of W, where $V(p)$ is the set of scenarios in which p holds.

1.3 Quantitatives

There are maybe ways to quantify various abstract ideas, such as probability, credence, and utility. Recall that there are many interpretations of probability, however, the core and widely accepted understanding are that probability is a numerical description of how likely an event is to occur. The three axioms are summarized to be: non-negative real numbers, disjoint (mutually exclusive), and summing to 1. The axioms of probability with many of the concepts and requirements are also inherited by the notion of credence. Credence is the expression of how much an agent believes an event is to occur, or more traditionally, how likely a proposition is true. Credence is synonymous with personal probability, so strictly speaking if a probably is not grounded in fact, it is credence.

Utility is the expression of an agent's preference, pleasure, and desire, in an arbitrary unit called a "util". The utility function maps an ability set of elements to utils. This is useful in economics for showing the desirability of a combination of, perhaps, money and donuts, in a single quantity. There are two types of uncertainty, aleatoric, and epistemic. Aleatoric ("statistical") uncertainty is the result of the stochasticity of the system, for example, dice rolling or the precise trajectory of a paper airplane. Epistemic ("systematic") uncertainty, includes both errors from imperfect measurement and imperfect models, for example, a weight scale might lack precision, or a physics formula assuming a spherical cat or neglecting air resistance.

2 Decision theory

2.1 Expected utility theory

Von Neumann–Morgenstern utility theory, commonly referred to as utility theory, can be applied to calculate the expected utility of an event. This is an extension of expected value by employing a utility function, $u : \$ \to$ "utils", mapping dollars to utils, a hypothetical measure of benefit. For our examples, $u(x) = x$.

Suppose a shady three-card Monte dealer offers you a new game, \$1 to roll a pair of dice, and a payout of \$40 if you roll two ones, *snake eyes*.

$$E[u(x)] = u(\$0) * p(\neg\text{Snake eyes}) + u(\$40) * p(\text{Snake eyes})$$

$$E[u(x)] = u(\$0) * (35/36) + u(\$40) * (1/36) = 1.11$$

$E[u(\text{Gamble})] > u(\$1)$, means out expected payoff if greater than the certain cost of $\$1$. It is beneficial to gamble!

This means we would, in the long run, profit from this game. However, consider a similar game in the lottery context, suppose the same expected payoff and buy-in, there is an epsilon ($1e^{-8}$) chance of winning, but the payoff is very large.

This gives rise to the need to factor in variance and risk.[1] Risk-weighted expected utility is achieved by the addition of a risk function. If $r(p) < p$ for every p, then the function describes a risk-averse person. The risk function must satisfy the constraints: $r(0) = 0$, $r(1) = 1$, and $0 \leq r(p) \leq 1$. Consider, $r(p) = p^2$ with our previous example

$$E[u(x)] = u(\$0) * r(p(\neg \text{Snake eyes})) + r(u(\$40) * p(\text{Snake eyes}))$$

$$E[u(x)] = u(\$0) * r(35/36) + u(\$40) * r(1/36) = 0.031$$

Because $E[u(\text{Gamble})] < u(\$1)$, our risk function has advised contrary to pure expected utility, we should not engage in the bet.

In practice, designing the risk and utility functions is very hard as there are several considerations such as the law of diminishing returns and its derivative. In terms of the next steps, factoring in variance is important as illustrated in the lottery prompt. One relevant formula is:

$$E[u(x)] = -Var(x) + [E(x) - x]^2, \text{ where } Var[x] = E[x2] - (E[x])^2$$

Evidential decision theory substitutes the probability term in expected utility with conditional probability. Conditional probability considers context. For example, P(Snake eyes | ¬one) = 0 is interpreted as, *if the first dice is not a one, there is no point in considering the second dice*. Lastly, it is possible to replace probability and conditional probability with credence and conditional credence, respectively. Often probabilities are not available outside of structured games.

2.2 MiniMax

The MinMax family of decision rules are strategies for the pursuit of a goal such as, maximizing the minimum gain or maximizing the maximum gain. Expected utility theory requires probabilities, while MinMax does not. Though we can replace probability with credence, which is subjective probabilities, there are cases in which due to ignorance or high uncertainty, it may be ideal to give up probability.

3 Game theory

3.1 Classic game theory

A game is an interactive situation consisting of an interdependent decision problem between self-interested agents ("players"). According to Pacuit and Roy (2017), there are three parts to the formal description:

1. N, The set of players
2. S, The set actions or strategies for each player
3. u, The players' preferences for the outcome, represented as utility functions

The goal of a player is to maximize their utility at the terminal state.

3.2 Epistemic game theory

The field of games theory has been influenced by philosophy, giving rise to Epistemic game theory, a Bayesian perspective on decision-making in strategic situations. According to Robert Stalnaker:

[1]Contrary to Markowitz, I do not believe standard deviation and risk to be synonymous.

> "There is no special concept of rationality for decision making in a situation where the outcomes depend on the actions of more than one agent. The acts of other agents are, like chance events, natural disasters and acts of God, just facts about an uncertain world that agents have beliefs and degrees of belief about. The utilities of other agents are relevant to an agent only as information that, together with beliefs about the rationality of those agents, helps to predict their actions." (Stalnaker, 1996, p. 136)

The *doxastic state* is the description of the decision maker's beliefs about the state of the environment as a whole, including other players. There is a varying level of information in games,

- Imperfect information: some aspects of the game are hidden, like facedown cards
- Rewards information: unknown outcome such as competing for an uncounted prize pool
- Strategic information: how another player will act
- High order information: the beliefs and strategies of other players
- Game dynamics information: the next state is non-deterministic (i.e., rolling dice)

In response to all these various uncertainties, epistemology methods can provide benefits. Credences allow a quantification similar to that of probability, with the consideration of uncertainty. Some authors argue all incompleteness in information can be reduced to uncertainty about the payoffs. Similarly,

> "Contemporary epistemic game theory holds that, although assessment of decision in game situations may ultimately be reducible to strategic uncertainty, making higher-order uncertainty explicit can clarify a great deal of what interactive or strategic rationality means." (Pacuit & Roy, 2017)

In the epistemic game theory context, a "possible world" is the "possible (epistemic) state" of how a scenario may progress. An intuitive example is predicting which three moves your opponent in chess is most likely to make. Secondly in this context, a "proposition" is an "event". As in predicate modal logic, predicates can be added to epistemic game theory, though there is seldom a need.

The sub-goal is to maximize the expected utility given the agent's utility function and beliefs. The expected value is a function of the player's move, a, and the hypothesized set of the opponent's moves. The later set is labeled β and corresponds to a probability distribution. For example,

$$E(a, \beta) = \sum_{b \in \beta} P(b) * u(a, b)$$

Building the hypothesized set of the opponent's moves is difficult. Suppose a set of scenarios of how the opponent might behave, perhaps ranging from cooperative to competitive or friendly to hostile. To reduce the set to a single scenario, assemble each scenario as a row vector, apply a normalized weighting scheme, finally perform a column-wise summation to "collapse" the future counterfactuals (scenarios) into one hypothetical scenario.[2]

3.2.1 Example

Table 1: Sample table title

Scenario	Competative	Cooperative	Irrationally vengeful
S_1	0.15	0.80	0.05
S_2	0.45	0.10	0.45
S_{hyp}	0.30	0.45	0.25

In this example, there are two scenarios with three events each. S_f is the hypothetical scenario conjured by a column-wise summation after a uniformly distributed weighting scheme, 0.5 on both

[2]This strategy could be applied to the expected utility itself. It is perhaps what many people mentally do already when considering momentous decisions.

$scenarios$. Table 1 can be used to compute expected values across epistemic worlds. This section has is adapted from SEP's Epistemic Foundations of Game Theory (Pacuit & Roy, 2017), which provides examples after significant additional build-up.

4 Probability Logics

4.1 Propositional Probability Logics

Expanding upon the well-known propositional logic (PL), a probability logic enables granularity. To facilitate, we require a probability function, P, which maps members of L, the set of propositions, to a real number (Burgess, 1970). The probability function is of the form$P : L \to R$. Recall that in PL the valuation function would map to $0, 1$, here we map to a real number and are subject to three constraints (Demey, Kooi, & Sack, 2019).

- Non-negatively:$P(\psi) \geq 0$ for all $\psi \in L$
- Tautologies: $\models \psi \to P(\psi) = 1$
- Finite additivity: $\models \neg(\psi \wedge \phi) \to P(\psi \vee \phi) = P(\psi) + P(\phi)$

The $\models$ symbol denotes (semantic) validity, as in PL. All the logical operators have their equivalent in probabilistic logic.

Note "that probabilities cannot be seen as generalized truth values, because probability functions are not 'extensional'; for example, $P(\psi \wedge \phi)$ cannot be expressed as a function of $P(\psi)$ and $P(\phi)$" (Demey et al., 2019).

For example, suppose we have the following argument, $p \vee q$ and $p \to q$, and the conclusion, q. Provided the fact that $P(q) = P(p \vee q) + P(p \to q) - 1$. If the probability of the premises are, $P(p \vee q) = 6/7$ and $P(p \to q) = 5/7$, then the conclusion follows to be $P(q) = 4/7$.

Probability logic is surely useful on its own in the practical sense, but it is possible to extend the idea of probability logic into modal logic.

4.2 Qualitative Modal Probability Logics

A basic qualitative Modal Probability Logic can be used to represent uncertainty. This is achieved with the addition of the $\Box$ operator which is read as "probably", or in a more formal sense, "sufficiently high probability". The operator fails to satisfy $(\Box\psi \wedge \Box\phi) \to \Box(\psi \wedge \phi)$, making it not only a non-normal modal operator, it is also incompatible with Kripke semantics.

There are some potential use cases. First, perhaps as originally intended by the author, this system is useful for modeling arguments and their implications. It may be extended with the $\diamond$ operator denoting "probably not". Second, which is my conjecture, the combination of this system with the MinMax family of decision rules might be of benefit. Though MinMax does not factor in any notion of likelihood, it may be extended - perhaps as a guiding heuristic, for instance, explicitly avoiding a less-adverse "probable" event. Finally, past work (Burgess, 1970) has been done which combines this logic with other modal operators such as metaphysical necessity and knowledge.

However, simply on the virtue of this logic being qualitative, for practical purposes, I do not further discuss this Logic.

4.3 Modal Probability Logics

The time has arrived! After all the build-up of probability logic, we come across a system that will assist us in our goal for formally aiming decision making under uncertainty.

4.3.1 Spatial example

As stated earlier, modal operations iterate over "worlds", which can be informally thought of as a row of a truth table. This concept is extended to the probability distribution of each "world" or "state". For example, suppose we have the twenty-one municipalities of Metro Vancouver as members of our set of states, and rather than propositions we have probability of {Sunny, Rainy, Snowy}.

Table 2: Weather across municipalities

Municipalities	Sunny	Rainy	Snowy
Vancouver	0.20	0.75	0.05
N. Vancouver	0.05	0.85	0.10
W. Vancouver	0.15	0.80	0.05

This illustrates "worlds" in the spatial sense.[3] A Modal probability logic enables an additional dimension of data.

4.3.2 Epistemic-Behavioural example

Similarly, the additional dimension provided by modal logic can be applied to behavioural scenarios spanning from optimistic to pessimistic, or friendly to hostile, and the "propositions" are instead the probability of an event occurring. A modal probability logic enables expected utility to be computed across five diverse scenarios.

Table 3: Belief in Opponent's reaction

Action	Friendly	Hostile
Corporate	0.80	0.20
Defect	0.05	0.95

This illustrates "worlds" in the epistemic sense. This concept may be expanded by generating these scenarios from a decision tree, in which every move is in reaction to an opponent's move, which recurses down to depth d. This extension may require a tense modal to accommodate the sequential nature, unless only one depth, number of steps in the future, was considered.[4] Below is an illustration,

Table 4: Belief in Opponent's future reaction

Actions	Friendly	Hostile
CCCCC	0.97	0.03
CDCCD	0.60	0.40
DDDDD	0.01	0.99

4.3.3 Formalization

The basic finite modal probabilistic model is inspired by the Kripke frame. The model $M = (W, P, V)$. W in a finite set of possible worlds, states, or scenarios. P assigned a probability distribution to each world. Lastly, V is the valuation function assigning atomic propositions from set ψ to each world. This structure is the same as a directed graph, stochastic matrix, and state-transition matrix, depending on the particular field.[5] For modeling convenience, there is an index set A typical representation is the set of agents or actions in a game.

4.3.4 Formal example

In the following example is provided by (Demey et al., 2019), suppose we have an index set $A = \{a, b\}$, and a set $\phi = \{p, q\}$ of atomic propositions. Consider (W, P, V), where

- $W = \{w, x, y, z\}$
- $P_{a,w}$ and $P_{a,x}$ map w to 1/2, x to 1/2

[3]It may be of benefit to extend this with spatial modal logic that features an accessibility relation corresponding to the physical border. This notion is similar to possible world semantics.

[4]In theory one may do a Monto Carlo simulation if the system dynamics are complex and/or it is desirable to look several steps ahead. This is desirable in chess where both these issues arise.

[5]The latter two differ only in jargon, belonging to statistics and systems engineering respectively.

- $P_{a,y}$ and $P_{a,z}$ map y to 1/3, z to 2/3
- $P_{b,w}$ and $P_{b,y}$ map w to 1/2, y to 1/2
- $P_{b,x}$ and $P_{b,z}$ map x to 1/4, z to 3/4
- $V(p) = \{w, x\}$
- $V(q) = \{w, y\}$

In the following diagram, inside each circle is a labeling of the truth of each proposition letter for the world whose name is labeled right outside the circle. The arrows indicate the transition probabilities. For example, an arrow from world x to world z labeled by (b, 3/4) indicates that from x, the probability of z under label b is 3/4.

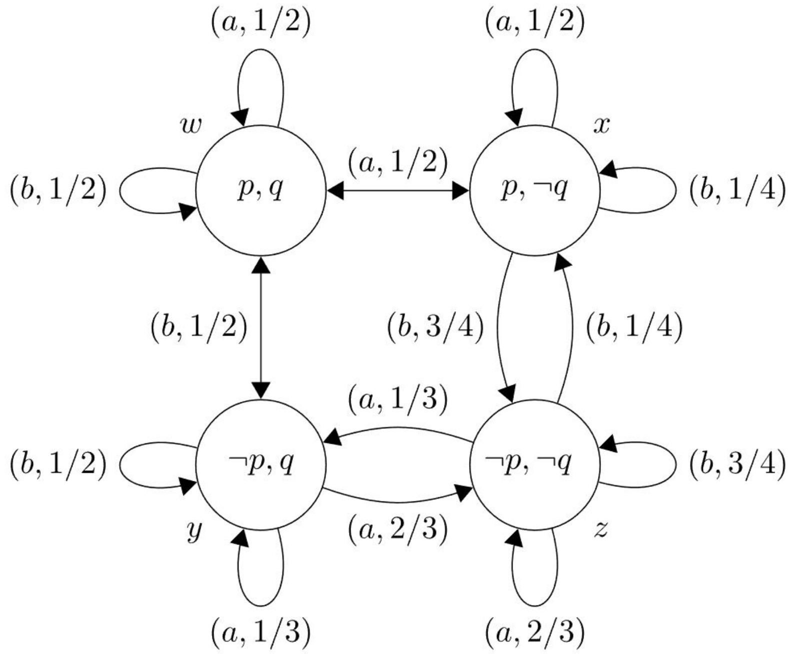

Figure 1: State transition diagram

Another example could be expect utility across several scenarios, similar to the minimal examples in table 1 & 4. A key feature is higher-order reasoning, which in this case is reasoning probabilities about probabilities. Reasoning can be done relative to the actions of another player or relative to the uncertainty resulting from action itself. Higher-order reasoning is discussed further by Demey et al. (2019).

4.4 Conditional probability logic

It is clear that there could be a modal conditional probability logic, and there indeed is. The conditional probability increases precision, potentially reducing the epistemic uncertainty. For instance incorporation of an observable or the last state in the weather context, P(snow) <= P(snow | snow yesterday). Conditional probability logic itself is very complex, there are several principles each with its requirements, assumptions, and potential failures. From Goldstein and Santorio (2021), for the sake of motivating future exploration, some of the central ideas are, ,

- Regarding the indicative conditional, Stalnaker's Thesis states, $Pr(A \to B) = Pr(B|A)$
- Conditionalization states, $P_A(B) = P(B|A)$
- Must Preservation states that, If$P(A) = 1$, then $p(\Box A) = 1$
- Might Contradiction states that, If$P(\ A) = 1$, then $P(\diamond A) = 0$
- Lukasiewicz's Principle states, $A \models \Box A$

Stalnaker's Thesis fails when sentences involving conditionals and modals are involved (Goldstein & Santorio, 2021, p .11). This has led to Robust Stalnaker which is more restrictive. The author has a lecture[6] and and set of notes[7] posted online.

5 Honorable mentions

First and foremost Dempster–Shafer theory which is a general framework for reasoning with epistemic uncertainty is an interesting direction to consider. Info-gap decision theory is diverse and spans domains, covering both decision rules and uncertainty, in the pursuit avoiding of failure under severe uncertainty. Osherson and Weinstein (2012, 2013) provide a quantified preference logic that combines utility theory and modal logic. Three-valued logic which under one interpretation adds the additional value, "indeterminate", perhaps combined with supervaluationism, could be of benefit, when in conjunction with a qualitative probability logic.

6 Conclusion

The purpose of the paper was to survey and provide a primer for the study of the more complex modal game theory logic, which is the intersection of decision theory, epistemic logic, and most importantly, probabilistic logic. To accomplish this I provided an introduction to decision theory and credence, then discussed epistemic game theory at a high level, which requires a probabilistic model logic for non-trivial examples. Finally, I discussed probabilistic model logic in some depth. This paper acts as a stepping stone to Bonanno (2002) which is a technically involved manner the end-goal for modal game theory logic.

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[6] www.youtube.com/watch?v=FU1lFsZNVhs

[7] paolosantorio.net/KS-NASSLLI2018.pdf