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Probability Logic for Type Spaces*

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Using a formal propositional language with operators "individual i assigns probability at least α" for countably many α, we devise an axiom system which is sound and complete with respect to the class of type spaces in the sense of Harsanyi (1967–1968, *Management Science*, **14,** 159–182). A crucial inference rule requires that degrees of belief be compatible for any two sets of assertions which are equivalent in a suitably defined natural sense. The completeness proof relies on a theorem of the alternative from convex analysis, and uses the method of filtration by finite sub-languages. *Journal of Economic Literature* Classification Numbers: D80, D82.

1. INTRODUCTION

Ever since Harsanyi's (1967–1968) seminal contribution, *type spaces* are the basic model in game theory and economics for describing asymmetric information in a group of interacting individuals. In a type space, each state of the world is associated with a *state of nature*, which describes the objective parameters relevant for the interaction, such as the payoff functions, and with a *type* for each individual, which is a probability measure on the space

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of states of the world. What is crucial about a type space is that one and the same set of states of the world both constitute the object of the individual's probabilistic beliefs and determines what these beliefs are. If necessary, the implicit circularity of this construction can be unfolded by specifying the beliefs of each individual about nature, about the other individuals' beliefs about nature, and so forth, in a recursive fashion.

When one abstracts from the quantitative aspects of the probabilistic beliefs, and replaces each type by the support of the probability measure, the result is a *multi-person Kripke structure*—a basic tool to describe certainty and knowledge. In a Kripke structure, each state of the world is associated not with a probability for each individual, but rather with a set of states that he considers as possible, without being able to tell which of them obtains.

Kripke structures were axiomatized with an epistemic modal logic by Kripke (1963), using a propositional language that has a knowledge operator K_i for each individual i. As Aumann (1995) in particular has shown, the logical formulation is of foundational importance because it enables an explicit construction of the Kripke structure from more primitive statements about the mutual knowledge of the individuals. These statements are most naturally expressed in a logical syntax. Thus, the logical approach attests to the generality and applicability of the model.

In this work, we strive for an analogous axiomatization of probabilistic type spaces. More precisely, we want to find an axiom system for type spaces, phrased in a simple language with probabilistic belief operators, which will be sound and complete with respect to the class of type spaces, such that the formulas which are valid in every type space will be exactly the theorems of the system.

Our logical language is thus a limited extension of the Kripkean modal syntax. All epistemic features are captured by belief operators L^i_α for rational $\alpha \in [0, 1]$, to be interpreted as "i assigns probability at least α." This syntax with indexed operators was suggested by Aumann (1995, Section 11). The set of axioms he states there is sound with respect to type spaces. We show, however, that it is not complete, i.e., that not every valid formula in the class of type spaces is a theorem provable from the axioms. The way we propose to complete Aumann's system might seem roundabout, and indeed it remains an open problem whether there is a simpler complete system. Nevertheless, the "complicated" extra inference rule that we need is closely related to a condition introduced to resolve two fundamental issues in probability theory.

The first is the existence of a probability compatible with a qualitative ordering of events. De Finetti (1951) stated a set of simple necessary conditions on the ordering to be compatible with a probability, and asked whether they are sufficient. Kraft *et al.* (1959) constructed a counterexample and introduced the missing condition for sufficiency—a condition to which our extra inference rule is very close in spirit.

The second issue is how to characterize those pairs of a super-additive lower probability and a sub-additive upper probability which can be separated by an (additive) probability. Suppes and Zanotti (1989, Theorem 1) introduced a necessary and sufficient condition, which is again related to the one we employ. In all of these cases, the condition is needed in order to use some version of the separation theorem or the theorem of the alternative in convex analysis.

As in ordinary modal logic, our language allows only for *finite* conjunctions and disjunctions. This approach leads to a well-recognized difficulty in investigating the probability calculus. This is the problem of "non-Archimedianity." For example, the set of formulas Σ which says that the probability of φ is at least $(1/2) - r$ for every rational r, is consistent with the formula ψ which says that the probability of φ is strictly smaller than 1/2, because every finite subset of $\Sigma \cup \{\psi\}$ is consistent. However, there is no real number for the probability of φ which would be compatible with the whole set $\Sigma \cup \{\psi\}$. When infinite conjunctions and disjunctions are permitted, Σ could be made to imply the negation of ψ, and thus avoid the problem. But in a finitary logic, such as Kripke's and ours, this cannot be done, so there is no hope of having *strong* completeness of the system.[1] Put differently, the canonical space of maximally consistent sets of formulas cannot be endowed with a probabilistic types structure compatible with the formulas that build the states, because in those states that contain the formulas of $\Sigma \cup \{\psi\}$, there is no suitable probability for the set of states containing φ. Still, we do succeed in finding a complete system, in which semantic truth in the family of type spaces, spelled out in the finitary language, is a theorem of the system.

We circumvent the difficulty of non-Archimedianity by employing a device which has led to successful completeness proofs of finitary axiomatizations of *common belief*. The idea is to choose suitable *filtrations* of the full language to sub-languages with finitely many formulas.

This goal of axiomatizing type spaces has been approached also in artificial intelligence and theoretical computer science. We will in particular discuss the connection of our work with that of Fagin and Halpern (1994) and Fagin *et al.* (1990). In contrast with our work, they use a very rich syntax, which expresses not only probabilities of formulas, but also valuations for *linear combinations of formulas*. Implicitly, this means that the individuals are required to assess directly not only the probability of statements but also the "integrals" of some simple real-valued functions. Here,

[1]That is, we cannot hope to have a finitary system for probability in which if ψ holds whenever a (possibly infinite) set of formulas Σ holds in a type space, then Σ proves ψ.

we show that an appropriate axiomatization is possible with a much simpler syntax, where individuals are required to express only their valuations for formulas.

The paper is organized as follows: Definitions appear in Section 2. Aumann's incomplete system is presented in Section 3. In Section 4 we introduce the inference rule needed to complete the system. Further axioms which express the individual's introspective capabilities and their implications are presented in Section 5. Section 6 concludes with a discussion and connections with the literature. The proofs appear in the appendix.

2. DEFINITIONS

For the entire discussion we fix a set I of individuals. The formal language $\mathcal{L}$ in is built in the familiar way from the following components: A set $\mathcal{P}$ of propositional variables, the connectives $\neg$ and $\wedge$, from which the other connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as usual, and the modal operators L_α^i for any rational α in $[0, 1]$, and every individual $i \in I$ with the intended meaning "individual i assigns probability at least α." The operator M_α^i—"i assigns probability at most α"—is an abbreviation defined by

$$M_\alpha^i \varphi \leftrightarrow L_{1-\alpha}^i \neg\varphi, \tag{Def. M}$$

and the operator E_α^i—"i assigns probability exactly α" is defined by

$$E_\alpha^i \varphi \leftrightarrow M_\alpha^i \varphi \wedge L_\alpha^i \varphi. \tag{Def. E}$$

(It follows that $\neg L_\alpha^i$ can be read as "i assigns probability strictly smaller than α," and $\neg M_\alpha^i$ as "i assigns probability strictly greater than α.")

The space $\mathcal{T}$ of *type spaces* that we aim to axiomatize has a typical element

$$\tau = \langle \Omega, \mathcal{A}, (T_i)_{i \in I}, v \rangle$$

where Ω is a non-empty set; $\mathcal{A}$ is a σ-field of subsets of Ω; for every $i \in I$, T_i is a measurable mapping from Ω to the space $\Delta(\Omega, \mathcal{A})$ of probability measures on Ω, which is endowed with the σ-field generated by the sets

$$\{\mu \in \Delta(\Omega, \mathcal{A}) : \ \mu(E) \geq \alpha\} \qquad \text{for all } E \in \mathcal{A} \text{ and rational } \alpha \in [0, 1],$$

and v is a mapping from $\Omega \times \mathcal{P}$ to $\{0, 1\}$, such that $v(\cdot, p)$ is measurable for every $p \in \mathcal{P}$.

The validation clauses of our logic are stated inductively in the usual way for the propositional connectives, and as follows for the modal operators L_α^i

$$\tau, \omega \models L_\alpha^i \varphi \quad \text{iff} \quad T_i(\omega)([\varphi]) \geq \alpha$$

where

$$[\varphi] = \{\omega \in \Omega : \tau, \omega \models \varphi\}.$$

We use the familiar abbreviations, $\tau \models \varphi$ for $[\forall \omega \in \Omega,\ \tau, \omega \models \varphi]$, and $\mathscr{T} \models \varphi$ for $[\forall \tau \in \mathscr{T},\ \tau \models \varphi]$.

To save on notation, from now on we omit the superscript i in the probability operators when only one individual needs to be considered.

3. AUMANN'S SYSTEM

A starting point for axiomatization is the following system Σ, which we have adapted from Aumann (1995, Section 11).[2] (The symbol $\vdash \varphi$ denotes that φ is a theorem of the system, i.e., provable from the axioms, and $\top$ and $\bot$ abbreviate $\varphi \vee \neg\varphi$, $\varphi \wedge \neg\varphi$, respectively.) Besides (Def. M) and (Def. E),

$$\text{any axiomatization of the propositional calculus} \tag{A0}$$

$$L_0\varphi \tag{A1}$$

$$L_\alpha \top \tag{A2}$$

$$L_\alpha(\varphi \wedge \psi) \wedge L_\beta(\varphi \wedge \neg\psi) \to L_{\alpha+\beta}\varphi, \qquad \alpha + \beta \leq 1 \tag{A3}$$

$$\neg L_\alpha(\varphi \wedge \psi) \wedge \neg L_\beta(\varphi \wedge \neg\psi) \to \neg L_{\alpha+\beta}\varphi, \qquad \alpha + \beta \leq 1 \tag{A4}$$

$$L_\alpha\varphi \to \neg L_\beta \neg\varphi \qquad \alpha + \beta > 1 \tag{A5}$$

$$\text{If } \vdash \varphi \leftrightarrow \psi \text{ then } \vdash L_\alpha\varphi \leftrightarrow L_\alpha\psi \tag{A6}$$

PROPOSITION 3.1. *From Σ the following axiom and inference rule schemata can be derived*:

$$\begin{array}{l}\text{If } \vdash \varphi \to \psi \text{ then } \vdash L_\alpha\varphi \to L_\alpha\psi \\ \text{and } \vdash M_\alpha\psi \to M_\alpha\varphi\end{array} \tag{A6+}$$

$$M_\alpha\neg\varphi \to \neg M_\beta\varphi \quad \alpha + \beta < 1 \tag{A5+}$$

$$\begin{array}{l}E_1\top \text{ and } \neg M_\alpha\top, \alpha < 1 \\ E_0\bot \text{ and } \neg L_\alpha\bot, \alpha > 0\end{array} \tag{A1-A2+}$$

[2]Note added in October 1999: In the published version of Aumann's notes (Aumann 1999), the system below does not appear anymore, and Aumann (1999, Section 15, p. 306) says that he did not succeed in developing a deductive logic for his probabilistic belief grammer (as opposed to the one for knowledge). The current work, therefore, fills this lacuna.

$$L_\alpha\varphi \rightarrow L_\beta\varphi \qquad \beta < \alpha \tag{A7}$$

$$M_\alpha\varphi \rightarrow M_\beta\varphi \qquad \beta > \alpha \tag{A7+}$$

$$\neg L_\alpha\varphi \rightarrow M_\alpha\varphi \tag{A8}$$

$$L_\alpha(\varphi \wedge \psi) \wedge \neg M_\beta(\varphi \wedge \neg\psi) \rightarrow \neg M_{\alpha+\beta}\varphi, \quad \alpha + \beta \leq 1 \tag{A9}$$

$$\neg M_\alpha(\varphi \wedge \psi) \wedge \neg M_\beta(\varphi \wedge \neg\psi) \rightarrow \neg M_{\alpha+\beta}\varphi, \quad \alpha + \beta \leq 1 \tag{A10}$$

$$E_\alpha\varphi \leftrightarrow E_{1-\alpha}\neg\varphi \tag{A11}$$

$$E_\alpha\varphi \rightarrow \neg E_\beta\varphi \quad \alpha \neq \beta \tag{A12}$$

The proof is sketched in the appendix. As he stated it, the probability logic part of Aumann's (1995) system consists of (A0)–(A5), (A6+) and (A7). This last schema has just been seen to be derivable from the others.

The system Σ involves a rendering of the complementation axiom of the probability calculus (cf. (A11)), as well as of the *uniqueness* of the probability values (cf. (A12)). What is not so easy to state in the finitary language of Σ is the *existence* of a probability value for each formula, as opposed to its uniqueness. Logical truths and contradictions do receive exact probability assignments (cf. (A1–A2+)), but it is not clear how other formulas do.

If the system Σ were a complete axiomatization of the class of type spaces, it should be able to express the following basic relations of the probability calculus: for any probability measure μ and any two disjoint sets A and B,

(i) $\mu(A) \geq \alpha,\ \mu(B) \geq \beta \Rightarrow \mu(A \cup B) \geq \alpha + \beta$

(ii) $\mu(A) > \alpha,\ \mu(B) > \beta \Rightarrow \mu(A \cup B) > \alpha + \beta$

(iii) $\mu(A) \geq \alpha,\ \mu(B) > \beta \Rightarrow \mu(A \cup B) > \alpha + \beta$

(iv) $\mu(A) \leq \alpha,\ \mu(B) \leq \beta \Rightarrow \mu(A \cup B) \leq \alpha + \beta$

(v) $\mu(A) < \alpha,\ \mu(B) < \beta \Rightarrow \mu(A \cup B) < \alpha + \beta$

(vi) $\mu(A) \leq \alpha,\ \mu(B) < \beta \Rightarrow \mu(A \cup B) < \alpha + \beta$

In the system Σ, the schemata (A3) and (A4) express (i) and (v), and theorems (A10) and (A9) express (ii) and (iii). But there appear to be no syntactical counterparts to the remaining properties (iv) and (vi) in the system Σ of Aumann. The next proposition formally confirms that Σ is not powerful enough to express them, and is thus incomplete.

PROPOSITION 3.2. *The following schemata* (*which express* (*iv*) *and* (*vi*), *respectively*) *are not entailed by* Σ:

$$M_\alpha(\varphi \wedge \psi) \wedge M_\beta(\varphi \wedge \neg\psi) \rightarrow M_{\alpha+\beta}\varphi, \qquad \alpha + \beta \leq 1 \tag{A13}$$

$$M_\alpha(\varphi \wedge \psi) \wedge \neg L_\beta(\varphi \wedge \neg\psi) \to \neg L_{\alpha+\beta}\varphi, \qquad \alpha + \beta \leq 1 \tag{A14}$$

We prove this proposition by constructing a model in which Σ holds but (A13) does not. (Schema (A14) is easily seen to be equivalent to (A13) given Σ.) Obviously, this model has to be a nonstandard one, with infinitesimal probability values, since any standard probability measure should satisfy at once all the relations (i)–(vi). The proof is in the appendix.

4. A COMPLETE SYSTEM

It would be enlightening to know whether the system Σ+(A13) is complete with respect to the family of type spaces. This question remains open for the time being. We do suggest, however, introducing an elaborate inference rule (B), which is sufficiently strong to imply—given a relevant part of Σ—all the syntactical schemata corresponding to (i)–(vi) *and* will be seen to lead to a complete axiomatization. This inference rule refers to semantic facts that are not so elementary as conditions (i)–(vi), and thus need explaining in some detail.

Recall that a probability measure μ on a space Ω defines the integral functional on the collection of characteristic functions (of measurable sets), and hence also on the semi-group of finite sums of such characteristic functions. In particular, if a function f in this semi-group can be written as a sum of characteristic functions in two different ways, then the two ways of calculating the integral with respect to μ will give the same result. Our supplementary inference rule is closely related to this semantic fact.

More precisely, suppose that f is the sum of the characteristic functions of $E_1, \dots, E_m$, and can also be written as the sum of the characteristic functions of $F_1, \dots, F_n$. This holds if and only if the points that belong to at least one of $E_1, \dots, E_m$ belong to at least one of $F_1, \dots, F_n$ and vice versa, and similarly for the points that belong to at least two sets, three sets, etc. Let us denote by $E^{(k)}$ the set of points that appear in at least k of the sets $E_1, \dots, E_m$, and by $F^{(k)}$ the points that appear in at least k of the sets $F_1, \dots, F_n$, i.e.

$$E^{(k)} = \bigcup_{1\leq \ell_1 < \cdots < \ell_k \leq m} (E_{\ell_1} \cap \cdots \cap E_{\ell_k})$$

$$F^{(k)} = \bigcup_{1\leq \ell_1 < \cdots < \ell_k \leq n} (F_{\ell_1} \cap \cdots \cap F_{\ell_k})$$

Using this notation and the convention that $E^{(k)} = \emptyset$ if $k > m$, and similarly $F^{(k)} = \emptyset$ if $k > n$, we clearly have that for f as above,

$$E^{(k)} = F^{(k)} \text{ for } 1 \leq k \leq \max(m, n).$$

The existence of a well-defined integral of f with respect to μ implies in particular that

$$\mu(E_i) \geq \alpha_i \text{ for } i = 1, \ldots m$$

and

$$\mu(F_j) \leq \beta_j \text{ for } j = 2, \ldots n,$$

entail that

$$\mu(F_1) \geq (\alpha_1 + \cdots + \alpha_m) - (\beta_2 + \cdots \beta_n).$$

Our supplementary inference rule (B), to be introduced now, is a syntactical rendering of this entailment. If $(\varphi_1, \ldots, \varphi_m)$ is a finite sequence of formulas, we use the notation $\varphi^{(k)}$ to refer to either the formula

$$\bigvee_{1 \leq \ell_1 < \cdots < \ell_k \leq m} (\varphi_{\ell_1} \wedge \cdots \wedge \varphi_{\ell_k}).$$

or to $\perp$ whenever $k > m$. If $(\psi_1, \ldots, \psi_n)$ is another sequence of formulas, the notation

$$(\varphi_1, \ldots, \varphi_m) \leftrightarrow (\psi_1, \ldots, \psi_n)$$

will refer to the formula

$$\bigwedge_{k=1}^{\max(m,n)} \varphi^{(k)} \leftrightarrow \psi^{(k)}.$$

(To illustrate how these conventions operate, consider the following example: $m = 2$, $n = 1$, $\varphi_1 = \varphi \wedge \psi$, $\varphi_2 = \neg\varphi \wedge \psi$, $\psi_1 = \psi$. Then $(\varphi_1, \varphi_2) \leftrightarrow (\psi_1)$ denotes $(\varphi^{(1)} \leftrightarrow \psi^{(1)}) \wedge (\varphi^{(2)} \leftrightarrow \psi^{(2)})$, that is to say

$$((\varphi_1 \vee \varphi_2) \leftrightarrow \psi_1) \wedge ((\varphi_1 \wedge \varphi_2) \leftrightarrow \perp),$$

which in the particular instance is a logical truth.)

The above semantic entailment can now be rendered by the inference rule

$$\text{If} \quad ((\varphi_1, \ldots, \varphi_m) \leftrightarrow (\psi_1, \ldots, \psi_n)) \quad \text{then}$$

$$\left(\left(\bigwedge_{i=1}^{m} L_{\alpha_i}\varphi_i\right) \bigwedge \left(\bigwedge_{j=2}^{n} M_{\beta_j}\psi_j\right) \to L_{(\alpha_1+\cdots+\alpha_m)-(\beta_2+\cdots+\beta_n)}\psi_1\right) \tag{B}$$

for $m, n \geq 1$ and $(\alpha_1 + \cdots + \alpha_m) - (\beta_2 + \cdots + \beta_n) \in [0, 1]$.

The following proposition shows how the schemata bearing on conditions (i)–(vi) can be recovered from (B), and that (B) admits of equivalent reformulations.

PROPOSITION 4.1. *In the presence of* (Def. M), (A0), (A1), (A2), (A5), (A6) *and* (A8),

4.1.1. (B) *implies* (A3), (A4), (A9), (A10), (A13),(A14)

4.1.2. (B) *is equivalent to* (B =), *which is* (B) *with* $m = n$

4.1.3. (B) *is equivalent to the following inference rule* (B′):

$$\text{If} \quad \big((\varphi_1, \ldots, \varphi_m) \leftrightarrow (\psi_1, \ldots, \psi_n)\big) \quad \text{then} \tag{B$'$}$$

$$\left(\neg M_{\alpha_1}\varphi_1 \wedge \left(\bigwedge_{i=2}^{m} L_{\alpha_i}\varphi_i\right) \wedge \left(\bigwedge_{j=2}^{n} M_{\beta_j}\psi_j\right) \to \neg M_{(\alpha_1+\cdots+\alpha_m)-(\beta_2+\cdots+\beta_n)}\psi_1\right)$$

We are now ready to state the main result of this paper. Define Σ^+ to be the system consisting of (A0), (A1), (A2), (A5), (A6), (A8) and (B), or more simply (though clearly redundantly) Σ+(B). Then,

THEOREM 4.2. *Σ^+ is a sound and complete axiomatization of $\mathcal{T}$, i.e.,*

$$\vdash_{\Sigma^+} \varphi \quad \Leftrightarrow \quad \mathcal{T} \models \varphi.$$

The proof of Theorem 4.2 employs the method of *filtration* (see e.g., Chellas (1980), p. 42), which has been used elsewhere in modal epistemic logic to prove the completeness of systems that are not necessarily *strongly* complete[3] (e.g., the Halpern and Moses (1992) or the Lismont and Mongin (1994) common belief logics). With this technique, completeness is proved "formula by formula": one fixes the formula φ for which the implication

$$\models \varphi \quad \Rightarrow \quad \vdash \varphi$$

should hold, and proceeds to construct the finite space of maximally consistent sets of formulas in the sub-language $\mathcal{L}[\varphi]$ generated by φ, up to some finite depth.

5. INTROSPECTION AND TRUTH OF BELIEFS

Up to now, we did not introduce any schemata concerning the beliefs of an individual regarding his own beliefs. A natural starting point for discussing this issue are the usual positive and negative introspection axioms (4) and (5) from epistemic logic, which are rephrased in our syntax as[4]

$$L_1^i\varphi \to L_1^i L_1^i\varphi \tag{4}$$

[3]An axiom system is strongly complete if for every formula φ and a set of formulas Γ, $\Gamma \models \varphi$ implies $\Gamma \vdash \varphi$ (i.e., if φ holds in every state of every model where all the formulas in Γ hold, then there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\psi_1 \wedge \cdots \wedge \psi_n \to \varphi$ is a theorem of the system).

[4]Note that $\vdash E_1\varphi \leftrightarrow L_1\varphi$.

$$\neg L_1^i \varphi \to L_1^i \neg L_1^i \varphi \tag{5}$$

Positive introspection, schema (4), states that if the individual is certain of φ, he is also certain that he is certain of φ; negative introspection, schema (5), states that if the individual is not certain of φ, he is also certain that he is not certain of φ (an axiom which has been the object of critical discussion in epistemic logic and artificial intelligence).

Schema (4) is valid in the subclass of *transitive type spaces* $\mathcal{T}_t$ satisfying

$$T_i(\omega)(\{\omega' \in \Omega : T_i(\omega') << T_i(\omega)\}) = 1 \quad \forall \omega \in \Omega, \; i \in I$$

where for probability measures μ and ν, the notation $\mu << \nu$ means that μ is absolutely continuous with respect to ν, that is $\nu(E) = 0$ implies $\mu(E) = 0$. In a transitive type space, each type excludes the possibility that he assigns a positive probability to events that he himself is sure did not happen. Thus, if he is sure that an event occurred, he is sure that he is sure it occurred.

Schema (5) is valid in the subclass of *Euclidean type spaces* $\mathcal{T}_e$ satisfying

$$T_i(\omega)(\{\omega' \in \Omega : T_i(\omega) << T_i(\omega')\}) = 1 \quad \forall \omega \in \Omega, \; i \in I$$

In a Euclidean type space, each type excludes the possibility that he assigns probability one to events to which he assigns in fact a probability less than one. Thus, if he is not sure that an event took place, he is sure that he is not sure that it did.

In fact, we have

THEOREM 5.1. *The system* $\Sigma^+ + (4)$ $(\Sigma^+ + (5)$, *respectively*) *is a sound and complete axiomatization of* $\mathcal{T}_t$ ($\mathcal{T}_e$, *respectively*).

Remark. We do know not if it is possible to formulate a general semantic counterpart for the *truth* axiom schema of epistemic logic[5]

$$L_1^i \varphi \to \varphi \tag{T}$$

Beyond the axioms (4), (5) and (T), known from the epistemic logic of knowledge, our rich syntax allows us to consider further introspection schemes. Consider the following generalization of (4) and (5):

$$L_\alpha^i \varphi \to L_1^i L_\alpha^i \varphi \tag{4$'$}$$

[5]The truth axiom is *not* generally valid in the subclass of type spaces in which ω is in the suppurt of $T_i(\omega)$ for every $\omega \in \Omega, \; i \in I$. We are grateful to Martin Meier for the following counterexample: $\Omega = [0, 1]$, φ holds everywhere in Ω except for the state 1/2, and the belief of player i in the state 1/2 is the Lebesgue measure. Then 1/2 is in the suppurt of $T_i(1/2)$, but the axiom (T) does not obtain in 1/2.

$$\neg L^i_\alpha \varphi \to L^i_1 \neg L^i_\alpha \varphi \tag{5'}$$

(See e.g., Gaifman (1986) and Samet (1997, 1998) for a discussion of higher-order probabilities.) If, in a type space, the set of states $[T_i(\omega)]$, where individual i has the same type as in ω

$$[T_i(\omega)] = \{\omega' \in \Omega : \ T_i(\omega') = T_i(\omega)\}$$

is measurable, and each type is certain of its type, i.e.,

$$T_i(\omega)[T_i(\omega)] = 1 \quad \forall \omega \in \Omega, \ i \in I$$

the type space is called a *Harsanyi type space* (Harsanyi, 1967–1968). The measurability of $[T_i(\omega)]$ is guaranteed when the σ-field $\mathcal{A}$ of the type space is generated by a countable subfield $\mathcal{A}_0 \subseteq \mathcal{A}$. Indeed, in this case, $[T_i(\omega)]$ is a countable intersection of measurable events:

$$\begin{aligned}[T_i(\omega)] &= \{\omega' \in \Omega \colon T_i(\omega') = T_i(\omega)\} \\ &= \bigcap_{\substack{T_i(\omega)(A) \geq \alpha,\ A \in \mathcal{A}_0 \\ \alpha \in [0,1] \text{ rational}}} \{\omega' \in \Omega : T_i(\omega')\,(A) \geq \alpha\}\end{aligned}$$

THEOREM 5.2. *The system* $\Sigma^+ + (4') + (5')$ *is a sound and complete axiomatization of the class of Harsanyi type spaces*.

6. DISCUSSION AND RELATED WORKS

The inference rule (B) (or its equivalent form (B=)) is very close in spirit to the sufficient condition first introduced by Kraft *et al*. (1959) for a "more or equally probable than" relation $\succeq$ on a finite algebra of events to be represented by a probability measure.[6] In Scott's (1964) convenient formulation, the condition says that if the sum of the characteristic functions of $E_1, \ldots, E_m$ equals that of $F_1, \ldots, F_m$ and

$$E_i \succeq F_i \quad i = 1, \ldots m-1$$

then $F_m \succeq E_m$. De Finetti had previously assumed that the simpler condition

$$E \succeq F \quad \Leftrightarrow \quad E \cup G \succeq F \cup G$$

for events G disjoint from both E and F, would suffice. However, a counterexample in Kraft *et al*. (1959) showed that de Finetti's condition is not sufficient for the existence of a probability representation.

[6]The relation $\succeq$ satisfies $\emptyset \not\succeq \Omega$ and $E \succeq \emptyset$ for every event $E \subseteq \Omega$, and every two events E and F are comparable—either $E \succeq F$ or $F \succeq E$.

Qualitative probability relations have been investigated in some logic papers. Following Segerberg (1971), Gärdenfors (1975) introduced the binary relation $\succeq$ into a propositional language, and was thus able to translate the theory of qualitative probability relations, including Scott's condition, into syntactical terms. He states a completeness theorem which shares a significant feature with ours: It is proved using a filtration device. This means that Gärdenfors's axiomatization, like ours, is proved to be complete, not *strongly* complete.

The inference rule (B) is also close in spirit to the necessary and sufficient condition found by Suppes and Zanotti (1989) for the existence of a probability μ between a pair (μ_*, μ^*) of lower and upper set functions $\mu_* \leq \mu^*$, of which the lower is super-additive and the upper sub-additive: For disjoint A and B,

$$\mu_*(A) + \mu_*(B) \leq \mu_*(A \cup B) \leq \mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B).$$

The necessary and sufficient condition for the existence of a probability μ satisfying $\mu_* \leq \mu \leq \mu^*$ is that if the sum of the characteristic functions of $E_1, \ldots, E_m$ equals that of $F_1, \ldots, F_n$, then

$$\sum_{i=1}^{m} \mu_*(E_i) \leq \sum_{j=1}^{n} \mu^*(F_j)$$

Without this condition, counterexamples by Walley (1981) and Papamarcou and Fine (1986) show that a separating μ need not exist.

All these technical conditions share a common feature, i.e., they make it possible to employ some version of the separation theorem or duality theorem of convex analysis. In our case as well, the proof of the theorem will use a general version of the theorem of the alternative.

The paper by Gärdenfors (1975) discussed above is an early example of the logical work that can be done on *reasoning about probability*—to borrow Fagin and Halpern's (1994) phrase. The defining feature of this area of probability logic is roughly that the probability concept (one way or another) belongs to the formal language; and although formulas are semantically assigned probability values, they receive only classical truth values. The probability logics of this kind are devised to investigate whether statements involving the probability concept are true or false. They stand in sharp contrast to those logics in which formulas are assigned probabilistic truth values, for instance in some of Lukasiewicz's (1970) multi-valued logics.

The logical work on reasoning about probability has received much impetus from artificial intelligence and distributed systems; for an overview, see Bacchus (1990). Fagin *et al*. (1990) and Fagin and Halpern (1994) provide elaborate examples of this recent work, as well as computer science applications. We will briefly relate these writers' axiomatizations to the present one. Both of their papers have the same rich language. It includes formulas like $w(\varphi) \geq 2/3$, to be read as "the weight of φ is greater or equal to 2/3," which are similar to ours, but also comparative statements such as $w(\varphi) > 2w(\psi)$, and more generally any statement of the form $a_1 w(\varphi_1) + \cdots + a_k w(\varphi_k) \geq c$, where $a_1, \ldots, a_k, c$ are integers. Their semantics has two cases, depending on whether or not all propositions are measurable, but in the measurable case, the weights $w(\cdot)$ call for a probabilistic interpretation, and the structures to be axiomatized are essentially like ours. (Fagin and Halpern (1994) add a Kripke relation to the probability structure in order to interpret an added knowledge operator in the language.) The main validation clause states that the formula $a_1 w(\varphi_1) + \cdots + a_k w(\varphi_k) \geq c$ is true at a world if and only if for the probability measure given at this particular world, the corresponding inequality holds. The axioms schemata reproduce the ordinary definition of a (finitely additive) probability measure, as well as some of the arithmetic rules for handling linear inequalities with integer coefficients.

The difference between this pioneering work and ours hinges on the expressive power of the language. We axiomatize the probability concept in terms of logical formulas that constitute a small subset of the set of formulas permitted by Fagin *et al*. (1990). Intuitively speaking, the problem of dealing syntactically with inequalities involving linear combinations of probability values, i.e. integrals, if you like, is resolved by introducing the powerful inference rule (B).

APPENDIX: PROOFS

Proof of Proposition 3.1.

- *Proof of* (A6+). If $\vdash \psi \to \varphi$, then $\vdash \varphi \wedge \psi \leftrightarrow \psi$ from (A0) and $\vdash L_\alpha(\varphi \wedge \psi) \leftrightarrow L_\alpha \psi$ from (A6), whence $\vdash L_\alpha \varphi \to L_\alpha \psi$ follows from (A0), (A1), and (A3) with $\beta = 0$. *In all subsequent proofs we normally do not mention the role of* (A0), (A6), (Def. M), and (Def. E).
- *Proof of* (A1–A2+). From (A1), (A2), (A5), and (A5+).
- *Proof of* (A7). If $\alpha > \beta$, then taking $\varphi = \psi$ and changing indices in (A4) leads to

$$\vdash \neg L_\beta \psi \wedge \neg L_{\alpha-\beta} \bot \to \neg L_\alpha \psi$$

and the result follows from (A1–A2+).

- *Proof of* (A8). Take $\varphi = \top$ and $\beta = 1 - \alpha$ in (A4) to get

$$\vdash \neg L_\alpha \varphi \wedge \neg L_{1-\alpha} \neg \varphi \to \neg L_1 \top$$

and apply (A2).

- *Proof of* (A9). The following rule can be derived from (A3):

If $\vdash \neg(\varphi \wedge \psi)$, then $\vdash L_\alpha\varphi \wedge L_\beta\psi \to L_{\alpha+\beta}(\varphi \vee \psi)$, $\alpha+\beta \leq 1$.

Hence

$$\vdash L_\alpha(\varphi \wedge \psi) \wedge L_{1-\alpha-\beta}\neg\varphi \to L_{1-\beta}((\varphi \wedge \psi) \vee \neg\varphi), \quad \alpha+\beta \leq 1$$

and

$$\vdash L_\alpha(\varphi \wedge \psi) \wedge M_{\alpha+\beta}\varphi \to M_\beta(\varphi \vee \neg\psi), \quad \alpha+\beta \leq 1.$$

- *Proof of* (A10). From (A8) and (A9).
- *Proof of* (A11). From (Def. E) and (Def. M).
- *Proof of* (A12). From (Def. M), (A5), and (Def. E). ■

Proof of Proposition 3.2. It is straightforward to prove that (A13) and (A14) are equivalent given Σ. We will show that Σ does not entail (A14). To this effect, we will consider an augmented class of "type spaces" $\widetilde{\mathcal{T}}$ for which Σ is sound (i.e., every theorem of Σ will be valid in all "type spaces" in $\widetilde{\mathcal{T}}$). Then, we will show an instance of a negation of (A14) which holds in one of these "type spaces." This will mean that (A14) is not entailed by Σ.

Consider that the ordered field $\mathbb{R}(\varepsilon)$ that results from adding an infinitesimal ε to $\mathbb{R}$ ($\mathbb{R}(\varepsilon)$ can be represented by the field of quotients of polynomials in x with the usual arithmetic, where ε is represented by $1/x$). Let $\widetilde{T}$ be the class of "type spaces"

$$\tilde{\tau} = \langle \Omega, \mathcal{A}, (\widetilde{T}_i)_{i\in I}, v \rangle$$

where each $\widetilde{T}_i(\omega)$ is a finitely additive measure with values in $\mathbb{R}(\varepsilon)$, which assigns to Ω a total mass of $1+\delta$, where δ is a positive infinitesimal or 0.

It is easy to verify that Σ is sound with respect to $\widetilde{\mathcal{T}}$. (Notice that the syntax remains unchanged, and in particular that the indexes α in the operators L_α continue to take only rational values). Consider now the following formula with the propositional variables φ and ψ

$$M_{\frac{1}{3}}(\varphi \wedge \psi) \wedge \neg L_{\frac{1}{3}}(\varphi \wedge \neg\psi) \wedge L_{\frac{2}{3}}\varphi \tag{3.2.1}$$

which is a negation of an instance of (A14). We will exhibit now a model $\tilde{\tau} \in \widetilde{\mathcal{T}}$ for (3.2.1). Ω will consist of three states

$$\Omega = \{a, b, c\}$$

with

$$[\varphi] = \{a, b\}, \qquad [\psi] = \{a\}.$$

There is a single individual i, whose "type" in one of the states—a, say—is

$$\widetilde{T}_i(a)\{a\} = \widetilde{T}_i(a)\{c\} = \frac{1}{3} + \varepsilon, \quad \widetilde{T}_i(a)\{b\} = \frac{1}{3} - \varepsilon.$$

It is easy to see (using (Def. M)!) that $\widetilde{\tau}$, a validates (3.2.1). Since (3.2.1) is a negation of an instance of (A14), (A14) is not a theorem of Ω—otherwise (3.2.1) would have been false in all the "type spaces" $\widetilde{\tau} \in \widetilde{\mathcal{T}}$, contrary to the above example. ■

Proof of Proposition 4.1. 4.1.1. To derive (A3) from (B), apply it with $m = 2$, $n = 1$, $\varphi_1 = \varphi \wedge \psi$, $\varphi_2 = \varphi \wedge \neg\psi$, $\psi_1 = \varphi$, using the logical truths $(\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \leftrightarrow \varphi$ and $(\varphi \wedge \psi) \wedge (\varphi \wedge \neg\psi) \leftrightarrow \bot$. To derive (A14) from (B), apply it with $m = 1$, $n = 2$, $\varphi_1 = \varphi$, $\psi_1 = \varphi \wedge \psi$, $\psi_2 = \varphi \wedge \neg\psi$, using the same logical truths as for (A3). In the presence of (A8), schema (A14) immediately implies (A4). Thus, given the assumptions listed in the proposition, (B) implies Σ. That it implies (A9), (A10), and (A13) then follows from already-known facts about Σ.

4.1.2. We have to prove that (B =) implies (B). We first observe that for all $m, n \geq 1$, if $m > n$

$$\vdash ((\varphi_1, \ldots, \varphi_m) \leftrightarrow (\psi_1, \ldots, \psi_n)) \leftrightarrow$$

$$((\varphi_1, \ldots, \varphi_m) \leftrightarrow (\psi_1, \ldots, \psi_n, \overbrace{\bot, \ldots, \bot}^{m-n \text{ times}}))$$

and if $m < n$

$$\vdash ((\varphi_1, \ldots, \varphi_m) \leftrightarrow (\psi_1, \ldots, \psi_n)) \leftrightarrow$$

$$((\varphi_1, \ldots, \varphi_m, \overbrace{\bot, \ldots, \bot}^{n-m \text{ times}}) \leftrightarrow (\psi_1, \ldots, \psi_n))$$

Now, if $m > n$, (A1-A2+) implies that

$$\vdash (L_{\alpha_1}\varphi_1 \wedge \cdots \wedge L_{\alpha_m}\varphi_m \wedge M_{\beta_2}\psi_2 \wedge \cdots \wedge M_{\beta_n}\psi_n)$$

$$\leftrightarrow (L_{\alpha_1}\varphi_1 \wedge \cdots \wedge L_{\alpha_m}\varphi_m \wedge M_{\beta_2}\psi_2 \wedge \cdots \wedge M_{\beta_n}\psi_n \wedge \overbrace{M_0\bot \wedge \cdots \wedge M_0\bot}^{m-n \text{ times}})$$

and we can apply (B =) to derive (B). A similar argument takes care of the case $m < n$.

4.1.3. Define (B′ =) to be (B′) with $m = n$. In view of 4.1.2. and a similarly proved equivalence between (B′) and (B′ =), it is enough to show that (B =) and (B′ =) are equivalent. To this effect, we can restate the inference rule (B =) as

$$\text{If}\quad ((\neg\varphi_1, \dots, \neg\varphi_m) \leftrightarrow (\neg\psi_1, \dots, \neg\psi_m))\quad \text{then}$$

$$\left(\left(\bigwedge_{i=1}^{m} L_{1-\alpha_i}\neg\varphi_i\right) \bigwedge \left(\bigwedge_{j=2}^{m} M_{1-\beta_j}\neg\psi_j\right) \to L_{1-(\alpha_1+\cdots+\alpha_m)+(\beta_2+\cdots\beta_m)}\neg\psi_1\right)$$

for $m \geq 1$ and $1-(\alpha_1+\cdots+\alpha_m)+(\beta_2+\cdots\beta_m) \in [0,1]$. Using (Def. M), (B=) can be further restated as

$$\text{If}\quad ((\neg\varphi_1, \dots, \neg\varphi_m) \leftrightarrow (\neg\psi_1, \dots, \neg\psi_m))\quad \text{then}$$

$$\left(\left(\bigwedge_{i=1}^{m} M_{\alpha_i}\varphi_i\right) \bigwedge \left(\bigwedge_{j=2}^{m} L_{\beta_j}\psi_j\right) \to M_{(\alpha_1+\cdots+\alpha_m)-(\beta_2+\cdots\beta_m)}\psi_1\right)$$

for $m \geq 1$ and $(\alpha_1+\cdots+\alpha_m)-(\beta_2+\cdots\beta_m) \in [0,1]$. Contraposing the consequent, interchanging the φ_i and ψ_j, and rewriting the indexes appropriately, (B=) is seen to be equivalent to

$$\text{If}\quad ((\neg\varphi_1, \dots, \neg\varphi_m) \leftrightarrow (\neg\psi_1, \dots, \neg\psi_m))\quad \text{then}$$

$$\left(\neg M_{\alpha_1}\varphi_1 \wedge \left(\bigwedge_{i=2}^{m} L_{\alpha_i}\varphi_i\right) \bigwedge \left(\bigwedge_{j=2}^{m} M_{\beta_j}\psi_j\right) \to \neg M_{(\alpha_1+\cdots+\alpha_m)-(\beta_2+\cdots\beta_m)}\psi_1\right).$$

Hence, the desired equivalence between (B =) and (B′ =) will hold if we prove the following theorem:

$$\vdash ((\varphi_1, \dots, \varphi_m) \leftrightarrow (\psi_1, \dots, \psi_m)) \leftrightarrow ((\neg\varphi_1, \dots, \neg\varphi_m) \\ \leftrightarrow (\neg\psi_1, \dots, \neg\psi_m))$$

This theorem is proved from the following purely propositional arguments. For $1 \leq p \leq m$,

$$\vdash \bigvee_{1\leq \ell_1\leq\cdots\leq \ell_p\leq m} (\varphi_{\ell_1} \wedge \cdots \wedge \varphi_{\ell_p}) \leftrightarrow$$

$$\neg\left(\bigvee_{1\leq k_1\leq\cdots\leq k_{m-p+1}\leq m} (\neg\varphi_{k_1} \wedge \cdots \wedge \neg\varphi_{k_{m-p+1}})\right)$$

or, more concisely,

$$\vdash \varphi^{(p)} \leftrightarrow \neg\left((\neg\varphi)^{(m-p+1)}\right).$$

This is because in the canonical space of 2^m truth valuations for $\varphi_{1,\dots,}\varphi_m$, a valuation belongs to at least p of the sets $[\varphi_i]$ ($[\varphi_i]$ is the set of valuations in which φ_i is true) iff it belongs to at most $m-p$ of the sets $[\neg\varphi_i]$, i.e., it does not belong to at least $m-p+1$ of the sets $[\neg\varphi_i]$. Therefore,

$$\vdash ((\varphi)^{(p)} \leftrightarrow (\psi)^{(p)}) \leftrightarrow ((\neg\varphi)^{(m-p+1)} \leftrightarrow (\neg\psi)^{(m-p+1)}),$$

so that

$$\vdash \bigwedge_{p=1}^{m} (\varphi^{(p)} \leftrightarrow \psi^{(p)}) \leftrightarrow \bigwedge_{p=1}^{m} ((\neg\varphi)^{(m-p+1)} \leftrightarrow (\neg\psi)^{(m-p+1)}),$$

which is the same as

$$\vdash \bigwedge_{p=1}^{m} (\varphi^{(p)} \leftrightarrow \psi^{(p)}) \leftrightarrow \bigwedge_{p=1}^{m} ((\neg\varphi)^{(p)} \leftrightarrow (\neg\psi)^{(p)}),$$

as was required to prove. ■

Proof of Theorem 4.2. We leave it for the reader to prove the soundness part. The only nonobvious step is to check that (B) is valid in every type space. The required argument is sketched in the paragraphs that precede the introduction of (B) in Section 4.

Now we turn to prove completeness, i.e., that for any formula ψ

$$\mathcal{T} \models \psi \quad \Rightarrow \quad \vdash_{\Sigma^+} \psi.$$

Fix the formula ψ, and consider the restricted language $\mathscr{L}[\psi]$ closed under the following conditions: It contains only the propositional variables appearing in ψ; it contains modal operators L^i_α (and the derived operators M^i_α and E^i_α) only for individuals i for which such operators appear in ψ; the indexes α in these operators belong to the finite set $A(\psi)$ of rational numbers in [0,1] of the form p/q, where q is the smallest common denominator of the indexes appearing in ψ; and it contains only formulas of depth smaller than or equal to that of ψ.

The *depth* of a formula φ, denoted by $\mathrm{dp}(\varphi)$, is defined as usual by

- if φ is a propositional variable, then $\mathrm{dp}(\varphi) = 0$;
- $\mathrm{dp}(\neg\varphi) = \mathrm{dp}(\varphi)$;
- $\mathrm{dp}(\varphi_1 \vee \varphi_2) = \max(\mathrm{dp}(\varphi_1), \mathrm{dp}(\varphi_2))$, and similarly for the other binary connectives;
- $\mathrm{dp}(L_\alpha\varphi) = \mathrm{dp}(M_\alpha\varphi) = \mathrm{dp}(E_\alpha\varphi) = \mathrm{dp}(\varphi) + 1$.

The restricted language $\mathscr{L}[\psi]$ gives rise to a set Ω of maximally consistent subsets. Formally, $\Gamma \subseteq \mathscr{L}[\psi]$ is said to be $\mathscr{L}[\psi]$-maximally consistent if it is consistent for $\vdash_{\Sigma^+}$ and no formula of $\mathscr{L}[\psi]$ can be added to Γ without making it inconsistent. Equivalently, Ω is the set of the $\Gamma \cap \mathscr{L}[\psi]$ where Γ ranges over the $\mathscr{L}$-maximally consistent sets (for the definition and properties of maximally consistent sets, see Chellas (1980) or any text in modal logic).

In the sequel, for every $\varphi \in \mathscr{L}[\psi]$, the notation $[\varphi]$ refers to the set $\{\Gamma \in \Omega\colon \varphi \in \Gamma\}$. The following properties hold true of Ω and its subsets:

LEMMA A.1.

A.1.1. *The set Ω is finite.*

A.1.2. *All subsets of Ω can be described by a formula of $\mathscr{L}[\psi]$, that is: for every $E \subseteq \Omega$, there is $\varphi \in \mathscr{L}[\psi]$ such that $E = [\varphi]$.*

A.1.3. *For all $\varphi_1, \varphi_2 \in \mathscr{L}[\psi]$*

$$[\varphi_1] \subseteq [\varphi_2] \quad \textit{iff} \quad \vdash_{\Sigma^+} \varphi_1 \to \varphi_2$$

This lemma is adapted from a similar one in the common knowledge logic (e.g., Lismont and Mongin (1994)), and we will not prove it here. It is typical of the *filtration* device used to reduce a generally infinite language to a sub-language which is essentially finite. The sub-language $\mathscr{L}[\psi]$ has finitely many formulas *modulo* logical equivalence.

From now on we refer to a representative individual $i \in I$, and omit the superscript i in the modal operators. We state now relevant properties of the $\mathscr{L}[\psi]$-maximally consistent subsets. For any fixed $\Gamma \in \Omega$ and $\varphi \in \Gamma$, define

$$\tilde{\alpha} = \max\{\alpha\colon L_\alpha \varphi \in \Gamma\} \quad \text{and} \quad \tilde{\beta} = \min\{\beta\colon M_\beta \varphi \in \Gamma\}.$$

The maximum and the minimum are attained, because the set of indexes $A(\psi)$ in $\mathscr{L}[\psi]$ is finite. Then:

LEMMA A.2.

A.2.1. $\forall \gamma \in A(\psi)$, $\gamma \leq \tilde{\alpha} \Rightarrow L_\gamma \varphi \in \Gamma$ *and* $\gamma \geq \tilde{\beta} \Rightarrow M_\gamma \varphi \in \Gamma$

A.2.2. *There are only two cases—either $\tilde{\alpha} = \tilde{\beta}$ and $E_{\tilde{\alpha}} \varphi \in \Gamma$, while $E_\gamma \varphi \notin \Gamma$ for $\gamma \neq \tilde{\alpha}$, or $\tilde{\alpha} < \tilde{\beta}$, and $E_\gamma \varphi \notin \Gamma$, $\forall \gamma \in A(\psi)$.*

A.2.3. $\tilde{\beta} - \tilde{\alpha} \leq \frac{1}{q}$.

Proof. A.2.1. follows from (A7) and (A7+). To prove (2.2), let us assume that $\tilde{\alpha} > \tilde{\beta}$. Then, from A.2.1., $L_{\tilde{\beta}} \varphi \in \Gamma$ and $M_{\tilde{\alpha}} \varphi \in \Gamma$, and from (Def. E), $E_{\tilde{\alpha}} \varphi \in \Gamma$ and $E_{\tilde{\beta}} \varphi \in \Gamma$, contradicting (A12). Thus, $\tilde{\alpha} \leq \tilde{\beta}$. Now, if $\tilde{\alpha} = \tilde{\beta}$, we have that $E_{\tilde{\alpha}} \varphi \in \Gamma$ and from (A12) again, $E_\gamma \varphi \notin \Gamma$ for $\gamma \neq \tilde{\alpha}$. In the case where $\tilde{\alpha} < \tilde{\beta}$, the definition of $\tilde{\alpha}$ and $\tilde{\beta}$ implies that $E_\gamma \varphi \in \Gamma$ for no $\gamma \in A(\psi)$. To prove A.2.3., suppose that $\widetilde{\beta} - \widetilde{\alpha} > 1/q$. This would imply that there is $\alpha^* \in A(\psi) \cap (\widetilde{\alpha}, \widetilde{\beta})$. But then, $\neg L_{\alpha^*}\varphi \in \Gamma$ and $\neg M_{\alpha^*}\varphi \in \Gamma$, contradicting (A8). ■

Given $\Gamma \in \Omega$ and $\varphi \in \mathscr{L}[\psi]$ we define $\mathscr{I}_{\psi}^{\Gamma}$ to be either $\{\widetilde{\alpha}\}$ if $\widetilde{\alpha} = \widetilde{\beta}$ or the open interval $(\widetilde{\alpha}, \widetilde{\beta})$ if $\widetilde{\alpha} < \widetilde{\beta}$. A crucial step in the completeness proof will consist of defining a probability measure $T_i(\Gamma)$ on the subsets of Ω with the property that

$$\forall \varphi \in \mathscr{L}[\psi], \quad T_i(\Gamma)([\varphi]) \in \mathscr{I}_{\psi}^{\Gamma} \tag{P}$$

Suppose that this step has been achieved. Then, we introduce the type space

$$\tau = \langle \Omega, 2^{\Omega}, (T_i)_{i \in I}, v \rangle$$

with v defined by $v(\Gamma, p) = 1$ iff $p \in \Gamma$. The next lemma states (using the terminology of modal logic) that τ is a canonical model.

LEMMA A.3. *Assume that for each* $\Gamma \in \Omega$, $T_i(\Gamma)$ *satisfies* (*P*). *Then, if* $L_\alpha \varphi \in \mathscr{L}[\psi]$,

$$\tau, \Gamma \models L_\alpha \varphi \quad \textit{iff} \quad L_\alpha \varphi \in \Gamma,$$

and similarly for $M_\alpha \varphi$ *and* $E_\alpha \varphi$.

Proof. From left to right: $\tau, \Gamma \models L_\alpha \varphi$ implies that $T_i(\Gamma)([\varphi]) \geq \alpha$ (from the definition of a type space semantics). Hence $\widetilde{\alpha} \geq \alpha$ (because (P) and Lemma A.2.3. imply that either $\widetilde{\alpha} = T_i(\Gamma)([\varphi]) \geq \alpha$ or $\widetilde{\alpha} + (1/q) > T_i(\Gamma)([\varphi]) > \widetilde{\alpha}$, in which case we also have $\widetilde{\alpha} \geq \alpha$ since $\widetilde{\alpha}, \alpha \in A(\psi)$), and therefore $L_\alpha \varphi \in \Gamma$.

The implication from left to right is routine. The equivalences for $M_\alpha \varphi$ and $E_\alpha \varphi$ follow from the equivalence for $L_\alpha \varphi$ and (Def. M) and (Def. E). ■

Granting the conclusion of Lemma A.3., the desired property that

$$\mathscr{T} \models \psi \quad \Rightarrow \quad \vdash_{\Sigma^+} \psi$$

follows from the standard argument for completeness, as slightly modified for the filtration device. (If $\mathscr{T} \models \psi$, then in particular $\tau \models \psi$, which, using Lemma A.3., implies that $\psi \in \Gamma$ for all $\Gamma \in \Omega$, and the conclusion that ψ is a theorem follows from Lemma A.1.3.)

Thus, the only remaining step is to show that there are indeed probability measures $T_i(\Gamma)$ satisfying (P). This will be proved from the following version of the theorem of the alternative (see Rockafellar (1970, Theorem 22.6) or Rockafellar (1969, Theorem 3). The theorem is stated there for the real number field, but, as mentioned in Rockafellar (1969, end of Section 2), it holds for any ordered field, and in particular for the field $\mathbb{Q}$ of rational numbers.). The strength of the theorem comes from the fact that the term "interval" in it refers to any kind of interval—open, closed, closed on one end and open on the other, half a line or a single point. A linear combination of intervals is the interval defined by

$$a_1\mathcal{I}_1 + \cdots + a_k\mathcal{I}_k = \{a_1x_1 + \cdots + a_kx_k : \ x_1 \in \mathcal{I}_1, \ldots, x_k \in \mathcal{I}_k\}$$

THEOREM A.4 (Rockafellar). *Let L be a subspace of $\mathbb{Q}^N$, and let $\mathcal{I}_1, \ldots, \mathcal{I}_N$ be intervals in $\mathbb{Q}$. Then one and only one of the following alternatives holds*:

($*$) *There exists a vector $z = (\zeta_1, \ldots, \zeta_N) \in L$ such that*

$$\zeta_1 \in \mathcal{I}_1, \ldots, \zeta_N \in \mathcal{I}_N$$

($**$) *There exists a vector $\bar{z} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_N) \in L^{\perp}$[7] such that*

$$\bar{\zeta}_1\mathcal{I}_1 + \cdots + \bar{\zeta}_N\mathcal{I}_N > 0.$$

LEMMA A.5 (Main Lemma). *For each $\Gamma \in \Omega$ there is a probability measure $T_i(\Gamma)$ on 2^{Ω} satisfying condition* (P).

Proof. Let $\Gamma_1, \ldots, \Gamma_\ell$ be the points of Ω. Denote $N = |2^{\Omega}|$. Due to Lemma A.2.2., N is also the number of sets of the form $[\varphi]$ ranging over $\mathcal{L}[\psi]$. Select an order on these sets and a representative for each of them, so that they can be written as $[\varphi_1], \ldots, [\varphi_N]$. Since, by Lemma A.1.3., $\Omega = [\top]$, let $\varphi_N = \top$. Consider the incidence matrix M of size $\ell \times N$, where $M(j,k) = 1$ if $\Gamma_j \in [\varphi_k]$, and $M(j,k) = 0$ otherwise. The rows of this matrix span the ℓ-dimensional subspace L of $\mathbb{Q}^N$ of rational-valued signed measures on 2^{Ω}. Fixing $\Gamma \in \Omega$ and an individual $i \in I$, we consider the intervals $\mathcal{I}_1 = \mathcal{I}^{\Gamma}_{\varphi_1}, \ldots, \mathcal{I}_N = \mathcal{I}^{\Gamma}_{\varphi_N}$, and set out to apply Rockafellar's theorem. Since, by (A1–A2+), $\vdash E_1\top$ and we chose $\varphi_N = \top$, we know that $\mathcal{I}_N = \{1\}$; furthermore, by (A1) we know that all the intervals $\mathcal{I}_1, \ldots, \mathcal{I}_N$ are contained in the interval $[0, 1]$. Therefore, alternative ($*$) in Rockafellar's theorem states that there is a probability measure $T_i(\Gamma)$ on 2^{Ω} which satisfies condition (P).

The other alternative ($**$) says that there exists $\bar{z} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_N) \in \mathbb{Q}^N$ such that $M\bar{z} = 0$ and $\bar{\zeta}_1\mathcal{I}_1 + \cdots + \bar{\zeta}_N\mathcal{I}_N > 0$. Multiplying $\bar{z}$ by the least common denominator of its entries, we get a vector of integers with the same properties, so without loss of generality we can assume that $\bar{z}$ is a vector of integers. Let $\bar{z}^+$ be the positive part of $\bar{z}$ (with zeroes in place of the negative entries of $\bar{z}$), and $\bar{z}^-$ the negative part of $\bar{z}$ (i.e. the positive

[7] That is, $\bar{z}z = \bar{\zeta}_1\zeta_1 + \cdots + \bar{\zeta}_N\zeta_N = 0$ for every $z = (\zeta_1, \ldots, \zeta_N) \in L$.

part of $-\bar{z}$. $\bar{z}^-$ is thus a non-negative vector). Then alternative $(**)$ now reads (where $\mathbf{1}_{[\varphi_k]}$ is the characteristic function of $[\varphi_k]$):

$$\sum_{k=1}^{N} \bar{z}_k^+ \mathbf{1}_{[\varphi_k]} = \sum_{k=1}^{N} \bar{z}_k^- \mathbf{1}_{[\varphi_k]} \quad \text{and} \quad \sum_{k=1}^{N} \bar{z}_k^+ \mathcal{I}_{\varphi_k}^{\Gamma} > \sum_{k=1}^{N} \bar{z}_k^- \mathcal{I}_{\varphi_k}^{\Gamma}$$

or equivalently: There exist $\varphi_1, \ldots, \varphi_m$ and $\psi_1, \ldots, \psi_n$, possibly with repetitions among the formulas, such that

$$\sum_{k=1}^{m} \mathbf{1}_{[\varphi_k]} = \sum_{j=1}^{n} \mathbf{1}_{[\psi_j]} \quad \text{and} \quad \sum_{k=1}^{m} \mathcal{I}_{\varphi_k}^{\Gamma} > \sum_{j=1}^{n} \mathcal{I}_{\psi_j}^{\Gamma} \qquad (**')$$

which, because of Lemma A.1.3, amounts to

$$(\varphi_1, \ldots, \varphi_m) \leftrightarrow (\psi_1, \ldots, \psi_n) \quad \text{and} \quad \sum_{k=1}^{m} \mathcal{I}_{\varphi_k}^{\Gamma} > \sum_{j=1}^{n} \mathcal{I}_{\psi_j}^{\Gamma}. \qquad (**'')$$

Now, if $\mathcal{I}_{\varphi_k}^{\Gamma}$ $(\mathcal{I}_{\psi_j}^{\Gamma})$ is a singleton denote it by $\{\underline{\alpha}_k\}$ [resp. $\{\bar{\beta}_j\}$], and if it is an open interval denote it by $(\underline{\alpha}_k, \bar{\alpha}_k)$ [resp. $(\underline{\beta}_j, \bar{\beta}_j)$] (from Lemma A.2.2 we know that these are the only possibilities). To show that $(**'')$ does not hold, we distinguish three cases.

Case 1. All of the $\mathcal{I}_{\varphi_k}^{\Gamma}$ and $\mathcal{I}_{\psi_j}^{\Gamma}$ are singletons. Then, $\sum_{k=1}^{m} \mathcal{I}_{\varphi_k}^{\Gamma}$ is the singleton $\{\sum_{k=1}^{m} \underline{\alpha}_k\}$ and $\sum_{j=1}^{n} \mathcal{I}_{\psi_j}^{\Gamma}$ is the singleton $\{\sum_{j=1}^{n} \bar{\beta}_j\}$. Γ contains

$$L_{\underline{\alpha}_1} \varphi_1, \ldots, L_{\underline{\alpha}_m} \varphi_m \quad \text{and} \quad M_{\bar{\beta}_1} \psi_1, \ldots, M_{\bar{\beta}_n} \psi_n.$$

From inference rule (B) we conclude that Γ contains also $L_{(\underline{\alpha}_1 + \cdots + \underline{\alpha}_m) - (\bar{\beta}_2 + \cdots \bar{\beta}_n)} \psi_1$. Hence, necessarily that

$$(\underline{\alpha}_1 + \cdots + \underline{\alpha}_m) - (\bar{\beta}_2 + \cdots \bar{\beta}_n) \leq \bar{\beta}_1.$$

Since alternative $(**'')$ implies that

$$\underline{\alpha}_1 + \cdots + \underline{\alpha}_m > \bar{\beta}_1 + \cdots \bar{\beta}_n$$

it cannot hold in this case.

Case 2. At least one of the intervals $\mathcal{I}_{\varphi_k}^{\Gamma}$ is an open interval. Without loss of generality let this be $\mathcal{I}_{\varphi_1}^{\Gamma} = (\underline{\alpha}_1, \bar{\alpha}_1)$. Γ contains

$$\neg M_{\underline{\alpha}_1} \varphi_1, L_{\underline{\alpha}_2} \varphi_2, \ldots, L_{\underline{\alpha}_m} \varphi_m \quad \text{and} \quad M_{\bar{\beta}_1} \psi_1, \ldots, M_{\bar{\beta}_n} \psi_n$$

(notice the use of Lemma A.2 to conclude that $\neg M_{\underline{\alpha}_1} \varphi_1 \in \Gamma$). From inference rule (B′)

$$\neg M_{(\underline{\alpha}_1 + \cdots + \underline{\alpha}_m) - (\bar{\beta}_2 + \cdots \bar{\beta}_n)} \psi_1 \in \Gamma$$

Hence necessarily

$$(\underline{\alpha}_1 + \cdots + \underline{\alpha}_m) - (\bar{\beta}_2 + \cdots \bar{\beta}_n) < \bar{\beta}_1.$$

But alternative $(**'')$ now implies that

$$\underline{\alpha}_1 + \cdots + \underline{\alpha}_m \geq \bar{\beta}_1 + \cdots \bar{\beta}_n$$

so it cannot hold in this case either.

Case 3. At least one of the intervals $\mathcal{I}^{\Gamma}_{\psi_j}$ is an open interval. Without loss of generality let this be $\mathcal{I}^{\Gamma}_{\psi_1} = (\underline{\beta}_1, \bar{\beta}_1)$. Γ contains

$$L_{\underline{\alpha}_1}\varphi_1, \ldots, L_{\underline{\alpha}_m}\varphi_m \quad \text{and} \quad \neg L_{\bar{\beta}_1}\psi_1, M_{\bar{\beta}_2}\psi_2, \ldots, M_{\bar{\beta}_n}\psi_n$$

(notice the use of Lemma A.2 to conclude that $\neg L_{\bar{\beta}_1}\psi_1 \in \Gamma$). From inference rule (B) Γ must contain $L_{(\underline{\alpha}_1+\cdots+\underline{\alpha}_m)-(\bar{\beta}_2+\cdots\bar{\beta}_n)}\psi_1$, whence necessarily

$$(\underline{\alpha}_1 + \cdots + \underline{\alpha}_m) - (\bar{\beta}_2 + \cdots \bar{\beta}_n) < \bar{\beta}_1.$$

But alternative $(**'')$ now implies that

$$\underline{\alpha}_1 + \cdots + \underline{\alpha}_m \geq \bar{\beta}_1 + \cdots \bar{\beta}_n,$$

a contradiction in this case as well.

This concludes the demonstration that $(**'')$ cannot hold, hence from Rockafellar's theorem that alternative $(*)$ holds, which asserts the existence of a probability $T_i(\Gamma)$ with property (P), as required. ∎

Proof of Propositions 5.1 *and* 5.2 *(sketch).* The proof of these propositions involves a slight variation of the proof of Theorem 4.2. For (4) and (5), the sub-language $\mathcal{L}[\psi]$ will have to include also formulas with depth that exceed by 1 the depth of ψ, to ensure that $\mathcal{L}[\psi]$ contain formulas that express introspection regarding formulas with the same depth as that of ψ. For (4') and (5'), one has further to take care to choose $T_i[\Gamma] = T_i[\Gamma']$ if Γ and Γ' contain exactly the same formulas of the form $L_\alpha\varphi$, thus ensuring that Ω becomes a Harsanyi type space.

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