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Modal Logic

M Cresswell, The University of Auckland, Auckland, New Zealand

In classical propositional logic, all the operators are truth-functional. That is to say, the truth or falsity of a complex formula depends only on the truth or falsity of its simpler propositional constituents. Take the monadic operator $\sim$. This is interpreted to mean 'it is not the case that', and its meaning is defined by Table 1, called a truth table, in which 1 represents 'true' and 0 represents 'false'. This operator is truth-functional because the truth value of the complex proposition it forms is determined by the truth value of the proposition it starts with. If α has the value 1, then $\sim\alpha$ has 0. If α is 0, $\sim\alpha$ is 1. (I use standard operators for propositional logic: $\sim$ for negation, $\vee$ for (inclusive) disjunction, $\wedge$ for conjunction, $\supset$ for (material) implication and $\equiv$ for (material) equivalence.) Modal logic is concerned with understanding propositions about what **must** or about what **might** be the case, and it is not difficult to see how we might have two propositions alike in truth value, both true, say, where one is true and could not possibly be false and the other is true but might easily have been false. For instance it must be that $2+2=4$, but, although it is true that I am writing this entry, it might easily not have been. Modal logic extends the well-formed formulae (wff) of classical logic by the addition of a one-place sentential operator L (or $\Box$) interpreted as meaning 'it is necessary that'. Using this operator, an operator M (or $\Diamond$) meaning 'it is possible that' may be defined as $\sim L\sim$. The notation L and M for the necessity and possibility operators dates from Feys (1950) (for L) and Becker (1930) (for M). (For a history of notation, see Hughes and Cresswell, 1968: 347–349). The use of $\Box$ for the necessity operator is due to F. B. Fitch and first appears in Barcan (1946); the use of $\Diamond$ for M dates from Lewis and Langford (1932). In fact, any one of L or M can be taken as primitive and the other defined in terms of it. Another operator that may be defined in terms of L and M is the operator Q (or ∇), where $Q\alpha$ means that α is contingent, that is, neither necessary nor impossible. $Q\alpha$ may be defined as $\sim L\alpha \wedge M\alpha$. Many uses of *possible* in natural language suggest that what is possibly true is also possibly false, and those uses are better served by Q than by M.

The modern development of modal logic dates from 1912 when C. I. Lewis published the first of a series of articles and books (culminating in Lewis and Langford, 1932) in which he expressed dissatisfaction with the notion of material implication found in Whitehead and Russell's (1910) *Principia mathematica*. In the system of *Principia mathematica* – indeed in any standard system of propositional calculus (PC) – there are found the theorems:

(1) $p \supset (q \supset p)$

(2) $\sim p \supset (p \supset q)$

The sense of (1) is often expressed by saying that if a proposition is true, any proposition whatsoever implies it; the sense of (2) is often expressed by saying that if a proposition is false, it implies any proposition whatsoever. Lewis did not wish to reject these. He argued that they are "neither mysterious sayings, nor great discoveries, nor gross absurdities" (Lewis, 1912: 522), but merely reflect the truth-functional sense in which Whitehead and Russell were using the word *imply*. But he also maintained that there is a sense of *imply* in which when we say that *p implies q* we mean that '*q* follows from *p*' and that in this sense of *imply* it is not the case that every true proposition is implied by any proposition whatsoever or that every

Table 1 Truth table

$\sim\alpha$
01
10

false proposition implies any proposition whatsoever. A brief account of the early history of modal logic is found in Hughes and Cresswell (1968: pt. 3, with Lewis's contributions documented in Chaps. 12–13).

Until the early 1960s, modal logics were discussed almost exclusively as axiomatic systems without access to a notion of validity of the kind used, for example, in the truth-table method for determining the validity of wff of the classical propositional calculus. The semantical breakthrough came by using the idea that a necessary proposition is one that is true in all possible worlds. But whether another world counts as possible may be held to be relative to the point we are at. So an interpretation for a modal system would consist of a set W of possible worlds and a relation R of accessibility between them – where the worlds accessible from a given world are often described as the worlds that world can see. For any wff α and world w, $L\alpha$ will be true at w iff α itself is true at every w' such that $w\mathrm{R}w'$. Put formally, a frame is a structure $\langle \mathrm{W,R} \rangle$ in which W can be any class at all, although it members are often called 'worlds' or 'indices,' and R is a relation between them. A model can then be based on that frame by adding a value assignment V. V assigns to each variable p a truth value in each world. If we read $\mathrm{V}(p,w) = 1$ (0) as 'V assigns p the value true (false) in world w', then we may, for instance, define the truth table for $\sim$ so that $\mathrm{V}({\sim}\alpha,w) = 1$ if $\mathrm{V}(\alpha,w) = 0$, and 0 otherwise, and we may define the disjunction operator $\vee$ so that $\mathrm{V}(\alpha \vee \beta,w) = 1$ iff either $\mathrm{V}(\alpha,w) = 1$ or $\mathrm{V}(\beta,w) = 1$. $\mathrm{V}(L\alpha,w) = 1$ iff $\mathrm{V}(\alpha,w') = 1$ for every w' such that $w\mathrm{R}w'$. We define validity on a frame by saying that a wff α is valid on a frame $\langle \mathrm{W,R} \rangle$ iff, for every model $\langle \mathrm{W,R,V} \rangle$ based on $\langle \mathrm{W,R} \rangle$ and for every $w \in \mathrm{W}$, $\mathrm{V}(\alpha,w) = 1$. The ideas that underlie this account of validity appeared in the late 1950s and early 1960s in the works of Kanger (1957), Bayart (1958), Kripke (1959, 1963), Montague (1960), and Hintikka (1961). Anticipations can be found in Wajsberg (1933), McKinsey (1945), Carnap (1946, 1947), Meredith (1956), Thomas (1962), and other works. An algebraic description of this notion of validity is found in Jónsson and Tarski (1951), although the connection with modal logic was not made in that article.

An axiomatic basis for a logical system consists of a selected set of wff, known as axioms, together with a set of transformation rules, licensing various operations on the axioms and on wff obtained from the axioms by previous applications of the transformation rules. The wff obtained from the axioms in this way, together with the axioms themselves, are known as the theorems of the system. The system of modal logic whose theorems are precisely the wff valid on every frame is known as the system K. This name, which has now become standard, was given to the system in Lemmon and Scott (1977) in honor of Saul Kripke, from whose work the way of defining validity for modal logic is mainly derived. The word 'frame' in this sense seems to have been first used in print in Segerberg (1968), but the word was suggested to him by Dana Scott. The axioms of K consist of all valid wff of PC together with the modal wff **K**:

$$L(p \supset q) \supset (Lp \supset Lq)$$

and it has the following three transformation rules:

- US (The Rule of Uniform Substitution): The result of uniformly replacing any variable or variables $p_1, \ldots, p_n$ in a theorem by any wff $\beta_1, \ldots, \beta_n$, respectively, is itself a theorem.
- MP (The Rule of Modus Ponens, sometimes also called the Rule of Detachment): If α and $\alpha \supset \beta$ are theorems, so is β.
- N (The Rule of Necessitation): If α is a theorem, so is $L\alpha$.

Other modal systems may be obtained by adding extra axioms to K. Each of these will be a proper extension of K (i.e., it will contain not only all the theorems of K but other theorems as well). Modal systems that contain K (including K, itself) together with US, MP, and N are commonly known as normal modal systems.

A wff valid on every frame is called K-valid, and the theorems of the system K are precisely those wff that are K-valid. It is important to be clear that this is a substantive fact and not something that is true by definition. To be a theorem of K is to be derivable from the axioms of K by the transformation rules of K; to be K-valid is to be valid on every frame, and the fact that a wff is a theorem of K iff it is K-valid is something we have to **prove**, not something we can assume. Similar remarks hold for extensions of K because different systems of modal logic can represent different ways of restricting necessity. It can then happen that whether a principle of modal logic holds or not can depend on the properties of the accessibility relation. We may have an axiomatic modal system defined without any reference to an account of validity and a definition of validity formulated without any reference to theoremhood in a system, and yet the theorems of that system are precisely the wff that are valid by that definition. To show that there is a match of this kind between a system and a validity definition, we have to prove two things: (1) that every theorem of the system is valid by that definition and (2) that every wff valid by that definition is a theorem of the system. If the first holds, we say that the system is sound, and if the second holds we say that it is complete, in each case with respect to the validity-definition in question.

The wff $Lp \supset p$ is not K-valid because it is not valid on a frame in which there is a world that is not accessible from itself. (Put p false in this world, and true everywhere else.) We could, however, add it as an extra axiom to obtain a system stronger than K itself. The system obtained by adding $Lp \supset p$ as a single extra axiom to K is usually referred to as T, and it has had a long history in modal logic dating from Feys (1937). Feys's own name for the system is 't' (it was first called 'T' by Sobociński, 1953). Feys derived the system by dropping one of the axioms in a system devised by Gödel (1933), with whom the idea of axiomatizing modal logic by adding to PC originates. Sobociński showed that T is equivalent to the system M of von Wright (1951); for this reason 'M' was occasionally used as an alternative name for T. T is $K + \mathbf{T}$.

T: $Lp \supset p$

Although **T** is not valid on every frame it **is** valid on all frames in which R is reflexive, that is, frames in which every world is accessible from itself – where $w\mathrm{R}w$ for every $w \in \mathrm{W}$. It can be proved that the system T is sound and complete with respect to such frames. In any system containing T, $L\alpha$ is equivalent to $\alpha \wedge \sim Q\alpha$, and so necessity may be defined in terms of contingency. Without T, this cannot usually be done. An examination of when necessity can be defined in terms of contingency in systems that do not contain T may be found in Cresswell (1988).

If we interpret L as expressing obligatoriness (moral necessity), we are unlikely to want to regard $Lp \supset p$ as valid because it will then mean that whatever ought to be the case is in fact the case. There is, however, a formula that, like $Lp \supset p$, is not a theorem of K but that, under the moral interpretation, it **is** plausible to regard as valid, and that is the wff $Lp \supset Mp$. If Lp means that it is obligatory that p, then Mp will mean that it is permissible that p (not obligatory that not-p), and so $Lp \supset Mp$ will mean that whatever is obligatory is at least permissible. This interpretation of L is known as a deontic interpretation, and for that reason $Lp \supset Mp$ is often called **D**, and the system obtained by adding it to K as an extra axiom is known as the system D; that is, D is defined as $K + \mathbf{D}$.

D: $Lp \supset Mp$

D is sound and complete with respect to the class of frames in which R is serial – for every $w \in \mathrm{W}$, there is some w', perhaps w itself, as in T, but perhaps not, such that $w\mathrm{R}w'$.

In the early days of modal logic, disputes centered around the question of whether a given principle of modal logic was correct or not. Often these disputes involved formulae in which one modal operator occurs within the scope of another – formulae such as $Lp \supset LLp$. Sequences such as LL are known as iterated modalities, and it is tempting to consider the possibility that all iterated modalities might be equivalent to uniterated ones or to a small number of iterated ones. Consider the following:

(3) $Mp \equiv LMp$

(4) $Lp \equiv MLp$

(5) $Mp \equiv MMp$

(6) $Lp \equiv LLp$

None of these is a theorem even of T. Some of them can be derived from others. We can obtain (5) and (6) by adding $Lp \supset LLp$ to T; this system is known as the system S4. We can obtain all of (3)–(6) by adding $Mp \supset LMp$ to T; this system is known as the system S5. The names 'S4' and 'S5' derive from Lewis and Langford (1932: 501), in which systems deductively equivalent to these are the fourth and fifth in a series of modal systems. These systems too can be studied semantically. Suppose that R is required to be transitive; that is, suppose that, for any worlds w_1, w_2, and w_3, if $w_1\mathrm{R}w_2$ and $w_2\mathrm{R}w_3$ then $w_1\mathrm{R}w_3$. If so, then $Lp \supset LLp$ will be valid, but if nontransitive frames are permitted it need not be. S4 is sound and complete with respect to all reflexive and transitive frames. Suppose that in addition R is symmetrical, that is, if $w\mathrm{R}w'$ then $w'\mathrm{R}w$. S5 is sound and complete with respect to all reflexive, transitive, and symmetrical frames.

Not all normal modal logics are so well-behaved, however. There exist systems that can be proved to be incomplete in the sense that there is **no** class of frames such that their theorems are precisely the wff that are valid on every frame in the class. One simple example is the system $\mathrm{KH} = \mathrm{K} + \mathbf{H}$.

H: $L(Lp \equiv p) \supset Lp$

Every frame for H also validates $Lp \supset LLp$, but $Lp \supset LLp$ is not a theorem of KH. The incompleteness of this system is proved in Boolos and Sambin (1985). A simplified proof appears in Hughes and Cresswell (1996: Chap. 9). Much current research in propositional modal logic is devoted to the systematic study of large families of logics to examine the conditions under which they have, or lack, properties such as these.

Modal logic can be given a temporal interpretation. The propositional logic of linear time in which L means 'it is and always will be that ...' is called S4.3; it is S4 together with **D1**.

D1: $L(Lp \supset q) \vee L(Lq \supset p)$

The logic of discrete time, in which each moment has a unique successor, S4.3.1, is S4.3 + **N1**.

N1: $L(L(p \supset Lp) \supset p) \supset (MLp \supset Lp)$

This logic has a venerable history. In 1957, Arthur Prior conjectured that S4 was the logic of time with this structure, but later authors proved it to be S4.3.1. Prior's name for this was D (for Diodorus Cronos, not to be confused with D for the deontic system). The history is told in Prior (1967: Chap. 2). Another logic with a philosophical and mathematical interest is one variously called KW or G. It is K together with **W**.

W: $L(Lp \supset p) \supset Lp$

One of its interpretations is that it expresses the logic of provability – with L meaning, 'it is provable that'. From a purely modal point of view, it is sound and complete with respect to the class of all finite frames in which R is transitive and asymmetrical. The name W is from Segerberg (1971: 84); it is called G in Boolos (1979). For a more recent survey of the history of provability logic, see Boolos and Sambin (1990). The system dates at least from Löb (1966).

An important modal notion is that of entailment. By this we understand the converse of the relation of following logically from (when this is understood as a relation between propositions, not wff); that is, to say that a proposition, p, entails a proposition, q, is simply an alternative way of saying that q follows logically from p or that the inference from p to q is logically valid. In modal logic, we can say that p entails q iff $L(p \supset q)$. But then we are faced with the following valid wff.

(7) $L((p \wedge \sim p) \supset q)$

(8) $L((q \supset (p \vee \sim p)))$

With this interpretation, (7) means that any proposition of the form $(p \wedge \sim p)$ entails any proposition whatever, and (8) means that any proposition whatever entails any proposition of the form $(p \vee \sim p)$. Although this may seem strange, those who wish to reject either (7) or (8) have to face the following argument. The following principles seem intuitively to be valid:

1. Any conjunction entails each of its conjuncts.
2. Any proposition, p, entails $(p \vee q)$, no matter what q may be.
3. The premises $(p \vee q)$ and $\sim p$ together entail the conclusion q (the principle of the disjunctive syllogism).
4. Whenever p entails q and q entails r, then p entails r (the principle of the transitivity of entailment).

Table 2 Derivation of propostions using the four principles

Principle	Proposition
	(i) $p \wedge \sim p$
From (i), by A	(ii) p
From (ii), by B	(iii) $p \vee q$
From (i), by A	(iv) $\sim p$
From (iii) and (iv), by C	(v) q
So by **D**, $(p \wedge \sim p)$ entails q	

Lewis showed long ago (Lewis and Langford, 1932: 250–251) that by using these principles we can always derive any arbitrary proposition, q, from any proposition of the form $(p \wedge \sim p)$, as shown in Table 2. This derivation shows that the price that has to be paid for denying that $(p \wedge \sim p)$ entails q is the abandonment of at least one of the principles 1–4. The most fully developed formal response to these paradoxes consists of abandoning principle 3, the principle of disjunctive syllogism. Logics that do this are called relevance logics. A survey of relevance logic is found in Dunn (1986) and Mares and Meyer (2001).

First-order predicate logic can also be extended by the addition of modal operators. (For more details of modal predicate logic see Hughes and Cresswell, 1996: pt. 3; Garson, 1984.) The most interesting consequences of such extensions are those that affect mixed principles, principles that relate quantifiers and modal operators and that cannot be stated at the level of modal propositional logic or nonmodal predicate logic. Thus, (9) is valid, but (10) is not.

(9) $\exists xL\alpha \supset L\exists x\alpha$

(10) $L\exists x\alpha \supset \exists xL\alpha$

(Even if a game must have a winner there need be no one who must win.) In some cases the principles of the extended system will depend on the propositional logic on which it is based. An example is a formula studied by Ruth Barcan Marcus, who first considered combining modal logic with first-order predicate logic and introduced the formula in Barcan (1946). The Barcan formula (BF) is:

$\forall xL\alpha \supset L\forall x\alpha$

which is provable in some modal systems but not in others. If both directions are assumed, so that we have $\forall xL\alpha \equiv L\forall x\alpha$, then this formula expresses the principle that the domain of individuals is held constant over all possible worlds. A temporal version of the Barcan formula might make its interpretation clearer. Let us assume that everyone now in existence will die before 2154. Then, if L means 'it will always be the case that' and α is the wff 'x dies before 2154', then $\forall xL\alpha$ is true. But it is not at all likely that it will always be the case that everyone will die before 2154 because people who do not now exist may well live beyond then, and so $L\forall x\alpha$ will be false. This assumes that a quantifier only ranges over the individuals that exist at the time or in the world at which the wff is being evaluated, but indicates why BF has been a matter of controversy.

It is possible to have complete predicate logics whose predicate extensions are incomplete. One such is S4.2, which is S4 + **G1**.

G1: $MLp \supset LMp$

S4.2 is characterized by frames that are reflexive, transitive, and convergent, in the sense that, if a world w_1 can see two worlds w_2 and w_3 then there is a world w_4 that both w_2 and w_3 can see. But the predicate extension of S4.2 + BF is not characterized by any class of frames. Perhaps more interesting is the case of the systems S4.3.1 and KW (or G) previously mentioned, the logics of discrete time and of provability. It is established in Cresswell (1997) that the predicate logic characterized by all frames for these systems is not recursively axiomatizable because they, together with a whole family of systems containing N1 and $Lp \supset LLp$, when combined with first-order logic, enable the expression of second-order arithmetic.

When identity is added, even more questions arise. The usual axioms for identity easily allow the derivation of LI:

$(x = y) \supset L(x = y)$

but should we really say that all identities are necessary? Take:

(11) the composer of 'Threnody for Mrs S' = the composer of 'Salm'

An important feature of (11) is that it uses the phrases *the composer of 'Threnody for Mrs S'* and *the composer of 'Salm'*. Such phrases are often called definite descriptions, and they pose problems even in nonmodal predicate logic. One of the first to see this was Bertrand Russell, whose celebrated example was

(12) the present king of France is bald

Because there is no present king of France, it would seem that (12) is false. But then it would seem that (13) is true.

(13) the present king of France is not bald

But (13) does not seem true either, for the same reason as (12). Even worse is sentence (14).

(14) the present king of France does not exist

Russell claimed that the phrase *the present king of France* does not function like a name at all, and his account of the matter can help with (11). If we follow Russell, (11) makes five claims:

1. At least one person composed 'Threnody for Mrs S.'
2. At most one person composed 'Threnody for Mrs S.'
3. At least one person composed 'Salm.'
4. At most one person composed 'Salm.'
5. $\forall x \forall y$((x composed 'Threnody for Mrs S' $\wedge$ y composed 'Salm') $\supset x = y$).

Look carefully at claim 5. This claim is true but not necessarily true, so putting L in front of it gives us a false sentence. But LI does not license us to put L in front of claim 5. What LI does allow is for us to move from claim 5 to claim 6.

6. $\forall x \forall y$((x composed 'Threnody for Mrs S' $\wedge$ y composed 'Salm') $\supset L(x = y)$).

And it is less clear that claim 6 is at all objectionable.

Suppose, nevertheless, that we still wish to abandon LI. To falsify LI, we let the values of the variables be strings of objects that may coincide in some worlds but not in others. In the present example, letting x mean 'The composer of "Threnody for Mrs S"' would mean requiring its value in a world w be whoever in w composed 'Threnody for Mrs S,' while y could stand for whoever composed 'Salm' in w. But there is a problem. Allowing all such strings would make the wff (10) valid, and it was remarked that (10) could be false in a game in which it is necessary that some player will win, although there is no individual player who is bound to win. There is, however, one way in which we could make (10) sound plausible. Consider, for example, the expression *the governor-general of New Zealand*, as it may occur in a constitutional context. The law may specify that at a certain point the signature of the governor-general is required before an act of parliament becomes law; yet on one occasion the governor-general may be Michael Hardie-Boys and on another it may be Sylvia Cartwright. Thus, the phrase *the governor-general of New Zealand* does not designate any particular individual (as an assemblage of flesh and blood); yet we can in a sense think of it as standing for a single object, contrasted with *the prime minister* and so forth.

Such objects are often called intensional objects or individual concepts. In a logic in which the individual variables range over all intensional objects, (10) would be valid because, if it must be the case that there is someone who must sign acts into law, then although no individual person must be that someone, yet there is someone (viz. the governor-general) whose signature is required. An adequate semantics for the contingent identity systems that do not validate (10) would therefore have to place restrictions on allowable strings of objects and neither require that only strings consisting of the same member of D in each world should count as objects—for that would validate LI – nor that any string whatever of members of D should count as an object – for that would validate (10).

All the systems of modal logic considered so far have assumed a single necessity operator, with its possibility dual. An important class of systems with more than one is the class of tense logics. (For a survey see Prior, 1967.) A tense logic has two operators, L_1 and L_2, where L_1 means 'it always will be the case that' and L_2 means 'it always has been the case that'. In frames for a tense logic, the associated relations R_1 and R_2 are so related that one is the converse of the other, that is, wR_1w' iff $w'R_2w$. Another way of introducing families of modal operators is suggested by a possible interpretation of modal logic in computer science. In this interpretation, the worlds are states in the running of a program. If π is a computer program, then $[\pi]\alpha$ means that, after program π has been run, α will be true. If w is any world, then $wR_\pi w'$ means that state w' results from the running of program π. This interpretation of modal logic is called dynamic logic. What gives dynamic logic its interest is the possibility of combining simple programs to get more complex ones. Thus, if π_1 and π_2 are two programs, then the expression $\pi_1;\pi_2$ refers to the program 'first do π_1 and then do π_2', and $[\pi_1;\pi_2]\alpha$ means that α will be true if this is done. The relation corresponding to $[\pi_1;\pi_2]$ may be defined to hold between w and w' iff $\exists u(wR_{\pi 1}u \wedge uR_{\pi 2}w')$. Other computing operations can generate similar modal operators, with appropriate conditions on their accessibility relations (see Goldblatt, 1987: Chap. 10). It is also possible to develop dynamic predicate logic. For an introductory survey, see Goldblatt (1987: pt. 3).

The most general kind of possible-worlds semantics for propositional operators is based on that idea of the truth set of a formula. In any model, we can define $|\alpha|$ as $\{w \in W: V(\alpha,w) = 1\}$. In evaluating $L\alpha$ in a world w, all the input that we require is to know which set of worlds forms the truth set of α. Whatever L means, what it has to do is to declare $L\alpha$ true at w for some truth sets and false for others. So the meaning of L must specify which sets of worlds form acceptable truth sets in world w. These sets of worlds are called the neighborhoods of w, and a neighborhood frame for a language of modal propositional logic is a pair $\langle W,R\rangle$ in which W is a set (of worlds) and R is a neighborhood relation. (For some remarks on the history of neighborhood semantics, see Segerberg, 1971: 72.) A neighborhood relation is a relation between a world w and a subset A of W, and A is a neighborhood of w iff wRA. We then say that $V(L\alpha,w) = 1$ iff $wR|\alpha|A$. A frame of the kind assumed earlier, in which R is a relation between worlds, is often called a relational frame, and it is not difficult to see that every relational frame is a special case of a neighborhood frame. To be precise, a relational frame is a neighborhood frame in which for every $w \in W$ there is a set B of those and only those worlds that are accessible to w (i.e., B is the set of worlds w can 'see') and wRA iff $B \subseteq A$. What this means is that α's truth set is a neighborhood of w iff it contains all the worlds accessible from W, which is of course precisely what the truth of $L\alpha$ in a relational frame amounts to.

Neighborhood semantics can be devised for systems with operators taking more than one argument. A philosophically important example here is the logic of counterfactuals as developed in the late 1960s and early 1970s. We present here a version of the semantics found in Lewis (1973), but the same idea is also found in Stalnaker (1968) and Åqvist (1973). Counterfactual logic is based on a dyadic operator $\Box\!\!\rightarrow$, where $\alpha \Box\!\!\rightarrow \beta$ means that 'if α were to be the case then β would be the case'. Lewis's idea is that, given a possible world w, some worlds are closer to w than others. If we write $w' <_w w''$ to mean that 'w' is closer to w than w'' is', then the semantics for $\Box\!\!\rightarrow$ will be that $\alpha \Box\!\!\rightarrow \beta$ is true in w iff either α is not true at any world or there is a world w' at which α and β are both true that is closer to w than any world w'' at which α is true but β is not. A counterfactual frame can be described as a neighborhood frame in the following way. Because $\Box\!\!\rightarrow$ is dyadic, its neighborhood relation R will relate worlds to pairs $\langle A,B\rangle$ where $A \subseteq W$ and $B \subseteq W$. The standard rule for a dyadic operator δ using an associated neighborhood relation R is (15).

(15) $V(\delta\alpha\beta,w) = 1$ iff $wR\langle|\alpha|, |\beta|\rangle$

A frame $\langle W,R\rangle$ will be a counterfactual frame iff it is based on a nearness relation $<$ in such a way that $wR\langle A,B\rangle$ iff either (16) or (17).

(16) $A = \varnothing$

(17) There is some w' such that $w' \in A \cap B$ and for every w'', if $w'' \in A \cap -B$ then $w' <_w w''$

Which counterfactual logic we get will depend on what kind of conditions we put on $<$. For instance, under the plausible assumption that the closest world to w is w itself, we get a frame that validates the wff in (18).

(18) $p \supset ((p \mathrel{\Box\!\!\rightarrow} q) \equiv q)$

By contrast, on any plausible account of nearness, many wff that are valid for $\supset$ fail for $\Box\!\!\rightarrow$. For instance in standard systems of counterfactual logic, neither (19) nor (20) is valid.

(19) $((p \mathrel{\Box\!\!\rightarrow} q) \wedge (q \mathrel{\Box\!\!\rightarrow} r)) \supset (p \mathrel{\Box\!\!\rightarrow} r)$

(20) $(p \mathrel{\Box\!\!\rightarrow} q) \supset (\sim q \mathrel{\Box\!\!\rightarrow} \sim p)$

Questions such as these bring us to the boundary between modal logic, metaphysics, and semantics and remind us of the rich potential that the theory of possible worlds has for illuminating such issues. The possible-worlds semantics can be generalized to deal with any operators whose meanings are operations on propositions as sets of possible worlds and form a congenial tool for those who think that the meaning of a sentence is its truth conditions and that these be taken literally as a set of possible worlds – the worlds in which the sentence is true.

Further information on modal logic, including more detailed bibliographical references, may be found in a number of works. See, for instance, Blackburn *et al.* (2001), Bull and Segerberg (1984), Chargrov and Zaharyaschev (1997), Chellas (1980), and Hughes and Cresswell (1996).

See also: Boole, George (1815–1864); Carnap, Rudolf (1891–1970); Conditionals; Fictional Discourse: Philosophical Aspects; Hintikka, Jaakko (b. 1929); Indexicality: Philosophical Aspects; Inference: Abduction, Induction, Deduction; *De Dicto* versus *De Re*; Kripke, Saul (b. 1940); Mood and Modality in Grammar; Negation; Propositional Attitudes; Quantifiers: Semantics; Semantic Value; Tense and Time: Philosophical Aspects.

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Modality Issues in Signed and Spoken Language

R P Meier, University of Texas at Austin, TX, USA

The modality of a language is the means by which that language is produced and perceived. Linguists long believed that human language was bound to a particular modality; language was necessarily spoken and heard (e.g., Hockett, 1960). Reading and writing were not overlooked, but these visual representations are largely derivative of a primary spoken language. Research since 1960 has shown that there are two modalities in which naturally evolved human languages are expressed: not only the oral–aural modality of spoken languages, but also the visual–gestural modality of signed languages. There may even be a third possibility: specifically the tactile–gestural modality that is used in deaf–blind signing (see Quinto-Pozos, 2002). However, we know of no independent languages that have emerged within the tactile–gestural modality.

Signed and spoken languages are encoded using different articulators, and decoded using different sensory-perceptual systems. Beyond such obvious facts as the irrelevance of voicing to signed languages and of handshape to spoken languages, there are interesting differences between the two language modalities; see **Table 1** and **Table 2** (adapted from Meier, 2002b). For example, in speech, the oral articulators are largely hidden from view; thus the failure of lip reading as a means of understanding speech. Although hearing addressees are not blind to the visual configuration of the oral articulators (as demonstrated by the McGurk effect), addressees must perceive the acoustic consequences of oral articulation if they are to understand the speaker. The relationship between auditory percept and articulator movement is exceedingly complex, a fact that

Table 1 Some properties of the articulators in sign and speech

Sign	*Speech*
Light source external to signer	Sound source internal to speaker
Manual articulation not coupled (or loosely coupled) to respiration	Oral articulation tightly coupled to respiration
Manual articulators move in a transparent space	Oral articulators largely hidden
Manual articulators relatively massive	Oral articulators relatively small
Manual articulators paired	Oral articulators not paired
No predominant oscillator associated with sign syllables	Mandible is predominant oscillator associated with speech syllables