Hierarchical Bayesian analysis using Stan - From a binary logit to advanced learning models

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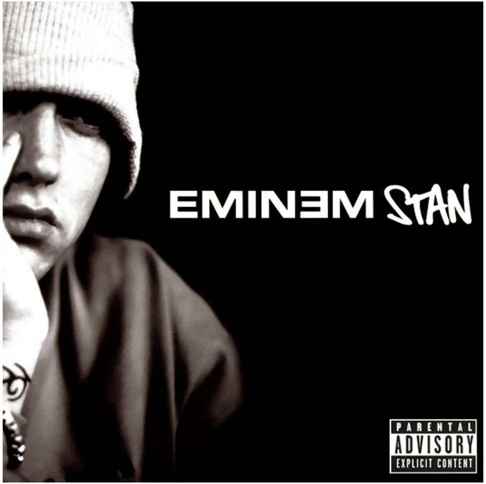
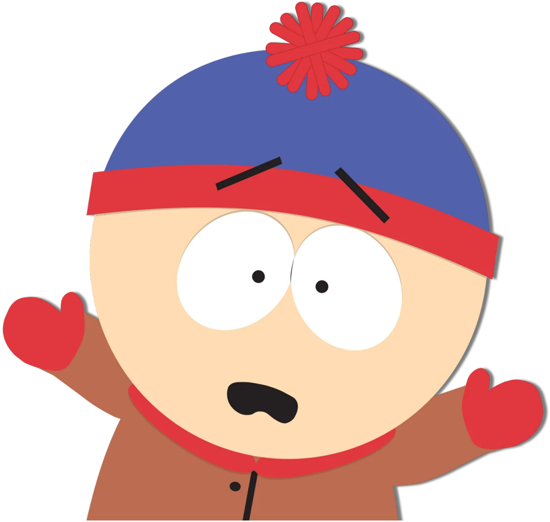
Stan



Stan



Stan

Probabilistic programming language

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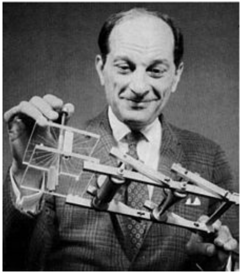

Stanislaw Ulam
(1909–1984)
Inventor of the Monte Carlo method

Bayes' theorem — quick recap

Bayes' theorem:

$$p(\theta|y,X) \propto p(y|\theta,X) \times p(\theta)$$

where:

- $p(\theta|y,X)$: posterior distribution;
- $p(y|\theta,X)$: model's likelihood;
- $p(\theta)$: prior distribution.

Bayesian workflow

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- Exploratory data analysis

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- Write out a model (full probability model)

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- Model fitting and algorithm diagnostics
- *Posterior* predictive checking
- Model comparison (e.g., via cross-validation)

Stan language

Stan program (with R, Python, Matlab, Julia, Stata, CmdStan interface)

- declares data and (constrained) parameter variables
- defines log posterior (or penalized likelihood)

Stan inference

- Hamiltonian Monte Carlo for full Bayesian estimation
- Variation inference for approximate Bayes
- Optimization for (penalized) Maximum Likelihood

Stan implementation

A Stan model is comprised of *code blocks*:

- *data:* declares the data.
- *transformed data:* makes transformations of the data, including any restrictions on their values.
- *parameters:* declares the parameters.
- *transformed parameters:* makes transformations or restrictions on the parameters.
- *model:* define the full *probability* model here.
- *generated quantities:* outputs from the model (posterior predictions, forecasts).

```
EWA_stan="
data {
......
}

parameters {
.....
}

transformed parameters {
....
}

model {
.....
}

generated quantities{
......
}
"
```

Hierarchical binary logit example: The data

A choice model of buying decisions given price and promotions:

- 3 product attributes (including intercept), indexed by j
- 500 consumers, indexed by i
- 10 purchase occasions each, indexed by t

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	User_ID	Observation	Y	Intercept	Price	Promotion
1	1	1	1	1	0.4242656054	0.429378508
2	1	2	0	1	0.8425126909	0.274047020
3	1	3	1	1	0.3764421751	0.512474872
4	1	4	1	1	0.9115300649	0.355274114
5	1	5	1	1	0.2585383623	0.127846915
6	1	6	1	1	0.2032627692	0.294366778
7	1	7	1	1	0.2246137869	0.161435480
8	1	8	1	1	0.9146362843	0.506660538
9	1	9	1	1	0.8902309975	0.377715006
10	1	10	1	1	0.2795407828	0.185361522
11	2	1	0	1	0.5262798222	0.636969226
12	2	2	0	1	0.9985261350	0.641892214
13	2	3	0	1	0.9664521548	0.662417121
14	2	4	0	1	0.4240157611	0.409965415
15	2	5	0	1	0.2575837581	0.383950177
16	2	6	1	1	0.9943112358	0.431254078

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Declare data in *stan*

```
"data {
int<lower=1> nvar; // number of parameters in the logit regression
int<lower=0> N; // number of observations
int<lower=1> nind; // number of individuals
int<lower=0,upper=1> y[N];
int<lower=1,upper=nind> ind[N]; // indicator for individuals
row_vector[nvar] x[N];
}
```

Hierarchical binary logit example: The model

A choice model of buying decisions given price and promotions:

- Buy/ Not buy a product with j product attributes;
- i consumers with t purchase occasions each.

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$$y_{ijt} \sim \mathcal{B}(p_{ijt})$$

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$$
\begin{aligned}
y_{ijt} &\sim \mathcal{B}(p_{ijt}) \\
logit(p_{ijt}) &\sim \mathcal{N}(x_{ijt}\beta_{ij}) \\
\beta_i &\sim MultiNormal(z_i\delta, \Sigma)
\end{aligned}
$$

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\delta &\sim \mathcal{N}(\delta_0, \sigma)
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\end{aligned}
$$

Non-conjugate prior

$$
\begin{aligned}
\Sigma &= diag(\tau)\,\Omega\,diag(\tau) \\
\tau &\sim \Gamma(a, b) \\
\Omega &\sim LKJcorr(\nu)
\end{aligned}
$$

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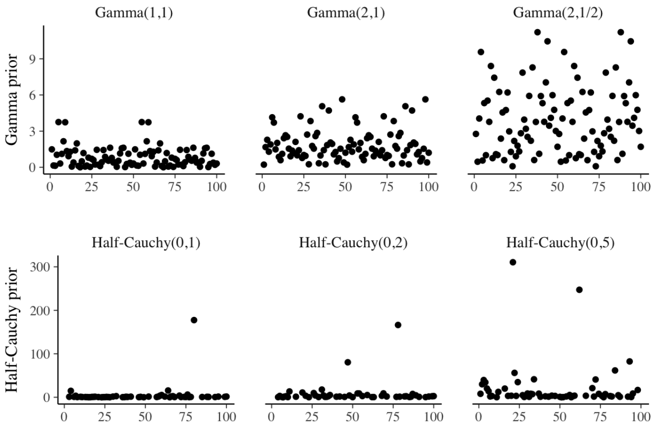
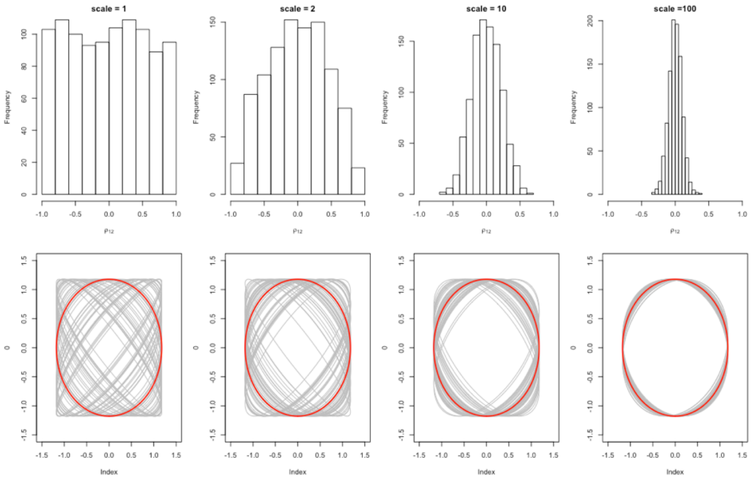
```
hierarchical_binlogit_fullcov="data {
....
}

parameters {
vector[nvar] delta;
vector<lower=0>[nvar] tau;
vector[nvar] beta[nind];
corr_matrix[nvar] Omega; // Vbeta - prior correlation
}

model {
to_vector(delta) ~ normal(0, 5);
to_vector(tau) ~ gamma(2, 0.5);
Omega ~ lkj_corr(2);

for (h in 1:nind)
beta[h]~multi_normal(delta, quad_form_diag(Omega, tau));

for (n in 1:N)
y[n] ~ bernoulli_logit(x[n] * beta[ind[n]]);
}

generated quantities {
corr_matrix[nvar] Omega_corr;
int z[N];
real log_lik[N];

Omega_corr=Omega;
for (n in 1:N){
z[n] = bernoulli_logit_rng(x[n] * beta[ind[n]]);
log_lik[n]= bernoulli_logit_lpmf(y[n]|x[n] * beta[ind[n]]);
}

}
"
```

Choice of prior distribution of the variance components



Choice of the LKJ prior distribution



Checking model fit

- Lack of mixing.
- Stationarity.
- Autocorrelation.
- Divergent transitions.

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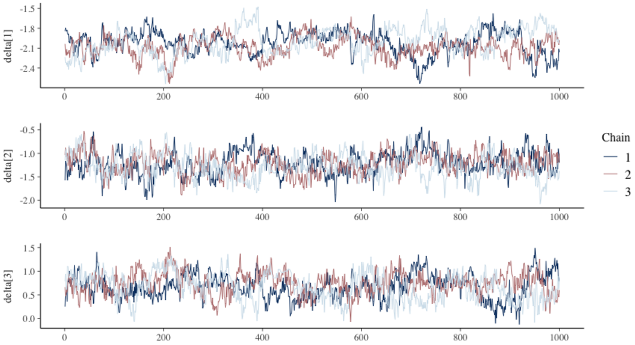

Figure 1: Traceplot of δ parameters (after burnin), using package *bayesplot*

Summary statistics

Effective sample size and convergence properties

	mean	se_mean	sd	2.5%	97.5%	n_eff	Rhat
delta[1]	-2.0226636	0.01805745	0.1661046	-2.3656187	-1.7061194	84.61558	1.034883
delta[2]	-1.2509968	0.02373062	0.2545043	-1.7638505	-0.7585503	115.01969	1.019059
delta[3]	0.6711088	0.01967228	0.2450191	0.1780753	1.1415407	155.12809	1.019018

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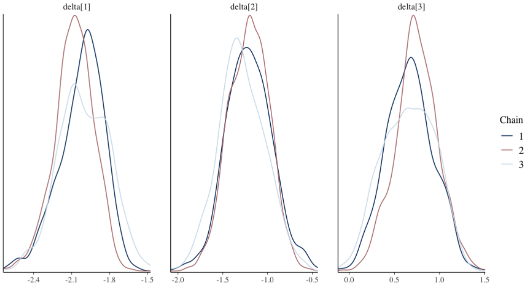


Figure 2: Density plot of δ parameters (after burnin), using package *bayesplot*

Noncentered (Re)Parameterization - the "Matt Trick"

Consider a model with a diagonal variance-covariance matrix

- Assume our intercept model: $\beta_i \sim \mathcal{N}(\delta, \tau)$
- We can decompose that into: $\mathcal{N}(\delta, \sigma) = \delta + \tau\mathcal{N}(0, 1)$
- The trick applies to other distributions in the location-scale family
- The transformation:
    1. declare α_i in the parameters block and β_i in the transformed parameters block
    2. draw $\alpha_i \sim \mathcal{N}(0, 1)$ & $\tau \sim \Gamma(a, b)$
    3. compute $\beta_i = \delta + \tau\alpha_i$

Noncentered (Re)Parameterization - the multivariate case

- Assume our full model: $\beta_{\mathbf{i}} \sim MultiNormal(\delta, \Sigma)$
  - If Σ_{jj} is small, then β_{ij} needs to fall into a small range, the sampler needs a small step size
  - If Σ_{jj} is large, then β_{ij} can fall into a wide range, the sampler needs a large step size or lots of small steps
- The transformation:
  1. declare α_i in the parameters block and β_i in the transformed parameters block.
  2. draw $\alpha_i \sim \mathcal{N}(0,1)$ & $\tau \sim \Gamma(a,b)$.
  3. compute $\beta_{\mathbf{i}} = \delta + \tau \mathbf{L} \alpha_{\mathbf{i}} \sim \mathcal{N}(\delta, \tau^2 \mathbf{L}\mathbf{L}^T)$,
  4. where $\tau \mathbf{L}$ is the Cholesky factor of $\Sigma = \tau^2 \mathbf{L}\mathbf{L}^T$, and τ is the standard deviation of the errors.

Noncentered (Re)Parameterization - the implementation

$$y_{ijt} \sim \mathcal{B}(p_{ijt})$$

$$logit(p_{ijt}) \sim \mathcal{N}(x_{ijt}\beta_{ij})$$

$$\beta_{\mathbf{i}} = \delta + \tau \mathbf{L} \alpha_{\mathbf{i}}$$

$$\alpha_i \sim \mathcal{N}(0, 1)$$

$$\delta \sim \mathcal{N}(\delta_0, \sigma)$$

Non-conjugate prior

$$\tau \sim \Gamma(a, b)$$

$$\Omega \sim LKJcorr(\nu)$$

```
hierarchical_binlogit_fullcov_noncentered="data {
.....
}

parameters {
matrix[nvar, nind] alpha; // nvar*H parameter matrix
row_vector[nvar] delta;
vector<lower=0>[nvar] tau;
cholesky_factor_corr[nvar] L_Omega;
}


transformed parameters{
row_vector[nvar] beta[nind];
matrix[nind,nvar] Vbeta_reparametrized;
Vbeta_reparametrized = (diag_pre_multiply(tau, L_Omega)*alpha)';

for (h in 1:nind)
beta[h]=delta+Vbeta_reparametrized[h];
}

model {
L_Omega~lkj_corr_cholesky(2);
to_vector(delta) ~ normal(0, 5);
to_vector(tau) ~ gamma(2, 0.5);
to_vector(alpha)~ normal(0,1);

for (n in 1:N)
y[n] ~ bernoulli_logit(beta[ind[n]]*x[n]);
}

generated quantities {
corr_matrix[nvar] Omega;
int z[N];
real log_lik[N];

Omega=L_Omega*L_Omega';

for (n in 1:N){
z[n] = bernoulli_logit_rng(beta[ind[n]]*x[n]);
log_lik[n]= bernoulli_logit_lpmf(y[n]|beta[ind[n]]*x[n]);
}

}
"
```

Traceplots of model parameters

The noncentered reparametrization helps tremendously

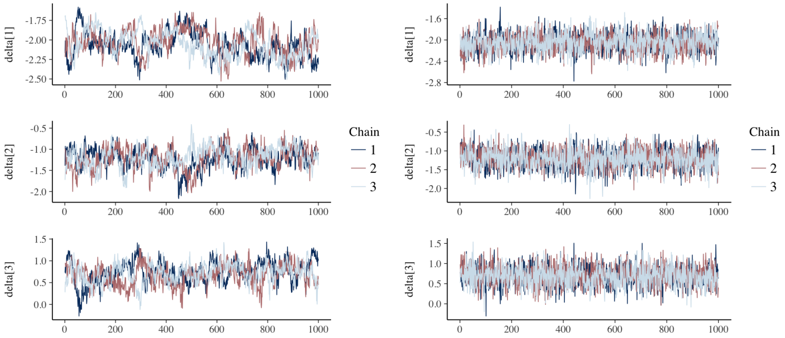
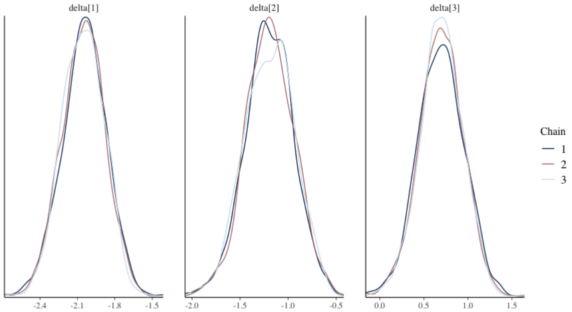


Figure 3: Traceplot of δ parameters (after burnin), estimated via the centered vs. noncentered parametrization

Summary statistics

Effective sample size and convergence properties

```
$summary
                mean     se_mean        sd       2.5%      97.5%    n_eff      Rhat
delta[1] -2.0508997 0.005138191 0.1701534 -2.3861202 -1.7288254 1096.632 1.0009678
delta[2] -1.1951041 0.006205318 0.2535605 -1.7052461 -0.7068838 1669.688 0.9998616
delta[3]  0.6966904 0.006432588 0.2470562  0.2011148  1.1660746 1475.095 1.0001598
```



Individual level parameters

Most parameters are within the 95% highest density intervals

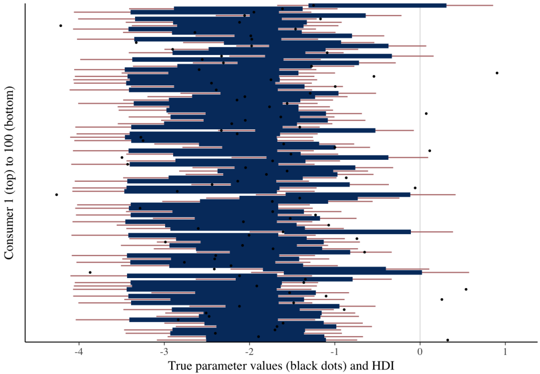


Figure 5: True values (black dots) and the 80% and 95% highest density intervals for the intercept, for the first 100 consumers

Model checking

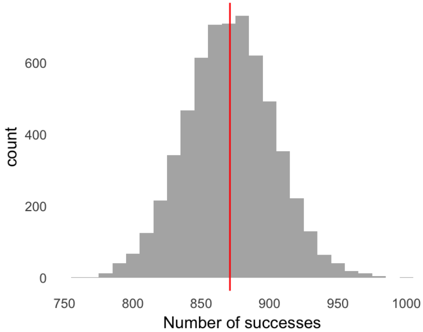
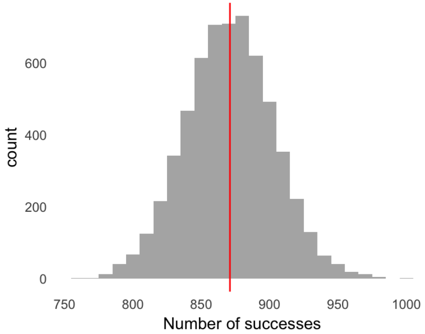


Figure 6: Number of successes: posterior replications vs. true value

Model checking



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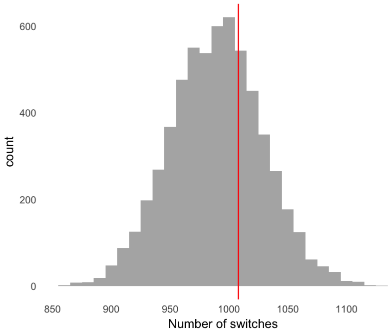


Figure 7: Switches between buying/ not buying: posterior replications vs. true value

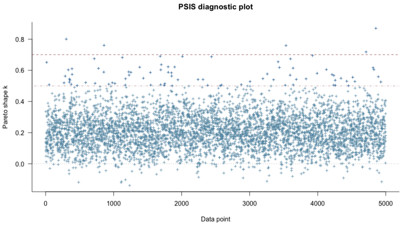
Compute hit rates and MSEs based on posterior replications

Model comparison

Likelihood-based measures: *Leave-one-out cross-validation*

Table 1: Model comparison based on LOO-CV, using package *loo*

	Variance model		*Full covariance model*		*NCP model*	
	Estimate	SE	Estimate	SE	Estimate	SE
elpd_loo	-1874.3	43.1	-1871.9	43.2	-1871.2	43
p_loo	398.6	12.5	364.4	11.8	363.7	11.8
looic	3748.6	86.2	3743.7	86.4	3742.4	86.5



Stan resources

Stan ecosystem

- lang, math library (C++)
- interfaces and tools (R, Python, many more)
- documentation (example model repo, user guide & reference manual, case studies, R package vignettes)
- online community (Stan Forums on Discourse)

Libraries implementing Stan

- rstanarm: complex hierarchical models
- hBayesDM: behavioral decision making models (stan codes on GitHub)
- bayesplot: data visualization

More Stan resources

StanCon 2018 talks: Link

Books:

- Bayesian Data Analysis: aka the Bible:)
  Link
- Bayesian Statistics using Stan
  Link
- Statistical Rethinking
  Link

github.com/alinafere/eQMW_stan_tutorial