Biomedical Imaging Exercise – Week 2

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1. Plane Waves

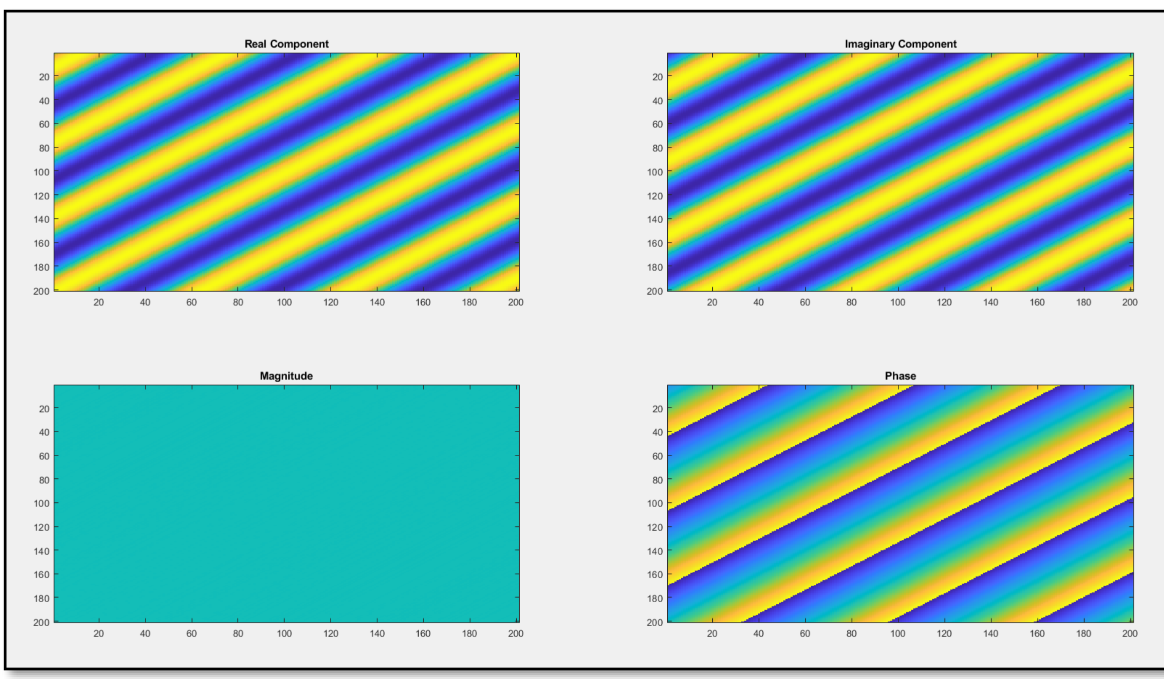
a) The following function is written in MATLAB to calculate the 2D plane waves (planewaves.mlx). The function takes as inputs the values of k_x and k_y and returns the 2D plane wave f(x,y).

This function calculates 2D plane waves for any given kx and ky.

```
function [fxy] = planewaves(kx,ky)
x = -10:0.1:10;
[X,Y] = meshgrid(x);
exponent = 1i*((X.*kx)+(Y.*ky));
fxy = exp(exponent);

figure;
subplot(2,2,1), imagesc(real(fxy)), axis tight, title('Real Component');
subplot(2,2,2), imagesc(imag(fxy)), axis tight, title('Imaginary Component');
subplot(2,2,3), imagesc(abs(fxy)), axis tight, title('Magnitude');
subplot(2,2,4), imagesc(angle(fxy)), axis tight, title('Phase');

end
```

b) For a selection of k_x and k_y such as $k_x = 1$ and $k_y = 1$, the function plots the graphs of the real component, the imaginary component, the magnitude and the phase.

```
>> [fxy] = planewaves(1,1)
```



c) The direction of the wave is determined by the vector $\vec{k}$, which has magnitude $\sqrt{{k_x}^2 + {k_y}^2}$ and angle $\tan^{-1}(\frac{k_y}{k_x})$.

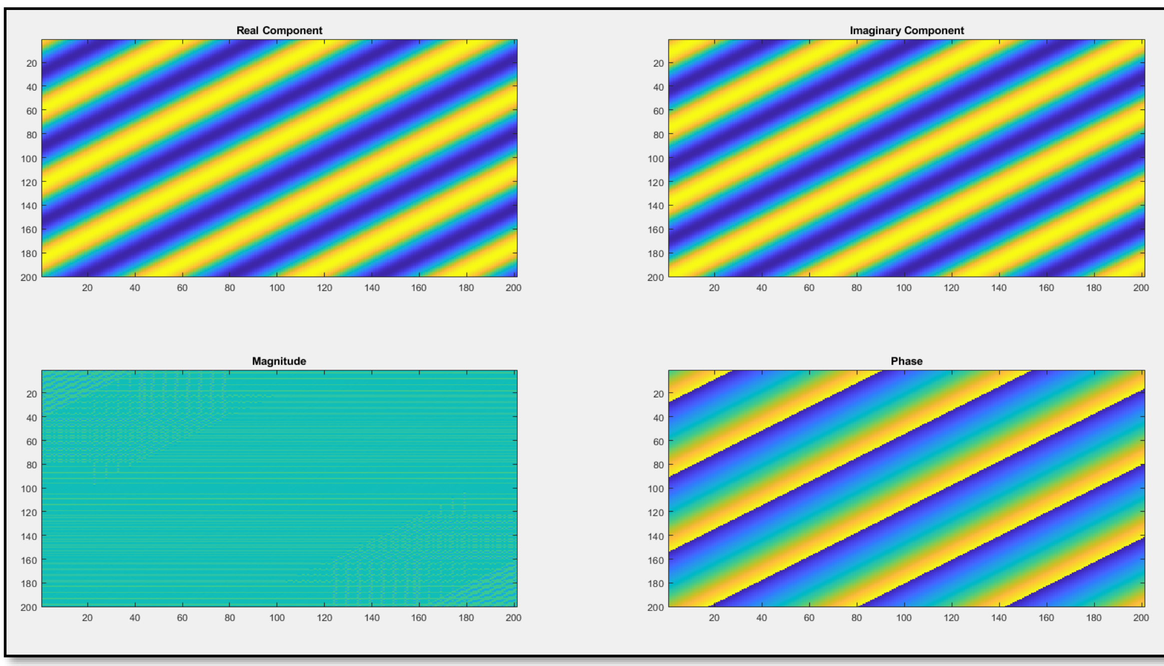
d) The wavelength of the wave is inversely proportional to the values of k_x and k_y and it depends on the combination of their values so we could say that:

$$\lambda \propto \frac{1}{\sqrt{{k_x}^2 + {k_y}^2}}$$

e) A linear, shift-invariant operation that can be applied is differentiation. If we take the example of a plane wave f(x,y) with $k_x = 1$ and $k_y = 1$ as in b), we can differentiate that by using diff(fxy) on MATLAB. The resulting graphs of the real component, the imaginary component, the magnitude and the phase are shown below:

```
>> fxy_diff = diff(fxy)
```

```
>> figure;
subplot(2,2,1), imagesc(real(fxy_diff)), axis tight, title('Real Component');
subplot(2,2,2), imagesc(imag(fxy_diff)), axis tight, title('Imaginary Component');
subplot(2,2,3), imagesc(abs(fxy_diff)), axis tight, title('Magnitude');
subplot(2,2,4), imagesc(angle(fxy_diff)), axis tight, title('Phase');
```



f) As we can see from the plots in b) and e), the resulting waves are the same. That is because the formula we are using to calculate the 2D plane wave contains complex exponentials and as it is known the eigenfunctions of linear, shift-invariant (LSI) systems are complex exponentials. Therefore, the system is linear and shift-invariant and that is why when applying an LSI operation such as differentiation, there is no difference in the resulting plots.

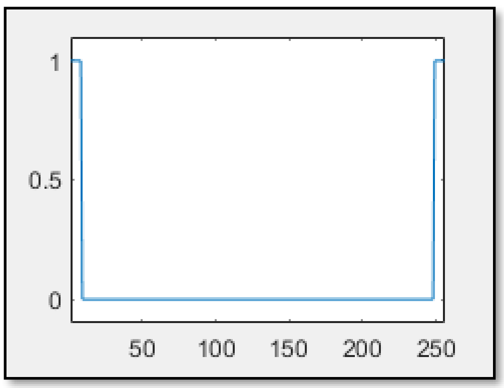
2. Fast Fourier Transform (FFT)

a) When calculating the Fourier Transform for a continuous signal, this would be an integral calculated from $-\infty$ to $+\infty$. Therefore, if the rectangle was centered around zero, the result would be a continuous and symmetric spectrum around zero. However in the case of the FFT algorithm in MATLAB, the Fourier Transform is calculated with discrete values and it is a sum starting from n=0 up to n=N-1, where N is the number of points of the transform. Therefore, the signal is represented in a discrete space from 0 to N-1 samples. As a result, the symmetry of continuous space is lost and the result is two halves of the Fourier Transform which are not however correctly centered due to the misplacement of the origin in the original domain.

b) Using fftshift, the shift of the origin occurs in the frequency domain and not in the original domain. Therefore, because the origin in the original domain is not positioned at the center of symmetry of the rectangle, there is still a phase apparent, which can be noticed from the oscillation in the imaginary part of the FFT.

c) The shift of the origin now occurs on the original domain, by shifting the origin at the center of symmetry of the rectangle. This occurs by multiplying the FFT with e^{-ikx_0}. Therefore, this shift prevents any phase from being apparent which is justified by the imaginary component of the FFT being zero.

d) This does not work fully because the fftshift on the rectangle at the original domain will shift the left and the right halves of the rectangle resulting in the following result:



Then taking the FFT of this signal and then performing fftshift in the frequency domain will not completely solve the problem, because we do not perform the shift of the origin in the original domain but in the frequency domain. Also, the phase correction is not applied in the original domain. As a result, there is still a phase due to the misplacement of the origin in the original domain as seen by observing the oscillation in the imaginary part of the FFT.

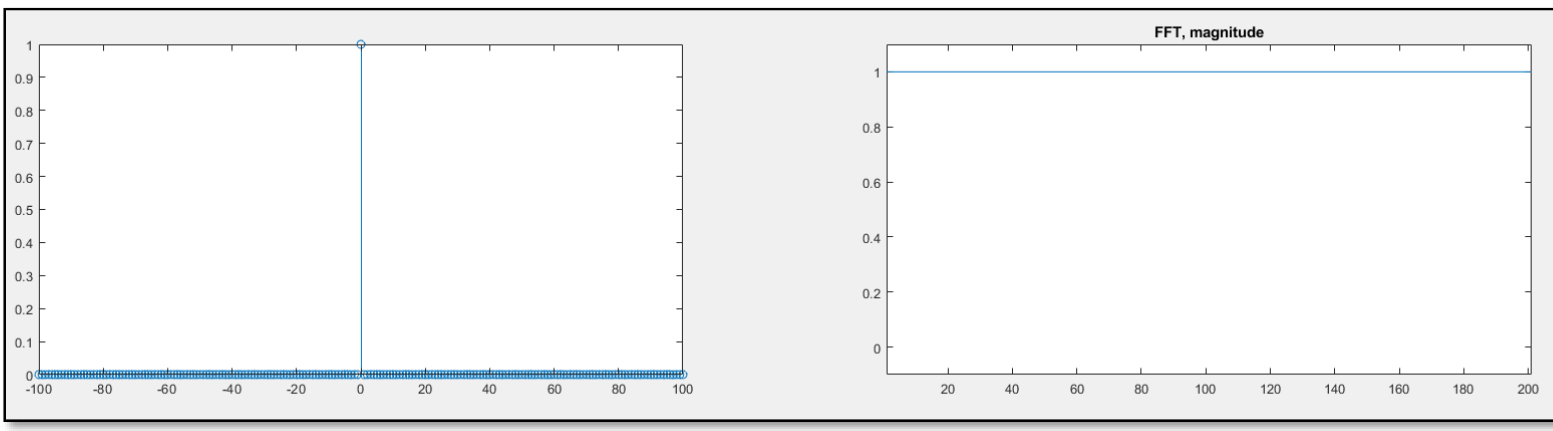
3. Building a Comb

a) A function is created on MATLAB which enables the user to specify the number of impulses and the spacing Δx between the impulses (comb_function.mlx). It is defined as:

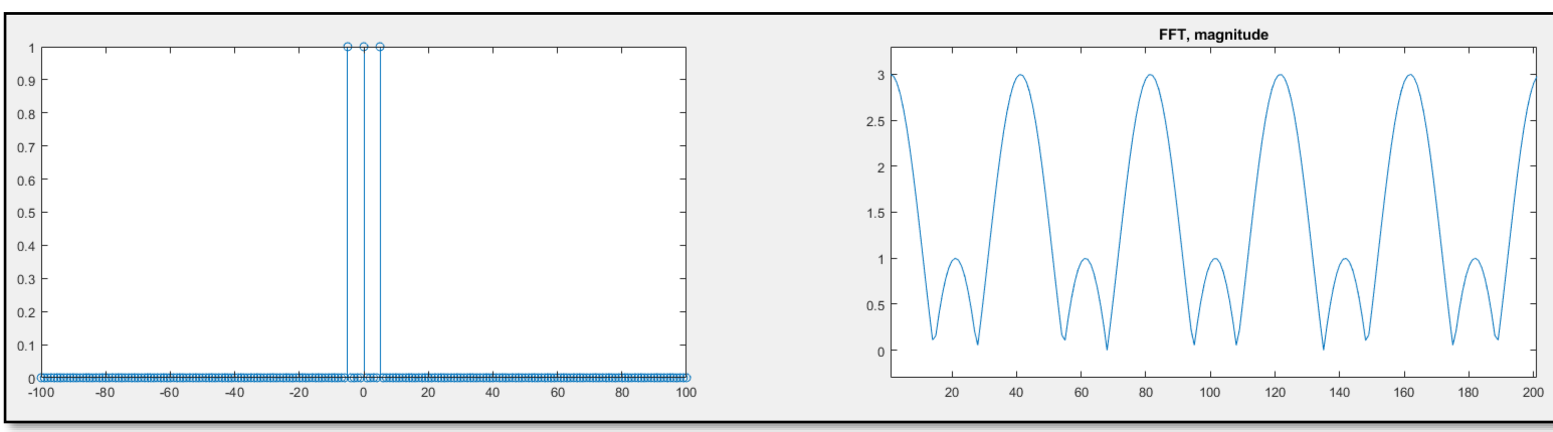
This function constructs a 1D comb function.

```
function [comb] = comb_function(number_of_extra_impulses,dx)
x = -100:1:100;
comb = 0;

if mod(number_of_extra_impulses,2) ~= 0
   disp("Invalid number! Needs to be even number.");
else
    comb = dirac(x);
    for a = number_of_extra_impulses:-2:0
        comb = comb + dirac(x-(a/2)*dx) + dirac(x+(a/2)*dx);
    end
    idx = comb == Inf;
    comb(idx) = 1;
    fourier_tr = fft(comb);
    figure;
    range2 = [-0.1 1.1]*max(abs(fourier_tr));
    subplot(2,2,1), stem(x,comb);
    subplot(2,2,2), plot(abs(fourier_tr)), axis tight, ylim(range2), title('FFT, magnitude');

end
end
```

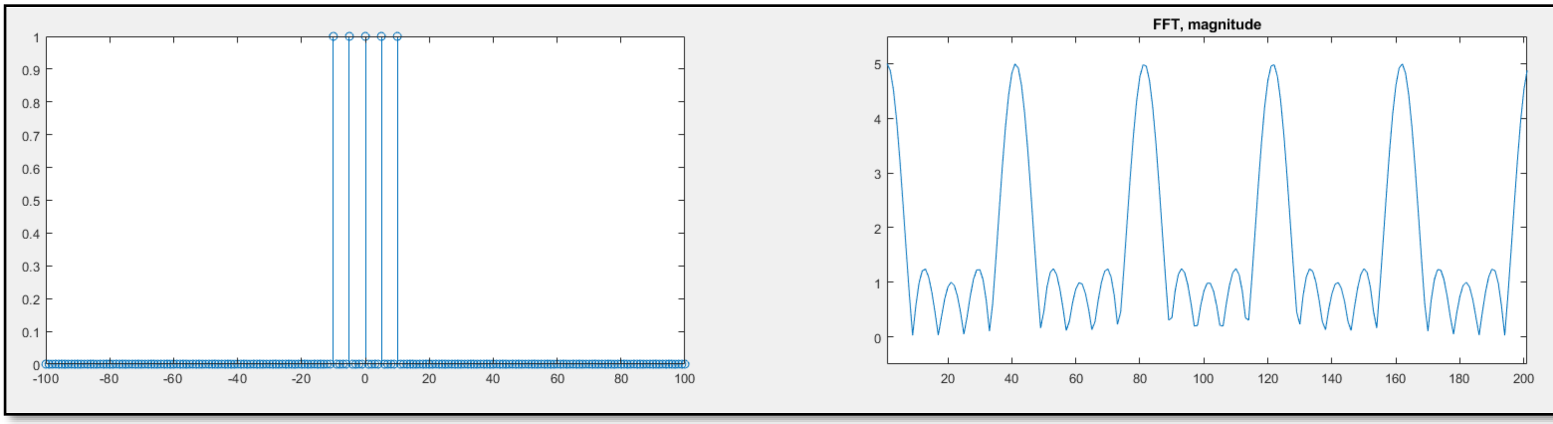
In the case, where only one impulse at the center exists, then the Fourier Transform is as shown below:



If two impulses are added, one on the right and one on the left, then the Fourier Transform becomes:

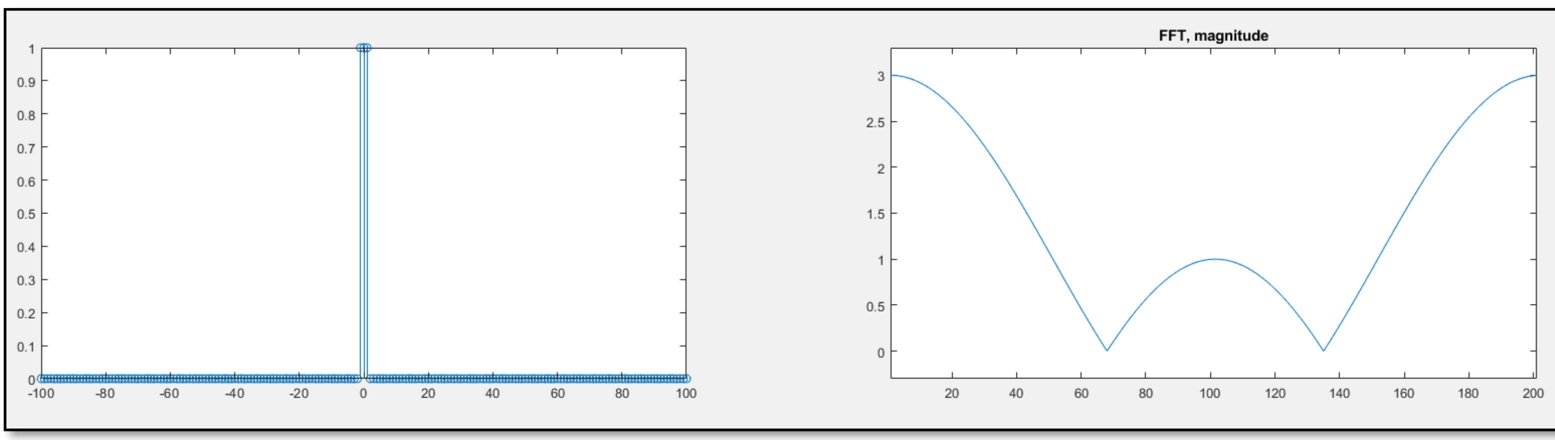


If two more impulses are added, one on the right and one on the left, then the Fourier Transform looks like the following:

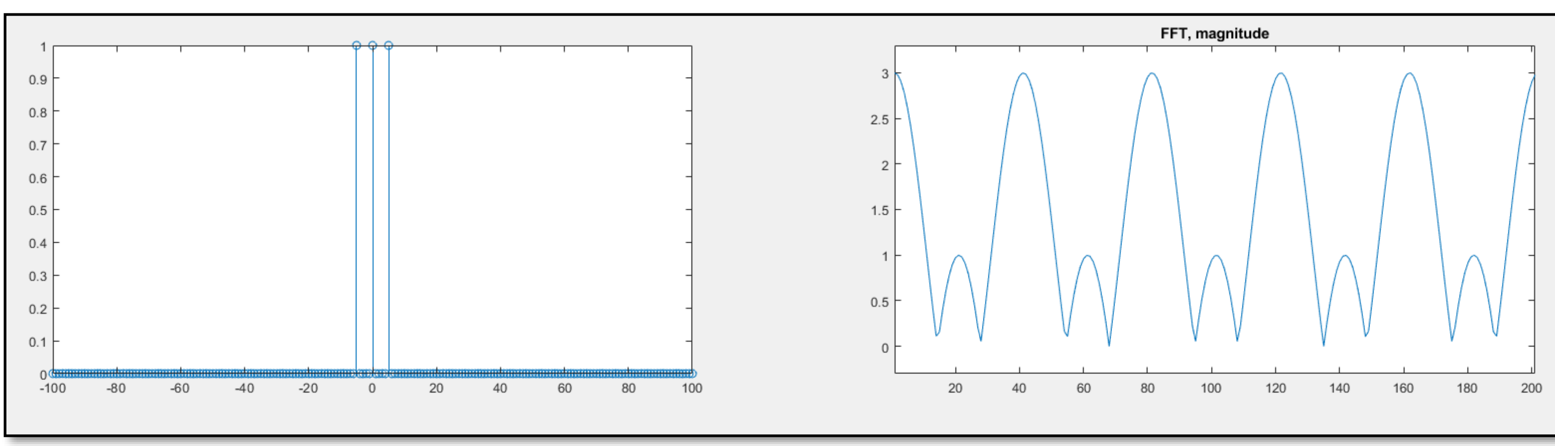


By observing the plots, it is obvious that by adding more impulses, more side lobes are added in the FFT magnitude plot but their magnitude stays constant. Furthermore, by adding more impulses the width of the main lobes decreases and their magnitude is increased. This means that by adding more samples into our system, the resolution of the image will be better and the image quality will become better because the SNR is increased.

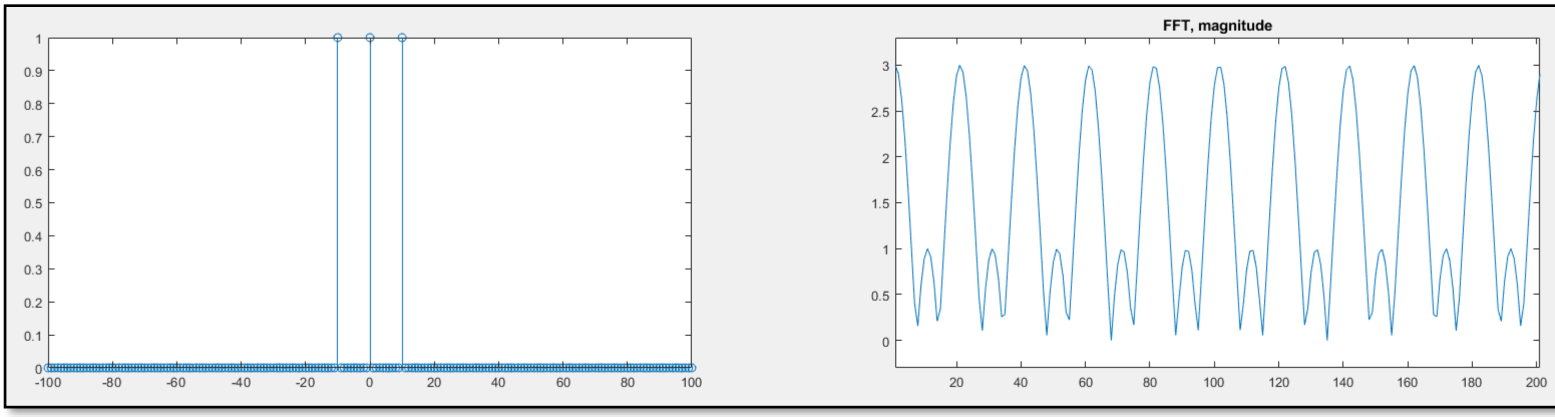
b) As an example, we take the case with three impulses, one at the center, one at the right and one at the left. With a spacing of Δx = 1, the following response is extracted:



By increasing Δx to 5, the magnitude of the FFT becomes:



If Δx becomes 10, then the magnitude of the FFT is the following:



From the above plots, it can be noticed that as Δx is increased more main lobes are added in the response and their width becomes smaller. Furthermore, the number of side lobes stays the same and the magnitude of these side lobes remains constant. Therefore, by increasing the spacing Δx the SNR stays constant and therefore the image quality does not improve.