VE216 RC 4

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Overview

1. Solution for Quiz 2.

2. Chap3 Fourier Series

3. Exercise

Quiz 2 : Given $u(t) \xrightarrow{T} s(t) = (3-t)\cdot \text{rect}\left(\frac{t-1}{2}\right)$

1) Solve : $\delta(t) \rightarrow h(t) = ?$

2) solve : $x(t) = \text{rect}\left(\frac{t-1}{2}\right) \longrightarrow y(t) = ?$

A better way to solve 2) :

$x(t) = \text{rect}\left(\frac{t-1}{2}\right) = u(t) - u(t-2) \xrightarrow{T} y(t) = s(t) - s(t-2) = \cdots$ done !

Strategy

- decompose input signals into a linear combination of basic signals
- To study LTI system : (two brilliant ways !)

– delayed impulse $\Leftarrow$ convolution ✓ last week

if $h(t)$ known $\Rightarrow$ $y(t)$ known

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow \boxed{h} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * \boxed{h(t)}$$

– complex exponentials ($e^{j\omega t}$) $\Leftarrow$ Fourier Analysis ✓ today.

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot e^{jk\omega_0 t} \rightarrow \boxed{H} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} C_k \cdot \underline{H(jk\omega_0)} \cdot e^{jk\omega_0 t}$$

$\uparrow$ comb. of sin / cos.

$\uparrow$ Upper Case. Freq. resp.

$\uparrow$ Scaled version of corresponding $e^{jk\omega_0 t}$

$\leftarrow$ Why need FS ?

- At least, easier to compute than convolution

1st Q: why need?   2nd Q: $x(t) = \sum xxx\, e^{jwt}$ possible?

Motivation

- For s complex $s = \sigma + jw$, $\sigma, w \in \mathbb{R}$

$$x(t) = e^{st} \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Fs Need LTI!

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \cdot e^{-s\tau} \cdot e^{st}\, d\tau$$

Fascinating result!

$$= \underbrace{e^{st}}_{x(t)} \cdot \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s)}$$

- scaled input

$H(s)$ - Laplace Trans, study later

- For s imag. $s = jw$.

Diff?: periodic!

$$x(t) = e^{jwt} \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t) = \underbrace{e^{jwt}}_{x(t)} \cdot H(jw)$$

Fourier Trans.

- Want: For "Any" Periodic input $x(t)$.

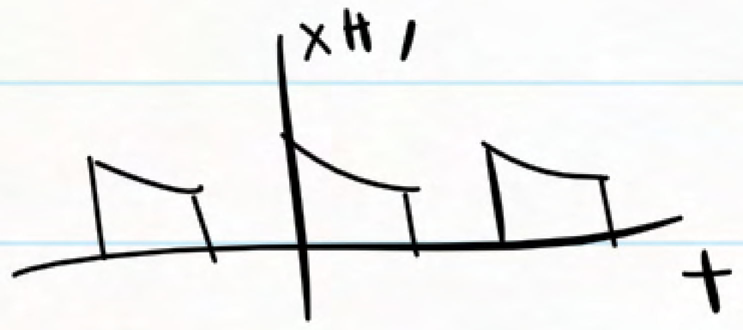
$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot e^{jkw_0t} \rightarrow \boxed{LTI\ h(t)} \rightarrow y(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot H(jkw_0)\, e^{jkw_0t}$$

Linear comb of e^{jkw_0t}

Need to find out C_k, w_0, wait a minute on how.
Try to interpret the fact first

Interpretation: $x(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot e^{jk\omega_0 t} \xrightarrow{\tau} y(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot H(jk\omega_0)\, e^{jk\omega_0 t}$

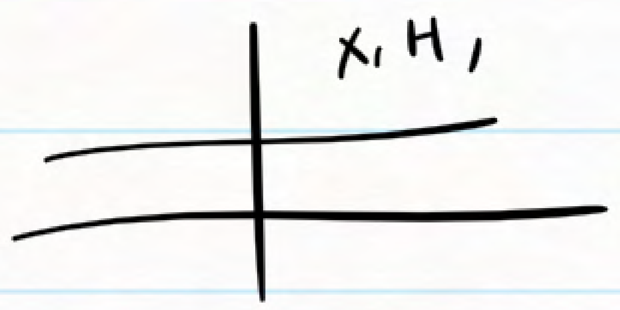
$\boxed{x(t)}$ $\xrightarrow{* h(t)}$ $\boxed{y(t)}$



$y(t) = x(t) * h(t)$ — con: - calculation - physical meaning?

FS: focus of today.

$\Downarrow$



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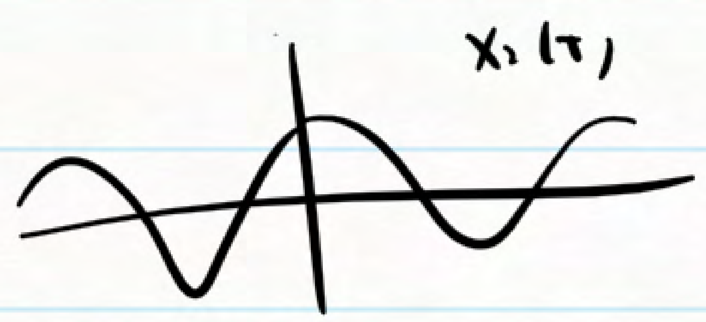

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$\Longrightarrow$ $y(t) = \underset{\text{scales}}{H}(xxx) \cdot X_1(t) + H(xxx) + X_2(t) + \cdots$ pro: - easy for $y(t)$ (wrt $x(t)$) - ✓ physical meaning

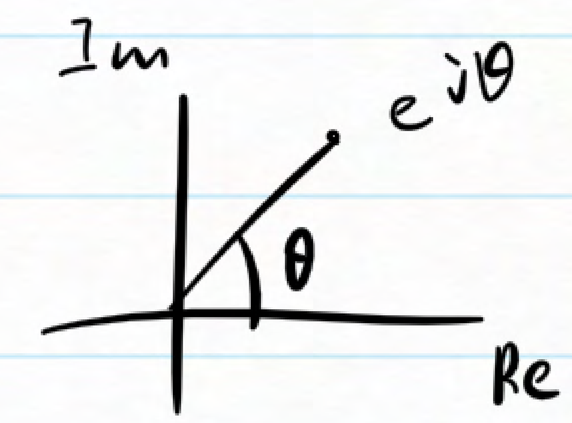
Next:

- How to find ω_0, C_k?
- First, focus on "signal" part - FS

Then, big picture of the "system", $H(\omega)$, $y(t)$ - next time.

Important Formula :

- $e^{j\theta} = \cos\theta + j \cdot \sin\theta$, $\theta \in \mathbb{R}$ $\Rightarrow$ $e^{-j\theta} = \cos\theta - j\sin\theta$:

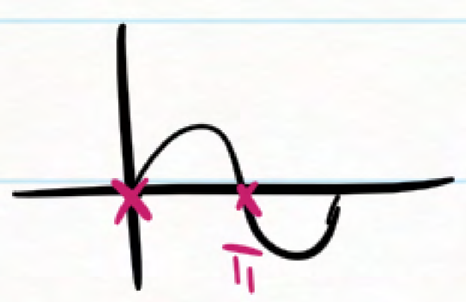


"rotates"

- $\begin{cases} \cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \\ \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \end{cases}$

- Period vs (Angular) Freq : $T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$

- $\sin(k\pi) = 0$, $\cos(k\pi) = (-1)^k$. $k \in \mathbb{Z}$



Fourier Series

- For Periodic Signal $x(t)$ with fundamental period T_0.

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \cdot e^{jk\omega_0 t} \quad (\omega_0 = \frac{2\pi}{T_0}) \qquad C_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}$$

- k pos. neg $\Rightarrow$ combine to sin / cos.
- Freq : 0, $\pm\omega_0$, $\pm 2\omega_0$, $\pm$ ... . Why neg ? $\because$ $e^{jk\omega_0 t}$, not $\cos(\omega_0 t)$
- C_k: careful when $k=0$
- Use table look up whenever you can (rather than the formula $C_k = \int xxx$)

Exercise

$$x(t) = \sum_k a_k \cdot e^{jk\omega_0 t}$$

$$\begin{cases} \cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \\ \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \end{cases}$$

1. Find FS. coeff. a_k of $x(t) = \sin(3\pi t) + \cos(4\pi t)$ - Q3

$T_1 = \frac{2\pi}{3\pi} = \frac{2}{3}$ $\quad T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$

- want $\omega_0 \Rightarrow$ find T_0 : $n_1 \cdot T_1 = n_2 T_2$, $n_1, n_2 \in \mathbb{Z}$

$n_1 \cdot \frac{2}{3} = n_2 \cdot \frac{1}{2}$

$\frac{n_1}{n_2} = \frac{3}{4} \Rightarrow \begin{cases} n_1 = 3 \\ n_2 = 4 \end{cases} \Rightarrow T_0 = n_1 T_1 = n_2 T_2 = 2 \Rightarrow \omega_0 = \frac{2\pi}{2} = \pi$

- $x(t) = \frac{1}{2j}\left(e^{j3\pi t} - e^{-j3\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right)$

$$= \underbrace{\frac{1}{2}}_{C_{-4}} \cdot e^{-j4\pi t} - \underbrace{\frac{1}{2j}}_{C_{-3}} e^{-j3\pi t} + \underbrace{\frac{1}{2j}}_{C_3} e^{j3\pi t} + \underbrace{\frac{1}{2}}_{C_4} e^{j4\pi t}$$

- Check : Hermitian Symmetry: If $x(t)$ real, then $C_{-k} = C_k^*$

$C_{-4} = C_4 = C_4^*$ - real

$C_{-3} = -C_3 = C_3^*$ - imag. ✓

odd harmonic : $\omega_0, 3\omega_0$

2. $x(t)$ period T : $\quad x(t) = \sum_{odd} a_k \cdot e^{jk\frac{2\pi}{T}t} \iff x(t) = -x(t+\frac{T}{2})$ Q7

$\Rightarrow$: $x(t+\frac{T}{2}) = \sum_{odd} a_k \cdot e^{jk\frac{2\pi}{T}(t+\frac{T}{2})} = \sum_{odd} a_k \cdot e^{jk\frac{2\pi}{T}t} \cdot e^{jk\pi}$

Recall : $e^{jk\pi} = \cos(k\pi) + j\sin(k\pi)$, $\sin(k\pi) = 0$, $\cos k\pi = (-1)^k$

$\rightarrow e^{jk\pi} = (-1)^k$. $\quad k$ odd : $e^{jk\pi} = -1$

$\rightarrow x(t+\frac{T}{2}) = \left(\sum_{odd} a_k \cdot e^{jk\frac{2\pi}{T}t}\right) \cdot (-1) = -x(t)$

Conti.

$$x(t) = \sum_{odd} a_k \cdot e^{jk\frac{2\pi}{T}t} \Leftarrow x(t) = -x(t+\frac{T}{2}) = -x(t-\frac{T}{2})$$

$\Leftarrow$: Want $a_k = 0$ for k even.

Eg:  $\sin(t) = -\sin(t-\pi)$ — How does the signal look like

let $t = t' + \frac{T}{2} \Rightarrow t' = t - \frac{T}{2}$

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\left[\int_0^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t}\, dt + \int_{\frac{T}{2}}^{T} x(t)\cdot e^{-jk\omega_0 t}\, dt\right)$$

$$= \text{xxx} + \int_0^{\frac{T}{2}} x(t'+\frac{T}{2})\cdot e^{-jk\omega_0 (t'+\frac{T}{2})}\, dt'$$

$$\text{xxx} + \int_0^{\frac{T}{2}} (-1)\cdot x(t')\cdot e^{-jk\omega_0 t'}\cdot e^{-jk\pi}\, dt'$$

$$= \frac{1}{T}\int_0^{\frac{T}{2}} x(t)\, e^{-jk\omega_0 t}\cdot\left(1 - e^{-jk\pi}\right) dt$$

Recall : $e^{-jk\pi} = (-1)^k \Rightarrow k$ even , $a_k = 0$

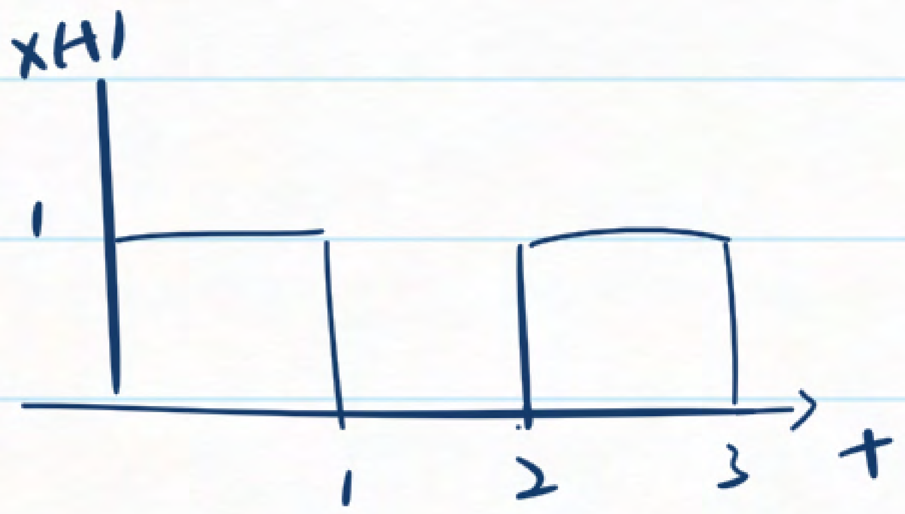
"Odd harmonic"

Recall : In Lecture Slides & MIT Video : $x(t) = \sum \text{rect}(t - \frac{1}{2} - 2n)$ $\Rightarrow$ $c_k = \begin{cases} \frac{1}{2} & k=0 \\ \frac{1}{jk\pi} & , k \text{ odd.} \end{cases}$

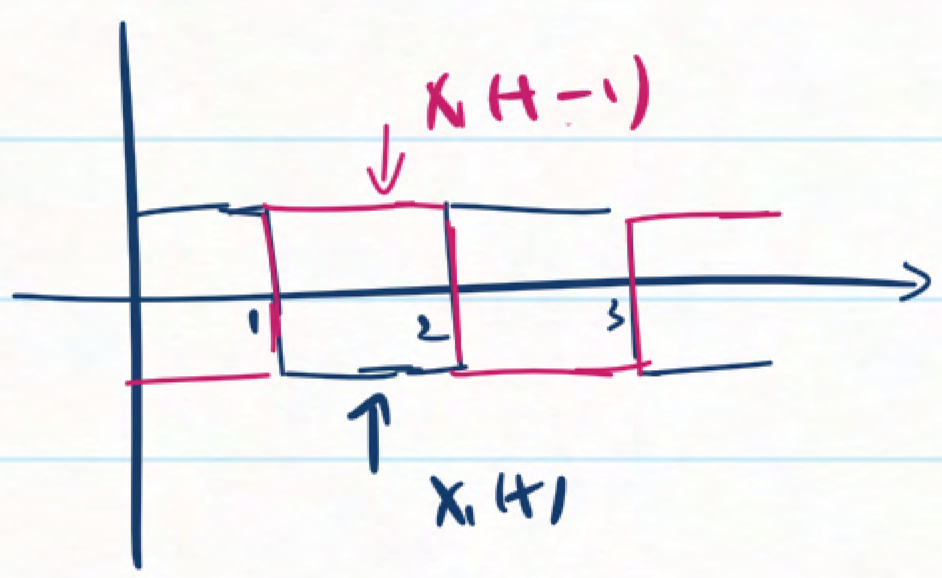
$T_0 = 2$.

$\Downarrow$

$$x(t) = \frac{1}{2} + \sum_{k=1,\text{odd}}^{+\infty} \frac{2}{k\pi} \sin(k\pi t)$$



$\Downarrow$ $x(t) - \frac{1}{2} =: x_1(t)$



• $x_1(t) = -x_1(t - \frac{T_0}{2}) = -x_1(t-1)$ ✓

• $x_1(t) = \sum_{k=1,\text{odd}} \frac{2}{k\pi} \sin(k\pi t)$ ✓

$\Downarrow$

$$x(t) = x_1(t) + \underset{\frac{1}{2}}{\underset{\shortparallel}{c_0}}$$

Summary

- Big Picture :

$$x(t) \Downarrow \Sigma \Rightarrow y(t)$$

- Fourier Series - Focus on Signal

- Don't use brute force when calculating C_k
  - Use table look-up - next week