VE216 Recitation Class 6

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2020 Summer

Overview

1. Chapter 4: Fourier Transform – Physical Meaning
   - FS vs FT
2. Chap.6: Filtering – All covered in Chap. 3. 4.
   - FS: Filtering
   - FT: Filtering
3. Summary

FS vs FT: for periodic signal

FOURIER TRANSFORM OF A PERIODIC SIGNAL $\tilde{x}(t)$

$\tilde{x}(t) \longleftrightarrow a_k$ — Fourier series coefficients

$\tilde{x}(t) \overset{\mathcal{F}}{\longleftrightarrow} \tilde{X}(\omega)$ — Fourier transform

$$\boxed{\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi\, a_k\ \delta(\omega - k\omega_0)}$$

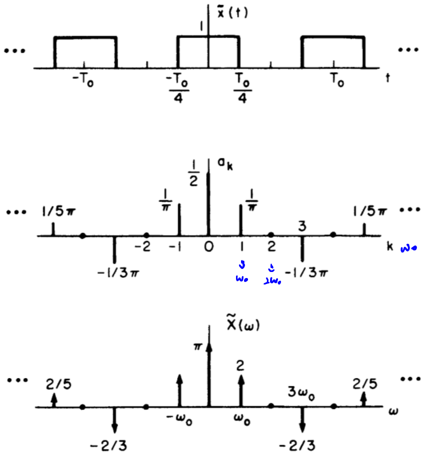
$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{X}(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} 2\pi\, a_k \underbrace{\int_{-\infty}^{+\infty} \delta(\omega - k\omega_0)\, e^{-j\omega t}\, d\omega}_{e^{-jk\omega_0}}$$

FS vs FT: for periodic signal - Example

$\tilde{X}(w) = \sum_k 2\pi \cdot a_k \cdot \delta(w - kw_0)$

Symmetric square wave



FS vs FT: definition

Fourier Series: for periodic signals

easier to understand.

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} && \text{synthesis} \\ a_k &= \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t} && \text{analysis} \end{aligned}}$$

Fourier Transform: for "all" signals, often aperiodic

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega && \text{synthesis} \\ X(\omega) &= \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt && \text{analysis} \end{aligned}}$$

$\Rightarrow$ also a summation.

$$T_0 a_k = X(\omega)\big|_{\omega = k\omega_0}$$

$$\Rightarrow \tilde{x}(t) = \sum_k a_k \cdot e^{jk\omega_0 t}$$

$$= \sum_k \frac{1}{T_0} X(k\omega_0) \cdot e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_k X(k\omega_0) \cdot e^{jk\omega_0 t} \cdot \omega_0$$

$$= \frac{1}{2\pi} \int_\omega X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$X(\omega) = Re\{X(\omega)\} + j\, Im\{X(\omega)\}$$
$$= |X(\omega)|\, e^{j\sphericalangle X(\omega)}$$

FS: Filtering

$\times H(j\cdot 0\cdot \omega_0) \Rightarrow$

$\times H(j\cdot 1\cdot \omega_0) \Rightarrow$

$\times H(j\cdot k\cdot \omega_0) \Rightarrow$

$x(t),\ h(t) \rightarrow y(t) = h(t) * x(t)$

$\downarrow$

$a_k,\ H(\omega)$

Input-Output Relation:

$$\color{red}{x(t) = \sum_k c_k e^{jk\omega_0 t}} \rightarrow \boxed{\text{LTI } h(t)} \rightarrow y(t) = \sum_k c_k \color{blue}{H(jk\omega_0)} e^{jk\omega_0 t}$$

multiplying a scaling factor for any $e^{jk\omega_0 t}$

$C_k \sim e^{jk\omega_0 k} \sim \omega_0 k$ rather than k

rad/s.

Example:

periodic square

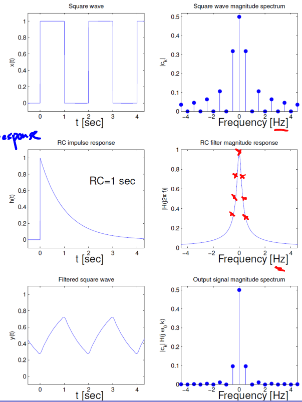
$x(t)$

impulse response

$h(t)$

$y(t) = x(t) * h(t)$

hard to compute



$\frac{\omega_0 k}{2\pi}$

$H(\omega) \sim H(f) \Rightarrow H(jk\omega_0)$

rad/s Hz $k \in \mathbb{Z}$

$C_k \cdot H(jk\omega_0)$ - amp.

$\Downarrow$

$y(t)$: only simple multiplication.

FT: Filtering

Previous, assume $H(\omega)$ given
⇓
but what is $H(\omega)$?

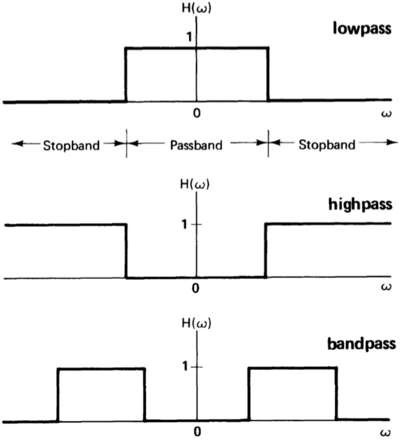
$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = h(t) * x(t)$: time domain, nothing special.
$X(\omega), H(\omega) \longrightarrow Y(\omega) = H(\omega) \cdot X(\omega)$

simple circuit
⇓
But in freq. domain, act as a lowpass filter!

- Convolution Property: $h(t) * x(t) \stackrel{\mathscr{F}}{\longleftrightarrow} H(\omega)X(\omega)$

why called "filter"?

make $|Y(\omega)| = 0$ for $\omega > \omega_0$ ⇐

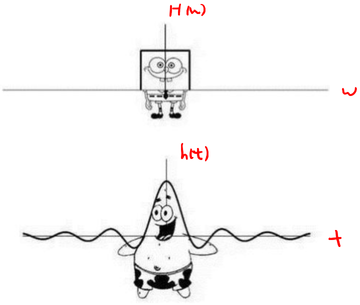


- LTI systems can be viewed as "filters" in frequency domain

seems hard to believe, but it's the power of LTI system and Fourier Analysis.

e.g. filter out high freq. component.

Lowpass Filter



$X(\omega) \cdot H(\omega)$ ✓

$x(t) * h(t) \Rightarrow$?

Patrick Star $\xleftrightarrow{FT}$ SpongeBob SquarePants

Exercise: FS Filtering - HW3 Q11

$x(t) = \sum_k a_k e^{jk w_0 t} \longrightarrow y(t) = \sum_k a_k \cdot H(jk w_0) \cdot e^{jk w_0 t}$

11. [5] Consider a continues-time ideal lowpass filter S whose frequency response is

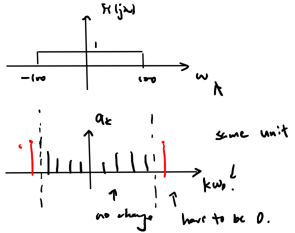
$$H(j\omega) = \begin{cases} 1 & , |\omega| \leq 100 \\ 0 & , |\omega| > 100 \end{cases}$$ – lowpass

When the input to this filter is a signal $x(t)$ with fundamental period $T = \pi/6$ and Fourier series coefficients a_k, it is found that

$w_0 = \frac{2\pi}{T} = 12$

$$x(t) \longrightarrow y(t) = x(t)$$ – after passing through Low pass filter, signal remain the same.

For what value of k it is guaranteed that a_k must be zero?



Need: when $|w_0 k| > 100$, $a_k = 0$.

Find k: $|k| > \frac{100}{12}$

$\Rightarrow |k| \geq 9$.

Exercise - FT: Filtering

HW4.

12. [10] Shown in the figure 0403 is the frequency response $H(j\omega)$ of a continuous-time filter referred to as a lowpass differentiator. For each of the input signals x(t) below, determine the filter output signal $y(t)$.

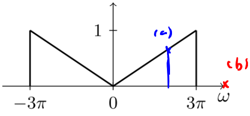
    (a) $x(t) = \cos(2\pi t + \theta)$

    (b) $x(t) = \cos(4\pi t + \theta)$

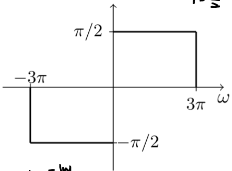
First step: find $H(j\omega)$

$H(\omega) = \boxed{J} + j \cdot c$

$|H(j\omega)| = \frac{|\omega|}{3\pi}$



$\angle H(j\omega) = \begin{cases} \frac{\pi}{2}, & \omega>0 \\ -\frac{\pi}{2}, & \omega<0. \end{cases}$



$H(j\omega) = j \cdot \frac{\omega}{3\pi}$ $\begin{cases} \omega<0 & = j \cdot \frac{-|\omega|}{3\pi} = \frac{|\omega|}{3\pi} \cdot (-j) \to -\frac{\pi}{2} \\ \omega>0 & = \dots \end{cases}$ ✓.

Hint:

$$x(t) = \cos(\omega t + \phi) \to \boxed{\text{LTI } h(t)} \to y(t) = |H(j\omega)| \cos(\omega t + \phi + \angle H(j\omega))$$

Exercise - FT: Filtering

13. [10] A power signal with the power spectral density shown in figure 0405 is the input of a linear system with the frequency response shown in figure 0406. Calculate and sketch the power spectral density of the system's output signal.

$P_f(\omega) = |X(\omega)|^2$

$y(t) = x(t) * h(t) \leftrightarrow Y(\omega) = X(\omega) \cdot H(\omega)$

$\Downarrow$

$|Y(\omega)|^2 = P_f(\omega) \cdot |H(\omega)|^2$

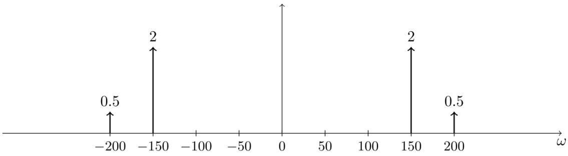


Figure: 0405

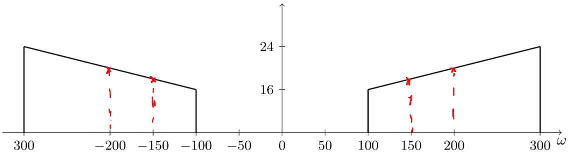


Figure: 0406

Summary

- FS vs FT
- Physical meaning of FT
- The place we are in the big picture

The End