VE216 RC3

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Overview

Chap. 1. lots of def., properties. Now focus on LTI

1. Chap 2 : Conti. Time LTI system

- Convolution

2. Exercise : - focused. ∴ × latex
✓ handwriting

selected good ones from HW. Quiz. ...

Strategy :

- decompose input signal $x(t)$ into a linear combination of basic signals
- To study LTI system : 2 billiant ways !

  delayed impulse $\delta(t-t_0)$ $\Leftrightarrow$ convolution – Focus of today

  Complex exponentials $e^{j\omega t}$ $\Leftrightarrow$ Fourier Analysis

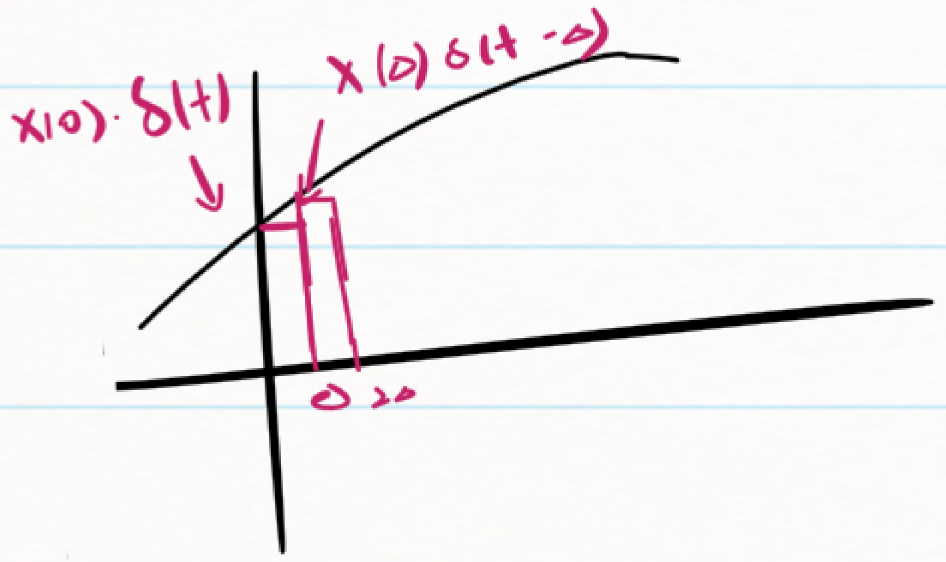
Now, let's consider the details

System : Denote : $\delta(t) \xrightarrow{T} h(t)$

$\delta_\tau(t) = \delta(t-\tau) \xrightarrow{T} h_\tau(t)$ ← don't know its form at this time

Any input signal:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\cdot\delta(t-\tau)\,d\tau \xrightarrow{\text{if Linear}} y(t) = \int_{-\infty}^{\infty} x(\tau)\, h_\tau(t)\,d\tau$$ — ∵ linear comb : $\sum x(t) \to \sum y(t)$.



$$\xrightarrow[h_\tau(t)=h(t-\tau)]{\text{if TI}} y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\,d\tau =: x(t) * h(t)$$

Conclusion : For LTI system, only need to know 1 pair : $\delta(t) \xrightarrow{T} h(t)$

↳ Denote : $x(t) \to \boxed{h} \to y(t)$

Tips : • $\tau + (t-\tau) = t$ — in case can't remember the formula

• $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\,d\tau$

Ex 1.

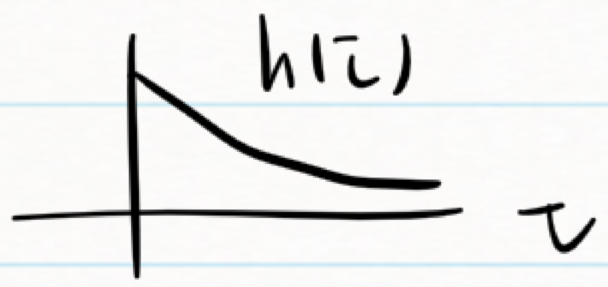
Graphical Understanding of : $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

- Recall : time transformation $h\left(\frac{t-t_0}{w}\right)$ — 1st : time scale by w
  2nd : time shift by t_0.

- $h(t-\tau)$ — A function of τ $\because \int \cdot\cdot\, d\tau$

$= h\left(\frac{\tau-t}{-1}\right) = h(-(\tau-t))$ — First, "flip" according to y-axis
— Then, time shift by t

Eg: $h(\tau) = e^{-\tau} \cdot u(\tau)$



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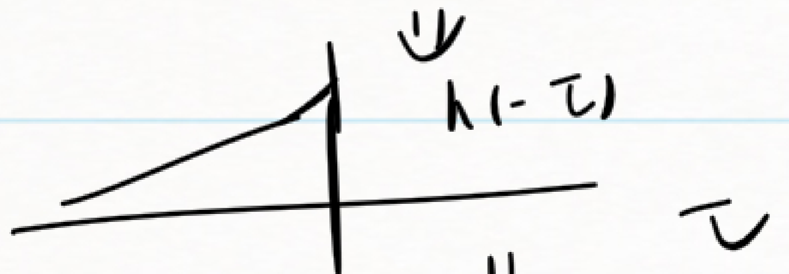
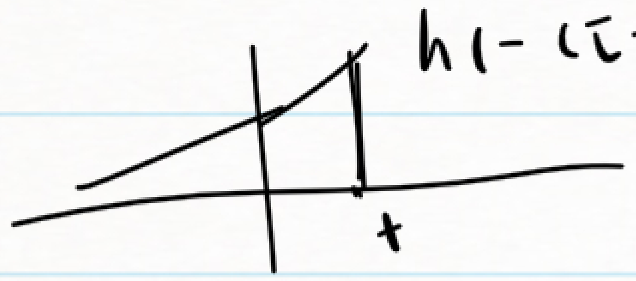


$\Downarrow$

$h(-(\tau-t))$ - shift right



$\Rightarrow$ At any instant t, $y(t) = \text{xxx}$

↑ need to evaluate all ts. "running integration"

Convolution is tricky : Part of Q_3

Claim 1: If $y(t) = h(t) * x(t)$, let $t' = t-3$, then $y(t') = h(t') * x(t')$

$$y(t-3) = h(t-3) * x(t-3) \quad ?$$

No, in fact $y(t-3) = h(t) * x(t-3)$.

- Prove by def: $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$ — math prob.
- Understand by: $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$ — physical meaning.

$$x(t-3) \rightarrow \boxed{h(t)} \rightarrow y(t-3) = h(t) * x(t-3)$$

$$x(t-3) \rightarrow \boxed{h(t)} \rightarrow \boxed{t-3} \rightarrow y(t-6) = h(t-3) * x(t-3)$$

Quiz 1. prob. 2. (later)

Convolution is <u>tricky</u> :

Claim 2 : $\text{rect}(t) * \text{rect}(t) = \text{tri}(t)$. let $t' = \frac{t}{2}$, then $\text{rect}(t') * \text{rect}(t') = \text{tri}(t')$

i.e. $\text{rect}(\frac{t}{2}) * \text{rect}(\frac{t}{2}) = \text{tri}(\frac{t}{2})$?

No. $\frac{1}{2}\,\text{rect}(\frac{t}{2}) * \text{rect}(\frac{t}{2}) = \text{tri}(\frac{t}{2})$

Proof : let $x(t) = \text{rect}(\frac{t}{2})$.

$$x(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \text{rect}(\frac{\tau}{2}) \cdot \text{rect}(\frac{t-\tau}{2})\, d\tau$$

let $\tau' = \frac{\tau}{2}$ $$= \int_{-\infty}^{\infty} \text{rect}(\tau') \cdot \text{rect}(\frac{t}{2} - \tau') \cdot 2\, d\tau'$$

let $t' = \frac{t}{2}$ $$= 2 \cdot \int_{-\infty}^{\infty} \text{rect}(\tau') \cdot \text{rect}(t' - \tau') \cdot d\tau'$$

$$= 2\ \text{rect}(t')$$

$$= 2 \cdot \text{tri}(t') = 2\,\text{tri}(\frac{t}{2})$$

in general $x(\frac{t}{a}) * h(\frac{t}{b})$. a, b may < 0

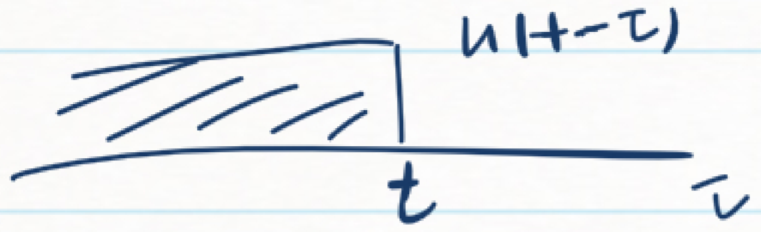
Conclusion : Try to avoid $x(\frac{t}{a}) * h(\frac{t}{b})$. convert to $x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

Exercise:

1. Know: LTI, Want: $h(t)$. — Q_{13} (part of)

1(a) $y(t) = \int_{-\infty}^{t} (t-\tau)\, e^{-(t-\tau)}\, x(t)\, d\tau$ — $u(t)$

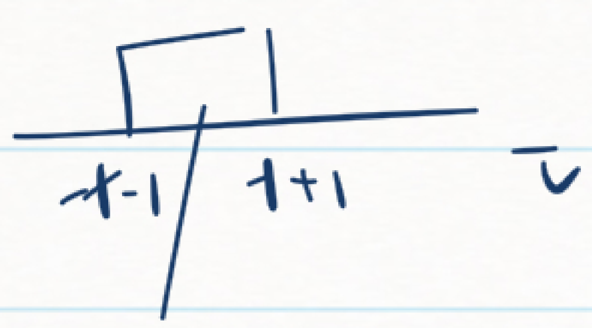


$$= \int_{-\infty}^{\infty} u(t-\tau)\cdot(t-\tau)\cdot e^{-(t-\tau)}\cdot x(t)\, d\tau$$

$u(\tau) \rightarrow u(-\tau) \rightarrow u(-(\tau-t))$

$$= \left(u(t)\cdot t\cdot e^{-t}\right) * x(t)$$

(b). $y(t) = \int_{t-1}^{t+1} e^{-2(t-\tau)}\cdot x(\tau)\, d\tau$ — $\text{rect}(t)$

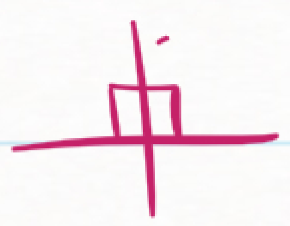


$$= \int_{-\infty}^{\infty} \text{rect}\left(\frac{t-\tau}{2}\right) e^{-2(t-\tau)}\cdot x(\tau)\, d\tau$$

$\text{rect}\left(\frac{\tau-t}{2}\right) = \text{rect}\left(\frac{t-\tau}{2}\right)$

$$= \left(\text{rect}\left(\frac{t}{2}\right)\cdot e^{-2t}\right) * x(\tau)$$

Recall: $\text{rect}(t) = \text{rect}(-t)$



Think about

$$y(t) = \int_{-2}^{3} \tau^2\, x(t-\tau)\, d\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2}\, x(\tau)\, d\tau$$

2. $y(t) = \left(\int_{t-T}^{t} x(\tau)\, d\tau \right) * u(t)$, $T > 0$. Time invariant ?

- Sol 1: Def: $y(t - t_0) = \left(\int_{t-T-t_0}^{t-t_0} x(\tau)\, d\tau \right) * u(t)$ (not $u(t-t_0)$)

  $x_d(t) = x(t - t_0) \rightarrow y_d(t) = \cdots = y(t - t_0)$

- Sol 2: Interconnection of LTI: $x(t) \rightarrow \boxed{S_1} \rightarrow \boxed{S_2} \rightarrow y(t)$

  $h(t)$

  prove: $\int_{t-T}^{t} x(\tau)\, d\tau$ is LTI

- Sol 3: Represent using convolution: using the method shown in the last page.

  $y(t) = x(t) * \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right) * u(t)$ $\qquad x(t_0) * h(t_0) \overset{?}{=} 0$ ∵ convolution isn't pointwise

- Stable ? $h(t) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right) * u(t) = [u(t) - u(t-T)] * u(t) = t\,u(t) - (t-T)\,u(t-T)$ careful.

  $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$. $\Rightarrow$ Not Stable

Summary:

- convolution – inspiring, compare with Vv256/286
- $h(t)$! – only thing need to specify an LTI system
- From $y(t) = \int_{-\infty}^{\infty} \dots x(\tau)\, d\tau$ $\Rightarrow$ Find $h(t)$ directly