VE216 Recitation Class 5

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VE216 SU20 Teaching Group

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Overview :

Focus on concept, next week / few weeks, focus on application

1. Chapter 3: Fourier Series
   - Properties
   - Common FS Pairs

Content very similar.

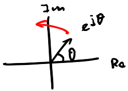
2. Chapter 4: Fourier Transform
   - FS vs FT
   - Common FT Pairs
   - Properties

3. Summary

$x(t) = \sum c_k e^{jk\omega_0 t}$

Properties

- FS : understand physically



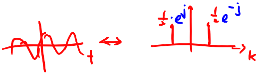
$$\begin{cases} |e^{j\theta}| = 1 \\ \angle e^{j\theta} = \theta \end{cases}$$

$\omega_0 = 1$

E.g. : $x(t) = \cos(t) = \frac{1}{2}(e^{-jt} + e^{jt})$

" rotates "
while amp = const.

$x(t) = \cos(t-1)$, $t_0 = 1$



- Understand $e^{j\theta}$ $= \cos\theta + j \cdot \sin\theta$
- Time shift: (time domain : variable + - in FT)

$$x(t - t_0) \longleftrightarrow c_k \boxed{e^{-jk\omega_0 t_0}}$$

$e^{-jk\omega_0 t_0} = e^{-jk}$

- Differentiation:

$$y(t) = \frac{d}{dt}x(t) \longleftrightarrow jk\omega_0 c_k$$

↑ . when $k\omega_0$ ↑

- Etc ...

why? : $\frac{d}{dt}$

$\frac{d}{dt}$

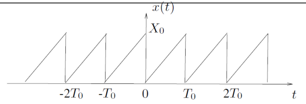
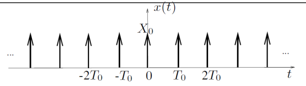
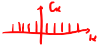
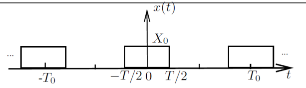
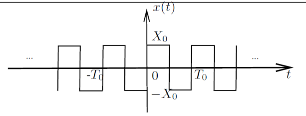
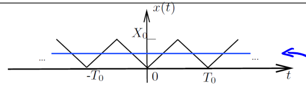
⇒ ↑

Common FS Pairs : in summary.pdf

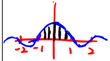
$x(t) = \sum_k c_k e^{jk\omega_0 t}$

be careful (∵ $\frac{1}{\pi k}$)

Table of Fourier Series for Common Signals

Name	Waveform	c_0	$c_k, k \neq 0$	Comments
Sawtooth		$\frac{X_0}{2}$	$j\frac{X_0}{2\pi k}$	
Impulse train		const. w.r.t k : $\frac{X_0}{T_0}$	$= \frac{X_0}{T_0}$	equal energy for all freq: $c_k e^{jk\omega_0 t}$ 
Rectangular wave		$\frac{TX_0}{T_0}$	sinc(x)= $\frac{\sin \pi x}{\pi x}$ (x=0, L'Hopital) $\frac{TX_0}{T_0}\,\text{sinc}(\frac{Tk\omega_0}{2\pi})$	$\sin \pi x = 0$, $x \in \mathbb{Z}$ $\frac{Tk\omega_0}{2\pi} = \frac{Tk}{T_0}$
Square wave		0	$-j\frac{2X_0}{\pi k}$	show MIT video L7 36min ($c_k \neq 0$, k odd) $c_k = 0$, k even
Triangular wave sine		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$c_k = 0$, k even

Know how to draw



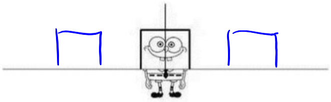
type "non-zero"

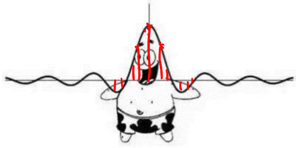
Recall: HW3 Q7 "odd" Harmonic

$x(t) = -x(t - \frac{T}{2})$

Approx. odd harmonic

To help remember





square

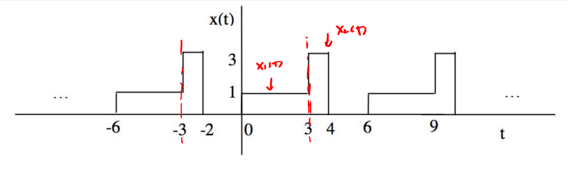
SpongeBob SquarePants $\xleftrightarrow{FT}$ Patrick Star

Periodic SpongeBob SquarePants $\xleftrightarrow{FS}$ Samples of Patrick Star

details : Chap. 7 Sampling - see this picture again

Exercise - Use Table Lookup: Quiz3

Find the Fourier series for the following signal [10]. Also, sketch the approximation if a large number of terms are kept in the series (say $N = 40$) [10].



$T_0 = 6$

for both $x_1(t)$ and $x_2(t)$

0. $x(t) = \sum \text{rect}\left(\frac{t-6n}{3}\right) \longleftrightarrow a_k$ ← table look up.

$x_1(t) = x(t - \frac{3}{2}) \longleftrightarrow a_k \cdot e^{-jk\omega_0 t_0}$. $t_0 = \frac{3}{2}$, $\omega_0 = \frac{2\pi}{T_0}$

$x(t) = 3 \times \sum \text{rect}(t - 6n) \longleftrightarrow c_k$

$x_2(t) = x(t - \frac{7}{2}) \longleftrightarrow c_k \cdot e^{-jk\omega_0 t}$, $t_0 = \frac{7}{2}$, $\omega_0 = \frac{2\pi}{T_0}$

$\oplus \rightarrow$ result.

FS vs FT: definition

Fourier Series: for periodic signals

decompose into $e^{jk\omega_0 t}$: freq of $0, \omega_0, 2\omega_0, 3\omega_0 \ldots$

multiple, discrete

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad \text{synthesis}$$

$$a_k = \frac{1}{T_0} \int_{\boxed{T_0}} x(t) e^{-jk\omega_0 t} \quad \text{analysis}$$

$a_k \sim e^{jk\omega_0 t}$

a_k, k ⇒ a_k, $k\omega_0$

dimension rad/s → ω

Fourier Transform: for "all" signals, often aperiodic

decompose into conti. freq: ω. $\omega+\delta$. $\omega+2\delta$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega \quad \text{synthesis}$$

$$X(\omega) = \int_{\boxed{-\infty}}^{+\infty} x(t) e^{-j\omega t} dt \quad \text{analysis}$$

← infinite

$x(t)$, t — time — $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$ — freq. domain: w.r.t ω. — $X(\omega)$, ω

: represent the same thing. different view

complex → $X(\omega) = Re\{X(\omega)\} + j\, Im\{X(\omega)\}$

$= \underbrace{|X(\omega)|}_{\geq 0}\, e^{j\underbrace{\sphericalangle X(\omega)}_{\text{phase}}}$

Im, $X(\omega)$, θ, Re

FS vs FT: for periodic signal

FT: $X(\omega) = \int_{-\infty}^{+\infty} x(t)\cdot e^{-j\omega t}\,dt$ — unwise, not integrable

FS: $a_k = \int_{T_0} x(t)\cdot e^{-jk\omega_0 t}\,dt$

FOURIER TRANSFORM OF A PERIODIC SIGNAL $\tilde{x}(t)$

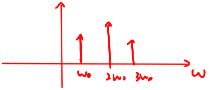
$\tilde{x}(t) \longleftrightarrow a_k$ **Fourier series coefficients**

$\tilde{x}(t) \overset{\mathcal{F}}{\longleftrightarrow} \tilde{X}(\omega)$ **Fourier transform**

Therefore, claim:

FS coeff.

$$\boxed{\tilde{X}(\omega) \triangleq \sum_{k=-\infty}^{+\infty} 2\pi \boxed{a_k}\, \delta(\omega - k\omega_0)}$$



Although $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$.

In fact, prove $x(t) \overset{\mathcal{F}}{\longleftarrow} X(\omega)$

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{X}(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} 2\pi a_k \underbrace{\int_{-\infty}^{+\infty} \delta(\omega - k\omega_0)\, e^{-j\omega t}\, d\omega}_{e^{-jk\omega_0 t}}$$

$= \sum_k a_k \cdot e^{-jk\omega_0 t}$ — FS

$= x(t)$ ✓ checked.

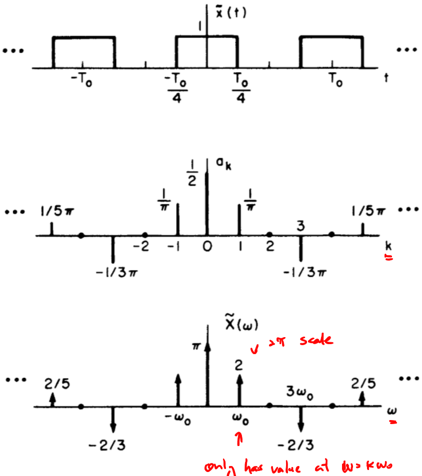
FS vs FT: for periodic signal - Example

$$\tilde{X}(w) = \sum_k 2\pi \cdot a_k \cdot \delta(w - kw_0)$$

Symmetric square wave

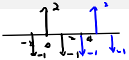
FS:

FT



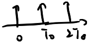
FS vs FT: for periodic signal - Quiz4 Q1

quite in advance, fine if you understand before final



[10] Find the FT of the following signal: $x(t) = \sum_{n=-\infty}^{\infty} 2\delta(t-6n) - \delta(t-6n-2) - \delta(t-6n+2)$. sketch the magnitude of the spectrum.

1st : FS table. $\sum_n x_0 \delta(t-nT_0) \leftrightarrow C_k = \frac{x_0}{T_0}$



$x_1(t) = \sum_n \delta(t-6n) \leftrightarrow C_k = \frac{1}{6}$ $\quad w_0 = \frac{2\pi}{T_0} = \frac{\pi}{3}$

$x_1(t-2) = \sum_n \delta(t-6n-2) \leftrightarrow C_k \cdot e^{-jkw_0 t_0} = C_k \cdot e^{-jk\cdot\frac{\pi}{3}\cdot 2}$

$x_1(t+2) = \sum_n \delta(t-6n+2) \leftrightarrow C_k \cdot e^{-jkw_0 t_0} = C_k \cdot e^{+jk\frac{\pi}{3}\cdot 2}$

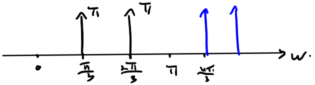
$\Rightarrow \quad x(t) = 2x_1(t) - x_1(t-2) - x_1(t+2) \leftrightarrow a_k = C_k\left(2 + e^{-jk\frac{2\pi}{3}} + e^{+jk\frac{2\pi}{3}}\right)$

$= \frac{1}{3}\left[1 - \cos\left(k\frac{2\pi}{3}\right)\right]$

2nd : FT for periodic :

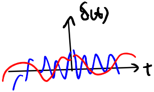
$$X(w) = \sum_k 2\pi \cdot a_k \cdot \delta(w - kw_0)$$

$$= \sum_k 2\pi \times \frac{1}{3} \times \left[1 - \cos\left(k\frac{2\pi}{3}\right)\right] \cdot \delta\left(w - k\cdot\frac{\pi}{3}\right)$$



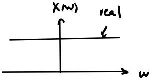
Common FT Pairs

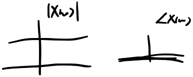
$x(t)$ $X(\omega)$

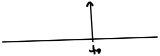


$$\delta(t) \overset{\mathscr{F}}{\longleftrightarrow} 1$$

consists of all freq.

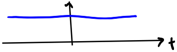






$$\delta(t-t_0) \overset{\mathscr{F}}{\longleftrightarrow} e^{-jk\omega t_0}$$

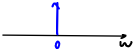
not much diff. only phase change. $|X(\omega)|$ same. $\angle X(\omega)$



const. →

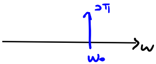
$$1 \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega)$$

only DC term ($\omega = 0$)



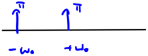
$$e^{j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$$

freq only at ω_0



freq only at $\pm\omega_0$.

$$\cos(\omega_0 t) \overset{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



Important in Chap. 8

Properties

- Time shift:

$$f(t-\tau) \stackrel{\mathscr{F}}{\longleftrightarrow} F(\omega)e^{-j\omega\tau}$$ – only phase change

- Time reversal:

$$f(-t) \stackrel{\mathscr{F}}{\longleftrightarrow} F(-\omega)$$

- Time Differentiation:

$$\frac{d^n}{dt^n}f(t) \stackrel{\mathscr{F}}{\longleftrightarrow} (j\omega)^n F(\omega)$$ – $\omega^n F(\omega)$

- Freq. Differentiation:

$t^n \cdot f(t)$ ~ $$(-jt)^n f(t) \stackrel{\mathscr{F}}{\longleftrightarrow} \frac{d^n}{d\omega^n}F(\omega)$$

- Hermitian: If f(t) real, then

$$F(-\omega) = F^*(\omega)$$

Exercise - Use FT Table

↓ hasn't been released

HW4 Q3

$x(t)$
$\shortparallel$

Find the FT of $t^2 e^{-(t/2)^2}$

Hint:

- $e^{-bt^2} \overset{\mathscr{F}}{\longleftrightarrow} \sqrt{\pi/b}e^{-\omega^2/4b}$
- $(-jt)^n f(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{d^n}{d\omega^n} F(\omega)$

$f(t) = e^{-\frac{t^2}{4}}$, $F(\omega) = \sqrt{4\pi} \cdot e^{-\omega^2}$

$b = \frac{1}{4}$: $e^{-\frac{t^2}{4}} \overset{F}{\longleftrightarrow} \sqrt{4\pi} \cdot e^{-\omega^2}$

$n=2$: $j^2 t^2 \cdot f(t) \overset{F}{\longleftrightarrow} \frac{d^2}{d\omega^2} F(\omega)$

$\Rightarrow$ $t^2 \cdot f(t) \overset{F}{\longleftrightarrow} -\frac{d^2}{d\omega^2} F(\omega)$

$\underset{x(t)}{\shortparallel}$ $\underset{X(\omega)}{\shortparallel}$,

Then, $X(\omega) = -\frac{d^2}{d\omega^2}\left(\sqrt{4\pi} \cdot e^{-\omega^2}\right)$

Summary

- FS vs FT — similarity
- FS Table, FT Table — Table look up
- Next week:
  - Closure of FT
  - Focus on Filtering (problem-solving) $x(t),\ h(t) \rightarrow y(t)$
- The place we are in the big picture $X(\omega)$. — next week
- Actually, we will spend a whole month on the applications related to FS and FT

Up to now. lots of concepts on FS. FT, then Chap. 6~8. applications

The End