VE216 Recitation Class 8

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VE216 SU20 TA Group

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Overview

1. Chapter 7: Sampling
   - Sampling Theorem
   - Reconstruction via Interpolation
2. Conclusion

Sampling Theorem

Sampling Theorem

- Equally Spaced Samples of $x(t)$

  $x(nT) \quad n = 0, \pm 1, \pm 2 \ldots$

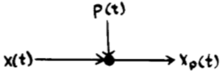
- $x(t)$ Band limited — composed of low freq.

  $X(\omega) = 0 \quad |\omega| > \omega_M$

sample fast enough

If $\frac{2\pi}{T} \triangleq \omega_s > 2\omega_M$

Then $x(t)$ uniquely recoverable



$$\rightarrow x_p(t) = x(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t - nT)$$

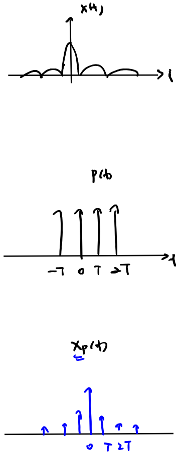
$$X_p(\omega) = \frac{1}{2\pi}\left[X(\omega) * P(\omega)\right]$$

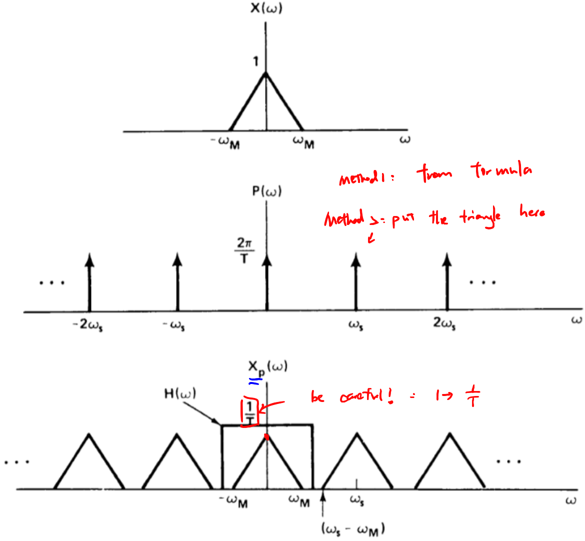
$$P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$\rightarrow X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$

Recoverable

$x(t) = \text{sinc}^2(\cdots)$



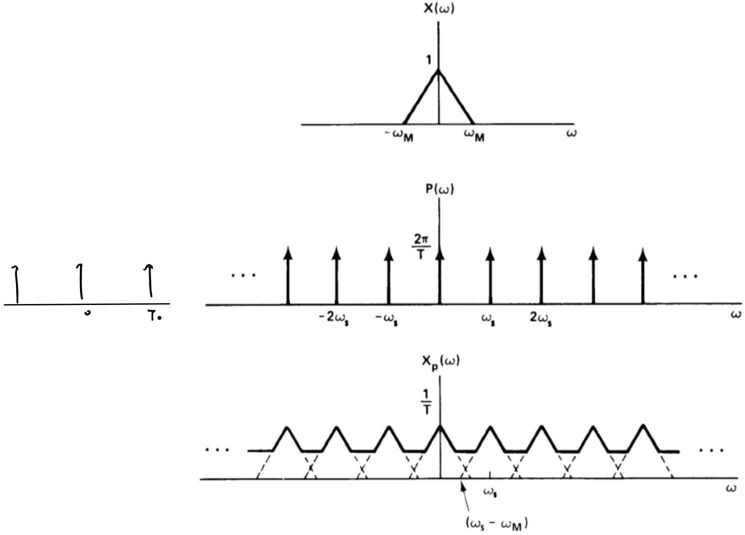


Method 1: from formula

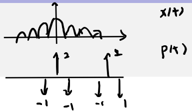
Method 2: put the triangle here

be careful! : $1 \to \frac{1}{T}$

Aliasing



Exercise - Quiz 4 Q2



$x(t) = \mathrm{sinc}^2(xxx)$

A signal $x(t)$ with spectrum $X(\omega) = (1 - 4|\omega|)\,\mathrm{rect}(2\omega)$ is modulated by the following modified impulse train: $p(t) = \sum_{n=-\infty}^{\infty} 2\delta(t-5n) - \delta(t-5n-1) - \delta(t-5n+1)$. Determine and sketch the magnitude spectrum of the resulting signal.

1. FS: $T_s = 5$, $\omega_s = \frac{2\pi}{5}$

$$\sum_n \delta(t-5n) \leftrightarrow C_k = \frac{1}{5}$$

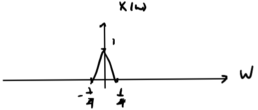
$$\sum_n \delta(t-5n-1) \leftrightarrow C_k \cdot e^{-jk\omega_s t_0} = C_k \cdot e^{-jk\frac{2\pi}{5}\cdot 1}$$

$$\sum_n \delta(t-5n+1) \to \cdots = C_k \cdot e^{jk\frac{2\pi}{5}}$$

$$\Rightarrow C_k = C_k\left(2 - e^{-jk\frac{2\pi}{5}} - e^{jk\frac{2\pi}{5}}\right) = \frac{2}{5}\left(1 - \cos k\frac{2\pi}{5}\right)$$

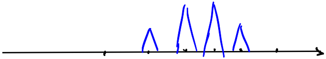
2. FS to FT: $P(\omega) = \sum_k 2\pi \cdot C_k \cdot \delta(\omega - k\omega_0) = \sum_k \frac{4\pi}{5}\left(1 - \cos k\frac{2\pi}{5}\right) \cdot \delta\left(\omega - k\cdot\frac{2\pi}{5}\right)$ – period: see plot

3. Plot:



$X(\omega) = |1 - 4|\omega|| \cdot \mathrm{rect}\left(\frac{\omega}{\frac{1}{2}}\right)$

$P(\omega)$



$Y(\omega) = \frac{1}{2\pi}\left[X(\omega) * P(\omega)\right]$

be careful

put triangles on to $P(\omega)$

Exercise - Quiz 6

$X(w) = 2\pi\cdot\delta(w+45) + 2\pi\delta(w+35) + 2\pi\delta(w+25) + 2\pi\delta(w+15) + \cdots + 2\pi\delta(w-45)$

- FT table

Consider the input signal $x(t)$ below

$$x(t) = e^{-j45t} + e^{-j35t} + e^{-j25t} + e^{-j15t} + e^{-j5t} + e^{j5t} + e^{j15t} + e^{j25t} + e^{j35t} + e^{j45t}.$$

- symmetric/even

The signal $x(t)$ is first input to an analogy filter with impulse response

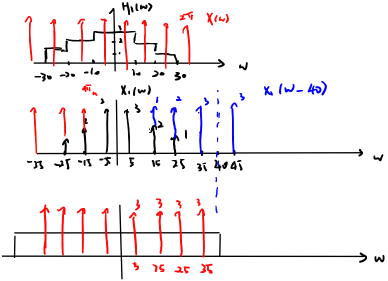
tedious. be careful!

$$h_1(t) = \frac{\sin(10t)}{\pi t} + \frac{\sin(20t)}{\pi t} + \frac{\sin(30t)}{\pi t} \quad \overset{FT}{\longleftrightarrow} \quad H_1(w) = \mathrm{rect}\left(\frac{w}{20}\right) + \mathrm{rect}\left(\frac{w}{40}\right) + \mathrm{rect}\left(\frac{w}{60}\right)$$

to form an output $x_1(t)$, and then $x_1(t)$ is sampled at a rate of $\omega_s = 40$ to form a sampled signal $x[n]$. The signal $x[n]$ thus obtained is then input to a lowpass filter with impluse response $h_2(t) = \frac{\sin(40t)}{\pi t}$ to reconstruct a single $z(t)$.

$\longleftrightarrow$ $H_2(w) = \mathrm{rect}\left(\frac{w}{80}\right)$

- [10 points] Find the Fourier Transform $x_1(t)$.
- [10 points] Write your expression for $z(t)$. Simply your result when possible.



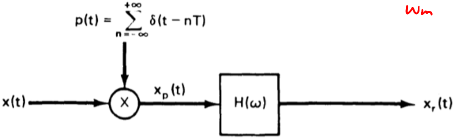
$X_p(w)$

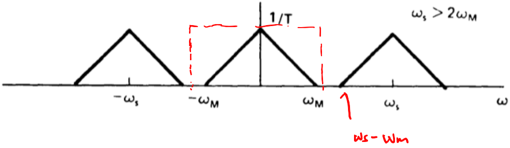
$Z(w) \longleftrightarrow z(t) = \frac{3}{2\pi}\left(e^{-j35t} + \cdots + e^{j35t}\right)$

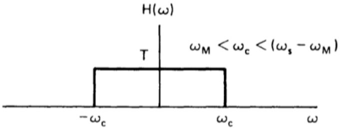
Ironically, lowpass of $x(t)$ after so many steps!

Sampling & Reconstruction

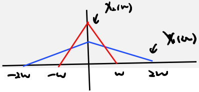
Wm vs Ws vs. Wc







Exercise - Sampling



6. [12]Consider the system in Figure 0506.

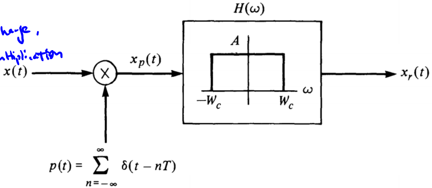
If $X_1(\omega) = 0$ for $|\omega| > 2W$ and $X_2(\omega) = 0$ for $|\omega| > W$. For the following inputs $x(t)$, find the ranges for the cutoff frequency W_c in terms of T and W and find the maximum values of T and A, such that $x_r(t) = x(t)$.

(a) $x(t) = x_1(t - \pi/2) + x_2(t)$

$X(\omega) = X_1(\omega) \cdot e^{-j\omega\cdot\frac{\pi}{2}} + X_2(\omega)$

ω_m don't change, ∵ pointwise -multiplication

$f(t-\tau) \Leftrightarrow F(\omega)\cdot e^{-j\omega\tau}$



$$\begin{cases} 2W < W_c < W_s - 2W \\ T = A \\ W_s > 2\times(2W) \end{cases}$$

$W_s = \frac{2\pi}{T}$

Sinc interplation (Ideal lowpass filter)

Review: Freq. domain → lowpass

Now: understand in time domain

Sampling: $x_p(t) \quad x(t)\,p(t)$

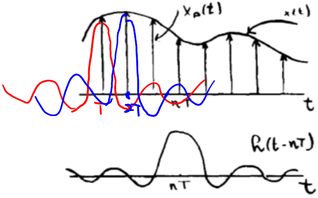
$$= x(t)\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t-nT)$$

reconstruction: $x_r(t) = x_p(t) * h(t)$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,h(t-nT)$$

Sum of h(t)s



For $H(\omega)$ an ideal lowpass filter with cutoff frequency ω_c,

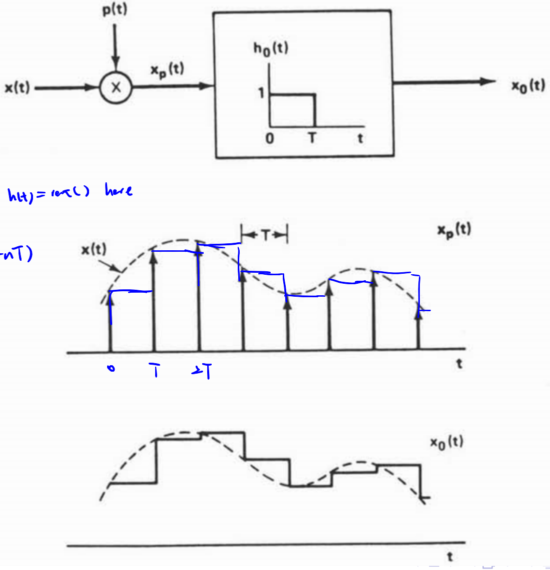
$$h(t) = T\frac{\omega_c}{\pi}\,\mathrm{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

Nearest neighbor interplation (zero-order hold filter)

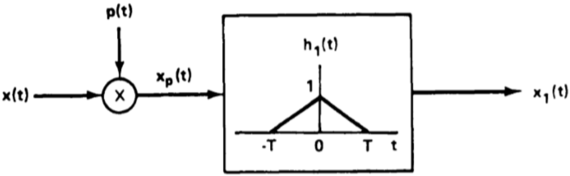
plug in h(t) = rect( ) here

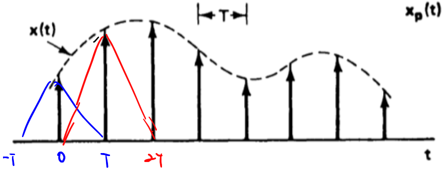
↓

$x_r(t) = \sum_n x(nT)\cdot h(t-nT)$



Linear interplation (first-order hold filter)





Add two lines together

⇕

connect two points

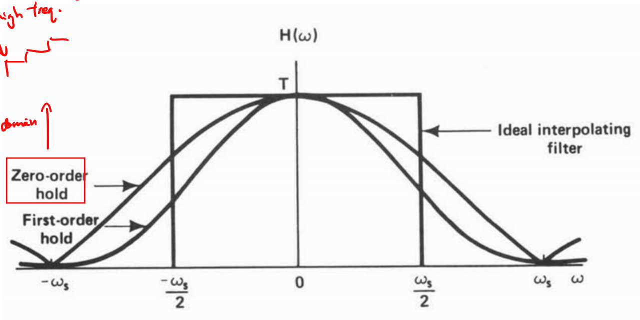
(verify by calculation)

Comparison in Frequency Domain

makes sense !

edge : high freq.

time domain



Exercise - Interplation

7. [5]Suppose we have the system in Figure 0507(a) and 0507(b), in which $x(t)$ is sampled with an impulse train. Sketch $x_p(t)$, $y(t)$ and $w(t)$. State your reasoning.

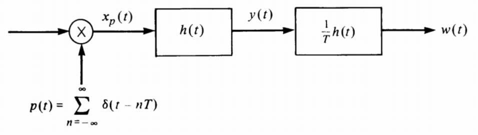
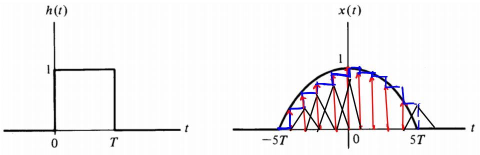


Figure 0507(a).



1st: sample.

2nd: zero-order hold.

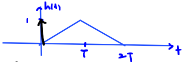
3rd: view in this way:

$h(t) * \frac{1}{T} h(t)$:

$\mathrm{rect}(t) * \mathrm{rect}(t) = \mathrm{tri}(t)$

$\Downarrow$

$\mathrm{rect}(t-\frac{T}{2}) * \mathrm{rect}(t-\frac{T}{2}) = \mathrm{tri}(t-T)$



$\delta(t) * \mathrm{tri}(t) = \mathrm{tri}(t)$

$\Downarrow$

$\delta(t) * \mathrm{tri}(t-T) = \mathrm{tri}(t-T)$

Conclusion

- Both sampling and reconstruction are indispensible
- Consider eye (watching the wheels) and ear (ultrasonic)
- For sampling & reconstruction-related problems, I prefer to view them graphically (often in freq. domain).
- Questions in HW5 are very interesting (or, tricky) - not much math involved, rather requires understanding

The End