EDS 212 - Essential Math in Environmental Data Science

Day 4 Part 1 - Linear algebra continued

Part 1: Linear algebra continued

- Refresher: vectors and working with them
- Matrices: notation, language, basic algebra
- Representing systems of equations with matrices
- Linear algebra in environmental science

Matrices

A matrix is a table of values (multiple vectors in combination). A vector, therefore, can be thought of as a matrix with a single column.

- Dimensions: the size of the matrix, in rows x columns (*m* x *n*)
- Elements: values in a matrix, often denoted symbolically with a subscript where the first number is the *row* and the second number is the *column* (e.g. a_{34} indicates the element in row 3, column 4)

Matrix algebra

Addition & subtraction

Add or subtract the corresponding elements (by matrix position) to create a new matrix of the same dimensions.

$$\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 5 & 8 \end{bmatrix}$$

Scalar multiplication

To multiply a matrix by a *scalar*, multiply each element in the matrix by the scalar to get a scaled matrix of the same dimensions.

For example:

$$6 \times \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -18 \\ 0 & 12 \end{bmatrix}$$

Recall: dot product

The dot product of two vectors is the sum of their elements multiplied:

For $\vec{a} = [1, 5]$ and $\vec{b} = [2, -3]$:

$$\vec{a} \cdot \vec{b} = (1)(2) + (5)(-3) = -13$$

Matrix multiplication

We find the *dot product* of row · column vectors:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

Practice problem:

$$\begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 5 \\ 6 & -3 \end{bmatrix} = ?$$

Critical thinking: Matrices with unequal dimensions

What do you think the output matrix would contain if you were multiplying the following?

Diagonal matrix

A **diagonal matrix** is (almost always) a square matrix (\$m\$ = n) where only elements on the diagonal are non-zero values.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

What happens when we multiply a matrix by a diagonal matrix?

A diagonal matrix is also called a *scaling matrix* because it scales rows proportionally, but not by the same value:

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \times \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 8 & 4 \end{bmatrix}$$

Matrices as systems of equations

Often in environmental data science, we have multiple equations representing processes. Matrices give us a way to express these *systems of equations* in data structures that are easy to store and work with in data science. For example, let's say we have a system:

$$3x - 8y = 5$$

$$x + 2y = 10$$

How can we write this using matrices?

Rewriting in matrix form:

$$3x - 8y = 5$$

$$x + 2y = 10$$

The matrix form of this system of equations looks as follows:

$$\begin{bmatrix} 3 & -8 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Example: matrices and linear algebra in environmental science

Leslie Matrix: Population ecology

A matrix model that accounts for survival / fecundity rates at different life stages for a species.

Overview:

- Define life stages
- Estimate probability of survival / reproduction at different life stages to create a matrix over time

Writing estimates as equations:

- For our species, each adult female will lay ~600 eggs during each cycle (let's say that's a year). Which means that the eggs at time $t+1$ can be estimated by the number of adult females * 600:

$$E_{t+1} = 600 * F_t$$

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We also estimate that 20% of eggs survive to reach larval stage:

$$L_{t+1} = 0.2 * E_t$$

$$E_{t+1} = 600 * F_t$$

$$L_{t+1} = 0.2 * E_t$$

We also estimate that 8% of those that reach larval stage will survive to become reproducing female adults:

$$F_{t+1} = 0.08 * L_t$$

$$E_{t+1} = 600 * F_t$$

$$L_{t+1} = 0.2 * E_t$$

$$F_{t+1} = 0.08 * L_t$$

How can we write this in matrix form?

$$\begin{bmatrix} E_{t+1} \\ L_{t+1} \\ F_{t+1} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} \text{Egg stage} & \text{Larval stage} & \text{Adult stage} \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \end{array} \begin{bmatrix} E_t \\ L_t \\ F_t \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc}\text{Egg stage} & \text{Larval stage} & \text{Adult stage}\end{array} \\ \begin{bmatrix}E_{t+1}\\ L_{t+1}\\ F_{t+1}\end{bmatrix} = & \begin{bmatrix}0 & 0 & 600\\ 0.2 & 0 & 0\\ 0 & 0.08 & 0\end{bmatrix}\begin{bmatrix}E_t\\ L_t\\ F_t\end{bmatrix}\end{array}$$

See also: Bison model

https://www.google.com/url?
sa=t&rct=j&q=&esrc=s&source=web&cd=&ved=2ahUKEwjy2ZDKuOPxAhUJ6J4K