1. **LT751** 10 numbers, $a_1, a_2, \cdots, a_{10}$ are written in a row on a blackboard. For any integer s in the range $1 \leq s \leq 10$, the average of the first s numbers is equal to s^2. What is $a_5 + a_{10}$?

   Written by: Linus Tang

   Answer: $\boxed{332}$

   The sum of the first n numbers would be $n \cdot n^2 = n^3$, so we have

   $$a_5 = (a_5 + a_4 + ... + a_1) - (a_4 + a_3 + a_2 + a_1) = 5^3 - 4^3 = 61$$

   and

   $$a_{10} = (a_{10} + a_9 + ... + a_1) - (a_9 + a_8 + ... + a_1) = 10^3 - 9^3 = 271$$

   The final answer is hence $61 + 271 = \boxed{332}$.

2. **LT269** Cameron the Colt went galloping every day over the last 7 days. He galloped at least 3 miles every day and at most 10 miles over any 2 consecutive days. What is the maximum possible number of miles that Cameron the Colt could have galloped over all 7 days?

   Written by: Linus Tang

   Answer: $\boxed{37}$

   For convenience, call the days Sunday, Monday, Tuesday, ..., Saturday. Since Cameron the Colt galloped at most 10 miles over any 2 consecutive days, he galloped at most 30 miles in 6 consecutive days, Monday through Saturday.

   Also, Cameron the Colt galloped at most 10 total miles on Sunday and Monday, at least 3 of which were on Monday. Thus, he galloped at most 7 miles on Sunday.

   Adding 7 miles on Sunday to 30 miles on the rest of the week, the maximum possible number of miles galloped is $\boxed{37}$.

   This is achieved by galloping 7,3,7,3,7,3, and 7 miles each day.

3. **LT293** What is the value of $\dfrac{11 \times 12 \times 13 \times 14 \times \cdots \times 20}{1 \times 3 \times 5 \times 7 \times \cdots \times 19}$?

   Written by: Linus Tang

   Answer: $\boxed{1024}$

   The numerator is the expression for 20!, except with factors $1, 2, 3, \cdots, 10$ missing. The denominator is the expression for 20!, except with factors $2, 4, 6, \cdots, 20$ missing.

   We can now rewrite the fraction:

   $$\frac{11 \times 12 \times 13 \times 14 \times \cdots \times 20}{1 \times 3 \times 5 \times 7 \times \cdots \times 19} = \frac{20!/(1 \times 2 \times 3 \times 4 \times \cdots \times 10)}{20!/(2 \times 4 \times 6 \times 8 \times \cdots \times 20)} = \frac{2 \times 4 \times 6 \times 8 \times \cdots \times 20}{1 \times 2 \times 3 \times 4 \times \cdots \times 10}$$

$$= \frac{2 \times 2 \times 2 \times 2 \times \cdots \times 2}{1 \times 1 \times 1 \times 1 \times \cdots \times 1} = 2^{10} = \boxed{1024}$$

4. **LT741** A cricket starts at the point $(1, 1)$ on the coordinate plane. Every minute, if the cricket is on the point (x, y) and its distance to the origin $(0, 0)$ is r, it hops to the point $(x + y - r, x + y + r)$. How far is the cricket from the origin after 9 hops?

   Written by: Linus Tang

   Answer: $\boxed{32\sqrt{1023}}$

   The key observation is that both the sum and the product of the cricket's coordinates double with each hop. To confirm this, we see that

$$(x + y - r) + (x + y + r) = 2(x + y),$$

   where (x, y) are the coordinates before a given hop and $(x + y - r, x + y + r)$ are the coordinates after the hop. Also,

$$(x + y - r)(x + y + r) = (x + y)^2 - r^2 = (x + y)^2 - x^2 - y^2 = 2xy.$$

   This confirms that the sum and product of the coordinates indeed double with each hop. Thus, after nine hops, the sum $x + y$ of the cricket's coordinates is $2 \cdot 2^9 = 1024$ and the product xy is $1 \cdot 2^9 = 512$.

   Therefore, the cricket's distance from the origin is $\sqrt{x^2 + y^2} = \sqrt{(x + y)^2 - 2xy} = \sqrt{1024^2 - 2(512)} = \boxed{32\sqrt{1023}}$.

5. **LT274** How many ways are there to arrange the letters in $BANANAS$ such that two A's never appear next to each other?

   Written by: Linus Tang

   Answer: $\boxed{120}$

   Notice that if we first replace all consonants with C's, we have that we are trying to rearrange $CACACAC$ such that no two A's are next to each other. This is equivalent to making sure there is at least 1 C between each A. We can first place the four C's, leaving five gaps for three A's to be placed. The A's must go in different gaps, otherwise we willl have two consecutive A's. Thus, there are $\binom{5}{3} = 10$ ways to place the A's.

   Now, there are $\frac{4!}{2} = 12$ ways to substitute the C's with the original consonants B, N, N, and S, so our answer is $12 \times 10 = \boxed{120}$.

6. **MK800** Gerald rolls a standard six-sided die and lands on the number k. He then rerolls the dice k times. What is the expected number of primes that Gerald rolls, including his initial roll?

   Written by: Matthias Kim

Answer: $\boxed{\dfrac{9}{4}}$

Since there are 3 primes out of 6 faces on the die, the probability of rolling a prime number on a single roll is $\frac{3}{6} = \frac{1}{2}$.

The expected value of the first roll, k, is $\frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$. This means that the expected number of rolls left after the first roll is $\frac{7}{2}$.

Since Gerald rolls the die $k+1$ times in total, the expected number of rolls is $\frac{7}{2}+1 = \frac{9}{2}$. The probability of a prime on each roll is $\frac{1}{2}$, so the expected number of primes rolled is $\frac{9}{2} \cdot \frac{1}{2} = \boxed{\dfrac{9}{4}}$.

7. **CY592** At a party, $\frac{7}{17}$ of the people are wearing green jackets. An additional 300 people arrive on a bus. Now, $\frac{7}{12}$ of the people are wearing green jackets. What is the smallest possible number of people that were wearing green jackets on the bus?

   Written by: Carsten Yeung

   Answer: $\boxed{210}$

   If there is x people initially in the party, then

   $$\frac{7}{12}(x+300) - \frac{7}{17}x = x\frac{35}{12 \cdot 17} + 175$$

   people wearing green jackets on the bus. For this expression to be an integer and minimized, $x = 12 \cdot 17$, leading our answer to be $35 + 175 = \boxed{210}$.

8. **LT286** A 5 by 5 grid of square cells is initially empty. How many ways can four identical kings be placed on the grid such that every empty cell shares at least one corner with a cell occupied by a king? Rotations and reflections are considered distinct.

   Written by: Linus Tang

   Answer: $\boxed{79}$

   For the sake of clarity, put this 5x5 grid such that the origin is on the center square. Notice that there must be one chess piece next to each corner. This means that each chess piece must lie strictly in each quadrant. Now let the points be $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ for the first through fourth quadrant. Notice that the axis must also be covered. Thus, at least one of x_1, x_2 is right next to the positive y-axis, at least one of y_1, y_4 must be next to the positive x-axis, one of y_2, y_3 must be next to the negative x-axis, and at least one of x_3, x_4 is next to the negative y-axis. For each pair, notice that there is exactly one solution that doesn't work: $(-2, 2)$. Thus there are $2^2 - 1 = 3$ ways for each pair so that each axis is covered. Altogether, $3^4 = 81$ ways.

   But there are two cases such that the center isn't covered: $(2,1), (-1,2), (-2,-1), (1,-2)$ and $(1,2), (-2,1), (-1,-2), (2,-1)$. Thus, our answer becomes $\boxed{79}$.

9. **MLI317** In square $ABCD$, midpoints E and F are drawn on $\overline{BC}$ and $\overline{CD}$ respectively. Compute the ratio between the area of $\triangle AEF$ to $ABCD$.

Written by: Michael Liu

Answer: $\boxed{\frac{3}{8}}$

Let side length $AB = x$. Therefore, we know the area of $ABCD = x^2$. Note that the area of $\triangle AEF$ is equal to $[ABCD] - [ADF] - [ABE] - [EFC]$. Since E and F are midpoints of sides of the square, we know that $DF = FC = CE = BE = \frac{x}{2}$. With all this information, we can now determine the areas of triangles in terms of x. We can now find that $[ADF] = [ABE] = x \cdot \frac{x}{2} \cdot \frac{1}{2} = \frac{x^2}{4}$ and $[EFC] = \frac{x}{2} \cdot \frac{x}{2} \cdot \frac{1}{2} = \frac{x^2}{8}$. Tberefore, $[AEF] = x^2 - 2 \cdot \frac{x^2}{4} - \frac{x^2}{8} = \frac{3x^2}{8}$, so our final answer is $\frac{\frac{3x^2}{8}}{x^2} = \boxed{\frac{3}{8}}$.

10. **TK959** Let Ω be a circle of radius 21. Circles ω_1 of radius 6 and ω_2 of radius 8 are internally tangent to Ω and externally tangent to each other. A chord of Ω with length L is tangent to both ω_1 and ω_2, which are on opposite sides of the chord. What is L?

Written by: Tristan Kay

Answer: $\boxed{24\sqrt{3}}$

Let O, O_1, and O_2 be the centers of circles Ω, ω_1, and ω_2, respectively.

Using the radii of the three circles along with the given tangencies between them, we can derive the following lengths: $O_1O_2 = 6 + 8 = 14$ because ω_1 and ω_2 are externally tangent. $OO_1 = 21-6 = 15$ because Ω and ω_1 are internally tangent. $OO_2 = 21-8 = 13$ because Ω and ω_2 are internally tangent. Using this information, we can find the distance between O and the chord.

Let P be the common point of tangency between ω_1, ω_2, and the chord. Let ℓ_1 be a line passing through O perpendicular to O_1O_2 and ℓ_2 be the line containing the chord. These lines are parallel to each other, since both are perpendicular to O_1O_2.

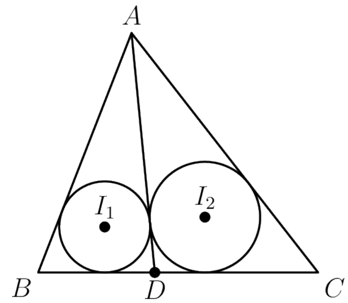
Since OO_1O_2 is a triangle with side lengths $13, 14$, and 15, it is known that the altitude ℓ_1 divides side O_1O_2 into parts of length 9 and 5. Also, ℓ_2 divides side O_1O_2 into parts of length 6 and 8. Thus, the distance between parallel lines ℓ_1 and ℓ_2 is 3.

Thus, the distance between O (which lies on ℓ_1) and the chord (which lies along ℓ_2) is 3. Since Ω has radius 21, the length of the chord is $2\sqrt{21^2 - 3^2} = \boxed{24\sqrt{3}}$.

11. **LT733** Let $\triangle ABC$ be a triangle with $AB = 11$, $BC = 12$, and $CA = 13$. Let D be a point on side $\overline{BC}$, and let I_1 and I_2 be the incenters (centers of the circles inscribed in) triangles ADB and ADC. If $\overline{I_1I_2}$ is perpendicular to $\overline{AD}$, what is the length of $\overline{BD}$?

Written by: Linus Tang

Answer: $\boxed{5}$



Since $\overline{I_1 I_2}$ is perpendicular to $\overline{AD}$, the incircles of ADB and ADC are tangent to AD at the same point, X.

Focusing on $\triangle ABD$, we have $DX = \frac{AD+BD-AB}{2}$. Focusing on $\triangle ACD$, we have $DX = \frac{AD+CD-AC}{2}$. Equating these expressions and multiplying by 2, $BD - AB = CD - AC$.

Now we have $BD - CD = AB - AC = 11 - 13 = -2$ and $BD + CD = BD = 12$, so $BD = \boxed{5}$ and $CD = 7$.

12. **YS964** Let $\triangle ABC$ have $AB = 13$, $BC = 14$, and $AC = 15$. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$. If a point P is uniformly randomly selected inside $\triangle ABC$, what is the expected area of $\triangle MNP$?

Written by: Yuuki Sawanoi

Answer: $\boxed{\dfrac{21}{2}}$

Since triangles AMN and ABC are similar in a $1:2$ ratio, this means that $MN = \frac{1}{2}BC = 7$.

Now we need to find the expected height h of $\triangle MNP$. Clearly, the minimum possible value is 0, and its maximum value is half the length of the altitude from A to BC. Using Heron's Formula, we find that $[ABC] = 84$, so the altitude has a length of $\frac{2[ABC]}{BC} = \frac{2(84)}{14} = 12$. This means the height of $\triangle MNP$ ranges on the interval $[0, 6]$. We will prove each value on this interval is equally likely.

For each $h \in [0, 6]$, the region that P can lie is defined by two horizontal line segments across the triangle, a distance h from the median on either side. The average length of these two lines will always be the length of the median, so we can conclude that each possible value of h is equally likely. This means the expected height is the midpoint of the interval, which is 3.

Thus, the expected area is $\frac{1}{2} \cdot 7 \cdot 3 = \boxed{\dfrac{21}{2}}$.

13. **TK893** The *importance* of a point (x, y) in the coordinate plane is $4x - 3y$. How

many lattice points (a, b) with $1 \leq a, b \leq 100$ are there with importance values that are positive factors of 100?

Written by: Tristan Kay

Answer: $\boxed{225}$

There are two main observations that need to be made. The first is that if (a, b) has an importance value of k, all points with importance value of k can be parameterized as $(a + 3n, b + 4n)$ for some integer n. The other observation is that (k, k) has an importance value of k. Thus, all points with importance value k are of the form $(k+3n, k+4n)$ for some $n \in \mathbb{Z}$. It follows that there are 25 lattice points with x and y coordinates between 1 and 100 with importance value k, as one exists for every $y \equiv k \pmod 4$. There are 9 positive factors of 100, so our answer is $9 \times 25 = \boxed{225}$.

14. **LT268** Let a and b be not necessarily distinct positive divisors of 42. What is the sum of the distinct possible values of $\frac{a}{b}$?

Written by: Linus Tang

Answer: $\boxed{\dfrac{247}{2}}$

First, we prime factorize $42 = 2^1 \cdot 3^1 \cdot 7^1$. The possible values of $\frac{a}{b}$ are all rational numbers of the form $2^a 3^b 7^c$ with (not necessarily nonnegative) integer values of a, b, c such that $|a| \leq 1$, $|b| \leq 1$, and $|c| \leq 1$. The sum of these values is given by the factorization $N = (2^{-1} + 2^0 + 2^1)(3^{-1} + 3^0 + 3^1)(7^{-1} + 7^0 + 7^1)$. To see why, notice that each of the 3^3 terms of the form $2^a 3^b 7^c$ described above appears exactly once when this product is expanded.

Now, evaluate N:

$$\begin{aligned} N &= \left(\frac{1}{2} + 1 + 2\right)\left(\frac{1}{3} + 1 + 3\right)\left(\frac{1}{7} + 1 + 7\right) \\ &= \left(\frac{7}{2}\right)\left(\frac{13}{3}\right)\left(\frac{57}{7}\right) \\ &= \frac{13 \cdot 19}{2} \\ &= \frac{247}{2} \end{aligned}$$

Therefore, the sum of all possible values of $\frac{a}{b}$ is $\boxed{\dfrac{247}{2}}$.

15. **YS936** What digit O makes the 5-digit number $2O23O$ divisible by every answer choice except O?

Written by: Yuuki Sawanoi

Answer: $\boxed{4}$

One can simply try each possibility to get $\boxed{4}$. Note that 7 is clearly false as $7|2023$ and 3 and 6 are impossible due to logic as both are multiples of 3.

16. **TK892** A *meaningful* number is a number whose prime factors sum to 42. What is the sum of the two smallest meaningful numbers?

Written by: Tristan Kay

Answer: $\boxed{407}$

If there are only two prime factors of a meaningful number, the meaningful number is minimized when they are as far apart as possible. The pair of primes that are most far apart that sum to 42 is 5 and 37. The next pair of primes after that is 11 and 31.

If there are three prime factors, notice one prime factor must be even. In other words, 2. Then we need a pair of primes that sum to 40, for which the most furthest apart they can be is with 3 and 37.

Looking at the above numbers, the sum of the two smallest is $5 \times 37 + 2 \times 3 \times 37 = 11 \times 37 = \boxed{407}$.