Factors

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1 Introduction

Every number can be written as a product of groups of positive (and negative) factors. However, negative factors are usually not commonly used as they are nearly identical to positive ones. For example,

$$\begin{aligned} 8 &= 1 \cdot 8 = 2 \cdot 4 = 2 \cdot 2 \cdot 2 \\ 15 &= 1 \cdot 15 = 3 \cdot 5 \\ 24 &= 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6 \end{aligned}$$

These factors can be very useful, but not in their current form as composite numbers. Prime factors are much more useful than composite factors, as they give cannot be broken down into smaller factors.

2 Prime Factorization

All numbers can also be written as a product of its prime factors. This is known as a number's **Prime Factorization**.

> **Definition 2.1** (Prime Factor)
> A prime factor is a factor of a number which has no other divisors besides 1 and itself.

For example, of the factors of 12, only 2 and 3 are prime factors, since they have no other divisors besides 1 and themselves. These prime factors are very important for building prime factorizations.

> **Definition 2.2** (Prime Factorization)
> A number's prime factorization is the set of prime numbers that multiply together to equal the original number.

Using the previous numbers, we can prime factorize them:

$$\begin{aligned} 8 &= 2^3 \\ 15 &= 3 \cdot 5 \\ 24 &= 2^3 \cdot 3 \end{aligned}$$

> **Definition 2.3** (Fundamental Theorem of Arithmetic)
> Every positive integer greater than 1 can be represented by a **unique** product of prime numbers, differing only in the order.

This theorem is simple, yet important. It tells us that after factoring a number into primes, it cannot be broken down any further, nor can be it factored into different products of prime factors. These prime factorizations will prove to be very useful in many number theory problems. For instance, we can use them to find the number of factors a number has.

3 Number of Factors

Theorem 3.1

A positive integer n which has the prime factorization $p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_m^{e_m}$ has $(e_1+1)(e_2+1)...(e_m+1)$ factors, where $p_1, p_2, \ldots p_m$ are its prime factors, and $e_1, e_2, \ldots e_m$ are the corresponding powers of each prime factor.

Proof. The proof for this is rather intuitive. For any number n, we can create a factor by selecting powers of the prime factors of n and multiplying them together. For example, if we have a number $n = 2^3 \cdot 3^2 \cdot 5$, we can select a power of 2 less than or equal to 2^3, including 1, to be a part of our factor. We can also do the same for the other factors. The $+1$ in the formula comes from letting the power of the prime be 1, giving the whole power an extra case. Using the number $108 = 2^2 \cdot 3^3$, we can create factors by taking a $1, 2,$ or 4 to be the power of 2. Then, we can do the same for the power of 3.

$$\begin{gathered}1 \cdot 1 \\ 2 \cdot 1 \\ 4 \cdot 1 \\ 1 \cdot 3 \\ 2 \cdot 3 \\ 4 \cdot 3 \\ \vdots\end{gathered}$$

$\square$

We can use this theorem on any positive integer, as long as we can factor it into its prime factors. This can be done by testing for divisibility by every prime number less than or equal to $\sqrt{n}$. (This is because if a factor of n is larger than the $\sqrt{n}$, there will be factor smaller than $\sqrt{n}$ for which the two factors multiply to equal n.)

Example 3.1

Find the number of positive factors of 36.

Solution. We can prime factorize 36 to get $36 = 2^2 \cdot 3^2$. Applying **Theorem 3.1** gives the number of factors to be $(2+1)(2+1) = \boxed{9}$. Specifically, they are $1, 2, 3, 4, 6, 9, 12, 18, 36$. †

Example 3.2

Find the smallest positive integer that has 6 factors.

Solution. We know that the number of factors of a number can be represented in the form $(e_1+1)(e_2+1)...(e_m+1)$ where e_i are the exponents of its prime factors. The prime factor doesn't matter here, only the exponent does. This means, we should make the primes as small as possible to find the smallest integer. Since we need 6 factors, we get the equation

$$(e_1+1)(e_2+1)...(e_m+1) = 6.$$

We don't yet know how many e_i there are, but we know that since the product equals 6, the possible values can only be $6 \cdot 1$ or $3 \cdot 2$. However, having a 1 redundant since we would have

$$e_i + 1 = 1 \Rightarrow e_i = 0 \Rightarrow p_i^{e_i} = 1$$

for all primes p_i.

This means we only have the cases where

$$(e_1+1)(e_2+1) = 3 \cdot 2 \Rightarrow e_1 = 2, e_2 = 1$$

and

$$(e_1+1) = 6 \Rightarrow e_1 = 5.$$

Note that the order of the exponents and primes do not matter. Since we want to find the smallest value, we use the smallest primes for the largest powers e_i, so we let $p_1 = 2$ for both cases and $p_2 = 3$ for the first case. This gives us the 2 values:

$$n = 2^2 \cdot 3^1 = 12$$
$$n = 2^5 = 32.$$

Clearly, $\boxed{12}$ is the smaller value and we are done. †

Example 3.3

For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

Solution. Let $110n^3 = p_1^{e_1} \cdot p_2^{e_2} \cdot ... \cdot p_m^{e_m}$. We know that $110 = 2 \cdot 5 \cdot 11$. $110n^3$ has prime factors $2, 5, 11$, so it must have at least 3 prime factors. However, since it also has $110 = 2 \cdot 5 \cdot 11 = (e_1+1)(e_2+1)\dots(e_m+1)$ divisors, we know that the maximum amount of primes that the number can have in its prime factorization is 3. (We can't have any ones in our product for the number of divisors. This means the number of divisors of $110n^3$ is

$$110 = (e_1 + 1)(e_2 + 1)(e_3 + 1) = 2 \cdot 5 \cdot 11.$$

We can see that $e_1 = 1, e_2 = 4, e_3 = 10$, and the only 3 prime factors of $110n^3$ are $2, 5, 11$. (remember that order doesn't matter). Let

$$n = 2^{f_1} \cdot 5^{f_2} \cdot 11^{f_3}$$
$$n^3 = 2^{3 \cdot f_1} \cdot 5^{3 \cdot f_2} \cdot 11^{3 \cdot f_3}$$

$110n^3$ has

$$((3f_1 + 1) + 1)((3f_2 + 1) + 1)((3f_3 + 1) + 1) = 2 \cdot 5 \cdot 11$$

divisors. Solving for f_i, we get $0, 1,$ and 3.
Plugging in for the number of divisors of n^4, we get

$$(4 \cdot f_1 + 1) \cdot (4 \cdot f_2 + 1) \cdot (4 \cdot f_3 + 1) = 65$$

Since we are looking for the number of divisors of $81n^4 = 3^4 \cdot n^4$, we multiply by $4 + 1 = 5$ to get $\boxed{325}$. †

4 Sum of Factors

Let's see if we can come up with theorem for the sum of factors of a number.

Example 4.1

Find the sum of the factors of 48.

Solution. We can start by listing the factors of 48. We have

$$1, 2, 3, 4, 6, 8, 12, 16, 24, 48.$$

We can notice that 5 of the factors are a multiple of 3, while the rest aren't. Factoring out the 3 from the multiples of 3 gives:

$$3, 6, 12, 24, 48 \Rightarrow 1, 2, 4, 8, 16$$

The new set of numbers are all factors of 48! This motivates us to look at taking a product of smaller factors to form the larger factors. For example, we can sum all of the above factors with the expression:

$$(1 + 2 + 4 + 8 + 16)(1 + 3),$$

since this gives us

$$\begin{aligned} & 1 \cdot (1 + 2 + 4 + 8 + 16) + 3 \cdot (1 + 2 + 4 + 8 + 16) \\ =& 1 + 2 + 4 + 8 + 16 + 3 + 6 + 12 + 24 + 48. \qquad = 124 \end{aligned}$$

This expression gives us the sum of all factors of 48, which happens to be $\boxed{124}$. †

Theorem 4.1

Given a number $n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \ldots$, the sum of its factors is equal to $(1+p_1^1+p_1^2 \ldots p_1^{e_1})(1+p_2^1+p_2^2 \ldots p_2^{e_2})(1+p_3^1+p_3^2 \ldots p_3^{e_3}) \ldots$

Expanding out the whole expression just gives the sum of the prime-factored factors. The sums within the parentheses can be simplified with the geometric sequence sum formula, but it is not necessary.

Example 4.2

Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

Solution. First, we can prime factorize N by prime factorizing each of the terms that are given, to get

$$\begin{aligned} N &= 2 \cdot 17 \cdot 2 \cdot 17 \cdot 3^2 \cdot 7 \cdot 3^3 \cdot 2 \cdot 5 \\ &= 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2. \end{aligned}$$

Next, we can separate even or odd by noticing that even factors come from the 2s. In the formula for the sum, if we let the exponent in the $p_1 = 2$ term be equal to 1, we force the sum to only contain the odd divisors:

$$(1)(1+3+9\dots)(1+5)\cdots = 1+3+9+\cdots+5+15+45\dots$$

Similarly, if we let the exponent of $p_1 = 2$ term be equal to the sum of all possible divisor powers of 2 greater than 1, we force all factors in the sum to be even:

$$(2+4\dots)(1+3+9\dots)\cdots = 2+6+18+\cdots+4+12+36\dots$$

Therefore, we get the sum of the odd divisors is equal to

$$(1)(1+3+9+27+81+243)(1+5)(1+7)(1+17+289) = ?$$

Similarly, sum of the even divisors is equal to

$$(2+4+8)(1+3+9+27+81+243)(1+5)(1+7)(1+17+289) = ?$$

These expressions doesn't look like fun to calculate, but notice that we don't need to calculate it. We only need the ratio between the sums. This means that the common term

$$(1+3+9+27+81+243)(1+5)(1+7)(1+17+289)$$

cancels out, and we are left with $1 : (2+4+8) = \boxed{1 : 14}$. †

5 Practice Problems

Exercise 5.1. (AIME I 2009) Let $\tau(n)$ denote the number of positive integer divisors of n (including 1 and n). Find the sum of the six least positive integers n that are solutions to $\tau(n) + \tau(n+1) = 7$.

Exercise 5.2. (AMC 12B 2021) For n a positive integer, let $f(n)$ be the quotient obtained when the sum of all positive divisors of n is divided by n. For example,

$$f(14) = (1 + 2 + 7 + 14) \div 14 = \frac{12}{7}$$

What is $f(768) - f(384)$?

Exercise 5.3. (AMC 12B 2017) The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

Exercise 5.4. (AIME I 2000) Let S be the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 1000. What is the greatest integer that does not exceed $S/10$?

Exercise 5.5. (AIME II 2004) How many positive integer divisors of 2004^{2004} are divisible by exactly 2004 positive integers?

Exercise 5.6. (AIME I 2010) Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 5.7. (AMC 12B 2005) A positive integer n has 60 divisors and $7n$ has 80 divisors. What is the greatest integer k such that 7^k divides n?