1. Bill can drive to his favorite beach resort on either a 60-mile-long northern route or a 40-mile-long southern route. If Bill drives on the northern route, he travels on average 20 miles per hour faster and reaches the resort in half the time it takes him if he drives on the southern route. What is Bill's average speed in miles per hour on the northern route?

   Answer: 30

   Solution: Let $s+20$ be the speed Bill drives along the northern route and t be the amount of time it takes Bill to drive along the northern route. That means s is the speed Bill drives along the southern route, and $2t$ is the amount of time it takes Bill to drive along the southern route. Using this, we can create two distance-speed-time equations for each route: $t(s+20) = 60$ and $2st = 40$. These equations are satisfied when $t = 2$ and $s = 10$. Thus, Bill's average speed on the northern route is $s + 20 = \boxed{30}$ mph.

2. What is the base-10 value of

$$121_{2022} - 121_{2021} + 121_{2020} - 121_{2019} + \cdots + 121_{24} - 121_{23}$$

   where subscripts denote bases?

   Answer: 2047000

   Solution: First, observe that $121_b = b^2 + 2b + 1 = (b+1)^2$. Thus, the given expression reduces to

$$2023^2 - 2022^2 + 2021^2 - 2020^2 + \cdots + 25^2 - 24^2$$

   By difference of squares, $n^2 - (n-1)^2 = (n + (n-1))(n - (n-1)) = 2n - 1$. Applying this to the above expression gives

$$(2 \cdot 2023 - 1) + (2 \cdot 2021 - 1) + \cdots + (2 \cdot 25 - 1)$$

   Since there are 1000 terms in this series, it's not hard to compute that the sum of the arithmetic series is

$$\frac{4095 + 49}{2} \cdot 1000 = \boxed{2047000}$$

3. A geometric sequence with n^{th} term a_n satisfies $a_3 = a_4 \cdot a_5 \cdot a_6 = 8$. What is $a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot a_7 \cdot a_8 \cdot a_9$?

   Answer: 512

   Solution: Since a_n is a geometric sequence, $a_4 \cdot a_6 = (a_5)^2$. Thus $8 = (a_5)^3$, and $a_5 = 2$. $a_1 \cdot a_9$, $a_2 \cdot a_8$, $a_3 \cdot a_7$ must consequently also be equivalent to $(a_5)^2$. Hence, the whole expression is equivalent to $(a_5)^9 = \boxed{512}$.

4. Andrew and David are having a contest to see who can make a bigger number. At $t = 0$ minutes, Andrew starts with 512 and David starts with 9. In every subsequent minute ($t = 1, 2, \ldots$), Andrew squares his current number and David cubes his current number. After how many minutes will David have a greater number than Andrew for the first time?

   Answer: 3

   Solution:

   Observe $512 = 2^9$ and $9 = 3^2$. Let A_n and D_n be Andrew's number and David's number, respectively at time $t = n$. We are trying to find the smallest number n such that $A_n < D_n$. We observe that D_n will be growing double-exponentially faster than A_n, which encourages the idea of just checking the first few terms. For $t = 0$:

$$A_0 = 2^9 > 3^2 = D_0$$

For $t = 1$:

$$\begin{aligned} 2^3 &= 8 > 3 \\ (2^3)^6 &> (3)^6 \\ A_1 = 2^{18} &> 3^6 = D_1 \end{aligned}$$

For $t = 2$:

$$\begin{aligned} 2^2 &= 4 > 3 \\ (2^2)^{18} &> (3)^{18} \\ A_2 = 2^{36} &> 3^{18} = D_2 \end{aligned}$$

For $t = 3$:

$$\begin{aligned} 2^4 = 16 &< 27 = 3^3 \\ (2^4)^{18} &< (3^3)^{18} \\ A_3 = 2^{72} &< 3^{54} = D_3 \end{aligned}$$

Hence our answer is $\boxed{3}$.

5. Let the two roots of $x^2 - 9x + 16$ be the leg lengths of a right triangle. What is the perimeter of the triangle?

   Answer: 16

   Solution: Let the roots be a and b. Then, by the Pythagorean Theorem, the perimeter can be expressed as

$$a + b + \sqrt{a^2 + b^2}$$

   Using $(a + b)^2 = a^2 + 2ab + b^2$, we write our perimeter as

$$a + b + \sqrt{(a + b)^2 - 2ab}$$

   By Vieta's, we can plug in $a + b = 9$ and $ab = 16$ to get our desired answer

$$\begin{aligned} a + b + \sqrt{(a + b)^2 - 2ab} &= 9 + \sqrt{9^2 - 2 \cdot 16} \\ &= 9 + \sqrt{49} \\ &= \boxed{16} \end{aligned}$$

6. What is the number of digits in $20^{22} \cdot 5^{28}$?

   Answer: 49

   Solution: $20^{22} \cdot 5^{28} = \left(10^{22} \cdot 2^{22}\right) \cdot \left(5^{22} \cdot 5^6\right) = 10^{22} \cdot 10^{22} \cdot 5^6 = 15625 \cdot 10^{44}$. Hence, as the quantity is 15625 with 44 trailing zeroes, there are $\boxed{49}$ digits.

7. Let E and O be the sum of the even and odd factors of 210, respectively. Compute $E - O$.

   Answer: 192

   Solution: $210 = 2 \cdot 3 \cdot 5 \cdot 7$. We can simply apply the formula for the sum of factors but change the sign of 2^0:

$$(2^1 - 2^0)(3^1 + 3^0)(5^1 + 5^0)(7^1 + 7^0) = \boxed{192}$$

8. What is the remainder when $19 \cdot 20 \cdot 22 \cdot 23$ is divided by 21?

Answer: 4

Solution: Using properties of modular arithmetic, we have

$$19 \cdot 20 \cdot 22 \cdot 23 \equiv (-2) \cdot (-1) \cdot 1 \cdot 2 \equiv \boxed{4} \pmod{21}$$

.

9. What is the only three-digit prime number whose digits multiply to 30?

Answer: 523

Solution: The digits either consist of $\{1, 5, 6\}$ or $\{2, 3, 5\}$. Since $1 + 5 + 6 = 12$, any three-digit permutation of $\{1, 5, 6\}$ is divisible by 3. For $\{2, 3, 5\}$, the number must have a 3 as its units digit (by the divisibility rule for 2 and 5). Since $253 = 23 \cdot 11$, the only other possibility is $\boxed{523}$.

10. What positive integer x satisfies $\text{lcm}(20, x) + \text{lcm}(22, x) = 528$?

Answer: 88

Solution: Rewrite the given equation as

$$20a + 22b = 528$$

for some positive integers a and b (you may notice that $a = \frac{x}{\gcd(20,x)}$ and $b = \frac{x}{\gcd(22,x)}$). Since $22b$ and 528 are both divisible by 11, $20a$ must also be divisible by 11, which implies that a is a multiple of 11. Letting $a = 11c$, the equation becomes

$$\begin{aligned} 220c + 22b &= 528 \\ 10c + b &= 24 \end{aligned}$$

Thus, we only need to consider $(c, b) = (1, 14), (2, 4)$.

For the first pair, if $b = 14$, then x must be a multiple of 7 (from $b = \frac{x}{\gcd(22,x)}$) which would also make a a multiple of 7 (from $a = \frac{x}{\gcd(20,x)}$). However, this contradicts $c = 1 \implies a = 11$ thus this pair does not work.

For the second pair, $c = 2 \implies a = 22$. Examining multiples of 22 gives us that $x = 88$ satisfies $a = 22 = \frac{x}{\gcd(20,x)}$. Since $x = 88$ also satisfies $b = 4 = \frac{x}{\gcd(22,x)}$, our desired answer is $\boxed{88}$. Note that $x = 440$ also works for $a = 22$ but $b \neq 4$ in that case.

11. Kaity constructs a 3-letter string by selecting letters uniformly at random from $\{M, A, T, H\}$ with replacement. If the probability that his string is a rearrangement of MMT can be expressed as a reduced common fraction $\frac{a}{b}$, compute $a + b$. Note that MMT also counts as a rearrangement.

Answer: 67

Solution: Since there are $4^3 = 64$ ways to construct a string and 3 ways to rearrange MMT, our desired answer is $\frac{3}{64} \implies 3 + 64 = \boxed{67}$.

12. Picachu the modern Picasso is plotting points on a coordinate plane. He first draws $A_1 = (12, 23)$ and $A_2 = (10, 25)$. For $n \geq 3$, he draws A_n such that A_{n-1} is the midpoint of $\overline{A_n A_{n-2}}$. Given that X is the sum of the x-coordinates of $A_1, A_2, A_3, \ldots, A_{20}$ and Y is the sum of the y-coordinates of $A_1, A_2, A_3, \ldots, A_{20}$, compute $X + Y$.

Answer: 700

Solution: Note that as A_{n+1} is always the midpoint of A_n and A_{n+2}, all the points are collinear. Hence observe that this means all of Picachu's points will always lie on the line $x + y = 35$. It follows that the answer is $35 \cdot 20 = \boxed{700}$.

13. How many Pythagorean triples of positive integers (a, b, c), where $1 \leq a, b, c \leq 100$ and $a^2 + b^2 = c^2$, satisfy $c - b = 1$?

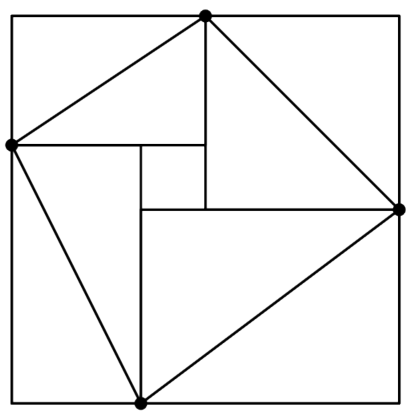
Answer: 6

Solution: $c - b = 1$ implies that $a^2 = c^2 - b^2 = (c + b)(c - b) = c + b$ must be odd, which means a is odd. For any odd a, some experimentation or algebra shows that $b = \frac{a^2-1}{2}$ and $c = \frac{a^2+1}{2}$ satisfy the problem statement. Since we need $c = \frac{a^2+1}{2} \leq 100$, the maximal value of a is $13 < \sqrt{199}$. Therefore, a can be any odd number between 3 and 13, which implies that the answer is $\boxed{6}$.

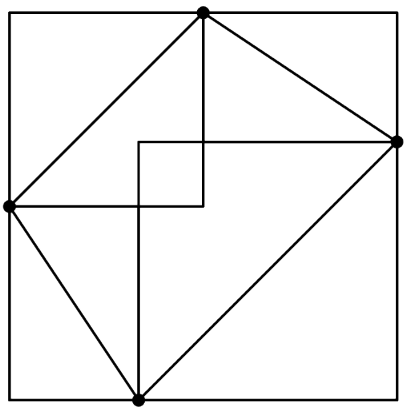
14. The square with vertices $(0, 0)$, $(6, 0)$, $(6, 6)$, and $(0, 6)$ is drawn on the coordinate plane. Justin constructs a convex quadrilateral by selecting a non-corner lattice point uniformly at random from each side of the square as its vertices. If the probability that the area formed by the quadrilateral is exactly half the area of the square can be expressed as a reduced common fraction $\frac{a}{b}$, compute $a + b$.

Answer: 34

Solution: We claim that the area of the quadrilateral is half the area of the square if and only if at least one of its diagonals is parallel to the x or y-axes. Let the points on the square be $(a, 0), (0, b), (c, 6)$, and $(6, d)$. Let's claim that there does exist a quadrilateral such that it is half the area of the square but $a \neq c$ and $b \neq d$. WLOG assume that $a < c$. Observe that double the area outside the square must equal the area of the square. Then we have two cases:
Case 1: $b > d$



In this case, as shown in the diagram above, double the outside area is the square's area minus a rectangle with area $|a - c||b - d|$, hence there are no cases that work.
Case 2: $b < d$



In this case, as shown in the diagram above, double the outside area is the square's area plus a rectangle with area $|a-c||b-d|$, hence there are no cases that work.

Thus this means our claim is correct. Hence, we may complementary count the probability both of the diagonals are not parallel to the x or y-axes. For each diagonal, fixing one of its endpoints shows that there is always a $\frac{4}{5}$ chance that it is not parallel to an axis. Since there are two diagonals, our desired probability is

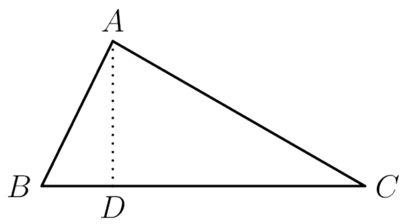
$$1-\frac{4}{5}\cdot\frac{4}{5}=\frac{9}{25}$$

Hence, our answer is $9+25=\boxed{34}$.

15. Let D be on $\overline{BC}$ such that $\overline{AD}\perp\overline{BC}$. If $AB=5$, $AC=9$, and $BC=10$, $CD-BD$ can be expressed as a reduced common fraction $\frac{a}{b}$. Compute $a+b$.

Answer: 33

Solution:



Observe

$$\begin{aligned}AB^2&=AD^2+BD^2\\AC^2&=AD^2+CD^2\end{aligned}$$

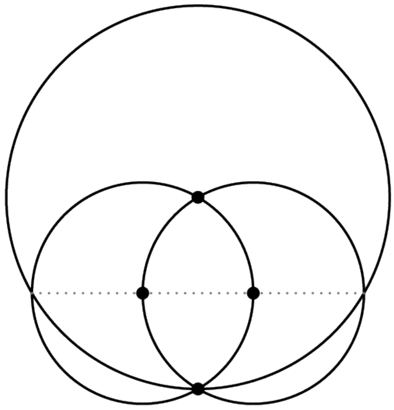
Subtracting the first equation from the second, we get

$$\begin{aligned}CD^2-BD^2&=AC^2-AB^2\\(CD+BD)(CD-BD)&=AC^2-AB^2\\(BC)(CD-BD)&=AC^2-AB^2\\CD-BD&=\frac{AC^2-AB^2}{BC}\\&=\frac{81-25}{10}\\&=\frac{28}{5}\implies 28+5=\boxed{33}\end{aligned}$$

16. Let O_1, O_2 be two circles of radius 8 with each circle passing through the other's center. Let their intersections be A and B. Let O_3 be a circle with center A and radius $\overline{AB}$. If O_3 intersects O_1 and O_2 at distinct points C and D which are not B, compute CD.

    Answer: 24

    Solution:

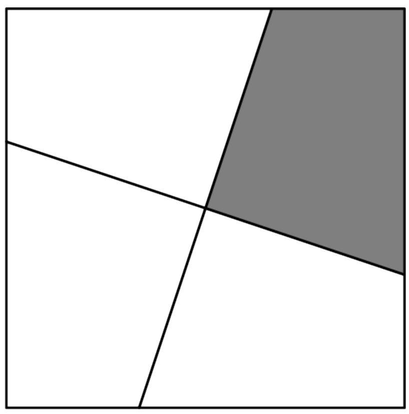


First, $\triangle CAB$ is a equilateral triangle. Note that by symmetry, triangle $\triangle CAD$ is congruent to $\triangle CBD$. Note that this symmetry also implies that $\overline{CD}$ goes through the center of both circles, implying the answer of $3 \cdot 8 = \boxed{24}$.

17. Let $ABCD$ be a square with area 144. Let points E, F, G, and H be on line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively such that $AE = BF = CG = DH = 4$. Let M be the intersection between $\overline{EG}$ and $\overline{FH}$. What is the area of $AEMH$?

    Answer: 36

    Solution:

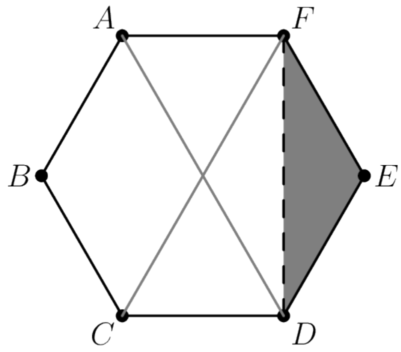


The square is split into 4 congruent parts by $\overline{EG}$ and $\overline{FH}$. Hence, the answer is $\frac{144}{4} = \boxed{36}$.

18. In regular hexagon $ABCDEF$, a random point P is chosen in its interior. If the probability that the area of quadrilateral $ABCP$ covers at least half of the area of $ABCDEF$ can be expressed as a reduced common fraction $\frac{a}{b}$, compute $a + b$.

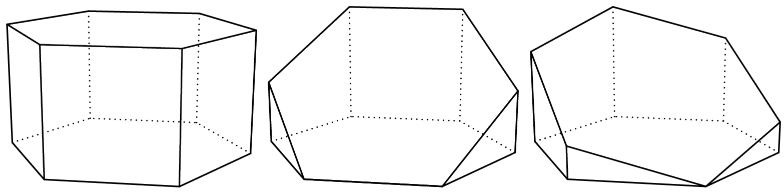
    Answer: 7

    Solution:



First, observe that $[ABCP] = [ABC] + [ACP]$. Hence, the area of the quadrilateral varies with the distance to $\overline{AC}$. Second, note that P landing on D or F makes the quadrilateral exactly half of $ABCDEF$. These two facts together implies that if P lands in $\triangle DEF$, the problem statement is fulfilled. Hence, as $\frac{[DEF]}{[ABCDEF]} = \frac{1}{6}$, our answer is $1 + 6 = \boxed{7}$.

19. Axel gave his students Yuuki and Tristan identical right hexagonal prisms of clay, with all the edges being 3 inches. He then told them to cut the clay in half. While Tristan cut it in half by cutting into opposite edges (center diagram), Yuuki cut it in half by cutting into opposite vertices (right diagram). Each of their cuts made a hexagonal cross-section across their clay. If the ratio between the areas of Tristan's cross-section to Yuuki's cross-section is R, R^2 can be expressed as a reduced common fraction $\frac{a}{b}$. Compute $a + b$.



Answer: 31

Solution: The primary observation is each cross-section could be made by a vertical stretch from the base. Hence, we just need to find how much each are stretched by. The stretch on Tristan's block of clay can be found by looking at the distance between opposite edges on the base of the clay compared to the most distant pair of edges in the cross-section. Hence, the stretch factor is

$$\frac{\sqrt{(3\sqrt{3})^2 + (3)^2}}{3\sqrt{3}} = \frac{2}{\sqrt{3}}$$

The stretch on Yuuki's block of clay can be found by looking at the distance between opposite vertices compared to the distance between the most distant pair of vertices in the cross-section. Hence, the stretch factor in Yuuki's block is

$$\frac{\sqrt{6^2 + 3^2}}{6} = \frac{\sqrt{5}}{2}$$

Let the area of the base be A. Then, we have our ratio

$$R = \frac{\frac{2}{\sqrt{3}}A}{\frac{\sqrt{5}}{2}A} = \frac{4}{\sqrt{15}} = \frac{4\sqrt{15}}{15}$$

Hence our answer is $R^2 = \frac{16}{15} \implies 16 + 15 = \boxed{31}$.

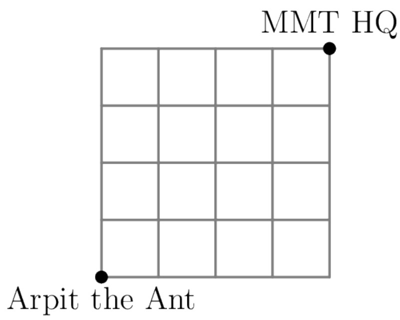
20. In the cutest auto battler, every pet has a health stat and an attack stat. Feeding various food items changes the pet's stats as shown in the table below. If a hummingbird starts out with 2 Attack and 2 HP, what is the minimum number of food items that Justin must feed it such that it reaches exactly 50 Attack and 50 HP? The pet must have a non-negative health and attack stat at all times.

Food	**Attack**	**HP**
Peach	$+0$	$+2$
Fried Shrimp	$+5$	-3
Broccoli	-1	$+3$

Answer: 48

Solution: The collective stat gain from each food item will give 2 stats, so it's forced to be $\frac{100-4}{2} = \boxed{48}$ food items. We now need to show this is possible. By giving 24 peaches, then alternating between fried shrimp and broccoli 12 times, we get to a 50/50 stat-line.

21. Arpit the Ant is looking for a path to go back to MMT HQ. If Arpit only travels on the grid lines drawn below, may only move left, right, and upwards, and never goes back to a point where he's already been, how many paths are there for Arpit to travel?



Answer: 625

Solution: Observe that if we choose where Arpit decides to move up, it entirely forces what path Arpit takes. Hence, as there are 5 locations where Arpit could move up on each row, and 4 rows where Arpit has to move up, our answer is $5^4 = \boxed{625}$.

22. Grace rolls a fair 6-sided die twice. If the probability that on the first roll, Grace gets a 3 or lower, but the sum of the two die rolls is greater than 6 can be expressed as a reduced common fraction $\frac{a}{b}$, compute $a + b$.

Answer: 7

Solution: There are only 6 ways this could happen (1,6), (2,5), (2,6), (3,4), (3,5), (3,6), so the answer is $\frac{6}{36} = \frac{1}{6} \implies 1 + 6 = \boxed{7}$.

23. Yash the STONK master is buying and selling stocks. Each day, he has a 50% chance to make 300 dollars and 50% chance to lose 100 dollars. If the probability that Yash does not have a net loss after 8 days can be expressed as a reduced common fraction $\frac{a}{b}$, compute $a+b$.

Answer: 503

Solution: Using complementary counting, if Yash loses money, then he made money on only one day or on no days. Hence, the probability is

$$1 - \frac{\binom{8}{1}}{2^8} - \frac{\binom{8}{0}}{2^8} = 1 - \frac{8}{2^8} - \frac{1}{2^8} = \frac{247}{256}$$

It follows that our answer is $247 + 256 = \boxed{503}$.

24. Aileen is making a fruit bowl to gift Anna for Mother's Day. Aileen tosses in an odd number of oddly-shaped apples, some number of pairs of pears (an even number of pears), and some number of bananas that come in bunches of 5. Anna counts that there are 200 pieces of fruit in the bowl. However, because Anna just pulled an all-nighter, her count might be off by up to 2. Given that Aileen tossed in at least one of each fruit, how many different combinations of apples, pears, and bananas may be in the bowl?

Answer: 4851

Solution: Observe that because of Anna's sleepiness, each possible number of fruits corresponds to a unique residue modulo 5. Since bananas come in bunches of 5, it follows that the combinations of apples and pears have a one-to-one correspondence with the total combinations. Hence it suffices to count the number of possible combinations of apples and pears. Note that this situation can be modeled by, for non-negative A and P,

$$(2A + 1) + (2P + 2) \leq 197$$

$$2A + 2P \leq 194$$

$$A + P \leq 97$$

Now let us define an arbitrary variable $X = 97 - A - P$. Note then we have

$$A + P + X = 97$$

and any solution (A, P, X) to the above equation corresponds to a unique number of apples and pears. Using stars and bars, there are $\binom{99}{2}$ solutions which means $\boxed{4851}$ is our answer.