1. **LT751** 10 numbers, $a_1, a_2, \cdots, a_{10}$ are written in a row on a blackboard. For any integer s in the range $1 \le s \le 10$, the average of the first s numbers is equal to s^2. What is $a_5 + a_{10}$?

   Written by: Linus Tang

   Answer: $\boxed{332}$

   The sum of the first n numbers would be $n \cdot n^2 = n^3$, so we have

   $$a_5 = (a_5 + a_4 + ... + a_1) - (a_4 + a_3 + a_2 + a_1) = 5^3 - 4^3 = 61$$

   and

   $$a_{10} = (a_{10} + a_9 + ... + a_1) - (a_9 + a_8 + ... + a_1) = 10^3 - 9^3 = 271$$

   The final answer is hence $61 + 271 = \boxed{332}$.

2. **SV428** Suppose x and y are distinct positive real numbers such that $x^3 - 20x = y^3 - 20y$ and $xy = 6$. Compute the value of $x + y$.

   Written by: Sriram Venkatesh

   Answer: $\boxed{\sqrt{26}}$

   Since x and y are distinct, we may rewite the given equation as

   $$x^3 - y^3 = 20x - 20y$$

   $$\frac{x^3 - y^3}{x - y} = 20 \implies x^2 + xy + y^2 = 20.$$

   Adding $xy = 6$ to both sides yields $x^2 + 2xy + y^2 = 26$, so $x + y = \boxed{\sqrt{26}}$. One can check that this does indeed yield a valid pair of (x, y).

3. **LT293** What is the value of $\dfrac{11 \times 12 \times 13 \times 14 \times \cdots \times 20}{1 \times 3 \times 5 \times 7 \times \cdots \times 19}$?

   Written by: Linus Tang

   Answer: $\boxed{1024}$

   The numerator is the expression for 20!, except with factors $1, 2, 3, \cdots, 10$ missing. The denominator is the expression for 20!, except with factors $2, 4, 6, \cdots, 20$ missing.

   We can now rewrite the fraction:

   $$\frac{11 \times 12 \times 13 \times 14 \times \cdots \times 20}{1 \times 3 \times 5 \times 7 \times \cdots \times 19} = \frac{20!/(1 \times 2 \times 3 \times 4 \times \cdots \times 10)}{20!/(2 \times 4 \times 6 \times 8 \times \cdots \times 20)} = \frac{2 \times 4 \times 6 \times 8 \times \cdots \times 20}{1 \times 2 \times 3 \times 4 \times \cdots \times 10}$$
   $$= \frac{2 \times 2 \times 2 \times 2 \times \cdots \times 2}{1 \times 1 \times 1 \times 1 \times \cdots \times 1} = 2^{10} = \boxed{1024}$$

4. **TK890** Let $7a + 2b = 54$ and let $4a + 7b = 43$. Evaluate $a + 12b$.

Written by: Tristan Kay

Answer: $\boxed{32}$

$$2(4a + 7b) - (7a + 2b) = 2 \times 43 - 54 = \boxed{32}$$

5. **LT274** How many ways are there to arrange the letters in $BANANAS$ such that two A's never appear next to each other?

   Written by: Linus Tang

   Answer: $\boxed{120}$

   Notice that if we first replace all consonants with C's, we have that we are trying to rearrange $CACACAC$ such that no two A's are next to each other. This is equivalent to making sure there is at least 1 C between each A. We can first place the four C's, leaving five gaps for three A's to be placed. The A's must go in different gaps, otherwise we willl have two consecutive A's. Thus, there are $\binom{5}{3} = 10$ ways to place the A's.

   Now, there are $\frac{4!}{2} = 12$ ways to substitute the C's with the original consonants B, N, N, and S, so our answer is $12 \times 10 = \boxed{120}$.

6. **LT272** A standard deck of 52 cards is shuffled into a random order. Given that the top card is a king, what is the probability that the bottom card is the king of diamonds?

   Written by: Linus Tang

   Answer: $\boxed{\dfrac{1}{68}}$

   We are given that the top card is a king. With probability $\frac{1}{4}$, the top card is the king of diamonds, so the bottom card cannot possibly be the king of diamonds. With probability $\frac{3}{4}$, the top card is one of the other kings. In this case, the king of diamonds is in one of the 51 positions other than the top. It is at the bottom with probability $\frac{1}{51}$. Therefore, the overall probability that the king of diamonds is at the bottom is $\frac{3}{4} \cdot \frac{1}{51} = \boxed{\dfrac{1}{68}}$.

7. **TK960** Gerald rolls a standard six-sided die and lands on the number k. He then rerolls the dice k times. What is the expected number of primes that Gerald rolls including the initial roll?

   Written by: Tristan Kay

   Answer: $\boxed{\dfrac{9}{4}}$

Since there are 3 primes out of 6 faces on the die, the probability of rolling a prime number on a single roll is $\frac{3}{6} = \frac{1}{2}$.

The expected value of the first roll, k, is $\frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$. Since Gerald rolls the die $k+1$ times in total, the expected number of rolls is $\frac{7}{2} + 1 = \frac{9}{2}$.

The probability of a prime on each roll is $\frac{1}{2}$, so the expected number of primes rolled is $\frac{9}{2} \cdot \frac{1}{2}$, giving us an answer of $\boxed{\dfrac{9}{4}}$.

8. **SS369** A box has 20 balls that are marked by the numbers 1 to 20. If 3 balls are randomly taken from the box of balls without replacement, what is the probability that one of them is the average of the other two?

   Written by: Saahil Shah

   Answer: $\boxed{\dfrac{3}{38}}$

   For there to be an average of 2 integers, they either both need to be odd, or they both need to be even.

   The number of ways for both integers to be odd is $\binom{10}{2} = 45$, which we double to account for even pairs to get $45 \cdot 2 = 90$. There is only 1 average for each of these pairs of integers, so we multiply by 1 to get $1 \cdot 90 = 90$. The number of ways to pick 3 balls is $\binom{20}{3} = 1140$.

   This means the final answer is $\dfrac{90}{1140} = \boxed{\dfrac{3}{38}}$.

9. **LT275** Let Ω be a circle of radius 21. Circles ω_1 of radius 6 and ω_2 of radius 8 are internally tangent to Ω and externally tangent to each other. A chord of Ω with length L is tangent to both ω_1 and ω_2, which are on opposite sides of the chord. What is L?

   Written by: Linus Tang

   Answer: $\boxed{24\sqrt{3}}$

   Let O, O_1, and O_2 be the centers of circles Ω, ω_1, and ω_2, respectively.

   Using the radii of the three circles along with the given tangencies between them, we can derive the following lengths: $O_1O_2 = 6 + 8 = 14$ because ω_1 and ω_2 are externally tangent. $OO_1 = 21-6 = 15$ because Ω and ω_1 are internally tangent. $OO_2 = 21-8 = 13$ because Ω and ω_2 are internally tangent.

   Using this information, we can find the distance between O and the chord.

Let P be the common point of tangency between ω_1, ω_2, and the chord. Let ℓ_1 be a line passing through O perpendicular to O_1O_2 and ℓ_2 be the line containing the chord. These lines are parallel to each other, since both are perpendicular to O_1O_2.

Since OO_1O_2 is a triangle with side lengths $13, 14$, and 15, it is known that the altitude ℓ_1 divides side O_1O_2 into parts of length 9 and 5 (this step is also shown in more detail at the end of the solution). Also, ℓ_2 divides side O_1O_2 into parts of length 6 and 8. Thus, the distance between parallel lines ℓ_1 and ℓ_2 is 3.

Thus, the distance between O (which lies on ℓ_1) and the chord (which lies along ℓ_2) is 3. Since Ω has radius 21, the length of the chord is $2\sqrt{21^2 - 3^2} = \boxed{24\sqrt{3}}$.

Going back to provide details on an earlier step, we wish to figure out how the altitude ℓ_1 divides side O_1O_2. Letting X be the foot of the altitude and $x = O_1X$, we have that the length of the altitude is $\sqrt{15^2 - x^2} = \sqrt{13^2 - (14-x)^2}$ by the Pythagorean Theorem on OXO_1 and OXO_2. Solving,

$$15^2 - x^2 = 13^2 - (14-x)^2$$
$$252 = 28x$$
$$x = 9$$

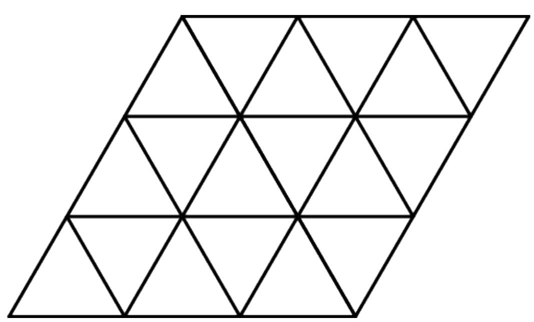
Thus, the altitude divides side O_1O_2 into parts of size 9 and 5.

10. **YS941** Tristan constructs a shape by gluing together 18 equilateral triangles of side length 1 with no overlap. What is the smallest possible perimeter of Tristan's shape?

Written by: Yuuki Sawanoi

Answer: $\boxed{12}$

There are multiple configurations of perimeter 12 that work. One is shown below:



11. **YS948** Which of the following is not a possible area for a triangle with perimeter 60?

Written by: Yuuki Sawanoi

Answer: $\boxed{225}$

Let's say there is a triangle with area A. Notice that every other area below A can be achieved by squishing the triangle without changing the perimeter. For example, a $20-20-20$ triangle can be squished to a $29-29-2$ triangle, which preserves a perimeter of 60 while decreasing the area. Thus, the answer must be the largest answer choice, else all the areas are achievable.

To solve this problem properly, the maximum area of a triangle with perimeter 60 is when the triangle is equilateral. Thus, the side length of the triangle is 20, and the area is $\frac{(20)^2\sqrt{3}}{4} = 100\sqrt{3}$, which is less than the largest answer choice of $\boxed{225}$.

12. **SG297** Equilateral triangle $\triangle ABC$ with side length 60 is cut into 3600 smaller equilateral triangles with side length 1. Point D is chosen on segment $\overline{AC}$ such that $AD = 20$. If a bug starts at point B and travels in a straight line path to point D, find the total number of triangles the bug passes through the interior of.

Written by: Sambhu Ganesan

Answer: $\boxed{80}$

We can place triangle $\triangle ABC$ on the coordinate plane with vertex B at the origin. This means that the coordinates of D are $(40, 20\sqrt{3})$.

We see that the line passes through points $(2i, i\sqrt{3})$, where $i \leq 30$. Note that between $(0,0)$ and $(2,\sqrt{3})$, the bug passes through 4 triangles and between $(2,\sqrt{3})$ and $(4, 2\sqrt{3})$ the bug passes through 4 triangles. Hence, the bug crosses 4 triangles between $(2(i-1), (i-1)\sqrt{3})$ and $(2i, i\sqrt{3})$.

Since the minimum value is $i = 1$ and the maximum value is $i = 20$, we have 20 such intervals. Thus our answer is $4 \cdot 20 = \boxed{80}$.

13. **DG399** How many digits does the base-16 number 3421_{16} have in base-4?

Written by: Daniel Ge

Answer: $\boxed{7}$

Sice $4^2 = 16$, two digits in base-4 make up one digit in base-16. Thus the conversions from base-16 to base-4 would be:

$$\begin{gathered} 1 \to 01, \\ 2 \to 02, \\ 4 \to 10, \\ \text{and } 3 \to 03. \end{gathered}$$

Together, our base-4 conversion would be 03100201. If we drop the leading zero, we have $\boxed{7}$ digits.

14. **LT268** Let a and b be not necessarily distinct positive divisors of 42. What is the sum of the distinct possible values of $\frac{a}{b}$?

Written by: Linus Tang

Answer: $\boxed{\dfrac{247}{2}}$

First, we prime factorize $42 = 2^1 \cdot 3^1 \cdot 7^1$. The possible values of $\frac{a}{b}$ are all rational numbers of the form $2^a 3^b 7^c$ with (not necessarily nonnegative) integer values of a, b, c such that $|a| \le 1$, $|b| \le 1$, and $|c| \le 1$. The sum of these values is given by the factorization $N = (2^{-1} + 2^0 + 2^1)(3^{-1} + 3^0 + 3^1)(7^{-1} + 7^0 + 7^1)$. To see why, notice that each of the 3^3 terms of the form $2^a 3^b 7^c$ described above appears exactly once when this product is expanded.

Now, evaluate N:

$$\begin{aligned}
N &= \left(\frac{1}{2} + 1 + 2\right)\left(\frac{1}{3} + 1 + 3\right)\left(\frac{1}{7} + 1 + 7\right) \\
&= \left(\frac{7}{2}\right)\left(\frac{13}{3}\right)\left(\frac{57}{7}\right) \\
&= \frac{13 \cdot 19}{2} \\
&= \frac{247}{2}
\end{aligned}$$

Therefore, the sum of all possible values of $\frac{a}{b}$ is $\boxed{\dfrac{247}{2}}$.

15. **YS936** What digit O makes the 5-digit number $2O23O$ divisible by every answer choice except O?

Written by: Yuuki Sawanoi

Answer: $\boxed{4}$

One can simply try each possibility to get $\boxed{4}$. Note that 7 is clearly false as $7|2023$ and 3 and 6 are impossible due to logic as both are multiples of 3.

16. **TK892** A *meaningful* number is a number whose prime factors sum to 42. What is the sum of the two smallest meaningful numbers?

Written by: Tristan Kay

Answer: $\boxed{407}$

If there are only two prime factors of a meaningful number, the meaningful number is minimized when they are as far apart as possible. The pair of primes that are most far apart that sum to 42 is 5 and 37. The next pair of primes after that is 11 and 31.

If there are three prime factors, notice one prime factor must be even. In other words, 2. Then we need a pair of primes that sum to 40, for which the most furthest apart they can be is with 3 and 37.

Looking at the above numbers, the sum of the two smallest is $5 \times 37 + 2 \times 3 \times 37 = 11 \times 37 = \boxed{407}$.