1. **LT897** Let $a \diamond b$ represent the value $ab + 100a$. What is the value of $18 \diamond 17 - 17 \diamond 18$?

   Written by: Linus Tang

   Answer: $\boxed{100}$

   $18 \diamond 17 - 17 \diamond 18 = (18 \cdot 17 + 100(18)) - (17 \cdot 18 + 100(17))$. After the $18 \cdot 17$ cancels out with $-17 \cdot 18$, this is $100(18) - 100(17) = \boxed{100}$.

2. **LT651** The average of the six numbers $1, 3, 5, 7, 9, N$ is equal to the average of the five numbers $2, 4, 6, 8, 10$. What is N?

   Written by: Linus Tang

   Answer: $\boxed{11}$

   The average of the five numbers $2, 4, 6, 8, 10$ is equal to $\frac{2+4+6+8+10}{5} = 6$.

   If the average of six numbers is 6, then their sum is 36.

   $1 + 3 + 5 + 7 + 9 + N = 36$

   $25 + N = 36$

   $N = \boxed{11}$

3. **LT903** Let $n!$ denote the product of the positive integers from 1 to n. Given that $8! = 40320$, what is the value of $8! + 9! + 10!$?

   Written by: Linus Tang

   Answer: $\boxed{4032000}$

   Notice that $9! = 9 \cdot 8!$ and $10! = 10 \cdot 9 \cdot 8!$. Thus, we can factor out $8!$ from the sum, getting

   $$8! + 9! + 10! = 8!(1 + 9 + 10 \cdot 9) = 8!(100) = 40320 \cdot 100 = \boxed{4032000}.$$

4. **LT885** There are 2023 numbers written on a blackboard. The first number is $\frac{1}{2023}$. The second number is $\frac{1}{2023} + \frac{1}{2022}$. The third number is $\frac{1}{2023} + \frac{1}{2022} + \frac{1}{2021}$, and so on until the last number, $\frac{1}{2023} + \frac{1}{2022} + \frac{1}{2021} + \cdots + \frac{1}{3} + \frac{1}{2} + \frac{1}{1}$. What is the average of the 2023 numbers?

   Written by: Linus Tang

   Answer: $\boxed{1}$

   The sum of the 2023 numbers is:

   $(\frac{1}{2023}) + (\frac{1}{2023} + \frac{1}{2022}) + (\frac{1}{2023} + \frac{1}{2022} + \frac{1}{2021}) + \cdots + (\frac{1}{2023} + \frac{1}{2022} + \frac{1}{2021} + \cdots + \frac{1}{3} + \frac{1}{2} + \frac{1}{1})$.

Note that the fraction $\frac{1}{2023}$ appears in all 2023 terms, the fraction $\frac{1}{2022}$ appears in 2022 terms (all but the first term), and so on. In general, the fraction $\frac{1}{k}$ appears k times for each $1 \leq k \leq 2023$.

So, the sum can be rewritten as

$2023 \cdot \frac{1}{2023} + 2022 \cdot \frac{1}{2022} + 2021 \cdot \frac{1}{2021} + \cdots + 2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{1}$

$= 1 + 1 + 1 + \cdots + 1 + 1$

$= 2023.$

Since the sum of the 2023 numbers is 2023, their average is $\boxed{1}$.

5. **LT713** Let a and b be real numbers, and define the function $f(x) = ax + b$. Given that $f(f(f(0))) = 2023$ and $f(f(f(1))) = 2031$, what is $f(0)$?

   Written by: Linus Tang

   Answer: $\boxed{289}$

   We compute $f(f(f(0)))$ in terms of a and b:

   $2023 = f(f(f(0)))$

   $= f(f(a(0) + b))$

   $= f(f(b))$

   $= f(a(b) + b)$

   $= a(a(b) + b) + b$

   $= a^2b + ab + b$

   $= (a^2 + a + 1)b.$

   Note that $f(f(f(x)))$ is a linear function with a slope of a^3. This slope can also be written as $\frac{f(f(f(1)))-f(f(f(0)))}{1-0} = 2031 - 2023 = 8$.

   Thus, $a^3 = 8$ and $a = 2$.

   Now, $2023 = (a^2 + a + 1)b$

   $2023 = 7b$

   $b = 289.$

   The answer to the problem is $f(0) = a(0) + b = b = \boxed{289}$.

6. **SR454** A photograph is 5 inches wide and 8 inches tall. It is mounted in a frame with a non-zero border x inches wide on all sides. If the border's width were doubled, the area of the frame would increase by 150%. What is the original width x? Express your

answer as a common fraction.

Written by: Shak Ragoler

Answer: $\boxed{\frac{13}{6}}$

We can find the areas of the original and expanded frames by taking the area of the photograph with the frame and subtracting the area of the portion of the photograph without the frame.

The area of the original frame is $(2x+8)(2x+5)-40$, while the area of the expanded frame is $(4x+8)(4x+5)-40$. This gives us $(4x+8)(4x+5)-40 = 2.5[(2x+8)(2x+5)-40]$.

Solving, we get $x = \boxed{\frac{13}{6}}$ inches.

7. **DG896** What is the sum of all possible unique 4-digit integers that can be formed using the digits $\{1, 3, 3, 7\}$?

   Written by: Daniel Ge

   Answer: $\boxed{46662}$

   Let's look at each digit.

   We can start by looking at the sum of all the unit digits. Tracking how often each digit appears, we find:

   - If the last digit is 3, there are 6 ways to rearrange the first three digits, and hence 3 appears 6 times $\rightarrow 3 \cdot 6 = 18$.
   - If the last digit is 1 or 7, there are 3 ways to arrange the other digits in the first three position for a total of $(1+7) * 3 = 24$.

   The total is hence $18 + 24 = 42$. By symmetry, we'll get the same result for the tens, hundreds, and thousands digits, so the final sum would be $a = 42 + 42 \cdot 10 + 42 \cdot 100 + 42 \cdot 1000 = \boxed{46662}$.

8. **RS387** Let r_1, r_2, and r_3 be the roots of $x^3 - x + 1$. What is the value of the expression below? Express your answer as a common fraction.

$$\frac{1}{1-r_1^3} + \frac{1}{1-r_2^3} + \frac{1}{1-r_3^3}$$

   Written by: Rohan Parthipan

   Answer: $\boxed{\frac{11}{7}}$

Plugging in r_1 into the polynomial, we get $r_1^3 - r_1 + 1 = 0$, so $r_1^3 = r_1 - 1$. Therefore, $1 - r_1^3 = 1 - (r_1 - 1) = 2 - r_1$. We can do the same for the other roots, so our sum is now:

$$\frac{1}{2-r_1} + \frac{1}{2-r_2} + \frac{1}{2-r_3}.$$

Simplifying this fraction gives:

$$\frac{(2-r_1)(2-r_2) + (2-r_2)(2-r_3) + (2-r_1)(2-r_3)}{(2-r_1)(2-r_2)(2-r_3)}.$$

Then, we can expand the numerator and denominator to get:

$$\frac{r_1r_2 + r_2r_3 + r_1r_3 - 4(r_1 + r_2 + r_3) + 12}{-r_1r_2r_3 + 2(r_1r_2 + r_2r_3 + r_1r_2) - 4(r_1 + r_2 + r_3) + 8}.$$

We can now use Vieta's, which gives $r_1 + r_2 + r_3 = 0$, $r_1r_2 + r_2r_3 + r_1r_3 = -1$, and $r_1r_2r_3 = -1$. Plugging these into our fraction gives:

$$\frac{-1+12}{1-2+8} = \boxed{\frac{11}{7}}.$$

9. **LT926** Sami swaps 2 (not necessarily adjacent) letters in the word $HOOFS$ to form a different string of letters. How many different strings of letters can result?

Written by: Linus Tang

Answer: $\boxed{9}$

Counting systematically, there are 10 pairs of letters in the five-letter word:

$HO_1, HO_2, HF, HS,$
$O_1O_2, O_1F, O_1S,$
$O_2F, O_2S,$
$FS,$

where O_1 is the first O and O_2 is the second. However, Sami cannot swap O_1O_2, since the result is required to be a different string of letters than the original word.

So, there are $10 - 1 = \boxed{9}$ different swaps that can be made, and each of them results in a different string of letters.

Remark: the answer could also be more quickly computed as $\binom{5}{2} - 1$.

10. **LT901** Bob uses a random number generator to pick 3 (not necessarily distinct) digits from 0 to 9. What is the probability that the sum of the digits is 27? Express your answer as a common fraction.

Written by: Linus Tang

Answer: $\boxed{\dfrac{1}{1000}}$

The only way to get a sum of 27 is when all three digits are the maximum value of 9, since any other triple would have a sum less than 27. Each digit has a $\frac{1}{10}$ probability of being a 9, thus the total probability is $\left(\frac{1}{10}\right)^3 = \boxed{\dfrac{1}{1000}}$.

11. **SS377** The outside of a $10 \times 10 \times 10$ cube is painted blue. It is then chopped into 1000 unit cubes. One of these unit cubes is chosen at random, and it is rolled like a die so that each face is equally likely to come up. What is the probability that the face that comes up is blue? Express your answer as a common fraction.

Written by: Saahil Shah

Answer: $\boxed{\dfrac{1}{10}}$

There are 10^2 little blue cube faces on each of the 6 faces, so there are $6 \cdot 10^2$ little blue cube faces in total.

Without regard to color, there are $6 \cdot 10^3$ little cube faces in total, meaning our answer is $\frac{6 \cdot 10^2}{6 \cdot 10^3} = \boxed{\dfrac{1}{10}}$.

12. **LT278** Angela is in charge of scheduling 5 meetings for her math club. The first meeting will be on Tuesday, September 3. The gap between adjacent meetings must be between 4 and 10 days, inclusive. Furthermore, meetings cannot be scheduled for Saturday nor Sunday. How many ways can Angela schedule the remaining 4 meetings? For example, this means that the second meeting must be on a weekday between September 7 and September 13.

Written by: Linus Tang

Answer: $\boxed{625}$

After each meeting, there must be 4 to 10 days until the next meeting. Based on this rule alone, there is a streak of 7 consecutive days in which Angela can schedule the next meeting. However, exactly 2 of these 7 days are Saturday and Sunday, so there are only 5 days for Angela to schedule the next meeting.

The day of the first meeting is fixed, and there are 5 ways to schedule each of the 4 subsequent meetings. Thus, there are $5^4 = \boxed{625}$ ways to schedule the remaining meetings.

13. **LT730** Emil adds 2 joker cards to a standard deck of 52 playing cards and shuffles the 54 total cards thoroughly. He then discards every card that lies between the two jokers. Given that the Ace of Spades, Ace of Clubs, and Ace of Diamonds were discarded, what is the probability that the Ace of Hearts was not discarded? Express your answer as a

common fraction.

Written by: Linus Tang

Answer: $\boxed{\frac{1}{3}}$

Since the other three aces were discarded, the two jokers and three other aces must have been in the following order, ignoring the other cards:

J A A A J

Now, the Ace of Hearts, which will be denoted A', can be in any of six positions with respect to the five critical cards mentioned above, each with equal probability:

A' J A A A J
J A' A A A J
J A A' A A J
J A A A' A J
J A A A A' J
J A A A J A'

In two of the six scenarios, the Ace of Hearts is not discarded, so the probability is $\frac{2}{6} = \boxed{\frac{1}{3}}$.

14. **LT957** How many ways are there to color each vertex of a 10-sided polygon red, green, or blue such that among any three consecutive vertices, exactly two distinct colors appear? Note that reflections and rotations are considered distinct.

Written by: Linus Tang

Answer: $\boxed{1026}$

Assign each color to a number: red is 0, green is 1, and blue is 2. Let the numbers corresponding to the colors of the beads be $a_1, a_2, a_3, \ldots, a_{10}$, in order around the polygon.

Additionally, let $b_i = a_{i+1} - a_i \pmod 3$ for each $1 \le i \le 10$, where $a_{11} = a_1$. Note that each b_i is an integer in $\{0, 1, 2\}$.

The condition that three consecutive beads have exactly two distinct colors is actually equivalent to the condition that $b_i \ne b_{i+1}$ for all $1 \le i \le 10$, where $b_{11} = b_1$. Of the 3^{10} possible tuples $(b_1, b_2, \ldots, b_{10})$, we will use recursion to find how many satisfy this condition.

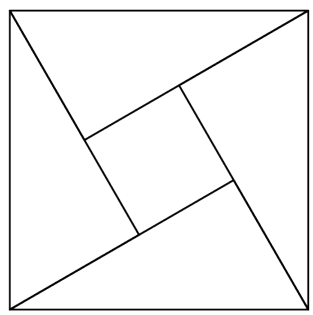
Let $f(n)$ denote how many tuples $(b_1, b_2, \ldots, b_n)$ satisfy $b_i \ne b_{i+1}$ for all $1 \le i \le n$. Note that for our base cases $n = 2$ and $n = 3$, we are essentially counting the number of pairs and triples that can be formed such that all elements are distinct, so $f(2) = f(3) = 6$.

For $n \geq 3$, we have two cases: either $b_{n-1} = b_1$ or $b_{n-1} \neq b_1$. For the first case, there are $f(n-2)$ possibilities for $(b_1, b_2, \ldots, b_{n-1})$ followed by two possibilities for b_n. For the second case, there are $f(n-1)$ possibilities for $(b_1, b_2, \ldots, b_{n-1})$ followed by one possibility for b_n. Thus, our recurrence relation is $f(n) = f(n-1) + 2f(n-2)$, which we use to find $f(10) = 1026$.

However, since $b_1 + b_2 + b_3 + \cdots + b_{10} \equiv 0 \pmod 3$, it follows that only $\frac{1}{3}(1026) = 342$ of these 1026 tuples are possible.

There are 3 choices for a_1, and after choosing a_1 and fixing $(b_1, b_2, \ldots, b_{10})$, the rest of the values a_i are also fixed. Thus, the number of possible ways to select the colors $(a_1, a_2, \ldots, a_{10})$ is $3 \cdot 342 = \boxed{1026}$.

15. **LT649** In the diagram below, 4 congruent right-angled triangles and a small square are arranged to form a large square. If the area of the small square is 5 and the area of each triangle is 19, what is the side length of the large square?



Written by: Linus Tang

Answer: $\boxed{9}$

The area of the large square is the sum of the areas of the five pieces, or

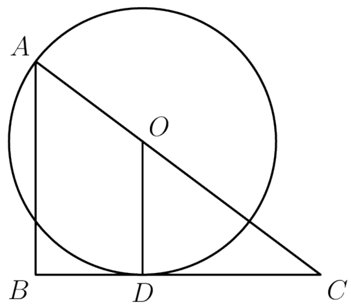
$5 + 19 + 19 + 19 + 19 = 81.$

If s is the side length of the large square, then its area is $s^2 = 81$. So, s is the square root of 81, which is $\boxed{9}$.

16. **LT732** Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Let ω be a circle whose center lies on $\overline{CA}$. If ω passes through A and is tangent to $\overline{BC}$, what is the radius of ω? Express your answer as a common fraction.

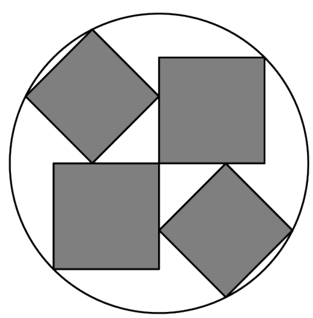
Written by: Linus Tang

Answer: $\boxed{\dfrac{15}{8}}$



Let the radius of ω be r and the altitude from O onto BC be D . Then, notice that $AO = r$ and that by similar triangles, $\frac{OD}{OC} = \frac{AB}{AC} = \frac{3}{5}$, so $OC = \frac{5}{3}AO = \frac{5r}{3}$. Thus, we have that $AC = AO + OC = r + \frac{5r}{3} = \frac{8r}{3} = 5$. Thus, we have that $r = \frac{3}{8} \times 5 = \boxed{\frac{15}{8}}$

17. **MM858** Four squares are inscribed in a circle as shown in the diagram. If the circle's radius is 1, what is the sum of the areas of all 4 squares? Express your answer as a common fraction.

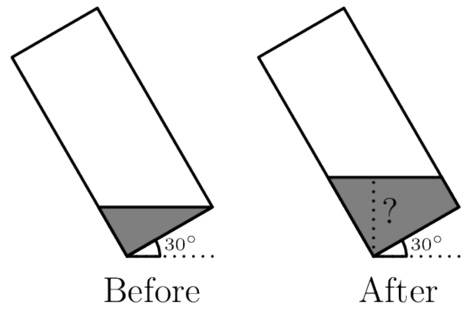


Written by: Markus

Answer: $\boxed{\frac{9}{5}}$

The larger two squares can be quickly seen to have diagonals of length 1, therefore the sides have length $\frac{1}{\sqrt{2}}$, so the area of each square is $\frac{1}{2}$. For the smaller squares, notice that if we draw dotted lines from the sides of one square to the corresponding sides of the other, we will have created 3 congruent squares. To find the side length, we notice that the diagonal across the squares is the diameter, 2. We can write the legs in terms of the side length s. Using the Pythagorean theorem we see that $(s)^2 + (3s)^2 = 10s^2 = 4$, so $s = \frac{2}{\sqrt{10}}$. The area of one of those squares is then $\frac{2}{5}$. Thus the total area of the squares is $2\left(\frac{1}{2} + \frac{2}{5}\right) = \boxed{\frac{9}{5}}$.

18. **SR456** A hollow rectangular prism has a square base that is 6 inches by 6 inches and has some water in it. When the prism is tilted on its edge by 30 degrees, the water is exactly deep enough to entirely cover the base of the prism, as shown in the diagram. If the amount of water in the prism is then doubled, what would the new depth of the water be? Express your answer as a common fraction.



Written by: *Shak Ragoler*

Answer: $\boxed{\dfrac{9}{2}}$

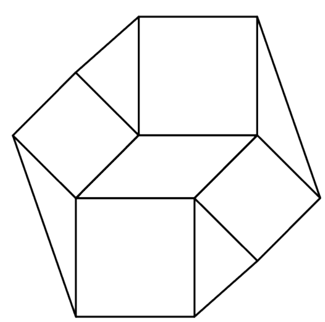
We can consider this problem from the side view and use area. Notice that the water before it is doubled forms a triangle, and the water added when the volume is doubled forms a parallelogram on top of the triangle.

Because the triangle and parallelogram have the same area, we can set them equal to one another, with b=base, h_p=parallelogram height, and h_t=triangle height. $h_p \cdot b = \frac{h_t \cdot b}{2} \Rightarrow h_p = \frac{h_t}{2}$. This means the height of the parallelogram is be half the height of the triangle.

We can find the height of the triangle by dropping an altitude from the bottom right corner of the rectangle to the ground, forming a 30-60-90 triangle. Since the hypotenuse is 6, the altitude from the bottom right corner of the rectangle to the ground is equal to 3, so the height of the triangle containing the water is also 3.

Thus, the final depth of the water is $3 + (3 \cdot \frac{1}{2}) = \boxed{\dfrac{9}{2}}$ inches.

19. **LT836** In the diagram below, squares of length 6 and 8 have been constructed outside a parallelogram. The eight outer vertices are then connected to form an octagon with area of 302. What is the area of parallelogram?



Written by: *Linus Tang*

Answer: $\boxed{34}$

By SAS congruence, each of the four outer triangles is congruent to half of the parallelogram. Therefore, the total area of the four triangles and the parallelogram is $3x$, where x is the area of the parallelogram. The area of the octagon is now

$$2(6^2 + 8^2) + 32 = 302.$$

Solving for x, we get $x = \boxed{34}$.

20. **LT753** Let $\triangle ABC$ be an equilateral triangle and P be a point on the incircle (circle inscribed in) $\triangle ABC$. If the distances from P to $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are $2, 5$, and a, what is the product of the possible values of a?

Written by: Linus Tang

Answer: $\boxed{9}$

Let P' be the point inside the triangle such that the distances from P' to AB and AC are 5 and 2, respectively. By symmetry, P' also lies on the incircle of $\triangle ABC$. Also, extending PP' to hit AB and AC at B' and C', we have $AB' = AC'$ by symmetry and $\angle B'AC' = \angle BAC = 60°$. So, $AB'C'$ is an equilateral triangle. By Viviani's Theorem on point P, $AB'C'$ has an altitude of length $2+5+0 = 7$. Thus, by ratios in a 30-60-90 triangle, we can calculate $AB' = \frac{14}{\sqrt{3}}$.

Letting T be the tangency point between AB and the incircle of $\triangle ABC$, which is also the midpoint of AB, we have by Power of a Point that $B'T = \sqrt{B'P \cdot B'P'} = \sqrt{(2 \cdot \frac{2}{\sqrt{3}})(5 \cdot \frac{2}{\sqrt{3}})} = \frac{2\sqrt{10}}{\sqrt{3}}$.

Thus, $AT = AB' \pm B'T = \frac{14 \pm 2\sqrt{10}}{\sqrt{3}}$. Now AB is twice of that, or $\frac{28 \pm 4\sqrt{10}}{\sqrt{3}}$. The height of $\triangle ABC$ is $\frac{\sqrt{3}}{2} \cdot AB = 14 \pm 2\sqrt{10}$.

Finally, we apply Viviani's Theorem again and find that $a = 14 \pm 2\sqrt{10} - 5 - 2 = 7 \pm 2\sqrt{10}$. The product of the possible values of a is $(7 + 2\sqrt{10})(7 - 2\sqrt{10}) = \boxed{9}$.

21. **LT910** Albert wrote down a number N. Betty wrote down the number that equals the sum of the digits of N. Carol wrote down the sum of the digits of Betty's number. If Carol wrote the number 11, what is the smallest possible value N could have been?

Written by: Linus Tang

Answer: $\boxed{2999}$

Working backwards, the smallest number Betty could have written is 29, so the sum of the digits of N is at least 29.

Because of this, N must be at least 4 digits (or else Betty's number would be at most $9 + 9 + 9 = 27$).

To minimize N, we want to maximize the later digits, which leads to $N = \boxed{2999}$.

We can check when $N = 2999$ that Betty indeed writes 29 and Carol indeed writes 11.

22. **MLI320** Let x be a 2022 digit number where every digit is 5 ($5555\cdots5$). Compute the greatest common divisor of x and 45.

Written by: Michael Liu

Answer: $\boxed{15}$

Here, the problem wants us to find the value of $\gcd(555\cdots5, 45)$. First, we can factor out a 5 from both values to simplify our expression a bit, which gives us $5\gcd(111\cdots1, 9)$ where the first value has 2022 digits of 1s. To find the greatest common divisor of $111\cdots1$ and 9, we just need to find whether $111\cdots1$ is divisible by either 3 or 9 because those are the only factors of 9 greater than 1. Summing up the digits of $111\cdots1$ we get 2022, which is divisible by 3, but not 9 which means that $111\cdots1$ is divisible by 3 and not 9. This ultimately means that $\gcd(111\cdots1, 9) = 3$ and our final answer is $5\cdot3 = \boxed{15}$

23. **JAT824** There exists a four-digit positive integer $\overline{abcd}$ (a, b, c, d are not necessarily distinct) such that $\overline{abcd} + 2\cdot\overline{dcba} = 22221$. Find $\overline{abcd}$.

Written by: Jatloe

Answer: $\boxed{7347}$

Writing out $\overline{abcd}$ and $\overline{dcba} = 22221$, we get:

$$\begin{aligned} 1000a + 100b + 10c + d + 2\cdot(1000d + 100c + 10b + a) &= 22221 \\ 1002a + 120b + 210c + 2001d &= 22221 \end{aligned}$$

The maximum possible value of $2a + 120b + 210c + d$ occurs when a, b, c, and d are all 9, giving us $(2 + 120 + 210 + 1)\cdot9 = 2997$. Subtracting this from the above equation, we find that $1000(a + 2d)$ must be at least 19224. This means that $a + 2d \geq 20$.

Taking the above equation mod 10, we find $2a + d \equiv 1 \pmod{10}$. We know that $2a + d$ cannot be 1 or 11(since $1 + 2\cdot9 = 19 < 20$). Therefore, $2a + d = 21$ ($2a + d$ cannot be greater than 21 since the maximum, achieved when $a = d = 9$ is 27). Therefore, $100a + 12b + 21c + 200d = 22200$.

Taking this mod 3, we find that:

$$\begin{aligned} 100a + 200d &\equiv 0 \pmod 3 \\ a + 2d &\equiv 0 \pmod 3 \end{aligned}$$

Thus $a + 2d$ has to equal one of $15, 18, 21, 24$ (since we know $2a + d = 21$). However, remember from earlier than $a + 2d \geq 20$. Additionally $a + 2d < 22.221$ since $(a + 2d)\cdot 1000 < 22221$. Therefore, there is only one value for $a + 2d$, which is 21. Finally, we can get that $a = d = 7$, meaning that $4b + 7c = 40$. Therefore $b = 3$ and $c = 4$, making our answer $\boxed{7347}$.

24. **LT267** What is the smallest positive integer n such that n^n is divisible by 2023^{2023} but n is not divisible by 2023?

Written by: Linus Tang

Answer: $\boxed{4165}$

First, note that the prime factorization of 2023 is $7^1 \cdot 17^2$.

Thus, $2023^{2023} = 7^{2023} \cdot 17^{4046}$.

Since n^n is divisible by this number, n must be divisible by 7 and 17.

Furthermore, the exponent of 17 in the prime factorization of n must be exactly 1; if it were greater, then n would be divisible by 2023.

Thus, the exponent of 17 in the prime factorization of n^n is exactly n.

Recalling that n^n is divisible by $2023^{2023} = 7^{2023} \cdot 17^{4046}$, we now know $n \geq 4046$ based on the exponent of the prime 17.

We cannot have $n = 4046$ as that is divisible by 2023, so we check that the next smallest multiple of $7 \cdot 17$, which is $\boxed{4165}$. We check that this number works.

Indeed, $2023^{2023} = 7^{2023} \cdot 17^{4046} \mid (7 \cdot 17)^{4165} \mid 4165^{4165}$, so $n = 4165$ works.

25. **CY589** Compute the sum of all positive integers $n \leq 100$ such that $\frac{30^n}{n!}$ is also an integer.

Written by: Carsten Yeung

Answer: $\boxed{21}$

$$\frac{30^n}{n!} = \frac{30}{1} \cdot \frac{30}{2} \cdot \frac{30}{3} \cdots \frac{30}{n}$$

Thus, $n = 1, 2, 3, 4, 5, 6$ all work, since these values of n make every factor in the above expression an integer. However, n cannot be greater than or equal to 7 because 7 does not divide 30 (or any power of 30) evenly, which means $\frac{30^n}{n!}$ can never be an integer when $n \geq 7$. Therefore, $S = \{1, 2, 3, 4, 5, 6\}$, and the sum of the elements is $1 + 2 + 3 + 4 + 5 + 6 = \boxed{21}$.

26. **LT735** Find the sum of all integers n between 1 and 105, inclusive, such that 105 evenly divides $n^2 - 4n + 3$.

Written by: Linus Tang

Answer: $\boxed{331}$

Note that $n^2 - 4n + 3 = (n-2)^2 - 1$, so we can rewrite the condition as $(n-2)^2 \equiv 1 \pmod{105}$. By the Chinese Remainder Theorem, since $105 = 3 \cdot 5 \cdot 7$, it suffices to independently solve $(n-2)^2 \equiv 1 \pmod{p}$ for each $p \in \{3, 5, 7\}$. The solutions are $n \equiv 1$ and $n \equiv 3$ for each of these primes p.

Since there are two solutions for each of these primes, there are $2 \cdot 2 \cdot 2 = 8$ solutions modulo 105.

A key observation that can be used to solve the problem without computing all 8 solutions is that if n_0 is a solution, then so is $4 - n_0 \pmod{105}$. Thus, the solutions form pairs that sum to 4 or 109. The only pair that sums to 4 is $1 + 3$, so the sum of all 8 solutions is $4 + 109 + 109 + 109 = \boxed{331}$.