1. Alon uniformly selects two (not necessarily distinct) factors of 20^{22}. If the probability that at least one of these factors is divisible by the other can be expressed as a reduced common fraction $\frac{a}{b}$, compute the value of $a+b$.

   Answer: 1586

   Solution: Note: $v_p(n)$ denotes the power of p in the prime factorization of n.

   Let the two factors be a and b. We have $20^{22} = 2^{44} \cdot 5^{22}$. If $v_2(a) = v_2(b)$, then we already know that one of a and b divide each other. Since there are 45 different possible values of $v_2(a)$, the probability of this is $\frac{1}{45}$. Otherwise, assume without loss of generality that $v_2(a) < v_2(b)$. We now compute the probability that $v_5(a) \le v_5(b)$. There is a $\frac{1}{23}$ probability that they are equal. If not, there's equal probability of one being greater than the other, so the probability is $\frac{1}{23} + \frac{1}{2}\left(1 - \frac{1}{23}\right) = \frac{12}{23}$. The final probability is then

   $$\frac{1}{45} + \frac{44}{45} \cdot \frac{12}{23} = \frac{551}{1035},$$

   so the answer is $\boxed{1586}$.

2. Daniel has 2 stacks of sticks. In the first stack, there are 4 yellow sticks. In the second stack, there are 2 yellow sticks and 2 blue sticks.

   They repeat the following process until one stack is empty: they randomly choose a stack at random with equal probability, then they uniformly at random choose a stick from that stack to throw away.

   If the probability that the first stack is emptied while the second stack still contains both blue sticks can be expressed as a reduced common fraction $\frac{a}{b}$, evaluate $a+b$.

   Answer: 221

   Solution: Let the first stack be S_1, and the second stack be S_2. This lays the foundation for our casework:

   Assume k yellow sticks in S_2 were thrown away. There are $\binom{k+3}{3}$ ways to pick the stacks. Indeed, the last stick removed must have come from S_1, leaving us with 3 times S_1 is chosen, and k times S_2 is chosen, which we can order in $\binom{k+3}{3}$ ways. Then, the probability that in an arrangement of 2 yellow and 2 blue sticks, both blue sticks are in the final $4-k$ sticks is $\frac{\binom{4-k}{2}}{\binom{4}{2}} = \frac{\binom{4-k}{2}}{6}$, since there are $\binom{4-k}{2}$ ways to arrange the order of the remaining sticks and $\binom{4}{2}$ ways to choose 2 of the sticks. Finally, multiply by the probability that S_1 was chosen 4 times and S_2 was chosen k times, which is $\frac{1}{2^{4+k}}$. We sum $\frac{\binom{k+3}{3}\frac{\binom{4-k}{2}}{6}}{2^{k+4}}$ over $k = 0, 1, 2$, which gives $\frac{1}{16} + \frac{1}{16} + \frac{5}{192} = \frac{29}{192}$. Now adding the numerator and the denominator gives $\boxed{221}$.

3. Hongning has 9 slips of paper labeled with the numbers 1 through 9. Reese has 8 slips of paper labeled with the numbers 1 through 8. Hongning and Reese each pick a slip uniformly at random from their respective piles. The probability that Reese's strip of paper has a larger number on it than Hongning's can be expressed as a reduced common fraction $\frac{a}{b}$. What is $a+b$?

   Answer: 25

   Solution: If Reese pulls the number i, then there are $i-1$ numbers less than i that Hongning could pull. For example, if Reese pulls 5, there are 4 possibilities for Hongning to pull a smaller number. Therefore, there are

   $$0 + 1 + 2 + \cdots + 7 = 28$$

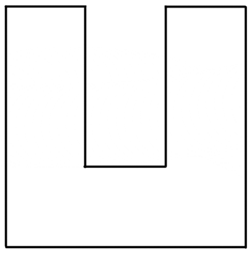
different ways for Reese to pull a number greater than Hongning. Since there are $9 \cdot 8$ total possibilities, the answer is $\frac{28}{72} = \frac{7}{18}$, and we add the numerator and the denominator to get $\boxed{25}$.

4. Consider an arithmetic sequence of real numbers with common difference -3. Let the sum of the first n terms be denoted by S_n. If $S_4 = 10$, compute S_7.

   Answer: -14

   Solution: Let the nth term be denoted by a_n. Then, $10 = S_4 = (a_1) + (a_1 - 3) + (a_1 - 6) + (a_1 - 9) = 4a_1 - 18$, which implies that $a_1 = 7$. Thus, we use the sum of terms formula to find that $S_7 = \boxed{-14}$.

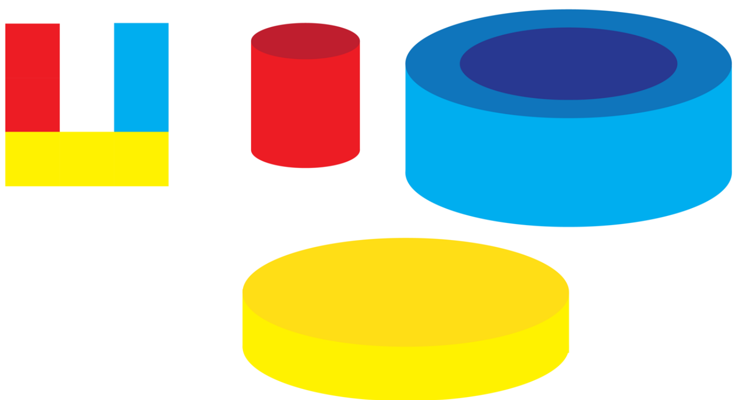
5. The following 2-dimensional shape is created by removing a 1 by 2 rectangle from the middle third of a 3 by 3 square:



   If the ratio of the volume of the figure created by rotating the shape 360 degrees around its left edge can be expressed in the form $a \cdot \pi$, what is a?

   Answer: 21

   Solution: We split up the shape into 3 sections: a 1 by 2 red rectangle, a 1 by 2 blue rectangle, and a 3 by 1 yellow rectangle as shown. We will rotate each of these individually.

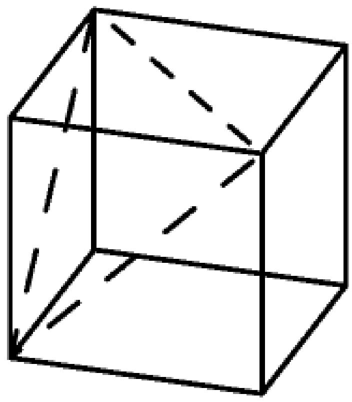


   The red rectangle is rotated into a cylinder with radius 1 and height 2, which has volume $(1^2)(2)\pi = 2\pi$. The yellow rectangle is rotated into a cylinder with radius 3 and height 1, which has volume $(3^2)(1)\pi = 9\pi$. The blue rectangle is rotated into a hollow cylinder. Its volume is the volume of a cylinder with radius 3 and height 2, minus the volume of a cylinder with radius 2 and height 2. This is $(3^2)(2)\pi - (2^2)(2)\pi = 18\pi - 8\pi = 10\pi$. So, our total volume is $(2 + 9 + 10)\pi = 21\pi$, so $a = \boxed{21}$.

6. Three vertices of a cube are selected uniformly at random, and a triangle is drawn connecting these selected vertices. If the probability that the resulting triangle is acute can be expressed as a reduced common fraction $\frac{a}{b}$, compute the value of $a+b$.

Answer: 8

Solution: If two of the vertices chosen are connected by an edge of the cube, then the triangle is a right triangle. Therefore the only possible acute triangle that can be formed from three vertices of a cube is an equilateral triangle in which the three selected vertices are all exactly one side length away from a common vertex, as shown in the diagram.



Since a cube has 8 vertices, there can only be 8 such triangles. Overall, there are $\binom{8}{3} = 56$ different ways to produce a triangle by selecting three vertices from the cube at random, so the probability that such a triangle would be acute is $\frac{8}{56} = \frac{1}{7}$. Taking the sum of the numerator and denominator results in $1 + 7 = \boxed{8}$.