1. The expression $\frac{1-1+1-1+\cdots-1+1}{2}$, where the numerator contains 111 ones, can be written as a reduced common fraction $\frac{a}{b}$. What is $a+b$?

   Answer: 3

   Solution: The first 110 ones all cancel out because each $1-1$ pair is equal to 0. This just leaves the last 1 at the end of the numerator, so we get 1 divided by 2 or $\frac{1}{2}$. Thus, our final answer is $\boxed{3}$.

2. 7 people were writing problems for Mustang Math Tournament. The test needs 240 problems in total, 77 of which have already been written. If each person writes 3 problems per day, how many (whole-number) days will it take to reach 240 problems?

   Answer: 8

   Solution: Since 77 problems have already been written, there are $240-77=163$ problems left to write. We also know there are $7 \cdot 3 = 21$ problems written per day, because we have 7 people writing at a rate of 3 problems per day. Thus, the answer is just $\frac{163}{21}$ days, or a bit over 7 days. Because we want a whole-number of days, we need to round up to our final answer of $\boxed{8}$ days.

3. Max the Mustang is waiting impatiently to attend Mustang Spirit Week. He is watching a clock with a minute hand of length 6 units and a hour hand of length 3 units, and must wait until the minute hand has swept out an area of 12π square units. The area that the hour hand sweeps out in the same amount of time can be expressed as $\frac{a}{b}\pi$ square units, where $\frac{a}{b}$ is a reduced common fraction. What is $a+b$?

   Answer: 5

   Solution: The minute hand sweeps out $\pi r^2 = \pi \cdot 6^2 = 36\pi$ square units in 60 minutes (a full revolution), so it will take a third of that or 20 minutes to sweep out 12π square units. The hour hand will undergo $\frac{1}{12}$ of a full revolution in 60 minutes (1 hour), so it must undergo $\frac{1}{36}$ of a revolution in 20 minutes. One revolution of the hour hand covers $\pi r^2 = \pi \cdot 3^2 = 9\pi$ square units, so in 20 minutes, the hour hand travels an area of $\frac{1}{36} \cdot 9\pi = \frac{\pi}{4}$. Thus, our answer is $1+4=\boxed{5}$.

4. Daniel is designing a rectangular jigsaw puzzle with 2300 square pieces, each of length 1 centimeter. What is the minimum perimeter, in centimeters, of the puzzle?

   Answer: 192

   Solution: To get the perimeter to a minimum, we need the two numbers (length and width) to be as close together as possible. Recognizing that $2300 = 100 \cdot 23 = 50 \cdot 2 \cdot 23 = 50 \cdot 46$, we get that the length and width are 50 and 46 (these are the two closest together numbers that multiply to 2300). From this, we get that the minimum perimeter is $2(50+46) = \boxed{192}$.

5. What is the smallest positive composite number that is not divisible by any of the 5 smallest positive primes?

   Answer: 169

   Solution: Any composite number is the product of at least two primes. Since we are looking for a small composite number, we want to multiply the two smallest primes we can. In this case, that means multiplying the 6th smallest prime with itself. The 6th smallest prime is 13, so our answer is just $13^2 = \boxed{169}$.

6. There are two wooden slabs, each with 9 cut-out holes. The two wooden slabs are placed on top of each other such that every hole on the first slab overlaps with a unique hole on the second slab. Then, one hole from each slab is randomly covered up and a ball is randomly

dropped from above one of the 9 original hole positions. If the probability that the ball will go through both slabs can be expressed as a reduced common fraction $\frac{a}{b}$, then what is $a+b$?

Answer: 145

Solution: There is an $\frac{8}{9}$ probability that the ball will go through the first slab (because there are 8 holes it can go through and 9 spots it can be dropped from). If the ball goes through the first slab, there is similarly an $\frac{8}{9}$ probability that it goes through the second slab. Thus, the probability is $\frac{8}{9} \cdot \frac{8}{9} = \frac{64}{81}$ and our final answer is $64 + 81 = \boxed{145}$

7. When the three numbers $109, 331$, and 479 are divided by a positive integer N, they have the same remainders. What is the largest possible N?

   Answer: 74

   Solution: If 109 and 331 have the same remainder when divided by N, then N must divide their difference $331 - 109 = 222$. Similarly, N must divide $479 - 331 = 148$. From here we find that the maximum value of N is the GCF of 222 and 148 which is $\boxed{74}$. Alternatively, since N divides both 222 and 148 it must divide $222 - 148 = 74$, and the largest number that divides 74 is just $\boxed{74}$.

8. A 6-digit number is called *rad* if it comes from writing a 3-digit number composed of 3 distinct digits twice in a row. For example, 123123 and 861861 are rad, but 122122 and 861816 are not. How many 6-digit rad numbers are divisible by 5?

   Answer: 136

   Solution: For the number to be divisible by 5, the units digit must be 5 or 0. After that, we just need to count the number of possibilities for the hundreds digit and tens digit (the other digits are determined by these ending 3 digits).

   If the units digit is 0, then there are 9 options for the hundreds digit (1-9) and 8 options for the tens digit (because the tens digit cannot equal the hundreds digit). So, there are $9 \cdot 8 = 72$ options in this case.

   If the units digit is 5, then there are only 8 options for the hundreds digit (1-4,6-9) as this digit cannot be 0 because it is equal to the leading digit. There are also 8 options for the tens digit since this digit can be zero but cannot be equal to the hundreds digit. So, there are $8 \cdot 8 = 64$ options in this case.

   Adding everything together, the final answer is $72 + 64 = \boxed{136}$.

9. How many quadratics of the form $f(x) = ax^2 + bx + c$ with positive integer coefficients are there such that $f(1) = 7$?

   Answer: 15

   Solution: Plugging in 1, we get $f(1) = a+b+c = 7$. Thus, we just need to find the number of ordered triples of positive integers (a, b, c) that add to 7. 7 is small enough that we could try to list out every single case, but there is a nicer solution. You can think of finding three positive integers instead as distributing seven 1s into three different groups. To count the number of possibilities for that, we can use sticks and stones. We want to place two dividers between the seven 1s so as to distribute them into three groups, and since we want positive integers, we can only place the dividers in the 6 spaces between the seven 1s (if we placed them on either side of the seven 1s we would end up a group of zero 1s, which isn't positive). So, the answer is $\binom{6}{2} = \boxed{15}$.

10. Let r be the answer to question 12. Alice and Bob are running around two concentric circular tracks of radii $60r$ and $90r$ respectively. They both run at 3 units/s. Assuming Alice and Bob run forever, what is the furthest distance they will ever be from each other?

Answer: 900

Solution: Since the two move at the same speed, Alice will have a greater rotational speed (i.e., she will complete a lap faster) since she runs on a smaller circle. Therefore, Alice and Bob will be opposite from each other at some point, so the maximum distance is $60r + 90r = \boxed{150r}$.

Plugging in r (from the algebra shown in the solution to problem 12), we get $150 \cdot 6 = \boxed{900}$.

11. Let a be the answer to question 10. Compute

$$\sqrt[3]{a\sqrt[3]{a\sqrt[3]{a\cdots}}}.$$

Answer: 30

Solution: Let $x = \sqrt[3]{a\sqrt[3]{a\sqrt[3]{a\cdots}}}$. Then, notice that we can substitute x into itself, giving $x = \sqrt[3]{ax} \implies x^3 = ax$. Thus, $x = \boxed{\sqrt{a}}$.

Alternatively, we can write

$$\sqrt[3]{a\sqrt[3]{a\sqrt[3]{a\cdots}}} = a^{1/3}a^{1/9}a^{1/3^3}\cdots = a^{\frac{1}{3}+\frac{1}{9}+\cdots}.$$

The geometric series in the exponent adds to $\frac{1/3}{1-1/3} = \frac{1}{2}$, giving $\boxed{\sqrt{a}}$ again.

Plugging in $a = 900$ (from the algebra shown in the solution to problem 12), we get $\sqrt{900} = \boxed{30}$.

12. Let n be the answer to question 11. In regular polygon $A_1 \ldots A_n$, compute the degree measure of angle $\angle A_1A_6A_n$.

Answer: 6

Solution: From the inscribed angle theorem, we know that the angle $A_1A_6A_n$ is half the measure of minor arc A_1A_n. The measure of the arc A_1A_n is $\frac{1}{n}$ of a whole circle or $\frac{360}{n}$ degrees. Combining these two facts together, we get that the answer is $\frac{360/n}{2} = \boxed{\frac{180}{n}}$.

Finally, to extract the final answers in this "circular round," we plug the answers into each other. From question 10, we have $a = 150r$, and by plugging that answer into 11 we get $n = \sqrt{a} = \sqrt{150r}$. Now, plugging that into 12, we get that $r = \frac{180}{n} = \frac{180}{\sqrt{150r}}$. So, squaring both sides we get

$$\frac{180^2}{150r} = r^2 \implies r^3 = 216.$$

Thus, $r = 6$, and plugging into the other answers, $a = 150r = 900$ and $n = \sqrt{a} = 30$. So, the answer to this question is $180/30 = \boxed{6}$.

13. The following expression is written out on a whiteboard

$$2022^{2022^{2022^{\cdot^{\cdot^{\cdot^{2022}}}}}}$$

where 2022 is written 2022 times. Evan evaluates this expression from the top exponent to the bottom base (i.e., right to left), while Serena evalues this expression from the bottom base to the top exponent (i.e., left to right). Evan gets x as his result and Serena gets y as her result. What is the sum of the units digits of x and y?

Answer: 12

Solution: The units digit for powers of 2022 will follow the same pattern as for powers of 2. That is, the units digit will repeat with a cycle of: $6, 2, 4, 8$; where a units digit of 6 is obtained when the exponent is divisible by 4. We can apply this to Evan's approach by realizing

$$2022^{2022^{2022^{\cdot^{\cdot^{\cdot^{2022}}}}}},$$

where 2022 is written 2021 times, is divisible by 4 when evaluated from right to left. This is because 2022 to the power of any $k \geq 2$ is divisible by 4, as $2022^k = 1011^k \cdot 2^k = 4 \cdot 1011^k \cdot 2^{k-2}$, and

$$2022^{2022^{2022^{\cdot^{\cdot^{\cdot^{2022}}}}}},$$

where 2022 is written 2020 times, is clearly greater than or equal to 2. As such, Evan's approach will end with taking 2022 to the power of a number that is divisible by 4, which will result in a units digit of 6.

If we instead apply Serena's approach and evaluate from left to right, then using exponent rules we can write:

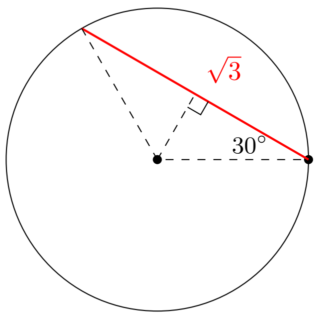
$$\left((2022)^{2022}\right)^{\cdot^{\cdot^{\cdot^{2022}}}} = 2022^{\overbrace{2022 \times \cdots \times 2022}^{}_{\text{2021 times}}}.$$

This exponent is clearly divisible by 4, so Serena also gets a units digit of 6. Adding the two, we get $6 + 6 = \boxed{12}$.

14. A horse is standing on the circumference of a circle with radius 1. The horse spins around so that it faces a random direction. After that, the horse walks $\sqrt{3}$ units forward and stops. The probability that the horse ends up inside the circle can be expressed as a reduced common fraction $\frac{a}{b}$. What is $a + b$?

Answer: 7

Solution: We want to find the maximum angle the horse's direction can deviate from facing directly towards the center of the circle. This would be when the walk of $\sqrt{3}$ units forward reaches the circumference of the circle as shown in the diagram below.



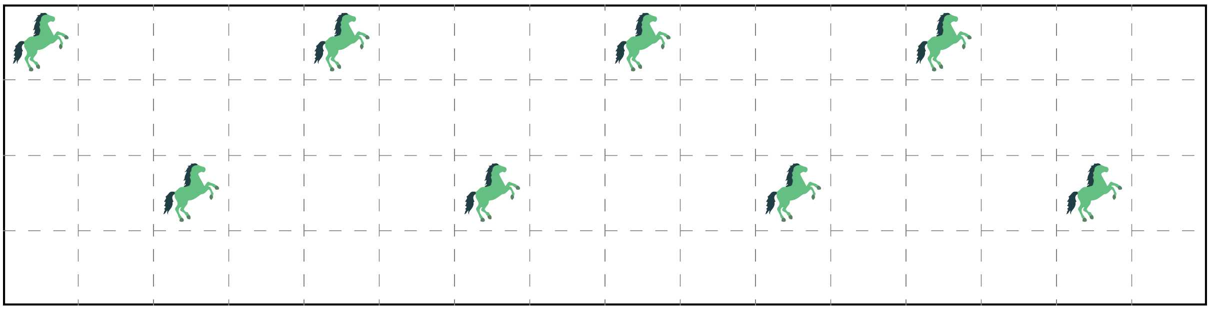
Dropping an altitude from the center of the circle to the chord reveals a 30-60-90 triangle (since the two sides, the radius and the half-chord, are in ratio $1 : \sqrt{3}/2$). Thus, the maximum angle the horse's direction can deviate from the central is $30°$. And this can be either above or below the center, creating an interval of $2 \cdot 30° = 60°$ that allows the horse to end up in the circle. There are a total of $360°$ in a full rotation, so the probability the horse ends up inside the circle is $\frac{60°}{360°} = \frac{1}{6}$. So, our answer is $1 + 6 = \boxed{7}$.

15. On each day n, Farmer Shak feeds his horse one carrot with n calories. His horse will continue eating these daily carrots until the total number of calories it has consumed is a multiple of 2022. How many days will Farmer Shak be able to feed his horse for?

Answer: 336

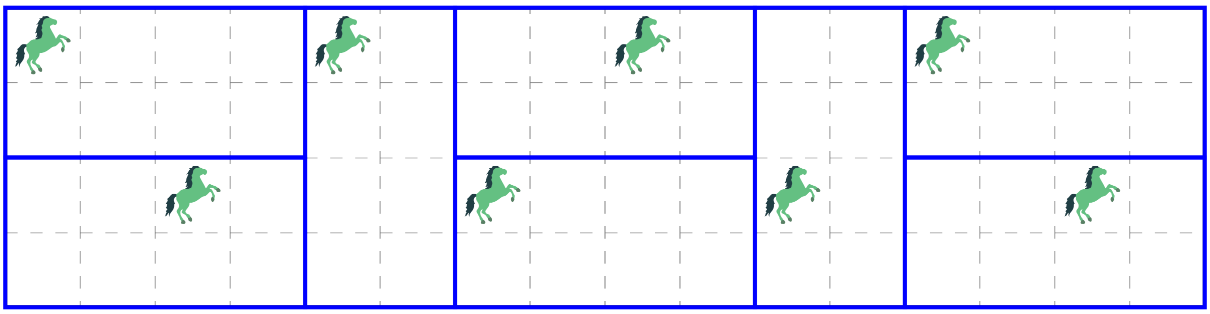
Solution: Using the sum formula, we know that on day n, Farmer Shak's horse would have consumed a total of $\frac{n(n+1)}{2}$ calories. Next, we get that the prime factorization of 2022 is $2 \cdot 3 \cdot 337$. To get a value of $n(n+1)$ that would work for this solution, we need a number that is both divisible by 2022 and further divisible by the denominator of 2. From this, we get that $n(n+1)$ needs to be divisible by $2 \cdot 3 \cdot 337 \cdot 2 = 2^2 \cdot 3 \cdot 337$. Since 337 is the largest prime there, we can start by trying it as the value for $n+1$. This works because $336 \cdot 337$ is divisible by the $2^2 \cdot 3 \cdot 337$ from earlier. Therefore, Farmer Shak will feed carrots to his horse for $\boxed{336}$ days.

16. Given the following 4×16 barn, Noel wants to separate his 8 mustangs into 8 equally-sized **rectangular** stables so that no two mustangs are in the same stable and no empty spaces remain - the stables cover the entire barn. Provided that the mustangs are fixed in their shown positions, and he only divides them along the dotted lines, how many ways can he do this?



Answer: 34

Solution 1: Note that the rectangular stables must be 8 units in area, and that none of the mustangs can individually fit into 1×8 stables. Therefore, we must use eight 2×4 rectangles to divide the mustangs. These rectangles can either be vertical or horizontal. Note that a horizontal stable forces the below/above horse to also have a horizontal stable. So, we can simplify this problem down to splitting the number 8 into an ordered set of 1s (a verticle stable) and 2s (two horizontal stables). For example, the diagram shown represents the case 21212.



In total, we have 5 cases:

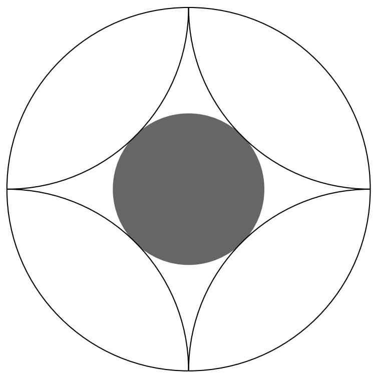
- Case 1: arrangements of 11111111 $\implies \binom{8}{0} = 1$
- Case 2: arrangements of 1111112 $\implies \binom{7}{1} = 7$
- Case 3: arrangements of 111122 $\implies \binom{6}{2} = 15$
- Case 4: arrangements of 11222 $\implies \binom{5}{3} = 10$

- Case 5: arrangements of 2222 $\implies \binom{4}{4} = 1$

This yields an answer of $1 + 7 + 15 + 10 + 1 = \boxed{34}$.

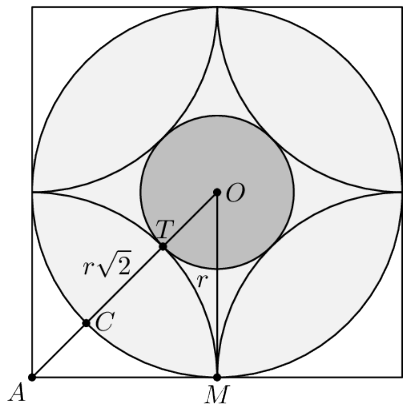
Solution 2: After the initial insights described above, we can use recursion to simplify the problem. Let $a(n)$ represent the number of ways to section off the n horses with a $4 \times 2n$ space. As our base case, we note that $a(0) = 1$ and $a(1) = 1$. For our recursive step, we consider how we can get to n horses from smaller numbers. We note that the only two cases are starting with $n - 1$ stables and sticking a single vertical stable onto the end, or starting with $n-2$ stables and sticking two horizontal stables onto the end. So, our recursive formula then becomes $a(n) = a(n-1) + a(n-2)$, which is just the fibonacci sequence. Working our way up, we have $a(0) = 1$, $a(1) = 1$, $a(2) = 2$, $a(3) = 3$, $a(4) = 5$, $a(5) = 8$, $a(6) = 13$, $a(7) = 21$ and $a(8) = \boxed{34}$.

17. In the circle below, four quarter circle arcs are drawn with endpoints starting on the circumference of the outer circle. The shaded circle is externally tangent to all four of the quarter circle arcs. If the ratio of the radius of the inner shaded circle to the radius of outer circle can expressed as $\sqrt{a} - b$, where a and b are positive integers. Compute $a + b$.

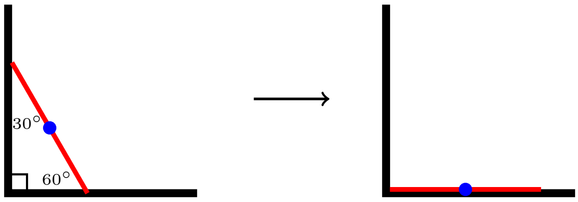


Answer: 3

Solution: Draw a square around the outer circle. Then, let O be the center of the circles, T be the point of tangency between the inner circle and a quarter arc, A be the corresponding corner of the square, and M be the midpoint of an adjacent side of the square, as shown below. Draw line segment $\overline{OA}$ which intersects the outer circle at point C. Then, draw line segment $\overline{OM}$ which has a length of $r =$ the radius of the outer circle. $\overline{OM}$ and $\overline{AM}$ are both half the side length of the outer square, so $AM = OM = r$. This gives us that triangle $\triangle OMA$ is a $45° - 45° - 90°$ triangle, so OA is just $r\sqrt{2}$. Furthermore, $\overline{AT}$ and $\overline{AM}$ are both radii of the given quarter circle, so $AT = AM = r$. From here, we see that the radius of the inner circle is $OT = OA - AT = r\sqrt{2} - r$, and the ratio of the radius of the inner circle to that of the outer circle is $\frac{r\sqrt{2}-r}{r} = \frac{r(\sqrt{2}-1)}{r} = \sqrt{2} - 1$. Thus, the answer is $1 + 2 = \boxed{3}$.

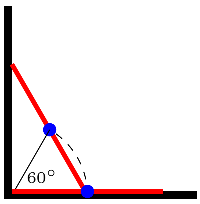


18. Sebi places a 6-inch stick with a pen attached to its midpoint in a corner such that the stick forms a 30-60-90 triangle, as shown below. He then slides the stick while keeping both ends in contact with the walls until it lies flat against the wall it previously made a 60° angle with. If the length of the path (in inches) that the pen traces out can be expressed as $a\pi$, where a is a positive integer, then what is the value of a?



Answer: 1

Solution:



The key observation here is that the path the pen traces out is a circle. This is just from the fact that the midpoint of the hypotenuse of any right triangle is equidistant from its 3 vertices, so it is always $6/2 = 3$ inches from the corner of the wall. We also know it traces through an angle of 60° because of the isoceles triangle formed by the stick and the radius above. So, the length is $\frac{60}{360}(2\pi \cdot 3) = \pi$ and the answer is $\boxed{1}$

19. A group of 2 boys and 2 girls are playing tag. A boy starts out as being "it." The probability that a girl is "it" after 6 random tags can be expressed as a reduced common fraction $\frac{a}{b}$. What is $a + b$?
(If you do not know, tag is a game where one person starts out as being "it." When they touch someone else, that person becomes "it" and the original person is no longer "it").

Answer: 1093

Solution 1: Let the probability that a girl is it after exactly k tags occurring be g_k. Then, $g_0 = 0$, and $g_k = \frac{2}{3}(1 - g_{k-1}) + \frac{1}{3}g_{k-1}$. We can see this because if a boy is it, then the probability a girl will be it next is $\frac{2}{3}$, and if a girl is it, then the probability a girl will be it next is $\frac{1}{3}$. Therefore, $g_1 = \frac{2}{3}$, $g_2 = \frac{4}{9}$, and $g_3 = \frac{14}{27}$. At this point, we see the pattern: if k is odd, then it is $\frac{3^k+1}{2\cdot 3^k}$, and if k is even, then it is $\frac{3^k-1}{2\cdot 3^k}$. Therefore, the probability that a girl is "it" is $\frac{3^6-1}{2\cdot 3^6}$, or $\frac{364}{729}$. So, our final answer is $364+729=\boxed{1093}$.

By the way, if we want to prove the formulas we used, we can do so using induction. Note that these formulas work for the first two cases ($\frac{3^0-1}{2\cdot 3^0}=0$ and $\frac{3^1+1}{2\cdot 3^1}=\frac{2}{3}$). From here, we can check that if the formula works for some even k, then it works for the following odd $k+1$. This is because $g_{k+1}=\frac{2}{3}(1-g_k)+\frac{1}{3}g_k=\frac{2}{3}(1-\frac{3^k-1}{2\cdot 3^k})+\frac{1}{3}\frac{3^k-1}{2\cdot 3^k}=\frac{3^{k+1}+1}{2\cdot 3^{k+1}}$. Similarly, if the formula works for some odd k then it works for the following even $k+1$ because $g_{k+1}=\frac{2}{3}(1-g_k)+\frac{1}{3}g_k=\frac{2}{3}(1-\frac{3^k+1}{2\cdot 3^k})+\frac{1}{3}\frac{3^k+1}{2\cdot 3^k}=\frac{3^{k+1}-1}{2\cdot 3^{k+1}}$. This is sufficient to prove that these formulas work for all even and odd k (you can research more about induction on google if you're interested).

Solution 2: Another approach to this problem is counting how many times the gender of the person who is it changes. Specifically, since we start with a boy but end up with a girl, we want an odd number of gender swaps to occur. Let's consider if just 1 gender swap occurs. For any given tag, there is a $\frac{2}{3}$ probability that a gender swap occurs and a $\frac{1}{3}$ probability that a gender swap does not occur. Since there are 6 tags and we are choosing 1 of them to be a gender swap, there are $\binom{6}{1}=6$ arrangements for the order of tags (in terms of when a gender swap occurs). Thus, the probability of 1 gender swap occurring is $6\cdot(\frac{2}{3})^1\cdot(\frac{1}{3})^5=\frac{12}{3^6}$ (this is just the binomial distribution if you are familiar with that). If 3 gender swaps occur, we can use similar logic to find a probability of $\binom{6}{3}\cdot(\frac{2}{3})^3\cdot(\frac{1}{3})^3=\frac{160}{3^6}$. Finally, if 5 gender swaps occur we get a probability of $\binom{6}{5}\cdot(\frac{2}{3})^5\cdot(\frac{1}{3})^1=\frac{192}{3^6}$. Thus, we get a total probability of $\frac{12+160+192}{3^6}=\frac{364}{729}$. So, our final answer is $364+729=\boxed{1093}$.

20. You are standing on the top right corner of a magical 2000 by 2022 grid of squares, all painted yellow. Then, you paint the square you're standing on green. Every second, you move one square to the left and one square down, and you paint the square you land on green (the magical grid teleports you to the rightmost column of squares if you cross the left edge, and it teleports you to the top row if you cross the bottom edge). Eventually, you get back to the same square you started at. How many squares have been painted green at that time?

Answer: 2022000

Solution: On each move, we paint exactly 1 square (painting a square at the start cancels out with not painting a square on the last move), so to see how many squares are painted once we return to the starting position we can count how many moves it takes to return to the starting position. After any multiple of 2000 moves, we will have gotten back to the right column. Similarly, after any multiple of 2022 moves we will have gotten back to the top row. We are at our starting position if and only if we are both on the top row and on the right column, so we will return to our starting position after a multiple of both 2000 and 2022 moves. The least such multiple is $\text{lcm}(2000, 2022)=\boxed{2022000}$.

21. If x and y are 2 positive integers chosen uniformly at random, then the probability that x^y and y^x have the same remainder when divided by 3 can be expressed as a reduced common fraction $\frac{a}{b}$. What is $a+b$?

Answer: 25

Solution: The first case is that x^y and y^x both have a remainder of 0 when divided by 3, which means that x and y must both be divisible by 3. This has a probability $\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$ of occurring.

The second case is that x^y and y^x both have a remainder of 2 when divided by 3. This means that x and y must both be odd and have a remainder of 2 when divided by 3. For both x and y, there is a $\frac{1}{2}$ chance of being odd and a $\frac{1}{3}$ chance of having a remainder of 2. So, we get a final probability of $(\frac{1}{2}\cdot\frac{1}{3})\cdot(\frac{1}{2}\cdot\frac{1}{3})=\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$ for this case.

The third case is that x^y and y^x both have a remainder of 1 when divided by 3, and there are three ways to do that. First, x and y can both have a remainder of 1 when divided by 3, which has probability $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ of occurring. Secondly, x or y is even and has a remainder of 1 whilst the other has a remainder of 2. There is a $\frac{1}{2}$ chance of the first number being even and a $\frac{1}{3}$ chance of it having a remainder of 1 while there is a $\frac{1}{3}$ chance of the second number having a remainder of 2. Furthermore, there are 2 ways to rearrange this (x is even and has a remainder of 1 whilst y has a remainder of 2, or vice versa), which means that the final probability for this option is $2 \cdot (\frac{1}{2} \cdot \frac{1}{3}) \cdot \frac{1}{3} = 2 \cdot \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{9}$. The last possible option is that x and y are both even and both have a remainder of 2 when divided by 3. Again, either number has a $\frac{1}{2}$ chance of being even and a $\frac{1}{3}$ chance of having a remainder of 2, which leads to a probability of $(\frac{1}{2} \cdot \frac{1}{3}) \cdot (\frac{1}{2} \cdot \frac{1}{3}) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ for this case.

Summing all of these values up, we can get that the final probability is $\frac{1}{9} + \frac{1}{36} + \frac{1}{9} + \frac{1}{9} + \frac{1}{36} = \frac{7}{18}$. So, our answer is $7 + 18 = \boxed{25}$.

22. Let a, b be positive integers that satisfy

$$2a\sqrt{3b} + 3b\sqrt{2a} - \sqrt{4044a} - \sqrt{6066b} + \sqrt{12132ab} = 2022.$$

Compute the sum of all possible values of a.

Answer: 338

Solution: Being by factoring the equation to be

$$2a\sqrt{3b} + 3b\sqrt{2a} - \sqrt{(2)(2022)a} - \sqrt{(3)(2022)b} + \sqrt{(6)(2022)ab} = 2022.$$

Let $x = \sqrt{2a}$, $y = \sqrt{3b}$ and $z = \sqrt{2022}$. Then

$$x^2y + xy^2 - xz - xy + xyz = z^2.$$

Rewriting this equation gives

$$x^2y + xy^2 + xyz = xz + yz + z^2$$

which factors as $xy(x + y + z) = z(x + y + z)$ and $xy = z$. So, $6ab = 2022 = (2)(3)(337)$ and $ab = 337$. Then, the sum of the possible values of a is just the sum of the factors of 337. 337 is a prime number, so the answer is just $1 + 337 = \boxed{338}$.

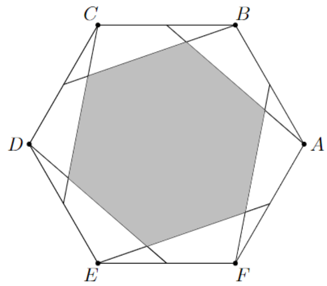
23. A sequence is constructed so that any given term of the sequence (except the first) is the perimeter of an equilateral triangle whose area is the preceding term. If the first three terms form a geometric sequence, and the value of the 3rd term can be expressed as $a\sqrt{b}$, find $a + b$

Answer: 15

Solution 1: Let s_0 be the first term. Then, letting a represent the side length of the equilateral triangle with area s_0, we use the area formula for an equilateral triangle to find: $s_0 = \frac{a^2\sqrt{3}}{4} \implies a = \frac{2\sqrt{s_0}}{\sqrt[4]{3}}$. The following term s_1 equals the perimeter of this triangle: $s_1 = 3a = 3 \cdot \frac{2\sqrt{s_0}}{\sqrt[4]{3}} = 2\sqrt[4]{27}\sqrt{s_0}$. Now, we find the common ratio of the geometric series is $\frac{2\sqrt[4]{27}}{\sqrt{s_0}}$, so the next term is, multiplying the second term by the common ratio, $4\sqrt[4]{27^2} = \boxed{12\sqrt{3}}$. Thus, the answer is $12 + 3 = \boxed{15}$.

Solution 2: Once we have $s_1 = 2\sqrt[4]{27}\sqrt{s_0}$, we use the same area-perimter relationship as before to compute $s_2 = 2\sqrt[4]{27}\sqrt{2\sqrt[4]{27}\sqrt{s_0}} = 2\sqrt{2}\sqrt[8]{27^3}\sqrt[4]{s_0}$. Because these terms are in a geometric sequence, $s_0 \cdot s_2 = s_1^2 \implies s_0 \cdot 2\sqrt{2}\sqrt[8]{27^3}\sqrt[4]{s_0} = 12\sqrt{3} \cdot s_0$. Solving yields $s_0 = 12\sqrt{3}$ (we disregard the option that $s_0 = 0$ because an equilateral triangle cannot have side length 0). Remember, $12\sqrt{3}$ is only the first term, but by computing the common ratio as $\frac{s_1}{s_0} = \frac{2\sqrt[4]{27}\sqrt{12\sqrt{3}}}{12\sqrt{3}} = 1$, we get the third term as $12\sqrt{3}$. Thus, the answer is $12 + 3 = \boxed{15}$

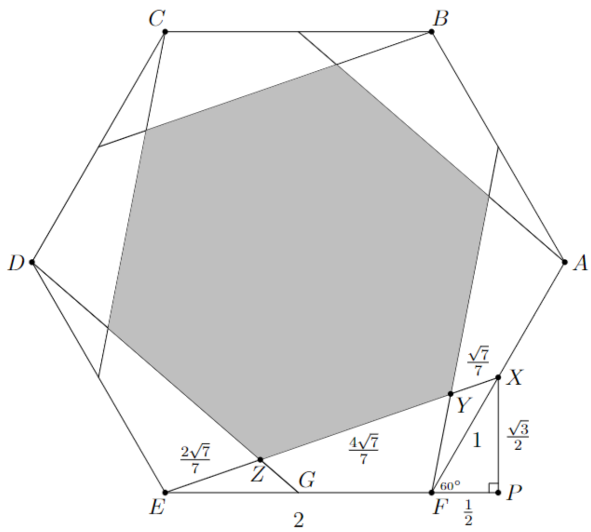
24. Consider regular hexagon $ABCDEF$. We form another regular hexagon by connecting each vertex of $ABCDEF$ to the midpoint of the side adjacent to it, as shown in the diagram. If the ratio of the area of the smaller hexagon to the area of $ABCDEF$ can be expressed as the reduced common fraction $\frac{a}{b}$, then what is $a+b$?



Answer: 11

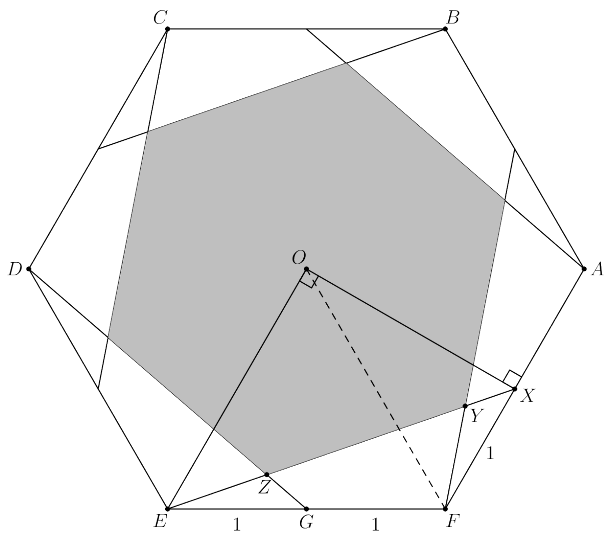
Solution 1: For simplicity, we assume the sidelength of $ABCDEF$ is 2. Label X, Y, and Z as shown in the diagram below (where X is the midpoint of $\overline{AF}$). We drop the perpendicular from X to the extension of $\overline{EF}$, intersecting at P, to form a $30-60-90$ triangle on the exterior of the hexagon (as $\angle EFA$ is $120°$). Using the properties of a $30-60-90$ triangle and the fact that $|\overline{FX}| = 1$, we can determine $|\overline{FP}| = \frac{1}{2}$ and $|\overline{XP}| = \frac{\sqrt{3}}{2}$. Then, using the pythagorean theorem on $\triangle EPX$, we determine $|\overline{EX}| = \sqrt{|\overline{EP}|^2 + |\overline{PX}|^2} = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{7}$.

Now, we note that $\triangle EZG \sim \triangle EFX$ by AAA similarity, as $\angle E \cong \angle E$, and $\angle EZG \cong \angle EFX = 120°$. The ratio of similarity for these two triangles is $\frac{\overline{EG}}{\overline{EX}} = \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7}$, so $|\overline{EZ}| = \frac{\sqrt{7}}{7} \cdot |\overline{EF}| = \frac{2\sqrt{7}}{7}$ and $|\overline{GZ}| = \frac{\sqrt{7}}{7} \cdot |\overline{XF}| = \frac{\sqrt{7}}{7}$. Note that $|\overline{XY}| = |\overline{GZ}| = \frac{\sqrt{7}}{7}$ by rotational symmetry, so $|\overline{YZ}| = |\overline{EX}| - |\overline{EZ}| - |\overline{XY}| = \sqrt{7} - \frac{2\sqrt{7}}{7} - \frac{\sqrt{7}}{7} = \frac{4\sqrt{7}}{7}$. Therefore, the ratio of the area of the inner hexagon to $ABCDEF$ is $\frac{|\overline{YZ}|^2}{|\overline{EF}|^2} = \frac{4}{7}$ and our answer is $4 + 7 = \boxed{11}$.

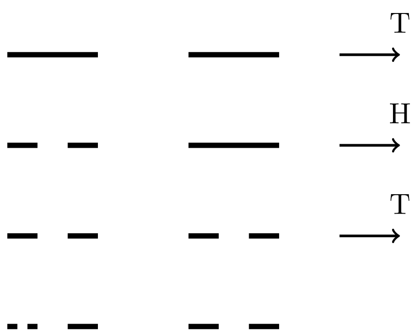


Solution 2: As before, we assume the sidelength of $ABCDEF$ is 2 and label X, Y, and Z as shown in the diagram below. Now, we connect the center of the two hexagons, labelled O, to points E and X to form a right triangle with right angle $\angle EOX$. We know $\angle EOX$

is right because $\angle EOF$ is $\frac{1}{6}$ of a full circle or $60°$ and $\angle FOX$ is half of that or $30°$, and $30 + 60 = 90$. Since $\overline{EO}$ is just the sidelength of equilateral $\triangle EOF$, $|\overline{EO}| = 2$. Similarly, either using $30 - 60 - 90$ identities on $\triangle FOX$, the formula for the apothem of a hexagon, or the pythagorean theorem on triangle $\triangle FOX$, we can get $|\overline{OX}| = \sqrt{3}$. Then, using the pythagorean theorem on $\triangle EXO$ yields $|\overline{EX}| = \sqrt{\sqrt{3}^2 + 2^2} = \sqrt{7}$. From here, the rest of the solution follows as in the second paragraph of solution 1. So, the ratio of the area of the inner hexagon to $ABCDEF$ is $\frac{|\overline{YZ}|^2}{|\overline{EF}|^2} = \frac{4}{7}$. Thus, our answer is $4 + 7 = \boxed{11}$.



25. Contor is playing with a rope of length L. He begins by cutting out the middle third. Then, every minute, he flips a coin. If it lands tails he cuts out the middle third of the leftmost segment of rope and if it lands heads he cuts out the middle third of the rightmost segment of rope. For example, the diagram below shows the result of the rope after the sequence of flips THT. The expected number of minutes until the length of the rope is less than or equal to $\frac{11L}{27}$ can be expressed as a reduced common fraction $\frac{a}{b}$. What is $a + b$?



Answer: 9

Solution 1:

The rope starts off with a length of $2L/3$ (because the middle third is removed), so we need to cut out at least $\frac{2L}{3} - \frac{11L}{27} = \frac{7L}{27}$. The fastest way to cut out $7L/27$ would be to do two cuts on one side and one cut on the other side,

$$\frac{L}{9} + \frac{L}{9} + \frac{L}{27} = \frac{7L}{27}.$$

However, we could also do more than two cuts on one side and one cut on the other side. In fact, there is no limit to how many cuts we do on one side before we cut the other side, because cutting only one side will never allow us to reach the threshold of $\frac{7L}{27}$ cut out. This can be seen with the following infinite geometric series:

$$L\left(\frac{1}{9}+\frac{1}{27}+\cdots\right)=L\frac{1/9}{2/3}=\frac{L}{6}<\frac{7L}{27}.$$

Thus, we can split this problem up into two cases: those where we reach the threshold in only 3 cuts, and those that take longer than 3 cuts (note, you could split up the cases in different ways, but the core logic of this solution should still apply). For our first case, there are 6 options:

$$\text{HHT},\text{HTH},\text{HTT},\text{THH},\text{THT},\text{TTH}$$

All of these options are specific sequences of 3 coin flips, so they all take 3 minutes and have probability $\frac{1}{2^3}=\frac{1}{8}$ of occurring. Since there are 6 such options, this first case contributes $6\cdot 3\cdot\frac{1}{8}=\frac{9}{4}$ to the exected number of minutes it takes to reach the threshold.

For the second case, there are 2 options:

$$\underbrace{\text{TTT}\ldots\text{TT}}_{k\geq 3\text{ times}}H,$$
$$\underbrace{\text{HHH}\ldots\text{HH}}_{k\geq 3\text{ times}}T$$

This case represents when the first 3 coin flips all have the same outcome. For now, let's consider the option where the first 3 coin flips are all tails. Then, we expect it to take 2 more coin flips until we get a heads. This is a well-known result (similar to the expected number of rolls until you get a 6 on a 6-sided die being 6), and can be derived by setting up a recursive formula (you could also use an arithmo-geometric series as shown in solution 2). For the recursive formula approach, if E represents the expected number of flips needed until you flip a heads, then $E=\frac{1}{2}E+1$. This is because after flipping the coin once (hence the 1), there is a $\frac{1}{2}$ chance that we get a tails and have to continue flipping (hence the $\frac{1}{2}E$). Solving this equation, we find $E=2$.

Going back to the problem at hand, there are 2 options for this second case, each of which are expected to take a total of $3+2=5$ minutes to reach the threshold and have a probability of $\frac{1}{8}$ of occurring (because each option is defined by a specific beginning sequence of 3 coin flips). So, this second case contributes $2\cdot 5\cdot\frac{1}{8}=\frac{5}{4}$ to the expected number of minutes it takes to reach the threshold. This leads us to an expected value of $\frac{9}{4}+\frac{5}{4}=\frac{14}{4}=\frac{7}{2}$ minutes. So, our final answer is $7+2=\boxed{9}$.

Solution 2:

This solution follows exactly as solution 1, except we calculate the expected number of flips until we get a heads (or a tails) differently. Specifically, let's consider the sequence:

$$\underbrace{\text{TT}\ldots\text{T}H}_{k\geq 1\text{ flips}}$$

We could also consider flips until we get a tails, but the problem is symmetric so the expected number of flips taken will just be the same. Note that in this set-up, the probability of the sequence being k flips long is simply $(\frac{1}{2})^k = \frac{1}{2^k}$. From here, we can sum up the probability of getting k flips multiplied by the time taken (which is just k) for all possible values of k, in order to find the expected number of flips it takes until we get a heads. This gives us the following arithmo-geometric series (which yields the same result as before):

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = (\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots) + (\frac{1}{2^2} + \frac{1}{2^3} + \cdots) + (\frac{1}{2^3} + \frac{1}{2^4} + \cdots) + \cdots = \sum_{k=1}^{\infty} \frac{1/2^k}{1 - 1/2} = 2$$

26. Consider a $4m$ by n grid, where m and n are positive integers chosen uniformly at random from the range $[1, 100]$, inclusive. Define the "Manhattan Distance" between two cells in a grid to be the sum of their horizontal distance and vertical distance (the cells $(1, 3)$ and $(2, 1)$ have a Manhattan distance of $|2 - 1| + |1 - 3| = 3$). Owen wants to use the colors blue and red to color some cells of the grid with the following restrictions:

- For any two distinct cells colored the same color, the Manhattan Distance between them must be even.
- For any two distinct cells colored different colors, the Manhattan Distance between them must be odd.
- It is ok for **none** of the cells to be colored or **all** of the cells to be colored.
- No cell can be colored with both colors.

Suppose $f(m, n)$ denotes the number of such possible colorings. Compute the expected value of k, where k is the largest integer such that $f(m, n) + 1$ is divisible by 2^k.

Answer: 10202

Solution: Suppose we color the $4m$ by n grid as a white and black checkerboard. The key observation is that all cells to be colored red must either all occupy black squares or all occupy white squares, and all cells to be colored blue must all occupy the opposite type of square. This condition is necessary and sufficient for the Manhattan Distance between two cells of the same color to be even and of different colors to be odd. Thus, given any set of the $4mn$ cells, there are 2 cases for coloring those cells red and blue: either have all the red cells on the white squares and blue cells on the black squares, or vice versa. For each of these cases, there are 2 options per square (either colored the appropriate color or not colored) giving 2^{4mn} options per case. However, we count the empty grid in both cases, so we need to subtract one. This gives us a final result that

$$f(m, n) = 2 \cdot 2^{4mn} - 1 = 2^{4mn+1} - 1 \implies f(m, n) + 1 = 2^{4mn+1}$$

So, we have $k = 4mn + 1$. Letting $E(k)$ denote the expected value of k, we get $E(k) = E(4mn + 1)$, but m and n are independent, so $E(k) = 4 \cdot E(m) \cdot E(n) + 1 = 4 \cdot \frac{101}{2} \cdot \frac{101}{2} + 1 = 101^2 + 1 = \boxed{10202}$.

27. Given a regular octagonal prism, we define a *diagonal* as any line segment connecting two distinct vertices of the prism. How many pairs of diagonals in a regular octagonal prism intersect (not counting intersections at a vertex of the prism)?

Answer: 240

Solution: First, we note that the problem is the same as finding the planes that go through 4 or more vertices of the prism. Once we have such a plane, we can choose any 4 vertices on this plane and there will be exactly one way to assign diagonals with those endpoints such that they intersect.

With this, we note that there are two planes that intersect 8 vertices each (namely, the top and bottom faces of our prism). For each of these faces we have $\binom{8}{4} = 70$ different ways to make diagonals, for a total of $2 * 70 = 140$ ways.

Then, we note that the only other category of ways to get a plane intersecting 4 or more vertices is to have a plane intersect exactly 4 vertices, 2 on the top face, and 2 on the bottom face. We additionally realize that this occurs if and only if the line between the two vertices on the top face is parallel to the line between the two vertices on the bottom face. So, the problem simplifies down to counting the number of ways that two not-necessarily-distinct lines can be parallel to each other in an octagon.

As you can see in the diagram below, we have two cases. One of them, shown in blue, is if the lines are parallel to the sides of the octagon, for which we find 4 lines. So, there are 4 possible lines on the top face times 4 possible lines on the bottom face, which gives us 16 options (for any one orientation). The other case, shown in red, is if the diagonals are parallel to the longest diagonal of the octagon, for which there are 3 lines so we have $3 * 3 = 9$ options (again, for any one orientation). Now, noting that we can orient these lines in 4 unique ways relative to the octagon, we have a total of $(16 + 9) * 4 = 100$ ways for this category. Therefore, our total number of pairs of diagonals is $140 + 100 = \boxed{240}$.