1. **LT651** The average of the six numbers $1, 3, 5, 7, 9, N$ is equal to the average of the five numbers $2, 4, 6, 8, 10$. What is N?

   Written by: Linus Tang

   Answer: $\boxed{11}$

   The average of the five numbers $2, 4, 6, 8, 10$ is equal to $\frac{2+4+6+8+10}{5} = 6$.

   If the average of six numbers is 6, then their sum is 36.

   $1 + 3 + 5 + 7 + 9 + N = 36$

   $25 + N = 36$

   $N = \boxed{11}$

2. **LT897** Let $a \diamond b$ represent the value $ab + 100a$. What is the value of $18 \diamond 17 - 17 \diamond 18$?

   Written by: Linus Tang

   Answer: $\boxed{100}$

   $18 \diamond 17 - 17 \diamond 18 = (18 \cdot 17 + 100(18)) - (17 \cdot 18 + 100(17))$. After the $18 \cdot 17$ cancels out with $-17 \cdot 18$, this is $100(18) - 100(17) = \boxed{100}$.

3. **DL409** The sequence $a_1, a_2, \ldots, a_{2023}$ is a geometric sequence. Compute the common ratio if $a_{2023} = a_{2018} \cdot 243$.

   Written by: Daniel Z Lu

   Answer: $\boxed{3}$

   Let the common ratio of this geometric sequence be r. Because there are 5 terms in between 2023 and 2018, we can write our given equation as: $a_{2023} = a_{2018} \cdot r^5 = a_{2018} \cdot 243$. This gives us the equation $r^5 = 243$. To solve for r, we can take the fifth root of both sides to get $r = 3$, so the common ratio is $\boxed{3}$.

4. **LT295** Walking at constant rates, it takes Alice the same amount of time to walk from the park to the library as it takes Bob to walk from the school to the supermarket. Walking at the same constant rates, it takes Bob 44% longer to walk from the park to the library than it takes Alice to walk from the school to the supermarket. What is the ratio of Alice's walking speed to Bob's walking speed? Express your answer as a common fraction in simplest form.

   Written by: Linus Tang

   Answer: $\boxed{\frac{6}{5}}$

Let a be Alice's speed, b be Bob's speed, p be the distance from the park to the library, and s be the distance from the school to the supermarket.

Applying the "Distance = Rate · Time" formula, we have $\frac{p}{a} = \frac{s}{b}$ and $\frac{p}{b} = 1.44\frac{s}{a}$.

Dividing the second equation by the first, we have $\frac{a}{b} = 1.44\frac{b}{a}$.

Multiplying by $\frac{a}{b}$ on both sides,
$(\frac{a}{b})^2 = 1.44$
$\frac{a}{b} = \sqrt{1.44} = 1.20$

So Alice is 120% as fast as Bob, giving us an answer of $1.20 = \boxed{\frac{6}{5}}$.

5. **SR454** A photograph is 5 inches wide and 8 inches tall. It is mounted in a frame with a non-zero border x inches wide on all sides. If the border's width were doubled, the area of the frame would increase by 150%. What is the original width x? Express your answer as a common fraction.

Written by: Shak Ragoler

Answer: $\boxed{\frac{13}{6}}$

We can find the areas of the original and expanded frames by taking the area of the photograph with the frame and subtracting the area of the portion of the photograph without the frame.

The area of the original frame is $(2x + 8)(2x + 5) - 40$, while the area of the expanded frame is $(4x+8)(4x+5)-40$. This gives us $(4x+8)(4x+5)-40 = 2.5[(2x+8)(2x+5)-40]$.

Solving, we get $x = \boxed{\frac{13}{6}}$ inches.

6. **TK891** What positive real value x satisfies $x^x = (2x)^{2x}$? Express your answer as a common fraction.

Written by: Tristan Kay

Answer: $\boxed{\frac{1}{4}}$

We can manipulate the equation using exponent rules

$$x^x = 2^{2x} \cdot x^{2x}$$
$$1 = 4^x \cdot x^x$$
$$1 = (4x)^x$$

Since 1 to the power of anything is 1, we want our exponent base to equal 1. Thus, we get that our answer is $\boxed{\frac{1}{4}}$.

7. **LT713** Let a and b be real numbers, and define the function $f(x) = ax + b$. Given that $f(f(f(0))) = 2023$ and $f(f(f(1))) = 2031$, what is $f(0)$?

Written by: Linus Tang

Answer: $\boxed{289}$

We compute $f(f(f(0)))$ in terms of a and b:

$2023 = f(f(f(0)))$

$= f(f(a(0) + b))$

$= f(f(b))$

$= f(a(b) + b)$

$= a(a(b) + b) + b$

$= a^2b + ab + b$

$= (a^2 + a + 1)b.$

Note that $f(f(f(x)))$ is a linear function with a slope of a^3. This slope can also be written as $\frac{f(f(f(1)))-f(f(f(0)))}{1-0} = 2031 - 2023 = 8$.

Thus, $a^3 = 8$ and $a = 2$.

Now, $2023 = (a^2 + a + 1)b$

$2023 = 7b$

$b = 289.$

The answer to the problem is $f(0) = a(0) + b = b = \boxed{289}$.

8. **YS961** Let r_1, r_2, and r_3 be the roots of $x^3 + x - 1$. What is value of the expression below?

$$\frac{1}{1-r_1^3} + \frac{1}{1-r_2^3} + \frac{1}{1-r_3^3}$$

Written by: Yuuki Sawanoi

Answer: $\boxed{1}$

Note that $r_1^3 + r_1 - 1 = 0 \implies r_1 = 1 - r_1^3$. The same equation will hold when we replace the variable with either r_2 or r_3. Therefore, we can substitute to get

$$\frac{1}{1-r_1^3} + \frac{1}{1-r_2^3} + \frac{1}{1-r_3^3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{r_1r_2 + r_2r_3 + r_1r_3}{r_1r_2r_3}.$$

The denominator is the product of the roots and the numerator is the pairwise sum, so using Vieta's formulas we have

$$\frac{r_1r_2 + r_2r_3 + r_1r_3}{r_1r_2r_3} = \frac{1}{1} = \boxed{1}.$$

9. **LT925** There are 10 lamps in a row from left to right, and all of them are initially off. One day, Mussy Mustang flips the switch of the first 4 lamps from the left. The next day, Mussy flips the switch of the first 5 lamps from the left. The next day, Mussy flips the switch of the first 6 lamps from the left, and so on. This process stops at the end of the day that Mussy flips the switch of all 10 lamps. At this point, how many lamps are on?

   Written by: Linus Tang

   Answer: $\boxed{7}$

   If a switch is flipped an odd number of times in total, then it will be on at the end, otherwise it will be off.

   We can list out the number of times each switch is flipped: 7, 7, 7, 7, 6, 5, 4, 3, 2, 1. Clearly, the $1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}, 4^{\text{th}}, 6^{\text{th}}, 8^{\text{th}}$, and 10^{th} switches are on, so our answer is $\boxed{7}$.

10. **YS944** Tristan flips a fair coin an infinite number of times and records the sequence of H (Heads) and T (Tails). What is the probability that the first instance of TT appears before the first instance of HT? Express your answer as a common fraction.

    Written by: Yuuki Sawanoi

    Answer: $\boxed{\frac{1}{4}}$

    Note that if Tristan ever flips H, it is impossible to get TT before HT. Hence, the only way for him to get TT is to flip two tails in a row on his first two flips, which occurs with probability $\boxed{\frac{1}{4}}$.

11. **LT901** Bob uses a random number generator to pick 3 (not necessarily distinct) digits from 0 to 9. What is the probability that the sum of the digits is 27? Express your answer as a common fraction.

    Written by: Linus Tang

    Answer: $\boxed{\frac{1}{1000}}$

    The only way to get a sum of 27 is when all three digits are the maximum value of 9, since any other triple would have a sum less than 27. Each digit has a $\frac{1}{10}$ probability of being a 9, thus the total probability is $\left(\frac{1}{10}\right)^3 = \boxed{\frac{1}{1000}}$.

12. **LT278** Angela is in charge of scheduling 5 meetings for her math club. The first meeting will be on Tuesday, September 3. The gap between adjacent meetings must be between 4 and 10 days, inclusive. Furthermore, meetings cannot be scheduled for Saturday nor Sunday. How many ways can Angela schedule the remaining 4 meetings? For example, this means that the second meeting must be on a weekday between September 7 and September 13.

    Written by: Linus Tang

    Answer: $\boxed{625}$

    After each meeting, there must be 4 to 10 days until the next meeting. Based on this rule alone, there is a streak of 7 consecutive days in which Angela can schedule the next meeting. However, exactly 2 of these 7 days are Saturday and Sunday, so there are only 5 days for Angela to schedule the next meeting.

    The day of the first meeting is fixed, and there are 5 ways to schedule each of the 4 subsequent meetings. Thus, there are $5^4 = \boxed{625}$ ways to schedule the remaining meetings.

13. **LT838** How many ways are there to arrange the letters in the word $MUSTANG$ such that there is no string of three or more consecutive consonants?

    Written by: Linus Tang

    Answer: $\boxed{720}$

    Given that there are five consonants, two vowels, and no three consecutive consonants, there are only 3 possible arrangements of vowels and consonants:

    $CCVCCVC$

    $CCVCVCC$

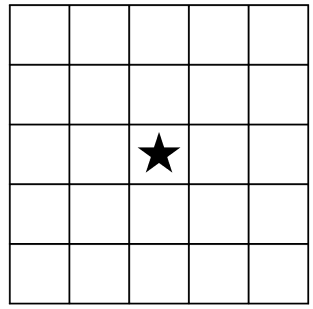
    $CVCCVCC$

    Where C denotes a consonant and V denotes a vowel.

    After deciding on one of these 3 arrangements, there are 5! ways to determine the locations of the 5 consonants and 2! ways to determine the locations of the 2 vowels.

    Thus, the total number of arrangements satisfying the given property is $3 \cdot 5! \cdot 2! = \boxed{720}$.

14. **LT832** In the 5 by 5 grid below, the center cell is marked with a star. Giorgio wants to color each of the 24 remaining cells red, blue, yellow, or green such that the cells of each color form a single rectangle. How many colorings are possible?



Written by: Linus Tang

Answer: $\boxed{384}$

Notice that the squares directly to the top, right, bottom, or to the left of the star must be part of different rectangles. Therefore, there are $4! = 24$ ways to assign each of these directions a color.

After these four squares have been colored, the colors of the squares on each edge are determined (for example, the top middle square must be the same color as the one directly below it).

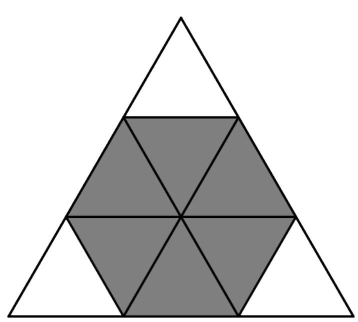
Now, each 2 by 2 chunk in the corner must be assigned the color of one of its neighboring edges. Thus, there are 2^4 ways to color these corner squares. The answer is $24 \times 16 = \boxed{384}$

15. **HN930** If a regular hexagon is inscribed in equilateral triangle such that 2 points of the hexagon lie on each side of the triangle, what is the ratio of the area of hexagon to the area of the triangle? Express your answer as a common fraction.

Written by: Harshil

Answer: $\boxed{\dfrac{2}{3}}$

The hexagon, denoted by the shaded area, covers $\frac{6}{9}$ of the equilateral triangles shown below, our answer is $\frac{6}{9} = \boxed{\dfrac{2}{3}}$



16. **SS917** If an isosceles triangle has vertex angle $12°$ larger than the other two, what is

the degree measure of the vertex angle?

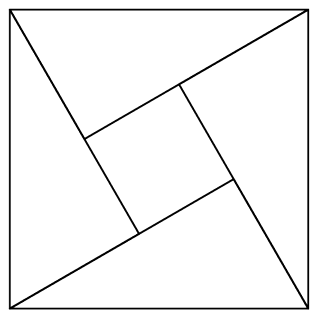
Written by: *Saahil Shah*

Answer: $\boxed{68}$

We can set the two non-vertex angles to be x and get $2x+(x+12)=180 \to 3x+12=180 \to 3x=168 \to x=56$.

However, our goal is to find the vertex angle, which is 12° larger than x, so our answer is $56+12=\boxed{68}$.

17. **LT649** In the diagram below, 4 congruent right-angled triangles and a small square are arranged to form a large square. If the area of the small square is 5 and the area of each triangle is 19, what is the side length of the large square?



Written by: *Linus Tang*

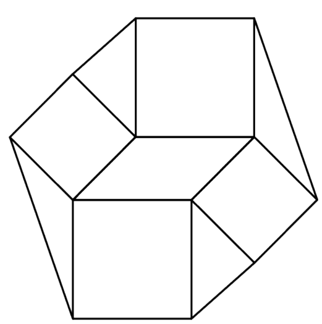
Answer: $\boxed{9}$

The area of the large square is the sum of the areas of the five pieces, or

$5+19+19+19+19=81.$

If s is the side length of the large square, then its area is $s^2=81$. So, s is the square root of 81, which is $\boxed{9}$.

18. **LT836** In the diagram below, squares of length 6 and 8 have been constructed outside a parallelogram. The eight outer vertices are then connected to form an octagon with area of 302. What is the area of parallelogram?



Written by: Linus Tang

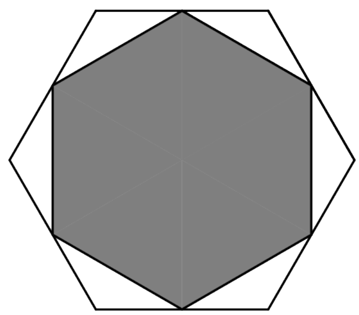
Answer: $\boxed{34}$

By SAS congruence, each of the four outer triangles is congruent to half of the parallelogram. Therefore, the total area of the four triangles and the parallelogram is $3x$, where x is the area of the parallelogram. The area of the octagon is now

$$2(6^2 + 8^2) + 32 = 302.$$

Solving for x, we get $x = \boxed{34}$.

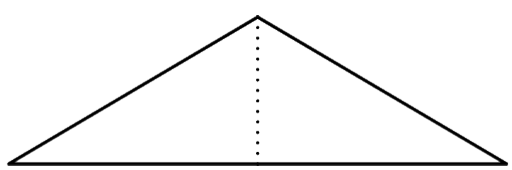
19. **DC774** A regular hexagon has side length 4. Then, a second hexagon is formed by connecting the midpoints of the first hexagon. Compute the positive difference between the values of the perimeter and area of the second hexagon. Express your answer in simplest radical form.



Written by: Daniel Chirakarn

Answer: $\boxed{6\sqrt{3}}$

Let's focus on one of the 6 white isosceles triangles in the diagram. These triangles have two legs of length 2 with an included angle of 120°.



We can drop altitude in this triangle to create two 30-60-90 triangles. Thus, using the special ratio, half of the side length is $\sqrt{3}$. Thus the side length of the inner hexagon is twice of that: $2\sqrt{3}$. Then, the perimeter is

$$P = 6 \cdot 2\sqrt{3} = 12\sqrt{3},$$

and the area is (by splitting the hexagon into 6 equilateral triangles and using the formula for an equilateral triangle: $A = \frac{\sqrt{3}s^2}{4}$)

$$A = 6 \cdot \frac{(2\sqrt{3})^2\sqrt{3}}{4} = 18\sqrt{3}.$$

So, the requested answer is $18\sqrt{3} - 12\sqrt{3} = \boxed{6\sqrt{3}}$.

20. **LT753** Let $\triangle ABC$ be an equilateral triangle and P be a point on the incircle (circle inscribed in) $\triangle ABC$. If the distances from P to $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ are $2, 5$, and a, what is the product of the possible values of a?

Written by: Linus Tang

Answer: $\boxed{9}$

Let P' be the point inside the triangle such that the distances from P' to AB and AC are 5 and 2, respectively. By symmetry, P' also lies on the incircle of $\triangle ABC$. Also, extending PP' to hit AB and AC at B' and C', we have $AB' = AC'$ by symmetry and $\angle B'AC' = \angle BAC = 60°$. So, $AB'C'$ is an equilateral triangle. By Viviani's Theorem on point P, $AB'C'$ has an altitude of length $2+5+0 = 7$. Thus, by ratios in a 30-60-90 triangle, we can calculate $AB' = \frac{14}{\sqrt{3}}$.

Letting T be the tangency point between AB and the incircle of $\triangle ABC$, which is also the midpoint of AB, we have by Power of a Point that $B'T = \sqrt{B'P \cdot B'P'} = \sqrt{(2 \cdot \frac{2}{\sqrt{3}})(5 \cdot \frac{2}{\sqrt{3}})} = \frac{2\sqrt{10}}{\sqrt{3}}$.

Thus, $AT = AB' \pm B'T = \frac{14 \pm 2\sqrt{10}}{\sqrt{3}}$. Now AB is twice of that, or $\frac{28 \pm 4\sqrt{10}}{\sqrt{3}}$. The height of $\triangle ABC$ is $\frac{\sqrt{3}}{2} \cdot AB = 14 \pm 2\sqrt{10}$.

Finally, we apply Viviani's Theorem again and find that $a = 14 \pm 2\sqrt{10} - 5 - 2 = 7 \pm 2\sqrt{10}$. The product of the possible values of a is $(7 + 2\sqrt{10})(7 - 2\sqrt{10}) = \boxed{9}$.

21. **SS374** The 3-digit number $\overline{ABC}$ is a perfect square and a multiple of 14. If $A+B+C \geq 18$, what is $\overline{ABC}$?

Written by: Saahil Shah

Answer: $\boxed{784}$

Since $\overline{ABC}$ is a multiple of 14, it must be even and a multiple of 7. $\overline{ABC}$ is also a perfect square, meaning it must be the perfect square of a number that is a multiple of 14, since 14 has no perfect square factors. The only 2 numbers that are perfect squares, multiples of 14, and have 3 digits are 14^2 and 28^2. $14^2 = 196$ does not satisfy the rule of $A + B + C$ being greater than or equal to 18, meaning our answer is $28^2 = \boxed{784}$.

22. **LT910** Albert wrote down a number N. Betty wrote down the number that equals the sum of the digits of N. Carol wrote down the sum of the digits of Betty's number. If Carol wrote the number 11, what is the smallest possible value N could have been?

Written by: Linus Tang

Answer: $\boxed{2999}$

Working backwards, the smallest number Betty could have written is 29, so the sum of the digits of N is at least 29.

Because of this, N must be at least 4 digits (or else Betty's number would be at most $9+9+9=27$).

To minimize N, we want to maximize the later digits, which leads to $N = \boxed{2999}$.

We can check when $N = 2999$ that Betty indeed writes 29 and Carol indeed writes 11.

23. **CY629** Positive integers x and y satisfy

$$\frac{\gcd(x,y)}{x} + \frac{1}{\operatorname{lcm}(x,y)} = \gcd(x,y)$$

Compute $x+y$.

Written by: Carsten Yeung

Answer: $\boxed{3}$

By multiplying each side by our denominators, we have

$$\operatorname{lcm}(x,y)\gcd(x,y) + x = \gcd(x,y)\operatorname{lcm}(x,y)x$$

Recall that $\gcd(x,y)\cdot\operatorname{lcm}(x,y) = xy$. Knowing that, we can manipulate the expression into:

$$\begin{aligned} xy + x &= x^2y \\ 1 + y &= xy \\ 1 &= xy - y \\ 1 &= y(x-1) \end{aligned}$$

The only integer factors of 1 are 1 and -1. Since x, y are positive integers, y and $x-1$ must both equal 1. Thus, $x = 2$ and $y = 1$ making the solution to the problem $\boxed{3}$.

24. **SR478** There is a pile of n coins, where n is an integer between 100 and 200 inclusive. Alice and Bob take turns playing a game, starting with Bob. Bob may remove either 3 or 5 coins on his turn, while Alice may remove either 2 or 4 coins on her turn. A player loses if, on their turn, they have no valid moves. For example, if it is Bob's turn and there are 2 coins on the stack, then Alice wins because there are not enough coins for Bob to remove either 3 or 5 of them. For how many values of n does Alice have a winning strategy, no matter how Bob plays?

Written by: Shak Ragoler

Answer: $\boxed{43}$

If there are at least 7 coins, then when one player removes x coins, the other player can remove $7 - x$ coins to prevent the module of 7 from being altered. Thus, we can simplify our solution by only considering starting seven cases where there are at least 7 coins for each modulo of 7.

If we start with 7, 8, or 9 coins ($7 \equiv 0 \pmod 7$, $8 \equiv 1 \pmod 7$, and $9 \equiv 2 \pmod 7$), then if Bob removes x coins, Alice would remove $7-x$ coins, leaving Bob with 0, 1, and 2 coins respectively and producing a victory for Alice.

If we start with 10 or 11 coins ($10 \equiv 3 \pmod 7$ and $11 \equiv 4 \pmod 7$), then Bob could remove 3 coins, have Alice remove x coins, remove another $7 - x$ coins himself, and leave Alice with 0 or 1 coins respectively and producing a victory for himself.

If we start with 12 or 13 coins ($12 \equiv 5 \pmod 7$ and $13 \equiv 6 \pmod 7$), then Bob could remove 5 coins, have Alice remove x coins, remove another $7 - x$ coins himself, and leave Alice with 0 or 1 coins respectively and producing a victory for himself.

Thus, if we start with x coins, Alice wins if $x \equiv 0 \pmod 7$, $x \equiv 1 \pmod 7$, or $x \equiv 2 \pmod 7$. In the interval from 100 to 200, this occurs exactly $\boxed{43}$ times.

25. **HN929** What is the greatest integer k such that 23^k evenly divides into $\gcd(2023! + 2025!, 2024! + 2026!)$?

Written by: Harshil

Answer: $\boxed{90}$

Let $n = 2023$. Our expression then becomes $\gcd(n! + (n+2)!, (n+1)! + (n+3)!)$.

Factoring out $n!$ from this expression, we get:

$$n!\cdot\gcd(1+(n+2)(n+1), (n+1)+(n+3)(n+2)(n+1)) = n!\cdot\gcd(n^2+3n+3, n^3+6n^2+12n+7).$$

When we divide $n^3 + 6n^2 + 12n + 7$ by $n^2 + 3n + 3$, we get a remainder of 2. So, by the GCD with remainder Theorem we can now write our equation as:

$$n! \cdot \gcd(n^2 + 3n + 3, 2).$$

We can now write $n^2 + 3n + 3$ as $(n+2)(n+1)+1$. Because $(n+2)(n+1)$ is the product of an even and an odd number, it will always be even. This means that $n^2 + 3n + 3$ will always be odd for when n is a non-negative integer. Therefore, $\gcd(n^2 + 3n + 3, 2) = 1$.

This means that the GCD of our original expression will be 2023!. Using Legendre's Formula, the power of 23 in the prime factorization of 2023! is:

$$\left\lfloor \frac{2023}{23} \right\rfloor + \left\lfloor \frac{2023}{23^2} \right\rfloor = 87 + 3 = 90.$$

Therefore, the greatest value of k is $\boxed{90}$

26. **LT267** What is the smallest positive integer n such that n^n is divisible by 2023^{2023} but n is not divisible by 2023?

Written by: Linus Tang

Answer: $\boxed{4165}$

First, note that the prime factorization of 2023 is $7^1 \cdot 17^2$.

Thus, $2023^{2023} = 7^{2023} \cdot 17^{4046}$.

Since n^n is divisible by this number, n must be divisible by 7 and 17.

Furthermore, the exponent of 17 in the prime factorization of n must be exactly 1; if it were greater, then n would be divisible by 2023.

Thus, the exponent of 17 in the prime factorization of n^n is exactly n.

Recalling that n^n is divisible by $2023^{2023} = 7^{2023} \cdot 17^{4046}$, we now know $n \geq 4046$ based on the exponent of the prime 17.

We cannot have $n = 4046$ as that is divisible by 2023, so we check that the next smallest multiple of $7 \cdot 17$, which is $\boxed{4165}$. We check that this number works.

Indeed, $2023^{2023} = 7^{2023} \cdot 17^{4046} \mid (7 \cdot 17)^{4165} \mid 4165^{4165}$, so $n = 4165$ works.