1. **LT651** The average of the six numbers $1, 3, 5, 7, 9, N$ is equal to the average of the five numbers $2, 4, 6, 8, 10$. What is N?

   Written by: Linus Tang

   Answer: $\boxed{11}$

   The average of the five numbers $2, 4, 6, 8, 10$ is equal to $\frac{2+4+6+8+10}{5} = 6$.

   If the average of six numbers is 6, then their sum is 36.

   $1 + 3 + 5 + 7 + 9 + N = 36$

   $25 + N = 36$

   $N = \boxed{11}$

2. **LT897** Let $a \diamond b$ represent the value $ab + 100a$. What is the value of $18 \diamond 17 - 17 \diamond 18$?

   Written by: Linus Tang

   Answer: $\boxed{100}$

   $18 \diamond 17 - 17 \diamond 18 = (18 \cdot 17 + 100(18)) - (17 \cdot 18 + 100(17))$. After the $18 \cdot 17$ cancels out with $-17 \cdot 18$, this is $100(18) - 100(17) = \boxed{100}$.

3. **DL409** The sequence $a_1, a_2, \ldots, a_{2023}$ is a geometric sequence. Compute the common ratio if $a_{2023} = a_{2018} \cdot 243$.

   Written by: Daniel Z Lu

   Answer: $\boxed{3}$

   Let the common ratio of this geometric sequence be r. Because there are 5 terms in between 2023 and 2018, we can write our given equation as: $a_{2023} = a_{2018} \cdot r^5 = a_{2018} \cdot 243$. This gives us the equation $r^5 = 243$. To solve for r, we can take the fifth root of both sides to get $r = 3$, so the common ratio is $\boxed{3}$.

4. **DL410** Compute the value of m so that the points $(-2m+1, -4)$ and $(-6m+8, m-6)$ lie on the same vertical line. Express your answer as a common fraction.

   Written by: Daniel Z Lu

   Answer: $\boxed{\frac{7}{4}}$

   For 2 points to lie on the same vertical line, their x-coordinates must be equal. Solving the equation $-2m + 1 = -6m + 8$ for m results in $m = \boxed{\frac{7}{4}}$.

5. **AR347** Alon and Reese are superhumans running on an arbitrarily long, straight path. Alon runs consistently 10 miles per minute. Reese runs consistenly at 35 miles per minute, however has very little stamina, so she needs to take a 6 minute break after every 2 minutes of running in order to rest (she immediately begins running again after every 6 minute break). How many times will they meet, not including the very beginning?

   Written by: Alon

   Answer: $\boxed{10}$

   Create a table with Alon and Reese's position at time t, where t is either $8k$ or $8k-6$ for some positive integer k. At time t, Alon's position is always $10t$, and Reese's position is $70k$ (where k is defined as before). Then, it is easy to see that Alon and Reese will meet at some point between time 2 and 8, 8 and 10, 10 and 16, and so on, all the way up to 34 and 40. Then, their positions will be exactly the same at 42, but beyond this, Alon will always be farther ahead than Reese, since his average speed is higher. Therefore, the answer is $\boxed{10}$.

6. **KB783** If a, b, and c are the roots of the polynomial $x^3-4x^2+12x+32$, compute $\frac{abc}{a+b+c}$.

   Written by: Krish Bhandari

   Answer: $\boxed{-8}$

   Note that by Vieta's formulas, $abc = -32$ and $a+b+c = 4$. Thus, the answer is $\frac{-32}{4} = \boxed{-8}$.

7. **TK891** What positive real value x satisfies $x^x = (2x)^{2x}$? Express your answer as a common fraction.

   Written by: Tristan Kay

   Answer: $\boxed{\dfrac{1}{4}}$

   We can manipulate the equation using exponent rules

$$x^x = 2^{2x} \cdot x^{2x}$$

$$1 = 4^x \cdot x^x$$

$$1 = (4x)^x$$

   Since 1 to the power of anything is 1, we want our exponent base to equal 1. Thus, we get that our answer is $\boxed{\dfrac{1}{4}}$.

8. **LT295** Walking at constant rates, it takes Alice the same amount of time to walk from the park to the library as it takes Bob to walk from the school to the supermarket. Walking at the same constant rates, it takes Bob 44% longer to walk from the park to

the library than it takes Alice to walk from the school to the supermarket. What is the ratio of Alice's walking speed to Bob's walking speed? Express your answer as a common fraction in simplest form.

Written by: Linus Tang

Answer: $\boxed{\frac{6}{5}}$

Let a be Alice's speed, b be Bob's speed, p be the distance from the park to the library, and s be the distance from the school to the supermarket.

Applying the "Distance = Rate · Time" formula, we have $\frac{p}{a} = \frac{s}{b}$ and $\frac{p}{b} = 1.44\frac{s}{a}$.

Dividing the second equation by the first, we have $\frac{a}{b} = 1.44\frac{b}{a}$.

Multiplying by $\frac{a}{b}$ on both sides,
$(\frac{a}{b})^2 = 1.44$
$\frac{a}{b} = \sqrt{1.44} = 1.20$

So Alice is 120% as fast as Bob, giving us an answer of $1.20 = \boxed{\frac{6}{5}}$.

9. **LT901** Bob uses a random number generator to pick 3 (not necessarily distinct) digits from 0 to 9. What is the probability that the sum of the digits is 27? Express your answer as a common fraction.

Written by: Linus Tang

Answer: $\boxed{\frac{1}{1000}}$

The only way to get a sum of 27 is when all three digits are the maximum value of 9, since any other triple would have a sum less than 27. Each digit has a $\frac{1}{10}$ probability of being a 9, thus the total probability is $\left(\frac{1}{10}\right)^3 = \boxed{\frac{1}{1000}}$.

10. **LT925** There are 10 lamps in a row from left to right, and all of them are initially off. One day, Mussy Mustang flips the switch of the first 4 lamps from the left. The next day, Mussy flips the switch of the first 5 lamps from the left. The next day, Mussy flips the switch of the first 6 lamps from the left, and so on. This process stops at the end of the day that Mussy flips the switch of all 10 lamps. At this point, how many lamps are on?

Written by: Linus Tang

Answer: $\boxed{7}$

If a switch is flipped an odd number of times in total, then it will be on at the end, otherwise it will be off.

We can list out the number of times each switch is flipped: 7, 7, 7, 7, 6, 5, 4, 3, 2, 1. Clearly, the $1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}, 4^{\text{th}}, 6^{\text{th}}, 8^{\text{th}}$, and 10^{th} switches are on, so our answer is $\boxed{7}$.

11. **LT926** Sami swaps 2 (not necessarily adjacent) letters in the word $HOOFS$ to form a different string of letters. How many different strings of letters can result?

Written by: Linus Tang

Answer: $\boxed{9}$

Counting systematically, there are 10 pairs of letters in the five-letter word:

$HO_1, HO_2, HF, HS,$
$O_1O_2, O_1F, O_1S,$
$O_2F, O_2S,$
$FS,$

where O_1 is the first O and O_2 is the second. However, Sami cannot swap O_1O_2, since the result is required to be a different string of letters than the original word.

So, there are $10 - 1 = \boxed{9}$ different swaps that can be made, and each of them results in a different string of letters.

Remark: the answer could also be more quickly computed as $\binom{5}{2} - 1$.

12. **YS944** Tristan flips a fair coin an infinite number of times and records the sequence of H (Heads) and T (Tails). What is the probability that the first instance of TT appears before the first instance of HT? Express your answer as a common fraction.

Written by: Yuuki Sawanoi

Answer: $\boxed{\frac{1}{4}}$

Note that if Tristan ever flips H, it is impossible to get TT before HT. Hence, the only way for him to get TT is to flip two tails in a row on his first two flips, which occurs with probability $\boxed{\frac{1}{4}}$.

13. **LT278** Angela is in charge of scheduling 5 meetings for her math club. The first meeting will be on Tuesday, September 3. The gap between adjacent meetings must be between 4 and 10 days, inclusive. Furthermore, meetings cannot be scheduled for Saturday nor Sunday. How many ways can Angela schedule the remaining 4 meetings? For example, this means that the second meeting must be on a weekday between September 7 and September 13.

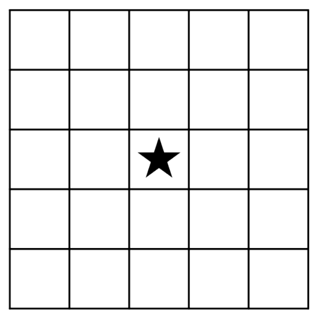
Written by: Linus Tang

Answer: $\boxed{625}$

After each meeting, there must be 4 to 10 days until the next meeting. Based on this rule alone, there is a streak of 7 consecutive days in which Angela can schedule the next meeting. However, exactly 2 of these 7 days are Saturday and Sunday, so there are only 5 days for Angela to schedule the next meeting.

The day of the first meeting is fixed, and there are 5 ways to schedule each of the 4 subsequent meetings. Thus, there are $5^4 = \boxed{625}$ ways to schedule the remaining meetings.

14. **LT832** In the 5 by 5 grid below, the center cell is marked with a star. Giorgio wants to color each of the 24 remaining cells red, blue, yellow, or green such that the cells of each color form a single rectangle. How many colorings are possible?



Written by: Linus Tang

Answer: $\boxed{384}$

Notice that the squares directly to the top, right, bottom, or to the left of the star must be part of different rectangles. Therefore, there are $4! = 24$ ways to assign each of these directions a color.

After these four squares have been colored, the colors of the squares on each edge are determined (for example, the top middle square must be the same color as the one directly below it).

Now, each 2 by 2 chunk in the corner must be assigned the color of one of its neighboring edges. Thus, there are 2^4 ways to color these corner squares. The answer is $24 \times 16 = \boxed{384}$

15. **SS917** If an isosceles triangle has vertex angle $12°$ larger than the other two, what is the degree measure of the vertex angle?

Written by: Saahil Shah

Answer: $\boxed{68}$

We can set the two non-vertex angles to be x and get $2x + (x + 12) = 180 \rightarrow 3x + 12 = 180 \rightarrow 3x = 168 \rightarrow x = 56$.

However, our goal is to find the vertex angle, which is $12°$ larger than x, so our answer is $56 + 12 = \boxed{68}$.

16. **SS916** The vertices of a triangle are $(1, 1), (4, 5)$, and $(9, 1)$ on a Cartesian coordinate plane. What is the area of the triangle formed by these three vertices?

Written by: Saahil Shah

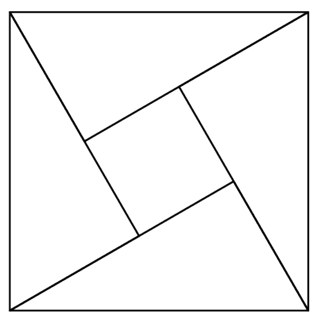
Answer: $\boxed{16}$

First, form a rectangular box around the triangle with vertices at (1,1), (1,5), (9,5) and (9,1).

The rectangle has an area of $8 \cdot 4 = 32$ and the two extraneous triangles have areas of $\frac{4 \cdot 3}{2} = 6$ and $\frac{5 \cdot 4}{2} = 10$.

We can subtract the area of the 2 extraneous triangles from the area of the rectangle to get the final answer of $32 - 6 - 10 = \boxed{16}$.

17. **LT649** In the diagram below, 4 congruent right-angled triangles and a small square are arranged to form a large square. If the area of the small square is 5 and the area of each triangle is 19, what is the side length of the large square?



Written by: Linus Tang

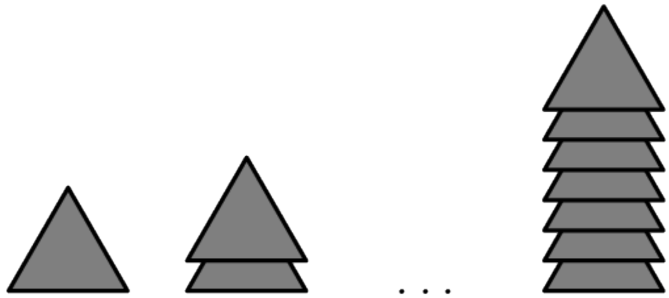
Answer: $\boxed{9}$

The area of the large square is the sum of the areas of the five pieces, or

$5 + 19 + 19 + 19 + 19 = 81$.

If s is the side length of the large square, then its area is $s^2 = 81$. So, s is the square root of 81, which is $\boxed{9}$.

18. **YS962** Yuuki is making a Christmas tree in April using 7 equilateral triangles of length 2. He layers each triangle such that it covers exactly half the area of the triangle below it, as shown in the diagram below. What is the total area of Yuuki's tree?



Written by: *Yuuki Sawanoi*

Answer: $\boxed{4\sqrt{3}}$

Yuuki's tree consists of 6 half triangles and a whole triangle. Letting the area of a whole triangle be A, the total area is

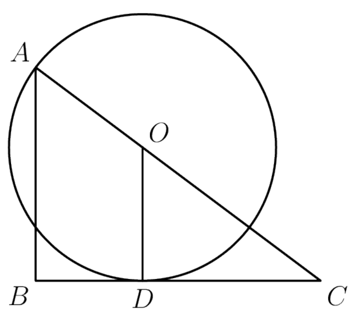
$$6(\frac{A}{2}) + A = 4A.$$

Since the equilateral triangles have a side length of 2, we have $A = \frac{2^2\sqrt{3}}{4} = \sqrt{3}$. Thus, the answer is $4A = \boxed{4\sqrt{3}}$.

19. **LT732** Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Let ω be a circle whose center lies on $\overline{CA}$. If ω passes through A and is tangent to $\overline{BC}$, what is the radius of ω? Express your answer as a common fraction.

Written by: *Linus Tang*

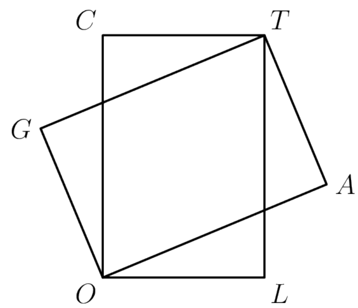
Answer: $\boxed{\frac{15}{8}}$



Let the radius of ω be r and the altitude from O onto BC be D . Then, notice that $AO = r$ and that by similar triangles, $\frac{OD}{OC} = \frac{AB}{AC} = \frac{3}{5}$, so $OC = \frac{5}{3}AO = \frac{5r}{3}$. Thus, we have that $AC = AO + OC = r + \frac{5r}{3} = \frac{8r}{3} = 5$. Thus, we have that $r = \frac{3}{8} \times 5 = \boxed{\frac{15}{8}}$

20. **LT420** Let $COLT$ and $GOAT$ be distinct congruent rectangles sharing the same

diagonal $\overline{OT}$, as shown below. Suppose each rectangle has side lengths of 6 and 9. What is the total area covered by the two rectangles?



Written by: Linus Tang

Answer: $\boxed{69}$

Without loss of generality, $CO = 9$ and $GO = 6$. Suppose we call the point where CO and GT intersect N. By symmetry, $CN = GN$, and we let k be this common length. Now $NO = 9 - k$.

By the Pythagorean Theorem of right angle $\angle OGN$, we have $GO^2 + GN^2 = NO^2$. Substituting, we have $6^2 + k^2 = (9 - k)^2$, and solving, we find that $k = \frac{5}{2}$. The total area covered by the two rectangles is given by $GOAT + 2\triangle CNT$, so we get that the area is $6 + 2(\frac{1}{2} \cdot k \cdot 6) = 54 + 6k = 54 + 15 = \boxed{69}$.

21. **SS374** The 3-digit number $\overline{ABC}$ is a perfect square and a multiple of 14. If $A+B+C \geq 18$, what is $\overline{ABC}$?

Written by: Saahil Shah

Answer: $\boxed{784}$

Since $\overline{ABC}$ is a multiple of 14, it must be even and a multiple of 7. $\overline{ABC}$ is also a perfect square, meaning it must be the perfect square of a number that is a multiple of 14, since 14 has no perfect square factors. The only 2 numbers that are perfect squares, multiples of 14, and have 3 digits are 14^2 and 28^2. $14^2 = 196$ does not satisfy the rule of $A + B + C$ being greater than or equal to 18, meaning our answer is $28^2 = \boxed{784}$.

22. **LT910** Albert wrote down a number N. Betty wrote down the number that equals the sum of the digits of N. Carol wrote down the sum of the digits of Betty's number. If Carol wrote the number 11, what is the smallest possible value N could have been?

Written by: Linus Tang

Answer: $\boxed{2999}$

Working backwards, the smallest number Betty could have written is 29, so the sum of the digits of N is at least 29.

Because of this, N must be at least 4 digits (or else Betty's number would be at most $9+9+9=27$).

To minimize N, we want to maximize the later digits, which leads to $N = \boxed{2999}$.

We can check when $N = 2999$ that Betty indeed writes 29 and Carol indeed writes 11.

23. **LT719** What fraction that lies strictly between $\frac{2}{3}$ and $\frac{3}{4}$ has the smallest positive integer denominator?

Written by: Linus Tang

Answer: $\boxed{\dfrac{5}{7}}$

Solution 1: Let the fraction be $\frac{p}{q}$. To minimize q, we try all of the positive integers q from smallest to largest until we find a solution.

In the work below, "too large" means that $\frac{p}{q}$ is greater than or equal to $\frac{3}{4}$ and "too small" means that $\frac{p}{q}$ is less than or equal to $\frac{2}{3}$.

Trying $q = 1$, $\frac{0}{1}$ is too small but $\frac{1}{1}$ is too large.

Trying $q = 2$, $\frac{1}{2}$ is too small but $\frac{2}{2}$ is too large.

Trying $q = 3$, $\frac{2}{3}$ is too small but $\frac{3}{3}$ is too large.

Trying $q = 4$, $\frac{2}{4}$ is too small but $\frac{3}{4}$ is too large.

Trying $q = 5$, $\frac{3}{5}$ is too small but $\frac{4}{5}$ is too large.

Trying $q = 6$, $\frac{4}{6}$ is too small but $\frac{5}{6}$ is too large.

When $q = 7$, $\frac{5}{7}$ is indeed between $\frac{2}{3}$ and $\frac{3}{4}$.

Thus, choosing $p = 5$ and $q = 7$ will minimize q. The answer is $\boxed{\dfrac{5}{7}}$

Solution 2:

Let the fraction be $\frac{p}{q}$. We are given the following condition, which we can manipulate

$$\frac{2}{3} < \frac{p}{q} < \frac{3}{4}$$
$$\frac{3}{2} > \frac{q}{p} > \frac{4}{3}$$

$$\frac{3p}{2} > q > \frac{4p}{3}$$
$$\frac{4p}{3} < q < \frac{3p}{2}$$

To minimize q, we must minimize p. Therefore, testing values of p starting from 1, we check if a positive integer solution in q exists.

After trial and error, we find that $p = 5$ yields $6\frac{2}{3} < q < 7\frac{1}{2}$, which has a positive integer solution of $q = 7$. The answer is $\boxed{\frac{5}{7}}$.

24. **YS937** 323323 is the product of n consecutive prime numbers. What is the value of n?

Written by: Yuuki Sawanoi

Answer: $\boxed{5}$

We can write the number 323323 as $323000 + 323$. Factoring 323 from this expression gives $323323 = 323(1000 + 1) = 323(1001)$. Since $323 = 17 \cdot 19$ and $1001 = 7 \cdot 11 \cdot 13$, this means that $323323 = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. Therefore, our answer is $\boxed{5}$.

25. **CY629** Positive integers x and y satisfy

$$\frac{\gcd(x, y)}{x} + \frac{1}{\operatorname{lcm}(x, y)} = \gcd(x, y)$$

Compute $x + y$.

Written by: Carsten Yeung

Answer: $\boxed{3}$

By multiplying each side by our denominators, we have

$$\operatorname{lcm}(x, y)\gcd(x, y) + x = \gcd(x, y)\operatorname{lcm}(x, y)x$$

Recall that $\gcd(x, y) \cdot \operatorname{lcm}(x, y) = xy$. Knowing that, we can manipulate the expression into:

$$\begin{aligned} xy + x &= x^2y \\ 1 + y &= xy \\ 1 &= xy - y \\ 1 &= y(x - 1) \end{aligned}$$

The only integer factors of 1 are 1 and -1. Since x, y are positive integers, y and $x - 1$ must both equal 1. Thus, $x = 2$ and $y = 1$ making the solution to the problem $\boxed{3}$.

26. **LT267** What is the smallest positive integer n such that n^n is divisible by 2023^{2023} but n is not divisible by 2023?

Written by: Linus Tang

Answer: $\boxed{4165}$

First, note that the prime factorization of 2023 is $7^1 \cdot 17^2$.

Thus, $2023^{2023} = 7^{2023} \cdot 17^{4046}$.

Since n^n is divisible by this number, n must be divisible by 7 and 17.

Furthermore, the exponent of 17 in the prime factorization of n must be exactly 1; if it were greater, then n would be divisible by 2023.

Thus, the exponent of 17 in the prime factorization of n^n is exactly n.

Recalling that n^n is divisible by $2023^{2023} = 7^{2023} \cdot 17^{4046}$, we now know $n \geq 4046$ based on the exponent of the prime 17.

We cannot have $n = 4046$ as that is divisible by 2023, so we check that the next smallest multiple of $7 \cdot 17$, which is $\boxed{4165}$. We check that this number works.

Indeed, $2023^{2023} = 7^{2023} \cdot 17^{4046} \mid (7 \cdot 17)^{4165} \mid 4165^{4165}$, so $n = 4165$ works.