1. Rithwick has a toilet paper roll in the shape of a hollow cylinder, where the hollowed part has a radius of 1 and the entire roll has a radius of 7. If Rithwick uses up half of the toilet paper roll, by volume, to achieve a new hollow cylinder still with inner radius 1, what is the new outer radius of his roll?

   Answer: 5

   Solution: Without loss of generality, assume the height of the toilet paper roll is 1. We can find the original volume of the toilet paper roll by finding the volume of the outer cylinder and then subtracting the volume of the hole. The outer cylinder has a volume of $\pi \cdot 7^2$ and the inner cylinder has a volume of $\pi \cdot 1^2$. Therefore, the original roll had a volume of $\pi \cdot 7^2 - \pi \cdot 1^2$. Similarly, the new roll has a volume of $\pi \cdot x^2 - \pi \cdot 1^2$. Because half of the toilet paper has been used, we equate the final volume to half of the inital volume of toilet paper:

$$\pi \cdot x^2 - \pi \cdot 1^2 = \frac{\pi \cdot 7^2 - \pi \cdot 1^2}{2}.$$

   By solving the equation, we obtain that $x = \sqrt{\frac{7^2+1^2}{2}} = \boxed{5}$.

2. Define the *confused factorial* of n (denoted $n?!$) as the arithmetic mean of $(n-1)!$ and $(n+1)!$, where $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$, and $0! = 1$. If the sum $\frac{1?!}{2!} + \frac{2?!}{3!} + \cdots + \frac{20?!}{21!}$ can be expressed as a reduced common fraction $\frac{a}{b}$, compute the value of $a+b$.

   Answer: 241

   Solution: We expand $k?! = \frac{(k-1)!+(k+1)!}{2} = (k-1)! \cdot \frac{1+k(k+1)}{2}$. Then, the inner term of the sum is

$$\frac{(k-1)! \cdot \frac{1+k(k+1)}{2}}{(k+1)!} = \frac{(k-1)! \cdot (1+k(k+1))}{2(k+1)!} = \frac{1+k(k+1)}{2k(k+1)}.$$

   We can expand the fraction as

$$\frac{1+k(k+1)}{2k(k+1)} = \frac{1}{2k(k+1)} + \frac{k(k+1)}{2k(k+1)} = \frac{1}{2k(k+1)} + \frac{1}{2}.$$

   We can expand the first term as $\frac{1}{2k(k+1)} = \frac{1}{2k} - \frac{1}{2(k+1)}$. However, this creates a telescoping series, and at the end we are left with $\frac{1}{2 \cdot 1} - \frac{1}{2 \cdot 21} = \frac{10}{21}$. Adding back in the $\frac{1}{2}$ term, our answer is $\frac{10}{21} + 20 \cdot \frac{1}{2} = \frac{220}{21}$. We add the numerator and the denominator to get $\boxed{241}$.

3. Alan has a square pyramid with volume V. If the base side lengths and the height are all increased by 50%, Alan obtains a new square pyramid with volume V'. If the ratio $\frac{V'}{V}$ can be expressed as a reduced common fraction $\frac{a}{b}$, compute $a+b$.

   Answer: 35

   Solution: Let the side length of the base be x and the height be y. Since the volume of a pyramid is given by $\frac{1}{3}bh$ where b and h are the area of the base and the height of the pyramid, the volume of the pyramid can be expressed as $\frac{1}{3}x^2y$. Then increasing all the dimensions of the pyramid by 50% amounts to having $x \mapsto \frac{3}{2}x$ and $y \mapsto \frac{3}{2}y$. We see that 3 new factors of $\frac{3}{2}$ are present in the volume. But $\frac{x^2y}{3} = V$, so the answer is $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$, and we add the numbers in this ratio to get $\boxed{35}$.

   In general, multiplying all dimensions of a 3-dimensional object by some constant c will scale the volume of the object by c^3.

4. Owen selects an ordered quadruplet of (not necessarily distinct) positive integers (a, b, c, d) uniformly at random such that $a \cdot b \cdot c \cdot d = 336$. If the probability that at least one of a, b,

c, and d is divisible by 14 can be expressed as a reduced common fraction $\frac{a}{b}$, compute the value of $a+b$.

Answer: 11

Solution: We factorize $336 = 2^4 \cdot 3 \cdot 7$. Without loss of generality, assume $7 \mid d$. Since the only factor of 7 was distributed to d, there must be at least one factor of 2 distributed to d to satisfy the problem statement. Hence, we can count the complement, the probability that there are no factors of 2 distributed to d. By sticks and stones, there are $\binom{7}{4}$ ways to distribute the factors of 2 to a, b, c, d (3 sticks and 4 stones) and $\binom{6}{4}$ ways to distribute the factors of 2 to a, b, c (2 sticks and 4 stones). Therefore, our desired answer is $\frac{\binom{7}{4}-\binom{6}{4}}{\binom{7}{4}} = \frac{20}{35} = \frac{4}{7}$. Adding the numerator and the denominator results in the final answer of $\boxed{11}$.

5. There exist two distinct positive integers a and b, both between 1 and 100 (exclusive), such that 2022 leaves a remainder of 1 when divided by either a or b. Compute the value of $\left|a^2 - b^2\right|$.

   Answer: 360

   Solution: If a and b are the two integers in question, then a and b must divide $2022 - 1 = 2021$. We know that 2021 has two factors smaller than 100: 43 and 47. The difference of their squares is $(47-43)(47+43) = \boxed{360}$.

6. Let $a_1, a_2, \ldots$ be an arithmetic sequence where $a_5 = 9$, $a_9 = 17$, and $2^{10} - 3$ is the n^{th} term of the sequence. Compute the value of n.

   Answer: 511

   Solution: Since the sequence is an arithmetic progression, $a_1 + a_9 = 2a_5$. From this, we get that $a_1 = 1$, from which we compute that the common difference is $d = 2$. Therefore the nth term is given by $a_n = a_1 + (n-1)d = 2n - 1$. Then $2^{10} - 3 = 2 \cdot (2^9 - 1) - 1$, so $n = 2^9 - 1 = \boxed{511}$.