1. **LT729** What is $16 \times 117 + 6 \times 8$?

   Written by: Linus Tang

   Answer: $\boxed{1920}$

   Solution 1:

   $16 \times 117 + 6 \times 8$

   $= 16 \times 125 - 16 \times 8 + 6 \times 8$

   $= 16 \times 125 - 10 \times 8$

   $= 2^4 \times 5^3 - 80$

   $= 2 \times 10^3 - 80$

   $= 2000 - 80$

   $= \boxed{1920}$

   Solution 2:

   Computing directly,

   $16 \times 117 + 6 \times 8$

   $= 1872 + 48$

   $= \boxed{1920}$

2. **LT731** Let m and n be positive integers. Albert calculates the quantity $1+2+3+\cdots+n$ and Betty calculates the quantity $5+6+7+\cdots+m$. Given that Albert and Betty get the same sum, what is this sum?

   Written by: Linus Tang

   Answer: $\boxed{45}$

   The sum of the first n positive integers is given by $\frac{n^2+n}{2}$.

   So, $\frac{n^2+n}{2} = 5 + 6 + 7 + \cdots + m$.

   $1 + 2 + 3 + 4 + \frac{n^2+n}{2} = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + m$

   $10 + \frac{n^2+n}{2} = \frac{m^2+m}{2}$

   $20 = m^2 + m - n^2 - n$

   $20 = m^2 - n^2 + m - n$

   $20 = (m - n)(m + n) + m - n$

$20 = (m-n)(m+n+1)$

Note that $m-n$ and $m+n+1$ have opposite parity, meaning that one of them is even and the other is odd.

The ways to factor 20 into numbers with opposite parity are (4)(5) and (1)(20).

The former gives $m = 4, n = 0$, which is not allowed since n is positive.

The latter gives $m = 10, n = 9$.

Thus, the answer is the common sum $1+2+3+\cdots+9 = 5+6+7+\cdots+10 = \boxed{45}$.

3. **TK919** Let $ABCD$ be a square. If the largest circle that can fit inside $ABCD$ has area 2023π, what is the area of $ABCD$?

Written by: Tristan Kay

Answer: $\boxed{8092}$

The area of a circle is given by πr^2, where r is the radius. Thus, $\pi r^2 = 2023\pi$, so $r = \sqrt{2023}$.

The side length of the square is twice the radius of the circle, or $2\sqrt{2023}$. Its area is $(2\sqrt{2}023)^2 = 2^2 \cdot \sqrt{2023}^2 = 4 \cdot 2023 = \boxed{8092}$.

4. **LT647** Let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x. For example, $\lfloor 3.5 \rfloor = 3$ and $\lfloor -1.3 \rfloor = -2$. Evaluate the following expression:

$$\left\lfloor \frac{10}{10} \right\rfloor + \left\lfloor \frac{10}{9} \right\rfloor + \cdots + \left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{1} \right\rfloor + \left\lfloor \frac{10}{-1} \right\rfloor + \left\lfloor \frac{10}{-2} \right\rfloor + \cdots + \left\lfloor \frac{10}{-9} \right\rfloor + \left\lfloor \frac{10}{-10} \right\rfloor$$

Written by: Linus Tang

Answer: $\boxed{-6}$

Solution 1:

Regroup the terms as follows:

$\left(\left\lfloor \frac{10}{10} \right\rfloor + \left\lfloor \frac{10}{-10} \right\rfloor\right) + \left(\left\lfloor \frac{10}{9} \right\rfloor + \left\lfloor \frac{10}{-9} \right\rfloor\right) + \cdots + \left(\left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{-2} \right\rfloor\right) + \left(\left\lfloor \frac{10}{1} \right\rfloor + \left\lfloor \frac{10}{-1} \right\rfloor\right)$

Note that each pair is of the form $\lfloor x \rfloor + \lfloor -x \rfloor$.

This is helpful because $\lfloor x \rfloor + \lfloor -x \rfloor = 0$ when x is an integer and $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ when x is not an integer.

It now suffices to count how many pairs above are equal to -1.

Note that $\frac{10}{10}, \frac{10}{5}, \frac{10}{2}$, and $\frac{10}{1}$ are integers and the other 6 positive fractions are not integers.

Thus, 6 of the pairs are equal to -1 and 4 of them are equal to 0. The overall sum is $\boxed{-6}$.

Solution 2:

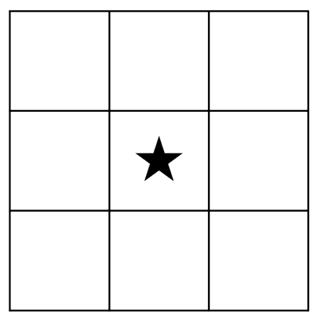
Evaluate each term individually and add them up:

$\left\lfloor \frac{10}{10} \right\rfloor + \left\lfloor \frac{10}{9} \right\rfloor + \cdots + \left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{1} \right\rfloor + \left\lfloor \frac{10}{-1} \right\rfloor + \left\lfloor \frac{10}{-2} \right\rfloor + \cdots + \left\lfloor \frac{10}{-9} \right\rfloor + \left\lfloor \frac{10}{-10} \right\rfloor$

$= 1+1+1+1+1+2+2+3+5+10+(-10)+(-5)+(-4)+(-3)+(-2)+(-2)+(-2)+(-2)+(-2)+(-1)$

$= \boxed{-6}$.

5. **LT831** In the 3 by 3 grid below, the center cell is marked with a star. How many rectangles created by the gridlines contain the star?



Written by: Linus Tang

Answer: $\boxed{16}$

Observe that a rectangle is bounded by a vertical line to its left, a vertical line to its right, a horizontal line above, and a horizontal line below. To choose a rectangle created by the gridlines that contains the star, there are two choices for each of these four bounding lines, so the number of such rectangles is $2^4 = 16$.

6. **ML804** There exist pairs of primes (p, q) that satisfy the equation $20^2 + p = 23^2 - q$. Compute the sum of all distinct possible values of pq.

Written by: Michael Li

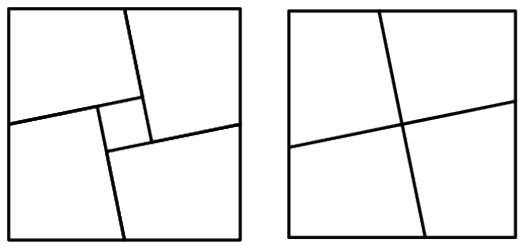
Answer: $\boxed{254}$

Rearranging terms, we get $23^2 - 20^2 = p + q$. Computing the left side gives $529 - 400 = 129 = p + q$.

Since $p + q$ is odd, one of p, q must be even. But p and q are prime, and 2 is the only even prime number. Thus, p or q is 2 and the other is $129 - 2 = 127$ (which is also prime). The answer is $2 \cdot 127 = \boxed{254}$.

7. **ARP564** A square of side length 5 is cut into four identical quadrilaterals and a

unit square, such that the four quadrilaterals can be moved and rotated to form a new square, as shown in the diagram below. What is the sum of the smallest and largest side lengths of one of the four quadrilaterals? Express your answer in simplest radical form.



Before After

Written by: Arpit Ranasaria

Answer: $\boxed{2\sqrt{6}}$

Notice that the smallest and largest sides of the quadrilateral together make a side of the smaller square. Hence the problem becomes finding the length of the smaller square.

Since the area of the larger square is $5 \cdot 5 = 25$ and we've removed a unit square to make the smaller one, so the area of the smaller square is $25 - 1 = 24$ and the side length is $\sqrt{24} = \boxed{2\sqrt{6}}$.

8. **AC883** Arpit needs to schedule a meeting with Anna. This week, both are available for 3 uniformly randomly selected days of the week (independent of each other). What is the probability that there is a day where both are available to meet?

Written by: Alex Chen

Answer: $\boxed{\dfrac{31}{35}}$

We solve using complementary counting.

Consider the four days during which Arpit is unavailable. The only scenario in which Arpit and Anna have no common availability is when all three of Anna's available days are among Arpit's four unavailalbe days. There are $\binom{4}{3} = 4$ possible triples of days that fall under this scenario, out of $\binom{7}{3} = 35$ possible triples in total.

Thus, the probability that Arpit and Anna have no common availability is $\frac{4}{35}$.

The probability that they have common availability is $1 - \frac{4}{35} = \boxed{\dfrac{31}{35}}$.

9. **TK922** Let $\triangle ABC$ and $\triangle ACD$ be similar right triangles with $\angle ABC = \angle ACD = 90^\circ$ and $\angle BAC = \angle CAD$. If $AB = 20$ and $AD = 23$, what is the ratio of the area of $\triangle ABC$ to $\triangle ACD$? Express your answer as a common fraction.

Written by: Tristan Kay

Answer: $\boxed{\frac{20}{23}.}$

Let the ratio between the side lengths of $\triangle ACD$ to $\triangle ABC$ be r. Then we have $\frac{AD}{AC} = \frac{AC}{AB} = r$. Since, $r^2 = \frac{AD}{AB} = \frac{23}{20}$, the wanted ratio is $\frac{1}{r^2} = \frac{20}{23}$. Thus our answer is $23 + 20 = \boxed{43}$.

Alternate slick solution: draw BD and let BD intersect AC at E. Then we know ABC to ACD is BE to ED which, by angle bisector theorem, is $\frac{AB}{AD} = \frac{20}{23}$.

10. **YS949** Let r be the answer to the third question in this set. Compute the sum of the digits of r.

Written by: Yuuki Sawanoi

Answer: $\boxed{18}$

Since $10^n \equiv 1 \pmod 9$ for any nonnegative integer n, the sum of the digits of a number has the same remainder as the original number when divided by 9. Thus, the only values we need to check are multiples of 9, and we get that $\boxed{18}$ is the only one that satisfies the cycle.

11. **MLI326** Let u be the answer to the first question in this set. Compute the sum of the two solutions to the quadratic $x^2 + (x-1)^2 + (x-2)^2 + \cdots + (x-u)^2 = 2023^2$.

Written by: Michael Liu

Answer: $\boxed{18}$

Solution 1: Let r be a root of the quadratic. If we plug in r to the quadratic, we get:

$$r^2 + (r-1)^2 + (r-2)^2 + \cdots + (r-u)^2 = 2023^2.$$

Note that by symmetry, we can also plug in $u - r$ into this quadratic to get:

$$(u-r)^2 + (u-r-1)^2 + (u-r-2)^2 + \cdots + (-r)^2 = 2023^2.$$

The left-hand sides of the two equations are the same, which means that the sum of the roots of the quadratic is simply u. After solving the first question in this set, we see that $u = \boxed{18}$.

Solution 2: First note that when expanded out, the equation we are trying to solve for is a quadratic, and the answer is the sum of the roots. So with Vieta's formulas, we only need to compute the coefficient of the x^2 term and the x term. After testing some small cases or using logic, we can see that the coefficient of the x^2 term is the amount of terms in the equation, which is $u+1$. And that the coefficient of the x term is negative two times the sum $1+2+\cdots+u$, or $-2(\frac{(u+1)\cdot(u)}{2}) = -(u+1)\cdot(u)$. So using Vieta's, the sum of the roots to the equation is $-(-\frac{u(u+1)}{u+1}) = \boxed{u}$. After solving the cycle, we realize $u = \boxed{18}$.

12. **YS950** Let n be the answer to the second question in this set. Arpit has a collection of n standard 6-sided dice. Compute the total number of dots on Arpit's dice.

Written by: Yuuki Sawanoi

Answer: $\boxed{378}$

There are $1+2+3+4+5+6 = 21$ dots on each dice. Thus, the answer is $21n$, which, when solving for cycle gives $\boxed{378}$.

13. **CY882** For a positive integer n, the function $f(n)$ switches the units digit and tens digit of n. For how many positive integers $n < 1000$ is $f(n)$ strictly greater than n? For example, $n = 156$ counts because $165 > 156$ but $n = 193$ does not because $139 < 193$. Note: If you have a single-digit number, assume the tens digit is 0. For example, 1 becomes 10 after switching digits.

Written by: Carsten Yeung

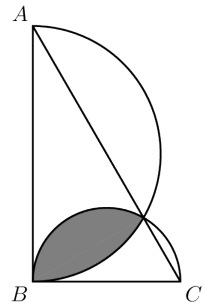
Answer: $\boxed{450}$

Notice that the digit in the hundreds digit will not be altered in any way by f, so there are 10 choices of digits in the hundreds place.

For $n < f(n)$, the units-digit must be strictly greater than the tens-digit. If we choose any two digits from $\{0, 1, 2, ..., 9\}$, there is exactly one way to order those two digits so that the greater one comes first. For example, if we choose 0 and 1, the ten's digit must be 1 and the one's digits must be 0. Thus the amount of digit pairs that satisfies this will be the amount of ways to choose two numbers unordered from 0 to 9 which is $\binom{10}{2}$.

Thus, our answer is $10\cdot\binom{10}{2} = 10\cdot\frac{10\cdot 9}{2} = \boxed{450}$.

14. **YS935** Right triangle ABC has $AB = 12\sqrt{3}$, $BC = 12$, and $AC = 24$. Let semicircles C_1 and C_2 have diameters $\overline{AB}$ and $\overline{BC}$, respectively. Compute the area of overlap between C_1 and C_2. Express your answer in simplest radical form in terms of π.



Written by: Yuuki Sawanoi

Answer: $\boxed{30\pi - 36\sqrt{3}}$

The key observation in this problem is that $\overline{AC} = 2\overline{BC}$ and $\overline{AB} = \overline{BC}\sqrt{3}$, which tells us that ABC is a 30-60-90 triangle. Specifically, $\angle ACB = 60^\circ$ and $\angle BAC = 30^\circ$.

Next, define O_1 and O_2 to be the centers of C_1 and C_2 respectively, define D to be the point of intersection between the two circles, and draw segments O_1D, O_2D, and BD.

This gives us a few new shapes: Triangles O_1DB and O_2DB, and sectors O_1DB and O_2DB. Now, the answer we are looking for is (Area of Sector O_1DB-Area of Triangle O_1DB)$+$ (Area of Sector $O_2DB +$ Area of Triangle O_2DB).

To find these areas individually, we can use the fact that the four line segments involved are radii and say that $\overline{O_1D} = \overline{O_1B} = 6\sqrt{3}$ and $\overline{O_2D} = \overline{O_2B} = 6$. In addition, angles $\angle BO_1D$ and $\angle BO_2D$, by the Inscribed Angle Theorem, are 60° and 120° respectively.

Now, since we know the radii and angles, we can calculated the areas of sectors O_1DB and O_2DB to be 18π and 12π respectively. We can also calculate the areas of the triangles with the sin area formula and get $[\triangle O_1DB] = 27\sqrt{3}$ and $[\triangle O_2DB] = 9\sqrt{3}$. Plugging these values to the equation from earlier, we get our answer to be $12\pi + 18\pi - 27\sqrt{3} - 9\sqrt{3} = \boxed{30\pi - 36\sqrt{3}}$.

15. **LT260** Let n be a positive integer with exactly 16 positive divisors. If n^2 has exactly 63 positive divisors, how many positive divisors does n^3 have?

Written by: Linus Tang

Answer: $\boxed{160}$

Recall the formula for the number of divisors of a number, which is the product of $e + 1$ over all exponents in the prime factorization.

Enumerating the ways that 16 can be written as such a product, namely $16 = 8 \cdot 2 = 4 \cdot 4 = 4 \cdot 2 \cdot 2 = 2 \cdot 2 \cdot 2 \cdot 2$, we find the corresponding possible prime factorizations p^{15}, p^7q, p^3q^3, p^3qr, or $pqrs$, where $p, q, r,$ and s are distinct primes.

With that, n^2 is equal to p^{30}, $p^{14}q^2$, p^6q^6, $p^6q^2r^2$, or $p^2q^2r^2s^2$, which have $31, 45, 49, 63,$ and 81 positive divisors, respectively.

We are given that n^2 has 63 factors, so $n = p^3qr$. Thus, $n^3 = p^9q^3r^3$, which has $\boxed{160}$ positive divisors.

16. **TK920** A convex polygon has distinct integer angle measure at each of its vertices. What is the maximum number of sides this polygon could have?

Written by: Tristan Kay

Answer: $\boxed{26}$

Utilize that the sum of all exterior angles is 360. If n is the number of sides, we must at least have

$$1 + 2 + 3 + \cdots n = \frac{n(n+1)}{2} < 360$$

Finding the maximum n yields $n = \boxed{26}$, and it is easy to check that this is achievable.

17. **LT755** Let $a_1, a_2, a_3, \cdots$ be a sequence of positive integers defined by $a_1 = 11$ and $a_{i+1} = a_i^2 - 2a_i + 2$ for all integers $i \geq 1$. What is the sum of the digits of the product $a_1 \times a_2 \times a_3 \times a_4 \times \ldots \times a_{10}$?

Written by: Linus Tang

Answer: $\boxed{1024}$

We can rewrite the equation for a_{i+1} by completing the square to get that $a_{i+1} = (a_i - 1)^2 + 1$. With this, it is much simpler to solve for future terms in the sequence. We get $a_2 = 101$, $a_3 = 10001$, and in general, we can see that $a_n = (10^{2^{n-1}} + 1)$.

Noting this, we can see that the result $a_1 \times a_2 \times a_3 \times a_4 \times \cdots \times a_{10}$ is equal to $11 \times 101 \times 10001 \times \cdots \times (10^{512} + 1)$ or

$$(10^1 + 1)(10^2 + 1)(10^4 + 1) \cdots (10^{512} + 1)$$

Using the distributive property, we can see that our value becomes $10^0 + 10^1 + 10^2 + \cdots + 10^{1023}$. Since this results in a 1024 digit number where each digit is 1, the sum of the digits of our result is $\boxed{1024}$.

18. **MLI909** The cubic $x^3 - 15x^2 - 41x + 119$ has roots a, b, and c. The cubic $4x^3 - 64x^2 - 85x + 289$ has roots c, d, and e. Given that the quintic $4x^5 - 56x^4 - 241x^3 + 567x^2 + 1173x - 2023$ has roots a, b, c, d, and e, compute c, the common root between all three polynomials.

Written by: Michael Liu

Answer: $\boxed{17}$

Using Vieta's to find the product of the roots in each polynomial, we know that $abc = -119$, $cde = -\frac{289}{4}$, and $abcde = \frac{2023}{4}$. Note that $c = \frac{abc^2de}{abcde} = \frac{(abc)(cde)}{abcde}$. Therefore, $c = \frac{(-119)(-\frac{289}{4})}{\frac{2023}{4}} = \frac{(7 \cdot 17)(17^2)}{7 \cdot 17^2} = \boxed{17}$

19. **AR758** Alon picks two distinct prime numbers below 50 uniformly at random. What is the probability they add up to a multiple of 6? Express your answer as a common fraction.

Written by: Alon

Answer: $\boxed{\frac{2}{5}}$

Clearly 2 and 3 cannot be one of the prime numbers as it requires the other prime being a multiple of 2 and a multiple of 3 respectively, where both possibilities are impossible. Then, notice that one of the prime numbers must be 1 mod 6, and the other must be 5 mod 6, because all prime numbers other than 2 and 3 fall into one of those categories, as they cannot be even, causing mod $0, 2, 4$ to be unavailable and cannot be divisible by 3, making mod 3 unavalible. Group the primes below 50 into these two groups, and it is pretty easy to see that there are 6 primes in the first group (7, 13, 19, 31, 37, and 43) and 7 primes in the second group (5, 11, 17, 23, 29, 41, and 47), where there are 15 primes total below 50. Therefore, the answer is $\frac{6\cdot 7}{\binom{15}{2}} = \boxed{\frac{2}{5}}$

20. **LT277** Six of the seven digits in the following set can be arranged to form a perfect square: $\{0, 3, 5, 6, 7, 8, 9\}$. Which digit is left out?

Written by: Linus Tang

Answer: $\boxed{7}$

The trick is to use the divisibility test for 9: Every positive integer is congruent to the sum of its digits mod 9.

Note that perfect squares can only be $0, 1, 4$, or 7 mod 9, which can be confirmed by computing $0^2, 1^2, 2^2, \ldots, 8^2$ mod 9.

If the digit 0 is left out, the sum of the remaining six digits is 38, which is congruent to 2 mod 9. Since this is not $0, 1, 4$, nor 7, the remaining 6 digits cannot be rearranged into a perfect square.

Similarly, we can rule out $3, 5, 6, 8$, and 9, as possibilities for the left-out digit, since they would leave sums congruent to $8, 6, 5, 3$, and 2 mod 9, respectively.

We conclude that the missing digit is $\boxed{7}$. Indeed, $916^2 = 839056$ and $713^2 = 508369$, which are congruent to 4 mod 9.

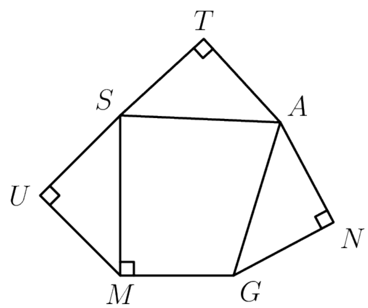
21. **DG392** In the empire of Oshenia, coins can be worth 5 different dollar values. Wynnston has two of each type of coin and discovered that he can combine some number of his coins to produce a sum of \$0, \$1, \$2, ..., and all integer dollar amounts up to \$$n$ (for example, if the values of the coins were $1, 2, 3, 4$, and 5, then Wynnston would be able to produce a combination of coins that sum to any integer between \$0 and \$30, inclusive). Find the maximum possible value of n.

Written by: Daniel Ge

Answer: $\boxed{\$242}$

By construction, there are 3 ways (don't use, use only 1, or use all 2) to use each type of coin. Since there are 5 types of coins, there is a maximum of $3^5 = 243$ ways. The possible sum of coins is $0, 1, 2, \ldots$ and the 243rd number on this list is \$242, thus 242 is the maximum value of n. This is achieved by using coins of \$1, \$3, \$9, \$27, \$81, therefore our answer is $\boxed{242}$.

22. **LT291** Convex heptagon $MUSTANG$ has all side lengths equal to 2. Furthermore, $\angle MUS = \angle STA = \angle ANG = \angle GMS = 90^\circ$. What is the area of $MUSTANG$? Express your answer in simplest radical form.



Written by: Linus Tang

Answer: $\boxed{2\sqrt{2} + \sqrt{15} + 6}$

By the Pythagorean Theorem, we have $MS = SA = AG = 2\sqrt{2}$. By the Pythagorean Theorem on triangle GMS, we have $GS^2 = GM^2 + MS^2 = 4 + 8 = 12$, so $GS = 2\sqrt{3}$.

We now turn our attention to isosceles triangle SAG with $SA = AG = 2\sqrt{2}$ and $GS = 2\sqrt{3}$. Drop perpendicular AX from A to GS. Since SAG is isosceles, X bisects GS. Thus, $SX = \sqrt{3}$. By Pythagorean Theorem on $\triangle SXA$, we find $AX^2 = SA^2 - SX^2 = 8 - 3 = 5$, so $AX = \sqrt{5}$.

The area of $\triangle SAG$ is $\frac{AX \cdot SG}{2} = \frac{\sqrt{5} \cdot 2\sqrt{3}}{2} = \sqrt{15}$.

Finally, the area of $MUSTANG$ is the sum of the areas of $\triangle GMS, \triangle SAG, \triangle MUS, \triangle STA$, and $\triangle ANG$: $2\sqrt{2} + \sqrt{15} + 2 + 2 + 2 = \boxed{2\sqrt{2} + \sqrt{15} + 6}$

23. **LT646** Let $\triangle ABC$ be a triangle with an area of 1 and a perimeter of 5. If side $\overline{BC}$ has length 2, what is the radius of the circle passing through A, B, and C? Express your answer as a common fraction.

Written by: Linus Tang

Answer: $\boxed{\frac{41}{40}}$

We use coordinates. Since $BC = 2$, we let $B = (-1, 0)$ and $C = (1, 0)$.

Since the area of ABC is 1 and the base BC has length 2, the height from A to BC must be 1. Thus, point A has a y-coordinate of ± 1. We can let $A = (x, 1)$.

Now, we use the information that the perimeter of ABC is 5 to eventually solve for x.

$AB + BC + CA = 5$

$AB + 2 + CA = 5$

$AB + CA = 3$

$\sqrt{(x+1)^2 + 1^2} + \sqrt{(x-1)^2 + 1^2} = 3$ (by the Euclidean distance formula or Pythagorean Theorem)

Squaring both sides,

$(x+1)^2 + 1 + 2\sqrt{((x+1)^2 + 1)((x-1)^2 + 1)} + (x-1)^2 + 1 = 9$

Expanding and collecting like terms,

$2x^2 + 4 + 2\sqrt{(x^2 + 2x + 2)(x^2 - 2x + 2)} = 9$

$2\sqrt{x^4 + 4} = 5 - 2x^2$.

Substituting $X = x^2$,

$2\sqrt{X^2 + 4} = 5 - 2X$

$4X^2 + 16 = 25 - 20X + 4X^2$

$20X = 9$

$X = \frac{9}{20}$.

We want to find the circumradius, R, of ABC, which is given by $R = \frac{AB \cdot BC \cdot CA}{4K}$, where $K = 1$ is the area of ABC. To do so, we need to calculate $AB \cdot CA$.

$AB \cdot CA$

$= \sqrt{(x+1)^2 + 1^2}\sqrt{(x-1)^2 + 1^2}$

$= \sqrt{(x^2 + 2x + 2)(x^2 - 2x + 2)}$

$= \sqrt{x^4 + 4}$

$= \sqrt{X^2 + 4}$

Since one of our immediate steps earlier was $2\sqrt{X^2 + 4} = 5 - 2X$, we have

$\sqrt{X^2 + 4}$

$= \frac{5-2X}{2}$

$= \frac{5-2\cdot\frac{9}{20}}{2}$

$= \frac{41}{20}$.

Thus, the answer is

$R = \frac{AB\cdot BC\cdot CA}{4K}$

$= \frac{2\cdot\frac{41}{20}}{4(1)}$

$= \boxed{\frac{41}{40}}$.

24. **DG398** A *cyclid* is a number such that, after repeatedly moving the last digit to the front some number of times, you will end up with a nondecreasing string of digits. How many six-digit cyclids are there (leading zeros are allowed)? Examples of six-digit cyclids include 044789 and 348012, while 268913 and 348210 are not cyclids.

Written by: Daniel Ge

Answer: $\boxed{29980}$

We first calculate the number of non-strictly increasing six-digit numbers allowed to start with 0. Let the six digits of the number be u, v, w, x, y, z in order. Using a common trick from stars and bars, we express the differences

$$
\begin{aligned}
u - 0 &= a \\
v - u &= b \\
w - v &= c \\
x - w &= d \\
y - x &= e \\
z - y &= f \\
9 - z &= g
\end{aligned}
$$

Note that when a, b, c, d, e, f, and g are non-negative integers satisfying the equations, we can find u, v, w, x, y, z making up a number with increasing digits (leading zeroes allowed). Furthermore, we know that the sum $a + b + c + d + e + f + g = 9$ by adding up the previous equations. We can use stars and bars to find the number of tuples of $a, b, \dots, g$ as $\binom{9+7-1}{9} = 5005$.

For most of these 5005 possibilities (let's say 023679), we can move the last digit to the front up to six times to get 5 more arrangements (in this case we also get 236790, 367902, 679023, 790236, 902367) for a total of $5005 + 5005 \cdot 5 = 30030$. However, the last tricky edge case is with 000000, 111111, ..., 999999, which do not produce 5 more cases due to being unchanged after the moves, so we take away $10 \cdot 5 = 50$ in total and end up with 30030–50 = $\boxed{29980}$

25. **TK951** Estimate the expression below to the nearest integer.

$$\frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \cdots \times \frac{2023}{2022}$$

Submit a positive integer N. If the correct answer is A, you will receive $\max(25 - |A - N|, 0)$ points (if you do not submit a positive integer, you will receive 0 points).

Written by: Tristan Kay

Answer: $\boxed{36}$

Let A be the exact value of the expression. Let

$$B = \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \cdots \times \frac{2022}{2021}$$

Then notice that $AB = 2023$. If we assume $A \approx B$, then we get $B \approx \sqrt{2023} = 45$, giving us 16 points. However, we also have

$$\frac{A}{B} = \frac{1 \times 3^2 \times 5^2 \times 7^2 \times \cdots \times 2021^2 \times 2023}{2^2 \times 4^2 \times 6^2 \times \cdots 2022^2}$$

We can approximate $3^2 \times 5^2 \times \cdots 2021^2 \approx 2 \times 4^2 \times 6^2 \times \cdots 2020^2 \times 2022$ yielding

$$\frac{A}{B} \approx \frac{1 \times 2023}{2 \times 2022} \approx \frac{1}{2}$$

Hence, we have $AB \times \frac{A}{B} = A^2 \approx \frac{2023}{2}$, thus $A \approx \sqrt{\frac{2023}{2}} \approx \frac{45}{\sqrt{2}} \approx 32$.

Refining this estimation of $\frac{A}{B}$ can continue to give us increasingly close answers.

26. **TK953** Anna the Anaconda initially has 1 gold coin. Every second, she has a 50% chance of doubling the number of coins she currently has. Else, she has a 50% chance of losing one coin. This process terminates whenever Anna loses all of her coins. The probability that Anna still has coins after 10^{2023} seconds is p. Estimate $\lfloor 100p \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Submit a positive integer N. If the correct answer is A, you will receive $\max(25-5|A-N|, 0)$ points (if you do not submit a positive integer, you will receive 0 points).

Written by: Tristan Kay

Answer: $\boxed{29}$

Let x_n denote the probability that Anna continues to have coins after 10^{2023} seconds when she has n coins. Then, we have that $x_0 = 0$, $x_n = \frac{x_{n-1}+x_{2n}}{2}$, and as n approaches infinity, $x_n = 1$. Using this, we can write

$$2x_1 = x_2$$

$$2x_2 = x_1 + x_4$$

$$2x_4 = x_3 + x_8$$
$$4x_4 = x_2 + x_6 + 2x_8$$
$$3x_1 = x_4$$
$$4x_4 = 2x_1 + x_6 + 2x_8$$
$$10x_1 = x_6 + 2x_8$$

Logically, notice that it is quite difficult for Anna to get back to zero gold coins from 6 gold coins (one way to think about this is that for her to lose 6 coins in a row, there is a $\frac{1}{2^6} \approx 2\%$ of that happening. Therefore, we can approximate both x_6 and x_8 as very close to 1, to get

$$10x_1 \approx 1 + 2(1) = 3$$
$$x_1 \approx 0.3$$

But notice how this slightly over estimates, giving us an answer of $\lfloor 100x_1 \rfloor = \boxed{29}$.

Monte Carlo simulations of this problem yield an approximate value of 29.5%.

27. **TK954** For nonnegative integers M, A, T, H, F, U, and N, let X be the number of possible quadruplets (M, A, T, H) and Y be the number of possible triplets (F, U, N) that satsfy the equations below. Estimate $\frac{X}{Y}$ to the nearest integer.

$$17M + 19A + 21T + 23H = 2023$$
$$9F + 10U + 11N = 1011$$

Submit a positive integer N. If the correct answer is A, you will receive $\max(25-(A-N)^2, 0)$ points for this question (if you do not submit a positive integer, you will receive 0 points).

Written by: Tristan Kay

Answer: $\boxed{18}$

If we first look at the number of solutions to

$$17M + 19A + 21T \leq 2023$$

assuming the number of lattice points are approximately equal to the total volume of the tetrahedron bounded by the plane and the xy-, yz-, and xz- axis, we get there are approximately $\frac{1}{6} \times \frac{2023}{17} \times \frac{2023}{19} \times \frac{2023}{21}$. Note that approximately $\frac{1}{23}$ of these lattice points will yield a solution $17M + 19A + 21T + 23H = 2023$. Thus, we have

$$X \approx \frac{2023^3}{6 \times 17 \times 19 \times 21 \times 23}$$

Now using similar logic for $9F + 10U + 11N = 1011$, there are

$$Y \approx \frac{1011^2}{2 \times 9 \times 10 \times 11}$$

Hence, our answer is around $\frac{2023^3 \times 2 \times 9 \times 10 \times 11}{1011^2 \times 6 \times 17 \times 19 \times 21 \times 23} \approx 2023 \times 2^2 \times \frac{1}{3} \times \frac{1}{2^3} \times \frac{1}{20} = \frac{2023}{6 \times 20} = 16.8$, close enough to give the vast majority of points.

More careful calculation of the above quantity yields 17.14. It can also be noted that the above is known to be an over estimate because the volume is strictly less than the number of lattice points.

The actual value of X and Y are 9379 and 532 respectively, so $\frac{X}{Y}$ is actually around 17.64 or 18.