Notes on the MVDM Model

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In our work, we use the $(+---)$ metric signature, and the natural units $\hbar = c = k_B = 1$.

1. The general formalism

We are interested in the case when the scalar field potential $U(\varphi)$ does not have a non-trivial minimum, and fermion masses are generated come from minimizing the total thermodynamic potential of the combined scalar-fermion. The thermodynamic evolution of the system is slow enough that the system is in equilibrium at a temperature $T(a)$ (till a phase transition). For the fermions, we can have two different cases:

1. $\mu = 0$, i.e., an equal number of fermions and antifermions, and,
2. $\mu \neq 0$, i.e., a slight excess of fermions.

If the fermions turn out to be *Majorana* particles, then one cannot have the second case above. For *Dirac* fermions, we can have $\mu = 0$ provided all $\psi - \bar{\psi}$ pairs are annihilated in the ground ($\beta \to \infty$) state. For reasons of simplicity (we are considering a toy model), we will take $\mu = 0$.

The calculations in Secs. 1.1 and 1.2 can be found in any standard textbook on thermal field theory; here we follow Ref. [1].

1.1. Scalar field

For a free scalar field φ with the Lagrangian

$$\mathcal{L}_B = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M_b^2\varphi^2, \tag{1}$$

the partition function at temperature $T = \dfrac{1}{\beta}$ is given by,

$$\mathcal{Z}_B \equiv \mathrm{Tr}\, e^{-\beta\hat{H}} = N\int_{\mathrm{PBC}} \mathcal{D}\varphi\, e^{S_B^E}, \tag{2}$$

where PBC stands for periodic boundary conditions, S_B^E is the Euclidean action

$$S_B^E \equiv \int_0^\beta d\tau \int d^3\mathbf{x}\,\sqrt{-g}\,\mathcal{L}_B^E = -\frac{1}{2}\int_0^\beta d\tau\int d^3\mathbf{x}\,a^3(\tau)\left[\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \left(\frac{\nabla\varphi}{a}\right)^2 + M_b^2\varphi^2\right] \tag{3}$$

with the imaginary time $\tau = it$, and N is an overall normalization constant that is irrelevant to the thermodynamics. Simplifying, we get

$$\begin{aligned} S_B^E &= -\frac{1}{2}\int_0^\beta d\tau\int d^3\mathbf{x}\,a^3\left[\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \left(\frac{\nabla\varphi}{a}\right)^2 + M_b^2\varphi^2\right] \\ &= -\frac{1}{2}\int_0^\beta d\tau\int d^3\mathbf{x}\; a^3\,\varphi\left[-\frac{\partial^2}{\partial\tau^2} - \left(\frac{\nabla}{a}\right)^2 + M_b^2\right]\varphi, \end{aligned} \tag{4}$$

where the second step is obtained upon integration by parts. We now take the ansatz

$$\varphi(\mathbf{x},\tau)=\sqrt{\frac{\beta}{V}}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{x}+\omega_n\tau)}\varphi_n(\mathbf{k}), \tag{5}$$

where $\underline{V \text{ is the } \textit{physical volume}}$, and $\mathbf{k}$ is the comoving momentum given by $\mathbf{k}=a\mathbf{p}$, and ω_n are the **Matsubara frequencies** given by $\omega_n=\dfrac{2n\pi}{\beta}$, so that $\varphi(\mathbf{x},\beta)=\varphi(\mathbf{x},0)$. Since $\varphi(\mathbf{x},\tau)$ is real, we must have,

$$\varphi_{-n}(\mathbf{k})=\varphi_n^{\star}(\mathbf{k}). \tag{6}$$

Putting Eq. (5) in Eq. (4), we get

$$\begin{aligned}
S_B^E&=-\frac{\beta}{2V}\sum_{m,n}\sum_{\mathbf{k},\mathbf{q}}\int_0^\beta d\tau\int d^3\mathbf{x}\,a^3\left(\omega_n^2+\frac{k^2}{a^2}+M_b^2\right)e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}e^{i(\omega_m-\omega_n)\tau}\varphi_m^{\star}(\mathbf{q})\varphi_n(\mathbf{k})\\
&=-\frac{\beta}{2V}\beta V\sum_{m,n}\sum_{\mathbf{k},\mathbf{q}}\delta_{m,n}\delta_{\mathbf{k},\mathbf{q}}\left(\omega_n^2+\frac{k^2}{a^2}+M_b^2\right)\varphi_m^{\star}(\mathbf{q})\varphi_n(\mathbf{k})\\
&=-\frac{\beta^2}{2}\sum_n\sum_{\mathbf{k}}\left(\omega_n^2+\frac{k^2}{a^2}+M_b^2\right)\varphi_n^{\star}(\mathbf{k})\varphi_n(\mathbf{k}).
\end{aligned} \tag{7}$$

The phase of the Fourier modes goes away, and we can just integrate over the amplitude $\mathcal{A}_n(\mathbf{k})=|\varphi_n(\mathbf{k})|$. So, taking $\epsilon(\mathbf{k})=\sqrt{\dfrac{k^2}{a^2}+M_b^2}=\sqrt{p^2+M_b^2}$, we can calculate the partition function

$$\begin{aligned}
\mathcal{Z}_B&=\int\mathcal{D}\mathcal{A}\,e^{-\frac{\beta^2}{2}\sum_n\sum_{\mathbf{k}}\left(\omega_n^2+\epsilon^2\right)\mathcal{A}_n^2(\mathbf{k})}\\
&=N\prod_n\prod_{\mathbf{k}}\left[\int_{-\infty}^{\infty}d\mathcal{A}_n(\mathbf{k})\,e^{-\frac{\beta^2}{2}\left(\omega_n^2+\epsilon^2\right)\mathcal{A}_n^2(\mathbf{k})}\right]=N\prod_n\prod_{\mathbf{k}}\sqrt{\frac{2\pi}{\beta^2\left(\omega_n^2+\epsilon^2\right)}}.
\end{aligned} \tag{8}$$

Absorbing the factors of 2π, we can write

$$\mathcal{Z}_B=\bar{N}\prod_n\prod_{\mathbf{k}}\left[\beta^2\left(\omega_n^2+\epsilon^2\right)\right]^{-1/2}, \tag{9}$$

and therefore

$$\boxed{\log\mathcal{Z}_B=-\frac{1}{2}\sum_n\sum_{\mathbf{k}}\log\left[\beta^2\left(\omega_n^2+\epsilon^2\right)\right].} \tag{10}$$

Now, we can write

$$\log\left[\left(2n\pi\right)^2+\beta^2\epsilon^2\right]=\int_1^{\beta^2\epsilon^2}\frac{d\theta^2}{\theta^2+\left(2n\pi\right)^2}+\underbrace{\log\left[\left(2n\pi\right)^2+1\right]}_{\beta-\text{independent}}, \tag{11}$$

and

$$\sum_{n=-\infty}^{\infty}\frac{1}{\theta^2+\left(2n\pi\right)^2}=\frac{1}{\theta}\left(\frac{1}{2}+\frac{1}{e^\theta-1}\right). \tag{12}$$

Therefore ignoring the β-independent part, we have

$$\log\mathcal{Z}_B=-\frac{1}{2}\sum_n\sum_{\mathbf{p}}\int_1^{\beta\epsilon}\frac{2\theta\,d\theta}{\theta^2+\left(2n\pi\right)^2}=-\sum_{\mathbf{p}}\int_1^{\beta\epsilon}d\theta\left(\frac{1}{2}+\frac{1}{e^\theta-1}\right). \tag{13}$$

In the limit of continuous momentum states, we can replace $\sum_{\mathbf{p}}$ by $\frac{V}{(2\pi)^3}\int d^3\mathbf{p}$ to get

$$\begin{aligned}\log \mathcal{Z}_B &= -\frac{V}{(2\pi)^3}\int d^3\mathbf{p}\left[\frac{1}{2}\beta\epsilon - \frac{1}{2} + 1 - \beta\epsilon + \log\left(e^{\beta\epsilon}-1\right) - \log(e-1)\right] \\ &= -\frac{V}{(2\pi)^3}\int d^3\mathbf{p}\left[-\frac{1}{2}\beta\epsilon + \log\left(e^{\beta\epsilon}-1\right)\right] \\ &= -\frac{V}{(2\pi)^3}\int d^3\mathbf{p}\left[\frac{1}{2}\beta\epsilon + \log\left(1-e^{-\beta\epsilon}\right)\right],\end{aligned} \tag{14}$$

where in the second line we have dropped the non-β terms. The **Helmholtz free energy** is

$$\boxed{F_B \equiv -\frac{1}{\beta V}\log\mathcal{Z}_B = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\left[\frac{1}{2}\epsilon + \frac{1}{\beta}\log\left(1-e^{-\beta\epsilon}\right)\right],} \tag{15}$$

and the **pressure** is

$$P_B \equiv \frac{1}{\beta}\frac{\partial}{\partial V}\log\mathcal{Z}_B = -F_B. \tag{16}$$

1.2. Fermion field

The Lagrangian for a free Dirac field ψ is

$$\mathcal{L}_F = \bar{\psi}(i\partial\!\!\!/ - \bar{m}_\psi)\psi = \psi^\dagger\gamma^0\left(i\gamma^0\frac{\partial}{\partial t} + i\boldsymbol{\gamma}\cdot\frac{\nabla}{a} - \bar{m}_\psi\right)\psi, \tag{17}$$

where $\bar{m}_\psi$ is the bare mass of the fermion. At temperature $T = \frac{1}{\beta}$ the corresponding partition function is (disregarding an overall normalization)

$$\mathcal{Z}_F \equiv \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{Q})} = \int_{\mathrm{APBC}} \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{S_F^E}, \tag{18}$$

where APBC denotes periodic boundary conditions, $\hat{Q}$ is the number operator

$$\hat{Q} = \int d^3\mathbf{x}\,\sqrt{-g}\,\psi^\dagger\psi, \tag{19}$$

and S_F^E is the Euclidean action

$$S_F^E \equiv \int_0^\beta d\tau\left\{\int d^3\mathbf{x}\,\sqrt{-g}\,\mathcal{L}_F^E + \hat{Q}\right\} = \int_0^\beta d\tau\int d^3\mathbf{x}\,a^3(\tau)\,\bar{\psi}\left[-\gamma^0\frac{\partial}{\partial\tau} + i\frac{\boldsymbol{\gamma}\cdot\nabla}{a} - \bar{m}_\psi + \mu\gamma^0\right]\psi. \tag{20}$$

As before, we get

$$\begin{aligned}S_F^E &= -\int_0^\beta d\tau\int d^3\mathbf{x}\,a^3\bar{\psi}\left[\gamma^0\frac{\partial}{\partial\tau} - i\frac{\boldsymbol{\gamma}\cdot\nabla}{a} + \bar{m}_\psi - \mu\gamma^0\right]\psi \\ &= -\int_0^\beta d\tau\int d^3\mathbf{x}\,a^3\psi^\dagger\left[\frac{\partial}{\partial\tau} - i\frac{\gamma^0\boldsymbol{\gamma}\cdot\nabla}{a} + \gamma^0\bar{m}_\psi - \mu\right]\psi.\end{aligned} \tag{21}$$

We now take the ansatz

$$\psi_\alpha(\mathbf{x},\tau) = \frac{1}{\sqrt{V}}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x}+\omega_n\tau)}\psi_{\alpha;n}(\mathbf{k}), \tag{22}$$

where α denotes the spinor indices, and the Matsubara frequencies are now given by $\omega_n = \dfrac{(2n+1)\pi}{\beta}$, so that $\psi(\mathbf{x},\beta) = -\psi(\mathbf{x},0)$. Putting Eq. (22) in Eq. (21), one gets

$$\begin{aligned} S_F^E &= -\frac{1}{V}\sum_{m,n}\sum_{\mathbf{k},\mathbf{q}}\sum_{\rho,\sigma}\int_0^\beta d\tau\int d^3\mathbf{x}\,a^3\left(i\omega_n + \frac{\gamma^0\boldsymbol{\gamma}\cdot\mathbf{k}}{a} + \gamma^0\bar{m}_\psi - \mu\right)_{\sigma\rho} e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{x}}\,e^{i(\omega_m-\omega_n)\tau}\,\psi^\dagger_{\sigma;m}(\mathbf{q})\psi_{\rho;n}(\mathbf{k}) \\ &= -\beta\sum_{m,n}\sum_{\mathbf{k},\mathbf{q}}\sum_{\rho,\sigma}\left(\big(i\omega_n-\mu\big) + \frac{\gamma^0\boldsymbol{\gamma}\cdot\mathbf{k}}{a} + \gamma^0\bar{m}_\psi\right)_{\sigma\rho}\delta_{m,n}\,\delta_{\mathbf{k},\mathbf{q}}\,\psi^\dagger_{\sigma;m}(\mathbf{q})\psi_{\rho;n}(\mathbf{k}) \\ &= -\beta\sum_{n}\sum_{\mathbf{k}}\sum_{\rho,\sigma}\left(\big(i\omega_n-\mu\big) + \frac{\gamma^0\boldsymbol{\gamma}\cdot\mathbf{k}}{a} + \gamma^0\bar{m}_\psi\right)_{\sigma\rho}\psi^\dagger_{\sigma;n}(\mathbf{k})\psi_{\rho;n}(\mathbf{k}). \end{aligned} \tag{23}$$

Putting this in Eq. (18), we have

$$\mathcal{Z}_F = \int_{\text{APBC}}\mathcal{D}\psi^\dagger\,\mathcal{D}\psi\,e^{S_F^E} = \left[\prod_n\prod_{\mathbf{k}}\prod_\alpha\int d\psi^\dagger_{\alpha;n}(\mathbf{k})\int d\psi_{\alpha;n}(\mathbf{k})\right]e^{-\beta\sum_n\sum_{\mathbf{k}}\sum_{\rho\sigma}D_{\sigma\rho}\,\psi^\dagger_{\sigma;n}(\mathbf{k})\psi_{\rho;n}(\mathbf{k})}, \tag{24}$$

where

$$D = \big(i\omega_n-\mu\big) + \frac{\gamma^0\boldsymbol{\gamma}\cdot\mathbf{k}}{a} + \gamma^0\bar{m}_\psi. \tag{25}$$

To evaluate this, we use the standard expression for Grassmann integrals,

$$\int d\eta_1^\dagger\,d\eta_1\cdots d\eta_N^\dagger\,d\eta_N\,e^{\eta^\dagger D\eta} = \det D \tag{26}$$

for an $N\times N$ matrix D. Looking at Eq. (26), we see that $\mathcal{Z}_F = \det\big(-\beta D\big)$, and so

$$\log\mathcal{Z}_F = \log\det\big(-\beta D\big) = \operatorname{Tr}\log\big(-\beta D\big), \tag{27}$$

where the trace is taken over α, n, and $\mathbf{k}$, and we have used that for a square matrix,

$$\log\big(\det A\big) = \operatorname{Tr}\big(\log A\big). \tag{28}$$

We can write

$$\operatorname{Tr}\log\big(-\beta D\big) = \frac{1}{2}\operatorname{Tr}\log\big(\beta^2D^2\big), \tag{29}$$

and since

$$D^2 = \underbrace{\left[\big(i\omega_n-\mu\big)^2 + \frac{k^2}{a^2} + \bar{m}_\psi^2\right]}_{d^2} I + \text{terms proportional to } \gamma^\mu \tag{30}$$

and the γ^μ are traceless, we have

$$\operatorname{Tr}\log\big(\beta^2D^2\big) = 4\sum_n\sum_{\mathbf{k}}\log\big(\beta^2d^2\big), \tag{31}$$

using the fact that if $\{\lambda_i\}$ are the eigenvalues of a square matrix A, the eigenvalues of $\log A$ are $\{\log\lambda_i\}$. We finally get, using Eqs. (27) and (29),

$$\boxed{\log\mathcal{Z}_F = 2\sum_n\sum_{\mathbf{k}}\log\left[\beta^2\left\{\big(i\omega_n-\mu\big)^2+\epsilon^2\right\}\right] = 2\sum_n\sum_{\mathbf{k}}\log\left[\beta^2\left\{\big(\omega_n+i\mu\big)^2+\epsilon^2\right\}\right]} \tag{32}$$

where $\epsilon(\mathbf{k}) = \sqrt{\dfrac{k^2}{a^2}+\bar{m}_\psi^2} = \sqrt{p^2+\bar{m}_\psi^2}$. Ignoring the imaginary part of the argument of the log, we can rewrite the above equation as

$$\log\mathcal{Z}_F = \sum_n\sum_{\mathbf{p}}\left\{\log\left[\beta^2\left(\omega_n^2+(\epsilon-\mu)^2\right)\right] + \log\left[\beta^2\left(\omega_n^2+(\epsilon+\mu)^2\right)\right]\right\}. \tag{33}$$

Analogous to Eqs. (11) and (12) we have,

$$\log\left[(2n+1)^2\pi^2+\beta^2(\epsilon\pm\mu)^2\right]=\int_1^{\beta^2(\epsilon\pm\mu)^2}\frac{d\theta^2}{\theta^2+(2n+1)^2\pi^2}+\underbrace{\log\left[(2n+1)^2\pi^2+1\right]}_{\beta-\text{independent}},\tag{34}$$

and

$$\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2\pi^2+\theta^2}=\frac{1}{\theta}\left(\frac{1}{2}-\frac{1}{e^\theta+1}\right);\tag{35}$$

therefore plugging Eqs. (34) and (35) into Eq. (33), and considering continuous momentum states gives us

$$\log\mathcal{Z}_F=\frac{2V}{(2\pi)^3}\int d^3\mathbf{p}\left[\beta\epsilon+\log\left(1+e^{-\beta\epsilon_-}\right)+\log\left(1+e^{-\beta\epsilon_+}\right)\right],\tag{36}$$

where $\epsilon_\pm=\epsilon\pm\mu$, and the μ- and β-independent terms are dropped. The Helmholtz free energy is,

$$\boxed{F_F=-\frac{1}{\beta V}\log\mathcal{Z}_F=-\frac{2}{(2\pi)^3}\int d^3\mathbf{p}\left[\epsilon+\frac{1}{\beta}\log\left(1+e^{-\beta\epsilon_-}\right)+\frac{1}{\beta}\log\left(1+e^{-\beta\epsilon_+}\right)\right],}\tag{37}$$

and as in the scalar case,

$$P_F=-F_F.\tag{38}$$

We define the zero-point energy and pressure,

$$F_0\equiv-\frac{2}{(2\pi)^3}\int d^3\mathbf{p}\,\epsilon(p)\equiv-P_0,\tag{39}$$

and rewrite the Helmholtz free energy of Eq. (37) as

$$\begin{aligned}F_F&=F_0-\frac{2}{(2\pi)^3\beta}\int d^3\mathbf{p}\left[\log\left(1+e^{-\beta\epsilon_-}\right)+\log\left(1+e^{-\beta\epsilon_+}\right)\right]\\&=F_0-\frac{1}{\pi^2\beta}\int_0^\infty dp\,p^2\left[\log\left(1+e^{-\beta\epsilon_-}\right)+\log\left(1+e^{-\beta\epsilon_+}\right)\right].\end{aligned}\tag{40}$$

We can simpify the above expression if we note that,

$$\begin{aligned}\int_0^\infty dp\,p^2\,\log\left(1+e^{-\beta\epsilon_\pm}\right)&=\cancel{\log\left(1-e^{-\beta\epsilon_\pm}\right)\frac{p^3}{3}\bigg|_0^\infty}+\beta\int_0^\infty\frac{dp\,e^{-\beta\epsilon_\pm}}{\epsilon(p)\left(1+e^{-\beta\epsilon_\pm}\right)}\frac{p^4}{3}\\&=\beta\int_0^\infty\frac{dp\,p^4}{3\epsilon(p)\left(1+e^{\beta\epsilon_\pm}\right)},\end{aligned}\tag{41}$$

which makes Eq. (40) of the form

$$\boxed{F_F=F_0-\frac{1}{3\pi^2}\int_0^\infty\frac{dp\,p^4}{\epsilon(p)}\left[n_F(\epsilon_+)+n_F(\epsilon_-)\right],}\tag{42}$$

where $n_F(x)$ is the Fermi distributon function,

$$n_F(x)=\frac{1}{e^{\beta x}+1}.\tag{43}$$

1.3. Coupled scalar and fermion

We take a Yukawa coupling of the form

$$\mathcal{L}_{\text{Yuk}}=-g_Y\bar{\psi}\varphi\psi\tag{44}$$

for some coupling constant g_Y. The total Euclidean action of a scalar and a Dirac fermion is

$$S^E_{\varphi\psi} = S^E_B + S^E_F - g_Y \int_0^\beta d\tau \int d^3\mathbf{x}\, a^3 \bar{\psi}\varphi\psi, \tag{45}$$

where the bosonic action has a potential $U(\varphi)$ where the mass term for the fermion will come from. This leads to the partition function

$$\begin{aligned} \mathcal{Z}_{\varphi\psi} &= \int \mathcal{D}\varphi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, e^{S^E_{\varphi\psi}} \\ &= \int \mathcal{D}\varphi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \exp\left\{ S^E_B - \int_0^\beta d\tau \int d^3\mathbf{x}\, a^3\, \bar{\psi} \left[\gamma^0 \frac{\partial}{\partial\tau} - i\frac{\boldsymbol{\gamma}\cdot\nabla}{a} + (g_Y\varphi + \bar{m}_\psi) - \mu\gamma^0 \right] \psi \right\}, \end{aligned} \tag{46}$$

where we have ignored the bare mass of the fermion, i.e., $\bar{m}_\psi = 0$[1]. We can *formally* integrate out the Dirac fields to get,

$$\mathcal{Z}_{\varphi\psi} = \int \mathcal{D}\varphi\, e^{S^E_{\text{eff}}(\varphi)} = \int \mathcal{D}\varphi\, e^{S^E_B + \log\det \hat{D}(\varphi)}, \tag{47}$$

where

$$\hat{D}(\varphi) = -\beta \left[\frac{\partial}{\partial\tau} - i\frac{\gamma^0\boldsymbol{\gamma}\cdot\nabla}{a} + \gamma^0 g_Y\varphi - \mu \right]. \tag{48}$$

Let us assume that the scalar at $\varphi = \varphi_m$ minimizes the action $S^E_{\text{eff}}(\varphi)$. This is the **vacuum expectation value** (vev), or the classical value of the field,

$$\varphi_m = \langle\varphi\rangle, \tag{49}$$

in which case we can precisely determine the term $\log\det\hat{D}(\varphi)$, and see that the fermion gets a mass

$$\boxed{m_\psi = g_Y\varphi_m.} \tag{50}$$

At $\varphi = \varphi_m$, we can write

$$\mathcal{Z}_{\varphi\psi} = \mathcal{Z}_F\, e^{-\beta V U(\varphi_m)}, \tag{51}$$

which leads to

$$F_{\varphi\psi}(\varphi_m) = U(\varphi_m) + F_F(\varphi_m). \tag{52}$$

This is equivalent to making a saddle-point approximation which is self-consistent if φ_m minimizes the free energy,

$$\left.\frac{\partial F_{\varphi\psi}(\varphi)}{\partial\varphi}\right|_{\mu,\beta;\varphi=\varphi_m} = 0, \qquad \left.\frac{\partial^2 F_{\varphi\psi}(\varphi)}{\partial\varphi^2}\right|_{\mu,\beta;\varphi=\varphi_m} > 0. \tag{53}$$

We take the derivative of both sides of Eq. (52) at φ_m, and we use Eq. (50) and the first of Eq. (53) to write the **fermionic mass equation**

$$\boxed{U'(\varphi_m) + g_Y\rho_s = 0,} \tag{54}$$

where ρ_s is the **scalar fermionic density** (or the **chiral condensate density**)

$$\rho_s \equiv \frac{\langle\hat{N}\rangle}{V} = \frac{\partial F_F}{\partial m_\psi}, \tag{55}$$

with $\hat{N} = \int d^3\mathbf{x}\,\sqrt{-g}\,\bar{\psi}\psi$. From Eqs. (39) and (40), and with $\bar{m}_\psi$ replaced in our case by $g_Y|\varphi_m|$, one gets

$$\rho_s = \frac{m_\psi}{\pi^2}\int_0^\infty \frac{dp\,p^2}{\epsilon(p)}\left[n_F(\epsilon_+) + n_F(\epsilon_-) - 1 \right]. \tag{56}$$

[1]We are interested in a situation where the bare mass $\bar{m}_\psi$ of the fermion is small compared to $g_Y\varphi_m$, and we can ignore it, i.e., we consider $m_\psi \approx g_Y\varphi_m$.

The equations of the model do not explicitly have time dependence, but the quantities like temperature and chemical potential depend on time through the scale factor $a(t)$, e.g., $T = T(a)$, $\mu = \mu(a)$, etc. The latter dependencies are determined from the **Friedmann continuity equation**

$$\dot{\rho}_{\rm tot} + \frac{3\dot{a}}{a}\big(\rho_{\rm tot} + P_{\rm tot}\big) = 0, \tag{57}$$

which is obtained from the two Friedmann equations

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\rm tot}, \qquad \dot{H}(t) + H^2(t) = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\big(\rho_{\rm tot} + 3P_{\rm tot}\big). \tag{58}$$

Also, the fermion mass $m_\psi \propto \varphi_m$ (up to a constant) is also time varying, since φ_m depends on T and μ through Eq. (54). We also redefine all the potentials wrt their vacuum values,

$$F_F \mapsto F_F - F_0, \qquad P_F \mapsto P_F - P_0, \qquad \rho_s \mapsto \rho_s - \rho_0, \tag{59}$$

where in Eq. (56), we have defined

$$\rho_0 \equiv -\frac{m_\psi}{\pi^2}\int_0^\infty \frac{dp\, p^2}{\epsilon(p)}. \tag{60}$$

These redefinitions amount to only keeping the finite-temperature parts of the correction[2], and getting rid of the zero-temperature parts; we will come back to this vacuum part later.

The dynamics of the system is interesting where $U(\varphi)$ does not have a non-trivial minimum, and fermion mass is generated, i.e., Eq. (54) is solved, by some interplay between the fermionic and scalar sectors. We consider here an equal number of fermions and antifermions (i.e., $\mu = 0$), and take $N_F = 1$ generation of fermions as a toy model.

1.4. High and low temperature regimes

Using the redefinitions of Eq. (59), the pressure in Eq. (38) becomes

$$P_F = -F_F = \frac{2N_F}{3\pi^2}\int_0^\infty \frac{dp\, p^4}{\epsilon(p)\big(e^{\beta\epsilon}+1\big)}. \tag{61}$$

Taking $z = \beta\epsilon = \beta\sqrt{p^2 + m_\psi^2}$, we get

$$\frac{dp\, p^4}{\epsilon(p)} = \frac{dz\,\left(z^2 - \beta^2 m_\psi^2\right)^{3/2}}{\beta^4}. \tag{62}$$

We can thus write the pressure as

$$\boxed{P_F = -F_F = \frac{2N_F}{3\pi^2\beta^4}\mathscr{I}_{3/2}(\kappa).} \tag{63}$$

and the chiral condensate density as

$$\boxed{\rho_s = \frac{2N_F m_\psi}{\pi^2\beta^2}\mathscr{I}_{1/2}(\kappa),} \tag{64}$$

where

$$\mathscr{I}_{1/2}(\kappa) \equiv \int_\kappa^\infty \frac{dz\,\left(z^2-\kappa^2\right)^{1/2}}{e^z+1}, \qquad \mathscr{I}_{3/2}(\kappa) \equiv \int_\kappa^\infty \frac{dz\,\left(z^2-\kappa^2\right)^{3/2}}{e^z+1}, \tag{65}$$

and we are introducing the dimensionless parameters

$$\Delta \equiv \frac{M}{T}, \qquad \kappa \equiv \frac{g_Y\varphi}{T}, \qquad F_R \equiv \frac{F_{\varphi\psi}}{M^4}, \tag{66}$$

[2] The correction F_F arises from computing the thermal 1-loop contributions, and have a finite-temperature and a zero-temperature part.

	$\boldsymbol{\kappa \ll 1}$ (high T)	$\boldsymbol{\kappa \gg 1}$ (low T)
$\mathscr{I}_{3/2}(\kappa)$	$\dfrac{7\pi^4}{120} - \dfrac{\pi^2}{8}\kappa^2 + \mathcal{O}(\kappa^4)$	$3\kappa^2 K_2(\kappa) + \mathcal{O}(e^{-2\kappa})$
$\mathscr{I}_{1/2}(\kappa)$	$\dfrac{\pi^2}{12} + \text{subleading}$	$\kappa K_1(\kappa) + \mathcal{O}(e^{-2\kappa})$
$\mathscr{I}_{n}(\kappa)$	$\dfrac{3}{2}\zeta(3) - \left(\dfrac{1}{2}\log 2\right)\kappa^2 + \mathcal{O}(\kappa^3)$	$\kappa^2 K_2(\kappa) + \mathcal{O}(e^{-2\kappa})$
$\mathscr{I}_{\epsilon}(\kappa)$	$\dfrac{7\pi^4}{120} - \dfrac{\pi^2}{24}\kappa^2 + \mathcal{O}(\kappa^4)$	$3\kappa^2 K_2(\kappa) + \kappa^3 K_1(\kappa) + \mathcal{O}(e^{-2\kappa})$
$\dfrac{d^2\mathscr{I}_{3/2}(\kappa)}{d\kappa^2}$	$-\dfrac{\pi^2}{4} - \dfrac{9}{4}\kappa^2\log\kappa + \mathcal{O}(\kappa^2)$	$3\sqrt{\dfrac{\pi}{2}}\kappa^{3/2}\, e^{-\kappa} + \mathcal{O}(e^{-2\kappa})$

Table 1: The asymptotic expressions for the various integrals we will encounter throughout this work.

with M being the overall mass scale of the scalar potential we will consider. The fermion (antifermion) number density is

$$\boxed{n_+ = n_- = \frac{N_F}{\pi^2}\int_0^\infty dp\, p^2\, n_F(\epsilon) = \frac{N_F}{\pi^2\beta^3}\mathscr{I}_n(\kappa),} \tag{67}$$

with

$$\mathscr{I}_n(\kappa) \equiv \int_\kappa^\infty \frac{dz\, z\left(z^2-\kappa^2\right)^{1/2}}{e^z+1}. \tag{68}$$

The asymptotic expressions for the integrals are given in Tab. 1. We look at two regimes of interest:

1. Low Temperature — $\kappa \gg 1$:

   We can expand $K_2(x)$ about ∞ to get

$$\begin{aligned} P_F &= \frac{2N_F}{3\pi^2\beta^4}\left[3\beta^2 m_\psi^2 K_2(\beta m_\psi) + \mathcal{O}(e^{-2\beta m_\psi})\right] \\ &= \frac{2N_F}{3\pi^2\beta^4}\left[3\left\{\sqrt{\frac{\pi}{m_\psi}}(\beta m_\psi)^{3/2} + \frac{15}{8}\sqrt{\frac{\pi}{m_\psi}}(\beta m_\psi)^{1/2} + \mathcal{O}\left(\sqrt{\frac{1}{x}}\right)\right\} e^{-\beta m_\psi} + \mathcal{O}(e^{-2\beta m_\psi})\right] \\ &\approx \frac{\sqrt{2}N_F}{\pi^{3/2}} T (T m_\psi)^{3/2} e^{-m_\psi/T}, \end{aligned} \tag{69}$$

   where in the last step, we have kept terms only up to the leading order. Similarly from Eq. (64), the scalar fermionic density is, to the leading order,

$$\rho_s \approx \frac{\sqrt{2}N_F}{\pi^{3/2}}(T m_\psi)^{3/2} e^{-m_\psi/T}, \tag{70}$$

   and from Eq. (67), we have

$$n_+ = n_- \approx \frac{N_F}{\sqrt{2}\pi^{3/2}}(T m_\psi)^{3/2} e^{-m_\psi/T}. \tag{71}$$

   We see from Eqs. (69)–(71) that

$$P_D \approx \left(n_+ + n_-\right)T \approx \rho_s T. \tag{72}$$

2. High Temperature — $\kappa \ll 1$:

Here, the relavant quantities are

$$P_F \approx \frac{N_F}{12} T^2 \left(\frac{7}{15} \pi^2 T^2 - m_\psi^2 \right), \tag{73}$$

$$\rho_s \approx \frac{N_F}{6} m_\psi T^2, \tag{74}$$

and,

$$n_\pm \approx \frac{3 N_F \zeta(3)}{2\pi^2} T^3. \tag{75}$$

Comments and Observations:

a) Since ρ_s is positive definite, $U(\varphi)$ cannot be monotonically increasing for Eq. (54) to have non-trivial solution, ruling out some popular quintessence potentials.

b) Monotonically decreasing potentials have "a window of parameters" allowing for a meaningful solution to Eq. (54). We can see from Eq. (70) that if $U(\varphi)$ has a minimum at $\varphi_m = \infty$, one has a solution (the "doomsday" vacuum), which is trivial; a non-trivial solution exists for some other minimum of $F_{\varphi\psi}(\varphi)$ that arises due to fermionic contributions. Since there is no zero-temperature fermionic contribution, a finite mass can only occur above a certain temperature.

1.5. The scalar mass

In the absence of fermionic contributions, the mass of a scalar field is given as $m_\varphi^2 = \left. \frac{\partial^2 U(\varphi)}{\partial \varphi^2} \right|_{\varphi=\varphi_m}$. Due to fermionic contributions, however, the effective potential of the scalar field φ is given by $F_{\varphi\psi}$ of Eq. (52). We thus define the scalar mass as

$$m_\varphi^2 \equiv \left. \frac{\partial^2 F_{\varphi\psi}(\varphi)}{\partial \varphi^2} \right|_{\varphi=\varphi_m} \tag{76}$$

Using Eqs. (52) and (63), we can rewrite this as,

$$\boxed{m_\varphi^2 = \left. \frac{\partial^2 U(\varphi)}{\partial \varphi^2} \right|_{\varphi=\varphi_m} - \frac{2 N_F g_Y^2}{3\pi^2 \beta^2} \left. \frac{d^2 \mathscr{I}_{3/2}(\kappa)}{d\kappa^2} \right|_{\kappa=\beta g_Y \varphi_m}.} \tag{77}$$

We use the Leibniz integral rule and the definition of $\mathscr{I}_{3/2}(\kappa)$ to get,

$$\frac{d^2 \mathscr{I}_{3/2}(\kappa)}{d\kappa^2} = 3 \left[\kappa^2 \int_\kappa^\infty \frac{dz}{\left(e^z + 1\right) \left(z^2 - \kappa^2\right)^{1/2}} - \mathscr{I}_{1/2}(\kappa) \right]. \tag{78}$$

The asymptotic expression for $\frac{d^2 \mathscr{I}_{3/2}(\kappa)}{d\kappa^2}$ is presented in Tab. 1.

1.6. Equation of state and sound speed

The **equation of state** $w_{\varphi\psi}$ for the $\varphi\psi$ fluid is defined as

$$\boxed{w_{\varphi\psi} \equiv \frac{P_{\varphi\psi}}{\rho_{\varphi\psi}},} \tag{79}$$

where $P_{\varphi\psi} = -F_{\varphi\psi}$, and $\rho_{\varphi\psi}$ is the total energy density

$$\rho_{\varphi\psi} = U(\varphi) + \frac{2N_F}{\pi^2} \int_0^\infty dp\, p^2\, n_F(\epsilon)\, \epsilon = U(\varphi) + \frac{2N_F}{\pi^2 \beta^4} \mathscr{I}_\epsilon(\kappa), \tag{80}$$

where

$$\mathscr{I}_\epsilon(\kappa) \equiv \int_\kappa^\infty \frac{dz\, z^2 \left(z^2 - \kappa^2\right)^{1/2}}{e^z + 1}. \tag{81}$$

The adiabatic **sound speed** $c_{s,\varphi\psi}$ is defined as

$$\boxed{c_{s,\varphi\psi}^2 \equiv \frac{dP_{\varphi\psi}}{d\rho_{\varphi\psi}}.} \tag{82}$$

In the *regimes where the gap equation has a solution*, we can write

$$\begin{aligned} c_{s,\varphi\psi}^2 = \frac{\frac{dP_{\varphi\psi}}{d\Delta}}{\frac{d\rho_{\varphi\psi}}{d\Delta}} &= \frac{\left.\left(\frac{\partial P_{\varphi\psi}}{\partial \Delta} + \frac{\partial P_{\varphi\psi}}{\partial \kappa}\frac{d\kappa}{d\Delta}\right)\right|_{\kappa=\beta g_Y \varphi_m}}{\left.\left(\frac{\partial \rho_{\varphi\psi}}{\partial \Delta} + \frac{\partial \rho_{\varphi\psi}}{\partial \kappa}\frac{d\kappa}{d\Delta}\right)\right|_{\kappa=\beta g_Y \varphi_m}} \\ &= \left.\frac{\frac{\partial P_{\varphi\psi}}{\partial \Delta}}{\frac{\partial \rho_{\varphi\psi}}{\partial \Delta} + \frac{\partial \rho_{\varphi\psi}}{\partial \kappa}\dot{\kappa}_m}\right|_{\kappa=\beta g_Y \varphi_m}, \end{aligned} \tag{83}$$

where κ_m and Δ are related through the mass equation Eq. (54), $\dot{\kappa}_m \equiv \left.\frac{d\kappa}{d\Delta}\right|_{\kappa=\beta g_Y \varphi_m}$, and in the last step we have used $\left.\frac{\partial P_{\varphi\psi}}{\partial \kappa}\right|_{\kappa=\beta g_Y \varphi_m} = 0$, a fact seen from the first of Eq. (53).

We will now write $c_{s,\varphi\psi}^2$ in a form more convenient for calculations. Let us start with the pressure and energy density normalized by M^4, i.e., $P_R \equiv \frac{P_{\varphi\psi}}{M^4}$ and $\rho_R \equiv \frac{\rho_{\varphi\psi}}{M^4}$. For a generic scalar field potential $u(\kappa,\Delta) \equiv \frac{U(\kappa,\Delta)}{M^4}$, we can write

$$P_R = -u(\kappa,\Delta) + \frac{2N_F}{3\pi^2\Delta^4}\mathscr{I}_{3/2}(\kappa), \qquad \rho_R = u(\kappa,\Delta) + \frac{2N_F}{\pi^2\Delta^4}\mathscr{I}_\epsilon(\kappa), \tag{84}$$

and the mass equation Eq. (54) becomes

$$\frac{\partial u}{\partial \kappa} + \frac{2N_F}{\pi^2\Delta^4}\kappa \mathscr{I}_{1/2}(\kappa) = 0. \tag{85}$$

The potential we will consider can be written as a function of $\frac{\kappa}{\Delta}$, for which it can be checked that $\frac{\partial u}{\partial \Delta} = -\frac{\kappa}{\Delta}\frac{\partial u}{\partial \kappa}$; thus

$$\frac{\partial u}{\partial \Delta} = -\frac{\kappa}{\Delta}\frac{\partial u}{\partial \kappa} = \frac{2N_F}{\pi^2\Delta^5}\kappa^2 \mathscr{I}_{1/2}(\kappa). \tag{86}$$

From the expressions in Eq. (84), we can write,

$$\begin{aligned} \frac{\partial P_R}{\partial \Delta} &= -\frac{2N_F}{\pi^2\Delta^5}\left(\kappa^2 \mathscr{I}_{1/2}(\kappa) + \frac{4}{3}\mathscr{I}_{3/2}(\kappa)\right), \\ \frac{\partial \rho_R}{\partial \Delta} &= \frac{2N_F}{\pi^2\Delta^5}\left(\kappa^2 \mathscr{I}_{1/2}(\kappa) - 4\mathscr{I}_\epsilon(\kappa)\right), \\ \frac{\partial \rho_R}{\partial \kappa} &= \frac{2N_F}{\pi^2\Delta^4}\left(\mathscr{I}'_\epsilon(\kappa) - \kappa \mathscr{I}_{1/2}(\kappa)\right). \end{aligned} \tag{87}$$

Thus we can rewrite Eq. (83) as,

$$c_{s,\varphi\psi}^2 = \left.\frac{\kappa^2 \kappa \mathscr{I}_{1/2}(\kappa) + \frac{4}{3}\mathscr{I}_{3/2}(\kappa)}{4\mathscr{I}_\epsilon(\kappa) - \kappa^2 \mathscr{I}_{1/2}(\kappa) + \left[\kappa \mathscr{I}_{1/2}(\kappa) - \mathscr{I}'_\epsilon(\kappa)\right]\kappa \frac{d\log_e \kappa}{d\log_e \Delta}}\right|_{\kappa=\beta g_Y \varphi_m}. \tag{88}$$

We can see some general properties from Eq. (88):

1. At very high temperatures, we have $\Delta \to 0$ and $\kappa \to 0$, and in this regime,
$$c^2_{s,\varphi\psi} \to \frac{1}{3}\frac{\mathscr{I}_{3/2}(0)}{\mathscr{I}_{\epsilon}(0)} = \frac{1}{3}. \tag{89}$$

2. *This point works for the potential we will consider, but cannot be generally extended to all classes of potentials.* We will see later that as the universe cools, the $\varphi\psi$ fluid undergoes a first-order phase transition at a critical temperature T_{PT}. As $T \to T^+_{\mathrm{PT}}$, the fermion mass approaches the critical mass $m_{\psi,\mathrm{PT}}$ with a critical exponent of $\frac{1}{2}$,
$$m_\psi - m_{\psi,\mathrm{PT}} \sim (T - T_{\mathrm{PT}})^{1/2} \qquad \Rightarrow \qquad \left.\frac{\partial m_\psi}{\partial T}\right|_{T_{\mathrm{PT}}} \sim \frac{1}{\sqrt{T - T_{\mathrm{PT}}}} \to \infty \tag{90}$$
This tells us that $\left.\frac{\partial \kappa}{\partial \Delta}\right|_{\kappa=\kappa_{\mathrm{PT}}} \to \infty$, and thus, the sound speed approaches 0 at the critical point.

2. The exponential scalar potential

This potential is of the form
$$\boxed{U(\varphi) = M^4 e^{-\lambda\varphi/M},} \tag{91}$$
where M was introduced previously in Eq. (66). As mentioned below, we will also set the bare mass of the fermion to zero, i.e., $\bar{m}_\psi = 0$. The dimensionless thermodynamic potential F_R, defined as in Eq. (66) becomes
$$\boxed{F_R = e^{-\lambda\kappa/g_Y\Delta} - \frac{2N_F}{3\pi^2\Delta^4}\mathscr{I}_{3/2}(\kappa),} \tag{92}$$
while the mass equation Eq. (54) becomes
$$\boxed{\frac{\lambda\pi^2}{2g_Y N_F}\Delta^3 = \underbrace{\kappa\, e^{\lambda\kappa/g_Y\Delta}\int_\kappa^\infty \frac{dz\,\left(z^2-\kappa^2\right)^{1/2}}{e^z+1}}_{\mathcal{I}_\Delta(\kappa)}.} \tag{93}$$

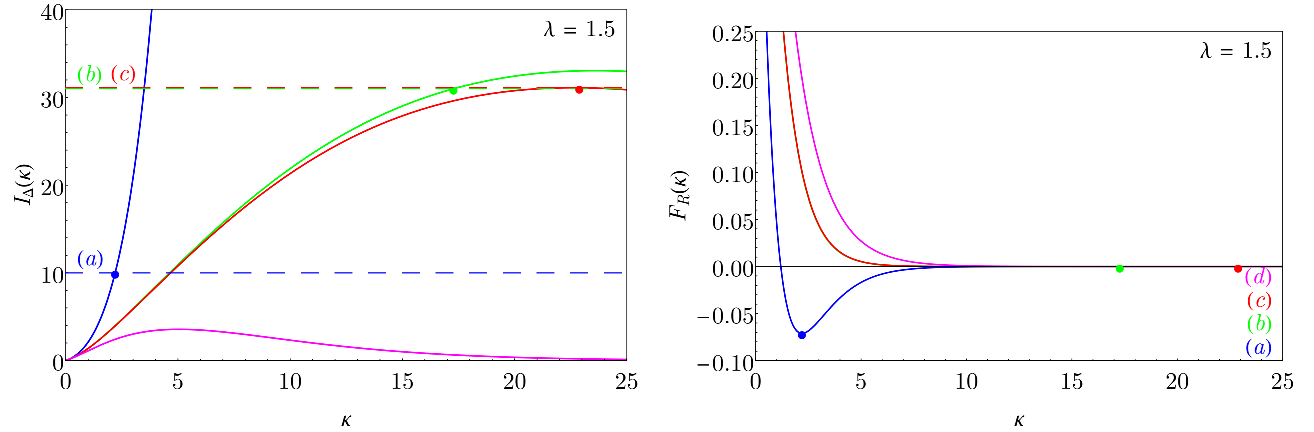
The evolution of the system is shown in Fig. 1a, where we plot the integral $\mathcal{I}_\Delta(\kappa)$ in solid curves for a chosen value of $\lambda/g_Y = 1.5$, and various values of Δ; the LHS of Eq. (93) for corresponding values of Δ is plotted with dashed lines of the same colour. In Fig. 1b, we show the normalized potential of the Eq. (92) for the corresponding cases. As expected, the equilibrium solutions to the mass equation minimize the potential F_R at the corresponding Δ.

As will be described in more details, the system goes through three phases.

a) When the universe is very hot, the $\varphi\psi$ fluid is in the ***stable phase***, where Eq. (93) has a non-trivial equilibrium solution κ_c (which minimizes F_R) and the pressure is positive, as seen for case '(a)' in Figs. 1a and 1b.

b) As the universe cools further, the system passes through the point of metastability (case '(b)' in Figs. 1a and 1b), when $\Delta = \Delta_o$ and the equilibrium solution to Eq. (93) is denoted κ_o, to go into the ***metastable phase*** where there is still a non-trivial solution to Eq. (93), but the pressure is now negative. The local minimum of F_R, which Eq. (93), is no longer the true minimum of F_R in the metastable regime.

c) Finally the system crosses the critical point $\Delta = \Delta_{\mathrm{PT}}$ (case '(c)' in Figs. 1a and 1b), where Eq. (93) is solved at $\kappa = \kappa_{\mathrm{PT}}$, to enter the ***unstable phase*** with no equilibrium solution to Eq. (93), as seen from case '(d)' in Figs. 1a and 1b. In this regime, we need to numerically solve the EOM of the $\varphi\psi$ system, along with the Friedmann equations. Qualitatively, the field φ slowly rolls towards the trivial solution $\varphi \to \infty$, known as the "doomsday vacuum" in dark energy contexts.



(a) Solutions of the mass equation, Eq. (93). (b) The thermodynamic potential of Eq. (92).

Figure 1: *Left*: Thermodynamicaly stable solutions (κ_m) of Eq. (93) for $\lambda/g_Y = 1.5$ and different values of Δ. The dashed line corresponding to the LHS of Eq. (93) for the purple $\mathcal{I}_\Delta$ curve lies beyond the range of the plot. *Right*: F_R for the cases in Fig. 1a. The equilibrium solutions are indicated by big filled dots.

We will now make an approximate analytic determination of the critical point using the fact that at the critical point, the solution to the mass equation also maximizes $\mathcal{I}_\Delta(\kappa)$. We expect that the critical point will be in the $\kappa \gg 1$ regime, and using the definition of $\mathcal{I}_\Delta(\kappa)$ in Eq. (93) and the form in Tab. 1, and recalling that for $x \gg 1$ we have $K_1(x) \approx e^{-x}\left[\sqrt{\frac{\pi}{2x}} + \mathcal{O}(x^{-3/2})\right]$, we have

$$\mathcal{I}_\Delta(\kappa) \approx \sqrt{\frac{\pi}{2}}\kappa^{3/2}\, e^{-\kappa\left(1-\frac{\lambda}{g_Y\Delta}\right)}. \tag{94}$$

The local maximum of $\mathcal{I}_\Delta$ at a given β is when $\mathcal{I}'_\Delta(\kappa) = 0$, which is satisfied when κ has the value

$$\kappa_m^\Delta = \frac{3}{2}\left(1 - \frac{\lambda}{g_Y\Delta}\right)^{-1}, \tag{95}$$

and the value of the integral $\mathcal{I}_\Delta$ at this point is

$$\mathcal{I}_\Delta(\kappa_m^\Delta) = \sqrt{\frac{\pi}{2}}\left[\frac{3}{2e}\left(1 - \frac{\lambda}{g_Y\Delta}\right)^{-1}\right]^{3/2}. \tag{96}$$

The critical value $\Delta_{\rm PT}$ is when this equals the LHS of Eq. (93),

$$\frac{\lambda\pi^2}{2g_Y N_F}\Delta_{\rm PT}^3 = \sqrt{\frac{\pi}{2}}\left[\frac{3}{2e}\left(1 - \frac{\lambda}{g_Y\Delta_{\rm PT}}\right)^{-1}\right]^{3/2}, \tag{97}$$

leading to

$$\boxed{\Delta_{\rm PT} = \frac{\lambda\left(1 + \sqrt{1 + \frac{Bg_Y^{8/3}}{\lambda^{8/3}}}\right)}{2g_Y}} \tag{98}$$

with $B = \dfrac{6}{\pi e}\left(2N_F^2\right)^{1/3}$. Thus from Eqs. (95) and (96), the critical value of κ and the corresponding $\mathcal{I}_\Delta$ are given by

$$\boxed{\kappa_{\rm PT} = \frac{3}{2}\frac{\sqrt{1+\frac{Bg_Y^{8/3}}{\lambda^{8/3}}}+1}{\sqrt{1+\frac{Bg_Y^{8/3}}{\lambda^{8/3}}}-1}, \qquad T_{\rm PT} = \frac{2g_Y M}{\lambda\left(1+\sqrt{1+\frac{Bg_Y^{8/3}}{\lambda^{8/3}}}\right)}.} \tag{99}$$

From Eq. (77), the scalar field mass is determined before the phase transition to be,

$$\boxed{\frac{m_\varphi^2}{g_Y^2 M^2} = \frac{\lambda^2}{g_Y^2}\, e^{-\lambda\kappa_m/g_Y\Delta} - \frac{2N_F}{3\pi^2\Delta^2}\mathscr{I}''_{3/2}(\kappa)\Big|_{\kappa=\kappa_m}.} \tag{100}$$

where κ_m solves Eq. (93).

We can work out the redshift $z_{\rm PT}$ at which the phase transition occurs by noting that the critical temperature $T_{\rm PT}$ can be related to the temperature today as $T_{\rm PT} = (1+z_{\rm PT})\,T_{\rm now}\left(\frac{g_{S,\rm now}}{g_{S,\rm PT}}\right)^{1/3}$, where g_S is the effective relativistic degrees of freedom. This gives

$$\boxed{1+z_{\rm PT} = \frac{2g_Y M}{\lambda T_{\rm now}\left(1+\sqrt{1+\frac{Bg_Y^{8/3}}{\lambda^{8/3}}}\right)}\left(\frac{g_{S,\rm PT}}{g_{S,\rm now}}\right)^{1/3}.} \tag{101}$$

2.1. The stable phase

In the stable phase, we can derive an approximate expression for the fermion mass by solving the mass equation Eq. (93). We make the (not so satisfactory) assumption that at very high temperatures, we can write $e^{\kappa\lambda/g_Y\Delta} \approx 1+\frac{\kappa\lambda}{g_Y\Delta}$, and using the $x \ll 1$ approximation of $\mathscr{I}_{1/2}(x)$, we can write $\mathcal{I}_\Delta(\kappa) \approx \frac{\pi^2}{12}\kappa$. This gives the high temperature solution

$$\kappa_c = \frac{6\lambda\Delta^3}{g_Y N_F}. \tag{102}$$

The normalized fermion mass is then given by

$$\boxed{\frac{m_\psi}{M} = \frac{\kappa_m}{\Delta} = \frac{6\lambda\Delta^2}{g_Y N_F} \sim T^{-2}.} \tag{103}$$

We can substitute this value of κ_c in Eq. (100) to get the scalar mass,

$$\boxed{\frac{m_\varphi^2}{g_Y^2 M^2} = \frac{\lambda^2}{g_Y^2}\left(1-\frac{6\lambda^2\Delta}{g_Y^2 N_F}\right) + \underbrace{\frac{N_F}{6\Delta^2} + \frac{54\lambda^2\Delta^4}{\pi^2 g_Y^2 N_F}\log_e\left(\frac{6\lambda\Delta^3}{g_Y N_F}\right)}_{\text{Fermionic contribution}}.} \tag{104}$$

We plot the scalar and fermion masses in Fig. 2 by numerically solving Eq. (93), and evaluating Eq. (100) for the obtained values of κ_m.

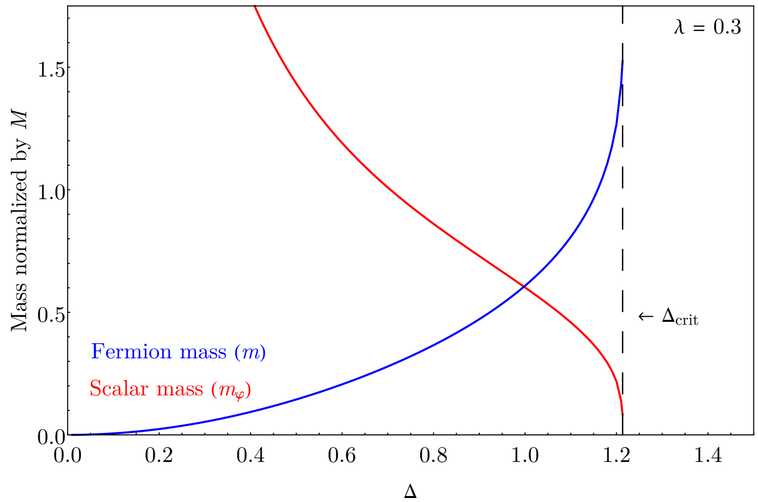


Figure 2: Fermion and scalar masses as a function of Δ for $\lambda = 0.3$. Note that these are exact numerical solutions, and not the approximate expresiions derived in Sec. 2.1.

2.2. The point of metastability

With the expectation that metastability occurs in a relatively low temperature regime, we use the low temperature expansion of $\mathscr{I}_{3/2}(\kappa)$ to write the thermodynamic potential as

$$F_R = e^{-\lambda\kappa/g_Y\Delta} - \frac{2N_F}{3\pi^2\Delta^4}\left(3\kappa^2 K_2(\kappa)\right), \tag{105}$$

while the mass equation Eq. (93) gives us

$$\frac{2N_F}{\pi^2\Delta^4} = \frac{\lambda e^{-\lambda\kappa_m/g_Y\Delta}}{g_Y\Delta\kappa_m\mathscr{I}_{1/2}(\kappa)} \approx \frac{\lambda e^{-\lambda\kappa_m/g_Y\Delta}}{g_Y\Delta\kappa_c^2 K_1(\kappa)}. \tag{106}$$

Putting these in Eq. (105), we get for an equilibrium solution point

$$F_R = e^{-\lambda\kappa_m/g_Y\Delta}\left[1 - \frac{\lambda}{3g_Y\Delta}\frac{3\kappa_m^2 K_2(\kappa_m)}{\kappa_m^2 K_1(\kappa_m)}\right] \approx e^{-\lambda\kappa_m/g_Y\Delta}\left[1 - \frac{\lambda}{g_Y\Delta}\left(1 + \frac{3}{2\kappa_m} + \frac{3}{8\kappa_m^2}\right)\right]. \tag{107}$$

The metastability point is when $F_R = 0$, which is achieved when

$$\boxed{1 - \frac{\lambda}{g_Y\Delta_o}\left(1 + \frac{3}{2\kappa_o} + \frac{3}{8\kappa_o^2}\right) = 0.} \tag{108}$$

The quantity Δ appears here, suggesting that this equation needs to be solved in conjuction with the mass equation with the low temperature approximation of $\mathcal{I}_\Delta$,

$$\frac{\lambda\pi^2}{2g_Y N_F}\Delta^3 = \sqrt{\frac{\pi}{2}}\kappa_m^{3/2}\, e^{-\kappa_m\left(1 - \frac{\lambda}{g_Y\Delta}\right)}. \tag{109}$$

2.3. Equation of state, speed of sound

The definition for the EoS of the $\varphi\psi$ fluid is given in Eq. (79), which can be equivalently expressed in terms of the normalized quantities defined in Eq. (84), i.e., $w_{\varphi\psi} \equiv \dfrac{P_R}{\rho_R}$; considering the FJ potential, we have

$$P_R = -e^{-\lambda\kappa/g_Y\Delta} + \frac{2N_F}{3\pi^2\Delta^4}\mathscr{I}_{3/2}(\kappa), \qquad \rho_R = e^{-\lambda\kappa/g_Y\Delta} + \frac{2N_F}{\pi^2\Delta^4}\mathscr{I}_\epsilon(\kappa). \tag{110}$$

The squared sound speed for the $\varphi\psi$ fluid is given by Eq. (88), where the only potential-dependent term is $\left.\dfrac{d\log_e\kappa}{d\log_e\Delta}\right|_{\kappa=\kappa_m}$ in the denominator. For the FJ potential, this evaluates to

$$\left.\frac{d\log_e\kappa}{d\log_e\Delta}\right|_{\kappa=\kappa_m} = \left.\frac{3+\frac{\lambda\kappa_m}{g_Y\Delta}}{1+\frac{\lambda\kappa_m}{g_Y\Delta}+\frac{\partial\log_e\mathscr{I}_2(\kappa)}{\partial\log_e\kappa}}\right|_{\kappa=\kappa_m}. \tag{111}$$

Asymptotic and critical behaviour:

a) High Temperature: At very high temperatures, in the stable phase, Eq. (103) shows that $\dfrac{\kappa_c}{\Delta}\sim T^{-2}$; also the exponential term in both P_R and ρ_R are almost 1, and so using the high T approximations for $\mathscr{I}_{3/2}(\kappa)$ and $\mathscr{I}_\epsilon(\kappa)$, we get the expected behavior

$$w\approx\frac{1}{3}. \tag{112}$$

As already seen from Eq. (89), in this regime, the squared sound speed also approaches $\dfrac{1}{3}$.

b) Low Temperature: For lower temperatures ($\kappa_m\gtrsim 1$) the pressure is obtained from Eq. (107), while the approximation for energy density is given by

$$\rho_R = e^{-\lambda\kappa_m/g_Y\Delta}\left[1+\frac{\lambda}{g_Y\Delta}\frac{3\kappa_m^2K_2(\kappa_m)+\kappa_m^3K_1(\kappa_m)}{\kappa_m^2K_1(\kappa_m)}\right]\simeq e^{-\lambda\kappa_m/g_Y\Delta}\left[1+\frac{\lambda}{g_Y\Delta}\left(\kappa_m+3+\frac{9}{2\kappa_m}+\frac{9}{8\kappa_m^2}\right)\right], \tag{113}$$

which leads to,

$$w\approx-\frac{1-\frac{\lambda}{g_Y\Delta}\left(1+\frac{3}{2\kappa_m}+\frac{3}{8\kappa_m^2}\right)}{1+\frac{\lambda}{g_Y\Delta}\left(\kappa_m+3+\frac{9}{2\kappa_m}+\frac{9}{8\kappa_m^2}\right)}, \tag{114}$$

where the relationship between κ_m and Δ is now given by Eq. (109).

c) The Critical Point: At the critical point, we use the expressions in Eq. (98) and (99) to get the pressure,

$$P_{R,\mathrm{PT}} = -e^{-\lambda\kappa_{\mathrm{PT}}/g_Y\Delta_{\mathrm{PT}}}\cdot\frac{R^2+1}{(R+1)^2}, \tag{115}$$

and the energy density,

$$\rho_{R,\mathrm{PT}} = e^{-\lambda\kappa_{\mathrm{PT}}/g_Y\Delta_{\mathrm{PT}}}\cdot\frac{R+10+\dfrac{6\lambda^{8/3}}{Bg_Y^{8/3}}(3-R)}{R+1}, \tag{116}$$

where $R=\sqrt{1+\dfrac{Bg_Y^{8/3}}{\lambda^{8/3}}}$. This gives us

$$w_{\mathrm{PT}} = -\frac{2+B\dfrac{g_Y^{8/3}}{\lambda^{8/3}}}{(R+1)\left[R+10+\dfrac{6\lambda^{8/3}}{Bg_Y^{8/3}}(3-R)\right]}. \tag{117}$$

Also seen previously is the fact that the sound speed approaches 0 at the critical point (see discussion below Eq. (90)).

2.4. Transition to the unstable regime

To understand the nature of the transition from the metastable to the unstable phases, we will expand the integral $\mathcal{I}_\Delta(\kappa)$ close to the critical point,

$$\mathcal{I}(\kappa_c) = \mathcal{I}(\kappa_{\rm PT}) + \frac{1}{2}\frac{\partial^2 \mathcal{I}_\Delta}{\partial \kappa^2}\bigg|_{\kappa_{\rm PT}} (\kappa_m - \kappa_{\rm PT})^2 + \mathcal{O}\left((\kappa_m - \kappa_{\rm PT})^3\right). \tag{118}$$

Using Eq. (93), we can write this as

$$1 - \left(\frac{\Delta}{\Delta_{\rm PT}}\right)^3 = -\frac{1}{2\mathcal{I}_\Delta(\kappa_{\rm PT})}\frac{\partial^2 \mathcal{I}_\Delta}{\partial \kappa^2}\bigg|_{\kappa_{\rm PT}} (\kappa_m - \kappa_{\rm PT})^2, \tag{119}$$

and with the form of $\mathcal{I}_\Delta$ in Eq. (94), we get

$$\frac{\partial \mathcal{I}_\Delta}{\partial \kappa} = \left(\frac{3}{2\kappa} - 1 + \frac{\lambda}{g_Y \Delta}\right)\mathcal{I}_\Delta, \qquad \frac{\partial^2 \mathcal{I}_\Delta}{\partial \kappa^2} = \left\{\left(\frac{3}{2\kappa} - 1 + \frac{\lambda}{g_Y \Delta}\right)^2 - \frac{3}{2\kappa^2}\right\}\mathcal{I}_\Delta. \tag{120}$$

As indicated by Eq. (95), the term in parentheses in the expression for $\dfrac{\partial^2 \mathcal{I}_\Delta}{\partial \kappa^2}$ above vanishes at the critical point, leaving us with

$$-\frac{1}{2\mathcal{I}_\Delta(\kappa_{\rm PT})}\frac{\partial^2 \mathcal{I}_\Delta}{\partial \kappa^2}\bigg|_{\kappa_{\rm PT}} = \frac{3}{4\kappa_{\rm PT}^2}. \tag{121}$$

Substituting back in Eq. (119) gives us

$$\frac{3}{4}\left(\frac{\kappa_m}{\kappa_{\rm PT}} - 1\right)^2 = 1 - \left(\frac{\Delta}{\Delta_{\rm PT}}\right)^3. \tag{122}$$

Thus near the critical point, we can expand

$$\frac{T}{T_{\rm PT}} = \left\{1 - \frac{3}{4}\left(\frac{\kappa_m}{\kappa_{\rm PT}} - 1\right)^2\right\}^{-1/3} \approx 1 + \frac{1}{4}\left(\frac{\kappa_m}{\kappa_{\rm PT}} - 1\right)^2, \tag{123}$$

Now, close to $T_{\rm PT}$, we can approximate,

$$\kappa_{\rm PT} - \kappa_m \equiv \frac{m_{\psi,\rm PT}}{T_{\rm PT}} - \frac{m_\psi}{T} \simeq \frac{m_{\psi,\rm PT} - m_\psi}{T_{\rm PT}}, \tag{124}$$

and so, dropping higher order terms we get

$$\boxed{m_{\psi,\rm PT} - m_\psi \sim \left(\frac{T}{T_{\rm PT}} - 1\right)^{1/2}.} \tag{125}$$

The system thus undergoes a first-order phase transition into the unstable phase, and the fermion mass approaches the critical mass $m_{\psi,\rm PT}$ with a critical exponent of $\dfrac{1}{2}$.

2.5. After the phase transition

After the system undergoes the phase transition and goes into the unstable regime, the equilibrium methods used before no longer apply, since Eq. (93) no longer has a solution. One therefore needs to go beyond the saddle-point approximation and solve the full equation

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial F_{\varphi\psi}}{\partial \varphi} = 0, \tag{126}$$

in conjunction with the Friedmann equations. For the fermions, we have $\rho_s \propto a^{-3}$; so for $a \geq a_{\rm PT}$ (i.e., $z < z_{\rm PT}$), we can write Eq. (126) using Eq. (52) as

$$\underbrace{\ddot{\varphi} + 3H\dot{\varphi}}_{X_1} = \underbrace{-\frac{\partial U}{\partial \varphi}}_{X_2} \underbrace{- g_Y \rho_{s,\rm PT} \left(\frac{a_{\rm PT}}{a}\right)^3}_{X_3}. \tag{127}$$

As a first step, we need to find an expression for $\rho_{s,\rm PT}$; we know from Eq. (64) that

$$\frac{\pi^2\beta^2}{2N_F m_\psi}\rho_s = \mathscr{I}_{1/2}(\kappa), \tag{128}$$

while Eq. (93) gives us

$$\frac{\lambda\pi^2\Delta^3}{2g_Y N_F \kappa} e^{-\lambda\kappa/g_Y\Delta} = \mathscr{I}_{1/2}(\kappa). \tag{129}$$

Equating these two, we get

$$\frac{\rho_s}{M^3} = \frac{\lambda e^{-\lambda\kappa/g_Y\Delta}}{g_Y}, \tag{130}$$

which becomes at the critical point,

$$\frac{\rho_{s,\rm PT}}{M^3} = \frac{\lambda e^{-\lambda\kappa_{\rm PT}/g_Y\Delta_{\rm PT}}}{g_Y}. \tag{131}$$

We can use the expressions in Eqs. (98) and (99) to get the critical ρ_s and the critical scalar energy density as

$$\boxed{\frac{\rho_{s,\rm PT}}{M^3} = \frac{\lambda e^{-3/(R\,-\,1)}}{g_Y}, \qquad \frac{\rho_{\varphi,\rm PT}}{M^4} = e^{-3/(R\,-\,1)}.} \tag{132}$$

We will solve Eq. (127) for $a \geq a_{\rm PT}$ by ignoring the LHS[3] to get,

$$e^{-\lambda\varphi/M} = e^{-3/(R\,-\,1)}\left(\frac{a_{\rm PT}}{a}\right)^3. \tag{133}$$

At the present time, the LHS of the above equation is just the normalized energy density in φ, i.e., $\rho_{\varphi,\rm now}/M^4$. ***The crucial thing to note from Eq.*** (133) ***is that the energy density ρ_φ of φ has a $T^3 \propto a^{-3}$ dependence, which makes it behave as pressureless matter after the phase transition.***

The absence of a stable solution to Eq. (93) also means that the expression Eq. (50) for the fermion mass is no longer valid in the dark-matter phase. However, analogous to Eq. (50), we define the fermion mass in this phase to be the current value of $\varphi(t)$, i.e.,

$$m_\psi(z) = g_Y\varphi(z) = \frac{3g_Y M}{\lambda}\left[\frac{1}{R-1} + \log_e \frac{1+z_{\rm PT}}{1+z}\right], \tag{134}$$

the last equality being obtained from Eq. (133). The fermion mass thus increases very weakly with redshift, and can be well approximated to be a constant in the dark-matter phase. At the present time, it is given by

$$\boxed{m_\psi^{\rm now} \equiv g_Y\varphi_{\rm now} = 3g_Y M\left[\frac{1}{R-1} + \log_e(1+z_{\rm PT})\right].} \tag{135}$$

Since this solution was obtained by setting $X_2 = X_3$ in Eq. (127), we need to justify that this is a valid assumption. This will be valid only if $|X_1| \ll |X_2|, |X_3|$ for our solution φ. In Fig. 3 we show the relative magnitude of X_1 with respect to X_2, after substituting the solution φ given by Eq. (134) in Eq. (127). We see that for all redshifts z, phase transition temperatures in the range of 10 MeV to 10^7 GeV yield $|X_1| \ll |X_2|, |X_3|$.

[3]This is justified below, as well as in Sec. 2.6.

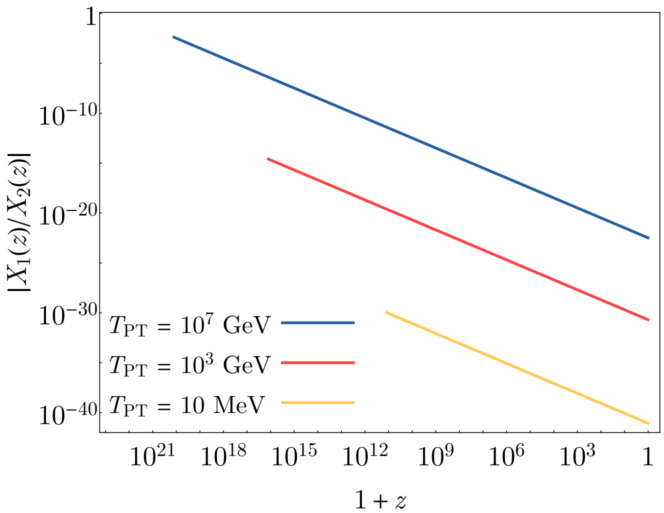


Figure 3: Relative magnitude of the term X_1 with respect to X_2 in Eq. (127), obtained by substituting in the solution of Eq. (134), as a function of redshift z for several choices of the phase transition temperature $T_{\rm PT}$. We see that $|X_1| \ll |X_2|$, and since $|X_2| = |X_3|$ by construction, then $|X_1| \ll |X_3|$.

The total dark matter energy density is a sum of the energy densities of both φ and ψ. Using Eq. (132) and noting that $\rho_\psi(z) = \frac{1}{2} m_\psi(z)\rho_s(z)$, we can write[4]

$$\begin{aligned}\rho_{\rm DM}(z) &= \rho_\psi(z) + \rho_\varphi(z)\\ &= \left[\frac{M^3\lambda m_\psi(z)}{2g_Y} + M^4\right] e^{-3/(R\,-\,1)} \left(\frac{1+z}{1+z_{\rm PT}}\right)^3,\end{aligned} \tag{136}$$

which can be simplified using Eq. (134) to get

$$\boxed{\rho_{\rm DM}(z) = M^4 e^{-3/(R\,-\,1)} \left[1 + \frac{3}{2}\left\{\frac{1}{R-1} + \log_e \frac{1+z_{\rm PT}}{1+z}\right\}\right] \left(\frac{1+z}{1+z_{\rm PT}}\right)^3.} \tag{137}$$

Numerical predictions

The model has two parameters M and λ in the potential $U(\varphi)$, and a coupling constant G_Y that can be tuned to make predictions. The parameter λ typically appears with g_Y together in the combination $\frac{g_Y}{\lambda}$. We will use two observational constraints, the phase transition temperature $T_{\rm PT}$ and the total DM energy density $\rho_{\rm DM,now}$ to fix these. We want the phase transition to happen around when the temperature of the universe is in the range $T_{\rm PT} \in \left[10\,\text{MeV}, 10^7\,\text{GeV}\right]$.

We undertake the following program to obtain the numerical predictions.

- Given a choice of $T_{\rm PT}$, we can solve for the corresponding value of λ/g_Y by constraining $\Omega_{\rm DM,now} \equiv \rho_{\rm DM,now}/\rho_{\rm tot,now} = 0.26$, and using the expression, obtained from Eq. (137) as

$$\frac{\rho_{{\rm DM},0} T_{\rm PT}^3 g_{S,{\rm PT}}}{M^4 T_0^3 g_{S,0}} = e^{-3/(R\,-\,1)} \left[1 + \frac{3}{2}\left\{\frac{1}{R-1} + \log_e(1+z_{\rm PT})\right\}\right], \tag{138}$$

  and using the values $g_{S,\rm now} = 3.91$ and $g_{S,\rm PT} = 100$.

- One can then calculate all the parameters in the model, and derive the mass of the DM now using Eq. (135).

[4] We note from Eqs. (70) and (71) that for $g_Y\varphi \gg T$, we have $\rho_s = 2n_\pm$.

In Tab. 2, we present an estimate of these quantities to check that the model is self-consistent and viable in describing DM. For all these cases, $\kappa_{\rm PT} \gtrsim 20$, which justifies the $\kappa \gg 1$ approximations used to derive the expressions; this is a test of self-consistency.

$T_{\rm PT}$ (GeV)	$m_\psi^{\rm now}$ (GeV)	λ/g_Y	M (GeV)
0.01	1.0	1.4448	1.56×10^{-2}
0.1	10	1.5185	1.63×10^{-1}
10	1.3×10^3	1.6479	1.74×10^1
10^3	1.5×10^5	1.7608	1.84×10^3
10^5	1.7×10^7	1.8618	1.94×10^5
10^7	1.9×10^9	1.9539	2.02×10^7
10^8	2.0×10^{10}	1.9968	2.06×10^8
10^9	2.1×10^{11}	2.0387	2.10×10^9

Table 2: The present masses $m_\psi^{\rm now}$ of the fermion for different choices of the phase transition temperature $T_{\rm PT}$. The relic dark matter energy density $\Omega_{\rm DM}^{\rm now}$ has been fixed to 0.264 to match the *Planck* data. The redshift $z_{\rm PT}$ for each case is given by $z_{\rm PT} \approx 1.25 \times 10^{12}(T_{\rm PT}/0.1\,{\rm GeV})$.

2.6. Is the mean field solution valid?

In Sec. 2.5, we solved Eq. (127) for $a \geq a_{\rm PT}$ by ignoring the LHS and claiming that the solution thus obtained is correct accurate enough; in this section, we provide a numerical estimate of this claim.

We first start by rewriting Eq. (127) to express all functions of t to functions of a, which makes the derivatives

$$\frac{d\varphi}{dt} = aH\frac{d\varphi}{da}, \qquad \frac{d^2\varphi}{dt^2} = aH\left[aH\frac{d^2\varphi}{da^2} + \left(a\frac{dH}{da} + H\right)\frac{d\varphi}{da}\right]. \tag{139}$$

Putting this in Eq. (127) and dividing throughout by a^2H^2, we write

$$\frac{d^2\varphi}{da^2} + \left(\frac{1}{H}\frac{dH}{da} + \frac{4}{a}\right)\frac{d\varphi}{da} = \frac{\lambda M^3}{a^2H^2}\left[e^{-\lambda\varphi/M} - e^{-3/(R\,-\,1)}\left(\frac{a_{\rm PT}}{a}\right)^3\right]. \tag{140}$$

We can do another change of variables from a to $x \equiv \log_e a - \log_e a_{\rm PT}$, and write Eq. (140) as

$$\frac{d^2\varphi}{dx^2} + \left(\frac{1}{H}\frac{dH}{dx} + 3\right)\frac{d\varphi}{dx} = \frac{\lambda M^3}{H^2}\left[e^{-\lambda\varphi/M} - e^{-3[1/(R\,-\,1)+x]}\right], \tag{141}$$

and the solution Eq. (134) as

$$\varphi(x) = \frac{3M}{\lambda}\left[\frac{1}{R-1} + x\right]. \tag{142}$$

The boundary conditions needed to solve Eq. (141) can be obtained from this solution, i.e.,

$$\varphi(x=0) = \varphi_{\rm PT} = \frac{3M}{\lambda}\left[\frac{1}{R-1} + x\right], \qquad \frac{d\varphi}{dx}(x=0) = \frac{3M}{\lambda}. \tag{143}$$

If we are in the very early times, i.e., when $x \approx x_{\rm PT}$, the universe is deep in the radiation dominated era where

$$H^2(a) = \frac{8\pi G}{3}\rho_{\rm rad}(a) = \frac{\Omega_{\rm rad,0}\rho_{\rm crit}}{3M_{\rm Pl}^2 a^4}, \tag{144}$$

which gives us

$$\log_e H = \log_e\sqrt{\frac{\Omega_{\rm rad,0}\rho_{\rm crit}}{3M_{\rm Pl}^2}} - 2\,(x + x_{\rm PT})\,, \tag{145}$$

and thus

$$\frac{1}{H}\frac{dH}{dx} = \frac{d}{dx}\log_e H = -2. \tag{146}$$

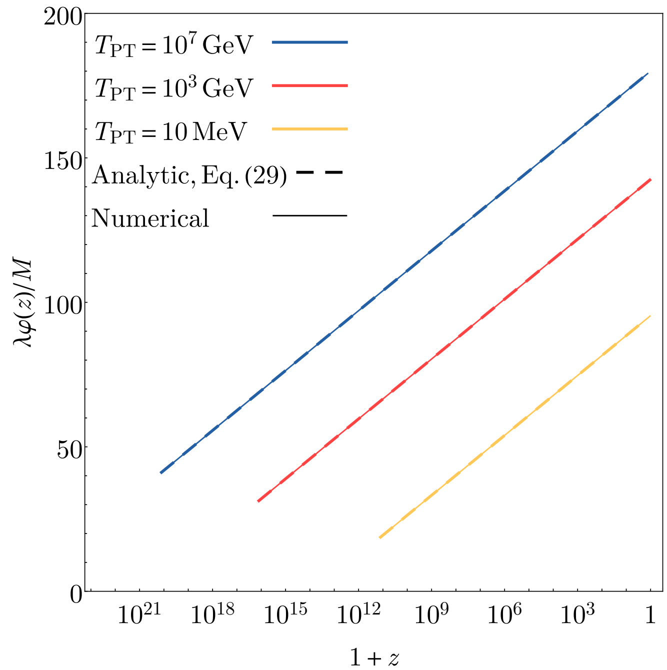


Figure 4: Comparison of the exact numerical solution of Eq. (127) with the analytic solution Eq. (134), as a function of redshift z for several choices of the phase transition temperature $T_{\rm PT}$; this shows the validity of Eq. (134) as an extremely accurate approximation to the true numerical solution.

Steps for numerical solution: We will use the *Runge-Kutta* method to solve Eq. (141) with a step size h. We define the dimensionless field $y \equiv \lambda\varphi/M$, and using Eq. (146) we can write Eq. (141) as

$$\begin{aligned} \frac{d^2y}{dx^2} + \frac{dy}{dx} &= \frac{M^2}{\tilde{G}_Y^2 H^2}\left[e^{-y} - Ae^{-3x}\right] \\ &= Ye^{4x}\left[e^{-y} - Ae^{-3x}\right]. \end{aligned} \tag{147}$$

where $A \equiv e^{-3/(R\,-\,1)}$, and $Y = \dfrac{3\lambda^2 M^2 M_{\rm Pl}^2 a_{\rm PT}^4}{\Omega_{\rm rad,0}\rho_{\rm crit}}$. Let us now define $z \equiv \dfrac{dy}{dx}$, and split the second-order ODE into two first-order ODEs with the help of two functions of (x, y, z),

$$\begin{aligned} \frac{dy}{dx} &\equiv f(x,y,z) = z \\ \frac{dz}{dx} &\equiv g(x,y,z) = -z + Ye^{4x}\left[e^{-y} - Ae^{-3x}\right]. \end{aligned} \tag{148}$$

The first step are the initial conditions, i.e.,

$$x_1 = 0, \qquad y_1 = \frac{3}{R-1}, \qquad z_1 = 3. \tag{149}$$

The Runge-Kutta steps are thus

$$y_{i+1} = y_i + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right], \qquad z_{i+1} = z_i + \frac{1}{6}\left[l_1 + 2l_2 + 2l_3 + l_4\right], \tag{150}$$

where at each step

$$
\begin{aligned}
k_1 &= hf(x_i, y_i, z_i), & l_1 &= hg(x_i, y_i, z_i), \\
k_2 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{l_1}{2}\right), & l_2 &= hg\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{l_1}{2}\right), \\
k_3 &= hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_2}{2}\right), & l_3 &= hg\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_2}{2}\right), \\
k_4 &= hf\left(x_i + h, y_i + k_3, z_i + l_3\right), & l_4 &= hg(x_i + h, y_i + k_3, z_i + l_3),
\end{aligned}
\tag{151}
$$

for the step size h.

We show in Fig. 4 that the analytic solution Eq. (134) is a good match to the exact numerical solution. The reason this is so is justified in the discussion following Eq. (135), where we show that the substituting the solution Eq. (134) into Eq. (127) makes the magnitude of the term X_1 many orders of magnitude smaller than that of X_2 and X_3. This means the relative error is assuming the guess solution Eq. (134) is negligible.

Appendices

A. Zero-T quantum corrections at the 1-loop level

This zero-temperature quantum correction arises from the fermionic one-loop contribution to the potential obtained by summing all one-particle irreducible diagrams with one fermionic loop to get

$$V_{1-\text{loop}}^{\text{fermion}}(\varphi) = -2 \int \frac{d^4p}{(2\pi)^4} \log_e \left[p^2 + m_\psi^2(\varphi)\right], \tag{152}$$

where $m_\psi(\varphi) = g_Y \varphi$ as in Eq. (134). We regularize this integral by introducing a UV cutoff, i.e., integrating from $p = 0$ to Λ. This gives [2], with $\rho \equiv p^2$,

$$\begin{aligned} V_{1-\text{loop}}^{\text{fermion}}(\varphi) &\to -\frac{1}{8\pi^2} \int_0^{\Lambda^2} d\rho\, \rho \log_e \left[\rho + (g_Y\varphi)^2\right] \\ &= -\frac{1}{16\pi^2} \left[\left\{\rho^2 - (g_Y\varphi)^4\right\} \log_e \left\{\rho + (g_Y\varphi)^2\right\} - \left\{\frac{\rho^2}{2} - (g_Y\varphi)^2\rho\right\}\right]_0^{\Lambda^2} \\ &= -\frac{1}{16\pi^2} \left[-\frac{\Lambda^4}{2} + (g_Y\varphi)^2\Lambda^2 + (g_Y\varphi)^4 \log_e \left(\frac{g_Y\varphi}{\Lambda}\right)^2 + \underbrace{\Lambda^4 \log_e \left\{\Lambda^2 + (g_Y\varphi)^2\right\}}_{\mathcal{X}}\right]. \end{aligned} \tag{153}$$

The term $\mathcal{X}$ can be further simplified in the $\Lambda \gg g_Y\varphi$ limit as

$$\begin{aligned} \mathcal{X} = \Lambda^4 \log_e \left\{\Lambda^2 + (g_Y\varphi)^2\right\} &= \Lambda^4 \left[\log_e \left\{1 + \left(\frac{g_Y\varphi}{\Lambda}\right)^2 + \log_e \Lambda^2\right\}\right] \\ &= \Lambda^4 \left[\left(\frac{g_Y\varphi}{\Lambda}\right)^2 - \frac{1}{2}\left(\frac{g_Y\varphi}{\Lambda}\right)^4 + \mathcal{O}\left(\frac{g_Y\varphi}{\Lambda}\right)^6 + \log_e \Lambda^2\right] \\ &= \Lambda^2 (g_Y\varphi)^2 - \frac{(g_Y\varphi)^4}{2} + \varphi\text{-independent terms}, \end{aligned} \tag{154}$$

and dropping the higher order terms. Substituting back in Eq. (153), we get

$$V_{1-\text{loop}}^{\text{fermion}}(\varphi) = -\frac{1}{16\pi^2} \left[2(g_Y\varphi)^2\Lambda^2 + (g_Y\varphi)^4 \left\{\log_e \left(\frac{g_Y\varphi}{\Lambda}\right)^2 - \frac{1}{2}\right\}\right]. \tag{155}$$

Adding this contribution to $U(\varphi)$, the effective potential for the scalar becomes[5]

$$U_{\text{effective}}(\varphi) = M^4 e^{-\lambda\varphi/M} - \frac{1}{16\pi^2} \left[2(g_Y\varphi)^2\Lambda^2 + (g_Y\varphi)^4 \left\{\log_e \left(\frac{g_Y\varphi}{\Lambda}\right)^2 - \frac{1}{2}\right\}\right] + \delta m (g_Y\varphi)^2 + \delta\lambda (g_Y\varphi)^4, \tag{156}$$

where δm and $\delta\lambda$ are the coefficients of the counterterms needed to absorb the Λ-dependence from the φ^2 and φ^4 terms. We can use the *renormalization condition* that the 1-loop contribution $V_{1-\text{loop}}^{\text{fermion}}(\varphi)$ vanishes at $\varphi = \varphi_{\text{PT}}$, which requires

$$\delta m = \frac{\Lambda^2}{8\pi^2}, \qquad \delta\lambda = \frac{1}{16\pi^2} \left\{\log_e \left(\frac{g_Y\varphi_{\text{PT}}}{\Lambda}\right)^2 - \frac{1}{2}\right\}, \tag{157}$$

which makes

$$\boxed{U_{\text{effective}}(\varphi) = M^4 e^{-\lambda\varphi/M} - \frac{(g_Y\varphi)^4}{16\pi^2} \log_e \left(\frac{\varphi}{\varphi_{\text{PT}}}\right)^2,} \tag{158}$$

and consequently

$$\boxed{\frac{\partial U_{\text{effective}}(\varphi)}{\partial\varphi} = -\lambda M^3 e^{-\lambda\varphi/M} - \frac{g_Y (g_Y\varphi)^3}{8\pi^2} \left[1 + 4\log_e \frac{\varphi}{\varphi_{\text{PT}}}\right].} \tag{159}$$

[5] We are looking at how $V_{1-\text{loop}}^{\text{fermion}}(\varphi)$ influences the scalar potential $U(\varphi)$, and thus do not consider the term $F_F(\varphi)$ in our analysis, which is always present and is crucial for our model.

Alternative renormalization condition: We can also demand that $\frac{\partial V_{1-\text{loop}}^{\text{fermion}}(\varphi)}{\partial \varphi}$ vanishes at $\varphi = \varphi_{\text{PT}}$, instead of $V_{1-\text{loop}}^{\text{fermion}}(\varphi)$ itself. In this case, the analog of Eq. (157) becomes

$$\delta m = \frac{\Lambda^2}{8\pi^2}, \qquad \delta\lambda = \frac{1}{16\pi^2}\left\{\log_e\left(\frac{g_Y \varphi_{\text{PT}}}{\Lambda}\right)^2 - \frac{3}{4}\right\}, \tag{160}$$

and thus

$$\boxed{\frac{\partial U_{\text{effective}}(\varphi)}{\partial \varphi} = -\lambda M^3 e^{-\lambda\varphi/M} - \frac{g_Y (g_Y \varphi)^3}{2\pi^2} \log_e \frac{\varphi}{\varphi_{\text{PT}}}.} \tag{161}$$

References

[1] J. I. Kapusta and C. Gale, *Finite-temperature field theory: Principles and applications*. Cambridge University Press, 2011.

[2] M. Quiros, *Finite temperature field theory and phase transitions*, in *ICTP Summer School in High-Energy Physics and Cosmology*, pp. 187–259, 1, 1999, `hep-ph/9901312`.