Local counter-terms for one loop corrections to $q + \overline{q} \to 2\gamma$

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1 List of local counter terms

The two photon emission at tree level is shown in Figure 1. All quarks are massless and

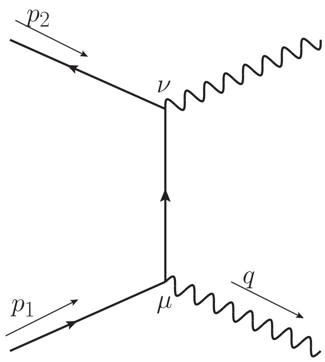


Figure 1: Tree level Feynman diagram

all external legs are onshell. The kinematics of the process is,

$$s = (p_1 + p_2)^2 \tag{1}$$
$$t = (p_1 - q)^2 \tag{2}$$
$$u = (p_2 - q)^2 = -s - t \tag{3}$$

In all subsequent expressions we truncate the photon polarization vectors as they are not important for our discussion. We use Fermi-Feynman gauge throughout. The tree level Feynman diagram expression is

$$\begin{aligned} M^{(0)} &= \overline{v}(p_2)(-ie\mu^{\epsilon}\gamma^{\nu})\frac{i}{\not{p}_1 - \not{q}}(-ie\mu^{\epsilon}\gamma^{\mu})u(p_1)\delta_{ij} \\ &= -\frac{ie^2\mu^{2\epsilon}}{t}\overline{v}(p_2)\gamma^{\nu}(\not{p}_1 - \not{q})\gamma^{\mu}u(p_1)\delta_{ij} \end{aligned} \tag{4}$$

Note the color structure is trivial, being an identity matrix. Henceforth in the counter-terms we will omit the identity color matrix. Figure 2 shows the one loop corrections to the amplitude. Denote the integrand of each Feyman diagram expression by Γ, the soft counter-term by S, the collinear counter-term with gluon collinear to the i-th leg by C_i, the UV counter-term by Γ_{UV}. Denote an IR regulator by μ_{IR}^2, mass scale of dimensional regularization by μ^2. We distinguish between the two mass scales as in numerical contour integrations the former is often set to a negative pure imaginary number. We present below the integrand and the local counter-terms for each Feynman diagram.

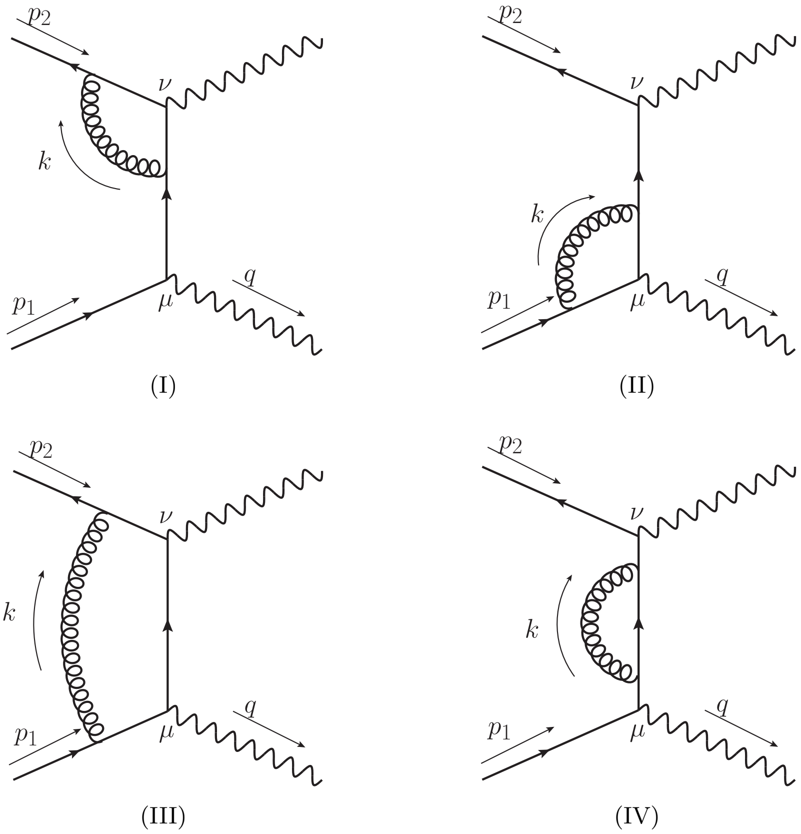


Figure 2: 1-loop corrections

<u>Diagram (I)</u>

$$\Gamma^{(\mathrm{I})} = -\frac{(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\frac{\overline{v}(p_2)\gamma^\lambda(-\not{p}_2-\not{k})\gamma^\nu(\not{p}_1-\not{q}-\not{k})\gamma_\lambda(\not{p}_1-\not{q})\gamma^\mu u(p_1)}{(p_2+k)^2k^2(p_1-q-k)^2} \tag{5}$$

$$C_2^{(\mathrm{I})} = -\frac{2(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\overline{v}(p_2)\gamma^\nu(\not{p}_1+\not{p}_2-\not{q})\gamma^\mu u(p_1)\left[\frac{1}{(p_2+k)^2k^2}-\frac{1}{(k^2-\mu_{IR}^2)^2}\right] \tag{6}$$

$$\Gamma_{\mathrm{UV}}^{(\mathrm{I})} = -\frac{(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\frac{\overline{v}(p_2)\gamma^\lambda(-\not{p}_2-\not{k})\gamma^\nu(\not{p}_1-\not{q}-\not{k})\gamma_\lambda(\not{p}_1-\not{q})\gamma^\mu u(p_1)}{(k^2-\mu_{IR}^2)^3} \tag{7}$$

The following expression is integrable in d=4,

$$\Gamma^{(\mathrm{I})}_{\mathrm{fin}} = \Gamma^{(\mathrm{I})} - C_2^{(\mathrm{I})} - \Gamma^{(\mathrm{I})}_{\mathrm{UV}} \tag{8}$$

<u>Diagram (II)</u>

$$\Gamma^{(\mathrm{II})} = -\frac{(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\frac{\overline{v}(p_2)\gamma^\nu(\not{p}_1-\not{q})\gamma^\lambda(\not{p}_1-\not{q}-\not{k})\gamma^\mu(\not{p}_1-\not{k})\gamma_\lambda u(p_1)}{(p_1-k)^2k^2(p_1-q-k)^2} \tag{9}$$

$$C_1^{(\mathrm{II})} = \frac{2(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\overline{v}(p_2)\gamma^\nu\not{q}\gamma^\mu u(p_1)\left[\frac{1}{(p_1-k)^2k^2} - \frac{1}{(k^2-\mu_{IR}^2)^2}\right] \tag{10}$$

$$\Gamma^{(\mathrm{II})}_{\mathrm{UV}} = -\frac{(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\frac{\overline{v}(p_2)\gamma^\nu(\not{p}_1-\not{q})\gamma^\lambda(\not{p}_1-\not{q}-\not{k})\gamma^\mu(\not{p}_1-\not{k})\gamma_\lambda u(p_1)}{(k^2-\mu_{IR}^2)^3} \tag{11}$$

The following expression is integrable in d=4,

$$\Gamma^{(\mathrm{II})}_{\mathrm{fin}} = \Gamma^{(\mathrm{II})} - C_1^{(\mathrm{II})} - \Gamma^{(\mathrm{II})}_{\mathrm{UV}} \tag{12}$$

<u>Diagram (III)</u>

$$\Gamma^{(\mathrm{III})} = -(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F\frac{\overline{v}(p_2)\gamma^\lambda(-\not{p}_2-\not{k})\gamma^\nu(\not{p}_1-\not{q}-\not{k})\gamma^\mu(\not{p}_1-\not{k})\gamma_\lambda u(p_1)}{(p_2+k)^2(p_1-k)^2k^2(p_1-q-k)^2} \tag{13}$$

$$S^{(\mathrm{III})} = 2(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F\frac{s}{t}\frac{\overline{v}(p_2)\gamma^\nu(\not{p}_1-\not{q})\gamma^\mu u(p_1)}{(p_2+k)^2(p_1-k)^2k^2} \tag{14}$$

$$C_1^{(\mathrm{III})} = -\frac{2(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\overline{v}(p_2)\gamma^\nu\not{p}_1\gamma^\mu u(p_1)\left[\frac{1}{(p_1-k)^2k^2} - \frac{1}{(k^2-\mu_{IR}^2)^2}\right] \tag{15}$$

$$C_2^{(\mathrm{III})} = \frac{2(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t}\overline{v}(p_2)\gamma^\nu\not{p}_2\gamma^\mu u(p_1)\left[\frac{1}{(p_2+k)^2k^2} - \frac{1}{(k^2-\mu_{IR}^2)^2}\right] \tag{16}$$

The following expression is integrable in d=4,

$$\Gamma^{(\mathrm{III})}_{\mathrm{fin}} = \Gamma^{(\mathrm{III})} - S^{(\mathrm{III})} - C_1^{(\mathrm{III})} - C_2^{(\mathrm{III})} \tag{17}$$

<u>Diagram (IV)</u>

$$\Gamma^{(\mathrm{IV})} = -\frac{(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t^2}\frac{\overline{v}(p_2)\gamma^\nu(\not{p}_1-\not{q})\gamma^\lambda(\not{p}_1-\not{q}-\not{k})\gamma_\lambda(\not{p}_1-\not{q})\gamma^\mu u(p_1)}{k^2(p_1-q-k)^2} \tag{18}$$

$$\Gamma^{(\mathrm{IV})}_{\mathrm{UV}} = -\frac{(e\mu^\epsilon)^2(g\mu^\epsilon)^2 C_F}{t^2}\frac{\overline{v}(p_2)\gamma^\nu(\not{p}_1-\not{q})\gamma^\lambda(\not{p}_1-\not{q}-\not{k})\gamma_\lambda(\not{p}_1-\not{q})\gamma^\mu u(p_1)}{[(k+\frac{q-p_1}{2})^2-\mu_{IR}^2]^2} \tag{19}$$

The following expression is integrable in d=4,

$$\Gamma^{(\mathrm{IV})}_{\mathrm{fin}} = \Gamma^{(\mathrm{IV})} - \Gamma^{(\mathrm{IV})}_{\mathrm{UV}} \tag{20}$$

Factorization of IR counter-terms

The soft and collinear counter-terms of different graphs when added up factorizes out a tree amplitude. The statement is general, while in our case there is only one soft counter-term.

$$S^{(\mathrm{III})} = M^{(0)}(2ig^2\mu^{2\epsilon}C_F)\frac{s}{k^2(p_2+k)^2(p_1-k)^2} \tag{21}$$

$$C_1^{(\mathrm{II})} + C_1^{(\mathrm{III})} = M^{(0)}(-2ig^2\mu^{2\epsilon}C_F)\left[\frac{1}{(p_1-k)^2k^2} - \frac{1}{(k^2-\mu_{IR}^2)^2}\right] \tag{22}$$

$$C_2^{(\mathrm{I})} + C_2^{(\mathrm{III})} = M^{(0)}(-2ig^2\mu^{2\epsilon}C_F)\left[\frac{1}{(p_2+k)^2k^2} - \frac{1}{(k^2-\mu_{IR}^2)^2}\right] \tag{23}$$

We chose the UV counter-terms so that they have the same numerator as the orginal expression. The sum of all finite integrands may be conveniently written as,

$$\begin{aligned}&(\Gamma^{(\mathrm{I})} - \Gamma^{(\mathrm{I})}_{\mathrm{UV}}) + (\Gamma^{(\mathrm{II})} - \Gamma^{(\mathrm{II})}_{\mathrm{UV}}) + \Gamma^{(\mathrm{III})} + (\Gamma^{(\mathrm{IV})} - \Gamma^{(\mathrm{IV})}_{\mathrm{UV}}) + \\ &\quad M^{(0)}(2ig^2\mu^{2\epsilon}C_F)\left[\frac{-s}{k^2(p_2+k)^2(p_1-k)^2} + \frac{1}{(p_1-k)^2k^2} + \frac{1}{(p_2+k)^2k^2} - \frac{2}{(k^2-\mu_{IR}^2)^2}\right]\end{aligned} \tag{24}$$

2 Check of the counter terms against Catani's formula

The Catani's formula for divergences of one-loop renormalised amplitude reads,

$$\boldsymbol{I}^{(1)}(\epsilon,\mu^2;\{p\}) = \frac{\alpha_s(\mu^2)}{4\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\sum_i\left(\frac{1}{\epsilon^2}+\frac{\gamma_i}{\boldsymbol{T}_i^2}\frac{1}{\epsilon}\right)\sum_{j\neq i}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\left(\frac{\pm\mu^2}{2p_i\cdot p_j}\right)^\epsilon \tag{25}$$

The minus sign is taken when both partons i and j are incoming or outgoing, otherwise the plus sign is taken. γ_E is Euler Gamma, p_i and p_j are the parton momenta and μ^2 is a mass regulator. The color operators $\boldsymbol{T}_i$ are defined by,

$$(\boldsymbol{T}_i)^a_{cb} = if_{cab} \qquad \text{, if i is gluon} \tag{26}$$

$$(\boldsymbol{T}_i)^a_{\alpha\beta} = t^a_{\alpha\beta} \qquad \text{, if i is initial-state antiquark or final-state quark} \tag{27}$$

$$(\boldsymbol{T}_i)^a_{\alpha\beta} = -t^a_{\beta\alpha} \qquad \text{, if i is initial-state quark or final-state antiquark} \tag{28}$$

The inner product of the color operators is only over the color index a, no contraction of the matrix indices are taken. Since the operator $\boldsymbol{I}^{(1)}$ operates on the tree amplitude by left multiplication, the right index of the matrix $(\boldsymbol{T}_i)_{\alpha\beta}$ or $(\boldsymbol{T}_i)_{ab}$ contracts with the color index of the tree amplitude. In addition, the notation $\boldsymbol{T}_i^2$ simply means the Casimir of representation i.

$$\boldsymbol{T}_q^2 = \boldsymbol{T}_{\bar{q}}^2 = C_F, \qquad \boldsymbol{T}_g^2 = C_A, \tag{29}$$

$$\gamma_q = \gamma_{\bar{q}} = \frac{3}{2}C_F, \qquad \gamma_g = \frac{11}{6}C_A - \frac{2}{3}T_R N_f \tag{30}$$

In our example of $q+\overline{q} \to 2\gamma$, the partons are one initial-state quark and one initial-state antiquark. The formula becomes,

$$\boldsymbol{I}^{(1)} = -\frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-s}\right)^\epsilon\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)t^a_{\alpha_2\beta_2}t^a_{\beta_1\alpha_1} \tag{31}$$

where summation over the color index a is understood. The subscripts β_1 and β_2 are to be contracted with the indices of tree amplitude at the quark and antiquark leg respectively. Act this operator on the tree amplitude equation (4), we get the one-loop divergences,

$$\begin{aligned} M^{(1)}_{\text{ren}} &= \boldsymbol{I}^{(1)}M^{(0)} \\ &= -\frac{\alpha_s}{2\pi}C_F\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-s}\right)^\epsilon\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)M^{(0)}+O(\epsilon^0) \end{aligned} \tag{32}$$

where the identity color matrix in $M^{(0)}$ contracts the two color matrices in $\boldsymbol{I}^{(1)}$and gives a C_F factor. The subscript of $M^{(1)}_{\text{ren}}$ indicates that it is renormalized amplitude. Before comparing this formula to our results, we review briefly the calculation of renormalized amplitude.

$$M_{\text{ren}} = \prod_i \sqrt{R_i}(M_{\text{bare}}+M_{\text{new}}) \tag{33}$$

where M_{bare} and M_{new} are respectively the connected amputated Feynman diagrams written without and with the new vertices produced by renormalization, R_i are the residues of full propagators of the external legs calculated with both the old and new vertices and the product i runs over all external legs. Upto order $O(\alpha_s)$, the expression can be expanded as,

$$M^{(0)}_{\text{ren}} = M^{(0)} \tag{34}$$

$$M^{(1)}_{\text{ren}} = \sum_i\left(\frac{R^{(1)}_i}{2}\right)M^{(0)}+M^{(1)}_{\text{bare}}+M^{(1)}_{\text{new}} \tag{35}$$

Note the superscripts denote the order in α_s, not the number of loops in the diagram. The Catani's formula thus gives an explicit expression to the divergent part of expression 35. Moreover, our method of local subtraction enables us to calculate the divergent and finite part of $M^{(1)}_{\text{bare}}$ separately,

$$M^{(1)}_{\text{bare}} = M^{(1)}_{\text{UV}}+M^{(1)}_{\text{IR}}+M^{(1)}_{\text{fin}} \tag{36}$$

where the first two terms are the integrated expressions of the UV and IR counter terms. Hence it suffices to compare to Catani's formula divergent part of the expression

$$\sum_i\left(\frac{R^{(1)}_i}{2}\right)M^{(0)}+M^{(1)}_{\text{UV}}+M^{(1)}_{\text{IR}}+M^{(1)}_{\text{new}} \tag{37}$$

The terms $M^{(0)}$, $M^{(1)}_{\mathrm{UV}}$ and $M^{(1)}_{\mathrm{IR}}$ are listed in Section 1. We now give analytic expressions after integration over loop momentum,

$$\int \frac{d^D k}{(2\pi)^D} \Gamma^{(\mathrm{I})}_{\mathrm{UV}} = \frac{\alpha_s}{(4\pi)^{1-\epsilon}} C_F \Gamma(1+\epsilon) \left(\frac{\mu^2}{\mu^2_{IR}}\right)^{\epsilon} \left[\frac{(1-\epsilon)^2}{\epsilon} M^{(0)} + \frac{i(e\mu^{\epsilon})^2}{t\mu^2_{IR}} \overline{v}(p_2)(\not{p}_1 - \not{q})\gamma^{\nu}\not{p}_2(\not{p}_1 - \not{q})\gamma^{\mu} u(p_1)\right] \tag{38}$$

$$\int \frac{d^D k}{(2\pi)^D} \Gamma^{(\mathrm{II})}_{\mathrm{UV}} = \frac{\alpha_s}{(4\pi)^{1-\epsilon}} C_F \Gamma(1+\epsilon) \left(\frac{\mu^2}{\mu^2_{IR}}\right)^{\epsilon} \left[\frac{(1-\epsilon)^2}{\epsilon} M^{(0)} + \frac{i(e\mu^{\epsilon})^2}{t\mu^2_{IR}} \overline{v}(p_2)\gamma^{\nu}(\not{p}_1 - \not{q})\not{p}_1\gamma^{\mu}\not{q} u(p_1)\right] \tag{39}$$

$$\int \frac{d^D k}{(2\pi)^D} \Gamma^{(\mathrm{IV})}_{\mathrm{UV}} = -\frac{\alpha_s}{(4\pi)^{1-\epsilon}} C_F \Gamma(1+\epsilon) \left(\frac{\mu^2}{\mu^2_{IR}}\right)^{\epsilon} \frac{1-\epsilon}{\epsilon} M^{(0)} \tag{40}$$

$$\int \frac{d^D k}{(2\pi)^D} S^{(\mathrm{III})} = -\frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} \frac{1}{\epsilon^2} M^{(0)} \tag{41}$$

$$\int \frac{d^D k}{(2\pi)^D} (C^{(\mathrm{II})}_1 + C^{(\mathrm{III})}_1) = -\frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{\mu^2_{IR}}\right)^{\epsilon} \frac{(4\pi)^{\epsilon}\Gamma(1+\epsilon)}{\epsilon} M^{(0)} \tag{42}$$

$$\int \frac{d^D k}{(2\pi)^D} (C^{(\mathrm{I})}_2 + C^{(\mathrm{III})}_2) = -\frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{\mu^2_{IR}}\right)^{\epsilon} \frac{(4\pi)^{\epsilon}\Gamma(1+\epsilon)}{\epsilon} M^{(0)} \tag{43}$$

Add up the divergent part of the expressions, we get,

$$M^{(1)}_{\mathrm{UV}} = \frac{\alpha_s}{4\pi} C_F \frac{1}{\epsilon} M^{(0)} + O(\epsilon^0) \tag{44}$$

$$M^{(1)}_{\mathrm{IR}} = -\frac{\alpha_s}{2\pi} C_F \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{-s}\right)^{\epsilon} \left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\right) M^{(0)} + O(\epsilon^0) \tag{45}$$

Before calculating the rest of the terms we first review the new vertices in renormalized perturbation theory. We renormalize the fields and coupling constants multiplicatively so that,

$$\psi_0 = \sqrt{Z_2}\psi_r \tag{46}$$

$$A^a_0 = \sqrt{Z_3} A^a_r \tag{47}$$

$$g_0 = Z_g g_r \mu^{\epsilon} \tag{48}$$

$$e_0 = Z_e e_r \mu^{\epsilon} \tag{49}$$

where the subscripts 0 and r are used to distinguish between bare and renormalized quantities. Photon field renormalization is omitted since it is supressed by powers of α_{QED}. Note all the constants present in the previous calculations are renormalized ones. The relevant part of the langrangian density reads,

$$\begin{aligned}
\mathcal{L} &= i\overline{\psi}_0 \not{\partial} \psi_0 + g_0 \overline{\psi}_0 \not{A}^a_0 t^a \psi_0 - e_0 \overline{\psi}_0 \not{A} \psi_0 \\
&= iZ_2 \overline{\psi}_r \not{\partial} \psi_r + Z_g Z_2 \sqrt{Z_3} g_r \mu^{\epsilon} \overline{\psi}_r \not{A}^a_r t^a \psi_r - Z_e Z_2 e_r \mu^{\epsilon} \overline{\psi}_r \not{A} \psi_r \\
&= i\overline{\psi}_r \not{\partial} \psi_r + g_r \mu^{\epsilon} \overline{\psi}_r \not{A}^a_r t^a \psi_r - e_r \mu^{\epsilon} \overline{\psi}_r \not{A} \psi_r \\
&\quad + i(Z_2 - 1)\overline{\psi}_r \not{\partial} \psi_r + (Z_g Z_2 \sqrt{Z_3} - 1) g_r \mu^{\epsilon} \overline{\psi}_r \not{A}^a_r t^a \psi_r - (Z_e Z_2 - 1) e_r \mu^{\epsilon} \overline{\psi}_r \not{A} \psi_r
\end{aligned} \tag{50}$$

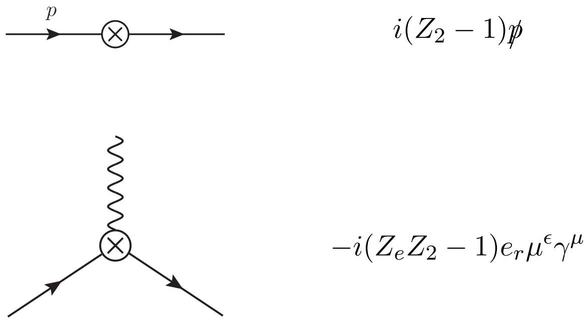
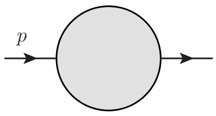
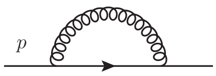
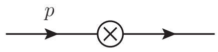


Figure 3: Feynman rules for new vertices

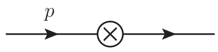
The new vertices relevant for our calculation are listed in Figure 3 To calculate the residue of quark propagator $R_2^{(1)}$, note by definition,

$R_2 = \lim_{p^2\to 0} -i\not{p}$  (51)

where the figure on RHS is the full quark propagator. Expand in powers of α_s we get,

$R_2 = \lim_{p^2\to 0} -i\not{p}\Big($  $+$  $+$  $\Big) + O(\alpha_s^2)$ (52)

But the second term contains a scaleless integral as $p^2 \to 0$, so it vanishes in dimensional regularization. Using the Feynman rules we obtain,

$R_2^{(1)} = \lim_{p^2\to 0} -i\not{p}$ 

$$= \lim_{p^2\to 0}(-i\not{p})\frac{i}{\not{p}}(iZ_2^{(1)}\not{p})\frac{i}{p} \tag{53}$$

$$= -Z_2^{(1)}$$

Note that we have expanded $Z_2 = 1 + Z_2^{(1)} + O(\alpha_s^2)$ and that the picture in the above equation is not amputated. It remains to calculate $M_{\text{new}}^{(1)}$, diagrams of which are shown in Figure 4. Applying Feynman rules we obtain,

$$M_{\text{new}}^{(1)} = (2Z_e^{(1)} + Z_2^{(1)})M^{(0)} \tag{54}$$

where we have likewise expanded $Z_e = 1 + Z_e^{(1)} + O(\alpha_s^2)$. Collecting all the terms in equation 37, we get the divergent part of one-loop $q + \overline{q} \to 2\gamma$,

$$-\frac{\alpha_s}{2\pi}C_F\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{-s}\right)^\epsilon\left(\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right)M^{(0)} + 2Z_e^{(1)}M^{(0)} \tag{55}$$

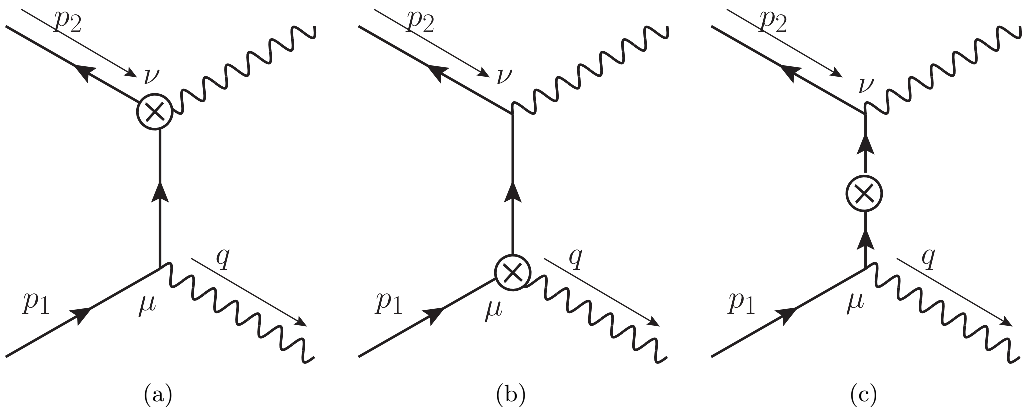


Figure 4: Diagrams involving new vetices that contribute to $M_{\text{new}}^{(1)}$

Moreover, we can view Z_e as renormalization of the composite operator $\overline{\psi}\gamma^\mu\psi$. Since conserved current operators receive no renormalization from QCD, which is the case for electric current, we deduce,

$$Z_e = 1 \tag{56}$$

Hence our above calculation exactly reproduces Catani's result equation 32.

3 Generalizations

3.1 emission of N_γ photons

The analysis can be generalized straightforwardly to the case of N_γ photons. We look at the effect of N_γ on each term of equation 37. Let V be number of photon quark vetices, P_q be the number of internal quark propagators, P_γ be the number of internal photon propagators and N_q be the number of external quarks and antiquarks. The tree graph topology satisfies,

$$V - P_\gamma - P_q = 1 \tag{57}$$

$$2V + N_q = 2(P_q + N_q) \tag{58}$$

$$V + N_\gamma = 2(P_\gamma + N_\gamma) \tag{59}$$

Using $N_q = 2$ and solving the above equations we get,

$$V = N_\gamma \tag{60}$$

$$P_q = N_\gamma - 1 \tag{61}$$

Thus in this case, there are N_γ local UV counter-terms for vertex radiative corrections as in Figure 2I and $N_\gamma - 1$ local UV counter-terms for propagator radiative corrections as in Figure 2IV. From equations 38 and 40, we see that the divergent parts of the two kinds of UV counter-terms factorize and exactly cancel. The net result is the same as equation 44. Moreover, more general analysis (see Babis' notes on January 16 for example) of the soft and colinear counter-terms using Ward identity shows that sum of these terms is independent of the number of colorless particles and is formally the same as equation 45. In addition, the value of $R_2^{(1)}$ is independent of N_γ. It remains to examine $M_{\text{new}}^{(1)}$. Each vertex in the tree diagram can be replaced by a new three point vertex (Figure 4a). The diagram thus obtained reads,

$$(Z_e^{(1)} + Z_2^{(1)})M^{(0)} \tag{62}$$

Similarly, each quark propagator in the tree diagram can be replaced by a new two point vertex (Figure 4c). Such a diagram evaluates to be,

$$-Z_2^{(1)}M^{(0)} \tag{63}$$

Since in the tree diagram there are N_γ vertices and $N_\gamma - 1$ quark propagators, sum of all diagrams with new vertices is,

$$M_{\text{new}}^{(1)} = (N_\gamma Z_e^{(1)} + Z_2^{(1)})M^{(0)} \tag{64}$$

Comparing equation 54, only the coefficient of $Z_e^{(1)}$ depends on N_γ. But we have argued $Z_e^{(1)} = 0$, so N_γ has no effect on $M_{\text{new}}^{(1)}$ as well. In summary, N_γ does not change the value of expression 37, apart from the factorized $M^{(0)}$. Neither does it change Catani's formula (equation 32) as it depends only on the number and species of colored particles. So the two expressions still match.

3.2 emission of other colorless particles

Suppose instead of photons the final state of the process are scalar particles coupled to quarks through Yukawa coupling (Figure 5). Calculations of the previous section showed that,

$$\sum_i \left(\frac{R_i^{(1)}}{2}\right) M^{(0)} + M_{\text{IR}}^{(1)} = M_{\text{ren}}^{(1)} + O(\epsilon^0) \tag{65}$$

$$M_{\text{UV}}^{(1)} + M_{\text{new}}^{(1)} = O(\epsilon^0) \tag{66}$$

By ward identities, one can show that the first line is still correct. It suffices to examine the second line, that is our choices of UV counter-terms cancel the contribution of new vertices due to renormalization. Let us first establish renormalization of the Yukawa term,

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} &= \lambda_0 \mu^\epsilon \phi \overline{\psi}_0 \psi_0 \\ &= Z_2 Z_\lambda \lambda_r \mu^\epsilon \phi \overline{\psi}_r \psi_r \\ &= \lambda_r \mu^\epsilon \phi \overline{\psi}_r \psi_r + (Z_2 Z_\lambda - 1)\lambda_r \mu^\epsilon \phi \overline{\psi}_r \psi_r \end{aligned} \tag{67}$$

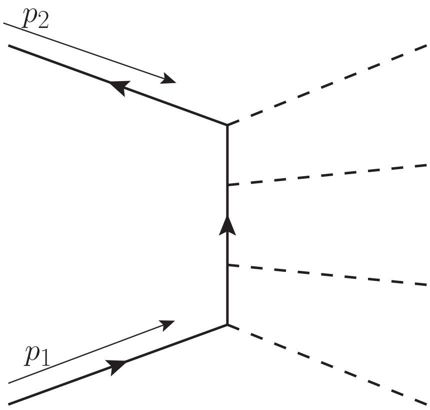
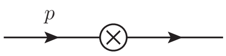
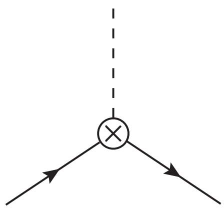


Figure 5: Emission of scalar particles



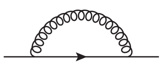
$$i(Z_2 - 1)\not{p}$$



$$i(Z_\lambda Z_2 - 1)\lambda_r \mu^\epsilon$$

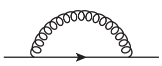
Figure 6: Feynman rules for new vertices in the case of scalar emission

Figure 6 shows the new two point and three point vertices. The two vertices should cancel the UV counter-terms for the two loop diagrams in Figure 7. The self energy correction (Figure 7a) reads,

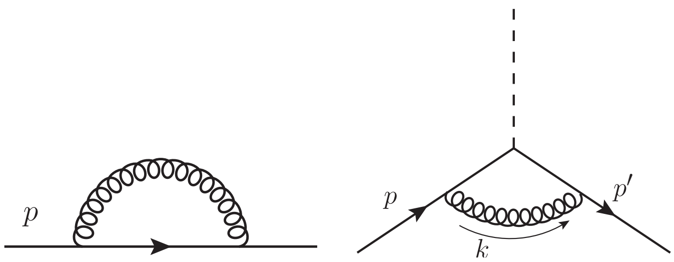


$$\begin{aligned} &= \int \frac{d^D k}{(2\pi)^D} (ig\mu^\epsilon t^a_{ik}\gamma^\mu) \frac{i}{\not{p} - \not{k}} (ig\mu^\epsilon t^a_{kj}\gamma_\mu) \frac{-i}{k^2} \\ &= -g^2 \mu^{2\epsilon} C_F \delta_{ij} \int \frac{d^D k}{(2\pi)^D} \frac{\gamma^\mu(\not{p} - \not{k})\gamma_\mu}{(p-k)^2 k^2} \end{aligned} \tag{68}$$

The Nagy and Soper prescription gives the local UV counter-term,


(UV)

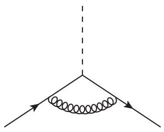
$$\begin{aligned} &= -g^2 \mu^{2\epsilon} C_F \delta_{ij} \int \frac{d^D k}{(2\pi)^D} \frac{\gamma^\mu(\not{p} - \not{k})\gamma_\mu}{[(k - \frac{p}{2})^2 - \mu_{IR}^2]^2} \\ &= i \frac{\alpha_s}{4\pi} C_F \delta_{ij} \left(\frac{\mu^2}{\mu_{IR}^2}\right)^\epsilon \frac{(4\pi)^\epsilon \Gamma(2+\epsilon)}{\epsilon} \not{p} \end{aligned} \tag{69}$$



(a) Self energy of quark (b) Radiative correction to Yukawa vertex

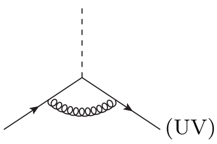
Figure 7: Radiative corrections in the case of scalar emission

Similarly, the vertex radiative correction (Figure 7b) reads,



$$
\begin{aligned}
&= \int \frac{d^D k}{(2\pi)^D} (ig\mu^\epsilon t^a_{ik}\gamma^\mu) \frac{i}{\not{p}' - \not{k}} (i\lambda\mu^\epsilon) \frac{i}{\not{p} - \not{k}} (ig\mu^\epsilon t^a_{kj}\gamma_\mu) \frac{-i}{k^2} \\
&= \lambda\mu^\epsilon g^2 \mu^{2\epsilon} C_F \delta_{ij} \int \frac{d^D k}{(2\pi)^D} \frac{\gamma^\mu(\not{p}' - \not{k})(\not{p} - \not{k})\gamma_\mu}{(p' - k)^2 (p-k)^2 k^2}
\end{aligned}
\tag{70}
$$

The simplest choice of local UV counter-term is,



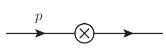
$$
\begin{aligned}
&= \lambda\mu^\epsilon g^2 \mu^{2\epsilon} C_F \delta_{ij} \int \frac{d^D k}{(2\pi)^D} \frac{\gamma^\mu(\not{p}' - \not{k})(\not{p} - \not{k})\gamma_\mu}{(k^2 - \mu_{IR}^2)^3} \\
&= i\lambda\mu^\epsilon \frac{\alpha_s}{4\pi} C_F \delta_{ij} (4\pi)^\epsilon \Gamma(1+\epsilon) \left(\frac{\mu^2}{\mu_{IR}^2}\right)^\epsilon \left[\frac{(2-\epsilon)^2}{\epsilon} - \frac{\gamma^\mu \not{p}' \not{p} \gamma_\mu}{2\mu_{IR}^2}\right]
\end{aligned}
\tag{71}
$$

The values Z_2 and Z_λ in $\overline{\mathrm{MS}}$ scheme are,

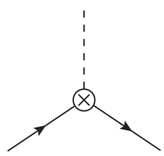
$$
Z_2 = 1 - \frac{\alpha_s}{4\pi} C_F \frac{(4\pi)^\epsilon}{\epsilon\Gamma(1-\epsilon)} + O(\alpha_s^2) \tag{72}
$$

$$
Z_\lambda = 1 - \frac{3\alpha_s}{4\pi} C_F \frac{(4\pi)^\epsilon}{\epsilon\Gamma(1-\epsilon)} + O(\alpha_s^2) \tag{73}
$$

Thus the two new vertices at order $O(\alpha_s)$ are,



$$
= -i\frac{\alpha_s}{4\pi} C_F \frac{(4\pi)^\epsilon}{\epsilon\Gamma(1-\epsilon)} \not{p} \tag{74}
$$



$$
= -i\lambda\mu^\epsilon \frac{\alpha_s}{\pi} C_F \frac{(4\pi)^\epsilon}{\epsilon\Gamma(1-\epsilon)} \tag{75}
$$

Comparing equation 69 with 74 and equation 71 with 75, we see that the divergent parts do cancel. Hence our method of local subtraction provides correct divergences for the renormalized amplitude in the case of scalar emission as well.