

# 18.704 Supplementary Notes: Simple Groups and Composition Series

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## Simple Groups

**Definition:** A *simple group* is a group with no proper normal subgroup.

Simple groups are the primitive building blocks of finite groups, in much the same way that primes are the building blocks of the integers. We can often learn about the structure of a finite group  $G$  by decomposing  $G$  into its simple factor (or quotient) groups. In order to decompose a finite group  $G$  into simple factor groups, we will need to work with quotient groups.

Recall that a normal subgroup  $N$  of a finite group  $G$  is a subgroup that is sent to itself by the operation of conjugation:  $\forall g \in N, x \in G, xgx^{-1} \in N$ . In particular, **the kernel of a homomorphism is a normal subgroup**.

**PROOF:** Recall that a homomorphism is compatible with the laws of composition. So if  $\psi$  is a homomorphism  $\psi: G \rightarrow G'$  with kernel  $N$ ,  $a \in N$ ,  $b \in G$ , then  $\psi(bab^{-1}) = \psi(b)\psi(a)\psi(b^{-1}) = \psi(b)(1)\psi(b)^{-1} = 1$ . Since  $b$  was arbitrary,  $N$  is normal.  $\square$

In order to decompose a finite group  $G$  into simple factor groups, we need to work with quotient groups. Recall that a coset of a subgroup  $N$  is set  $aN = \{g \in G | g = an, \text{ for some } n \in N\}$  or  $Na = \{g \in G | g = na, \text{ for some } n \in N\}$ . The set of left or right cosets of a subgroup partitions  $G$ , and if  $N$  is a normal subgroup, then  $aN = Na$  for all  $a \in G$ . The *quotient space*  $G/N$  is the set of all left cosets of  $N$  in  $G$ :  $G/N = \{aN | a \in G\}$  and the map  $\pi: G \rightarrow G/N, \pi(x) = xN$  is a homomorphism with kernel  $N$ . This coset space  $G/N$  is the quotient group of  $G$  mod  $N$ , with the induced law of composition  $(aN)(bN) = (ab)N$ . It is easy to

see that this is a well-defined group operation if we remember that  $xN = Nx$  for all  $x \in G$ . So every normal subgroup  $N$  of  $G$  is the kernel of the homomorphism from  $G$  to its quotient group  $G/N$ .

Quotient groups are often easier to work with than their parent groups, and it seems intuitive that we should be able to describe a group in some way using quotient groups. In particular, we can decompose a finite group  $G$  using quotients of  $G$  into a series of simple groups. If all of these simple groups happen to be abelian, we say that the group  $G$  is solvable. This concept of decomposition is useful because, as George Polya said,

"If there's a problem you can't figure out, there's a simpler problem you can figure out."

In order to describe this decomposition process, we first need a few definitions.

**Definition:** A *subnormal series* is a finite chain of subgroups  $G_n \subset G_{n-1} \dots \subset G_0 = G$  such that  $G_{i+1}$  is normal in  $G_i \forall i, 0 \leq i \leq n$ . The *factor groups* of the series are the quotients  $G_{i+1}/G_i$ . If  $G_n = 1$  and the factor groups are all simple, we say that it is a *composition series* of  $G$ . The *length* of a composition series is the number of subgroups in the chain, not including the identity.

The following theorem is essential to our understanding of group structures in relation to simple groups.

## The Jordan-Holder Decomposition Theorem.

**THEOREM:**

Every finite group  $G$  has a *composition series*, and any two composition series of  $G$  have the same *length* and the same *factor groups*, up to ordering. Our proof of the Jordan-Holder theorem will use the Second Isomorphism Theorem.

## Second Isomorphism Theorem.

Let  $G$  be a group, let  $H$  be a subgroup of  $G$ , let  $K$  be a normal subgroup of  $G$ . Define  $HK = \{hk \in G | h \in H, k \in K\}$ . Then  $HK$  is a subgroup of  $G$ ,  $K$  is a normal subgroup of  $HK$ ,  $(H \cap K)$  is a normal subgroup of  $H$ , and there is an isomorphism  $\psi: H/(H \cap K) \rightarrow (HK)/K$ .

*Supercomposition series:* this is a chain of subgroups satisfying

- (1) the subgroup  $G_{i+1}$  is normal in  $G_i$
- (2) the quotient group  $G_i/G_{i+1}$  is simple or trivial for  $i = 0, 1, \dots, n - 1$

This is similar to the composition series for  $G$  discussed earlier, but some of the quotients, like in a prime factorization of an integer, are factors of 1 (or  $E$  in the case of groups).

### Proof of the Jordan-Holder Theorem

Suppose:

$1 = G_q \subset G_{q-1} \subset \dots \subset G_1 \subset G$  and  $1 = G'_m \subset G'_{m-1} \subset \dots \subset G'_1 \subset G$  are two supercomposition series for  $G$ . Then the set of simple groups appearing in the list  $\{G_i/G_{i+1}\}$  is exactly the same as the set appearing in the list  $\{G'_j/G'_{j+1}\}$ .

Before beginning the proof, we need a lemma.

#### Lemma

Suppose  $1 = G_q \subset G_{q-1} \subset \dots \subset G_1 \subset G_0 = G$  is a composition series for  $G$ , and  $N$  is any normal subgroup of  $G$ .

- (1) The chain of subgroups  $N = G_0 \cap N \supset G_1 \cap N \supset \dots \supset G_n \cap N = 1$  is a supercomposition series for  $N$ .
- (2) The chain of subgroups  $G/N = G_0/(G_0 \cap N) \supset G_1/(G_1 \cap N) \supset \dots \supset G_n/(G_n \cap N) = 1$  is a supercomposition series for  $G/N$ .
- (3) For each  $i$ , exactly one of the composition factors  $G_i \cap N / (G_{i+1} \cap N)$ ,  $[G_i / (G_i \cap N)] / [G_{i+1} / (G_{i+1} \cap N)]$  is isomorphic to  $G_i = G_{i+1}$ , and the other is trivial.
- (4) With these particular choices of composition series, the set of simple composition factors of  $G$  (counted with multiplicity) is the union of the simple composition factors of  $N$  and the simple composition factors of  $G/N$ .

**Proof of theorem** By induction on the order of  $G$ . If  $G$  is simple, the result is trivial, so suppose  $G$  is not simple. Let  $N$  be a proper normal subgroup of  $G$ . By inductive hypothesis, the theorem is known for  $N$  and for  $G/N$ . Use the first composition series for  $G$  as in the above lemma to get the first kind of supercomposition series for  $N$  and  $G/N$ . The lemma says

- (A)  $G_i/G_{i+1} = (\text{1st kind of factors for } N) / (\text{1st kind of factors for } G/N)$   
the union taken with multiplicities.

Next use the second composition series for  $G$  to get second kinds of supercomposition series for  $N$  and  $G/N$ . The lemma says

(B)  $G'_j/G'_{j+1} = (\text{2nd kind of factors for } N) / (\text{2nd kind of factors for } G/N)$

The inductive hypothesis (applied to  $N$  and to  $G/N$ ) says that the right hand sides of (A) and (B) are the same. So the left sides are the same, which is what we wanted to show.  $\square$

**NOTE:** However, nonisomorphic groups may have the same simple decomposition factors.

## Solvable Groups

Next we turn to solvable groups. Recall that a group is solvable if the factor groups of its composition series are all abelian. As it happens, the only abelian simple groups are the cyclic groups of prime order, and so a solvable group has only prime-order cyclic factor groups.

**PROOF:** Let  $A$  be a non-zero finite abelian simple group. Since  $A$  is simple,  $A$  has no normal subgroups. But  $A$  is abelian, and every subgroup of an abelian group is normal. Thus,  $A$  has no proper subgroups. Now suppose the  $|A| = p$ , for some  $p \in \mathbb{Z}$ . Since  $A$  is finite, every  $x \in A$  has finite order, so  $\forall x \in A, x^q = 1$  for some  $q \in \mathbb{Z}$ , and  $q$  must divide  $p$ . Now consider the set  $\{1, x, \dots, x^{q-1}\}$  generated by  $x$ . This is clearly a subgroup of  $A$ , and since  $A$  has no proper subgroups, we find that  $q = p$ . Thus  $A$  is cyclic. Now suppose that  $p$  is not prime, so that  $p = nm$  for some  $n, m \in \mathbb{Z}$ . Then the set  $\{1, x^m, x^{2m}, \dots, x^{(n-1)m}\}$  generated by  $x^m$  is a proper subgroup of  $A$  with  $n$  elements. But  $A$  has no proper subgroups, and thus  $p$  must be prime. Thus  $A$  is a prime cyclic group.  $\square$

### Nice properties of solvable groups

If  $G$  is solvable, and  $H$  is a subgroup of  $G$ , then  $H$  is solvable.

If  $G$  is solvable, and  $H$  is a normal subgroup of  $G$ , then  $G/H$  is solvable.

If  $G$  is solvable, and there is a homomorphism  $\psi : G \rightarrow H$ , then  $H$  is solvable.

If  $H$  and  $G/H$  are solvable, then so is  $G$ .

There are some nice examples of simple groups and decompositions series involving some familiar groups.

### Example

#### Decomposition of $C_{12}$

In general, a group will have multiple, different composition series. For example, the cyclic group  $C_{12}$  has  $\{E, C_2, C_6, C_{12}\}$ ,  $\{E, C_2, C_4, C_{12}\}$ , and  $\{E, C_3, C_6, C_{12}\}$

as different composition series. However, the result of the Jordan-Hlder Theorem is that any two composition series of a group are equivalent, in the sense that the sequence of factor groups in each series are the same, up to permutations of their order in the sequence  $A_{i+1}/A_i$ . In the above example, the factor groups as follow, are isomorphic to  $\{C_2, C_3, C_2\}$ ,  $\{C_2, C_2, C_3\}$ , and  $\{C_3, C_2, C_2\}$ , respectively. (Wikipedia, "Solvable groups")

## Iwasawa's Theorem and its significance

To understand Iwasawa's Theorem we must first have a bit of background information.

### Commutators

A *commutator* of two elements  $a, b \in G$  is  $aba^{-1}b^{-1}$ . This commutator is clearly 1 if the group  $G$  is an abelian group.

### Commutator subgroups

A *commutator subgroup*, or derived subgroup is the smallest subgroup containing all commutators. In other words, the set of all commutators generates the commutator subgroup. We denote the commutator subgroups of  $G$  by  $G'$ .

For example, the commutator subgroup of  $S_n$  is  $A_n$ .

**The quotient group  $G/G'$  of the group with its commutator subgroup is abelian and is referred to as the abelianization of  $G$ .**

### Stabilizers

Recall a stabilizer is the action of a group  $G$  on a set  $S$ ,  $a \in S$  such that  $|Orb_G(a)| = [G : Stab_G(a)]$ .

## Iwasawa's Theorem

Suppose that  $G$  is faithful and primitive on  $S$  and  $G' = G$ . Fix  $s \in S$  and set  $H = Stab_G(s)$ . Suppose there is a solvable subgroup  $K \triangleleft H$  such that  $G = [\cup\{K^x : x \in G\}]$ . Then  $G$  is simple.

PROOF:

Suppose that  $1 \neq N \triangleleft G$ .  $N$  is transitive on  $S$ , and  $G = HN$ . Thus  $KN \triangleleft HN = G$ . If  $x \in G$  then  $K^x \leq (KN)^x = KN$ , so  $\cup\{K^x : x \in G\} \subseteq KN$  and hence  $KN = G$ . Since  $K$  is solvable  $K^m = 1$  for some  $K^m$  in the derived series of  $K$ . Check inductively that  $(KN)^l \leq K^l \forall l$ . Thus

$$G = G^m = (KN)^m N = N \text{ and } N = G. \square$$