

The Clifford theory of the n-qubit Clifford group

Kieran Mastel^{1,2}

¹⁾*Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

²⁾*Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada^{a)}*

(Dated: 7 November 2025)

The n-qubit Pauli group and its normalizer the n-qubit Clifford group have applications in quantum error correction and device characterization. Recent applications have made use of the representation theory of the Clifford group. We apply the tools of (the coincidentally named) Clifford theory to examine the representation theory of the Clifford group using the much simpler representation theory of the Pauli group. We find an unexpected correspondence between irreducible characters of the n-qubit Clifford group and those of the (n+1)-qubit Clifford group.

I. INTRODUCTION

The Pauli group and its normalizer, the Clifford group, are fundamental structures in quantum information theory. These groups have applications in quantum error correction¹ and randomized benchmarking². By the Gottesman-Knill theorem, quantum computation using Clifford unitaries is efficiently simulable on a classical computer^{3,4}. The Clifford group is a unitary 2-design⁵, in other words, ‘averages over the Clifford group approximate averages over the unitary group well.’ Generating random Clifford unitaries is less computationally expensive than sampling Haar random unitaries⁶. Thus, random Clifford elements have utility in performing randomized protocols. Recent applications of the Clifford group to randomized benchmarking and classical shadow estimation have utilized its representation theory^{2,7,8}. Determining the character table of the Clifford group, which classifies its irreducible representations, is a natural open problem prompted by these papers. Surprisingly, despite the usefulness of the representation theory of the Clifford group, its character table has not been determined.

The representation theory of the Pauli group is simple and explained in section III C. Thus, it would be advantageous to use our understanding of the representation theory of the Pauli group to examine that of the Clifford group. To do this, we can apply the tools of Clifford theory (which is named after Alfred H. Clifford, while William K. Clifford gives his name to the group). Clifford theory is the part of representation theory focused on relating representations of a normal subgroup N of G to representations of G . The inertia subgroup $I_G(\sigma)$ is the subgroup of G that maps a representation σ of N to an isomorphic representation under conjugation. The central result of Clifford theory is the *Clifford correspondence* between irreducible representations of the inertia subgroup and certain irreducible representations of G . When the inertia subgroup is understood, this simplifies the calculation of irreducible characters of G . Since any two nontrivial irreducible Pauli representations are conjugate in the Clifford group and conjugate representations have isomorphic inertia subgroups, we need only examine one inertia subgroup. In our first result, we determine the inertia subgroup of a nontrivial irreducible representation of the n -qubit Pauli group in the n -qubit Clifford group up to complex phases for $n \geq 2$.

Clifford theory does not fully calculate the character table of the Clifford group. The Clifford correspondence does not give us any information when the inertia subgroup $I_G(\sigma)$ is all of G . In particular, the Clifford correspondence does not help when σ is the trivial representation of N . For a group G , inflation produces a bijection between irreducible representations of the quotient group G/N and irreducible representations of G whose restriction to N is trivial. We can thus understand the case where the Clifford correspondence offers no information by examining the representation theory of the quotient group. For the n -qubit Clifford group and the n -qubit Pauli group, the quotient group is the symplectic group $Sp(2n, 2)$. The symplectic group is a finite group of Lie type, and thus its representation theory is calculated by the Deligne-Lusztig theory⁹, which we do not examine in this paper. Together with the representations calculated using Clifford theory, this accounts for all the irreducible representations of the Clifford group.

In section IV A, we show that the inertia quotient group $I_G(\sigma)/N$ of a nontrivial Pauli representation in the n -qubit Clifford group is a central extension of the affine symplectic group $Sp(2(n - 1), 2) \ltimes \mathbb{Z}_2^{2(n-1)}$ by \mathbb{Z}_2 . The Clifford group, in the literature on finite group extensions, is known as the unique non-split extension of $Sp(2n, 2)$ by \mathbb{Z}_2^{2n} . By examining the Clifford group from this perspective, Bernd Fischer showed in¹⁰ that the Clifford and affine symplectic

^{a)}Now at the Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada; Electronic mail: kmastel@uottawa.ca

groups have identical character tables. Combining these facts allows us to produce a surprising correspondence between irreducible characters of the n -qubit Clifford group \mathcal{C}_n and the $(n+1)$ -qubit Clifford group \mathcal{C}_{n+1} . Any irreducible character of \mathcal{C}_n can be viewed as an irreducible character of $Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}$ which inflates to an irreducible character of the inertia subgroup IN_{n+1} of a nontrivial representation of the $(n+1)$ -qubit Pauli group in the $(n+1)$ -qubit Clifford group. This representation induces an irreducible character of \mathcal{C}_{n+1} by the Clifford correspondence. The natural map of characters from \mathcal{C}_n to \mathcal{C}_{n+1} is induction. However, unlike the induction of characters, our correspondence maps irreducible characters to irreducible characters. Knowing the irreducible characters of \mathcal{C}_n allows us to calculate an equal number of the irreducible characters of \mathcal{C}_{n+1} .

II. PRELIMINARIES

A. Representation theory

In this section, we recall from¹¹ some basic facts about representation and character theory of finite groups. A **linear representation** of a finite group G is a homomorphism ρ from the group G into the group $GL(V)$, where V is a vector space over \mathbb{C} . If W is a vector subspace of V such that $\rho(g)x \in W$ for all $g \in G$ and $x \in W$, then the restriction $\rho^W(g)$ of $\rho(g)$ to W is a linear representation of G on W . W is called a **subrepresentation** of V . An **irreducible** representation is one where V is not 0 and has no nontrivial subrepresentations. It is a standard result that every representation is a direct sum of irreducible representations.

If ρ and σ are representations of a finite group G on the vector spaces V and W respectively then a linear map $\phi : V \rightarrow W$ is called an **intertwining map** of representations if $\phi(\rho(g)v) = \sigma(g)\phi(v)$ for all $g \in G$ and all $v \in V$. The vector space of all such G -linear maps between V and W is denoted by $\text{Hom}_G(\rho, \sigma)$ or $\text{Hom}_G(V, W)$. If ϕ is also invertible, it is said to be an **isomorphism** of representations. When we classify irreducible representations, we do so up to isomorphism. Isomorphic representations are sometimes called **equivalent** representations.

Let F be a field, a **projective representation** of a finite group G is a map $\Phi : G \rightarrow GL_n(F)$ such that for every $g, h \in G$, there exists a scalar $\alpha(g, h) \in F$ such that

$$\Phi(g)\Phi(h) = \Phi(gh)\alpha(g, h).$$

The set of values $\alpha(g, h)$ is called the **factor set**, and is uniquely determined by Φ . The notions of equivalence and irreducibility translate verbatim for projective representations. We refer the reader to section 7.2 of¹² for a more exhaustive discussion of projective representations.

Let $\rho : G \rightarrow GL(V)$ be a linear representation of a finite group G . For each $g \in G$, define

$$\chi_\rho(g) = \text{Tr}(\rho(g)),$$

with $\text{Tr}(\rho(g))$ being the trace of the operator $\rho(g) \in GL(V)$. The function χ_ρ on G is called the **character** of the representation ρ . If ρ is irreducible, we call χ_ρ an irreducible character. It is a standard result that two representations are isomorphic if and only if they have the same character. Note that, from properties of the trace, $\chi_\rho(h^{-1}gh) = \chi_\rho(g)$ and thus characters are constant on the conjugacy classes of groups. In other terms, characters are **class functions**.

Since characters form an orthonormal basis of the space of class functions, the number of inequivalent irreducible representations equals the number of conjugacy classes of G . If χ is the character of a representation (ρ, V) of G , and $e \in G$ is the identity, then $\chi(e) = \dim V$ and is called the **degree** of the character. If G is abelian, then every character is of degree 1.

The **character table** of a finite group G is the table with rows corresponding to inequivalent irreducible characters of G and columns corresponding to conjugacy classes of G . Entry (i, j) of the table is the value of the i^{th} irreducible character of G on the j^{th} conjugacy class of G .

Clifford theory deals with induced and restricted representations, which we will now define.

Definition II.1. *If ρ is a representation of G and H is a subgroup of G then we can define the **restriction** of ρ to H*

$$(\text{Res}_H^G \rho)(h) := \rho(h), \text{ for all } h \in H.$$

The restriction is a representation of H by definition. If χ is the character of ρ , we can also define the restriction of χ to H by

$$(\text{Res}_H^G \chi)(h) := \chi(h), \text{ for all } h \in H.$$

Notice that $\text{Res}_H^G \chi$ is the character of $\text{Res}_H^G \rho$.

Let ρ be a representation of G and ψ be a subrepresentation of the restriction $\text{Res}_H^G \rho$ of ρ to a subgroup H of G . Let V and W be the respective representation spaces of ρ and ψ . For $s \in G$ the vector space $\rho(s)W$ depends only on the left coset sH of s . Thus if γ is a left coset of H we can define the subspace W_γ of V to be $\rho(s)W$ for any $s \in \gamma$. Clearly, the W_γ are permuted by $\rho(s)$ for $s \in G$. This tells us that $\sum_{\gamma \in G/H} W_\gamma$ is a subrepresentation of V .

Definition II.2. We say that the representation ρ of G is **induced** by the representation ψ of H on W if V is equal to the direct sum of the W_σ for $\sigma \in G/H$.

Restriction and induction of representations do not preserve irreducibility in general. We state the following theorem from¹¹ without proof.

Theorem II.3. Let (W, ψ) be a representation of H , and H be a subgroup of G . There exists a linear representation of G induced by ψ which we denote $\text{Ind}_H^G \psi$ or $\text{Ind}_H^G W$. This induced representation is unique up to isomorphism.

B. Clifford theory

The objective of Clifford theory is to study the representation theory of a group via the representation theory of its normal subgroups. Here we review the central results of Clifford theory. In this section, we largely follow the outline of Clifford theory given in part 2 of¹³, and we refer the reader there for proofs and a more thorough exposition. In addition, we collect some results from¹², which will prove essential to our analysis.

Let G be a finite group and $N \trianglelefteq G$ be a normal subgroup of G . Let \widehat{G} and \widehat{N} denote the set of all irreducible representations of G and N , respectively, up to equivalence. For two representations ρ and σ we write $\sigma \succeq \rho$ to denote that ρ is a subrepresentation of σ and $\rho \sim \sigma$ to denote that ρ and σ are isomorphic representations.

Definition II.4. Let $\sigma \in \widehat{N}$ and $g \in G$. We define

$$\widehat{G}(\sigma) = \{\theta \in \widehat{G} : \text{Res}_N^G(\theta) \succeq \sigma\}.$$

The **g -conjugate** of σ is the representation ${}^g\sigma \in \widehat{N}$ defined by

$${}^g\sigma(n) = \sigma(g^{-1}ng), \quad (1)$$

for all $n \in N$. The **inertia subgroup** of $\sigma \in \widehat{N}$ is defined

$$I_G(\sigma) = \{g \in G : {}^g\sigma \sim \sigma\}.$$

Note that ${}^g\sigma$ is irreducible, since any subspace invariant under ${}^g\sigma$ is also invariant under σ . Since ${}^{gh}\sigma(n) = \sigma((gh)^{-1}n(gh)) = \sigma(h^{-1}(g^{-1}ng)h) = {}^g(h\sigma(n))$, eq. (1) defines an action of G on \widehat{N} , and thus $I_G(\sigma)$ is the stabilizer of σ in G . Notice that

$${}^m\sigma(n) = \sigma(m^{-1}nm) = \sigma(m)^{-1}\sigma(n)\sigma(m) \quad \text{for } m, n \in N.$$

If χ is the character of σ and χ_m is the character of ${}^m\sigma$, we have, for $n \in N$

$$\chi_m(n) = \text{tr}({}^m\sigma(n)) = \text{tr}(\sigma(m)^{-1}\sigma(n)\sigma(m)) = \text{tr}(\sigma(n)) = \chi(n).$$

Thus, we have ${}^m\sigma \sim \sigma$ for $m \in N$, and so $N \leq I_G(\sigma)$.

Lemma II.5. If σ and ${}^g\sigma$ are conjugate irreducible representations of a normal subgroup N of a finite group G , and $I_G(\sigma)$ and $I_G({}^g\sigma)$ are the respective inertia subgroups, then $I_G(\sigma)$ and $I_G({}^g\sigma)$ are conjugate subgroups of G . In particular, $I_G(\sigma)$ and $I_G({}^g\sigma)$ are isomorphic.

We can now recall from¹³ some central results of Clifford theory. Let R be a family of coset representatives for the left $I_G(\sigma)$ -cosets in G with $e_G \in R$, that is

$$G = \bigsqcup_{r \in R} rI_G(\sigma).$$

Then $\{{}^g\sigma : g \in G\} = \{{}^r\sigma : r \in R\}$ and the representations ${}^r\sigma$ are pairwise inequivalent.

Theorem II.6 (¹³ Theorem 2.1). Suppose that $N \trianglelefteq G$ and $\sigma \in \widehat{N}$ and $\theta \in \widehat{G}(\sigma)$. If we set $d = [I_G(\sigma) : N]$ and let l denote the multiplicity of σ in $\text{Res}_N^G \theta$, we have:

1. $\text{Hom}_G(\text{Ind}_N^G \sigma, \text{Ind}_N^G \sigma) \cong \mathbb{C}^d$ as vector spaces, and
2. $\text{Res}_N^G \theta \cong l \bigoplus_{r \in R} {}^r \sigma$.

The number $l = \dim(\text{Hom}_N(\sigma, \text{Res}_N^G \theta))$ is called the **inertia index** of $\theta \in \widehat{G}(\sigma)$ with respect to N .

Theorem II.7 (Clifford Correspondence). Let $N \trianglelefteq G$, $\sigma \in \widehat{N}$ and $I = I_G(\sigma)$, then

$$\widehat{I}(\sigma) \longrightarrow \widehat{G}(\sigma) : \psi \longmapsto \text{Ind}_I^G \psi$$

is a bijection. The inertia index of $\psi \in \widehat{I}(\sigma)$ with respect to N coincides with the inertia index of $\text{Ind}_I^G \psi$ with respect to N . In turn, the inertia index of $\text{Ind}_I^G \psi$ with respect to N is equal to the multiplicity m_ψ of ψ in $\text{Ind}_N^I \psi$. Furthermore,

$$\text{Res}_N^I \psi = m_\psi \sigma.$$

The following corollary of the Clifford correspondence is from section 8.1 of Serre's book¹¹.

Corollary II.8. If N is an abelian normal subgroup of G , the degree of each irreducible representation ρ of G divides the index $[G : N]$ of N in G .

Unfortunately, this correspondence does not tell us anything in the case where the inertia subgroup is all of G . The study of what happens in this case is known as stable Clifford theory and can be quite complicated¹⁴.

Definition II.9. Let ψ be a representation of G/N , the **inflation** $\tilde{\psi}$ of ψ is a representation of G defined by setting

$$\tilde{\psi}(g) = \psi(gN) \quad \text{for all } g \in G.$$

If χ and $\tilde{\chi}$ be characters of ψ and $\tilde{\psi}$ respectively, then the map $\chi \mapsto \tilde{\chi}$ is a bijection between the irreducible characters of G/N and the irreducible characters of G with N in their kernel (i.e. $\tilde{\chi}(n) = \deg(\tilde{\chi})$). Note that $\deg(\tilde{\chi}) = \deg(\chi)$.

Let σ be an irreducible representation of G and ρ_1 be the trivial representation of N (the representation mapping every $n \in N$ to 1). If we suppose $\text{Res}_N^G \sigma \succeq \rho_1$ then notice that ${}^h \rho_1(g) = \rho_1(h^{-1}gh) = 1 = \rho_1(g)$ for all $h, g \in G$, thus ${}^h \rho_1 \sim \rho_1$ for all $h \in H$. Combining this observation with part 2 of Theorem II.6 we see

$$\text{Res}_N^G \sigma \cong \bigoplus_{l=1}^{\deg(\sigma)} \rho_1.$$

So $N \leq \ker(\sigma)$ and thus σ is the inflation of an irreducible representation of G/N .

Definition II.10. Let $H \leq G$, and let σ be a representation of H . We call a representation σ' of G an **extension** of σ if $\text{Res}_H^G \sigma' = \sigma$.

We can now state a consequence of the Clifford correspondence that will prove very useful in our study of the Clifford group.

Theorem II.11 (The little group method;¹³, Theorem 5.1). Let G be a finite group with $N \trianglelefteq G$ a normal subgroup. Suppose that any $\sigma \in \widehat{N}$ has an extension σ' to its inertia group $I_G(\sigma)$. In \widehat{N} , define an equivalence relation \approx by setting $\sigma_1 \approx \sigma_2$ if there exists $g \in G$ such that ${}^g \sigma_1 \sim \sigma_2$. Let Σ be a set of representatives for the equivalence classes of \approx . For $\psi \in \widehat{I_G(\sigma)/N}$, let $\tilde{\psi}$ be its inflation to $I_G(\sigma)$. Then

$$\widehat{G} = \{\text{Ind}_{I_G(\sigma)}^G (\sigma' \otimes \tilde{\psi}) : \sigma \in \Sigma, \psi \in \widehat{I_G(\sigma)/N}\},$$

that is, the representations $\sigma' \otimes \tilde{\psi}$ form a complete list of irreducible representations of G and are pairwise inequivalent.

Definition II.12. Let Q , G , and N be groups. If we have an injective homomorphism $\iota : N \rightarrow G$, and a surjective homomorphism $\pi : G \rightarrow Q$, and if $\iota(N) = \ker(\pi)$, then we call G an **extension** of Q by N . If $\iota(N)$ is contained in the center of G , then we call G a **central extension**. A group extension G is often written as a short exact sequence

$$1 \rightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \rightarrow 1.$$

Theorem II.11 classifies the irreducible representations of a group extension G under the constraint that the irreducible representations σ of the normal subgroup N can always be extended to representations σ' of the corresponding inertia subgroup $I_G(\sigma)$.

Let $1 \rightarrow B \xrightarrow{\iota} G \xrightarrow{\pi} H \rightarrow 1$ be a central extension. When G is a central extension and $G \not\cong H \times B$, the little group method does not apply. To examine this case we require more specialized machinery. A **section** of the extension is a map $t : H \rightarrow G$ which is a right inverse for π , that is

$$\pi(t(h)) = h$$

for all $h \in H$. We call the section **normalized** if $t(e_H) = e_G$. For $h, k \in H$ we have

$$\pi[t(h)t(k)] = \pi(t(h))\pi(t(k)) = hk = \pi(t(hk)),$$

so there exists a unique $b(h, k) \in B$ such that

$$t(h)t(k) = t(hk)\iota[b(h, k)].$$

Let \widehat{H}^α denote all the irreducible projective representations of a finite group H with factor set α . We may now state a version of the little group method for central extensions.

Proposition II.13 (12, Proposition 7.24). *For every $\xi \in \widehat{B}$ we have $I_G(\xi) = G$. Let*

$$\eta(h, k) = \xi(b(h, k)),$$

Let $\bar{\eta}(h, k) = (\eta(h, k))^{-1}$, then the map

$$\widehat{H}^{\bar{\eta}} \longrightarrow \widehat{G}(\xi) : \Phi \longmapsto \Theta$$

is a bijection, with Θ defined by

$$\Theta(t(h)b) = \xi(b)\Phi(h) \tag{2}$$

for all $h \in H(\xi)$, $b \in B$. Finally,

$$\widehat{G} = \left\{ \Theta : \xi \in \widehat{B}, \Theta \text{ as in eq. (2), } \Phi \in \widehat{I_G(\xi)/B}^{\bar{\eta}} \right\}.$$

III. THE PAULI AND CLIFFORD GROUPS

A. Definitions

Here we recall the definitions of the Pauli and Clifford groups.

Definition III.1. *Let $U(d)$ be the set of d -by- d unitary matrices, where d is some power of 2. This has a standard representation on \mathbb{C}^d , the complex vector space¹⁵. Let v_0, v_1 be an orthonormal basis of \mathbb{C}^2 and define the linear operators X , Y and Z by*

$$Xv_l = v_{l+1}, \quad Zv_l = (-1)^l v_l, \quad Yv_l = -iZXv_l = (-1)^l i v_{l+1}$$

for $l \in \{0, 1\}$, with addition over indices being modulo 2. These operators are unitary. We define the **n-qubit Pauli group** \mathcal{P}_n as the subset of the unitary group $U(2^n)$ consisting of all n -fold tensor products of elements of $\mathcal{P}_1 := \langle X, Z, iI_2 \rangle$, where I_n is the identity on \mathbb{C}^n .

\mathcal{P}_1 is a group of order 16 with centre $|Z(\mathcal{P}_1)| = 4$. Since \mathcal{P}_n consists of n -fold tensor products of elements of \mathcal{P}_1 it is a central product of copies of \mathcal{P}_1 , and thus $|\mathcal{P}_n| = 4^{n+1}$. The operators X , Y , and Z can be written in matrix form with respect to the eigenbasis of Z as

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

These are known as the Pauli matrices.

Definition III.2. The ***n-qubit Clifford group*** $\text{Cliff}(n)$ is the normalizer of the *n-qubit Pauli group* in the unitary group

$$\text{Cliff}(n) = \{U \in U(2^n) : U\mathcal{P}_n U^\dagger \subseteq \mathcal{P}_n\}.$$

Since in quantum information theory global phases have no effect on measurement outcomes, it is common to define the Clifford group modulo phases. We denote this group

$$\mathcal{C}_n = \{U \in U(2^n) : U\mathcal{P}_n U^\dagger \subseteq \mathcal{P}_n\} / U(1),$$

and will call it the ***n-qubit projective Clifford group*** to differentiate it from other ways the Clifford group is defined in the literature.

We would like to understand the representation theory of the projective Clifford group.

The Clifford group $\text{Cliff}(n)$ is generated by the Hadamard (H) and Phase (S) gates on each qubit (i.e. on each tensor factor), and Controlled-Z (CZ) gate on each pair of qubits, along with phases. In matrix form these gates are

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Operators that are n -fold tensor products only of 2-by-2 matrices are said to consist only of single-qubit operations. If an operator has 4-by-4 matrices in its tensor decomposition, such as CZ that do not decompose further into tensor factors, it is said to contain multi-qubit operations.

Multi-qubit Pauli operators either commute or anticommute. Notice that the group $C_4 = \langle iI_n \rangle$ of phases in \mathcal{P}_n is the centre of \mathcal{P}_n .

Definition III.3. Define the ***n-qubit projective Pauli group*** to be $\tilde{\mathcal{P}}_n = \mathcal{P}_n / C_4$. Since C_4 contains the commutator subgroup $C_2 = \langle -I_n \rangle$ of \mathcal{P}_n , we have that $\tilde{\mathcal{P}}_n$ is abelian.

Notice, $\tilde{\mathcal{P}}_n$ is a normal subgroup of \mathcal{C}_n . Since $\tilde{\mathcal{P}}_n \cong \mathbb{Z}_2^{2n}$ we will often just write \mathbb{Z}_2^{2n} for the projective Pauli group.

B. The symplectic structure of the Clifford group

The quotient of the Clifford group by the Pauli group and phases is essential to the stable Clifford theory of the Clifford group. We will need the following proposition, which we present without proof.

Proposition III.4 (16, Prop 3.3). Let $\phi : \mathcal{P}_n \rightarrow \mathcal{P}_n$ be an automorphism of the Pauli group that fixes scalars. That is, $\phi(i^l I_{2^n}) = i^l I_{2^n}$. Then there exists $U \in \text{Cliff}(n)$ unique up to phases such that for all $P \in \mathcal{P}_n$ we have $UPU^\dagger = \phi(P)$.

Remark III.5. This says that $\mathcal{C}_n = \text{Aut}_{\langle i \rangle}(\mathcal{P}_n)$, that is, the Clifford group consists of the automorphisms of the Pauli group that fix the centre.

Following arguments from¹⁷ and¹⁸ we can show the following.

Theorem III.6. The quotient of the Clifford group $\text{Cliff}(n)$ by the Pauli group and phases is

$$\mathcal{C}_n / \tilde{\mathcal{P}}_n \cong Sp(2n, 2),$$

the symplectic group of degree $2n$ over \mathbb{Z}_2 .

Proof. For $\mathbf{x} = (\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{2n}$ define the **Weyl Operator**

$$W_{\mathbf{x}} = W_{\mathbf{p}, \mathbf{q}} = i^{-\mathbf{p} \cdot \mathbf{q}} (Z^{p_1} X^{q_1}) \otimes \cdots \otimes (Z^{p_n} X^{q_n}).$$

Clearly, all Weyl operators are elements of the Pauli group \mathcal{P}_n , and any element of the Pauli group is a Weyl operator up to a factor of i to some power. Weyl operators only depend on \mathbf{x} modulo 4, since

$$W_{\mathbf{x}+2\mathbf{z}} = (-1)^{[\mathbf{x}, \mathbf{z}]} W_{\mathbf{x}}, \tag{3}$$

where we have introduced the \mathbb{Z} -valued symplectic form $[\cdot, \cdot]$ on \mathbb{Z}^{2n}

$$[\mathbf{x}, \mathbf{z}] = [(\mathbf{p}, \mathbf{q}), (\mathbf{p}', \mathbf{q}')] = \mathbf{p} \cdot \mathbf{q}' - \mathbf{q} \cdot \mathbf{p}'.$$

We will use this form when $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_4^{2n}$, and interpret accordingly. For example, by direct computation we have

$$W_{\mathbf{x}} W_{\mathbf{y}} = i^{[\mathbf{x}, \mathbf{y}]} W_{\mathbf{x} + \mathbf{y}}.$$

Then for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_4^{2n}$, we have

$$W_{\mathbf{x}} W_{\mathbf{y}} = i^{[\mathbf{x}, \mathbf{y}]} W_{\mathbf{x} + \mathbf{y}} = i^{-[\mathbf{y}, \mathbf{x}]} W_{\mathbf{y} + \mathbf{x}} = i^{-2[\mathbf{y}, \mathbf{x}]} W_{\mathbf{y}} W_{\mathbf{x}} = (-1)^{[\mathbf{x}, \mathbf{y}]} W_{\mathbf{y}} W_{\mathbf{x}}.$$

Thus the commutation relation depends on $[\mathbf{x}, \mathbf{y}] \bmod 2$. By definition if $U \in \text{Cliff}(n)$ then for every $\mathbf{x} \in \mathbb{Z}_2^{2n}$, $UW_{\mathbf{x}}U^\dagger$ is proportional to a Weyl operator $W_{\mathbf{x}'}$ by some power of i and by eq. (3) we can take $\mathbf{x}' \in \mathbb{Z}_2^{2n}$. We define the function $g : \mathbb{Z}_2^{2n} \rightarrow \mathbb{Z}_4$ where $i^{g(\mathbf{x})} W_{\mathbf{x}'} = UW_{\mathbf{x}}U^\dagger$. Since conjugation preserves commutation relations, we have

$$(-1)^{[\mathbf{x}, \mathbf{y}]} W_{\mathbf{y}} W_{\mathbf{x}'} = W_{\mathbf{x}'} W_{\mathbf{y}'} = (-1)^{[\mathbf{x}', \mathbf{y}']} W_{\mathbf{y}'} W_{\mathbf{x}'}.$$

Thus the map $\Gamma : \mathbf{x} \mapsto \mathbf{x}'$ preserves the symplectic form $[\cdot, \cdot]$. Furthermore,

$$\begin{aligned} i^{g(\mathbf{x} + \mathbf{y}) + [\mathbf{x}, \mathbf{y}]} W_{(\mathbf{x} + \mathbf{y})'} &= U i^{[\mathbf{x}, \mathbf{y}]} W_{\mathbf{x} + \mathbf{y}} U^\dagger = UW_{\mathbf{x}} W_{\mathbf{y}} U^\dagger \\ &= UW_{\mathbf{x}} U^\dagger UW_{\mathbf{y}} U^\dagger = i^{g(\mathbf{x}) + g(\mathbf{y})} W_{\mathbf{x}'} W_{\mathbf{y}'} \\ &= i^{[\mathbf{x}, \mathbf{y}] + g(\mathbf{x}) + g(\mathbf{y})} W_{\mathbf{x}' + \mathbf{y}'}. \end{aligned}$$

Thus

$$i^{g(\mathbf{x} + \mathbf{y})} W_{(\mathbf{x} + \mathbf{y})'} = i^{g(\mathbf{x}) + g(\mathbf{y})} W_{\mathbf{x}' + \mathbf{y}'}.$$

Then

$$W_{(\mathbf{x} + \mathbf{y})'} = \pm W_{\mathbf{x}' + \mathbf{y}'}.$$

So by eq. (3), Γ is compatible with addition in \mathbb{Z}_2^{2n} . Since \mathbb{Z}_2 has only the scalars 0 and 1, we deduce that Γ is linear and thus an element of the symplectic group $Sp(2n, 2)$. Then for each $U \in \text{Cliff}(n)$ there is a $\Gamma \in Sp(2n, 2)$ and a function $g : \mathbb{Z}_2^{2n} \rightarrow \mathbb{Z}_4$ such that

$$UW_{\mathbf{x}}U^\dagger = i^{g(\mathbf{x})} W_{\Gamma(\mathbf{x})}.$$

Now notice that the n -qubit Pauli matrices form a basis of the vector space $M_{2^n}(\mathbb{C})$ of all 2^n -by- 2^n matrices. If we specify the action of $U \in \text{Cliff}(n)$ on a generating set of the \mathcal{P}_n , then we determine U up to a phase since $U' = e^{i\theta}U$ has the same action as U by conjugation. From Proposition III.4 we know that for any scalar fixing automorphism ϕ of \mathcal{P}_n there exists some $U \in \text{Cliff}(n)$ such that

$$\phi(P) = UPU^\dagger$$

for all $P \in \mathcal{P}_n$.

For any linear $\Gamma : \mathbb{Z}_4^{2n} \rightarrow \mathbb{Z}_4^{2n}$ that preserves the symplectic product modulo 4, we can define the map $\Phi : \mathcal{P}_n \rightarrow \mathcal{P}_n$ by

$$\Phi(W_{\mathbf{x}}) = W_{\Gamma(\mathbf{x})}, \quad \text{for all } \mathbf{x} \in \mathbb{Z}_4^{2n}.$$

To see this is well defined, notice that $W_{\Gamma(\mathbf{x})}$ is expressible as a linear combination of other Weyl operators only if $\pm W_{\Gamma(\mathbf{y})} = W_{\Gamma(\mathbf{x})}$ for some $\mathbf{y} \in \mathbb{Z}_4^{2n}$. Then by dimension counting and eq. (3) we have $\Gamma(\mathbf{y}) = \Gamma(\mathbf{x} + 2\mathbf{z}) = \Gamma(\mathbf{x}) + 2\Gamma(\mathbf{z})$. Thus the sign is given by $(-1)^{[\Gamma(\mathbf{x}), \Gamma(\mathbf{z})]} = (-1)^{[\mathbf{x}, \mathbf{z}]}$, so Φ is well defined. Furthermore, since

$$\Phi(W_{\mathbf{x}})\Phi(W_{\mathbf{y}}) = W_{\Gamma(\mathbf{x})}W_{\Gamma(\mathbf{y})} = i^{[\mathbf{x}, \mathbf{y}]} W_{\Gamma(\mathbf{x} + \mathbf{y})} = i^{[\mathbf{x}, \mathbf{y}]} \Phi(W_{\mathbf{x} + \mathbf{y}}) = \Phi(W_{\mathbf{x}}W_{\mathbf{y}}),$$

extending Φ by linearity defines an automorphism on \mathcal{P}_n that fixes scalars. We thus have a $U \in \text{Cliff}(n)$ such that

$$UW_{\mathbf{x}}U^\dagger = W_{\Gamma(\mathbf{x})}.$$

Let $\{\mathbf{e}_j\}_{j=1}^{2n}$ be a basis of \mathbb{Z}_2^{2n} . Fix $\Gamma \in Sp(2n, 2)$, and let

$$\Gamma \mathbf{e}_j = \mathbf{v}_j \text{ for all } j \in \{1, \dots, 2n\}.$$

In other words, let $C = [\mathbf{v}_1 \cdots \mathbf{v}_{2n}]$ be the matrix corresponding to the symplectic map Γ . Define $\bar{\mathbf{v}}_1 := \mathbf{v}_1$, and for each subsequent $j > 1$ define $\bar{\mathbf{v}}_j := \mathbf{v}_j + 2\mathbf{x}_j$ with $\mathbf{x}_j \in \mathbb{Z}_2^{2n}$ chosen such that

$$[\mathbf{v}_j, \bar{\mathbf{v}}_j] = 0 \pmod{4} \quad \text{and} \quad [\bar{\mathbf{v}}_h, \bar{\mathbf{v}}_j] = \delta_{h,n+j} - \delta_{j,n+h} \pmod{4} \quad \text{for } h < j,$$

where $\delta_{a,b}$ is the Kronecker delta. Notice that for all j we have $\bar{\mathbf{v}}_j \in \mathbb{Z}_4^{2n}$. Since Γ preserves the symplectic product mod 2 (and thus preserves commutators and anticommutators), we have both of these restrictions already satisfied mod 2. The matrix $\bar{C} = [\bar{\mathbf{v}}_1 \cdots \bar{\mathbf{v}}_{2n}]$ is symplectic modulo 4 and $\bar{C} = C \pmod{2}$. This means that for each $\Gamma \in Sp(2n, 2)$ there is a $\tilde{\Gamma} \in Sp(\mathbb{Z}_4^{2n})$ such that $\Gamma \mathbf{x} = \tilde{\Gamma} \mathbf{x} \pmod{2}$ thus we obtain $\tilde{\Gamma} \mathbf{x} - \Gamma \mathbf{x} = 2\mathbf{z}$ for some $\mathbf{z} \in \mathbb{Z}^{2n}$. If we define the function $f : \mathbb{Z}_2^{2n} \rightarrow \mathbb{Z}_2$ by $f(\mathbf{x}) = [\Gamma \mathbf{x}, \mathbf{z}] \pmod{2}$, we have

$$(-1)^{f(\mathbf{x})} W_{\Gamma \mathbf{x}} = W_{\Gamma \mathbf{x} + 2\mathbf{z}} = W_{\tilde{\Gamma} \mathbf{x}}.$$

This implies that for every $\Gamma \in Sp(2n, 2)$ there exists a $U \in \text{Cliff}(n)$ and a function $f : \mathbb{Z}_2^{2n} \rightarrow \mathbb{Z}_2$ such that for all $\mathbf{x} \in \mathbb{Z}_2^{2n}$

$$UW_{\mathbf{x}}U^\dagger = (-1)^{f(\mathbf{x})} W_{\Gamma \mathbf{x}}. \quad (4)$$

This U is determined uniquely up to phase since we have determined its action by conjugation. Thus we have a surjective correspondence $U \mapsto \Gamma$ between \mathcal{C}_n and $Sp(2n, 2)$, and we see that the quotient of $\text{Cliff}(n)$ by Paulis and phases is $Sp(2n, 2)$. \square

Note that this implies that \mathcal{C}_n is an extension of $Sp(2n, 2)$ by \mathbb{Z}_2^{2n} , but since we cannot specify that $f \equiv 0$ for all choices of $U \in \text{Cliff}(n)$ in eq. (4) the extension does not split for $n > 1$. For $n = 1$ we have $\mathcal{C}_1 \cong S_4 \cong Sp(2, 2) \ltimes \mathbb{Z}_2^2$. Since $|Sp(2n, 2)| = 2^{n^2} \prod_{j=1}^n (2^{2j} - 1)$, we obtain the following immediate corollary.

Corollary III.7. *The order of the Clifford group is*

$$|\mathcal{C}_n| = |\tilde{\mathcal{P}}_n| |Sp(2n, 2)| = 2^{n^2+2n} \prod_{j=1}^n (2^{2j} - 1).$$

C. The character table of the projective Pauli group

Since the n -qubit projective Pauli group is abelian it has only degree one irreducible characters. One-dimensional representations and characters coincide since the trace leaves 1-by-1 matrices invariant. Elements of the n -qubit projective Pauli group have order at most 2. Thus, the character of an element of $\tilde{\mathcal{P}}_n$ must be ± 1 . $\tilde{\mathcal{P}}_n$ is generated by $\{[X_1], [Z_1], \dots, [X_n], [Z_n]\}$, where $[A_j]$ is the equivalence class of the Pauli operator A acting on the j^{th} qubit

$$A_j = I_2^{\otimes j-1} \otimes A \otimes I_2^{\otimes n-j}.$$

So a character of $\tilde{\mathcal{P}}_n$ is fully determined by a choice of ± 1 for $[X_i]$ and $[Z_i]$ for each $i \in \{1, \dots, n\}$. The 4 choices for each qubit leave us with 4^n choices for the whole group. There are $4^n = |\tilde{\mathcal{P}}_n|$ characters since $\tilde{\mathcal{P}}_n$ is abelian and thus has only singleton conjugacy classes. Irreducible characters that disagree on any one element must be distinct, so this completely determines the character table of $\tilde{\mathcal{P}}_n$. Thus, the character table of $\tilde{\mathcal{P}}_n$ can be written by filling the first row and column of a 4^n -by- 4^n table with ones, then in the rest of each remaining row writing each permutation of $\frac{4^n}{2} - 1$ ones and $\frac{4^n}{2}$ negative ones.

IV. THE INERTIA SUBGROUP

To begin our study of the character theory of the n -qubit projective Clifford group, we examine the inertia subgroups of the representations of the n -qubit projective Pauli group in the n -qubit projective Clifford group.

Lemma IV.1. *Let σ and ρ be nontrivial irreducible representations of $\tilde{\mathcal{P}}_n$, then there exists $g \in \mathcal{C}_n$ such that ${}^g\sigma \sim \rho$. In other words, all nontrivial irreducible representations of $\tilde{\mathcal{P}}_n$ are conjugate in \mathcal{C}_n .*

Proof. We begin the proof by noticing that

$$\begin{aligned} HXH^{-1} &= Z \\ HZH^{-1} &= X \\ HYH^{-1} &= -Y. \end{aligned} \tag{5}$$

So we have that conjugation of Pauli matrices by H maps X to Z and vice versa, while mapping Y to $-Y$. Thus conjugation by $[H]$ maps $[X]$ to $[Z]$ and vice versa, while leaving $[Y]$ invariant. We can calculate

$$\begin{aligned} SXS^{-1} &= Y \\ SZS^{-1} &= Z \\ SYS^{-1} &= -X, \end{aligned} \tag{6}$$

thus conjugation by $[S]$ maps $[X]$ to $[Y]$ and vice versa, while leaving $[Z]$ invariant. Furthermore conjugation by $[H][S][H]$ maps $[Z]$ to $[Y]$ and vice versa, while leaving $[X]$ invariant. We see that we can permute the non-identity elements of the one-qubit projective Pauli group in any way via conjugation by elements of \mathcal{C}_n .

We now turn our attention to 2-qubit operators. Consider the swap gate, if A and B are any 2-by-2 matrices we have

$$(SWAP)(A \otimes B)(SWAP) = B \otimes A.$$

Our previous calculations for 1-qubit matrices tell us that any pair of nontrivial representations σ and ρ of $\tilde{\mathcal{P}}_n$ that have the same number of pairs of generators $([X_i], [Z_i])$ in their kernels, that is

$$|\{i \in \{1, \dots, n\} : \rho([X_i]) = \rho([Z_i]) = 1\}| = |\{i \in \{1, \dots, n\} : \sigma([X_i]) = \sigma([Z_i]) = 1\}|,$$

are conjugate in \mathcal{C}_n . Consider the two representations ρ and σ of $\tilde{\mathcal{P}}_2$ defined by

$$\begin{aligned} \sigma([X \otimes I]) &= \sigma([I \otimes Z]) = -1 \\ \sigma([Z \otimes I]) &= \sigma([I \otimes X]) = 1 \\ \rho([X \otimes I]) &= \rho([I \otimes X]) = \rho([Z \otimes I]) = 1 \\ \rho([I \otimes Z]) &= -1. \end{aligned}$$

Now we calculate

$$\begin{aligned} CZ(I \otimes X)CZ &= (Z \otimes X) \\ CZ(Z \otimes I)CZ &= (Z \otimes I) \\ CZ(X \otimes I)CZ &= (X \otimes Z) \\ CZ(I \otimes Z)CZ &= (I \otimes Z). \end{aligned}$$

Thus we have

$$\begin{aligned} {}^{CZ}\rho([X \otimes I]) &= \rho([X \otimes Z]) = -1 = \sigma([X \otimes I]) \\ {}^{CZ}\rho([Z \otimes I]) &= \rho([Z \otimes I]) = 1 = \sigma([Z \otimes I]) \\ {}^{CZ}\rho([I \otimes X]) &= \rho([Z \otimes X]) = 1 = \sigma([I \otimes X]) \\ {}^{CZ}\rho([I \otimes Z]) &= \rho([I \otimes Z]) = -1 = \sigma([I \otimes Z]). \end{aligned}$$

Thus nontrivial irreducible representations σ and ρ of $\tilde{\mathcal{P}}_2$ with differing numbers of $([X_i], [Y_i])$ pairs in their kernels, that is

$$|\{i \in \{1, 2\} : \rho([X_i]) = \rho([Z_i]) = 1\}| \neq |\{i \in \{1, 2\} : \sigma([X_i]) = \sigma([Z_i]) = 1\}|,$$

are conjugate in \mathcal{C}_2 . Since restricting irreducible representations of $\tilde{\mathcal{P}}_n$ to any two qubits gives an irreducible representation of $\tilde{\mathcal{P}}_2$, taking all the previous calculations together, we have that all nontrivial irreducible representations of $\tilde{\mathcal{P}}_n$ are conjugate in \mathcal{C}_n . \square

Since, by Lemma II.5, conjugate representations of normal subgroups have isomorphic inertia subgroups, we see that there is only one inertia subgroup to calculate for the nontrivial representations of the projective Pauli group in the Clifford group. We have the following immediate corollary.

Corollary IV.2. *If ρ is an irreducible representation of \mathcal{C}_n and σ an irreducible representation of $\tilde{\mathcal{P}}_n$ with $\text{Res}_{\tilde{\mathcal{P}}_n}^{\mathcal{C}_n} \rho \succeq \sigma$ then one of two cases holds:*

1. σ is trivial, and $\tilde{\mathcal{P}}_n$ is in the kernel of ρ . In this case, ρ is the inflation of an irreducible representation of $Sp(2n, 2)$, or
2. σ is nontrivial in which case we can apply Lemma IV.1 and Theorem II.6 to obtain

$$\text{Res}_{\tilde{\mathcal{P}}_n}^{\mathcal{C}_n} \rho = l \bigoplus_{\substack{\theta \in \text{Irr}(\tilde{\mathcal{P}}_n) \\ \theta \text{ nontrivial}}} \theta, \quad (7)$$

where $\text{Irr}(\tilde{\mathcal{P}}_n)$ is the set of irreducible representations of $\tilde{\mathcal{P}}_n$ and l is the inertia index of ρ with respect to $\tilde{\mathcal{P}}_n$.

Additionally, since $\tilde{\mathcal{P}}_n$ is an abelian normal subgroup of \mathcal{C}_n we have that the degree of ρ divides

$$[\mathcal{C}_n : \tilde{\mathcal{P}}_n] = 2^{n^2} \prod_{j=1}^n (2^{2j} - 1),$$

by Corollary II.8.

If we specialize to case 2, then eq. (7) and the fact that all irreducible representations of $\tilde{\mathcal{P}}_n$ have degree 1 imply that the degree of ρ is divisible by $4^n - 1$ (the number of nontrivial irreducible representations of $\tilde{\mathcal{P}}_n$). If χ is the character of ρ , then eq. (7) implies

$$\text{Res}_{\tilde{\mathcal{P}}_n}^{\mathcal{C}_n} \chi = l \sum_{\substack{\psi \in \text{IrrChar}(\tilde{\mathcal{P}}_n) \\ \psi \text{ nontrivial}}} \psi,$$

where $\text{IrrChar}(\tilde{\mathcal{P}}_n)$ is the set of irreducible characters of $\tilde{\mathcal{P}}_n$. In particular, if $g \in \tilde{\mathcal{P}}_n$ is a non-identity element then $\chi(g) = -l$, since for any such g the summand takes the value -1 a total of $2^{2(n-1)+1}$ times and takes the value 1 a total of $2^{2(n-1)+1} - 1$ times.

To understand case 2, we need to calculate the inertia subgroup $I_{\mathcal{C}_n}(\sigma)$ of a nontrivial representation σ of the Pauli group in the Clifford group. For a two qubit gate A , let $A_{i,j}$ denote A acting on the pair of qubits i and j . For $M = CX_{1,2}(Z_1 H_1 X_2)CX_{1,2}$, we have the following theorem.

Theorem IV.3. *For $n \geq 2$ the inertia subgroup of a nontrivial representation of $\tilde{\mathcal{P}}_n$ in \mathcal{C}_n is isomorphic to $IN_n := \langle \{[M], [H_1], [X_1], [I \otimes A] : \text{for } A \in \text{Cliff}(n-1)\} \rangle$.*

Proof. Notice that if σ is a nontrivial irreducible representation of $\tilde{\mathcal{P}}_n$, and ψ an irreducible representation of $I = I_{\mathcal{C}_n}(\sigma)$ with $\text{Res}_{\tilde{\mathcal{P}}_n}^I \psi \succeq \sigma$ then by the Clifford correspondence we have

$$m_\psi(2^{2n} - 1) = \deg m_\psi \bigoplus_{\substack{\theta \in \text{Irr}(\tilde{\mathcal{P}}_n) \\ \theta \text{ nontrivial}}} \theta = \deg \text{Ind}_I^{\mathcal{C}_n} \psi = [\mathcal{C}_n : I] \deg \psi,$$

where m_ψ is the inertia index of ψ with respect to $\tilde{\mathcal{P}}_n$. Additionally, by the Clifford correspondence,

$$[\mathcal{C}_n : I] \deg \psi = [\mathcal{C}_n : I] m_\psi \deg \sigma = m_\psi [\mathcal{C}_n : I],$$

Thus $[\mathcal{C}_n : I] = 2^{2n} - 1$ and

$$|I| = \frac{1}{2^{2n} - 1} |\mathcal{C}_n| = 2^{n^2 + 2n} \prod_{j=1}^{n-1} (2^{2j} - 1) = 2^{2n+1} |\mathcal{C}_{n-1}|.$$

So if for any particular σ we can find a subgroup of \mathcal{C}_n that preserves σ under conjugation and has this order, then we have found the inertia subgroup.

Consider the irreducible character σ_1 of $\tilde{\mathcal{P}}_n$ defined by $\sigma_1([X_1]) = \sigma_1([Z_1]) = -1$ and $\sigma_1([X_i]) = \sigma_1([Z_i]) = 1$ for all $i \in \{2, \dots, n\}$. We want to calculate $I_{\mathcal{C}_n}(\sigma_1) = \{g \in \mathcal{C}_n : {}^g\sigma_1 \sim \sigma_1\}$. So we want to find the elements of \mathcal{C}_n that preserve the presence of X or Z in the first tensor factor by conjugation. We immediately see that $[I \otimes A] \in I_{\mathcal{C}_n}(\sigma_1)$ for $A \in \text{Cliff}(n)$ since operations restricted to other qubits do not affect the first qubit. Similarly, conjugation by Pauli elements preserves σ_1 . Since conjugation by H simply exchanges X and Z , we also have $[H_1] \in I_{\mathcal{C}_n}(\sigma_1)$. Additionally, we have the operator

$$CX(ZH \otimes X)CX = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

The action of this matrix on \mathcal{P}_2 by conjugation is

$$\begin{aligned} CX(ZH \otimes X)CX(X \otimes I)CX(HZ \otimes X)CX &= -Z \otimes X \\ CX(ZH \otimes X)CX(Z \otimes I)CX(HZ \otimes X)CX &= X \otimes X \\ CX(ZH \otimes X)CX(I \otimes X)CX(HZ \otimes X)CX &= I \otimes X \\ CX(ZH \otimes X)CX(I \otimes Z)CX(HZ \otimes X)CX &= -ZX \otimes ZX. \end{aligned}$$

Notice that $\sigma_1([ZX \otimes ZX]) = \sigma_1([Z \otimes Z])\sigma_1([X \otimes X]) = (-1)^2 = \sigma_1([I \otimes Z])$. Thus the action of $[CX(ZH \otimes X)CX]$ preserves σ_1 . The conjugation action of IN_n leaves $[X_1Z_1]$ invariant, thus there are 2^{2n-1} possible images of the pair $([X_1], [Z_1])$. The order of IN_n is thus

$$(2^{2n-1})(2^{2n})|Sp(2(n-1), 2)| = 2^{2n+1}|\mathcal{C}_{n-1}|.$$

□

Note that there is no way to write $[X_1]$ in terms of the other generators of IN_n . Since $[X_1]^{-1} = [X_1]$, any reduction we perform on a word written in these generators preserves the parity of the number of $[X_1]$ s. For g a word in IN_n , let $n_{X_1}(g)$ be the number of times $[X_1]$ appears in g . The above analysis implies that the map

$$\begin{aligned} \sigma'_1 : IN_n &\longrightarrow \{1, -1\} \\ g &\longmapsto (-1)^{n_{X_1}(g)} \end{aligned}$$

is an irreducible character of IN_n . Furthermore, we have that $\text{Res}_{\tilde{\mathcal{P}}_n}^{IN_n} \sigma'_1 = \sigma_1$. Thus σ'_1 is an extension of σ_1 to IN_n , its own inertia subgroup. Since all nontrivial irreducible representations of the projective Pauli group \mathbb{Z}_2^{2n} are conjugate, we have that any irreducible representation σ of the projective Pauli group can be extended to a representation σ' of its own inertia subgroup, $I(\sigma)$, in the Clifford group. We apply the little group method to obtain the following.

Theorem IV.4. *The irreducible representations of the projective Clifford group are*

$$\widehat{\mathcal{C}_n} = \left\{ \text{Ind}_{IN_n}^{\mathcal{C}_n}(\sigma'_1 \otimes \tilde{\psi}) : \psi \in \widehat{IN_n/\mathbb{Z}_2^{2n}} \right\} \cup \left\{ \tilde{\psi} : \psi \in \widehat{Sp(2n, 2)} \right\},$$

where the $\tilde{\psi}$ in the left set is an inflation to an irreducible representation of IN_n and in the right set is an inflation to an irreducible representation of \mathcal{C}_n .

Theorem IV.4 gives a complete list of the irreducible representations of the n -qubit Clifford group. To actually calculate these representations, we would like to know the representations of $Sp(2n, 2)$ and those of the quotient group IN_n/\mathcal{P}_n . Using Theorem IV.4, we may calculate the following example character tables.

Example IV.5. *Table I is the character table of the 1-qubit projective Pauli group. Notice that the inertia group of the representation ψ_4 is just the subgroup $I(\psi_4) = \langle [H], [X], [Z] \rangle \subset \mathcal{C}_1$. The extension ψ'_4 of ψ_4 to $I(\psi_4)$ is achieved by defining the value of $\psi'_4([H]) = 1$. Table II is the character table of $I(\psi_4)/\mathbb{Z}_2^2$. Via GAP4 calculation, Table III is the character table of $Sp(2, 2)$. Then by Theorem IV.4 the character table of the 1-qubit Clifford group \mathcal{C}_1 is Table IV.*

| | $[I_2]$ | $[X]$ | $[Z]$ | $[Y]$ |
|----------|---------|-------|-------|-------|
| ψ_1 | 1 | 1 | 1 | 1 |
| ψ_2 | 1 | -1 | 1 | -1 |
| ψ_3 | 1 | 1 | -1 | -1 |
| ψ_4 | 1 | -1 | -1 | 1 |

TABLE I. The character table of the 1-qubit projective Pauli group.

| | $[[I_2]]$ | $[[H]]$ |
|----------|-----------|---------|
| ϕ_1 | 1 | 1 |
| ϕ_2 | 1 | -1 |

TABLE II. Character table of $I(\psi_4)/\mathbb{Z}_2^2$.

A. The representation theory of the inertia quotient group

To understand the irreducible representations of the Clifford group with nontrivial restriction to the Pauli group, we will now examine the irreducible representations of IN_n/\mathbb{Z}_2^{2n} . Notice that

$$H_1^M = (XZ \otimes X)H_1,$$

and that H_1 commutes with all other non-Pauli operators in the generating set of IN_n . From this we see that $\mathbb{Z}_2 \cong \langle [[I]], [[H_1]] \rangle$ forms an order 2 normal subgroup of IN_n/\mathbb{Z}_2^{2n} . For convenience we define the group $\mathcal{H}_{1,n} := \langle [H_1], \{[X_i], [Z_i], \text{ for } i = 1, \dots, n\} \rangle$. We are ready to prove the following lemma.

Lemma IV.6. *The inertia quotient group has the affine symplectic group as a quotient group, that is*

$$(IN_n/\mathbb{Z}_2^{2n})/\mathbb{Z}_2 \cong IN_n/\mathcal{H}_{1,n} \cong Sp(2(n-1), 2) \ltimes \mathbb{Z}_2^{2(n-1)}.$$

Proof. For $x \in \mathbb{Z}_2^{2(n-1)}$, let W_x be the Weyl operator defined in the proof of Theorem III.6. Consider operators of the form $X \otimes W_x$ and $Z \otimes W_x$, which we will call inertia Weyl operators since by definition n -qubit Weyl operators of this form are preserved by the inertia subgroup IN_n under conjugation. Since the n -qubit Pauli group is generated by these inertia Weyl operators, the action of $U \in IN_n$ by conjugation on these operators defines the action of U on the Pauli group.

From Theorem III.6, we know that conjugating an inertia Weyl operator by $I \otimes U$ for $U \in \text{Cliff}(n-1)$ will give us $X \otimes W_{\Gamma x}$ or $Z \otimes W_{\Gamma x}$ respectively for some $\Gamma \in Sp(2(n-1), 2)$ with potential phase factors. Furthermore, we know that any such Γ is realized by some $U \in \text{Cliff}(n-1)$. Conjugation by H_1 will exchange the X and Z on the first qubit. Conjugating by $MS_2H_2S_2^{-1}$ amounts to multiplication by X_2 on the left with a possible phase factor of -1 and a possible exchange of X and Z on the first qubit. Similarly conjugation by $H_1H_2MS_2H_2S_2^{-1}H_2$ amounts to multiplication by Z_2 on the left with a possible phase factor of -1 and a possible exchange of X for Z on the first qubit.

Notice that the actions by conjugation of the matrices we have examined generate the affine symplectic group $Sp(2(n-1), 2) \ltimes \mathbb{Z}_2^{2(n-1)}$ on the index x of a Weyl operator W_x , with an extra operator H that exchanges the X and Z on the first qubit. Since the equivalence classes of said matrices also generate IN_n , and the inertia Weyl operators along with H_1 generate $\mathcal{H}_{1,n}$, we have the result. \square

From the proof of this lemma, we see that the quotient group IN_n/\mathbb{Z}_2^{2n} is a central extension of $Sp(2(n-1), 2) \ltimes \mathbb{Z}_2^{2(n-1)}$ by \mathbb{Z}_2 . Through GAP4 calculation we have determined that in general, the extension will not be a direct product, although it is in the two qubit case. Fix a normalized section t of the central extension IN_n/\mathbb{Z}_2^{2n} of $Sp(2(n-1), 2) \ltimes \mathbb{Z}_2^{2(n-1)}$ by \mathbb{Z}_2 . Let $b(h, k) \in \mathbb{Z}_2$ be the corresponding factor set. The only nontrivial irreducible representation ξ of \mathbb{Z}_2 maps the non-identity element to -1. Let $\eta(h, k) = \xi(b(h, k))$. By applying Proposition II.13, we obtain

$$\widehat{IN_n/\mathbb{Z}_2^{2n}} = \left\{ \widetilde{\psi} : \psi \in \widehat{Sp(2(n-1), 2) \ltimes \mathbb{Z}_2^{2(n-1)}} \right\} \cup \left\{ \Theta : \Psi \in \widehat{(Sp(2(n-1), 2) \ltimes \mathbb{Z}_2^{2(n-1)})^\eta} \right\},$$

with Θ defined by $\Theta(t(h)b) = \xi(b)\Psi(h)$ for all $h \in Sp(2(n-1), 2) \ltimes \mathbb{Z}_2^{2(n-1)}$ and $b \in \mathbb{Z}_2$.

| | $[[I_2]]$ | $[[H]]$ | $[[SH]]$ |
|------------|-----------|---------|----------|
| θ_1 | 1 | 1 | 1 |
| θ_2 | 1 | -1 | 1 |
| θ_3 | 2 | 0 | -1 |

TABLE III. Character table of $Sp(2, 2)$

| | $[I_2]$ | $[H]$ | $[SH]$ | $[X]$ | $[S]$ |
|--|---------|-------|--------|-------|-------|
| $\tilde{\theta}_1$ | 1 | 1 | 1 | 1 | 1 |
| $\tilde{\theta}_2$ | 1 | -1 | 1 | 1 | -1 |
| $\tilde{\theta}_3$ | 2 | 0 | -1 | 2 | 0 |
| $\text{Ind}_{I(\psi_4)}^{C_1}(\psi'_4 \otimes \tilde{\phi}_1)$ | 3 | -1 | 0 | -1 | 1 |
| $\text{Ind}_{I(\psi_4)}^{C_1}(\psi'_4 \otimes \tilde{\phi}_2)$ | 3 | 1 | 0 | -1 | -1 |

TABLE IV. Character table of C_1

V. LIFTING IRREDUCIBLE CHARACTERS TO HIGHER DIMENSIONAL CLIFFORD GROUPS

We will now explain how irreducible characters of the n -qubit Clifford group can be used to explicitly construct characters of the $(n+1)$ -qubit Clifford group. First, we need to understand the representation theory of the affine symplectic group $Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}$. It is clear that if U acts on \mathbb{Z}_2^{2n} by $\Gamma \in Sp(2n, 2)$ then ${}^{(\mathbf{x}, \Gamma)}\sigma \sim {}^U\sigma$ for any $\sigma \in \widehat{\mathbb{Z}_2^{2n}}$ and $(\mathbf{x}, \Gamma) \in Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}$. Let σ_1 be the irreducible representation of \mathbb{Z}_2^{2n} defined in section IV, then it follows that $I_{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma_1)/\mathbb{Z}_2^{2n} \cong IN_n/\mathbb{Z}_2^{2n}$. Let σ''_1 be the extension of σ_1 to $I_{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma_1)$ via $\sigma''_1(x, \Gamma) = \sigma_1(x)$. By applying Theorem II.11, we immediately obtain the following.

Lemma V.1. *The irreducible representations of the affine symplectic group are*

$$\widehat{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}} = \left\{ \text{Ind}_{(IN_n/\mathbb{Z}_2^{2n}) \ltimes \mathbb{Z}_2^{2n}}^{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma'_1 \otimes \tilde{\psi}) : \psi \in \widehat{IN_n/\mathbb{Z}_2^{2n}} \right\} \cup \left\{ \tilde{\psi} : \psi \in \widehat{Sp(2n, 2)} \right\},$$

where $\tilde{\psi}$ in the left set is the inflation to $(IN_n/\mathbb{Z}_2^{2n}) \ltimes \mathbb{Z}_2^{2n}$ and in the right set is inflation to $Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}$.

We can now prove the following lemma which was first proven by Bernd Fischer using the technique of Fischer-Clifford matrices¹⁰.

Lemma V.2. *$Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}$ and C_n have identical character tables.*

Proof. This is trivially true if $n = 1$, as in that case the groups are isomorphic. For $n > 1$ we first notice that

$$(Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n})/\mathbb{Z}_2^{2n} \cong Sp(2n, 2) \cong C_n/\widetilde{\mathcal{P}}_n.$$

The irreducible characters that come from $Sp(2n, 2)$ are nothing but inflations of the irreducible characters of $Sp(2n, 2)$. Thus if χ is an irreducible character of $Sp(2n, 2)$ and $\tilde{\chi}$ and $\tilde{\chi}'$ are its inflations to C_n and $Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}$ respectively, we have

$$\tilde{\chi}(U) = \chi(\Gamma) = \tilde{\chi}'(\mathbf{x}, \Gamma) \tag{8}$$

for all $\mathbf{x} \in \mathbb{Z}_2^{2n}$ and $U \in C_n$ such that $UW_{\mathbf{x}}U^\dagger = (-1)^{f(\mathbf{x})}W_{\Gamma\mathbf{x}}$.

Fix a normalized section $t : Sp(2n, 2) \rightarrow C_n$ of the extension

$$1 \rightarrow \mathbb{Z}_2^{2n} \rightarrow C_n \rightarrow Sp(2n, 2) \rightarrow 1$$

such that $\sigma'_1(t(\Gamma)) = 1$ for all $\Gamma \in IN_n/\mathbb{Z}_2^{2n}$. Define the mapping $\phi : Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n} \rightarrow C_n$ by $\phi(\mathbf{x}, \Gamma) = W_{\mathbf{x}}t(\Gamma)$. It is clear that this mapping is one-to-one and onto, and $\sigma''_1(s) = \sigma'_1(\phi(s))$ for all $s \in I_{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma_1)$. Using the notation of eq. (8) we see that $\tilde{\chi}(\phi(s)) = \tilde{\chi}'(s)$ for all $s \in Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}$. Let ψ be an irreducible representation of IN_n/\mathbb{Z}_2^{2n} , and $\tilde{\psi}$ and $\tilde{\psi}'$ be its inflations to $I_{C_n}(\sigma_1)$ and $I_{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma_1)$ respectively. From the formula for induced characters, we have

$$\text{Ind}_{I_{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma_1)}^{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\tilde{\psi}' \otimes \sigma''_1)(s) = \frac{1}{|I_{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma_1)|} \sum_{\substack{r \in Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n} \\ r^{-1}sr \in I_{Sp(2n, 2) \ltimes \mathbb{Z}_2^{2n}}(\sigma_1)}} \tilde{\psi}' \otimes \sigma''_1(r^{-1}sr),$$

and

$$\text{Ind}_{I_{\mathcal{C}_n}(\sigma_1)}^{\mathcal{C}_n}(\tilde{\psi} \otimes \sigma'_1)(s) = \frac{1}{|I_{\mathcal{C}_n}(\sigma_1)|} \sum_{\substack{r \in \mathcal{C}_n \\ r^{-1}sr \in I_{\mathcal{C}_n}(\sigma_1)}} \tilde{\psi} \otimes \sigma'_1(r^{-1}sr).$$

Since the action by conjugation of $\phi(\mathbf{x}, \Gamma)$ depends only on Γ , we see that $\phi(r)^{-1}\phi(s)\phi(r) \in I_{\mathcal{C}_n(\sigma_1)}$ if and only if $r^{-1}sr \in I_{Sp(2n, 2) \times \mathbb{Z}_2^{2n}}(\sigma_1)$ for any $r, s \in Sp(2n, 2) \times \mathbb{Z}_2^{2n}$, and furthermore $\tilde{\psi}(\phi(r)^{-1}\phi(s)\phi(r)) = \tilde{\psi}'(r^{-1}sr)$. Finally, we obtain

$$\text{Ind}_{I_{\mathcal{C}_n}(\sigma_1)}^{\mathcal{C}_n}(\tilde{\psi} \otimes \sigma'_1)(\phi(s)) = \text{Ind}_{I_{Sp(2n, 2) \times \mathbb{Z}_2^{2n}}(\sigma_1)}^{Sp(2n, 2) \times \mathbb{Z}_2^{2n}}(\tilde{\psi}' \otimes \sigma''_1)(s)$$

for all $s \in Sp(2n, 2) \times \mathbb{Z}_2^{2n}$. By column orthogonality of character tables we have that $r, s \in Sp(2n, 2) \times \mathbb{Z}_2^{2n}$ are conjugate if and only if $\phi(r)$ and $\phi(t)$ are conjugate in \mathcal{C}_n . Thus the map ϕ respects conjugacy classes and the character tables are identical. \square

Taken together these lemmas imply a remarkable property of the Clifford group.

Theorem V.3. *Let $\phi : Sp(2n, 2) \times \mathbb{Z}_2^{2n} \rightarrow \mathcal{C}_n$ be the map defined in the proof of Lemma V.2. If χ is an irreducible character of the n -qubit Clifford group \mathcal{C}_n then $\text{Ind}_{IN_{n+1}}^{\mathcal{C}_{n+1}}(\widetilde{\chi \circ \phi}) \otimes \sigma'_1$ is an irreducible character of the $(n+1)$ -qubit Clifford group \mathcal{C}_{n+1} .*

Proof. By Lemma V.2 we see that every irreducible character χ of \mathcal{C}_n is also an irreducible character of $Sp(2n, 2) \times \mathbb{Z}_2^{2n}$ when precomposed with the bijection ϕ of the conjugacy classes of the two groups. We can then see by Lemma IV.6 that the irreducible character $\chi_\phi := \chi \circ \phi$ of $Sp(2n, 2) \times \mathbb{Z}_2^{2n}$ inflates to an irreducible character $\widetilde{\chi_\phi}$ of IN_{n+1} that contains $\mathcal{H}_{1,n+1}$ in its kernel. In Particular this means that $\widetilde{\mathcal{P}}_{n+1}$ will be contained in the kernel of $\widetilde{\chi_\phi}$, so we know that $\widetilde{\chi_\phi} \otimes \sigma'_1$ is an irreducible character of IN_{n+1} that has σ_1 in the decomposition of its restriction to $\widetilde{\mathcal{P}}_{n+1}$ into irreducible representations. Therefore, by the Clifford correspondence we obtain the result. \square

This gives a straightforward method for obtaining irreducible characters of the $(n+1)$ -qubit Clifford group from irreducible characters of the n -qubit Clifford group.

| | |
|----|---|
| 1 | $[I_4]$ |
| 2 | $[I_2 \otimes Z]$ |
| 3 | $[I_2 \otimes H]$ |
| 4 | $[I_2 \otimes HZ]$ |
| 5 | $[Z \otimes HZ]$ |
| 6 | $[H \otimes H]$ |
| 7 | $[H \otimes HZ]$ |
| 8 | $[HZ \otimes HZ]$ |
| 9 | $[(S^{-1}H \otimes SH)CZ(S^{-1}HSH \otimes ZH)]$ |
| 10 | $[(S^{-1}H \otimes SHS^{-1})CZ(ZHS \otimes HSH)]$ |
| 11 | $[I_2 \otimes SHS^{-1}XS^{-1}]$ |
| 12 | $[HS^{-1}XS^{-1}H \otimes S^{-1}H]$ |
| 13 | $[H \otimes SHS^{-1}XS^{-1}H]$ |
| 14 | $[S^{-1}XS^{-1}H \otimes S^{-1}H]$ |
| 15 | $[(SH \otimes SHS^{-1})CZ(ZHSH \otimes I_2)]$ |
| 16 | $[(SH \otimes ZHSH)CZ(SHS \otimes I_2)]$ |
| 17 | $[(I_2 \otimes SH)CZ(H \otimes I_2)CZ(SXSH \otimes I_2)]$ |
| 18 | $[(I_2 \otimes S^{-1}HSX)CZ(H \otimes I_2)CZ(H \otimes S)]$ |
| 19 | $[CZ(I_2 \otimes S)(H \otimes H)CZ(H \otimes H)CZ(ZX \otimes H)]$ |
| 20 | $[(S^{-1} \otimes I_2)CZ(H \otimes HSH)CZ(SXSH \otimes I_2)]$ |
| 21 | $[(I_2 \otimes S^{-1}XS^{-1})CZ(SH \otimes H)CZ(I_2 \otimes S)]$ |

TABLE V. Conjugacy class representatives for \mathcal{C}_2

| | ([I ₂], 0) | ([I ₂], 1) | ([H], 1) | ([H], 0) | ([SH], 1) | ([SH], 0) | ([X], 1) | ([X], 0) | ([S], 1) | ([S], 0) |
|--------------------------------------|------------------------|------------------------|----------|----------|-----------|-----------|----------|----------|----------|----------|
| $\mu_1 := \chi_1 \times \theta_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\mu_2 := \chi_1 \times \theta_2$ | 1 | -1 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 1 |
| $\mu_3 := \chi_2 \times \theta_2$ | 1 | -1 | 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 |
| $\mu_4 := \chi_2 \times \theta_1$ | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | -1 | -1 |
| $\mu_5 := \chi_3 \times \theta_2$ | 2 | -2 | 0 | 0 | 1 | -1 | -2 | 2 | 0 | 0 |
| $\mu_6 := \chi_3 \times \theta_1$ | 2 | 2 | 0 | 0 | -1 | -1 | 2 | 2 | 0 | 0 |
| $\mu_7 := \chi_4 \times \theta_2$ | 3 | -3 | -1 | 1 | 0 | 0 | 1 | -1 | 1 | -1 |
| $\mu_8 := \chi_5 \times \theta_2$ | 3 | -3 | 1 | -1 | 0 | 0 | 1 | -1 | -1 | 1 |
| $\mu_9 := \chi_4 \times \theta_1$ | 3 | 3 | -1 | -1 | 0 | 0 | -1 | -1 | 1 | 1 |
| $\mu_{10} := \chi_5 \times \theta_1$ | 3 | 3 | 1 | 1 | 0 | 0 | -1 | -1 | -1 | -1 |

TABLE VI. Character table of $\mathcal{C}_1 \times \mathbb{Z}_2$

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\widetilde{\psi_1}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\widetilde{\psi_2}$ | 1 | 1 | -1 | -1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | -1 | 1 | 1 | 1 |
| $\widetilde{\psi_3}$ | 5 | 5 | 3 | 3 | 3 | 1 | 1 | 1 | -1 | -1 | 2 | 2 | 0 | 0 | 1 | 1 | -1 | -1 | -1 | 0 | -1 |
| $\widetilde{\psi_4}$ | 5 | 5 | -3 | -3 | -3 | 1 | 1 | 1 | 1 | 2 | 2 | 0 | 0 | -1 | -1 | -1 | -1 | 1 | 0 | -1 | |
| $\widetilde{\psi_5}$ | 5 | 5 | -1 | -1 | -1 | 1 | 1 | 1 | 3 | 3 | -1 | -1 | -1 | 1 | 1 | -1 | -1 | 0 | 0 | 2 | |
| $\widetilde{\psi_6}$ | 5 | 5 | 1 | 1 | 1 | 1 | 1 | -3 | -3 | -1 | -1 | 1 | 1 | -1 | -1 | -1 | 0 | 0 | 0 | 2 | |
| $\widetilde{\psi_7}$ | 9 | 9 | -3 | -3 | -3 | 1 | 1 | -3 | -3 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | -1 | 0 | | |
| $\widetilde{\psi_8}$ | 9 | 9 | 3 | 3 | 3 | 1 | 1 | 3 | 3 | 0 | 0 | 0 | 0 | -1 | -1 | 1 | 1 | 0 | -1 | 0 | |
| $\widetilde{\psi_9}$ | 10 | 10 | -2 | -2 | -2 | -2 | -2 | -2 | 2 | 2 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | -1 | 0 | 1 |
| $\widetilde{\psi_{10}}$ | 10 | 10 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | -2 | 1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_1} \otimes \sigma'_1)$ | 15 | -1 | 1 | 5 | -3 | 3 | -1 | -1 | 1 | -3 | 3 | -1 | 1 | -1 | 1 | -1 | 1 | -1 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_2} \otimes \sigma'_1)$ | 15 | -1 | 1 | -7 | 1 | -1 | -1 | 3 | 1 | -3 | 3 | -1 | 1 | -1 | -1 | 1 | -1 | 1 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_3} \otimes \sigma'_1)$ | 15 | -1 | -1 | 7 | -1 | -1 | -1 | 3 | -1 | 3 | 3 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_4} \otimes \sigma'_1)$ | 15 | -1 | -1 | -5 | 3 | 3 | -1 | -1 | 1 | 3 | 3 | -1 | -1 | 1 | -1 | 1 | 1 | -1 | 0 | 0 | 0 |
| $\widetilde{\psi_{11}}$ | 16 | 16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -2 | |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_5} \otimes \sigma'_1)$ | 30 | -2 | -2 | 2 | 2 | 2 | -2 | 2 | -2 | 6 | -3 | 1 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_6} \otimes \sigma'_1)$ | 30 | -2 | 2 | -2 | -2 | 2 | -2 | 2 | 2 | -6 | -3 | 1 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_7} \otimes \sigma'_1)$ | 45 | -3 | 3 | -9 | -1 | -3 | 1 | 1 | -1 | 3 | 0 | 0 | 0 | 0 | 1 | -1 | 1 | -1 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_8} \otimes \sigma'_1)$ | 45 | -3 | -3 | 9 | 1 | -3 | 1 | 1 | -3 | 0 | 0 | 0 | 0 | -1 | 1 | 1 | -1 | 0 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_9} \otimes \sigma'_1)$ | 45 | -3 | -3 | -3 | 5 | 1 | 1 | -3 | 1 | -3 | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 | 0 | 0 | 0 |
| $Ind_{I_{\mathcal{C}_2}}^{C_2}(\sigma_1)(\widetilde{\mu_{10}} \otimes \sigma'_1)$ | 45 | -3 | 3 | 3 | -5 | 1 | 1 | -3 | -1 | 3 | 0 | 0 | 0 | 0 | -1 | 1 | -1 | 1 | 0 | 0 | 0 |

TABLE VII. Character table of \mathcal{C}_2

Example V.4. As an example, we demonstrate the lifting procedure from the 1-qubit to the 2-qubit Clifford group. In this case, because of the isomorphism $\mathcal{C}_1 \cong Sp(2, 2) \times \mathbb{Z}_2^2$, we know that the inertia quotient IN_2/\mathbb{Z}_2^4 group is a central extension of \mathcal{C}_1 by \mathbb{Z}_2 . Moreover, in this case, the extension splits and we have $IN_2/\mathbb{Z}_2^2 \cong \mathcal{C}_1 \times \mathbb{Z}_2$. Table VI is the character table of $\mathcal{C}_1 \times \mathbb{Z}_2$, where we denote the characters of \mathcal{C}_1 by χ_i for $i \in \{1, \dots, 5\}$ and denote the characters of \mathbb{Z}_2 by θ_1 and θ_2 , with θ_1 being the trivial representation. So from every character of \mathcal{C}_1 we will get two characters of \mathcal{C}_2 , and the character table of \mathcal{C}_2 is determined entirely by these characters, and inflated characters from $Sp(4, 2)$. Thus the character table of \mathcal{C}_2 is Table VII, where the $\widetilde{\psi}_i$ are inflated characters from $Sp(4, 2)$. The numbered conjugacy classes in the character table of \mathcal{C}_2 are represented by the elements in Table V.

ACKNOWLEDGMENTS

I thank William Slofstra, Benjamin Lovitz and Jack Davis for helpful discussions. I acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). In particular, this work was supported by an NSERC CGS D award.

- ¹D. Gottesman, “An introduction to quantum error correction and fault-tolerant quantum computation,” in *Quantum information science and its contributions to mathematics*, Proc. Sympos. Appl. Math., Vol. 68 (Amer. Math. Soc., Providence, RI, 2010) pp. 13–58.
- ²J. Helsen, J. J. Wallman, S. T. Flammia, and S. Wehner, “Multiqubit randomized benchmarking using few samples,” Physical Review A **100** (2019), 10.1103/physreva.100.032304.
- ³D. Gottesman, “The Heisenberg representation of quantum computers,” in *Group22: Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics, Hobart, July 13-17* (1999) pp. 32–43, arXiv:quant-ph/9807006 [quant-ph].
- ⁴S. Aaronson and D. Gottesman, “Improved simulation of stabilizer circuits,” Phys. Rev. A **70**, 052328 (2004).
- ⁵C. Dankert, R. Cleve, J. Emerson, and E. Livine, “Exact and approximate unitary 2-designs and their application to fidelity estimation,” Physical Review A **80** (2009), 10.1103/physreva.80.012304.
- ⁶R. Koenig and J. A. Smolin, “How to efficiently select an arbitrary Clifford group element,” Journal of Mathematical Physics **55**, 122202 (2014).
- ⁷J. Helsen and M. Walter, “Thrifty shadow estimation: Reusing quantum circuits and bounding tails,” Phys. Rev. Lett. **131**, 240602 (2023).
- ⁸J. Helsen, J. J. Wallman, and S. Wehner, “Representations of the multi-qubit Clifford group,” Journal of Mathematical Physics **59**, 1–21 (2018), arXiv:1609.08188.
- ⁹M. Geck and G. Malle, *The Character Theory of Finite Groups of Lie Type: A Guided Tour*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, 2020).
- ¹⁰B. Fischer, “Examples of groups with identical character tables,” Rendiconti del Circolo Matematico di Palermo. Supplemento **19**, 71–77 (1988).
- ¹¹J.-P. Serre, *Linear representations of finite groups.*, Graduate texts in mathematics, Vol. 42 (Springer, 1977) pp. I–X, 1–170.
- ¹²T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, *Representation Theory of Finite Group Extensions: Clifford Theory, Mackey Obstruction, and the Orbit Method*, Springer Monographs in Mathematics (Springer International Publishing, 2022).
- ¹³T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, “Clifford theory and applications,” Journal of Mathematical Sciences (United States) **156**, 29–43 (2009).
- ¹⁴D. A. Craven, *Representation Theory of Finite Groups: a Guidebook* (Springer International Publishing, 2019).
- ¹⁵W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics (Springer New York, 1991).
- ¹⁶P. Selinger, “Generators and relations for n-qubit Clifford operators,” Logical Methods in Computer Science **11** (2015), 10.2168/lmcs-11(2:10)2015.
- ¹⁷D. Gross, S. Nezami, and M. Walter, “Schur–Weyl duality for the Clifford group with applications: Property testing, a robust Hudson theorem, and de Finetti representations,” Communications in Mathematical Physics **385**, 1325–1393 (2021).
- ¹⁸N. de Beaudrap, “A linearized stabilizer formalism for systems of finite dimension,” Quantum Information and Computation **13**, 73–115 (2013).