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On invertible matrices over antirings ☆

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Abstract

An antiring is a semiring which is zerosumfree (i.e., $a + b = 0$ implies $a = b = 0$ for any a, b in this semiring). In this paper, the complete description of the invertible matrices over a commutative antiring is given and some necessary and sufficient conditions for a matrix over a commutative antiring to be invertible are obtained. Also, Cramer's rule over commutative antirings is presented. The main results in this paper generalize and develop the corresponding results for the Boolean matrices, the fuzzy matrices, the lattice matrices and the incline matrices.

AMS classification: 15A15; 15A18

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1. Introduction

A *semiring* is an algebraic system $(S, +, \cdot)$ in which $(S, +)$ is an Abelian monoid with identity element 0 and $(S, \cdot)$ is another monoid with identity element 1, connected by ring-like distributivity. Also, the element 0 is an absorbing element in S, i.e., $0r = r0 = 0$ for all $r \in S$. A semiring S is called *commutative* if $ab = ba$ for all $a, b \in S$.

Let S be a semiring. S is called an *antiring* if $a + b = 0$ implies that $a = b = 0$ for any $a, b \in S$ (see [23]). Antirings were studied in [8] under the name of zerosumfree semirings. All

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rings with identity are semirings, but not all rings are antirings. Antirings are quite abundant: for example, every Boolean algebra, the fuzzy algebra $([0, 1], \vee, T)$, where T is a t-norm (for t-norm, refer to [13]), every distributive lattice and any incline (see [3]) are commutative antirings. Also, the set Z^+ of nonnegative integers with the usual operations of addition and multiplication of integers is a commutative antiring. The same is true for the set Q^+ of all nonnegative rational numbers, for the set R^+ of all nonnegative real numbers. In addition, the max-plus algebra $(R \cup \{-\infty\}, \max, +)$ and the min-plus algebra $(R \cup \{+\infty\}, \min, +)$ are commutative antirings (see [4,28]).

The study of matrices over general semirings has a long history. In 1964, Rutherford [19] gave a proof of the Cayley–Hamiltion theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on theory of matrices over semirings were published (see e.g. [2,6,9,12,15–18]). In 1999, Golan described semirings and matrices over semirings in his work [8] comprehensively. The techniques of matrices over semirings have important applications in optimization theory, models of discrete event networks and graph theory. For further examples, see [1,5].

Invertible matrices are an important type of matrices. Since the beginning of the 1950s, many authors have studied this type of matrices for some special cases of antirings (see e.g. [3,7,11,14,20,25–27]). In 1952, Luce [14] showed that a matrix over a Boolean algebra of at least two elements is invertible if and only if it is an orthogonal matrix. Zhao [27] proved that a fuzzy square matrix is invertible if and only if it is a permutation matrix. Give'on [7] developed the theory of invertible lattice matrices, thus generalizing the result of Luce [14]. Zhao [25,26] discussed the conditions for invertibility of matrices over a kind of Brouwerian lattices and arbitrary distributive lattice, respectively. Skornyakov [20] gave an extensive description of the invertible lattice matrices. Cao et al. [3] were the first to study the condition for an incline matrix to be invertible and showed that the statement of Luce [14] holds for the incline matrices as well. In addition, Zarko [24] established Cramer's rule over general Boolean algebras. Tian et al. [22] presented Cramer's ruler over complete and completely distributive lattices. Recently, Han et al. [11] gave the complete description of the invertible incline matrices, they studied some necessary and sufficient conditions for an incline matrix to be invertible and presented Cramer's rule over inclines.

In the present work, we consider the invertible matrices over general commutative antirings. In Section 3, we give the complete description for the invertible matrices and obtain some necessary and sufficient conditions for a matrix over a commutative antiring to be invertible. In Section 4, we present Cramer's rule for a matrix equation over a commutative antiring. The main results in the present paper generalize and develop the corresponding results in the literature for Boolean matrices, fuzzy matrices, lattice matrices and incline matrices.

2. Definitions and preliminary lemmas

In this section, we give some definitions and preliminary lemmas. For convenience, we use $\underline{n}$ to denote the set $\{1, 2, \ldots, n\}$ for any positive integer n and use S_n to denote the symmetric group on the set $\underline{n}$. Also, we use $|X|$ to denote the cardinal number for any finite set X.

Let S be a semiring and $a \in S$. We denote by a^k the kth power of a and by ka the sum $a + a + \cdots + a$ (k times) for any positive integer k. For $a \in S$, a is called an *idempotent element* in S if $a^2 = a$. The set of all idempotent elements in S is denoted by $I(S)$, i.e., $I(S) = \{a \in S\colon a^2 = a\}$. An element $a \in S$ is called *invertible* in S if there exists an element $b \in S$ such that

$ab = ba = 1$. The element b is called an *inverse* of a in S. It is easily proved that the inverse of a in S is unique. The inverse of a in S is denoted by a^{-1}. Let $U(S)$ denote the set of all invertible elements in S. Then $U(S)$ forms a group with respect to the multiplication of the semiring S.

A commutative semiring S is called an *incline* if $a + 1 = 1$ for all $a \in S$ (see [3,8]). If S is an incline and $a, b \in S$ such that $a + b = 0$, then $a = a + 0 = a + (a + b) = (a + a) + b = (1 + 1)a + b = a + b = 0$ and so $b = 0$. Therefore, any incline is a commutative antiring. Also, if S is an incline and $a \in U(S)$, then there exists an element b in S such that $ab = 1$, and so $a = a \cdot 1 = a(1 + b) = a + ab = a + 1 = 1$. Then $U(S) = \{1\}$.

Let now S be a commutative semiring. We denote by $M_{m\times n}(S)$ and $V_n(S)$ the set of all $m \times n$ matrices over S and the set of all column vectors of order n over S, respectively. Especially, we denote by $M_n(S)$ the set of all square matrices of order n over S. It is clear that $V_n(S) = M_{n\times 1}(S)$.

For $A \in M_{m\times n}(S)$, we denote by a_{ij} or A_{ij} the element of S corresponding to the (i, j)th entry of A and denote by A_{*j} the jth column of A. Also, we denote by A^{T} the transposed matrix of A. For $A \in M_n(S)$, if $a_{ij} = 0$ for all i and j provided that $i \neq j$ then A is called a *diagonal matrix* and denoted by $\operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn})$, in particular, if $a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ for all $i, j \in \underline{n}$, then A is called the *identity matrix* and denoted by I_n; if $a_{ij} = 0$ for all $i, j \in \underline{n}$, then A is called the *zero matrix* and denoted by O_n.

Likewise, for $\alpha \in V_n(S)$, we denote by α_i the element of S corresponding to the ith coordinate of α. If $\alpha_i = 1$ for all $i \in \underline{n}$ then α is called the *universal vector* of $V_n(S)$ and denoted by e; if $\alpha_i = 0$ for all $i \in n$, then α is called the *zero vector* and denoted by 0. For any $i \in \underline{n}$, we denote by e_i the vector in $V_n(S)$ with 1 as the ith coordinate, 0 otherwise.

Given $A, B \in M_{m\times n}(S)$ and $C \in M_{n\times l}(S)$, we define:

$$A + B = (a_{ij} + b_{ij})_{m\times n};$$
$$AC = \left(\sum_{k\in\underline{n}} a_{ik}c_{kj}\right)_{m\times l};$$
$$\lambda A = (\lambda a_{ij})_{m\times n}, \quad \lambda \in S.$$

The following properties are derived immediately from these definitions.

(1) $M_n(S)$ is an Abelian monoid with the identity element O_n with respect to the matrix addition;
(2) $M_n(S)$ is another monoid with the identity element I_n with respect to the matrix multiplication;
(3) The distributivity holds, i.e., $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$ for any $A, B, C \in M_n(S)$.
(4) The absorption property holds, i.e., $O_nA = AO_n = O_n$ for all $A \in M_n(S)$.

Therefore, $(M_n(S), +, \cdot, O_n, I_n)$ is a semiring.

For $A \in M_n(S)$, the *powers* of A are defined as follows: $A^0 = I_n$, $A^l = A^{l-1} \cdot A$, where l is any positive integer. The (i, j)th entry of A^l is denoted by $a_{ij}^{(l)}$. A matrix A in $M_n(S)$ is said to be *right invertible* (*left invertible*) in $M_n(S)$ if $AB = I_n(BA = I_n)$ for some $B \in M_n(S)$. The matrix B is called a *right inverse* (*left inverse*) of A in $M_n(S)$. If A is both right and left invertible in $M_n(S)$ then it is called *invertible* in $M_n(S)$. Obviously, if A is invertible then its right inverse coincides with its left inverse which is called its *inverse*. The inverse of A is denoted by A^{-1}. Clearly, any permutation matrix P is invertible and its inverse is P^{T}. Let $GL_n(S)$ denote the set of all invertible matrices in $M_n(S)$. Then $GL_n(S)$ forms a group with respect to the multiplication of $M_n(S)$.

Definition 2.1. Let $A \in M_{m\times n}(S)$ and $m \leqslant n$. Then the *permanent* per A of A is defined by

$$\operatorname{per} A = \sum_{\sigma \in S_{m,n}} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{m\sigma(m)},$$

where $S_{m,n}$ is the set of all injective mappings from the set $\underline{m}$ to the set $\underline{n}$.

For $A \in M_n(S)$, we denote by $A(i_1,\ldots,i_r|\, j_1,\ldots,j_r)$ the $(n-r)\times(n-r)$ submatrix of A obtained from A by deleting rows $i_1,\ldots,i_r$ and columns $j_1,\ldots,j_r$, where $\{i_1,i_2,\ldots,i_r\}$ and $\{j_1,j_2,\ldots,j_r\} \subseteq \underline{n}$ with $i_s \neq i_t (s\neq t)$ and $j_s \neq j_t (s \neq t)$.

Definition 2.2. Let $A \in M_n(S)$. The *adjoint matrix* adj $A \in M_n(S)$ of A is the matrix whose (i, j)th entry is per $A(j|i)$ for all $i, j \in \underline{n}$.

The following lemmas are used.

Lemma 2.1 [18]. *Let $A, B \in M_n(S)$. If $AB = I_n$ then $BA = I_n$.*

Lemma 2.2

(1) *Let $a_i \in S$, $i = 1, 2, \ldots, n$. Then the matrix diag$(a_1, a_2, \ldots, a_n)$ is invertible in $M_n(S)$ iff a_i is invertible in S for each $i \in \underline{n}$.*
(2) *The inverse of any invertible diagonal matrix in $M_n(S)$ is a diagonal matrix.*

The proof is omitted.

Lemma 2.3. *Let $A \in M_n(S)$. Then*

(1) per $A = \sum_{j\in\underline{n}} a_{ij}$per $A(i|j)$ *for any* $i \in \underline{n}$;
(2) per $A = \sum_{i\in\underline{n}} a_{ij}$per $A(i|j)$ *for any* $j \in \underline{n}$.

The proof is omitted.

Lemma 2.4 ([10], Lemma 2.3). *If S is an incline and $I(S)$ is the set of all idempotent elements in S, then $I(S)$ is a distributive lattice.*

Lemma 2.5 ([7], Corollary 1.1). *If S is a distributive lattice and $A \in M_n(S)$, then there exist positive integers k and d such that $A^{k+d} = A^k$.*

3. Some conditions for invertible matrices over antirings

In this section, we give the complete description for the invertible matrices over an antiring and obtain some necessary and sufficient conditions for the matrices to be invertible. In this section and Section 4, S is always supposed to be a commutative antiring.

Proposition 3.1 ([8], Proposition 19.4). *Let $A, B \in M_n(S)$. If $AB = I_n$, then*

(1) $a_{ij}a_{ik} = a_{ji}a_{ki} = b_{ij}b_{ik} = b_{ji}b_{ki} = 0$ *for any* $i, j, k \in \underline{n}$ *with* $j \neq k$;
(2) $a_{ik}b_{kj} = a_{ki}b_{jk} = 0$ *for any* $i, j, k \in \underline{n}$ *with* $i \neq j$;
(3) $\left(\sum_{k\in\underline{n}} a_{ik}\right)\left(\sum_{l\in\underline{n}} b_{li}\right) = \left(\sum_{k\in\underline{n}} a_{kj}\right)\left(\sum_{l\in\underline{n}} b_{jl}\right) = 1$ *for any* $i, j \in \underline{n}$.

Proposition 3.2. *Let $A \in M_n(S)$. If A is invertible in $M_n(S)$, then AA^{T} and $A^{\mathrm{T}}A$ are invertible diagonal matrices.*

Proof. Suppose that A is invertible in $M_n(S)$. Then there exists a matrix B in $M_n(S)$ such that $AB = I_n$. By Proposition 3.1(3), we have $\left(\sum_{k\in\underline{n}} a_{ik}\right)\left(\sum_{l\in\underline{n}} b_{li}\right) = 1$ for all $i \in \underline{n}$, and so $\sum_{k\in\underline{n}} a_{ik} \in U(S)$ for all $i \in \underline{n}$. Let $u_i = \sum_{k\in\underline{n}} a_{ik}$ for each $i \in \underline{n}$. Then $u_i^2 = \left(\sum_{k\in\underline{n}} a_{ik}\right)^2 = \sum_{k\in\underline{n}} a_{ik}^2 + \sum_{1\leqslant k<l\leqslant n} 2a_{ik}a_{il} = \sum_{k\in\underline{n}} a_{ik}^2$ (by Proposition 3.1(1)) and $u_i^2 \in U(S)$. By Proposition 3.1(1), we have that for any i and j in $\underline{n}$, $(AA^{\mathrm{T}})_{ij} = \sum_{k\in\underline{n}} a_{ik}a_{jk} = \begin{cases} u_i^2 & i = j \\ 0 & i \neq j \end{cases}$. By Lemma 2.2, $AA^{\mathrm{T}} = \operatorname{diag}(u_1^2, u_2^2, \ldots, u_n^2)$ is an invertible matrix in $M_n(S)$.

Similarly, we can prove that $A^{\mathrm{T}}A$ is an invertible diagonal matrix in $M_n(S)$.

This completes the proof. □

Remark 3.1. The diagonal matrices AA^{T} and $A^{\mathrm{T}}A$ in Proposition 3.2 need not to be equal in general. For example, consider the matrix $A = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ over the antiring R^+. Then, it is clear that A is invertible in $M_2(R^+)$ and $AA^{\mathrm{T}} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$. But $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$.

By Proposition 3.2, we have:

Corollary 3.1. *If S satisfies $U(S) = \{1\}$ and $A \in GL_n(S)$, then $AA^{\mathrm{T}} = A^{\mathrm{T}}A = I_n$.*

Remark 3.2. Corollary 3.1 generalizes Theorem 3.2 in Han and Li [11] and Theorem 6 of Give'on [7] and Theorem 4.2 in Luce [14].

Proposition 3.3. *Let $A \in M_n(S)$. If A is right invertible in $M_n(S)$, then $A^{[n]}$ is an invertible diagonal matrix in $M_n(S)$, where $[n]$ denotes the least common multiple of the integers $1, 2, \ldots, n$.*

Proof. Let $A \in M_n(S)$ be right invertible. Then, by Lemma 2.1, A is invertible in $M_n(S)$, and so $A^{[n]}$ is invertible in $M_n(S)$. In the following we will prove that $A^{[n]}$ is a diagonal matrix.

Let $w = [n]$. If $n = 2$, then $w = [2] = 2$ and

$$\begin{aligned} A^w = A^2 &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^2 \\ &= \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{21}a_{22} & a_{21}a_{12} + a_{22}^2 \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^2 + a_{12}a_{21} & 0 \\ 0 & a_{21}a_{12} + a_{22}^2 \end{bmatrix} \quad \text{(by Proposition 3.1(1)).} \end{aligned}$$

Therefore A^2 is a diagonal matrix.

We now assume $n \geqslant 3$.

For any $i, j \in \underline{n}$. If $a_{ij}^{(w)} \neq 0$, then there exist $i_1, i_2, \ldots, i_{w-1} \in \underline{n}$ such that $a_{ii_1}a_{i_1i_2}\cdots a_{i_{w-1}j} \neq 0$, where $w = [n]$. In the following we shall prove $i = j$.

We prove it in four steps.

(1) If $i_s = i_t$ for some s, t with $0 \leqslant s < t < w$ (taking $i = i_0$, $j = i_w$), then $i_{s+1} = i_{t+1}$. In fact, if $i_{s+1} \neq i_{t+1}$, then $a_{i_si_{s+1}}a_{i_ti_{t+1}} = a_{i_si_{s+1}}a_{i_si_{t+1}} = 0$ (by Proposition 3.1(1)), and so $a_{ii_1}a_{i_1i_2}\cdots a_{i_{w-1}j} = a_{i_0i_1}\cdots a_{i_si_{s+1}}\cdots a_{i_ti_{t+1}}\cdots a_{i_{w-1}i_w} = 0$. This is a contradiction.
(2) If $i_s = i_t$ for some s, t with $0 < s < t \leqslant w$, then $i_{s-1} = i_{t-1}$. In fact, if $i_{s-1} \neq i_{t-1}$, then $a_{i_{s-1}i_s}a_{i_{t-1}i_t} = a_{i_{s-1}i_s}a_{i_{t-1}i_s} = 0$ (by Proposition 3.1(1)), and so $a_{ii_1}a_{i_1i_2}\cdots a_{i_{w-1}j} = a_{i_0i_1}\cdots a_{i_{s-1}i_s}\cdots a_{i_{t-1}i_t}\cdots a_{i_{w-1}i_w} = 0$. This is a contradiction.
(3) There exists a $d \in \underline{n}$ such that $i_0 = i_d$. In fact, since $i_0, i_1, \ldots, i_n \in \underline{n}$ (note that $w > n$), there exist some u, v in $\{0, 1, \ldots, n\}$ such that $i_u = i_v$ with $u < v$. If $u = 0$, then $i_0 = i_v$ and in this case, our statement is proved. If $0 < u$, then, by (2), we have $i_{u-1} = i_{v-1}$, and if $0 < u - 1$, then, again, we have $i_{u-2} = i_{v-2}$ (by (2)). Repeating this argument, we have $i_0 = i_{v-u}$. Taking $d = v - u$, we have $d \in \underline{n}$ and $i_0 = i_d$.
(4) Since $1 \leqslant d \leqslant n$, we have $d|w$. Let $w = gd$, where g is a positive integer. Then, by (1) and (3), we have $i = i_0 = i_d = i_{2d} = \cdots = i_{gd} = i_w = j$.

Consequently, we have that $a_{ij}^{(w)} = 0$ for all i, j in $\underline{n}$ with $i \neq j$. Thus, A^w is a diagonal matrix. This completes the proof. □

By Proposition 3.3, we have

Corollary 3.2. *If S satisfies $U(S) = \{1\}$ and A is a right invertible matrix in $M_n(S)$, then $A^{[n]} = I_n$.*

Remark 3.3. Corollary 3.2 generalizes Theorem 3.1 of Han and Li [11] and Theorem 7 in Give'on [7] and Proposition 5 of Skornyakov [20].

Let $A \in M_n(S)$. The mapping $f_A : V_n(S) \to V_n(S)$ is defined by $f_A(x) = Ax$ for $x \in V_n(S)$.

Proposition 3.4. *If $A \in M_n(S)$ and f_A is a surjective mapping, then A is right invertible in $M_n(S)$.*

Proof. Since f_A is a surjective mapping, there exist column vectors $(x_{1i}, x_{2i}, \ldots, x_{ni})^{\mathrm{T}} \in V_n(S)$ ($i \in \underline{n}$) such that

$$A\begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, A\begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, A\begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Put $X = (x_{ij})$. Then $X \in M_n(S)$ and $AX = I_n$, i.e., A is right invertible. □

Proposition 3.5. *Let $A \in M_n(S)$. If f_A is an injective mapping and for any $i, j \in \underline{n}$, $\sum_{s\in\underline{n}} a_{is}$, $\sum_{t\in\underline{n}} a_{tj} \in U(S)$, and $a_{ij}\left(\sum_{s\in\underline{n}} a_{is}\right) = a_{ij}\left(\sum_{t\in\underline{n}} a_{tj}\right) = a_{ij}^2$, then A is invertible in $M_n(S)$.*

Proof. For any $i, j \in \underline{n}$, take $b_{ij} = \left(\sum_{s\in\underline{n}} a_{is}\right)^{-1} a_{ij}$. Then $\sum_{s\in\underline{n}} b_{is} = 1$ and $b_{ij}^2 = b_{ij}$. Let $d_i = \sum_{s\in\underline{n}} a_{is}$ for $1 \leqslant i \leqslant n$ and $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ and $B = (b_{ij})$. Then D is invertible in $M_n(S)$ (by Lemma 2.2(1)) and $B = D^{-1}A$.

To show A is invertible in $M_n(S)$, it is sufficient to show that B is invertible in $M_n(S)$. We shall prove it in four steps.

(1) *f_B is an injective mapping*. In fact, suppose that $f_B(x) = f_B(y)$ for some $x, y \in V_n(S)$. Then $Bx = By$, i.e., $D^{-1}Ax = D^{-1}Ay$, and so $Ax = Ay$, i.e., $f_A(x) = f_A(y)$. Since f_A is an injective mapping, we have $x = y$. Therefore, f_B is an injective mapping.

(2) *For any $j \in \underline{n}$, we have*

$$\prod_{t\in\underline{n}} b_{tj} = 0. \tag{3.1}$$

Assume that $\prod_{t\in\underline{n}} b_{tj_0} \neq 0$ for some $j_0 \in \underline{n}$. Put $x = \prod_{t\in\underline{n}} b_{tj_0} e_{j_0}$, $y = \prod_{t\in\underline{n}} b_{tj_0} e$. Then

$$\begin{aligned}
Bx &= (b_{1j_0}^2 b_{2j_0} \cdots b_{nj_0}, b_{1j_0} b_{2j_0}^2 \cdots b_{nj_0}, \dots, b_{1j_0} b_{2j_0} \cdots b_{nj_0}^2)^{\mathrm{T}} \\
&= \left(\prod_{t\in\underline{n}} b_{tj_0}, \prod_{t\in\underline{n}} b_{tj_0}, \dots, \prod_{t\in\underline{n}} b_{tj_0}\right)^{\mathrm{T}} \quad (\text{because } b_{ij}^2 = b_{ij} \text{ for all } i, j \in \underline{n}) \\
&= \prod_{t\in\underline{n}} b_{tj_0} e = y
\end{aligned}$$

and

$$\begin{aligned}
By &= \prod_{t\in\underline{n}} b_{tj_0} Be \\
&= \prod_{t\in\underline{n}} b_{tj_0} \left(\sum_{s\in\underline{n}} b_{1s}, \sum_{s\in\underline{n}} b_{2s}, \dots, \sum_{s\in\underline{n}} b_{ns}\right)^{\mathrm{T}} \\
&= \prod_{t\in\underline{n}} b_{tj_0} (1, 1, \dots, 1)^{\mathrm{T}} \quad \left(\text{since } \sum_{s\in\underline{n}} b_{is} = 1 \text{ for all } i \in \underline{n}\right) \\
&= \prod_{t\in\underline{n}} b_{tj_0} e = y.
\end{aligned}$$

Thus $f_B(x) = Bx = By = f_B(y)$. But $x \neq y$. This contradicts the fact that f_B is an injective mapping.

(3) *For any $i, j, p \in \underline{n}$ with $i \neq j$, we have*

$$b_{ip} b_{jp} = 0 \tag{3.2}$$

Assume that $b_{i_0 p_0} b_{j_0 p_0} \neq 0$ for some $i_0, j_0, p_0 \in \underline{n}$ with $i_0 \neq j_0$.

Let M be a maximal subset of the set $\underline{n} \times \underline{n}$ such that $(i_0, p_0), (j_0, p_0) \in M$ and $\Delta = \prod_{(s,t)\in M} b_{st} \neq 0$, where $\underline{n} \times \underline{n} = \{(i, j) : i \in \underline{n}, j \in \underline{n}\}$. Then $M \neq \phi$ since (i_0, p_0) and $(j_0, p_0) \in M$, and $M \neq \underline{n} \times \underline{n}$ since $\prod_{t\in\underline{n}} b_{tj} = 0$ for any $j \in \underline{n}$ (by (3.1)).

Furthermore, for any $(i, j) \in \underline{n} \times \underline{n}$, we have

$$b_{ij}\Delta = \begin{cases} \Delta & \text{if } (i, j) \in M, \\ 0 & \text{if } (i, j) \notin M. \end{cases} \tag{3.3}$$

In fact, if $(i, j) \in M$, then

$$\begin{aligned} b_{ij}\Delta &= \left(\prod_{(s,t)\in M} b_{st}\right) b_{ij} = \left(\prod_{\substack{(s,t)\in M \\ (s,t)\neq(i,j)}} b_{st}\right) b_{ij}^2 \\ &= \left(\prod_{\substack{(s,t)\in M \\ (s,t)\neq(i,j)}} b_{st}\right) b_{ij} \quad (\text{since } b_{ij}^2 = b_{ij}) \\ &= \prod_{(s,t)\in M} b_{st} = \Delta, \end{aligned}$$

if $(i, j) \notin M$, then, by the definitions of M and Δ, we have $b_{ij}\Delta = 0$. Thus, (3.3) holds.

In the following, we will prove that for any $i \in \underline{n}$, there exists a $j \in \underline{n}$ such that $(i, j) \in M$. In fact, since $\sum_{s\in\underline{n}} b_{is} = 1$ for each $i \in \underline{n}$, we have $\Delta = \Delta\left(\sum_{s\in\underline{n}} b_{is}\right) = \sum_{s\in\underline{n}}(b_{is}\Delta) \neq 0$, and so $b_{ij}\Delta \neq 0$ for some $j \in \underline{n}$. By (3.3), we have $b_{ij}\Delta = \Delta$ and $(i, j) \in M$.

For each $i \in \underline{n}$, let $M(i) = \{s \in \underline{n} : (i, s) \in M\}$. It is clear that $M(i) \neq \phi$. Let $k_i = |M(i)|$. Then $1 \leqslant k_i \leqslant n$ and $\Delta = \Delta \cdot 1 = \Delta\left(\sum_{s\in\underline{n}} b_{is}\right) = \sum_{s\in\underline{n}} \Delta b_{is} = \sum_{(i,s)\in M} \Delta(\text{by}(3.3)) = \sum_{s\in M(i)} \Delta = k_i\Delta$. That is, $\Delta = k_i\Delta$ for each $i \in \underline{n}$.

In the following we shall prove that $k_1 = k_2 = \cdots = k_n = 1$.

To do this, we first prove that for any given $j \in \underline{n}$, there exists an $i \in \underline{n}$ such that $b_{ij}\Delta = 0$. In fact, if there exists some $j \in \underline{n}$ such that $b_{ij}\Delta \neq 0$ for all $i \in \underline{n}$, then $b_{ij}\Delta = \Delta$ for all $i \in \underline{n}$ (by (3.3)), and so $\prod_{t\in\underline{n}} b_{tj}\Delta = \Delta$. But $\prod_{t\in\underline{n}} b_{tj} = 0$ (by (3.1)), we have $\Delta = 0$. This contradicts the fact that $\Delta \neq 0$.

Suppose that there exists an $i' \in \underline{n}$ such that $k_{i'} \geqslant 2$. Taking $k = \max\{k_1, k_2, \ldots, k_n\}$, we have that $k \geqslant 2$ and $\Delta = k\Delta$ (since $\Delta = k_i\Delta$ for every k_i). Let $H = \{i \in \underline{n}, k_i \geqslant 2\}$. Then $|H| \geqslant 1$ since $i' \in H$.

We now choose an s_0 in $\underline{n}$ such that $b_{ts_0}\Delta = 0$ for all $t \in \underline{n}\backslash H$ (note that if $H = \underline{n}$ then s_0 may be chosen arbitrarily in $\underline{n}$), and let $x = \Delta\left(\sum_{\substack{s\in\underline{n} \\ s\neq s_0}} e_s\right) + (k-1)\Delta e_{s_0}$, $y = \Delta\left(\sum_{\substack{s\in\underline{n} \\ s\neq s_0}} e_s\right)$, $u = Bx$ and $v = By$.

For any $i \in \underline{n}$, if $b_{is_0}\Delta = 0$, then

$$\begin{aligned} u_i = (Bx)_i &= \sum_{s\in\underline{n}} b_{is}x_s \\ &= \left(\sum_{\substack{s\in\underline{n} \\ s\neq s_0}} b_{is}\right)\Delta + b_{is_0}(k-1)\Delta \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{\substack{s\in\underline{n}\\ s\neq s_0}} b_{is}\right)\Delta \quad (\text{since } b_{is_0}\Delta = 0)\\
&= (By)_i = v_i.
\end{aligned}
$$

If $b_{is_0}\Delta \neq 0$, then $i \in H$ (since $b_{ts_0}\Delta = 0$ for all $t \in \underline{n}\backslash H$) and $(i, s_0) \in M$ (or $s_0 \in M(i)$), and so $b_{is_0}\Delta = \Delta$ (by (3.3)). Then

$$
\begin{aligned}
u_i = (Bx)_i &= \sum_{s\in\underline{n}} b_{is}x_s\\
&= \left(\sum_{\substack{s\in\underline{n}\\ s\neq s_0}} b_{is}\Delta\right) + (k-1)\Delta = \left(\sum_{\substack{(i,s)\in M\\ s\neq s_0}} b_{is}\Delta\right) + (k-1)\Delta\\
&= \left(\sum_{\substack{(i,s)\in M\\ s\neq s_0}} \Delta\right) + (k-1)\Delta = \left(\sum_{\substack{s\in M(i)\\ s\neq s_0}} \Delta\right) + (k-1)\Delta\\
&= (k_i-1)\Delta + (k-1)\Delta \quad (\text{since } s_0 \in M(i)).
\end{aligned}
$$

Since $i \in H$, we have $k_i \geqslant 2$. If $k_i = 2$ then $u_i = \Delta + (k-1)\Delta = (1 + (k-1))\Delta = k\Delta = \Delta$ (since $\Delta = k\Delta$) $= (k_i - 1)\Delta$ and if $k_i \geqslant 3$ then

$$
\begin{aligned}
u_i &= (k_i-1)\Delta + (k-1)\Delta\\
&= (k_i-2)\Delta + \Delta + (k-1)\Delta \quad (\text{since } k_i \geqslant 3)\\
&= (k_i-2)\Delta + k\Delta = (k_i-2)\Delta + \Delta \quad (\text{since } \Delta = k\Delta)\\
&= (k_i-1)\Delta.
\end{aligned}
$$

Since

$$
v_i = (By)_i = \sum_{s\in\underline{n}} b_{is}y_s = \left(\sum_{\substack{s\in\underline{n}\\ s\neq s_0}} b_{is}\right)\Delta = \sum_{\substack{(i,s)\in M\\ s\neq s_0}} b_{is}\Delta = \sum_{\substack{(i,s)\in M\\ s\neq s_0}} \Delta = (k_i-1)\Delta,
$$

we have $u_i = v_i$.

Consequently, we have $u = v$, i.e., $f_B(x) = f_B(y)$. But $x \neq y$. This contradicts the fact that f_B is an injective mapping. Therefore $k_1 = k_2 = \cdots = k_n = 1$.

Since $k_1 = k_2 = \cdots = k_n = 1$, we have that for each $i \in \underline{n}$, there exists a unique $j \in \underline{n}$ such that $(i, j) \in M$. Since $(i_0, p_0), (j_0, p_0) \in M$ and $i_0 \neq j_0$, there exists a $q_0 \in \underline{n}$ such that $(i, q_0) \notin M$ for each $i \in \underline{n}$, that is, $b_{iq_0}\Delta = 0$ for each $i \in \underline{n}$. Then $a_{iq_0}\Delta = d_i b_{iq_0}\Delta = 0$ for all $i \in \underline{n}$, and so

$$
\left(\sum_{t\in\underline{n}} a_{tq_0}\right)\Delta = \sum_{t\in\underline{n}} a_{tq_0}\Delta = 0.
$$

But $\sum_{t\in\underline{n}} a_{tq_0} \in U(S)$, we have $\Delta = 0$. This contradicts the fact that $\Delta \neq 0$. Thus $b_{ip}b_{jp} = 0$ for any $i, j, p \in \underline{n}$ with $i \neq j$.

(4) Now

$$
\begin{aligned}
BB^{\mathrm{T}} &= \left(\sum_{s\in\underline{n}} b_{is}b_{js}\right)_{n\times n} \\
&= \mathrm{diag}\left(\sum_{s\in\underline{n}} b_{1s}^2, \ldots, \sum_{s\in\underline{n}} b_{ns}^2\right) \quad \text{(by (3.2))} \\
&= \mathrm{diag}\left(\sum_{s\in\underline{n}} b_{1s}, \ldots, \sum_{s\in\underline{n}} b_{ns}\right) \quad (\text{because } b_{ij}^2 = b_{ij} \text{ for all } i, j \in \underline{n}) \\
&= \mathrm{diag}(1, 1, \ldots, 1) \quad \left(\text{because } \sum_{s\in\underline{n}} b_{is} = 1 \text{ for all } i \in \underline{n}\right) = I_n.
\end{aligned}
$$

Then B is invertible in $M_n(S)$. The proof is completed. □

Proposition 3.6. *If $A \in M_n(S)$ is right (left) invertible and $a_{11}, a_{22}, \ldots, a_{nn} \in U(S)$, then A is a diagonal matrix.*

Proof. We assume that A is right invertible in $M_n(S)$, i.e., $AB = I_n$ for some $B \in M_n(S)$. Then for any $i, j \in \underline{n}$

$$
a_{ii}b_{ij} + \sum_{\substack{k\in\underline{n}\\ k\neq i}} a_{ik}b_{kj} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}
$$

Consequently $a_{ii}b_{ij} = 0$ for $i \neq j$, and so $b_{ij} = 0$ for $i \neq j$ (since $a_{ii} \in U(S)$). Therefore B is a diagonal matrix. Since $AB = I_n$, B is an invertible diagonal matrix, and so $A = B^{-1}$ is an invertible diagonal matrix (by Lemma 2.2(2)). If A is left invertible, then, by Lemma 2.1, A is right invertible, according to what we have proved, A is a diagonal matrix. □

By Proposition 3.6, we have:

Corollary 3.3. *If $A \in M_n(S)$ is right (left) invertible with $a_{11} = a_{22} = \cdots = a_{nn} = 1$, then $A = I_n$.*

Remark 3.4. Corollary 3.3 generalizes Proposition 4 of Skornyakov [20].

Let $x, y \in V_n(S)$, the *scalar product* of x and y is defined as $x^{\mathrm{T}}y \in S$.

Theorem 3.1. *Let $A \in M_n(S)$. Then the following statements are equivalent:*

(1) *A is right invertible.*
(2) *A is left invertible.*
(3) *A is invertible.*
(4) *AA^{T} is an invertible diagonal matrix.*
(5) *$A^{\mathrm{T}}A$ is an invertible diagonal matrix.*
(6) *AA^{T} and $A^{\mathrm{T}}A$ are invertible diagonal matrices.*

(7) *A^d is an invertible diagonal matrix for some positive integer d.*
(8) *f_A is a surjective mapping.*
(9) *f_A is a bijective mapping.*
(10) *There exist $u_1, u_2, \ldots, u_n \in U(S)$ such that $(f_A(x))^{\mathrm{T}} f_A(y) = \sum_{i\in\underline{n}} u_i x_i y_i$ for any $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in V_n(S)$.*
(11) *f_A is an injective mapping and for any $i, j \in \underline{n}$, $\sum_{s\in\underline{n}} a_{is}$, $\sum_{t\in\underline{n}} a_{tj} \in U(S)$ and $a_{ij}\left(\sum_{s\in\underline{n}} a_{is}\right) = a_{ij}\left(\sum_{t\in\underline{n}} a_{tj}\right) = a_{ij}^2$.*
(12) *f_A is an injective mapping and $A^{k+d} = A^k \cdot D$ for some invertible diagonal matrix D and positive integers k and d.*

Proof. By Lemma 2.1 and Propositions 3.2 and 3.3, we have that statements (1)–(7) are equivalent. Also, the implications (3) $\Longrightarrow$ (9) and (9) $\Longrightarrow$ (8) are obvious.

(8) $\Longrightarrow$ (1). It is Proposition 3.4.

(7) $\Longrightarrow$ (12). It is obvious.

(12) $\Longrightarrow$ (7). Since f_A is an injective mapping, f_{A^k} is also an injective mapping. Since $A^k \cdot A^d = A^{k+d} = A^k \cdot D$ for some invertible diagonal matrix D and positive integers k and d, we have $A^k \cdot ((A^d)_{*j}) = A^k \cdot (D_{*j})$ for all $j \in \underline{n}$, i.e., $f_{A^k}((A^d)_{*j}) = f_{A^k}(D_{*j})$. Then $(A^d)_{*j} = D_{*j}$ for all $j \in \underline{n}$ (since f_A is an injective mapping), and so $A^d = D$.

(11) $\Longrightarrow$ (3). It is Proposition 3.5.

(3) $\Longrightarrow$ (11). By Proposition 3.1(1) and (3).

(5) $\Longrightarrow$ (10). Let $A^{\mathrm{T}}A = D$, where D is an invertible diagonal matrix. Then for any $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in V_n(S)$, we have $(f_A(x))^{\mathrm{T}} f_A(y) = (Ax)^{\mathrm{T}}(Ay) = x^{\mathrm{T}}(A^{\mathrm{T}}A)y = x^{\mathrm{T}}Dy$. Let $D = \mathrm{diag}(u_1, u_2, \ldots, u_n)$. Then $u_1, u_2, \ldots, u_n \in U(S)$ and $(f_A(x))^{\mathrm{T}} f_A(y) = \sum_{i\in\underline{n}} u_i x_i y_i$.

(10) $\Longrightarrow$ (5). Suppose that (10) holds. Then for any $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in V_n(S)$, we have $x^{\mathrm{T}}A^{\mathrm{T}}Ay = (Ax)^{\mathrm{T}}(Ay) = (f_A(x))^{\mathrm{T}} f_A(y) = \sum_{i\in\underline{n}} u_i x_i y_i$. Let $D = \mathrm{diag}(u_1, u_2, \ldots, u_n)$. Then D is an invertible diagonal matrix in $M_n(S)$ and $x^{\mathrm{T}}(A^{\mathrm{T}}A)y = x^{\mathrm{T}}Dy$. Therefore, $e_i^{\mathrm{T}}(A^{\mathrm{T}}A)e_j = e_i^{\mathrm{T}}De_j$ for any i and $j \in \underline{n}$, i.e.,

$$\sum_{s\in\underline{n}} a_{si}a_{sj} = \begin{cases} u_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence $A^{\mathrm{T}}A = D$ is an invertible diagonal matrix in $M_n(S)$. The proof is completed. □

Corollary 3.4 ([11], Theorem 3.4). *If S is an incline and $A \in M_n(S)$. Then the following statements are equivalent*:

(1) *A is right invertible.*
(2) *A is left invertible.*
(3) *A is invertible.*
(4) *$AA^{\mathrm{T}} = I_n$.*
(5) *$A^{\mathrm{T}}A = I_n$.*
(6) *$AA^{\mathrm{T}} = A^{\mathrm{T}}A = I_n$.*
(7) *$A^k = I_n$ for some positive integer k.*
(8) *f_A is a surjective mapping.*
(9) *f_A is a bijective mapping.*

(10) f_A *preserves the scalar products, that is, for any* $x, y \in V_n(S)$, $(f_A(x))^{\mathrm{T}} f_A(y) = x^{\mathrm{T}} y$.
(11) $A \in M_n(I(S))$ *and* f_A *is an injective mapping, where* $M_n(I(S))$ *denotes the set of all* $n \times n$ *matrices over the set* $I(S)$.
(12) f_A *is an injective mapping and* $A^{k+d} = A^k$ *for some integers* k *and* d.

Proof. Since S is an incline, we have $U(S) = \{1\}$ and so any invertible diagonal matrix in $M_n(S)$ is the identity matrix I_n. Therefore, by Theorem 3.1, the statements (1)–(10) and (12) are equivalent.

(3) $\Longrightarrow$ (11). If (3) holds, then by Theorem 3.1 and the condition $U(S) = \{1\}$, we have that for any $i \in \underline{n}$, $\sum_{s\in\underline{n}} a_{is} = \sum_{t\in\underline{n}} a_{tj} = 1$ and that for any $i, j \in \underline{n}$, $a_{ij}^2 = a_{ij}$, and so $A \in M_n(I(S))$. It is clear that f_A is an injective mapping. Thus the statement (11) holds.

(11) $\Longrightarrow$ (12). Since $I(S)$ is a distributive lattice (by Lemma 2.4) and $A \in M_n(I(S))$, it follows from Lemma 2.5 that $A^{k+d} = A^k$ for some positive integers k and d. Since f_A is an injective mapping, (12) holds. This completes the proof. $\square$

A set $\{a_1, a_2, \ldots, a_m\}$ of elements in the antiring S is called a *decomposition* of 1 in S if $\sum_{i\in\underline{m}} a_i = 1$; the set $\{a_1, a_2, \ldots, a_m\}$ is called *orthogonal* if $a_i a_j = 0$ holds for any i and j provided that $i \neq j$; a set of elements in S is called an *orthogonal decomposition* of 1 in S if it is orthogonal and a decomposition of 1 in S.

Let $\{A_1, A_2, \ldots, A_m\} \subseteq M_n(S)$. A matrix $A \in M_n(S)$ is called an *orthogonal combination* of the matrices $A_1, A_2, \ldots, A_m$ if there exists an orthogonal decomposition of 1 in S, $\{a_1, a_2, \ldots, a_m\}$, such that $A = \sum_{i\in\underline{m}} a_i A_i$.

Proposition 3.7. *If* S *satisfies* $U(S) = \{1\}$ *and* $A \in M_n(S)$, *then the following statements are equivalent*:

(1) A *is invertible*.
(2) A *is an orthogonal combination of some permutation matrices of order* n.

Proof. (1) $\Longrightarrow$ (2). If A is invertible in $M_n(S)$, then, by Theorem 3.1, we have $A^{\mathrm{T}}A = AA^{\mathrm{T}} = I_n$ (since $U(S) = \{1\}$). Therefore $a_{ij}a_{ik} = a_{ji}a_{ki} = 0$ for all i, j, k in $\underline{n}$ with $j \neq k$ and $\sum_{s\in\underline{n}} a_{is}^2 = \sum_{t\in\underline{n}} a_{tj}^2 = 1$ for all $i, j \in \underline{n}$, and so $\left(\sum_{s\in\underline{n}} a_{is}\right)^2 = \sum_{s\in\underline{n}} a_{is}^2 + \sum_{s_1\neq s_2} a_{is_1}a_{is_2} = \sum_{s\in\underline{n}} a_{is}^2 = 1$ and $\left(\sum_{t\in\underline{n}} a_{tj}\right)^2 = \sum_{t\in\underline{n}} a_{tj}^2 + \sum_{t_1\neq t_2} a_{t_1 j}a_{t_2 j} = \sum_{t\in\underline{n}} a_{tj}^2 = 1$. Thus, $\sum_{s\in\underline{n}} a_{is} = 1$ and $\sum_{t\in\underline{n}} a_{tj} = 1$, that is, each row and each column of A is an orthogonal decomposition of 1 in S. Also, for any $i, j \in \underline{n}$, $a_{ij} = a_{ij}\left(\sum_{s\in\underline{n}} a_{is}\right) = a_{ij}^2 + \sum_{s\neq j} a_{is}a_{ij} = a_{ij}^2$, that is, each entry of A is idempotent.

Since $\sum_{s\in\underline{n}} a_{is} = 1$ for all $i \in \underline{n}$, we have $1 = \prod_{i\in\underline{n}}\left(\sum_{s\in\underline{n}} a_{is}\right) = \sum_{1\leqslant s_1,s_2,\ldots,s_n\leqslant n} a_{1s_1}a_{2s_2}\cdots a_{ns_n}$. If $s_i = s_j$ for some $i, j \in \underline{n}$ with $i \neq j$, then $a_{is_i} \cdot a_{js_j} = 0$, and so $a_{1s_1}a_{2s_2}\cdots a_{ns_n} = 0$. Therefore $1 = \sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$. Let $a_\sigma = a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$ for $\sigma \in S_n$ and $G = \{\sigma : \sigma \in S_n, a_\sigma \neq 0\}$. Then, it is clear that $\sum_{\sigma\in G} a_\sigma = 1$ and $a_\sigma a_\tau$ for any $\sigma, \tau \in G$ with $\sigma \neq \tau$, and so the set $\{a_\sigma : \sigma \in G\}$ is an orthogonal decomposition of 1 in S. Also, for any $\sigma \in G$

$$a_\sigma a_{ij} = \begin{cases} a_\sigma & \text{if } j = \sigma(i) \\ 0 & \text{if } j \neq \sigma(i) \end{cases} \quad i, j \in \underline{n}.$$

Therefore, there exists a unique (0,1) matrix $P_\sigma \in M_n(S)$ such that $a_\sigma A = a_\sigma P_\sigma$ holds. It follows that $a_\sigma^2 P_\sigma P_\sigma^{\mathrm{T}} = (a_\sigma P_\sigma)(a_\sigma P_\sigma)^{\mathrm{T}} = (a_\sigma A)(a_\sigma A)^{\mathrm{T}} = a_\sigma^2 AA^{\mathrm{T}} = a_\sigma^2 I_n$, and so $P_\sigma P_\sigma^{\mathrm{T}} = I_n$. Then P_σ is a permutation matrix for each $\sigma \in G$. Clearly,

$$\sum_{\sigma\in G} a_\sigma P_\sigma = \sum_{\sigma\in G} a_\sigma A = \left(\sum_{\sigma\in G} a_\sigma\right) A = A$$

i.e., A is an orthogonal combination of some permutation matrices.

(2) $\Longrightarrow$ (1). If A is such combination, say

$$A = \sum_{s\in \underline{m}} a_s P_s,$$

then

$$AA^{\mathrm{T}} = \left(\sum_{s\in\underline{m}} a_s P_s\right)\left(\sum_{t\in\underline{m}} a_t P_t\right)^{\mathrm{T}} = \sum_{s\in\underline{m}}\sum_{t\in\underline{m}} a_s a_t P_s P_t^{\mathrm{T}} = \sum_{s\in\underline{m}} a_s^2 I_n$$
$$= \sum_{s\in\underline{m}}\sum_{t\in\underline{m}} a_s a_t I_n \quad (\text{since } a_s a_t = 0 \text{ with } s \neq t) = \left(\sum_{s\in\underline{m}} a_s\right)^2 I_n = I_n.$$

Thus A is invertible in $M_n(S)$. This completes the proof. □

Remark 3.5. Proposition 3.7 generalizes Theorem 8 in Give'on [7].

4. Cramer's rule over commutative antirings

In this section, we will present Cramer's rule for a matrix equation over a commutative antiring S.

The following lemmas are used.

Lemma 4.1. *Let $A \in M_{m\times n}(S)(m \leqslant n)$, and $U \in GL_m(S)$ and $V \in GL_n(S)$. Then*

$$\mathrm{per}(UAV) = \mathrm{per}\, U\, \mathrm{per}\, A\, \mathrm{per}\, V.$$

Proof. We first prove that $\mathrm{per}(UA) = \mathrm{per}\, U \mathrm{perA}$ for any $A \in M_{m\times n}(S)$ and U in $GL_m(S)$.

Let $W = UA$. Then for any $i \in \underline{m}$ and $j \in \underline{n}$, $w_{ij} = \sum_{k=1}^m u_{ik}a_{kj}$. Thus

$$\mathrm{per}(UA) = \mathrm{per}\, W = \sum_{\sigma\in S_{m,n}} \prod_{i\in\underline{m}} w_{i\sigma(i)}$$
$$= \sum_{\sigma\in S_{m,n}} \prod_{i\in\underline{m}} \left(\sum_{k\in\underline{m}} u_{ik}a_{k\sigma(i)}\right)$$
$$= \sum_{\sigma\in S_{m,n}} \sum_{1\leqslant k_1,k_2,\ldots,k_m\leqslant m} u_{1k_1}a_{k_1\sigma(1)}u_{2k_2}a_{k_2\sigma(2)}\cdots u_{mk_m}a_{k_m\sigma(m)}$$
$$= \sum_{\sigma\in S_{m,n}} \sum_{1\leqslant k_1,k_2,\ldots,k_m\leqslant m} (u_{1k_1}u_{2k_2}\cdots u_{mk_m})(a_{k_1\sigma(1)}a_{k_2\sigma(2)}\cdots a_{k_m\sigma(m)}).$$

If $k_s = k_t$ for some $s, t \in \underline{m}$ with $s \neq t$, then $u_{sk_s}u_{tk_t} = u_{sk_s}u_{tk_s} = 0$ (by Proposition 3.1(1)) and so

$$(u_{1k_1}u_{2k_2}\cdots u_{mk_m})(a_{k_1\sigma(1)}a_{k_2\sigma(2)}\cdots a_{k_m\sigma(m)}) = 0.$$

Therefore

$$\begin{aligned}
\operatorname{per}(UA) &= \sum_{\sigma\in S_{m,n}} \sum_{\substack{1\leqslant k_1,k_2,\ldots,k_m\leqslant m\\ k_s\neq k_t\,(s\neq t)}} (u_{1k_1}u_{2k_2}\cdots u_{mk_m})(a_{k_1\sigma(1)}a_{k_2\sigma(2)}\cdots a_{k_m\sigma(m)})\\
&= \sum_{\sigma\in S_{m,n}} \sum_{\rho\in S_m} (u_{1\rho(1)}u_{2\rho(2)}\cdots u_{m\rho(m)})(a_{\rho(1)\sigma(1)}a_{\rho(2)\sigma(2)}\cdots a_{\rho(m)\sigma(m)})\\
&= \sum_{\rho\in S_m} u_{1\rho(1)}u_{2\rho(2)}\cdots u_{m\rho(m)} \left(\sum_{\sigma\in S_{m,n}} a_{\rho(1)\sigma(1)}a_{\rho(2)\sigma(2)}\cdots a_{\rho(m)\sigma(m)}\right)\\
&= \sum_{\rho\in S_m} u_{1\rho(1)}\cdots u_{m\rho(m)} \left(\sum_{\sigma\in S_{m,n}} a_{\rho(1),\sigma(\rho^{-1}(\rho(1)))}\cdots a_{\rho(m),\sigma(\rho^{-1}(\rho(m)))}\right)\\
&= \sum_{\rho\in S_m} u_{1\rho(1)}\cdots u_{m\rho(m)} \left(\sum_{\sigma\in S_{m,n}} a_{1\sigma(\rho^{-1}(1))}\cdots a_{m\sigma(\rho^{-1}(1))}\right)\\
&\quad \text{(because } \rho \text{ is a bijection of the set } \underline{m}\text{)}\\
&= \sum_{\rho\in S_m} u_{1\rho(1)}\cdots u_{m\rho(m)} \left(\sum_{\sigma\rho^{-1}\in S_{m,n}} a_{1(\sigma\rho^{-1})(1)}\cdots a_{m(\sigma\rho^{-1})(m)}\right)\\
&= \sum_{\rho\in S_m} u_{1\rho(1)}\cdots u_{m\rho(m)} \left(\sum_{\sigma\in S_{m,n}} a_{1\sigma(1)}\cdots a_{m\sigma(m)}\right) = \operatorname{per} U \operatorname{per} A.
\end{aligned}$$

Similarly, we can prove that $\operatorname{per}(AV) = \operatorname{per} A \operatorname{per} V$ for any $A \in M_{m\times n}(S)$ and $V \in GL_n(S)$. Therefore, $\operatorname{per}(UAV) = \operatorname{per} U \operatorname{per}(AV) = \operatorname{per} U \operatorname{per} A \operatorname{per} V$.

This completes the proof. □

Corollary 4.1. *If* $A \in GL_n(S)$, *then* $\operatorname{per} A \in U(S)$.

Proof. If $A \in GL_n(S)$, then there exists a $B \in M_n(S)$ such that $AB = I_n$. By Lemma 4.1, we have $1 = \operatorname{per} I_n = \operatorname{per}(AB) = \operatorname{per} A \operatorname{per} B$, and so $\operatorname{per} A \in U(S)$. □

By Lemma 4.1 and Corollary 4.1, we have

Corollary 4.2. *If* S *satisfies* $U(S) = \{1\}$, *then*

(1) *for any* $A \in GL_n(S)$, *we have* $\operatorname{per} A = 1$;
(2) *for any* $A \in M_{m\times n}(s) (m \leqslant n)$, *and* $U \in GL_m(S)$ *and* $V \in GL_n(S)$, *we have*

$$\operatorname{per}(UAV) = \operatorname{per} A.$$

Remark 4.1. Corollary 4.2(1) generalizes Theorem 4.1 of Han and Li [11] and Lemma 2.10(1) in Tan [21], Corollary 4.2(2) generalizes Lemma 4.2 of Han and Li [11].

Lemma 4.2. *Let* $A \in GL_n(S)$. *Then* $A^{-1} = (\text{per } A)^{-1}\text{adj}A$.

Proof. Let $A \in GL_n(S)$. Then $a_{ik}a_{il} = 0$ for any $i, k, l \in \underline{n}$ with $k \neq l$ (by Proposition 3.1(1)). Let $B = A\text{adj } A$. Then for any $i, j \in \underline{n}$, we have

$$b_{ij} = \sum_{k\in\underline{n}} a_{ik} \operatorname{per}(j|k).$$

If $i = j$, then $b_{ii} = \sum_{k\in\underline{n}} a_{ik} \operatorname{per} A(i|k) = \operatorname{per} A$ (by Lemma 2.3); if $i \neq j$, then

$$\begin{aligned} b_{ij} &= \sum_{k\in\underline{n}} a_{ik}\left(\sum_{\substack{l\in\underline{n}\\ l\neq k}} a_{il}\operatorname{per} A(ij|kl)\right) \quad \text{(by Lemma 2.3)}\\ &= \sum_{k\in\underline{n}}\sum_{\substack{l\in\underline{n}\\ l\neq k}} a_{ik}a_{il} \operatorname{per} A(ij|kl) = 0 \quad \text{(by Proposition 3.1(1))} \end{aligned}$$

Therefore, $B = A\text{adj } A = (\text{per } A)I_n$. Since $\text{per } A \in U(S)$ (by Corollary 4.1), $A^{-1} = (\text{per } A)^{-1}\text{adj } A$. The proof is completed. □

Corollary 4.3. *If S satisfies* $U(S) = \{1\}$ *and* $A \in GL_n(S)$, *then* $\text{adj } A = A^{\mathrm{T}}$.

Proof. By Corollary 3.4 and Corollary 4.2(1) and Lemma 4.2, we have $A^{\mathrm{T}} = A^{-1} = \text{adj } A$. □

Remark 4.2. Corollary 4.3 generalizes Theorem 4.2 in Han and Li [11] and Lemma 2.10(2) in Tan [21].

The following theorem is Cramer's rule for a matrix equation over a commutative antiring.

Theorem 4.1. *Let* $A \in M_n(S)$ *and* $b = (b_1, b_2, \ldots, b_n)^{\mathrm{T}} \in V_n(S)$. *If A is invertible in* $M_n(S)$, *then the matrix equation* $Ax = b$ *has a unique solution* $x = (d^{-1}d_1, d^{-1}d_2, \ldots, d^{-1}d_n)^{\mathrm{T}} \in V_n(S)$, *where* $d = \text{per } A$ *and*

$$d_j = \operatorname{per}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}, \quad j = 1, 2, \ldots, n.$$

Proof. It is clear that the equation $Ax = b$ has a solution $x = A^{-1}b$. Let $y \in V_n(S)$ be any solution of this equation. Then $Ay = b$, and so $y = I_n y = (A^{-1}A)y = A^{-1}(Ay) = A^{-1}b$, which means that the equation $Ax = b$ has a unique solution.

Let now $(\text{adj } A)b = (d_1, d_2, \ldots, d_n)^{\mathrm{T}}$. Then for any j in $\underline{n}$, we have $d_j = \sum_{i\in\underline{n}} \operatorname{per} A(i|j)b_i$. On the other hand, we have

$$\operatorname{per}\begin{bmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}$$
$$= \sum_{i \in \underline{n}} b_i \operatorname{per} A(i|j) \quad \text{(by Lemma 2.3).}$$

Therefore

$$x = A^{-1}b = (\operatorname{per} A)^{-1}(\operatorname{adj} A)b \quad \text{(by Lemma 4.2)}$$
$$= d^{-1}(\operatorname{adj} A)b = (d^{-1}d_1, d^{-1}d_2, \ldots, d^{-1}d_n)^{\mathrm{T}}.$$

This completes the proof. □

By Theorem 4.1, we have

Corollary 4.4. *If S satisfies $U(S) = \{1\}$ and $A \in GL_n(S)$, and $(b_1, b_2, \ldots, b_n)^{\mathrm{T}} \in V_n(S)$, then the matrix equation $Ax = b$ has a unique solution $x = (d_1, d_2, \ldots, d_n)^{\mathrm{T}} \in V_n(S)$, where*

$$d_j = \operatorname{per}\begin{bmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{bmatrix}, \quad j = 1, 2, \ldots, n. \qquad \square$$

Remark 4.3. Corollary 4.4 generalizes Theorem 4.3 in Han and Li [11].

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