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Computing in the Monster

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We discuss the feasibility of a general technique for computing in the Fischer–Griess Monster, and provide information on some of its subgroups which illustrates the use of computational techniques in solving a particular problem in this group.

1. Introduction

In recent years many computational techniques for handling groups have been developed. We start with a summary of how specific finite simple groups can be handled. All notation is as in the ATLAS (Conway *et al.*, 1985).

What does it mean to be able to compute in a given group? If the group is defined as a set of elements of given type (e.g. matrices), we need to be able to multiply or invert them, and an optional extra is to distinguish elements of the group from non-elements. However, even if there is no explicit algorithm for the latter, it can usually be solved in practice by multiplying by random group elements and determining the order of the product.

Alternatively, one can define a group as the set of words in given generators subject to an equivalence relation; then one needs to be able to determine exactly when two words are equivalent (the word problem).

Most group theoretic computational techniques deal with either permutations or matrices. Multiplication or inversion of these is straightforward. And as it happens, almost all finite simple groups can be defined as permutation or matrix groups, with parameters small enough to allow explicit computation.

Ignoring the trivial Abelian case, the classification theorem divides finite simple groups into four types: alternating, classical type, exceptional Lie type, and sporadic. By definition an alternating group consists of all even permutations on a given set of points, and a group of classical type is (a central quotient of) a group of matrices over a finite field with determinant 1, which may be required to preserve a unitary, symplectic or orthogonal form and (in the orthogonal case) to have spinor norm 1. So multiplication, inversion, and distinguishing elements from non-elements are all straightforward.

Exceptional groups of Lie type can all be written as matrix groups. To be specific, groups of type ${}^2B_2$, ${}^2G_2$, G_2, ${}^2F_4$, F_4, E_6, E_7 and E_8 over $GF(q)$ have (possibly projective) representations of respective degrees 4, 7, 7, 26, 26, 27, 56 and 248 over the same field; and groups of type ${}^2E_6$ and ${}^3D_4$ have representations of degree 27 and 8 over

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Table 1. Sporadic group representations.

Group	Perm Rep	Matrix Rep	Field
M_{11}	11	5	$GF(3)$
M_{12}	12	6	$GF(3)$
J_1	266	7	$GF(11)$
M_{22}	22	6	$GF(4)$
J_2	100	6	$\mathbb{Q}(\sqrt{5})$
M_{23}	23	11	$GF(2)$
HS	100	22	$\mathbb{Q}$
J_3	6156	9	$GF(4)$
M_{24}	24	11	$GF(2)$
McL	275	22	$\mathbb{Q}$
He	2058	51	$\mathbb{Q}(\sqrt{-7})$
Ru	4060	28	$\mathbb{Q}(i)$
Suz	1782	12	$\mathbb{Q}(\sqrt{-3})$
$O'N$	122760	45	$GF(7)$
Co_3	276	23	$\mathbb{Q}$
Co_2	2300	23	$\mathbb{Q}$
Fi_{22}	3510	27	$GF(4)$
HN		133	$\mathbb{Q}(\sqrt{5})$
Ly		111	$GF(5)$
Th		248	$\mathbb{Q}$
Fi_{23}	31671	253	$GF(3)$
Co_1	98280	24	$\mathbb{Q}$
J_4		112	$GF(2)$
Fi'_{24}	306936	783	$\mathbb{Q}(\sqrt{-3})$
B		4371	$\mathbb{Q}$
$\mathbb{M}$		196883	$\mathbb{Q}$

$GF(q^2)$ and $GF(q^3)$, respectively. Only for E_8 is the degree not "small", and even E_8 is well within current computational capability. So this just leaves the sporadic groups.

1.1. SPORADIC GROUPS

Table 1 lists the sporadic groups in increasing order of size, together with degrees of small permutation and (possibly projective) matrix representations. The smallest permutation representation is shown whenever the degree is less than 10^6. For matrix representations, there are two cases: some groups have a particularly low-degree representation over a specific characteristic, while for others the smallest ordinary character gives the least degree representation over any characteristic (possibly after splitting off up to two copies of the trivial character). We show the degree and ground field of the smallest modular (in the first case) or ordinary (in the second case) character. Note that in one case (J_2) the representation cannot actually be written over that field, but in all cases the character value irrationalities determine the *finite* fields over which it can be written.

We see from this that all sporadic groups can be represented by permutations of degree up to 31671, or matrices of degree up to 248, except for Fi'_{24}, B and $\mathbb{M}$. Fi'_{24} is well within computational reach and B is just so.

For the Monster, a pair of generating matrices (of degree 196882 over $GF(2)$) has been obtained (Linton *et al.*, 1998). However, the multiplication of matrices of this size is impossible within a reasonable amount of time, so the only applications of these matrices so far have been for computations that only require the image of *vectors* by matrices.

2. The Monster

A practical method of computing in the Monster would be valuable for several reasons: "because it's there", i.e. to tidy up the last case; to settle certain problems about groups attacked by means of the classification of finite simple groups; to enumerate the maximal subgroups of the Monster; and to study some of its special properties.

As an example of the last, we mention the theory of *nets.* These are referred to in Norton (1996a, 1998a) under the name *footballs,* and defined and discussed under their present name in Norton (1998b). Further information may be obtained in the references cited in that paper. Here is a brief summary of what we need for present purposes.

The Generalized Moonshine Conjecture is stated in page 208 of Norton (1987). (Note: insert "projective" before "character of $C_{\mathbb{M}}(g)$" in condition 2.) Roughly speaking it asserts that pairs of commuting elements in $\mathbb{M}$ lead to genus zero modular functions.

The action of the modular group Γ on pairs of commuting elements of any group G is related to the action of the 3-string braid group on triples of involutions of G, which is generated by $(a, b, c) \mapsto (b, a^b, c)$ and $(a, b, c) \mapsto (a, c, b^c)$ and whose central quotient is Γ. This leads to a geometrical structure called a *quilt* (Hsu, 1994, 1998; Conway and Hsu, 1995). A *net* is a quilt with $G = \mathbb{M}$ whose involutions lie in class $2A$ (i.e. the class of involution whose centralizer is the double cover of the Baby Monster). The significance of this is this class satisfies the 6-transposition property (i.e. the product of any two $2A$-elements has order at most 6), which implies that most nets have genus zero. This leads to the question, first raised in Norton (1987), of whether there is a link between the two occurrences of the "genus zero" motif. No such link has yet been found, but the study of nets has led to many interesting observations which are fully discussed in Norton (1998b).

This explains why a method of computing in the Monster would be of great interest. Is there any hope of developing such a method? Here are three possible strategies.

(1) Brute force: wait for computer technology to improve to the point where one can multiply 196882×196882 matrices.
(2) Reflection groups: see below.
(3) Griess–Conway algebra: see below.

2.1. Reflection groups

This strategy was used to develop the first program for working in the Fischer group $Fi_{24} = Fi'_{24}.2$ (Pritchard, 1990; Conway and Pritchard, 1992). For many computations in that group the reflection group method is more efficient than permutations or matrices.

Take a 13-dimensional space consisting of vectors of type

$$\mathbf{a} = (a_{-1}, a_0 | a_1, \ldots, a_{12}) \qquad \text{where} \quad 6(a_{-1} + a_0) = \sum_{i=1}^{12} a_i$$

with norm

$$(\mathbf{a}, \mathbf{a}) = -5(a_{-1} + a_0)^2 + \sum_{i=-1}^{12} a_i^2$$

and the associated inner product. The space is Lorentzian, i.e. the norm form is indefinite, but positive definite on the orthogonal complement of a vector of negative norm.

Considered as a $\mathbb{Z}$-module the space has a submodule, say N, of vectors with integral coordinates. If we define *reflection* in a vector t as the operation

$$v \mapsto v - \frac{2(v,t)}{(t,t)} t$$

then N is closed under reflections by norm 2 vectors.

Let k_i $(1 \le i \le 11)$ be the reflection in the vector with value -1 on coordinate i, 1 on coordinate $i+1$ and 0 elsewhere. The effect of k_i is to interchange coordinates i and $i+1$, so the k_i form a diagram of type A_{11} (i.e. a chain of 11 nodes) and generate the symmetric group S_{12}.

Let k_0 and k_{-1} be the reflections in $(0, 1|0^6, 1^6)$ and $(1, -1|0^{12})$. Then $k_0 k_i$ has order 3 for $i = 6$ (or -1) and 2 for $1 \le i \le 5$ or $7 \le i \le 11$. This means that adjoining k_0 to our A_{11} gives a diagram with three arms of length 5, 5 and 1 attached to a common "middle" node.

Following pages 232–3 of the ATLAS, and some of the other references cited here, we may define a diagram of type Y_{lmn} as a graph with arms of length l, m and n attached to a common node. Each such diagram gives rise to a *reflection group* which corresponds to the diagram in the same way that the group generated by the k_i corresponds to Y_{551}. It also gives rise to a *Y-group,* which we may also denote by Y_{lmn} without ambiguity, which can be defined as a given quotient of the reflection group.

By reading N mod 2, it can be seen that the reflection group corresponding to Y_{551} has $O_{10}^-(2).2 = Y_{551}$ as a quotient. If we now adjoin k_{-1}, which commutes with k_i $(1 \le i \le 11)$, the resulting group has a Y_{552} diagram and, therefore, $3.Fi_{24} = 3.Fi_{24}'.2 = Y_{552}$ as a quotient.

Fi_{24} has 306936 3-transpositions (i.e. involutions belonging to the class $2C$, which have the property that the product of any two of them has order at most 3). Each of them is the image of infinitely many reflections under the map from the Y_{552} reflection group to Fi_{24}. One may use a set of canonical reflections to label the transpositions, or their 3×306936 inverse images in $3.Fi_{24}$. Furthermore, one can write a program which, given two transpositions (in either group), chooses some corresponding reflections (not necessarily the canonical ones), conjugates one of them by the other and then transforms the result into canonical form.

This gives a program for conjugating transpositions of Fi_{24} or $3.Fi_{24}$. This leads to a solution of the word problem, as two words in the generators correspond to the same element of Fi_{24} (or $3.Fi_{24}$) if and only if they conjugate each generating transposition in the same way.

Can this idea be extended to $Y_{533} = 2 \times 2.B$, $Y_{553} = 2 \times \mathbb{M}$ or $Y_{555} = M \wr 2$ to give algorithms for working in the Baby Monster, Monster and Bimonster? Possibly, but the much greater number of transpositions in these groups makes the task much harder.

2.2. the Griess–Conway algebra

In Conway (1985) the author modifies the (non-associative) algebra constructed in Griess (1982) in a way that simplifies the construction and leads to some new properties of interest, discussed in the former paper. We call the modified algebra the *Griess–Conway algebra.* The construction is summarized in the ATLAS and further properties of the algebra are examined in Norton (1996a).

A key feature of the construction is that conjugation by a transposition can be expressed in terms of algebra multiplication by a corresponding *axis vector*. This means that a program for multiplying in the algebra, or in the corresponding algebra over a finite field such as $GF(3)$, can be used to conjugate transpositions. As described above this leads to a solution of the word problem.

Anyone who wishes to try to solve the problem of computing in the Monster by this means should consult the references cited above for more information.

3. Subgroups of the Monster

While at present there is no practical algorithm for most computations in the Monster, there are many problems involving $\mathbb{M}$ where all the relevant elements are in fact contained in a proper subgroup, so one can use the computational methods applicable to that group. To make the most of this procedure, it is important to correlate the generators of the relevant subgroup with words in generators of the Monster.

3.1. the monster and the projective plane

The "projective plane" provides a system of generators for the Bimonster $\mathbb{M} \wr 2$ and Monster as first described in Conway *et al.* (1988) and further developed in Norton (1990, 1992). They are highly versatile in that many subgroups are easy to define. We summarize the main results.

We start with $Y_{552} = 3.Fi_{24}$. Within the Bimonster we adjoin three nodes generating the S_4 centralizing $Y_{551} = O_{10}^-(2).2$, two of which generate the S_3 centralizing Y_{552}. This gives a Y_{555} diagram. This has three subdiagrams of type A_{11} (chains of 11 points). The symmetric group S_{12} corresponding to each such diagram contains an element which extends the diagram to the corresponding affine diagram $\tilde{A}_{11}$, a dodecagon. It can be shown that the closure of the Y_{555} diagram under the process of adjoining such nodes is the incidence graph of the projective plane of order 3, with 26 nodes, half corresponding to the points of the plane and half to the lines.

Using X, E, T as abbreviations for 10, 11, 12, one may number the points and lines from P_0 to P_T and from L_0 to L_T in such a way that a point lies on a line (so is joined to it in the diagram) just when the sum of their suffices is 0, 1, 3 or 9 mod 13.

Take two (group elements corresponding to) points and conjugate their product by the line containing them. This gives a *cog* (Norton, 1990). The group generated by all cogs is $2^{1+26}.2^{24}.Co_1$ where, as in the ATLAS, 2^{1+26} denotes an extraspecial group of order 2^{27}. The cog obtained from points a and b is denoted by $[ab]$. (Note: in Norton (1990) (ab) was used, but here we wish to reserve this symbol for an S_{12}-transposition.)

The 2^{1+26} subgroup is generated by the points and *stars*. The star corresponding to a point is the central element of the group generated by any line containing the point, and the three *other* points on that line (which form a D_4 diagram which the point extends to a $\tilde{D}_4$). It can be shown that this definition does not depend on the choice of line containing our point. All points and stars commute except a point and its own star, and their commutator, which we call π, is independent of the choice of point, and commutes with all points, stars and cogs.

Omitting the middle node of a Y_{555} diagram gives three disjoint diagrams of shape A_5, each of which generates an S_6. It is known that there is an involution in the Bimonster which normalizes each of these S_6's and whose product with the middle node has order 5. We call such an element a *duality*.

A subgroup of the Bimonster lies in the Monster if and only if it is even (lies in $\mathbb{M}\times\mathbb{M}$) and centralizes an odd involution such as a node.

3.2. NETS OF ORDER UP TO 7

Recall that a *net* is a geometric structure associated with three 6-transpositions in the Monster, say a, b and c. The *class* and *order* of a net are the class and order of abc. So far all nets of order up to 7 have been classified; the results are summarized in Norton (1998b) and given in full in Norton (1996b). This classification involved working in many subgroups of $\mathbb{M}$, some of which we now list.

S_{12} (natural permutation representation)—one of the most useful.

$O_{10}^-(2).2$ (permutation representation on 528 3-transpositions)—again very useful. Obtained by coset enumeration using the Y_{551} presentation.

Fi_{22} (permutation representation on 3510 3-transpositions). Obtained by coset enumeration using the Y_{332} presentation.

Fi_{23} (permutation representation on 31671 3-transpositions). Obtained by coset enumeration using the Y_{532} presentation.

$2.O_8^+(3)$ (natural matrix representation). The central quotient of this group corresponds to $Y_{522}{}' = O_8^+(3)$. Note that there are three conjugacy classes of involutions in this group which correspond to $2A$-involutions of $\mathbb{M}$, of which only one is easily expressible in "reflection" terms; that is one reason why we prefer to use matrices.

X_{3111} (see page 233 of the ATLAS for definition), a group of order $2^{20}.3^2.5$. Omitting one generator gives a D_6 subdiagram generating $2^5.S_6$; coset enumeration over this group was used to obtain a permutation representation on 2048 points.

HN (133×133 matrices over $GF(5)$).

He (subgroup of Fi_{24} with permutation representation on 2058 points).

${}^2F_4(2)'$ (subgroup of Fi_{22}).

$2^{1+24}.Co_1$ (permutation representation on 196560 points). This is the centralizer in $\mathbb{M}$ of an involution of class $2B$.

$3.{}^2E_6(2)$ (27×27 matrix representation over $GF(4)$). The central quotient corresponds to that of $Y_{333} = 2 \times 2^2.{}^2E_6(2)$. This is one of the most useful groups to work in, and a correspondence with another of its embeddings in the Monster has been used—see below.

We discuss the last four of the above groups in more detail. For the Harada–Norton group HN, 133×133 matrices over $GF(5)$ generating $HN.2$, including a subset generating

S_{12}, were obtained (Ryba and Wilson, 1994), and a matrix was found within this group for the duality mentioned earlier, which extends S_{12} to $HN.2$.

For the Held group He, here is a correspondence between the generators of the presentation of $He.2$ in the ATLAS (page 104) and products of transpositions of Fi_{24}. We use (ab) $(1 \le a, b \le 12)$ to denote the S_{12}-transposition that interchanges coordinates a and b, i.e. the reflection in the vector with value -1 at a, 1 at b and 0 elsewhere. Recall that k_{-1} is the reflection in $(1, -1|0^{12})$.

$$
\begin{aligned}
a &= (12)(36) \\
b &= (12|001011332322)(44|010227464886) \\
c &= (9T)(XE) \\
d &= (12)(36)(7X)(9E) \\
e &= (12)(36)(01|001111110000)(01|001001110110) \\
f &= k_{-1}(13)(26)(45) \\
g &= (12)(36)(78)(9T)
\end{aligned}
$$

It is convenient to record here some further information not in the ATLAS, some from Soicher (1991) and some new.

$$
\begin{aligned}
He.2 \cap \langle k_i | 0 \le i \le 11 \rangle &= \langle a, a^b, c, d, e, g, (bcdg)^7 \rangle \cong S_4(4).2 = S, \quad \text{say, with} \\
a^b &= (16)(23)(45)(79)(8T)(XE) \qquad \text{and} \qquad (bcdg)^7 = (142)(356), \\
a^{f^{edcb}degdca} &= (17)(28)(39)(4X)(5E)(6T) \in S \qquad \text{and} \\
&\langle S, b(cd)^2(fe)^2c^d efb \rangle \cong S_4(4).4.
\end{aligned}
$$

The Tits group ${}^2F_4(2)'$ is generated by $(34)(1,1|0^4,1^4,2^4)$, $(12)(37)(45)(6E)(8T)(9X)$ and $(12)(35)(46)(79)(8T)(XE)$, which lie inside the $2.Fi_{22}$ centralizing k_{-1} and $k_1 = (12)$. To be precise, the above elements generate $2 \times {}^2F_4(2)'$ where $k_{-1}k_1$ is central.

For the group $2^{1+24}.Co_1$ we have a presentation that leads to a permutation representation of the central quotient $2^{24}.Co_1$ on 196560 points; this is of particular interest because the permutation character for the latter representation contains the degree 98280 constituent of the degree 196883 character of $\mathbb{M}$ restricted to $2^{1+24}.Co_1$. The presentation was obtained from the relations of Lemma 4 of Norton (1990, p. 598), after showing that the "element of K" in relation (5) is actually $P_{137}(= P_1P_3P_7)$.

Our group has five generators: $a = P_{0X}$, $b = [3X]$ (recall that this means the cog ${P_{3X}}^{L_6}$), $c = [23]$, d and e. The last two are projective plane automorphisms whose actions on the points are $(19)(25)(4T)(78)$ and $(93)(56)(X4)(E7)$ respectively. The product of these permutations is the "tripler" $(139)(265)(4TX)(78E)$. The subgroup of index 196560 is $\langle b^e, c, c^e, c^{de}, d \rangle \cong 2^{1+23}.Co_2$. The relators are:

$$
\begin{aligned}
&a^2, b^2, c^2, d^2, e^2, (ab)^4, (ac)^2, (ad)^2, (ae)^4, (bc)^3, (bd)^2, (be)^4, (cd)^4, (ce)^6, (de)^3, \\
&(ab^e)^2, (ac^e)^2, (bc^e)^2, (cd^e)^4, (de^c)^6, (a^ec^d)^2, (b^ec^d)^2, ((cd)^2e)^6, ((cd)^2c^e)^3, \\
&d^c((cd)^{2.(c^ed)^2}(b^ed)^2)^2((a(a^bc)^2)^ed)^2, c^{de}c^{ec^d}(c^ec^{db})^2(a(a^bc)^2)^d, (cd(bced)^2)^{28}.
\end{aligned}
$$

A base and strong generating set has been obtained.

Finally, some comments on ${}^2E_6(2)$ including an account of a particular computation. Matrices for the Y_{333} formulation of the group are shown in Norton (1998a). However, another rather different formulation can be obtained by starting with $Y_{551} = O_{10}^-(2).2$, which, in the Bimonster Y_{555}, centralizes the S_4 generated by two points and a line of the Y_{555}.

If we adjoin π to Y_{551} the resulting group still centralizes the two points and the corresponding cog, which together generate a D_8 inside the S_4, and is therefore $2^2.{}^2E_6(2).2$. Note that this group is not conjugate in the Bimonster to $Y_{333} = 2 \times 2^2.{}^2E_6(2)$. However, their even parts (intersections with $\mathbb{M} \times \mathbb{M}$) *are* conjugate, as a duality takes the even part of the above D_8 to the group of even products of the three points on that arm of the Y_{555}.

27×27 matrices for the even part of this group can be obtained by taking the direct sum of the 1-, 10- and 16-dimensional representations of $O_{10}^-(2)$, and using standard techniques to obtain a matrix for π. One can also obtain a matrix $\mathbf{m}$ that takes this $2^2.{}^2E_6(2)$ to the one obtained in the Y_{333} formulation. (Note: $\mathbf{m}$ does not actually represent the duality-like element mentioned above, or even correspond to any element of $\mathbb{M}$.)

Here is an example of how the above was used to solve a specific problem, namely to obtain generators for a particular net of class $6B$. Recall that a net is determined by three $2A$-elements, which we call a, b and c. We choose a $6B$-element inside A_{12}, namely $(123)(475869)(XE)$; this is the product of $a = (47)(58)(69)(XE)$, and a $3A$-element that is to be the product of b and c, namely $(123)(456)$. We have $C_{\mathbb{M}}\langle a, bc\rangle \cong U_5(2)$. There are 306936 extensions of bc to an S_3 in which b and c can be found. It was easy to find representatives of each $U_5(2)$-orbit on these extensions except for one case, where the stabilizer is $Q_8 \; o \; 2A_4$.

The strategy eventually adopted was this. The permutations a and bc lie in $A_{12} < O_{10}^-(2)$ and so correspond to 27×27 matrices in the second formulation of $2^2.{}^2E_6(2)$. The matrix $\mathbf{m}$ transforms this to a Y_{333} in which bc is the product of two nodes. All 306936 extensions of such a product to S_3 are easy to describe—they correspond to the transpositions of a Y_{552} that commutes with the two nodes. Thus, all one needs to do is to search the transpositions that lie inside our Y_{333} for one in the right orbit.

This procedure gave 27×27 matrices for a, b and c which generated a subgroup of $2^2.{}^2E_6(2)$ easily seen to be different from any previously obtained, and therefore in the missing orbit. But what subgroup is it? The author acknowledges the help of Eamonn O'Brien who, by computing the order by means of MAGMA (Bosma *et al.*, 1997), confirmed a conjecture that the group is $2^{1+10}.3^4.D_{10}$.

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