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FINITE p-GROUPS WITH SMALL AUTOMORPHISM GROUP

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Abstract

For each prime p we construct a family $\{G_i\}$ of finite p-groups such that $|\mathrm{Aut}(G_i)|/|G_i|$ tends to zero as i tends to infinity. This disproves a well-known conjecture that $|G|$ divides $|\mathrm{Aut}(G)|$ for every nonabelian finite p-group G.

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1. Introduction

A well-known question (see, for example, [23, Problem 12.77]) asks whether it is true that $|G|$ divides $|\mathrm{Aut}(G)|$ for every nonabelian finite p-group G. It is not clear who raised this question first explicitly; the first result in this direction that we have found in the literature is due to Schenkman [28], and it is more than 50 years old. In that paper, Schenkman showed that this is true for finite nonabelian p-groups of class 2 (the proof has a gap, which is corrected by Faudree in [13]). Later, it was also established for p-groups of exponent p in [26], for p-groups of maximal class in [25], for p-groups with center of order p in [15], for metacyclic p-groups when p is odd in [4], for central-by-metacyclic p-groups when p is odd in [7], for p-abelian p-groups in [5] (see also [30]), for finite modular p-groups in [8], for some central products in [3, 18], for p-groups with center of index at most p^4 in [6], for p-groups with cyclic Frattini subgroup in [11], for p-groups of order at most p^6 in [6, 12], for p-groups of order at most p^7 in [16], for p-groups of coclass 2 in [14] (see also a related result in [10]), and for p-groups G such that $(G, Z(G))$ is a Camina pair in [31].

All these partial results indicate that a counterexample to the problem should have a large size, and that it will be difficult to present it explicitly. In this paper we use pro-p techniques, and we are able to show the following.

THEOREM 1.1. *For each prime p there exists a family of finite p-groups $\{U_i\}$ such that*

$$\lim_{i\to\infty} |U_i| = \infty \quad \textit{and} \quad \limsup_{i\to\infty} \frac{|\mathrm{Aut}\, U_i|}{|U_i|^{40/41}} < \infty.$$

In particular, for every prime p, there exists a nonabelian finite p-group G such that $|\mathrm{Aut}(G)| < |G|$.

Let us briefly explain our construction. It consists of two parts.

(1) First, we take an infinite finitely generated pro-p group U such that $\mathrm{Aut}(U)$ is 'smaller' than U. Of course, to be smaller for infinite groups does not refer to the order. In our construction, U will be a uniform p-adic pro-p group, and so we can speak about $\dim U$. Recall that $\dim U$ is defined as $\dim_{\mathbb{Q}_p} \mathbf{L}(U)$, where $\mathbf{L}(U)$ is the Lie $\mathbb{Q}_p$-algebra associated with U. Since U is compact p-adic analytic, $\mathrm{Aut}(U)$ is also a p-adic analytic profinite group. Thus, saying that $\mathrm{Aut}\, U$ is 'smaller' than U will simply mean that $\dim \mathrm{Aut}(U) < \dim(U)$.

(2) Second, U can be written as an inverse limit $U = \varprojlim U_i$ of finite p-groups $U_i = U/U^{p^i}$. Since $\mathrm{Aut}(U) = \varprojlim \mathrm{Aut}(U_i)$, we may hope that $\mathrm{Aut}(U_i)$ is smaller than U_i when i is large (compare with Lemma 2.2).

In order to construct U from the first step, we notice that, if U is a uniform pro-p group, then

$$\dim \mathrm{Aut}(U) = \dim_{\mathbb{Q}_p} \mathrm{Der}(\mathbf{L}(U)),$$

where $\mathrm{Der}(\mathbf{L}(U))$ is the algebra of $\mathbb{Q}_p$-derivations of $\mathbf{L}(U)$. Examples of Lie algebras L with $\dim \mathrm{Der}(L) < \dim L$ are known to exist, and they were first constructed by Luks [22] and Sato [27]. In Sato's example, the algebra is constructed over $\mathbb{Q}$, it has dimension 41, its center has dimension 1, and its derived algebra consists only of inner derivations (and so it has dimension 40). This is the explanation for the numbers which appear in Theorem 1.1.

The realization of the second step of our proof is based on an analysis of the first cohomology groups $H^1(U, L_i)$, where $L_i = \mathbf{log}(U)/p^i\mathbf{log}(U)$ and $\mathbf{log}(U)$ is the

Lie ring corresponding to a uniform pro-p group U by Lazard's correspondence. It turns out that, since $\mathrm{Der}(\mathbf{L}(U)) = \mathrm{Inn}(\mathbf{L}(U))$, $\mathrm{Der}(\mathbf{log}(U))$ is finite, and so

$$H^1_{\mathrm{cts}}(U, \mathbf{log}(U)) \cong \mathrm{Der}(\mathbf{log}(U))$$

is finite. This implies the existence of a uniform upper bound for $|H^1(U, L_i)|$. As a consequence, we obtain a uniform upper bound for $|\mathrm{Aut}(U_i) : \mathrm{Inn}(U_i)|$, where $U_i = U/U^{p^i}$, that finishes the proof.

The organization of the paper is as follows. In Section 2, we describe the basic facts about p-adic analytic groups, introduce continuous cohomology groups of pro-p groups, and establish the uniform upper bound for $|H^1(U, L_i)|$. In Section 3, we present the proof of Theorem 1.1. We discuss a possible direction for future work in Section 4.

2. Uniform pro-p groups and their cohomology groups

2.1. Uniform pro-p groups. Let L be a Lie $\mathbb{Z}_p$-algebra. We say that L is *uniform* if, for some k, $L \cong \mathbb{Z}_p^k$ as a $\mathbb{Z}_p$-module and $[L, L] \subseteq 2pL$. Analogously, we say that a pro-p group U is *uniform* if it is torsion free, finitely generated, and $[G, G] \subseteq G^{2p}$.

One can define the functors **exp** and **log** between the categories of uniform Lie $\mathbb{Z}_p$-algebras and uniform pro-p-groups which are isomorphism of categories (see [9, Section 4]). There is a relatively easy way to define the functor **log**. If U is a uniform pro-p group, then $\mathbf{log}(U)$ is the Lie $\mathbb{Z}_p$-algebra, whose underlying set coincides with U, and the Lie operations are defined as follows:

$$a + b = \lim_{i\to\infty} (a^{p^i} b^{p^i})^{1/p^i}, \quad [a, b]_L = \lim_{i\to\infty} [a^{p^i}, b^{p^i}]^{1/p^{2i}}, \quad a, b \in U. \tag{1}$$

If $f : U \to V$ is a homomorphism between two uniform pro-p groups, then $\mathbf{log}(f) = f$ is a homomorphism of Lie $\mathbb{Z}_p$-algebras. In particular, the conjugation turns $\mathbf{log}(U)$ into a U-module.

LEMMA 2.1. *Let U be a uniform pro-p group. Let $i, j \in \mathbb{N}$ be such that $i \leqslant j \leqslant 2i + 1$. Then U^{p^i}/U^{p^j} is abelian, and*

$$U^{p^i}/U^{p^j} \cong \mathbf{log}(U)/p^{j-i}\mathbf{log}(U)$$

as U-modules (U acts on U^{p^i}/U^{p^j} by conjugation).

Proof. The lemma is a consequence of the definition of sum in (1). □

Let G be a p-adic analytic profinite group. Then it contains a uniform open subgroup U. The Lie algebra $\mathbf{L}(G)$ of G is a Lie $\mathbb{Q}_p$-algebra defined as $\mathbf{L}(G) = \mathbf{log}(U) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The definition does not depend on the choice of U. We put

$\dim G = \dim_{\mathbb{Q}_p} \mathbf{L}(G)$. For a p-adic pro-p group we have the following internal characterization of the dimension.

LEMMA 2.2 [20, Proposition III.3.1.8], [9, Lemma 4.10]. *Let G be a p-adic analytic pro-p group. Denote by $G_i = G^{p^i}$ the subgroup of G generated by p^ith powers of the elements of G. Then there are positive constants c_1 and c_2 such that*

$$c_1 p^{i \dim G} \leqslant |G : G_i| \leqslant c_2 p^{i \dim G}.$$

Moreover, if G is uniform, then $|G : G_i| = p^{i \dim G}$.

2.2. Continuous cohomologies of pro-p groups. In this subsection, we present the results about continuous cohomology groups of pro-p groups that we will need in this paper. More details and omitted proofs can be found in [24, 29].

Let G be a pro-p group. We say that A is a *topological* G-module if A is an abelian Hausdorff topological group which is endowed with the structure of an abstract left G-module such that the action $G \times A \to A$ is continuous. In this paper, A will be one the following three types: a finite abelian group, a profinite abelian group, and a finite-dimensional vector space over $\mathbb{Q}_p$. We denote by $G^{(i)}$ the Cartesian product of i copies of G. We put

$$\mathcal{C}^i_{\mathrm{cts}}(G, A) = \{f : G^{(i)} \to A \mid f \text{ is a continuous function}\}$$

and denote the coboundary operator $\partial^{i+1}_A : \mathcal{C}^i_{\mathrm{cts}}(G, A) \to \mathcal{C}^{(i+1)}_{\mathrm{cts}}(G, A)$ by means of

$$\begin{aligned}(\partial^{i+1}_A f)(g_1, \ldots, g_{i+1}) = g_1 \cdot f(g_2, \ldots, g_{i+1}) \\ + \sum_{j=1}^{i} (-1)^j f(g_1, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \ldots, g_{i+1}) \\ + (-1)^{i+1} f(g_1, \ldots, g_i).\end{aligned}$$

Now, we set

$$\mathcal{Z}^i_{\mathrm{cts}}(G, A) = \ker \partial^{i+1}_A \quad \text{and} \quad \mathcal{B}^i_{\mathrm{cts}}(G, A) = \operatorname{im} \partial^i_A$$

and define the *i-th continuous cohomology group* $H^i_{\mathrm{cts}}(G, A)$ of G with coefficients in A by

$$H^i_{\mathrm{cts}}(G, A) = \mathcal{Z}^i_{\mathrm{cts}}(G, A)/\mathcal{B}^i_{\mathrm{cts}}(G, A).$$

If A is a finite p-group, then $H^i_{\mathrm{cts}}(G, A)$ coincides with the usual definition of $H^i(G, A)$, and it is equal to $\mathrm{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, A)$ (see [29, Ch. 3.2]).

If $\alpha : A \to B$ is a continuous homomorphism of topological G-modules, then we have the induced homomorphism of complexes

$$\tilde{\alpha} : (\mathcal{C}^*_{\mathrm{cts}}(G, A), \partial) \to (\mathcal{C}^*_{\mathrm{cts}}(G, B), \partial), (\tilde{\alpha} f)(g_1, \ldots, g_i) = \alpha(f(g_1, \ldots, g_i)).$$

Hence, $\tilde{\alpha}$ extends to the homomorphisms of the homology groups of this complexes $\alpha^*_i : H^i_{\mathrm{cts}}(G, A) \to H^i_{\mathrm{cts}}(G, B)$.

By [24, Lemma 2.7.2], we have the following long exact sequence in cohomology.

LEMMA 2.3. *Let G be a pro-p group, and let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be a short exact sequence of left $\mathbb{Z}_p[[G]]$-modules with C finite. Then there exists a canonical boundary homomorphism

$$\delta : H^1_{\mathrm{cts}}(G, C) \to H^2_{\mathrm{cts}}(G, A)$$

such that

$$H^1_{\mathrm{cts}}(G, A) \xrightarrow{\alpha^*_1} H^1_{\mathrm{cts}}(G, B) \xrightarrow{\beta^*_1} H^1_{\mathrm{cts}}(G, C) \xrightarrow{\delta} H^2_{\mathrm{cts}}(G, A) \xrightarrow{\alpha^*_2} H^2_{\mathrm{cts}}(G, B)$$

is exact.

We say that G is of type FP_∞ if the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ has a free resolution over $\mathbb{Z}_p[[G]]$ whose terms are finitely generated. For example, if G is p-adic analytic, then $\mathbb{Z}_p[[G]]$ is Noetherian ([20, Proposition V.2.2.4], [9, Corollary 7.25]), and so G is FP_∞.

In the case when G is a FP_∞ pro-p group and A is a topological pro-p G-module, $H^i_{\mathrm{cts}}(G, A)$ coincides with $\mathrm{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, A)$ (see [29, Theorem 3.7.2]). Hence, if A is finitely generated as a $\mathbb{Z}_p$-module, then $H^i_{\mathrm{cts}}(G, A)$ are also finitely generated as $\mathbb{Z}_p$-modules.

2.3. The first cohomology groups of a uniform group. In this subsection, we consider a uniform pro-p group U such that $\mathbf{L}(U)$ has only inner derivations, and try to understand its first cohomology groups with coefficients in some natural modules. First, we consider $H^1_{\mathrm{cts}}(U, \mathbf{log}(U))$.

PROPOSITION 2.4. *Let U be a uniform pro-p group. Assume that the Lie algebra $\mathbf{L}(U)$ has only inner derivations. Then $H^1_{\mathrm{cts}}(U, \mathbf{log}(U))$ is finite.*

Proof. Since U is a finitely generated pro-p group and $\mathbf{log}(U)$ is finitely generated as a $\mathbb{Z}_p$-module, $H^1_{\mathrm{cts}}(U, \mathbf{log}(U))$ is also finitely generated as a

$\mathbb{Z}_p$-module. Hence it is enough to show that $H^1_{\text{cts}}(U, \mathbf{log}(U))$ is a torsion module; that is, that

$$H^1_{\text{cts}}(U, \mathbf{log}(U)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is equal to 0. By [29, Theorem 3.8.2],

$$H^1_{\text{cts}}(U, \mathbf{log}(U)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1_{\text{cts}}(U, \mathbf{L}(U))$$

and, by [29, Theorem 5.2.4],

$$H^1_{\text{cts}}(U, \mathbf{L}(U)) \cong H^1(\mathbf{L}(U), \mathbf{L}(U)).$$

Note that, by definition of $H^1(\mathbf{L}(U), \mathbf{L}(U))$,

$$H^1(\mathbf{L}(U), \mathbf{L}(U)) = \operatorname{Der}(\mathbf{L}(U))/\operatorname{Inn}(\mathbf{L}(U))$$

and, by our hypotheses, it is equal to zero. □

REMARK 2.5. There is an alternative way to prove the previous proposition. One can show directly that $H^1_{\text{cts}}(U, \mathbf{log}(U)) \cong \operatorname{Der}(\mathbf{log}(U))/\operatorname{Inn}(\mathbf{log}(U))$ and conclude that, since $\mathbf{L}(U)$ has only inner derivations, $\operatorname{Der}(\mathbf{log}(U))/\operatorname{Inn}(\mathbf{log}(U))$ is finite.

Now, we can bound $|H^1_{\text{cts}}(U, \mathbf{log}(U)/p^i\mathbf{log}(U))|$ uniformly in i.

PROPOSITION 2.6. *Let U be a uniform pro-p group. Assume that the Lie algebra $\mathbf{L}(U)$ has only inner derivations. Then there exists a constant C such that*

$$|H^1_{\text{cts}}(U, \mathbf{log}(U)/p^i\mathbf{log}(U))| \leqslant C$$

for every i.

Proof. Let $\alpha : \mathbf{log}(U) \to \mathbf{log}(U)$ be a multiplication by p^i. Then

$$\alpha_2^* : H^2_{\text{cts}}(U, \mathbf{log}(U)) \to H^2_{\text{cts}}(U, \mathbf{log}(U))$$

is also the multiplication by p^i. Hence $\ker \alpha_2^*$ is contained in the torsion part of $H^2_{\text{cts}}(U, \mathbf{log}(U))$.

Since U is FP_∞ and $\mathbf{log}(U)$ is finitely generated as a $\mathbb{Z}_p$-module, we have that $H^2_{\text{cts}}(U, \mathbf{log}(U))$ is also finitely generated as a $\mathbb{Z}_p$-module. Hence the torsion subgroup T of $H^2_{\text{cts}}(U, \mathbf{log}(U))$ is finite. Applying Lemma 2.3, we conclude that

$$|H^1_{\text{cts}}(U, \mathbf{log}(U)/p^i\mathbf{log}(U))| \leqslant |H^1_{\text{cts}}(U, \mathbf{log}(U))||T|.$$

Thus, Proposition 2.4 implies Proposition 2.6. □

3. Proof of Theorem 1.1

The next example is the basis of our construction.

PROPOSITION 3.1 [27]. *There exists a Lie $\mathbb{Q}$-algebra M of dimension 41 such that* $\dim Z(M) = 1$ *and* $\mathrm{Der}(M)$ *consists only of inner derivations.*

This proposition allows us to construct uniform pro-p groups considered in Section 2.3. It is done in the following way.

The algebra M has a subring M_0, such that $M = M_0 \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $L = p^2(M_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. Then L is a uniform Lie $\mathbb{Z}_p$-algebra. If we put $U = \mathbf{exp}(L)$, then $L = \mathbf{log}(U)$ and $\mathbf{L}(U) \cong L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong M \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

LEMMA 3.2. *The Lie $\mathbb{Q}_p$-algebra $\mathbf{L}(U)$ is of dimension* 41, *its center has dimension* 1, *and* $\mathrm{Der}(\mathbf{L}(U))$ *consists only of inner derivations.*

Proof. This easily follows from the fact $\mathbf{L}(U) \cong M \otimes_{\mathbb{Q}} \mathbb{Q}_p$. □

Let $U_i = U/U^{p^i}$. Denote by $\rho_{i,j} : \mathrm{Aut}(U_i) \to \mathrm{Aut}(U_j)$ (for $i \geqslant j$) the map

$$\rho_{i,j}(\alpha)(uU^{p^j}) = \alpha(uU^{p^i})U^{p^j} \quad \text{for } \alpha \in \mathrm{Aut}(U_i), \quad u \in U.$$

Now, we are ready to present the main step in our proof.

PROPOSITION 3.3. *There exists a constant k such that, for all $i \geqslant 2k$,*

$$\ker \rho_{i,k} \leqslant \mathrm{Inn}(U_i) \ker \rho_{i,i-k}.$$

Proof. By Proposition 2.6, there exists k such that

$$p^k H^1_{\mathrm{cts}}(U, \mathbf{log}(U)/p^i\mathbf{log}(U)) = 0$$

for all i. We will prove the proposition by induction on i. When $i = 2k$, the proposition is clear. Assume that we have shown the proposition for i; let us prove it for $i + 1$.

Let $\phi \in \ker \rho_{i+1,k}$. Since $\rho_{i+1,i}(\phi) \in \ker \rho_{i,k}$, the inductive assumption implies that $\phi \in \ker \rho_{i+1,i-k} \operatorname{Inn} U_{i+1}$. Thus, without loss of generality, we may assume that $\phi \in \ker \rho_{i+1,i-k}$.

Define the following function $c : U \to U^{p^{i-k}}/U^{p^{i+1}}$:

$$c(u) = \phi(uU^{p^{i+1}})u^{-1}.$$

Then

$$c(u_1u_2) = \phi(u_1u_2U^{p^{i+1}})u_2^{-1}u_1^{-1} = c(u_1)u_1c(u_2)u_1^{-1}.$$

Thus $c \in \mathcal{Z}^1_{\mathrm{cts}}(U, U^{p^{i-k}}/U^{p^{i+1}})$.

By Lemma 2.1, $U^{p^{i-k}}/U^{p^{i+1}}$ is abelian, and it is isomorphic to $\mathbf{log}(U)/p^{k+1}\mathbf{log}(U)$ as a U-module. In particular,

$$p^k H^1_{\mathrm{cts}}(U, U^{p^{i-k}}/U^{p^{i+1}}) = 0.$$

Consider the following exact sequence of U-modules:

$$1 \to U^{p^{i-k+1}}/U^{p^{i+1}} \xrightarrow{\alpha} U^{p^{i-k}}/U^{p^{i+1}} \xrightarrow{\beta} U^{p^i}/U^{p^{i+1}} \to 1,$$

where α is the inclusion and β is the p^kth power map. Then β_1^* is multiplication by p^k and so, by Lemma 2.3, $\operatorname{im}\alpha_1^* = H^1_{\mathrm{cts}}(U, U^{p^{i-k}}/U^{p^{i+1}})$. Hence there are $c' \in \mathcal{Z}^1_{\mathrm{cts}}(U, U^{p^{i-k+1}}/U^{p^{i+1}})$ and $v \in U^{p^{i-k}}/U^{p^{i+1}}$ such that

$$c(u) = c'(u)vuv^{-1}u^{-1} \quad \text{for every } u \in U.$$

Thus, we obtain that

$$\phi(uU^{p^{i+1}}) = c(u)u = c'(u)vuv^{-1} = c'(u)vuU^{p^{i+1}}v^{-1} \quad \text{for every } u \in U.$$

But this implies that $\phi \in \ker \rho_{i+1,i+1-k} \operatorname{Inn}(U_{i+1})$, and we are done. □

COROLLARY 3.4. *There exists a constant D such that*

$$|\operatorname{Aut}(U_i) : \operatorname{Inn}(U_i)| \leqslant D \quad \textit{for all } i.$$

Proof. By the previous proposition, we have that

$$\begin{aligned}|\operatorname{Aut}(U_i) : \operatorname{Inn}(U_i)| &\leqslant |\operatorname{Aut}(U_i) : \operatorname{Inn}(U_i)\ker\rho_{i,i-k}||\operatorname{Inn}(U_i)\ker\rho_{i,i-k} : \operatorname{Inn}(U_i)| \\ &\leqslant |\operatorname{Aut}(U_i) : \ker\rho_{i,k}||\ker\rho_{i,i-k}| \leqslant |\operatorname{Aut}(U_k)||\ker\rho_{i,i-k}|.\end{aligned}$$

Since the number of generators of U_i is 41 and $|U^{p^{i-k}}/U^{p^i}| = p^{41k}$, we obtain that $|\ker\rho_{i,i-k}| \leqslant p^{(41)^2k}$. This finishes the proof of the corollary. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.2,

$$|U_i| = p^{41i}.$$

Since $Z(U)$ is one dimensional, $\dim U/Z(U) = 40$. Hence

$$|\operatorname{Inn}(U_i)| \leqslant |U/U^{p^i}Z(U)| = p^{40i}.$$

Now, the theorem follows from Corollary 3.4. □

4. Final remarks

Let ϕ be the Euler totient function. It is not difficult to show that, for a finite abelian group A, $|\operatorname{Aut}(A)| \geqslant \phi(|A|)$. In [23, Problem 15.43], Deaconescu has asked if the same is true for an arbitrary finite group. The examples from [1, 2] show that $|\mathrm{Aut}(G)|/\phi(|G|)$ can be made arbitrarily small when G is a soluble or perfect finite group. Our examples show that in fact

$$\frac{|\mathrm{Aut}(G)|}{\phi(|G|)^{(40/41)+\epsilon}}$$

can be made arbitrarily small for every $\epsilon > 0$ when G is a finite nilpotent group.

This also provides a counterexample to a conjecture from [2] that says that, for a finite nonnilpotent supersoluble group G, $|\mathrm{Aut}(G)| > \phi(|G|)$. For this, simply take a family $\{U_i\}$ of five groups from Theorem 1.1, and consider the following family of finite nonnilpotent supersoluble groups: $\{\Sigma_3 \times U_i\}$.

As a consequence of the previous discussion, we would like to raise the following question.

QUESTION 4.1. *Does there exist a constant* $\alpha > 0$ *such that, for every finite group* $|G|$,

$$|\mathrm{Aut}(G)| \geqslant \phi(|G|)^{\alpha}?$$

By a classical result of Ledermann and Neumann [21], there exists a function $g(h)$ having the property that $|\mathrm{Aut}(G)|_p \geqslant p^h$ whenever $|G|_p \geqslant p^{g(h)}$. A quadratic upper bound for g was established by Green [17], and until now only not very important improvements of Green's bound have been obtained (see, for example, [19]).

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