Octonion Quantum Chromodynamics

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Abstract

Starting with the usual definitions of octonions, an attempt has been made to establish the relations between octonion basis elements and Gell-Mann λ matrices of $SU(3)$ symmetry on comparing the multiplication tables for Gell-Mann λ matrices of $SU(3)$ symmetry and octonion basis elements. Consequently, the quantum chromo dynamics (QCD) has been reformulated and it is shown that the theory of strong interactions could be explained better in terms of non-associative octonion algebra. Further, the octonion automorphism group $SU(3)$ has been suitably handled with split basis of octonion algebra showing that the $SU(3)_C$ gauge theory of colored quarks carries two real gauge fields which are responsible for the existence of two gauge potentials respectively associated with electric charge and magnetic monopole and supports well the idea that the colored quarks are dyons.

Key Words: Octonions, Quantum Chromodynamics (QCD), $SU(3)$ symmetry, λ matrices.

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1 Introduction

In spite of the symmetry, conservation laws and gauge fields describe elementary particle in terms of their field quanta and interactions. Nevertheless, the role of number system (hyper complex numbers) has been an important factor in understanding the various theories of physics from macroscopic to microscopic level. In fact , there has been a revival in the formulation of natural laws in terms of numbers. So, according to celebrated Hurwitz theorem there exists [1] four division algebra consisting of $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions) [2, 3] and $\mathbb{O}$ (octonions) [4, 5, 6]. All these four algebra's are alternative with totally anti symmetric associators. Real and complex numbers are limited only up to two dimensions, quaternions are extended to four dimensions (one real and three imaginaries) while octonions represent eight dimensions (one real and seven imaginaries). Real and complex numbers are commutative and associative. Quaternion are associative but not commutative while its next generalization to octonions is neither commutative nor associative. Rather, the laws of alternatively and distributivity are obeyed by octonions. Quaternions and octonions are extensively used in the various branches of physics and mathematics. The octonion analysis has also played an important role in the context of various physical problems [7, 8, 9, 10, 11, 12, 13, 14] of higher dimensional supersymmetry, super gravity and super strings etc while the quaternions have an important role to unify [15, 16, 17] electromagnetism and weak forces to represent the electroweak $SU(2) \times U(1)$ sector of standard model. Likewise, the octonions are extensively studied [18, 19] for the description of color quarks and played an important role for unification programme of fundamental interactions in terms of successful gauge theories. Furthermore, the quaternionic formulation of Yang–Mill's field equations and octonion reformulation of quantum chromo dynamics (QCD) has also been developed [20] by taking magnetic monopoles [21, 22, 23] and dyons (particles carrying electric and magnetic charges) [24, 25, 26, 27] into account. It is shown that the three quaternion units explain the structure of Yang-Mill's field while the seven octonion units provide the consistent structure of $SU(3)_C$ gauge symmetry of quantum chromo dynamics. Keeping in view the potential importance of monopoles and dyons and their possible role in quark confinement, in this paper, we have made an attempt to construct $SU(3)$ gauge theory suitably handled with octonions for colored quarks. So, starting from the usual definitions of octonions, we have established the suitable connection between octonion basis elements and $SU(3)$ symmetry after comparing the multiplication tables for Gell-Mann λ matrices of $SU(3)$ symmetry and octonion basis elements. Consequently, the quantum chromo dynamics (QCD) has been reformulated and it is shown that the theory of strong interactions could be explained better in terms of non-associative octonion algebra. Further, the octonion automorphism group $SU(3)$ has been reconnected to the split basis of octonion algebra and it is shown that the $SU(3)_C$ gauge theory of colored quarks describes two real gauge fields identified as the gauge strengths of two types of chromo charges showing the presence of electric charge and and magnetic monopoles. So, it is concluded that the present formalism of colored quarks suitably describes the existence of dyons:particles carry the simultaneous existence of electric charge and magnetic monopoles.

2 Octonion Definition

An octonion x is expressed [28, 29, 30] as

$$x = (x_0, x_1, ..., x_7) = x_0 e_0 + \sum_{A=1}^{7} x_A e_A \quad (A = 1, 2, ..., 7) \tag{1}$$

where $e_A(A = 1, 2, ..., 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The octet $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is known as the octonion basis and its elements satisfy the following multiplication rules

$$e_0 = 1, \quad e_0 e_A = e_A e_0 = e_A, \quad e_A e_B = -\delta_{AB} e_0 + f^{ABC} e_C. \ (\forall A, B, C = 1, 2, ..., 7) \tag{2}$$

The structure constants f^{ABC} are completely antisymmetric and take the value 1. i.e. $f^{ABC} = +1\{\forall(ABC) = (123), (471), (257), (165), (624), (543), (736)\}$. Here the octonion algebra $\mathcal{O}$ is described over the algebra of real numbers having the vector space of dimension 8. Octonion algebra [14] is non associative and multiplication rules for its basis elements given by equations (2) are then generalized in the following table as [28, 29, 30];

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_7	$-e_6$	e_5	$-e_4$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_5$	e_4	e_7	$-e_6$
e_4	$-e_7$	$-e_6$	e_5	-1	$-e_3$	e_2	e_1
e_5	e_6	$-e_7$	$-e_4$	e_3	-1	$-e_1$	e_2
e_6	$-e_5$	e_4	$-e_7$	$-e_2$	e_1	-1	e_3
e_7	e_4	e_5	e_6	$-e_1$	$-e_2$	$-e_3$	-1

Table1- Octonion Multiplication table.

Hence, we get the following relations among octonion basis elements i.e.

$$[e_A, e_B] = 2f^{ABC} e_C; \quad \{e_A, e_B\} = -\delta_{AB} e_0; \quad e_A(e_B e_C) \neq (e_A e_B) e_C; \tag{3}$$

where brackets [ ] and { } are respectively used for commutation and the anti commutation relations while δ_{AB} is the usual Kronecker delta-Dirac symbol. Octonion conjugate is thus defined as,

$$\bar{x} = x_0 e_0 - \sum_{A=1}^{7} x_A e_A \quad (A = 1, 2, ..., 7). \tag{4}$$

An Octonion can be decomposed in terms of its scalar $(Sc(x))$ and vector $(Vec(x))$ parts as

$$Sc(x) = \frac{1}{2}(x + \bar{x}) = x_0; \quad Vec(x) = \frac{1}{2}(x - \bar{x}) = \sum_{A=1}^{7} x_A e_A. \tag{5}$$

3

Conjugates of product of two octonions as well as its own conjugate are defined as

$$(\overline{xy}) \quad = \quad \bar{y}\,\bar{x} \;\; ; \quad \overline{(\bar{x})} \; = x; \tag{6}$$

while the scalar product of two octonions is defined as

$$\langle x\,,\,y\,\rangle \quad = \sum_{\alpha=0}^{7} \quad x_\alpha y_\alpha = \frac{1}{2}(x\,\bar{y} + y\,\bar{x}) = \frac{1}{2}(\bar{x}\,y + \bar{y}\,x); \tag{7}$$

which can be written in terms of octonion units as

$$\langle e_A\,,\,e_B\,\rangle \quad = \quad \frac{1}{2}(e_A\overline{e_B} + e_B\overline{e_A}) = \frac{1}{2}(\overline{e_A}e_B + \overline{e_B}e_A) = \delta_{AB}. \tag{8}$$

The norm of the octonion $N(x)$ is defined as

$$N(x) = \overline{x}x \quad = x\,\bar{x} \quad = \sum_{\alpha=0}^{7} x_\alpha^2 e_0; \tag{9}$$

which is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) \quad = N(x)N(y) \quad = N(y)N(x). \tag{10}$$

As such, for a nonzero octonion x , we define its inverse as

$$x^{-1} \quad = \quad \frac{\bar{x}}{N(x)} \tag{11}$$

which shows that

$$x^{-1}x \quad = xx^{-1} \quad = 1.e_0; \;\; (xy)^{-1} = y^{-1}x^{-1}. \tag{12}$$

3 SU(3) Generators (Gell-Mann Matrices)

The Gell-Mann λ matrices are used for the representations of the infinitesimal generators of the special unitary group called $SU(3)$. This group consists eight linearly independent generators $G_A \forall (A = 1, 2, 3, ...8)$ which satisfy the following commutation relation as

$$[G_A\,,G_B] = iF^{ABC}G_C \tag{13}$$

where F^{ABC} is the structure constants. It is completely antisymmetric (i.e. $F^{123} = +1$; $F^{147} = F^{165} = F^{246} = F^{257} = F^{354} = F^{367} = \frac{1}{2}$ and $F^{458} = F^{678} = \frac{\sqrt{3}}{2}$). Independent generators $G_A (\forall A = 1, 2, 3, ...8)$ of $SU(3)$ symmetry group are related with the 3×3 Gell-Mann λ matrices as

$$G_A = \frac{\lambda_A}{2} \tag{14}$$

where $\lambda_A (\forall A = 1, 2, 3, ...8)$ are defined as

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \tag{15}$$

and satisfy the following properties

$$(\lambda_A)^\dagger = \lambda_A;$$
$$Tr(\lambda_A) = 0 \quad Tr(\lambda_A \lambda_B) = 2\delta_{AB};$$
$$[\lambda_A, \lambda_B] = 2iF^{ABC}\lambda_C \ (\forall A, B, C = 1, 2, 3.....8). \tag{16}$$

As such, we may summarize the multiplication rules for the generators (in terms of λ matrices) of $SU(3)$ symmetry in the following table as;

$\cdot$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
λ_1	Λ_1	$i\lambda_3$	$-i\lambda_2$	$\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_6$	$\frac{i}{2}\lambda_5$	$-\frac{i}{2}\lambda_4$	$\frac{1}{\sqrt{3}}\lambda_1$
λ_2	$-i\lambda_3$	Λ_2	$i\lambda_1$	$\frac{i}{2}\lambda_6$	$\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_4$	$-\frac{i}{2}\lambda_5$	$\frac{1}{\sqrt{3}}\lambda_2$
λ_3	$i\lambda_2$	$-i\lambda_1$	Λ_3	$-\frac{i}{2}\lambda_5$	$\frac{i}{2}\lambda_4$	$\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_6$	$\frac{1}{\sqrt{3}}\lambda_3$
λ_4	$-\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_6$	$\frac{i}{2}\lambda_5$	Λ_4	$-\frac{i}{2}\lambda_3$	$\frac{i}{2}\lambda_2$	$\frac{i}{2}\lambda_1$	$-\frac{\sqrt{3}}{2}i\lambda_5$
λ_5	$\frac{i}{2}\lambda_6$	$-\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_4$	$\frac{i}{2}\lambda_3$	Λ_5	$-\frac{i}{2}\lambda_1$	$\frac{i}{2}\lambda_2$	$\frac{\sqrt{3}}{2}i\lambda_4$
λ_6	$-\frac{i}{2}\lambda_5$	$\frac{i}{2}\lambda_4$	$-\frac{i}{2}\lambda_7$	$-\frac{i}{2}\lambda_2$	$\frac{i}{2}\lambda_1$	Λ_6	$\frac{i}{2}\lambda_3$	$-\frac{\sqrt{3}}{2}i\lambda_7$
λ_7	$\frac{i}{2}\lambda_4$	$\frac{i}{2}\lambda_5$	$\frac{i}{2}\lambda_6$	$-\frac{i}{2}\lambda_1$	$-\frac{i}{2}\lambda_2$	$-\frac{i}{2}\lambda_3$	Λ_7	$\frac{\sqrt{3}}{2}i\lambda_6$
λ_8	$-\frac{1}{\sqrt{3}}\lambda_1$	$-\frac{1}{\sqrt{3}}\lambda_2$	$-\frac{1}{\sqrt{3}}\lambda_3$	$\frac{\sqrt{3}}{2}i\lambda_5$	$-\frac{\sqrt{3}}{2}i\lambda_4$	$\frac{\sqrt{3}}{2}i\lambda_7$	$-\frac{\sqrt{3}}{2}i\lambda_6$	Λ_8

Table 2- Multiplication table for Gell-Mann λ matrices of $SU(3)$ symmetry.

where the $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6, \Lambda_7, \Lambda_8$ are described in terms of 3×3 matrices as

$$\Lambda_1 = \Lambda_2 = \Lambda_3 = \Lambda_{123} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies (\lambda_1)^2 = (\lambda_2)^2 = (\lambda_3)^2 = \Lambda_{123};$$

$$\Lambda_4 = \Lambda_5 = \Lambda_{45} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (\lambda_4)^2 = (\lambda_5)^2\,;$$

$$\Lambda_6 = \Lambda_7 = \Lambda_{67} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (\lambda_6)^2 = (\lambda_7)^2\,;$$

$$\Lambda_8 = \frac{4}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (\lambda_8)^2 = \frac{4}{3}\widehat{1}. \tag{17}$$

where $\widehat{1}$ is 3×3 unit matrix.

4 Relation between Octonions and $SU(3)$ Generators

Comparing Table-1 and Table-2, we may observe a resemblance between the octonions and the Gell-Mann λ matrices of $SU(3)$ symmetry on using simultaneously the relations (2) and (17) in the following table as

Octonions basis	SU(3) generators
$e_1 \longmapsto$	$i\lambda_1$
$e_2 \longmapsto$	$i\lambda_2$
$e_3 \longmapsto$	$i\lambda_3$
$e_4 \longmapsto$	$\frac{i}{2}\lambda_4$
$e_5 \longmapsto$	$\frac{i}{2}\lambda_5$
$e_6 \longmapsto$	$-\frac{i}{2}\lambda_6$
$e_7 \longmapsto$	$-\frac{i}{2}\lambda_7$
$e_0 \longmapsto$	$\frac{\sqrt{3}}{2}\lambda_8$

Table 3- Relation between Octonion basis and SU(3) generators.

As such, we have the freedom to establish [20] a connection between the octonion basis elements e_A and 3×3 Gell-Mann λ matrices of $SU(3)$ in the following manner i.e.

$$e_1 \Rightarrow i\lambda_1,\ e_2 \Rightarrow i\lambda_2,\ e_3 \Rightarrow i\lambda_3 \longmapsto e_A \Longleftrightarrow i\lambda_A; \quad (\forall\, A = 1, 2, 3);$$
$$e_4 \Rightarrow \frac{i}{2}\lambda_4,\ e_5 \Rightarrow \frac{i}{2}\lambda_5, \Longleftrightarrow e_A = \frac{i}{2}\lambda_A; \quad (\forall\, A = 4, 5,);$$
$$e_6 \Rightarrow \frac{i}{2}\lambda_6,\ e_7 \Rightarrow -\frac{i}{2}\lambda_7, \Longleftrightarrow e_A = -\frac{i}{2}\lambda_A; \quad (\forall\, A = 6, 7,);$$
$$e_0 \Longleftrightarrow \frac{\sqrt{3}}{2}\lambda_8. \tag{18}$$

These results are similar to the those derived earlier by Günaydin Gürsey [18] for octonion units and λ matrices of $SU(3)$ symmetry. Here, equation (18) satisfies the Cayley algebra followed by the octonion

multiplication rule $e_A \cdot e_B = -\delta_{AB} + f_{ABC} e_C$. So, we have the freedom to establish the following relations among structure constants of octonions and $SU(3)$ symmetry as

$$
\begin{aligned}
F^{ABC} &\Rightarrow f^{ABC} \ (\forall \ ABC = 123); \\
F^{ABC} &\Rightarrow \frac{1}{2} f^{ABC} \ (\forall \ ABC = 147,\, 246,\, 257,\, 435,\, 516,\, 673); \\
F^{ABC} &\Rightarrow \frac{\sqrt{3}}{2} f^{ABC} \ (\forall \ ABC = 458,\, 678).
\end{aligned}
\tag{19}
$$

Hence, we get

$$
\begin{aligned}
[e_A, e_B] &\Rightarrow i\,[\lambda_A, \lambda_B] & (\forall \ ABC = 123); \\
[e_A, e_B] &\Rightarrow \frac{i}{2}\,[\lambda_A, \lambda_B] & (\forall \ ABC = 147,\, 246,\, 257,\, 435,\, 516, 673); \\
[e_A, e_B] &\Rightarrow \frac{\sqrt{3}}{2} i\,[\lambda_A, \lambda_B] & (\forall \ ABC = 458,\, 678);
\end{aligned}
\tag{20}
$$

which are the commutation relations among octonions basis elements and Gell-Mann λ matrices of $SU(3)$ symmetry the so called Eight fold way. The benefit to write the octonions in terms of Gell-Mann λ matrices of $SU(3)$ symmetry may be described as

$\diamond$ Non-associativity of octonions does not effect the invariance of the symmetry group $SU(2)$ spin (or isospin) multiplets for the given values of structure constants f^{ABC}.

$\diamond$ It is better to describe the $SU(3)$ symmetry in terms of compact notations of octonions. Accordingly, the theory of strong interactions could be described better in terms of non associative Cayley algebra.

$\diamond$ The eighth Gell-Mann λ matrix could be designated in terms of hyper charge which may have the direct link with the scalar octonion unit e_0.

$\diamond$ It may be concluded that the algebra of strong interactions corresponds to the $SU(3)$ automorphisms of the octonion algebra which is in support of the results obtained earlier by Günaydin [19].

5   Octonions and QCD

The color group $SU(3)$ corresponds to the local symmetry whose gauging gives rise to Quantum Chromodynamics (QCD).There are two different types of $SU(3)$ symmetry. The first one is the symmetry that acts on the different colors of quarks. This symmetry is an exact gauge symmetry mediated by the gluons. Other $SU(3)$ symmetry is a flavor symmetry which rotates different flavors of quarks to each other, or flavor $SU(3)$. Flavor SU(3) is an approximate symmetry of the vacuum of QCD, and is not a fundamental symmetry at all. It is an accidental consequence of the small mass of the three lightest quarks. Here, we are interested in exact $SU(3)$ symmetry of colors in terms of octonion algebra. For this, let us substitute the values of octonion units e_A in terms of λ_A from equation (18) in to equation (1) so that we may express an octonion x as

$$
x = x_0 \left(\frac{\sqrt{3}}{2} \lambda_8 \right) + x_1 \left(i\lambda_1 \right) + x_2 \left(i\lambda_2 \right) + x_3 \left(i\lambda_3 \right) + x_4 \left(\frac{i}{2} \lambda_4 \right) + x_5 \left(\frac{i}{2} \lambda_5 \right) + x_6 \left(-\frac{i}{2} \lambda_6 \right) + x_7 \left(-\frac{i}{2} \lambda_7 \right)
\tag{21}
$$

which can further be reduced as

$$x = x_0 O_0 + x_1 \mathcal{O}_1 + x_2 \rangle \mathcal{O}_2 + x_3 \mathcal{O}_3 + x_4 \mathcal{O}_4 + x_5 \mathcal{O}_5 + x_6 \mathcal{O}_6 + x_7 \mathcal{O}_7 \tag{22}$$

where $\mathcal{O}_0 \to \frac{\sqrt{3}}{2}\lambda_8$, $\mathcal{O}_1 \to i\lambda_1$, $\mathcal{O}_2 \to i\lambda_2$, $\mathcal{O}_3 \to i\lambda_3$, $\mathcal{O}_4 \to \frac{i}{2}\lambda_4$, $\mathcal{O}_5 \to \frac{i}{2}\lambda_5$, $\mathcal{O}_6 \to -\frac{i}{2}\lambda_6$, $\mathcal{O}_7 \to -\frac{i}{2}\lambda_7$. Thus the octonion conjugate be written as as

$$\bar{x} = x_0 \mathcal{O}_0 - x_1 \mathcal{O}_1 - x_2 \mathcal{O}_2 - x_3 \mathcal{O}_3 - x_4 \mathcal{O}_4 - x_5 \mathcal{O}_5 - x_6 \mathcal{O}_6 - x_7 \mathcal{O}_7. \tag{23}$$

Here the new octonion units $\mathcal{O}_A$ associated with the $SU(3)$ symmetry satisfy the octonion algebra

$$\mathcal{O}_A \cdot \mathcal{O}_B = -\delta_{AB} + f_{ABC}\, \mathcal{O}_C. \tag{24}$$

As such, we may reformulate the theory of strong interactions, the quantum Chromodynamics (QCD) based on colour $SU(3)_c$ whose generators satisfy the non-associative algebra of octonions. Let us consider the triplet (u, d, s) (i.e. the up, down, and strange) flavors of quarks as the three objects of the group namely the $SU(3)$ group of flavor symmetry. Each quark has been described in terms of three colors namely the red, blue and green. The dynamics of the quarks and gluons are controlled by the quantum Chromodynamics Lagrangian. The gauge invariant QCD Lagrangian is described as

$$\begin{aligned} \mathcal{L} &= \bar{\psi}_j (i\gamma^\mu (D_\mu)_{jk} - m\,\delta_{jk})\psi_k - \frac{1}{4}\, G_{\mu\nu}^a G_a^{\mu\nu} \\ &= \bar{\psi}_j (i\gamma^\mu \partial_\mu - m\,)\psi_j - g\, G_\mu^a \bar{\psi}_j \gamma^\mu T_{jk}^a \psi_k - \frac{1}{4}\, G_{\mu\nu}^a G_a^{\mu\nu} \end{aligned} \tag{25}$$

where $(j, k = 1, 2, 3)$ are labeled for three quark fields associated with three colors (namely the red, blue and green) so that we have

$$\psi_j = \begin{pmatrix} \psi_R \\ \psi_B \\ \psi_G \end{pmatrix}; \qquad \bar{\psi}_j = (\bar{\psi}_R, \bar{\psi}_B, \bar{\psi}_G) \tag{26}$$

which is a dynamical function of space-time, in the fundamental representation of the $SU(3)$ gauge group, indexed by $(j, k = 1, 2, 3)$. In equation (25) G_μ^a is the octet of gluon fields which is also a dynamical function of space-time in the adjoint representation of the $SU(3)$ gauge group, indexed by $a, b, \dots = 1, 2, \dots, 8$; the γ^μ are the Dirac matrices connecting the spinor representation to the vector representation of the Lorentz group; and T_{jk}^a are the generators connecting the fundamental, anti-fundamental and adjoint representations of the $SU(3)$ gauge group. In our case, the octonion units connecting to Gell-Mann λ matrices provide one such representation for the generators of $SU(3)$ gauge group. In equation (25), the symbol $G_{\mu\nu}^a$ represents the gauge invariant gluon field strength tensor, analogous to the electromagnetic field strength tensor $F_{\mu\nu}$ in Electrodynamics. It is described by

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g\, f^{abc}\, G_\mu^b G_\mu^c \ \ (\forall a, b, c = 1, 2, 3, \dots, 8) \tag{27}$$

where f^{abc} are the structure constants of $SU(3)$ groups as described above in terms of octonions. In equation (25), the constants m and g control the quark mass and coupling constants of the theory, subject to renormalization in the full quantum theory. Here we may introduce a local phase transformation in color space. Under $SU(3)$ symmetry the spinor ψ transforms as

$$\psi \longrightarrow \psi^{'} = U\psi = \exp\{i\lambda.\alpha(x)\}\psi; \qquad (\lambda = 1, 2, ..., 8) \tag{28}$$

where

$$\lambda.\alpha(x) = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3 + \lambda_4\alpha_4 + \lambda_5\alpha_5 + \lambda_6\alpha_6 + \lambda_7\alpha_7 + \lambda_8\alpha_8 \tag{29}$$

which on using Table-3, may directly be written in the in the following form in terms of octonions i.e.

$$\lambda.\alpha(x) = -ie_1\alpha_1 - ie_2\alpha_2 - ie_3\alpha_3 - 2ie_4\alpha_4 - 2ie_5\alpha_5 - 2ie_6\alpha_6 - 2ie_7\alpha_7 + \frac{2}{\sqrt{3}}e_0\alpha_8. \tag{30}$$

As such, the the quantum Chromodynamics (QCD) may be reformulated in terms of octonions and non-associative algebra in order to explain its interesting consequences like

◇ Quarks confinement

◇ Color blindness of nature

◇ Asymptotic freedom

◇ Calculation for the masses of mesons and baryons etc.

6 Split octonions

The split octonions [18, 19, 20, 29] are a non associative extension of split quaternions. They differ from the octonion in the signature of quadratic form. The split octonion have a signature $(4, 4)$ whereas the octonions have positive signature $(8, 0)$. The Cayley algebra of octonions over the field of complex number is visualized as the algebra of split octonions with its following basis element,

$$u_0 = \frac{1}{2}(e_0 + ie_7), \qquad u_0^{\star} = \frac{1}{2}(e_0 - ie_7),$$
$$u_j = \frac{1}{2}(e_j + ie_{j+3}), \qquad u_j^{\star} = \frac{1}{2}(e_j - ie_{j+3})\,(\forall j = 1, 2, 3) \tag{31}$$

where $(\star)$ denotes the complex conjugation and $(i = \sqrt{-1})$, the imaginary unit, commutes with all $e_A (\forall A = 1, 2, 3, ..., 7)$. In equation (31) $u_0, u_0^{\star}, u_j, u_j^{\star}$ are defined as the bi-valued representations of quaternion units e_0 , e_1, e_2, e_3 which satisfy the following [31] multiplication rule

$$e_je_k = -\delta_{jk} + \epsilon_{jkl}e_l\,(\forall j, k, l = 1, 2, 3) \tag{32}$$

where ϵ_{jkl} are the three index Levi-Civita symbols. The split octonion basis elements $u_0, u_0^{\star}, u_j, u_j^{\star}$ satisfy the following multiplication rule

$$u_i u_j = \epsilon_{ijk} u_k^\star; \qquad u_i^\star u_j^\star = -\epsilon_{ijk} u_k^\star \qquad (\forall\, i,j,k = 1,2,3)$$

$$u_i u_j^\star = -\delta_{ij} u_0; \qquad u_i u_0 = 0; \qquad u_i^\star u_0 = u_i^\star$$

$$u_i^\star u_j = -\delta_{ij} u_0; \qquad u_i u_0^\star = u_0; \qquad u_i^\star u_0^\star = 0$$

$$u_0 u_i = u_i; \qquad u_0^\star u_i = 0; \qquad u_0 u_i^\star = 0$$

$$u_0^\star u_i^\star = u_i; \qquad u_0^2 = u_0; \qquad u_0^{\star 2} = u_0^\star; \qquad u_0 u_0^\star = u_0^\star u_0 = 0. \tag{33}$$

So, we may introduce a convenient realization for the basis elements $(u_0,\, u_0^\star,\, u_j,\, u_j^\star)$ in term of Pauli's spin matrices as

$$u_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \qquad u_0^\star = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

$$u_j = \begin{bmatrix} 0 & 0 \\ e_j & 0 \end{bmatrix}; \qquad u_j^\star = \begin{bmatrix} 0 & -e_j \\ 0 & 0 \end{bmatrix}. \qquad (\forall\, j = 1,2,3) \tag{34}$$

The Cayley's split octonion algebra may also be expressed via 2×2 Zorn's vector matrix realizations as

$$A = a u_0^\star + b u_0 + x_i u_i^\star + y_i u_i = \begin{pmatrix} a & -\overrightarrow{x} \\ \overrightarrow{y} & b \end{pmatrix} \tag{35}$$

where a and b are scalars and $\overrightarrow{x}$ and $\overrightarrow{y}$ are $3-$vectors with the product defined [18, 32] as

$$\begin{pmatrix} a & \overrightarrow{x} \\ \overrightarrow{y} & b \end{pmatrix} \begin{pmatrix} c & \overrightarrow{u} \\ \overrightarrow{v} & d \end{pmatrix} = \begin{pmatrix} ac + \overrightarrow{x} \cdot \overrightarrow{v} & a\overrightarrow{u} + d\overrightarrow{x} - \overrightarrow{y} \times \overrightarrow{v} \\ c\overrightarrow{y} + b\overrightarrow{v} - \overrightarrow{x} \times \overrightarrow{u} & bd + \overrightarrow{y} \cdot \overrightarrow{u} \end{pmatrix}. \tag{36}$$

Hence, the split octonion conjugate equation may be written via 2×2 Zorn's vector matrix realizations as

$$\overline{A} = a u_0 + b u_0^\star - x_i u_i^\star - y_i u_i = \begin{pmatrix} b & \overrightarrow{x} \\ -\overrightarrow{y} & a \end{pmatrix}. \tag{37}$$

So, the norm of A is defined as

$$A\overline{A} = \overline{A}A = (ab + \overrightarrow{x} \cdot \overrightarrow{y})\hat{1} \tag{38}$$

where $\hat{1}$ as the unit matrix of order 2×2.

7 Split-Octonion $SU(3)$ Gauge Theory

The automorphism group of the Octonion algebra is the 14-dimensional exceptional G_2 group that admits a $SU(3)$ subgroup leaving invariant the idempotents u_0 and $u_0^\star$ described by equation (31). This $SU(3)_C$ was identified as the color group acting on the quark and anti-quark triplets [19, 32]. As such, the automorphism group $SU(3)$ of the quantum mechanical Hilbert space should be considered as an exact

symmetry and can not be identified as the symmetry of broken unitary spin gauge group. It is like the $SU(3)_C$ color gauge group of quantum chromo-dynamics (QCD). Therefore, in order to describe the $SU(3)$ gauge theory suitably handled with split octonions, let us start with the split octonion equivalent of any four vector A_μ and its conjugate in terms of the following 2×2 Zorn matrix realization as

$$Z(A) = \begin{bmatrix} x_4 & -\overrightarrow{x} \\ \overrightarrow{y} & y_4 \end{bmatrix}; \qquad Z(\overline{A}) = \begin{bmatrix} x_4 & \overrightarrow{x} \\ -\overrightarrow{y} & y_4 \end{bmatrix}. \tag{39}$$

Rather, the Octonion covariant derivative or $\mathcal{O}$- derivative of an octonion K is defined [32, 33, 34] as

$$K_{\|\mu} = K_{,\mu} + [\Im_\mu, K] \tag{40}$$

where $\Im_\mu$ is the Octonion affinity. It is the object that makes $K_{\|\mu}$ transform like an octonion under $\mathcal{O}$ transformations i.e.

$$\begin{aligned} K' &= U K U^{-1} \\ K'_{\|\mu} &= U K_{\|\mu} U^{-1} \\ \Im'_\mu &= U \Im_\mu U^{-1} - \frac{\partial U}{\partial x^\mu} U^{-1} \end{aligned} \tag{41}$$

where the $U(x)$ are octonions which define local (Octonion) unitary transformations and are isomorphic to the rotation group $O(3)$. Thus, equation (41) describes $SU(2)$ nature of octonion $\mathcal{O}$ transformations resulting to the octonion affinity (gauge potential) $\Im_\mu$ of Yang-Mill's type field and is expressed as.

$$\Im_\mu = -L_{\mu j} u_j^\star - K_{\mu j} u_j = \begin{bmatrix} 0_2 & L_{\mu i} e_j \\ -K_{\mu j} e_j & 0_2 \end{bmatrix} (\forall j = 1, 2, 3) \tag{42}$$

where the quaternion units $e_j = -i\sigma_j$ are suitably handled [31] with Pauli spin matrices σ_j. Now, we have the freedom to extend $SU(2)$ gauge theory to the case of $SU(3)$ Yang Mills gauge theory of colored quarks by replacing the Pauli spin matrices to Gellmann λ matrices. So, from equation (42), it is clear that octonion covariant derivative (40) is subjected by two real (or one complex) gauge potential transformations. Hence, G_μ^a the octet of gluon fields describing Lagrangian (25) is either a complex gauge field or comprises the order pair of two real gauge fields. So, we may write the covariant derivative D_μ for $SU(3)$ Lagrangian (25) as

$$D_\mu = \partial_\mu + \mathbb{V}_\mu \tag{43}$$

where $\mathbb{V}_\mu$ is the octonion form of generalized four potential described as

$$\mathbb{V}_\mu = e_0 \left(A_\mu^\alpha e_\alpha \right) + i e_7 \left(B_\mu^\alpha g_\alpha \right) \qquad (\forall \mu = 0, 1, 2, 3; \alpha = 1, 2, ..., 8.) \tag{44}$$

The beauty of the equation (44) reinforces the $SU(3)$ symmetry of colored quarks with two gauge potentials as the consequence of automorphism group of split octonions in terms of 2×2 Zorn vector matrix realization. Here, the two gauge potentials A_μ^α and $B_\mu^\alpha (\forall \mu = 0, 1, 2, 3; \alpha = 1, 2, ..., 8.)$ may be identified as the gauge potentials for two chromo charges supposed to be responsible for the existence of electric

and magnetic chromo-charges. It may, therefore, be concluded that octonion colored quarks are dyons: the particles which carry the simultaneous existence of electric and magnetic charges [24, 25, 26, 27]. Substituting the value of $SU(3)_C$ octonion gauge potential $\mathbb{V}_\mu$ (44) in to the equation (43), we may write the covariant derivative D_μ as

$$
\begin{aligned}
D_\mu &= \partial_\mu + e_0 \left(A_\mu^\alpha e_\alpha \right) + ie_7 \left(B_\mu^\alpha g_\alpha \right) \\
&= u_0^* \left(\partial_\mu + A_\mu^\alpha e_\alpha + B_\mu^\alpha g_\alpha \right) + u_0 \left(\partial_\mu + A_\mu^\alpha e_\alpha - B_\mu^\alpha g_\alpha \right)
\end{aligned}
\tag{45}
$$

which may is equivalently [34] be written as

$$
D_\mu = \begin{pmatrix} \partial_\mu + \left(e_\alpha A_\mu^\alpha + g_\alpha B_\mu^\alpha \right) & 0 \\ 0 & \partial_\mu + \left(e_\alpha A_\mu^\alpha - g_\alpha B_\mu^\alpha \right) \end{pmatrix}.
\tag{46}
$$

It yields to

$$
[D_\mu, D_\nu] = \begin{pmatrix} G_{\mu\nu}^\alpha e_\alpha + \mathsf{G}_{\mu\nu}^\alpha g_\alpha & 0 \\ 0 & G_{\mu\nu}^\alpha e_\alpha - \mathsf{G}_{\mu\nu}^\alpha g_\alpha \end{pmatrix} \longmapsto \mathbb{G}_{\mu\nu}^\alpha
\tag{47}
$$

where

$$
\begin{aligned}
G_{\mu\nu}^\alpha &= \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + e_\alpha \left[A_\mu^\alpha, A_\nu^\alpha \right] \longmapsto E_{\mu\nu}^\alpha; \\
\mathsf{G}_{\mu\nu}^\alpha &= \partial_\mu B_\nu^\alpha - \partial_\nu B_\mu^\alpha + g_\alpha \left[B_\mu^\alpha, B_\nu^\alpha \right] \longmapsto H_{\mu\nu}^\alpha;
\end{aligned}
\tag{48}
$$

are two $SU(3)$ non-Abelian gauge field strengths in term of electric $\left(E_{\mu\nu}^\alpha \right)$ and magnetic field $\left(H_{\mu\nu}^\alpha \right)$ field strengths obtained respectively from electric A_μ^α and magnetic B_μ^α gauge potentials. Operating the covariant derivative D_μ (46) to the the generalized field strength $\mathbb{G}_{\mu\nu}^\alpha$ of dyons (47), we get

$$
D_\mu \mathbb{G}^\alpha{}_{\mu\nu} = \begin{pmatrix} \partial_\mu G_{\mu\nu}^\alpha e_\alpha + \partial_\mu \mathsf{G}_{\mu\nu}^\alpha g_\alpha & 0 \\ 0 & \partial_\mu G_{\mu\nu}^\alpha e_\alpha - \partial_\mu \mathsf{G}_{\mu\nu}^\alpha g_\alpha \end{pmatrix} = \mathbb{J}_\nu^\alpha
\tag{49}
$$

where $\mathbb{J}_\nu^\alpha$ describes the generalized octonion gauge current of dyons in term of 2×2 Zorn matrix realization of split octonion $SU(3)$ gauge theory. It also comprises the electric and magnetic four currents of dyons as

$$
\mathbb{J}_\nu^\alpha = \begin{pmatrix} J_\nu^\alpha e_\alpha + K_\nu^\alpha g_\alpha & 0 \\ 0 & J_\nu^\alpha e_\alpha - K_\nu^\alpha g_\alpha \end{pmatrix}.
\tag{50}
$$

Here $J_\nu^\alpha = \partial_\mu G_{\mu\nu}^\alpha$ and $K_\nu^\alpha = \partial_\mu \mathsf{G}_{\mu\nu}^\alpha$ are the four currents respectively associated with the presence of electric and magnetic charges. So, it is concluded concluded that split octonion $SU(3)$ gauge theory of colored quarks describes dyons which are the particles carrying the simultaneous existence of electric and magnetic monopoles.

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