Second Order System

Transfer Function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n > 0$$

Poles

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

where ζ is the **damping ratio** and ω_n is the **undamped natural frequency.**

Underdamped System: $0 \geq \zeta < 1$

Critically Damped System: $\zeta = 1$

Overdamped System: $\zeta > 1$

Underdamped & Critically Damped System, $0 \geq \zeta \geq 1$

Define poles at,

$$s = -\sigma \pm j\omega_d$$

such that,

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

where ω_d is the **damped natural frequency.**

The transfer function becomes:

$$H(s) = \frac{\omega_n^2}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d}$$

Step Response

$$y(t) = 1 - e^{-\sigma t}(cos(\omega_d t) + \frac{\sigma}{\omega_d} sin(\omega_d t))$$

Overdamped System, $\zeta > 1$

Poles are at,

$$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The transfer function becomes:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}$$

Step Response

$$y(t) = 1 - k_1 e^{(-\zeta\omega - \omega_n\sqrt{\zeta^2 - 1})t} - k_2 e^{(-\zeta\omega + \omega_n\sqrt{\zeta^2 - 1})t}$$

For some constants k_1 and k_2.

Performance Measures

Rise Time, t_r

Rise time is the time required to go from 10% to 90% of the final value.

$$t_r \approx \frac{2.16\zeta + 0.6}{\omega_n}$$

which is good for $0.3 > \zeta > 0.8$.

Cruder approximation:

$$t_r \approx \frac{1.8}{\omega_n}$$

Peak Time, t_p

Peak time is the time at which the maximum value is reached.

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1 - \zeta^2}}$$

Peak Value, M_p

Peak value is the maximum value the output reaches.

$$M_p = 1 + e^{-\sigma\frac{\pi}{\omega_d}} = 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Overshoot Percentage, $\%OS$

Overshoot percentage is the percentage by which the response overshoots from the steady state value in the first peak.

$$\%OS = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Settling Time, t_s

Settling time is the time required to get within 2% of the final value and stay there.

$$t_s \approx \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$$