

Fourier Series

Definition

Exponential Form

Let $x(t)$ be a complex-valued periodic signal with period $T > 0$ and angular frequency $\omega_0 = \frac{2\pi}{T}$. Then,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

where,

$$X_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} dt, \quad n \in \mathbb{Z}$$

and $\tilde{x}(t)$ is the **Fourier Series** of $x(t)$.

Special Case for Real Periodic Signals

Let $x(t)$ be a real-valued periodic signal with period $T > 0$ and angular frequency $\omega_0 = \frac{2\pi}{T}$. Then,

$$\tilde{x}(t) = X_0 + 2 \sum_{n=1}^{\infty} \text{Re}(X_n e^{jn\omega_0 t})$$

where,

$$X_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} dt, \quad n \in \mathbb{Z}$$

and $\tilde{x}(t)$ is the **Fourier Series** of $x(t)$.

Trigonometric Form

Let $x(t)$ be a complex-valued periodic signal with period $T > 0$ and angular frequency $\omega_0 = \frac{2\pi}{T}$. Then,

$$\begin{aligned} \tilde{x}(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)) \\ \tilde{x}(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)) \end{aligned}$$

where,

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

$$a_n = \frac{2}{T} \int_T x(t) \cos(n\omega_0 t) dt = \frac{2}{T} \int_T x(t) \cos(2\pi n f_0 t) dt$$

$$b_n = \frac{2}{T} \int_T x(t) \sin(n\omega_0 t) dt = \frac{2}{T} \int_T x(t) \sin(2\pi n f_0 t) dt$$

and $\tilde{x}(t)$ is the **Fourier Series** of $x(t)$.

Fourier Transform

Definition

Let $x(t)$ be a complex-valued signal and $\omega = 2\pi f$. Then,

$$\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad \text{for all } -\infty < \omega < \infty$$

$$\mathcal{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt, \quad \text{for all } -\infty < f < \infty$$

and,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad \text{for all } -\infty < t < \infty$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df, \quad \text{for all } -\infty < t < \infty$$

where $\mathcal{F}\{x(t)\} = X(\omega)$ and $\mathcal{F}\{x(t)\} = X(f)$ are **Fourier Transforms** of $x(t)$.

Properties

Linearity

- $\mathcal{F}\{c_1 x_1(t) + c_2 x_2(t)\} = c_1 \mathcal{F}\{x_1(t)\} + c_2 \mathcal{F}\{x_2(t)\}$

Duality

- $\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$
- $\mathcal{F}\{X(t)\} = x(-f)$

Complex Conjugate

- $\mathcal{F}\{x^*(t)\} = X^*(-\omega)$
- $\mathcal{F}\{x^*(t)\} = X^*(-f)$

Symmetry

- $X(-\omega) = X^*(\omega)$
- $X(-f) = X^*(f)$

Time Scaling

- $\mathcal{F}\{x(\alpha t)\} = \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$
- $\mathcal{F}\{x(\alpha t)\} = \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$

Time Shift

- $\mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0} X(\omega)$
- $\mathcal{F}\{x(t - t_0)\} = e^{-j2\pi f t_0} X(f)$

Frequency Shift

- $\mathcal{F}\{e^{j\omega t} x(t)\} = X(\omega - \omega_0)$
- $\mathcal{F}\{e^{j2\pi f t} x(t)\} = X(f - f_0)$
- $\mathcal{F}\{\cos(\omega_0 t) x(t)\} = \frac{X(\omega + \omega_0) + X(\omega - \omega_0)}{2}$
- $\mathcal{F}\{\cos(2\pi f_0 t) x(t)\} = \frac{X(f + f_0) + X(f - f_0)}{2}$
- $\mathcal{F}\{\sin(\omega_0 t) x(t)\} = j \frac{X(\omega + \omega_0) - X(\omega - \omega_0)}{2}$
- $\mathcal{F}\{\sin(2\pi f_0 t) x(t)\} = j \frac{X(f + f_0) - X(f - f_0)}{2}$

Derivatives

- $\mathcal{F}\{x^{(n)}(t)\} = (j\omega)^n X(\omega)$
- $\mathcal{F}\{x^{(n)}(t)\} = (j2\pi f)^n X(f)$

Integrals

- $\mathcal{F}\left\{\int_{-\infty}^t x(\tau) d\tau\right\} = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
- $\mathcal{F}\left\{\int_{-\infty}^t x(\tau) d\tau\right\} = \frac{X(f)}{j2\pi f} + \frac{1}{2} X(0)\delta(f)$

Convolution

- $\mathcal{F}\{x(t) * y(t)\} = X(\omega)Y(\omega)$
- $\mathcal{F}\{x(t) * y(t)\} = X(f)Y(f)$

Multiplication

- $\mathcal{F}\{x(t)y(t)\} = \frac{1}{2\pi} (X(\omega) * Y(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)Y(\omega - \lambda)d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\lambda)X(\omega - \lambda)d\lambda$
- $\mathcal{F}\{x(t)y(t)\} = X(f) * Y(f) = \int_{-\infty}^{\infty} X(\lambda)Y(f - \lambda)d\lambda = \int_{-\infty}^{\infty} Y(\lambda)X(f - \lambda)d\lambda$

Parseval's Theorem

- $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y^*(\omega)d\omega$
- $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$

Fourier Transform Table

Signal $x(t)$ for $t \geq 0$	Fourier Transform $X(\omega)$	Fourier Transform $X(f)$
$\delta(t)$	1	1
1	$2\pi\delta(\omega)$	$\delta(f)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$	$e^{-j2\pi f t_0}$
$\cos(\omega_0 t), \cos(2\pi f_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$\frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$
$\sin(\omega_0 t), \sin(2\pi f_0 t)$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$\frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\Pi(t)$	$\pi \text{sinc}(\frac{\omega}{2\pi})^*$	$\text{sinc}(f)^*$
$\Lambda(t)$	$\pi^2 \text{sinc}^2(\frac{\omega}{2\pi})^*$	$\text{sinc}^2(f)^*$
$\text{sgn}(t)$	$\frac{2}{j\omega}$	$\frac{1}{j\pi f}$

$$^* \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$