Fourier Series

Definition

Exponential Form

Let $x(t)$ be a complex-valued periodic signal with period $T > 0$ and angular frequency $\omega_0 = \frac{2\pi}{T}$. Then,

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

where,

$$X_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} dt$$

and $\tilde{x}(t)$ is the **Fourier Series** of $x(t)$.

Special Case for Real Periodic Signals

Let $x(t)$ be a real-valued periodic signal with period $T > 0$ and angular frequency $\omega_0 = \frac{2\pi}{T}$. Then,

$$\tilde{x}(t) = X_0 + 2 \sum_{n=1}^{\infty} Re(X_n e^{jn\omega_0 t})$$

where,

$$X_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_0 t} dt$$

and $\tilde{x}(t)$ is the **Fourier Series** of $x(t)$.

Trigonometric Form

Let $x(t)$ be a complex-valued periodic signal with period $T > 0$ and angular frequency $\omega_0 = \frac{2\pi}{T}$. Then,

$$\tilde{x}(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n cos(n\omega_0 t) + b_n sin(n\omega_0 t))$$

$$\tilde{x}(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n cos(2\pi n f_0 t) + b_n sin(2\pi n f_0 t))$$

where,

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

$$a_n = \frac{2}{T} \int_T x(t) cos(n\omega_0 t) dt = \frac{2}{T} \int_T x(t) cos(2\pi n f_0 t) dt$$

$$b_n = \frac{2}{T} \int_T x(t) sin(n\omega_0 t) dt = \frac{2}{T} \int_T x(t) sin(2\pi n f_0 t) dt$$

and $\tilde{x}(t)$ is the **Fourier Series** of $x(t)$.

Fourier Transform

Definition

Let $x(t)$ be a complex-valued signal and $\omega = 2\pi f$. Then,

$$\mathcal{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \qquad for\, all -\infty < \omega < \infty$$
$$\mathcal{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt, \qquad for\, all -\infty < f < \infty$$

and,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \qquad for\, all -\infty < t < \infty$$
$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df, \qquad for\, all -\infty < t < \infty$$

where $\mathcal{F}\{x(t)\} = X(\omega)$ and $\mathcal{F}\{x(t)\} = X(f)$ are **Fourier Transforms** of $x(t)$.

Properties

Linearity

- $\mathcal{F}\{c_1x_1(t) + c_2x_2(t)\} = c_1\mathcal{F}\{x_1(t)\} + c_2\mathcal{F}\{x_2(t)\}$

Duality

- $\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$
- $\mathcal{F}\{X(t)\} = x(-f)$

Complex Conjugate

- $\mathcal{F}\{x^*(t)\} = X^*(-\omega)$
- $\mathcal{F}\{x^*(t)\} = X^*(-f)$

Symmetry

- $X(-\omega) = X^*(\omega)$
- $X(-f) = X^*(f)$

Time Scaling

- $\mathcal{F}\{x(\alpha t)\} = \frac{1}{|a|}X(\frac{\omega}{\alpha})$
- $\mathcal{F}\{x(\alpha t)\} = \frac{1}{|a|}X(\frac{f}{\alpha})$

Time Shift

- $\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(\omega)$
- $\mathcal{F}\{x(t-t_0)\} = e^{-j2\pi ft_0}X(f)$

Frequency Shift

- $\mathcal{F}\{e^{j\omega_0 t}x(t)\} = X(\omega - \omega_0)$
- $\mathcal{F}\{e^{j2\pi f_0 t}x(t)\} = X(f - f_0)$
- $\mathcal{F}\{cos(\omega_0 t)x(t)\} = \frac{X(\omega+\omega_0)+X(\omega-\omega_0)}{2}$
- $\mathcal{F}\{cos(2\pi f_0 t)x(t)\} = \frac{X(f+f_0)+X(f-f_0)}{2}$
- $\mathcal{F}\{sin(\omega_0 t)x(t)\} = j\frac{X(\omega+\omega_0)-X(\omega-\omega_0)}{2}$

- $\mathcal{F}\{sin(2\pi f_0 t)x(t)\} = j\frac{X(f+f_0)-X(f-f_0)}{2}$

Derivatives

- $\mathcal{F}\{x^{(n)}(t)\} = (j\omega)^n X(\omega)$
- $\mathcal{F}\{x^{(n)}(t)\} = (j2\pi ft)^n X(f)$

Integrals

- $\mathcal{F}\{\int_{-\infty}^{t} x(\tau)d\tau\} = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
- $\mathcal{F}\{\int_{-\infty}^{t} x(\tau)d\tau\} = \frac{X(f)}{j2\pi ft} + \frac{1}{2}X(0)\delta(f)$

Convolution

- $\mathcal{F}\{x(t) * y(t)\} = X(\omega)Y(\omega)$
- $\mathcal{F}\{x(t) * y(t)\} = X(f)Y(f)$

Multiplication

- $\mathcal{F}\{x(t)y(t)\} = \frac{1}{2\pi}(X(\omega) * Y(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)Y(\omega-\lambda)d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda)X(\omega-\lambda)d\lambda$
- $\mathcal{F}\{x(t)y(t)\} = X(f) * Y(f) = \int_{-\infty}^{\infty} X(\lambda)Y(f-\lambda)d\lambda = \int_{-\infty}^{\infty} Y(\lambda)X(f-\lambda)d\lambda$

Multiplication by t

- $\mathcal{F}\{tx(t)\} = jX'(\omega)$
- $\mathcal{F}\{t^n x(t)\} = j^n X^{(n)}(\omega)$
- $\mathcal{F}\{tx(t)\} = \frac{j}{2\pi}X'(f)$
- $\mathcal{F}\{t^n x(t)\} = (\frac{j}{2\pi})^n X^{(n)}(f)$

Parseval's Theorem

- $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)d\omega$
- $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$

Fourier Transform Table

Signal $x(t)$ for $t \geq 0$	Fourier Transform $X(\omega)$	Fourier Transform $X(f)$
$\delta(t)$	1	1
1	$2\pi\delta(\omega)$	$\delta(f)$
$\delta(t-t_0)$	$e^{-j\omega t_0}$	$e^{-j2\pi f t_0}$
$cos(\omega_0 t),\ cos(2\pi f_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$
$sin(\omega_0 t),\ sin(2\pi f_0 t)$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\prod(t)$	$\pi sinc(\frac{\omega}{2\pi})$*	$sinc(f)$*
$\Lambda(t)$	$\pi^2 sinc^2(\frac{\omega}{2\pi})$*	$sinc^2(f)$*
$sgn(t)$	$\frac{2}{j\omega}$	$\frac{1}{j\pi f}$
$e^{-\alpha t}u(t),\ [\alpha > 0]$	$\frac{1}{\alpha+j\omega}$	$\frac{1}{\alpha+j2\pi f}$
$e^{-\alpha t}u(-t),\ [\alpha > 0]$	$\frac{1}{\alpha-j\omega}$	$\frac{1}{\alpha-j2\pi f}$
$e^{-\alpha\lvert t\rvert},\ [\alpha > 0]$	$\frac{2\alpha}{\alpha^2+\omega^2}$	$\frac{2\alpha}{\alpha^2+(2\pi f)^2}$
$e^{-\alpha t^2}$	$\sqrt{\frac{\pi}{\alpha}}e^{\frac{-\omega^2}{4\alpha}}$	$\sqrt{\frac{\pi}{\alpha}}e^{\frac{-(2\pi f)^2}{4\alpha}}$
$te^{-\alpha t}u(t),\ [\alpha > 0]$	$\frac{1}{(\alpha+j\omega)^2}$	$\frac{1}{(\alpha+j2\pi f)^2}$
$t^n e^{-\alpha t}u(t),\ [\alpha > 0]$	$\frac{n!}{(\alpha+j\omega)^{n+1}}$	$\frac{n!}{(\alpha+j2\pi ft)^{n+1}}$
$\sum\limits_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\omega_0\sum\limits_{n=-\infty}^{\infty}\delta(\omega-n\omega_0),\ \omega_0=\frac{2\pi}{T_0}$	$\frac{1}{T_0}\sum\limits_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_0})$

$$*sinc(t) = \frac{sin(\pi t)}{\pi t}$$