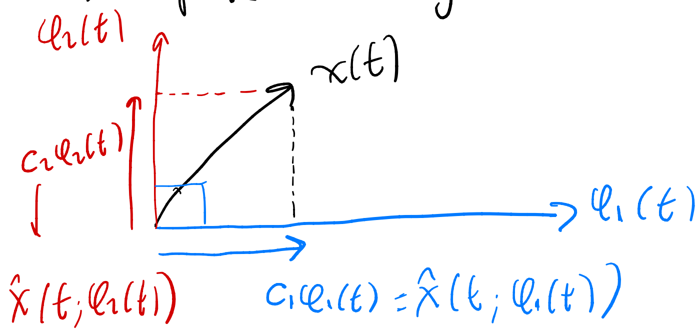
Proceso Digital de señales

- Serie de Fourier: $x(t) \in \mathbb{R}$ ó $\mathbb{C}$; $t \in [t_i, t_f]$

↳ Representación generalizada de Fourier → funciones ortogonales

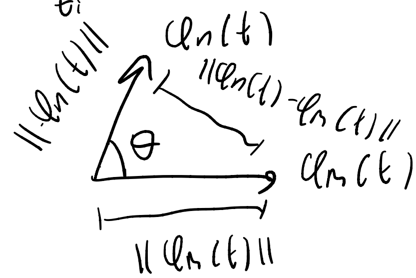


$$\hat{x}(t) = \sum_{n \in \mathbb{Z}} c_n \varphi_n(t)$$

Proveer conjunto de bases ortogonales $\{\varphi_n(t) \in \mathbb{R}$ ó $\mathbb{C}\}_{n\in\mathbb{Z}}$

Bajo restricción de ortogonalidad:

$$\bar{\varepsilon}^2(c_n) = \frac{1}{T}\int_{t_i}^{t_f} |x(t) - \hat{x}(t;c_n)|^2 \partial t; \quad \langle \varphi_n(t), \varphi_m(t)\rangle = \int_{t_i}^{t_f} \varphi_n(t)\varphi_m^*(t)\,\partial t = \begin{cases} 0 ; & n \neq m \\ E_n ; & n = m \end{cases}$$



$T = t_f - t_i$

$$\bar{\varepsilon}^2(c_n) = \frac{1}{T}\int |x(t) - \sum_n c_n\varphi_n(t)|^2 \partial t$$

$$\cos(\theta) = \frac{\langle \varphi_n(t), \varphi_m(t)\rangle}{\|\varphi_n(t)\|\,\|\varphi_m(t)\|}$$

$$\frac{\partial \bar{\varepsilon}^2(c_n)}{\partial c_n} = 0$$

$$c_n = \frac{\langle x(t), \varphi_n(t)\rangle}{\|\varphi_n(t)\|} = \frac{\int_{t_i}^{t_f} x(t)\varphi_n^*(t)\,\partial t}{\int_{t_i}^{t_f} |\varphi_n(t)|^2 \partial t}$$

$$\varphi_n(t) = e^{jn\omega_0 t} = \cos(n\omega_0 t) + j\,\mathrm{Sen}(n\omega_0 t)$$

$$\omega_0 = \frac{2\pi}{T}; \qquad c_n = \frac{\int_{t_i}^{t_f} x(t)\,e^{-jn\omega_0 t}}{T} = \frac{1}{T}\int_{t_i}^{t_f} x(t)\,e^{-jn\omega_0 t}\,\partial t.$$

$$\int_{t_i}^{t_f} e^{jn\omega_0 t} e^{-jn\omega_0 t}\,\partial t = \int_{t_i}^{t_f} e^0 = t_f - t_i = T$$

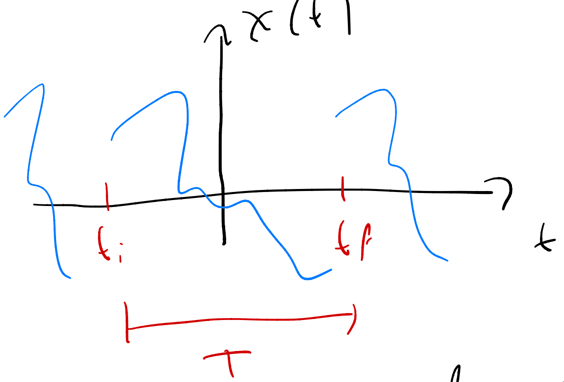
$$\hat{x}(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t};$$

c_n: espectro (frecuencia); n-ésimo armónico

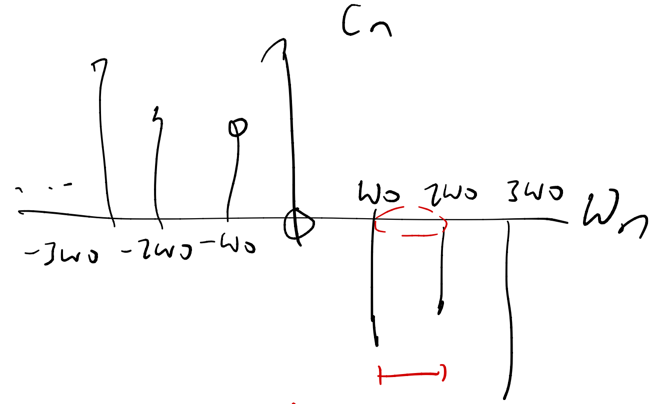
$\omega_n = n\omega_0$

Serie Fourier: tiempo continuo → intervalo finito → Periódica

espectro discreto → $w_n = n w_o$.



Serie →



$\Delta W_n = W_o$

$$W_o = \frac{2\pi}{t_f - t_i} = \frac{2\pi}{T}; \quad \lim_{T\to\infty} \frac{2\pi}{T} \to 0$$

Si $T \to \infty$; $\Delta W_n = W_o \to 0$

espectro discreto a espectro denso (continuo)

Transformada de Fourier

$$\hat{x}(t) = \sum_n C_n e^{jnw_ot} \qquad C_n = \frac{1}{T}\int_{-t_i}^{t_f} x(t)\, e^{-jnw_ot}\, \partial t$$

$$X(W_n) = T C_n; \quad W_n = n W_o$$

$$\hat{x}(t) = \sum_n \frac{T C_n}{T} e^{jnw_ot}$$

$$\hat{x}(t) = \sum_n \frac{X(W_n)}{T} e^{jW_nt}$$

$$W_o = \frac{2\pi}{T} \to T = \frac{2\pi}{W_o} = \frac{2\pi}{\Delta W_n}$$

$$\hat{x}(t) = \sum_{n\in\mathbb{Z}} \frac{X(W_n)}{\frac{2\pi}{\Delta W_n}} e^{jW_nt}$$

$$\hat{x}(t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} X(W_n) e^{jW_nt}\, \Delta W_n$$

$$\lim_{\substack{T\to\infty \\ \Delta W_n \to 0}} \hat{x}(t) = \lim_{\substack{T\to\infty \\ \Delta W_n \to 0}} \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} X(W_n) e^{jW_nt}\, \Delta W_n$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, \partial w$$

densidad $\quad \partial w \to 0$

Transf. Fourier → Rep. integral (densa)

$$x(t) = F^{-1}\{X(w)\}$$

$X(w)$: espectro → función densidad espectral en frecuencia

$$X(W_n) = T C_n = \int_T x(t)\, e^{-jW_nt}\, \partial t \to X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, \partial t = F\{x(t)\}$$

Laplace: Obligar $x(t)$ a cumplir Dirichlet:

$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

$$x(t)e^{-\sigma t} = \tilde{x}(t)$$

$$X(\omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{\tilde{x}(t)\} = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)\,e^{-t(\sigma+j\omega)}\,dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt = \mathcal{L}\{x(t)\}$$

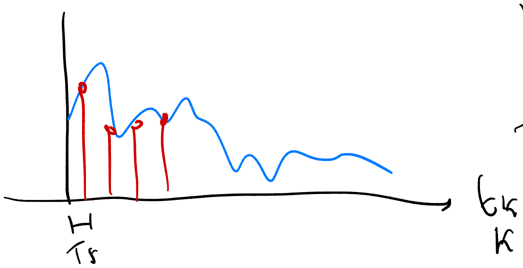
→ tiempo discreto, frec. discreta → cuantizado → DIGITAL.

Transformada de Fourier en tiempo discreto.

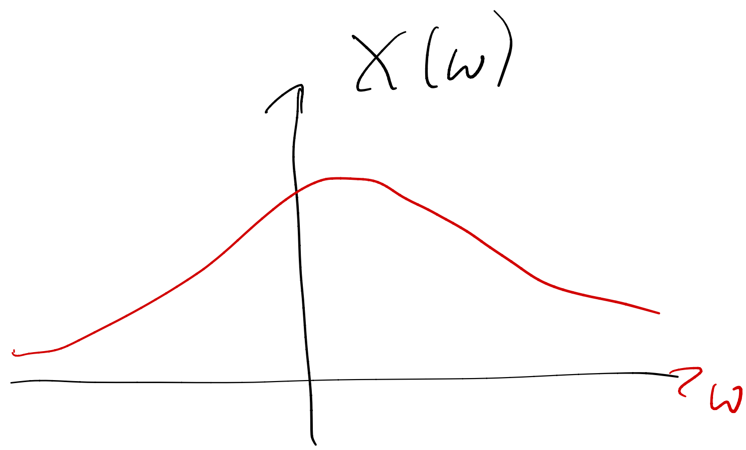
Discrete Time Fourier Transform → DTFT.

$x[t_k] \in \mathbb{R}$ ó $\mathbb{C}$

$x[k] \in \mathbb{R}$ ó $\mathbb{C}$; k: #muestras



$\xrightarrow{\text{DTFT}}$



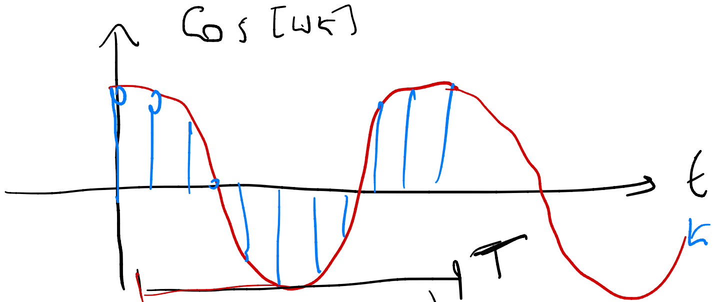
¿Qué pasa con bases ortogonales de Fourier?

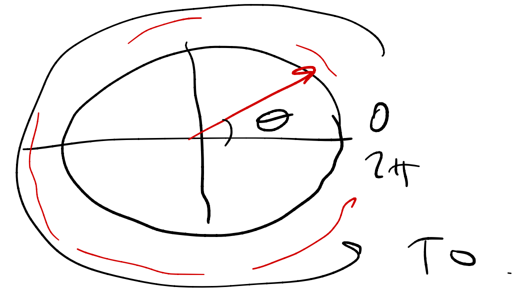
$e^{jn\omega_0 t}$ → Serie ; $T \to \infty$, $\Delta\omega_n \to 0$; $e^{j\omega t}$ → Transformada

$\cos(n\omega_0 t) + j\,\mathrm{Sen}(n\omega_0 t)$; $\cos(\omega t) + j\,\mathrm{Sen}(\omega t)$

$t \to k$

$$e^{j\omega k} = \cos[\omega k] + j\,\mathrm{Sen}[\omega k]$$





Periodicidad en tiempo y en frecuencia angular

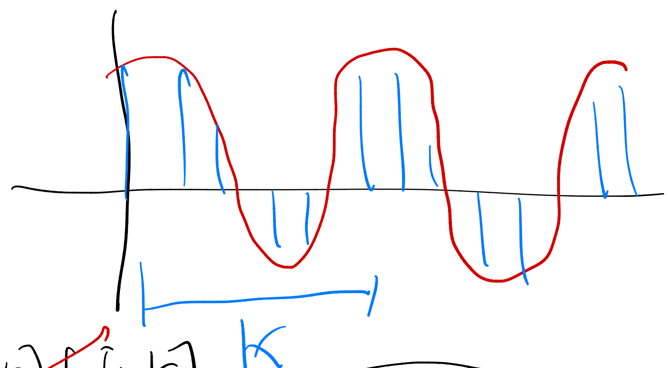
$\cos(\omega(t \pm T)) = \cos(\omega t)$; $T = ?$ único y el más pequeño.

$\cos(\omega t \pm \omega T)$

$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

$\cos(\omega t)\underbrace{\cos(\omega T)}_{1} \mp \cancel{\sin(\omega t)\underbrace{\sin(\omega T)}_{0}} = \cos(\omega t)$

$\omega T = \frac{2\pi}{\cancel{T_0}}\cancel{T} \rightarrow T = T_0$

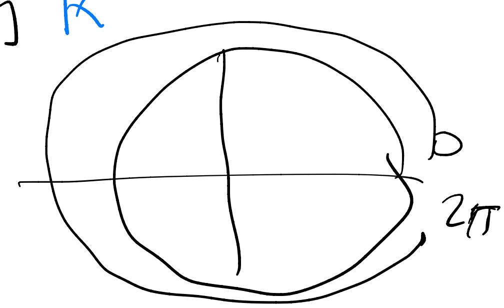


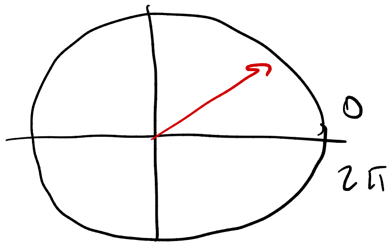
$\cos[\omega(k \pm K)] = \cos[\omega k]$

$K \rightarrow T$ periodo en # muestras

$\cos[\omega k \pm \omega K] = \cos[\omega k]\underbrace{\cos[\omega K]}_{1} \mp \cancel{\sin[\omega k]\underbrace{\sin[\omega K]}_{0}}$

$\omega = \frac{2\pi}{\cancel{K_0}}\cancel{K}$ $\qquad K = K_0$





$\cos(\theta) = \cos(\omega t)$; $\theta = \omega t$

posición, vel., tiempo

$r \in \mathbb{Z}$.

$\cos(\theta \pm r2\pi) = \cos(\theta)\cos(\cancel{r2\pi}) \mp \sin(\theta)\cancel{\sin(r2\pi)}$ ✓

$\cos((\omega \pm r2\pi)t) = \cos(\omega t) = ?$

$\cos(\omega t)\cos(r2\pi t) \mp \sin(\omega t)\sin(r2\pi t) \neq \cos((\omega \pm r2\pi)t)$

$r2\pi t = \underbrace{rt}_{\in\mathbb{R}}\,2\pi$; $r \in \mathbb{Z}$, $t \in \mathbb{R}$

NO ES ENTERA

$r = 2$

$t = 0.253$

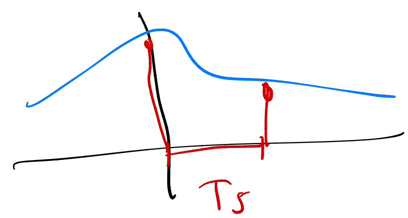
$\cos[(\omega \pm r2\pi)k] = \cos[\omega k] = ?$

$\cos[\omega k]\cos[r2\pi k] \mp \sin[\omega k]\sin[r2\pi k]$

$r2\pi k = \underbrace{rk}_{\in\mathbb{Z}}\,2\pi$; $r \in \mathbb{Z}$, $k \in \mathbb{Z}$

VUELTA ENTERA

$\cos[(\omega \pm r2\pi)k] = \cos[\omega k]$



Muestreo de señales cosenoidales

$\cos(\omega t) \rightarrow$ DISCRETIZAR. $\cos[\omega k]$; $t = kT_s$; $F_s = \frac{1}{T_s}$

$\cos\left[\frac{2\pi}{T}kT_s\right] = \cos\left[2\pi k \frac{F}{F_s}\right]$; $F = \frac{1}{T}$.

$f = \frac{F}{F_s}$; $F \rightarrow$ continuo ; $F_s \rightarrow$ muestreo

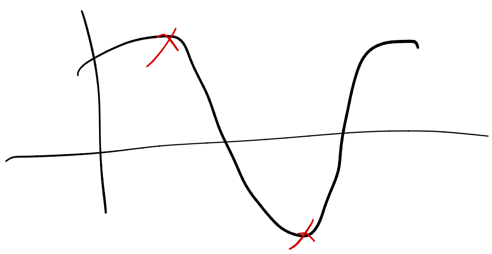
$\cos[2\pi f k] = \cos[\omega_d k]$, $\omega_d = 2\pi f$.

Periodicidad en ω_d de $\pm r2\pi$ → escoger 1 vuelta (ciclo) de ω_d.

$\omega_d \in [0, 2\pi)$; $\omega_d \in [-\pi, +\pi)$ → En discreto cos/sen originales restringen $\omega_d \in [0, 2\pi)$ $[-\pi, +\pi)$

$\omega_d = 2\pi f \in [-\pi, +\pi]$

$= 2\pi \frac{F}{F_s}$



Información en continuo se preserve en discreto

DTFT: $e_n(t) \to e^{j\omega t}$

$e^{j\omega k} \to \omega \in [-\pi, +\pi]$

$[0, 2\pi]$

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

$$x[k] = DTFT^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\omega) e^{j\omega k} \, d\omega; \quad \omega \in [-\pi, +\pi]$$

$\omega \in [0, 2\pi)$

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$$X(\omega) = DTFT\{x[k]\} = \sum_{k \in \mathbb{Z}} x[k] e^{-j\omega k}$$

$-\pi \le \omega_d \le \pi$

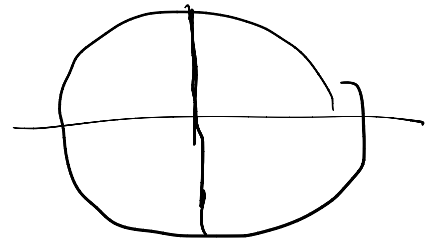
$-\pi \le 2\pi f \le \pi$

$-\frac{1}{2} \le f \le \frac{1}{2}$

$1/2$

1 ciclo

$f = \frac{F}{F_s} \left[\frac{1/s}{\text{muest}/s} \right] \left[\frac{\text{ciclos}}{\text{muestra}} \right]$



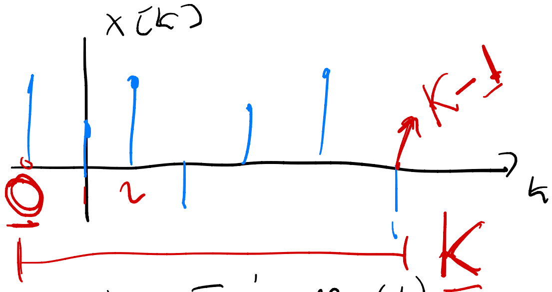
$-\frac{1}{2} \le \frac{F}{F_s} \le \frac{1}{2}$

$2F \le F_s \to F_s \ge 2F$

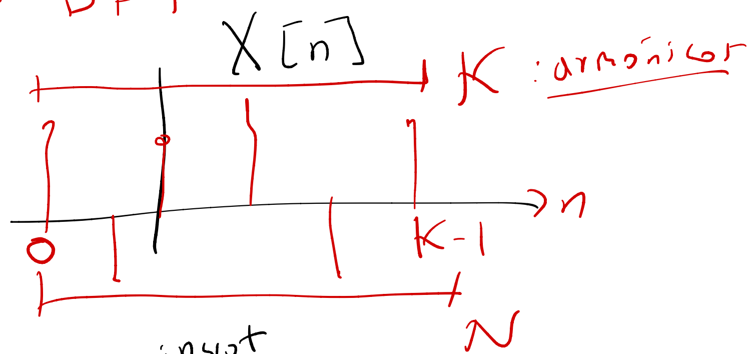
Teorema Nyquist

Transformada Discreta de Fourier

Discrete Fourier Transform → DFT



DFT



$$x(t) = \sum_{n \in \mathbb{Z}} c_n \varphi_n(t)$$

$$x(t) = \sum_{n \in \mathbb{Z}} c_n e^{jn\omega_0 t} \quad \omega_n = n\omega_0 = n\frac{2\pi}{K}$$

$$c_n = \int_{t_i}^{t_f} \frac{x(t)\varphi_n^*(t)}{\int |\varphi_n(t)|^2 dt}$$

$$T c_n = \int_{-t_i}^{t_f} x(t) e^{-jn\omega_0 t} dt.$$

$$X(\omega) = \sum_k x[k] e^{j\omega k} \qquad x[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{-j\omega k} d\omega$$

$$x[k] = \sum_{n=0}^{K-1} X[n] e^{jn\frac{2\pi}{K}k} = DFT^{-1}\{X[n]\}$$

$n\omega_0 = \omega_n$ → frec. angular en discreto

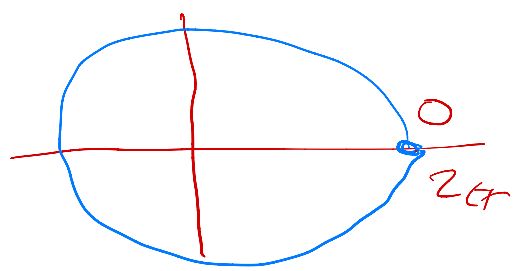
$\omega_n \in [-\pi, +\pi]$

$[0, 2\pi)$

$n = 0$

$n\frac{2\pi}{K} = 0 \cdot \frac{2\pi}{K} = 0$

$(K-1)\frac{2\pi}{K} = 2\pi$



$$X[n] = \sum_{k=0}^{K-1} x[k] e^{-jn\frac{2\pi}{K}k} = DFT\{x[k]\}$$

NOTA: DTFT → efecto fuga

DFT