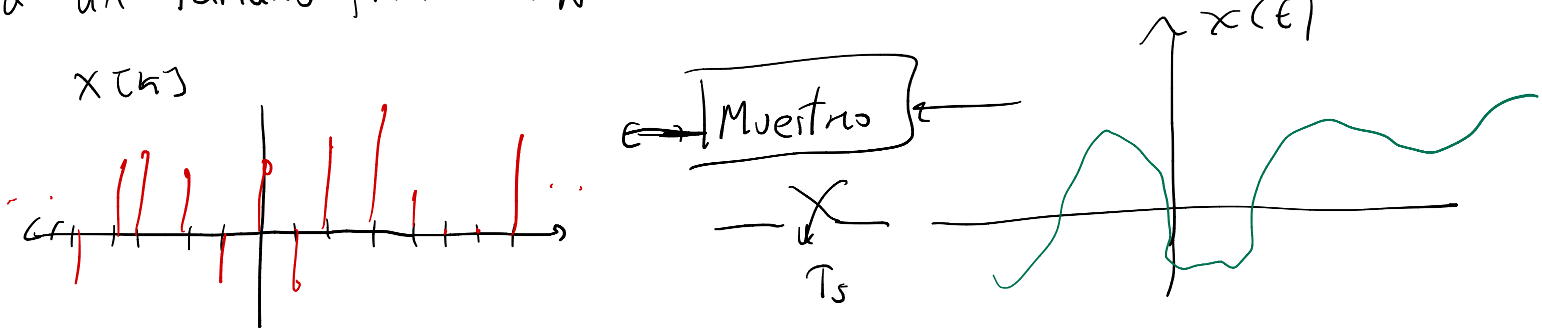
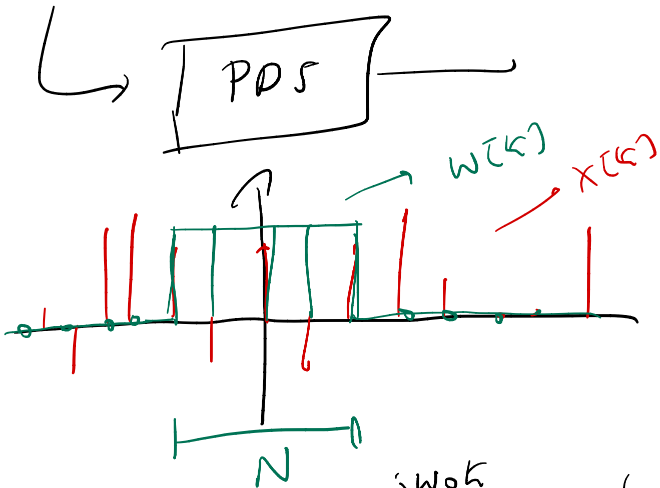
Efecto de fuga: consecuencia de restringir señal discreta $x[k]$ a un tamaño finito $x_N[k]$





$x_N[k] = x[k] \cdot w[k]$ → ventaneo

Ej: $x[k] = e^{j\omega_0 k}$ → tono simple (armónico simple)

$\varphi_n(t) = \{ e^{jn\omega_0 t} \in \mathbb{C} \}_{n=-\infty}^{\infty}$ — múltiplo entero de una frecuencia fundamental

$x[k] = A_1 e^{j\omega_1 k} + A_2 e^{j\omega_2 k}$ → mezcla de armónicos (tonos)

En transformada de Fourier:

$$X(\omega) = \mathcal{F}\{e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t}\, \partial t =$$

$$\mathcal{F}\{\mathcal{F}^{-1}\{X(\omega \pm \omega_0)\}\} = \mathcal{F}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega \pm \omega_0) e^{j\omega t}\, \partial\omega\right\}, \quad \omega' = \omega \pm \omega_0, \quad \omega = \omega' \mp \omega_0, \quad \partial\omega' = \partial\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega') e^{jt(\omega' \mp \omega_0)}\, \partial\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega') e^{\mp jt\omega_0} e^{jt\omega'}\, \partial\omega'$$

$$= e^{\mp jt\omega_0} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega') e^{jt\omega'}\, \partial\omega' = e^{\mp jt\omega_0} \mathcal{F}^{-1}\{X(\omega)\} = e^{\mp jt\omega_0} x(t)$$

$$\mathcal{F}\{e^{\mp jt\omega_0} x(t)\} = X(\omega \pm \omega_0)$$

$X(\omega) = \mathcal{F}\{1\cdot e^{j\omega_0 t}\} = X(\omega - \omega_0)$

$\mathcal{F}\{1\} = ?$; $\quad \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, \partial t = e^{-\omega(0)} = e^{0} = 1$

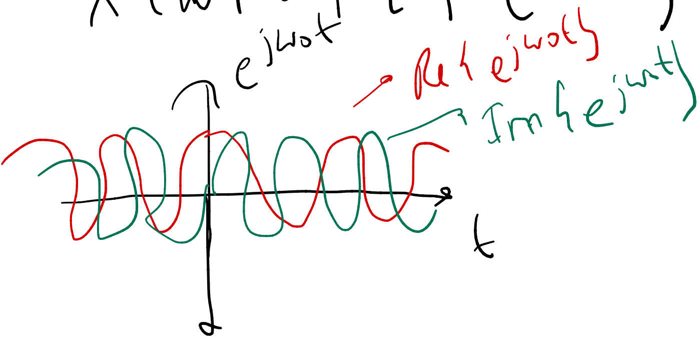
$\int_{-\infty}^{\infty} x(t)\, \delta(t \pm t_0)\, \partial t = x(\mp t_0)$

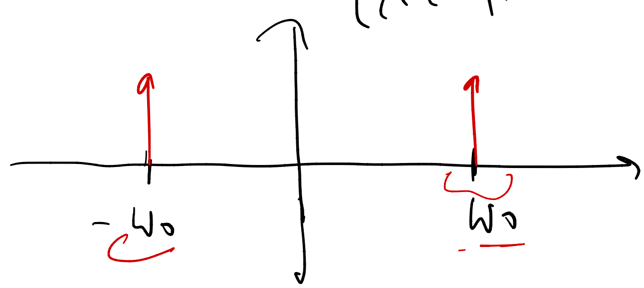
$\left.\begin{array}{l} \mathcal{F}\{x(t)\} = X(\omega) \\ \mathcal{F}\{X(t)\} = 2\pi\, x(-\omega) \end{array}\right\}$ Dualidad / simetría

$\mathcal{F}\{\delta(t)\} = 1$

$\mathcal{F}\{1\} = 2\pi \underbrace{\delta(-\omega)}_{\text{PAR}} = 2\pi\, \delta(\omega) = \int_{-\infty}^{\infty} 1\cdot e^{-j\omega t}\, \partial t = \int_{-\infty}^{\infty} e^{-j\omega t}\, \partial t$

$X(\omega) = \mathcal{F}\{1\cdot e^{j\omega_0 t}\} = 2\pi\, \delta(\omega - \omega_0)$ $\quad |X(\omega)|$





$e^{j\omega_0 t} = \cos(\omega_0 t) + j\, \mathrm{sen}(\omega_0 t)$

$x[k] = e^{j\omega_0 k}$; $\quad X(\Omega) = \mathrm{DTFT}(x[k])$; $\quad k \in \mathbb{Z}$ No longitud finita

$X(\Omega) = \sum_{k \in \mathbb{Z}} x[k]\, e^{-j\Omega k}$; $\quad \left.\begin{array}{l} \Omega \in [0, 2\pi) \\ \Omega \in [-\pi, +\pi) \end{array}\right\}$ periodicidad en w para tiempo discreto.

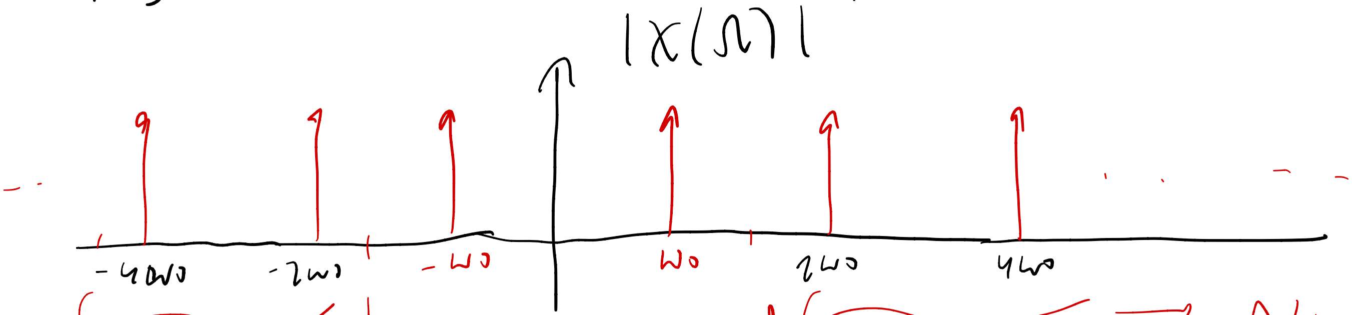
Aliasing

$\cos[\Omega k] = \cos[(\Omega \pm 2m\pi)k]$

$\cos(\omega t) \neq \cos((\omega \pm 2m\pi)t)$

$$X(\Omega) = \sum_{k\in\mathbb{Z}} e^{jw_0 k} e^{-j\Omega k} = \sum_{k\in\mathbb{Z}} e^{jk(w_0-\Omega)} = \sum_{n\in\mathbb{Z}} 2\pi\,\delta\left((\Omega-w_0)-2\pi n\right)$$

$$X(w) = \int e^{jw_0 t} e^{-jwt}\,\partial t = 2\pi\,\delta(w-w_0)$$



Alias

$0, 2\pi$

$-\pi, +\pi$

Una vuelta en Ω

Alias

Ventaneo: $x_N[k] = x[k]\cdot w[k]$

atenua $x[k]$
selecciona muestras (finitas)

$X_N(\Omega) = \mathrm{DTFT}\{x_N[k]\}$

$k = \{0, 1, \dots, N-1\}$

conv cíclica

$$X_N(\Omega) = \mathrm{DTFT}\{x[k]\cdot w[k]\} = \frac{1}{2\pi} X(\Omega) \circledast W(\Omega)$$

$$\mathcal{F}\{x(t)\cdot y(t)\} = \int_{-\infty}^{\infty} x(t)\,y(t)\,e^{-jwt}\,\partial t = \int_{-\infty}^{\infty} \mathcal{F}^{-1}\{X(w)\}\,y(t)\,e^{-jwt}\,\partial t$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w')\,e^{jw't}\,\partial w'\; y(t)\,e^{-jwt}\,\partial t$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w') \int_{-\infty}^{\infty} y(t)\,e^{-jt(w-w')}\,\partial t\,\partial w' = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w')\,Y(w-w')\,\partial w'$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,\partial\tau$$

$\tau = w'$
$t = w$

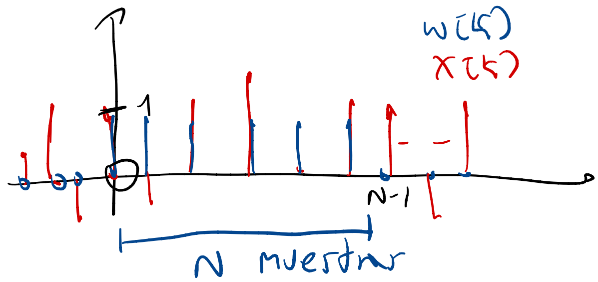
$$\mathcal{F}\{x(t)\,y(t)\} = \frac{1}{2\pi} X(w) * Y(w)$$

$x[k] = e^{j w_0 k}$; $x_N[k] = x[k] \cdot w[k]$

$$X(\Omega) = DTFT\{x_N[k]\} = \frac{1}{2\pi} X(\Omega) \circledast W(\Omega)$$

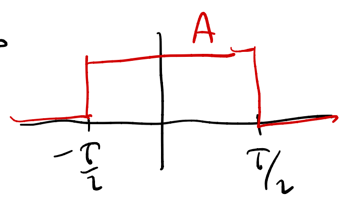
$$X(\Omega) = DTFT\{x[k]\} = \sum_{n \in \mathbb{Z}} 2\pi \delta(\Omega - w_0 - 2\pi n)$$

$w[k] = rect_N[k]$



$$\mathcal{F}\{A\, rect_T(t)\} = AT\, Sinc(\theta)$$

$$= \int_{-\infty}^{\infty} A\, rect_T(t)\, e^{-jwt} \partial t = \int_{-T/2}^{T/2} A e^{-jwt} \partial t = A \int_{-T/2}^{T/2} e^{-jwt} \partial t = \frac{A}{-jw} e^{-jwt} \Big|_{-T/2}^{T/2}$$



$$= \frac{A}{-jw} \left[e^{-jw\frac{T}{2}} - e^{+jwT/2} \right]$$

$$= \frac{A2}{w} \left[\frac{e^{jwT/2} - e^{-jwT/2}}{2j} \right] = \frac{2A}{w} Sen(wT/2)$$

$$Sen(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{\cancel{2} A T / \cancel{2}\; Sen(wT/2)}{wT/2} = AT\, Sinc(wT/2)$$

$$Sinc(\theta) = \frac{Sen(\theta)}{\theta}$$

$$W(\Omega) = DTFT\{rect_N[k]\}$$

$$= \sum_{k \in \mathbb{Z}} rect_N[k]\, e^{-j\Omega k} = \sum_{k=0}^{N-1} 1 \cdot e^{-j\Omega k} = \sum_{k=0}^{N-1} \left(e^{-j\Omega}\right)^k$$

$$\sum_{k=0}^{N-1} a r^k = a \frac{1 - r^N}{1 - r} \; ; \quad |r| \leq 1 \qquad |e^{-j\Omega}| \leq 1$$

Serie geométrica

$$\sum_{k=0}^{N-1} 1 \cdot \Big(\underbrace{e^{-j\Omega}}_{r}\Big)^k = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}}$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{9} < \infty$$

$$2 + 7 + 8 + 10 \to \infty$$

$$W(\Omega) = DTFT\{rect_N[k]\} = \frac{1-e^{-j\Omega N}}{1-e^{-j\Omega}} = \frac{e^{-j\Omega\frac{N}{2}}\left[e^{+j\Omega\frac{N}{2}} - e^{-j\Omega\frac{N}{2}}\right]}{e^{-j\frac{\Omega}{2}}\left[e^{+j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right]}$$

$$= \frac{e^{-j\Omega\frac{N}{2}}\cancel{2j}}{e^{-j\frac{\Omega}{2}}\cancel{2j}} \frac{\left[\frac{e^{j\Omega\frac{N}{2}} - e^{-j\Omega\frac{N}{2}}}{2j}\right]}{\left[\frac{e^{j\Omega/2} - e^{-j\Omega/2}}{2j}\right]} = \frac{e^{-j\Omega\frac{N}{2}}\ \mathrm{Sen}\left(\frac{\Omega N}{2}\right)}{e^{-j\frac{\Omega}{2}}\ \mathrm{Sen}(\Omega/2)}$$

$$\mathrm{Sen}(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$W(\Omega) = e^{-j\Omega\frac{N}{2}} e^{+j\frac{\Omega}{2}} \frac{\mathrm{Sen}\left(\Omega\frac{N}{2}\right)}{\mathrm{Sen}(\Omega/2)} = e^{-j\Omega\left[\frac{N}{2}-\frac{1}{2}\right]} \frac{\mathrm{Sen}(\Omega N/2)}{\mathrm{Sen}(\Omega/2)}$$

$$W(\Omega) = e^{-j\Omega\left(\frac{N-1}{2}\right)} \frac{\mathrm{Sen}(\Omega N/2)}{\mathrm{Sen}(\Omega/2)}; \quad X(\Omega) = \cancel{\sum_{n\in\mathbb{Z}}} 2\pi\delta(\Omega - w_0 - 2\pi \cancel{n})$$

$n=0$

$$X_N(\Omega) = DTFT\{x[k]\cdot w[k]\} = \frac{1}{2\pi} X(\Omega) \circledast W(\Omega)$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} X(\Omega') W(\Omega-\Omega')\, d\Omega' =$$

$\overset{\pi}{}$ $-\pi$

Tomar 1 ciclo de $\circledast$

$n=0$

$$X_N(\Omega) = \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\, \delta(\Omega' - w_0)\, W(\Omega-\Omega')\, d\Omega' = W(\Omega - w_0)$$

$$\boxed{X_N(\Omega) = e^{-j(\Omega - w_0)\left(\frac{N-1}{2}\right)} \frac{\mathrm{Sen}\left((\Omega - w_0)N/2\right)}{\mathrm{Sen}\left(\frac{\Omega - w_0}{2}\right)}}$$