Matching pursuits with time-frequency dictionaries

A.M. Alvarez-Meza, Ph.D.

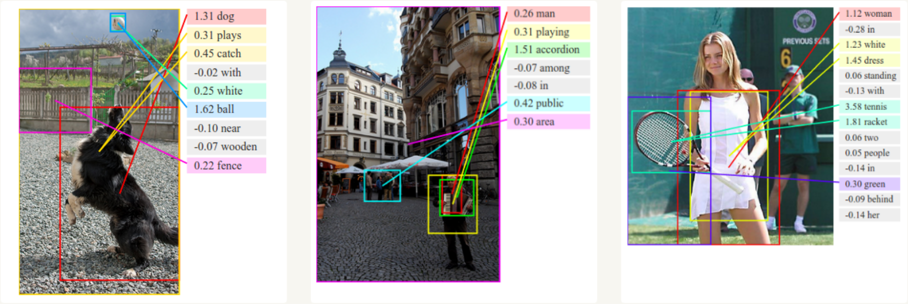
amalvarezme@unal.edu.co

DIEEyC
Universidad Nacional de Colombia

Outline

Signal representation

- Natural languages have large vocabularies (words with close meanings).
- Low level signal representations provide explicit information.
- Numerical parameters should offer compact characterizations of the elements we are looking for.
- A suitable representation gives simple cues to differentiate patterns!



source: Computer vision example - Stanford vision Lab

The generalized Fourier series

Let Ψ be a set of square-integrable functions in $\mathcal{F}$

$$\Psi = \{\phi_n : [a, b] \to \mathcal{F}\}_{n=0}^{\infty},$$

which are pairwise orthogonal for the inner product

$$\langle f, g\rangle = \int_a^b f(x) g^*(x) w(x) dx$$

$w(x)$: weight function.

The generalized Fourier series with respect to Ψ, is then

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x) \tag{1}$$

where the coefficients are given by

$$c_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|_w^2}$$

If Ψ is a complete set, i.e., an orthonormal basis on $[a, b]$, $\sim$ becomes $=$ in the L^2 sense.

The Fourier transform

Let $f(t) \in L^2(R)$ be a function in the Hilbert space $L^2(R)$, the Fourier transform $f(\omega)$ is defined by:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{2}$$

The inverse Fourier transform is defined as:

$$\hat{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega.$$

The Short Time Fourier transform

- Employs a window function to localize the complex sinusoid as:

$$F(\omega, u) = \int_{-\infty}^{\infty} f(t)h(t-u)e^{-i\omega t}dt \quad (3)$$

- $u \in \mathbb{R}$ is a translation parameter and $h(t-u)$ is the window function which confines the complex sinusoid $e^{-i\omega t}$.
- Some Window shapes: Hanning, hamming, cosine, Gaussian, etc.
- Gaussian windowed STFT (Gabor transform):

$$h_{\xi,u}(t) = h(t-u)e^{i2\pi\xi t}, \quad (4)$$

$h(t) = \frac{1}{\sqrt{\sigma}\pi^{1/4}} e^{-\frac{1}{2}(t^2/\sigma^2)}$, $\sigma \in \mathbb{R}^+$, and $\xi \in \mathbb{R}$: modulating factor.

- $h_{\xi,u}(t)$ is known as a windowed Fourier atom.

The Short Time Fourier transform

- Convolving the atom with the signal:

$$
\begin{aligned}
F(\xi, u) &= \int_{-\infty}^{\infty} f(t) h_{\xi,u}^{*}(t) dt \\
&= \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{\sigma}\pi^{1/4}} e^{\frac{-1}{2}[(t-u)^2/\sigma^2]} e^{-i2\pi\xi t} dt
\end{aligned} \tag{5}
$$

The Wavelet transform

- To match a signal by moving and stretching/squezzing a wavelet function.

$$T(s,u) = \int_{-\infty}^{\infty} f(t)\psi_{s,u}^{*}(t)dt \tag{6}$$

- Example: Morlet wavelet $\psi(t) = \frac{1}{\pi^{1/4}} e^{i2\pi\xi_0 t} e^{-t^2/2}$, ξ_0: central frequency.
- The Morlet wavelet transform gives:

$$T(s,u) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}\pi^{1/4}} e^{\frac{-1}{2}[(t-u)^2/s^2]} e^{-i2\pi(\xi_0/s)(t-u)} dt \tag{7}$$

The Time-frequency atomic decomposition

- To **extract information from complex signals**, we need to adapt the time-frequency decomposition to the particular signal structures.
- A family of time-frequency atoms can be generated by scaling, translating, and modulating a single window function $g(t) \in L^2(R)$.
- $g(t)$ is continuously differentiable and $\|g(t)\| = 1$, $g(0) \neq 0$.
- For any scale $s > 0$, frequency modulating ξ, and translation u, we denote $\gamma = (s, u, \xi)$ and define:

$$g_\gamma(t) = \frac{1}{\sqrt{s}} g\left(\frac{t-u}{s}\right) e^{i\xi t} \tag{8}$$

where $\frac{1}{\sqrt{s}}$ normalizes $\|g_\gamma(t)\| = 1$.

- The Fourier transform of $g(t)$ yields:

$$g_\gamma(\omega) = \sqrt{s} g\left(s(\omega - \xi)\right) e^{-i(\omega-\xi)u}, \tag{9}$$

since $|g(\omega)|$ is even, $g_\gamma(\omega)$ is centered at frequency $\omega = \xi$.

- The family $\mathcal{D} = \{g_\gamma(t)\}_{\gamma\in\Gamma}$ is redundant ($\Gamma = \mathbb{R}^+ \times \mathbb{R}^2$).
- To represent properly $f(t)$ we must select a subset of atoms $\{g_{\gamma_n}(t)\}_{n\in N}$, with $\gamma_n = (s_n, u_n, \xi_n)$, so that:

$$f(t) = \sum_{n=-\infty}^{\infty} a_n g_{\gamma_n}(t) \tag{10}$$

The Fourier and wavelet transforms correspond
to different families of time-frequency atoms

The Fourier transform's atoms

$$F(\xi_n, u_n) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{\sigma}\pi^{1/4}} e^{\frac{-1}{2}[(t-u_n)^2/\sigma^2]} e^{-i2\pi\xi_n t} dt \quad (11)$$

- All the atoms have a constant scale $s_n = s_0$ ($s_0 \sim \sigma$ for Gabor window).
- If the main signal structures are localized over a time-scale of the order of s_0 and a_n gives important insights on their localization and frequency content.
- Window Fourier transform is not well adapted to describe structures that are much smaller or much larger than s_0.
- To analyze components of varying sizes, it is necessary to use time-frequency atoms of different scales.

The wavelet transform's atoms

$$T(s_n, u_n) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s_n}\pi^{1/4}} e^{\frac{-1}{2}[(t-u_n)^2/s_n^2]} e^{-i2\pi(\xi_0/s_n)(t-u_n)} dt \qquad (12)$$

- Decomposes signals over time-frequency atoms of varying scales, called wavelets.
- The resulting family is composed of dilations and translations of a single function, multiplied by complex phase parameter.
- The expansion coefficients a_n characterize the scaling behavior of signal structures.
- However, wavelet coefficients do not provide precise estimates of the frequency content of waveforms whose Fourier transforms is well localized (restriction on the frequency parameter $\xi_n = \xi_0/s_n$).

Matching pursuit in Hilbert spaces

- We want to compute a linear expansion of $f \in L^2(R)$ over a set of vectors selected from $\mathcal{D} = (g_\gamma)_{\gamma\in\Gamma}$ to best match its inner structures.
- Procedure: successive approximations of f with orthogonal projections on elements of $\mathcal{D}$.
- Let $g_{\gamma_0} \in \mathcal{D}$, then:

$$f = \langle f, g_{\gamma_0} \rangle g_{\gamma_0} + Rf, \tag{13}$$

  where Rf is the residual vector.
- Since g_{γ_0} is orthogonal to Rf (Exercise: proof it):

$$\|f\|^2 = \|\langle f, g_{\gamma_0} \rangle\|^2 + \|Rf\|^2. \tag{14}$$

Matching pursuit in Hilbert spaces

- To minimize $\|Rf\|$ we must choose $g_{\gamma_0} \in \mathcal{D}$ such that $|\langle f, g_{\gamma_0} \rangle|$ is maximum.
- It is possible to find a vector that is almost the best:

$$|\langle f, g_{\gamma_0} \rangle| \geq \alpha \sup_{\gamma \in \Gamma} |\langle f, g_\gamma \rangle|, \tag{15}$$

  where $0 < \alpha \leq 1$.
- Matching pursuit is an iterative algorithm that subdecomposes the residue Rf by projecting it on a vector of $\mathcal{D}$ that matches Rf almost at best.

Matching pursuit in Hilbert spaces

- Energy conservation equation:

$$\|f\|^2 = \sum_{n=0}^{m-1} |\langle R^n f, g_{\gamma_n}\rangle|^2 + \|R^m f\|^2. \tag{16}$$

- In practice, the algorithm is terminated either when the residual energy below a preset cut-off level:

$$\|R^n f\|^2 < \epsilon \|f\|^2; \quad \forall \epsilon \in \mathbb{R}^+. \tag{17}$$

- The original f is decomposed into a sum of dictionary elements, that are chosen to best match its residues.
- The decomposition is nonlinear, however, we maintain an energy conservation as if it was a linear orthogonal decomposition.
- Matching pursuit is a greedy algorithm that chooses at each iteration a waveform that is best adapted to approximate part of the signal.

Matching pursuit with time-frequency dictionaries

- For dictionaries of time-frequency atoms a matching yields an adaptive time-frequency transform.
- The function f is decomposed into a sum of complex time-frequency atoms that best match its residues.
- The matching pursuit decomposes f as:

$$f = \sum_{n=0}^{+\infty} \langle R^n f, g_{\gamma_n} \rangle g_{\gamma_n}, \tag{18}$$

where $\gamma_n = (s_n, u_n, \xi_n)$ and

$$g_{\gamma_n}(t) = \frac{1}{\sqrt{s_n}} g\left(\frac{t - u_n}{s_n}\right) e^{i\xi_n t}. \tag{19}$$

Matching pursuit with time-frequency dictionaries

- At each iteration the MP algorithm selects a vector g_{γ_n} that satisfies:

$$|\langle R^n f, g_{\gamma_n}| \geq \alpha \sup_{\gamma \in \Gamma} |\langle R^n f, g_\gamma \rangle|. \tag{20}$$

- From the decomposition of any $f(t)$ within a time-frequency dictionary a new time-frequency energy distribution is obtained by adding the Wigner distribution of each selected atom.
- The cross Wigner distribution of two functions $f(t)$ and $h(t)$ is defined by:

$$W[f,h](t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left(t + \frac{\tau}{2}\right) h^*\left(t - \frac{\tau}{2}\right) e^{-i\omega\tau} d\tau. \tag{21}$$

where $Wf(t,\omega) = W[f,f](t,\omega)$.

Matching pursuit with Gabor atoms

- The signal approximation reconstructed from N MP expansion coefficients is given by:

$$f_N(t) = \sum_{n=0}^{N-1} M_n h_{s_n,u_n,\xi_n}(t). \tag{22}$$

- The Gabor atom for the MP method is defined as:

$$h_{s_n,u_n,\xi_n}(t) = \frac{1}{\sqrt{s_n}} h\left(\frac{t-u_n}{s_n}\right) \tag{23}$$

Matching pursuit with Gabor atoms

- A Gaussian window is defined for the MP as:

$$h(t) = 2^{1/4} e^{-\pi t^2} \tag{24}$$

- The Gabor atom for MP can be written as:

$$h_{s_n,u_n,\xi_n,\phi_n}(t) = K_n \frac{2^{1/4}}{\sqrt{s_n}} e^{-\pi[(t-u_n)/s_n]^2} \cos(2\pi\xi_n t + \phi_n) \tag{25}$$

s_n : scale, u_n : localization factor for Gaussian envelope, ξ_n : frequency, ϕ_n : phase of the real sinusoid, K_n : normalization factor to preserve unit energy.

- The expansion coefficients are determined as follows:

$$M(s_n, u_n, \xi_n, \phi_n) = \int_{-\infty}^{\infty} f(t) K_n \frac{2^{1/4}}{\sqrt{s_n}} e^{-\pi[(t-u_n)/s_n]^2} \cos(-2\pi\xi_n t + \phi_n) dt \tag{26}$$

Summary: playing with signals (Gabor atoms example)

- Fourier:

$$F(\xi_n) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi\xi_n t} dt. \tag{27}$$

- Short-time Fourier transform:

$$F(\xi_n, u_n) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{\sigma}\pi^{1/4}} e^{\frac{-1}{2}[(t-u_n)^2/\sigma^2]} e^{-i2\pi\xi_n t} dt \tag{28}$$

- Wavelet transform:

$$T(s_n, u_n) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s_n}\pi^{1/4}} e^{\frac{-1}{2}[(t-u_n)^2/s_n^2]} e^{-i2\pi(\xi_0/s_n)(t-u_n)} dt \tag{29}$$

- Matching pursuit transform:

$$M(s_n, u_n, \xi_n, \phi_n) = \int_{-\infty}^{\infty} f(t) K_n \frac{2^{1/4}}{\sqrt{s_n}} e^{-\pi[(t-u_n)/s_n]^2} \cos(-2\pi\xi_n t + \phi_n) dt \tag{30}$$

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Matching pursuits with time-frequency dictionaries.
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