Assignment 1 Report

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1 BEZIER CURVE CREATION

Given a list of control points $P_0, P_1, P_2, P_3, ...P_N$ entered by the user, a piece-wise cubic Bezier curve is used to interpolate between these points. To compute a Bezier curve between any two adjacent points P_i and P_{i+1}, it necessary to create 2 additional *phantom* control points lying between P_i and P_{i+1} to enable the creation of a cubic curve. Two primary challenges are handled in the submitted implementation -

- The first is the selection of these phantom control points for every adjacent pair of given control points.
- The second is ensuring the curve is C^1 continuous everywhere. The challenge lies in ensuring this condition at the control points P_{i+1} since two different Bezier curves B_i and B_{i+1} share this point.

2 SELECTING PHANTOM CONTROL POINTS

Consider points P_{i-1},P_i and P_{i+1}. In this case each point exists in 2 dimensions and hence can be represented as $P_i = (x_i, y_i)$. Two vectors T and N are defined as follows -

$$T_{i,i+1} = (x_{i+1} - x_i, y_{i+1} - y_i)$$

$$N_{i,i+1} = (y_{i+1} - y_i, -(x_{i+1} - x_i))$$

Note that $T_{i,i+1}$ describes the line segment between P_i and P_{i+1}, while $N_{i,i+1}$ is orthogonal to it.

Further the two phantom points between any control points P_i and P_{i+1} are defined by R_i^1 and R_i^2. The ordering of the points while constructing the cubic Bezier curve is P_i , R_i^1 , R_i^2 , P_{i+1}.

For constructing the phantom points I use the vectors $N'_{i,i+1}$ and $T'_{i,i+1}$ which are the unit vectors along $N_{i,i+1}$ and $T_{i,i+1}$ respectively. The implementation defines a vector D and uniform random variable λ.

$$D_{i,i+1} = (1 - \lambda_i) \times N'_{i,i+1} + \lambda_i \times T'_{i,i+1} \quad \lambda_i \in [-0.7, 0.7]$$

The points R_i^1 and R_i^2 are now computed as -

$$R_i^1 = P_i - \alpha_{i-1} \times |T_{i-1,i}| \times D'_{i-1,i} \quad \alpha_{i-1} \in [0.35, 0.85]$$

$$R_i^2 = P_{i+1} + \alpha_i \times |T_{i,i+1}| \times D'_{i,i+1} \quad \alpha_{i-1} \in [0.35, 0.85]$$

Here α is another uniform random variable like λ while || is represents the magnitude of the vector.

To understand the significance of the above formulation let us first observe λ. Through this random variable it is possible to generate a diverse range of phantom control points thereby leading to a variety of different shapes

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of curves. By restricting λ to [-0.7,0.7], a prior is being induced upon the generated vector such that it more likely to have larger dot product with the vector $N_{i,i+1}$ as compared to $T_{i,i+1}$. The intuition behind this is that the closer the phantom control points to the line segment joining the user chosen control points the closer the Bezier curve will resemble a straight line segment. To harness the expressiveness of a cubic polynomial, the phantom points are kept further away from the $T_{i,i+1}$ vector to generate a more visible curvature in the curve.

The second key mechanism here is the term $\alpha_i \times |T_{i,i+1}|$. This essentially controls how far from the point P_{i+1} along the direction described by $D^{'}_{i,i+1}$ should the phantom point be plotted. This is computed relative to the length of the line segment between P_i and P_{i+1}, which is captured by $|T_{i,i+1}|$. The random variable α_i is used to control the distance by limiting displacement from P_{i+1} at 0.35 to 0.85 times to the length of the original line segment. This is done to ensure that the phantom control points are neither chosen too far off from the reference control points nor too close. Phantom points too close would lead small sharp bends at the original control points, while is they are too far away the Bezier curve will also bend far away from the original control points.

3 PROOF OF C^1 CONTINUITY OF THE IMPLEMENTATION

Given control points $P_0, P_1, P2$ and corresponding phantom control points $R_0^1, R_0^2, R_1^1, R_1^2$. These points describe bezier curves B_0 and B_1.

$$B_0 = (1-t)^3 P_0 + 3t(1-t)^2 R_0^1 + 3t^2(1-t) R_0^2 + t^3 P_1 \quad t \in [0,1]$$
$$B_1 = (1-t)^3 P_1 + 3t(1-t)^2 R_1^1 + 3t^2(1-t) R_1^2 + t^3 P_2 \quad t \in [0,1]$$

It is clear that each Bezier curve B_i is continuous for all $t \in (0,1)$ as in this domain it both the left and right limit will converge to the same value and the derivative will be defined. It is only at the control points P_i that it is necessary to show C^1 continuity. To do so it is necessary to show that for two consecutive Bezier curves $B_i(t)$ and $B_{i+1}(t)$,

$$B^{'}_i(t)_{t=1} = B^{'}_{i+1}(t)_{t=0}$$

Here $B^{'}(t)$ is the derivative with respect to t. The equation states that the tangent to the to curves at their point of intersection must be equal.

To prove this, let us compute the derivatives for each Bezier curve mentioned in the above equation.

$$B^{'}_i(t)_{t=1} = 3 \times (P_{i+1} - R_i^2)$$
$$B^{'}_{i+1}(t)_{t=0} = 3 \times (R_{i+1}^1 - P_{i+1})$$

Given that

$$R_i^2 = P_{i+1} + \alpha_i \times |T_{i,i+1}| \times D^{'}_{i,i+1}$$
$$R_{i+1}^1 = P_{i+1} - \alpha_i \times |T_{i,i+1}| \times D^{'}_{i,i+1}$$

Thus we are left with,

$$B^{'}_i(t)_{t=1} = -\alpha_i \times |T_{i,i+1}| \times D^{'}_{i,i+1}$$
$$B^{'}_{i+1}(t)_{t=0} = -\alpha_i \times |T_{i,i+1}| \times D^{'}_{i,i+1}$$

Hence,

$$B^{'}_i(t)_{t=1} = B^{'}_{i+1}(t)_{t=0}$$

This mathematically shows that the given implementation for finding phantom control points ensure C^1 continuity across the curve. Below are some figures showing some of the generated curves.

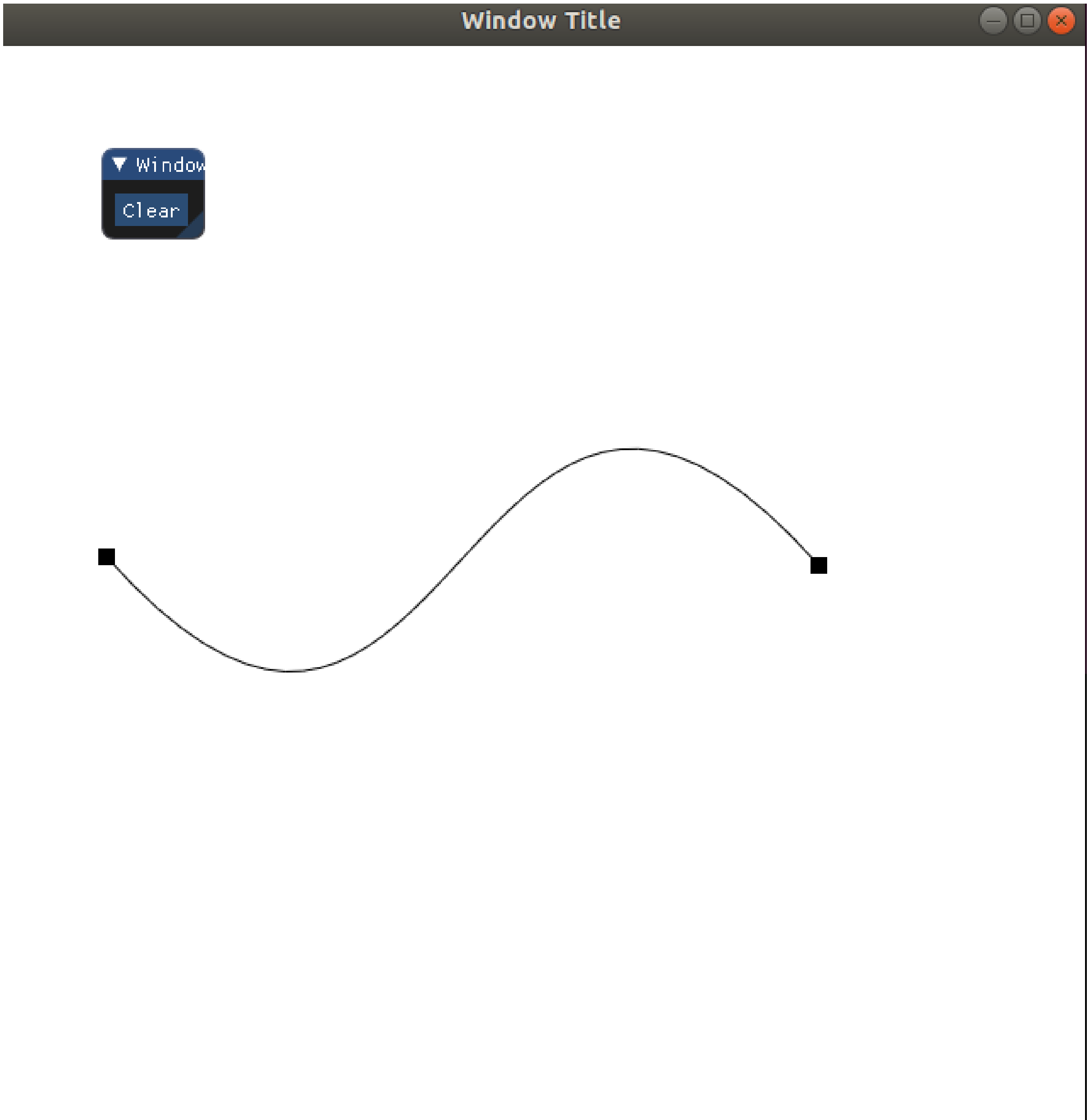

Fig. 1

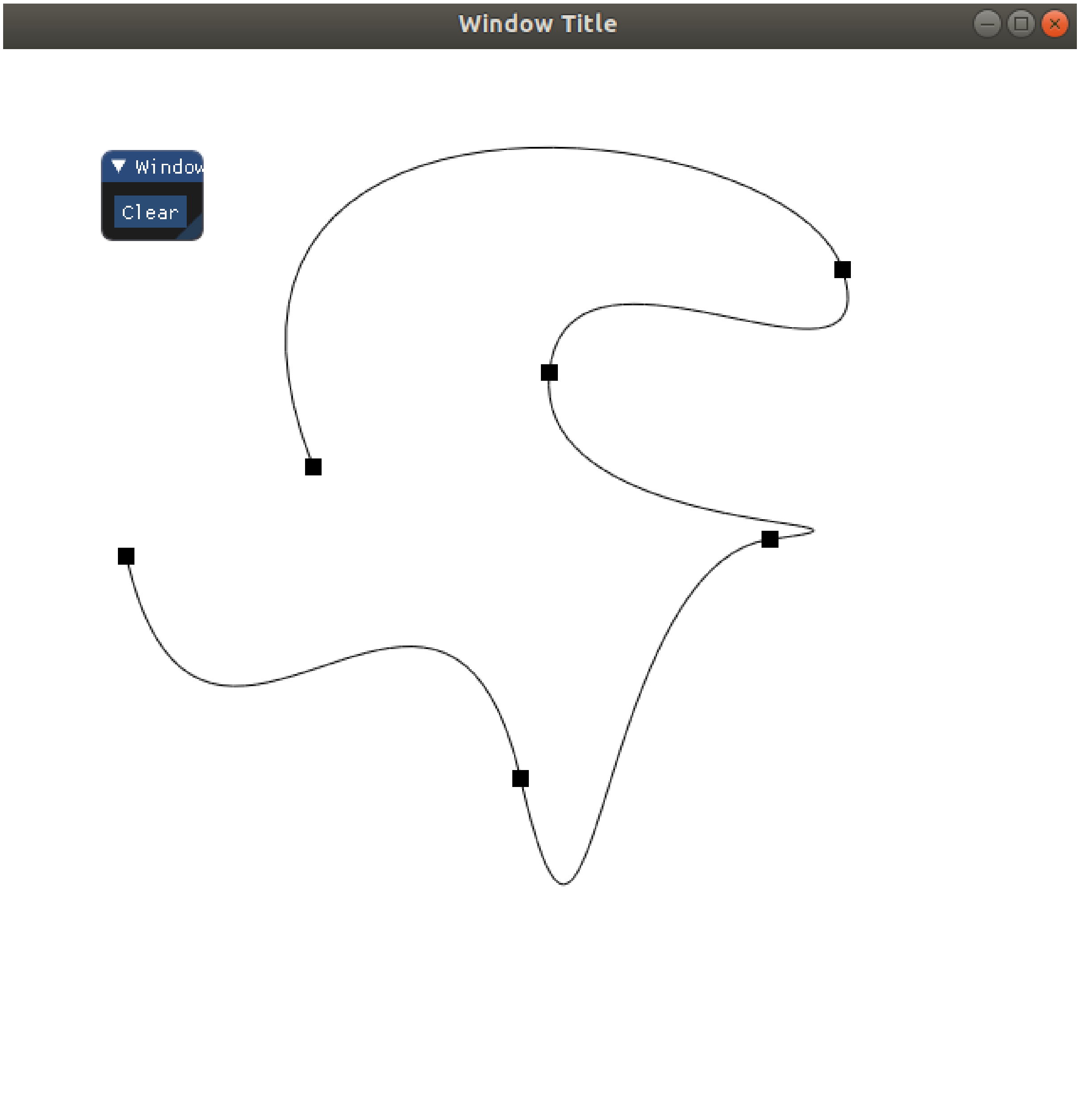

Fig. 2

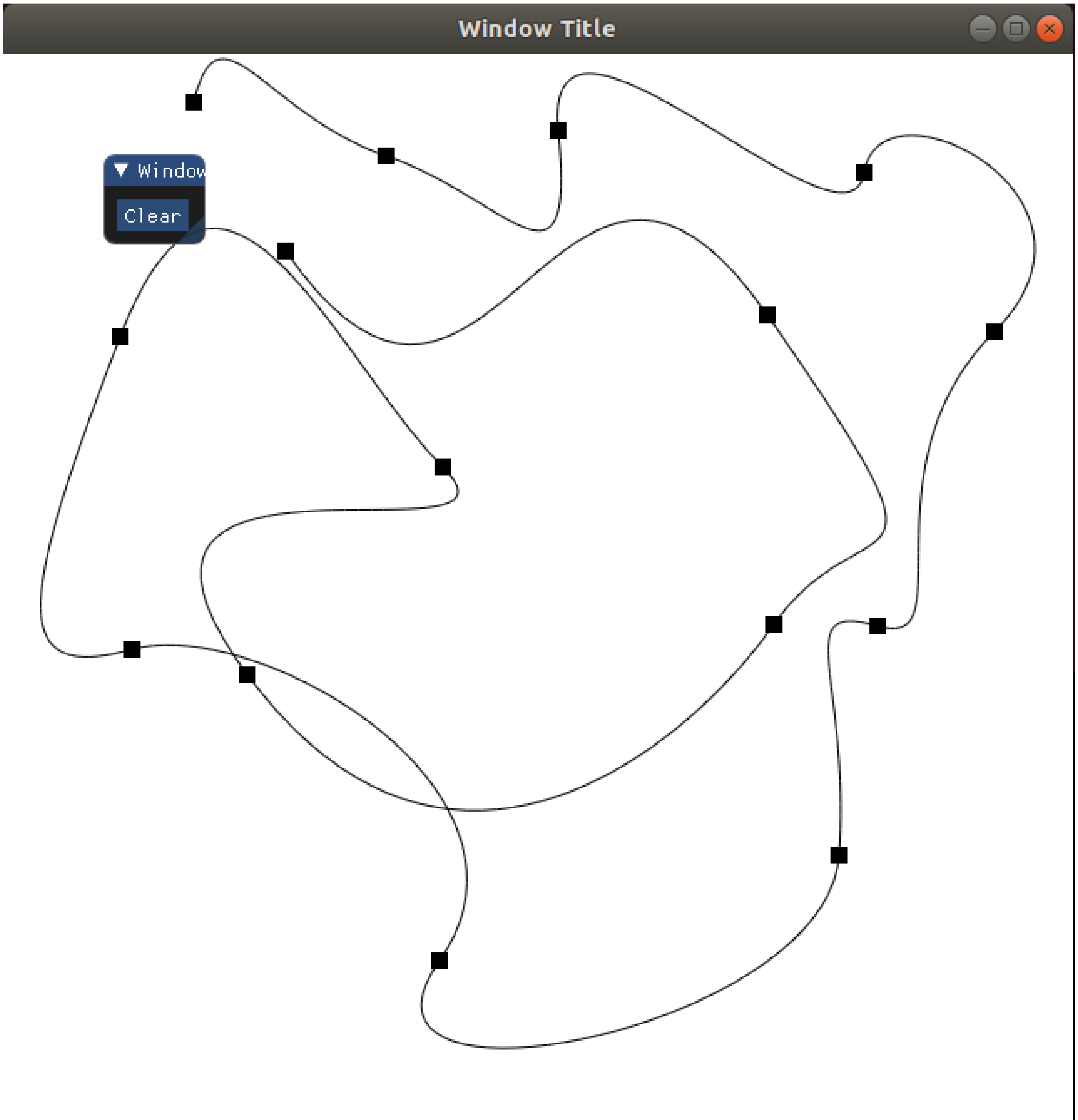

Fig. 3