Bayesian-Optimal Multi-Classification implies Abstract Representations

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February 9, 2024

Consider an intelligent agent making decisions in some environment. The agent receives noisy observations conditioned on the environment state, and must produce optimal decisions (i.e., learn a multi-classification objective). We show that the agent must represent an estimate of the de-noised environment state if it optimally estimates decision output using noisy observations.

1 Problem Statement

Noisy Multi-Classifier: Formalize the "environment state" as $X \sim P(X)$ with sample space $\mathcal{X}$ and a corresponding ground truth decision set $P(Y_i|X)$ for $i \in [N]$ (e.g., multi-classification on the environment state). Denote the i.i.d. noise process $X_i \sim P(\tilde{X}_i|X)$ from which observations $\tilde{X}_i$ are sampled. We consider optimal estimators of the ground truth readout Y given noisy measurements $\tilde{X}$ denoted $P(\hat{Y}|\tilde{X}_1, \dots, \tilde{X}_T)$.

$$\begin{tikzcd}
X \arrow[r, "\text{noise}"] \arrow[dr] & \{\tilde{X}_t\}_{t\in[T]} \arrow[r, "\text{agent}"] & \{\hat{Y}_i\}_{i\in[N]} \\
& \{Y_i\}_{i\in[N]} &
\end{tikzcd} \tag{1}$$

Geometry: Let X reside in a metric space $\mathcal{X}$. Let each Y_i be defined in terms of a binary discriminator $\phi_i : \mathcal{X} \to \{0, 1\}$. Let the equivalence classes of $\mathcal{X}$ under each discriminator ϕ_i be connected (i.e., $\{x|\phi_i(x) = 1, x \in \mathcal{X}\}$ is connected for each ϕ_i).

Claim: Under fairly general conditions,

$$I(Z(t); X) = I(\tilde{X}; X) \tag{2}$$

$$I(Z(t); X) = I(\tilde{X}_1 \dots \tilde{X}_T; X) \tag{3}$$

Proof Sketch:

- Derive $\hat{Y}_i \sim P(Y_i|\tilde{X}_t)$ using Bayes theorem. Due to independence, $P(Y_i|\tilde{X}_1, \dots, \tilde{X}_T)$ follows.
- Show that $P(Y_i|\tilde{X}_1, \dots, \tilde{X}_T)$ represents a distance between an implied $\hat{X} \sim P(X|\tilde{X}_1, \dots, \tilde{X}_T)$ and the boundary of the set $\{x|\phi(x) = 1, x \in \mathcal{X}\}$.
- Show that the set of N distances along with knowledge of the boundaries narrows $\hat{X}$ down to a point.

Results:

- Linear decision boundaries + Gaussian noise: proven in "Disentangling Representations in RNNs through Multi-task Learning".
- If X is continuously deformable to some $U = g(X)$ such that decision boundaries in U are linear, then the model must represent an optimal estimate of coordinates in U (new result).