Local P-adaptivity with the Discontinuous Galerkin Method

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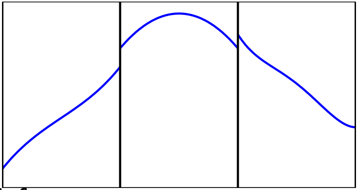
Scott Moe

March 13$^{\text{th}}$, 2015

1D DG-FEM



Conservation law weak form:

- Multiply by test function:

$$\frac{1}{2}\int_{-1}^{1} \varphi^{(k)} \left\{ q_{,t} + f(q)_{,x} \right\} d\xi = 0$$

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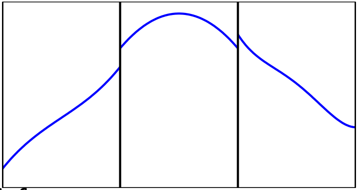

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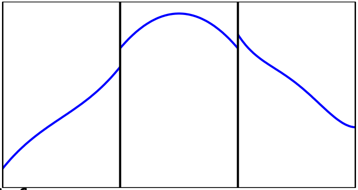

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- Galerkin ansatz:

$$q(t,\xi) \sim \sum_{k=0}^{N} Q^{(k)}(t)\, \varphi^{(k)}(\xi)$$

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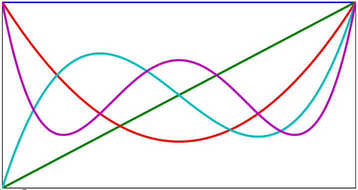

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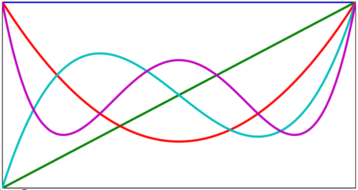

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$$\dot{Q}_i^{(k)} = \frac{2k+1}{\Delta x}\int_{-1}^{1} f\left(q^h\right)\,\varphi_\xi^{(k)}\,d\xi - \frac{2k+1}{\Delta x}\left[\varphi^{(k)}(1)\,F_{i+\frac{1}{2}} - \varphi^{(k)}(-1)\,F_{i-\frac{1}{2}}\right]$$

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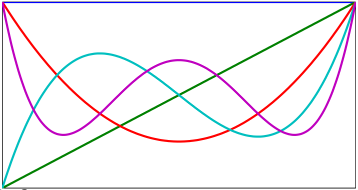

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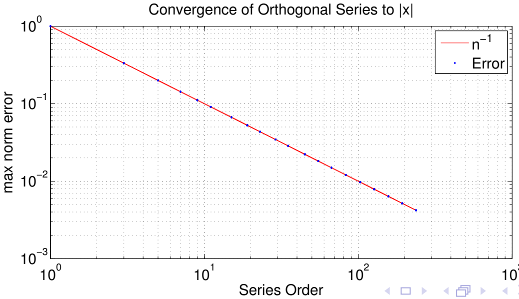
- For $k = 0$:

$$\dot{Q}_i^{(0)} = -\frac{1}{\Delta x}\left[F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right]$$

Orthogonal Series

- For continuous functions series coefficients converge.
- Coefficients give us a proxy for error.



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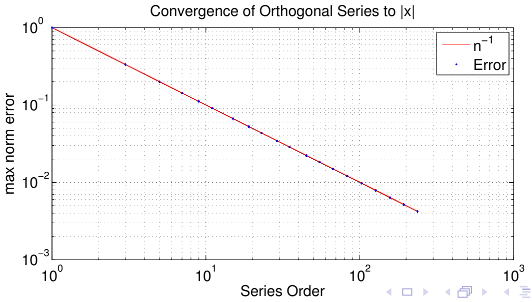

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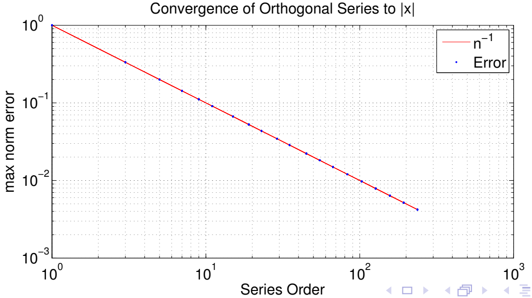
- If f has a derivative ν of bounded variation

$$\|f - f_p\| \leq \sum_{i=p+1}^{\infty} |c_k \varphi^{(k)}(\xi)| \leq \frac{1}{\nu(p-\nu)^{\nu}}$$



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- For smooth functions:
  higher order polynomials $\rightarrow$ **MUCH** better approximation

Constructing an Algorithm

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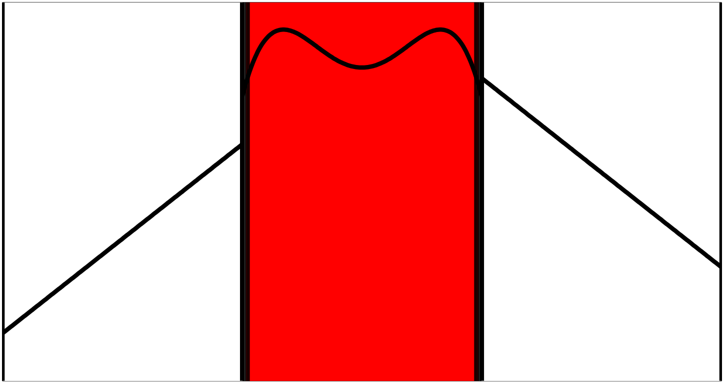
- Idea:
  1. Very small coefficients $\rightarrow$ can be ignored
  2. Large coefficients $\rightarrow$ need more resolution
- This is complicated by the PDE
  1. All PDEs propagate information
  2. Hyperbolic PDEs $\rightarrow$ propagation speed is finite
  3. Cells with high-order neighbors may soon become high-order

Buffer Regions



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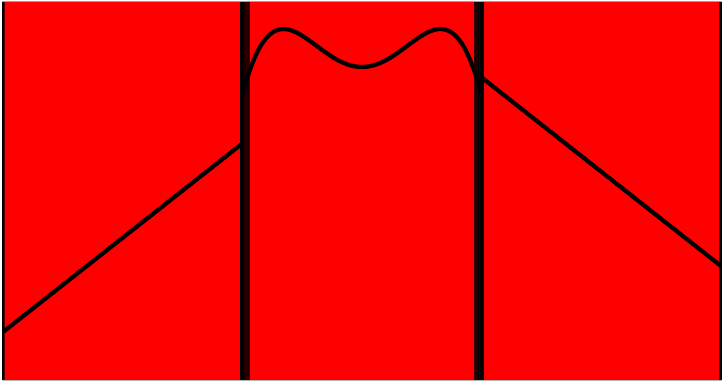

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Pick a maximum order p_M and project initial conditions

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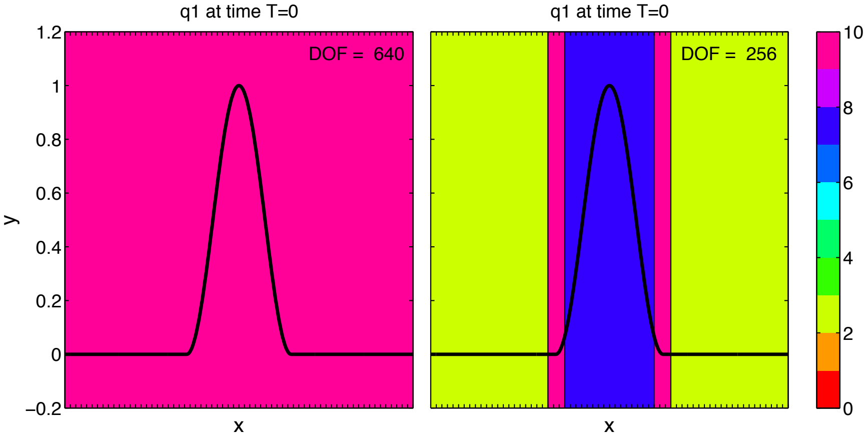
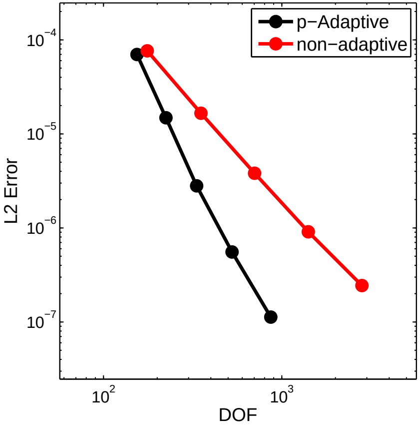
4. Evolve coefficients over the interval
5. Repeat until we have reached the final time

Cosbell Advection



Cosbell Advection

Both methods similarly accurate



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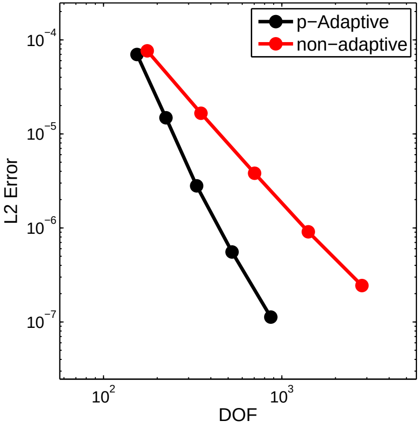

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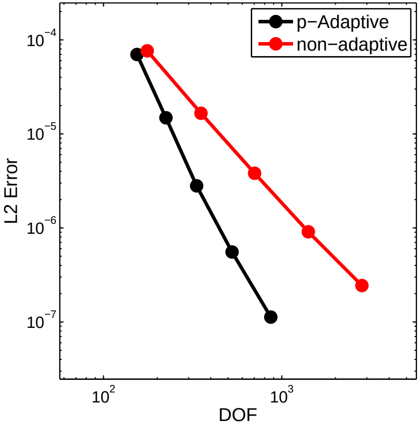
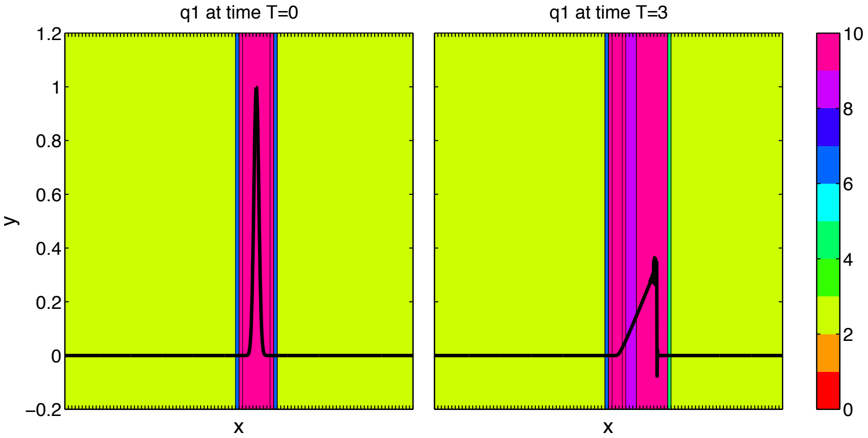
Less DOFs $\rightarrow$ Less CPU time



Burger's Equation



Conclusions

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   - Atmospheric modeling, Seismology, etc..

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   - Atmospheric modeling, Seismology, etc..
2. Works best where solution is smooth
   - Coefficients decay quickly
3. Not as great where solutions are discontinuous
   - Coefficients decay slowly

The Future

1. Local high order time stepping (Local Lax-Wendroff method)
2. Simultaneous *h*–*p* adaptivity
3. Limiting?
4. Adaptive quadrature

References

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[4] G. Tumolo, L. Bonaventura, and M. Restelli, "A semi-implicit, semi-lagrangian, p-adaptive discontinuous galerkin method for the shallow water equations," *Journal of Computational Physics*, vol. 232, no. 1, pp. 46–67, 2013.

Thank you!

Questions?