Adaptive Mesh Refinement for 1D Hyperbolic PDEs

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wave propagation method for Riemann Problem

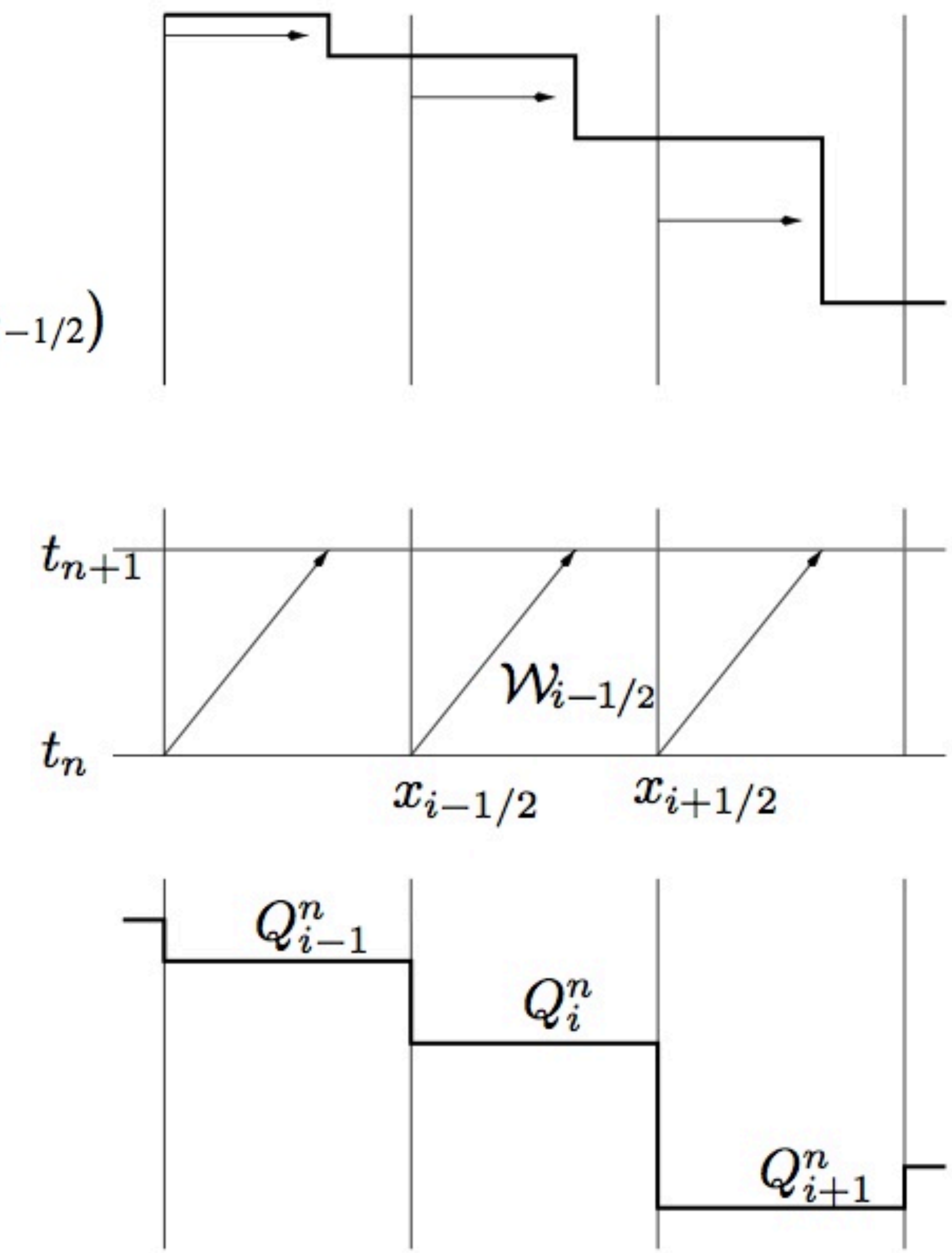
Godunov, “2nd order” correction, limiters

$$Q_i^{n+1} = Q_i - \frac{\Delta t}{\Delta x}\left(A^+\Delta Q_{i-1/2} + A^-\Delta Q_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right)$$

$$A^+\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^+ \mathcal{W}_{i-1/2}^p,$$

$$A^-\Delta Q_{i-1/2} = \sum_{p=1}^{m} (\lambda^p)^- \mathcal{W}_{i-1/2}^p,$$

$$\tilde{F}_{i-1/2} = \frac{1}{2}\sum_{p=1}^{m} |\lambda^p| \left(1 - \frac{\Delta t}{\Delta x}|\lambda^p|\right) \widetilde{\mathcal{W}}_{i-1/2}^p$$



color equation, variable coefficient Riemann problem

$$q(x,t)_t + u(x)q(x,t)_x = 0$$

$$u(x) = \begin{cases} u_{i-1}, & x < x_{i-\frac{1}{2}} \\ u_i, & x > x_{i-\frac{1}{2}} \end{cases} \qquad q(x,0) := \phi(x) = \begin{cases} q_{i-1}, & x < x_{i-\frac{1}{2}} \\ q_i, & x > x_{i-\frac{1}{2}} \end{cases}$$

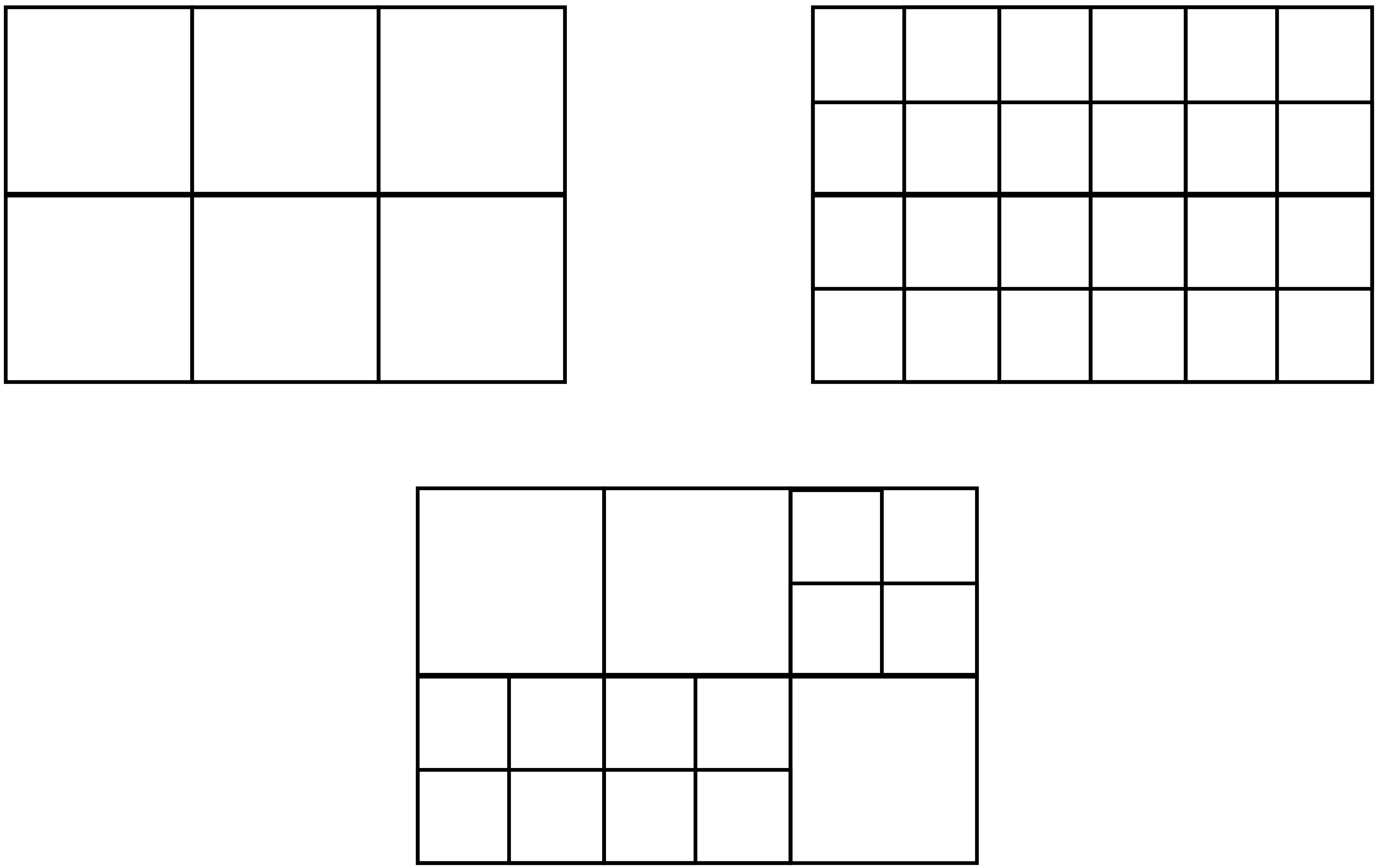
$$q_i^{n+1} = q_i^n - \frac{k}{h}(u_i^+(q_i^n - q_{i-1}^n) + u_i^-(q_{i+1}^n - q_i^n)) - \frac{k}{h}(\tilde{F}_{i+1} - \tilde{F}_i)$$

$$\tilde{F}_i = \frac{1}{2}|u_i|(1 - \frac{k}{h}|u_i|)\tilde{W}_i$$

$$\tilde{W}_i = \begin{cases} limiter(W_i, W_{i-1}), & u_i > 0 \\ limiter(W_i, W_{i+1}), & u_i < 0 \end{cases}$$

Why AMR

- resolve sharp discontinuities without all the overhead
- keep track of physically interesting regions



The remainder of the talk ...

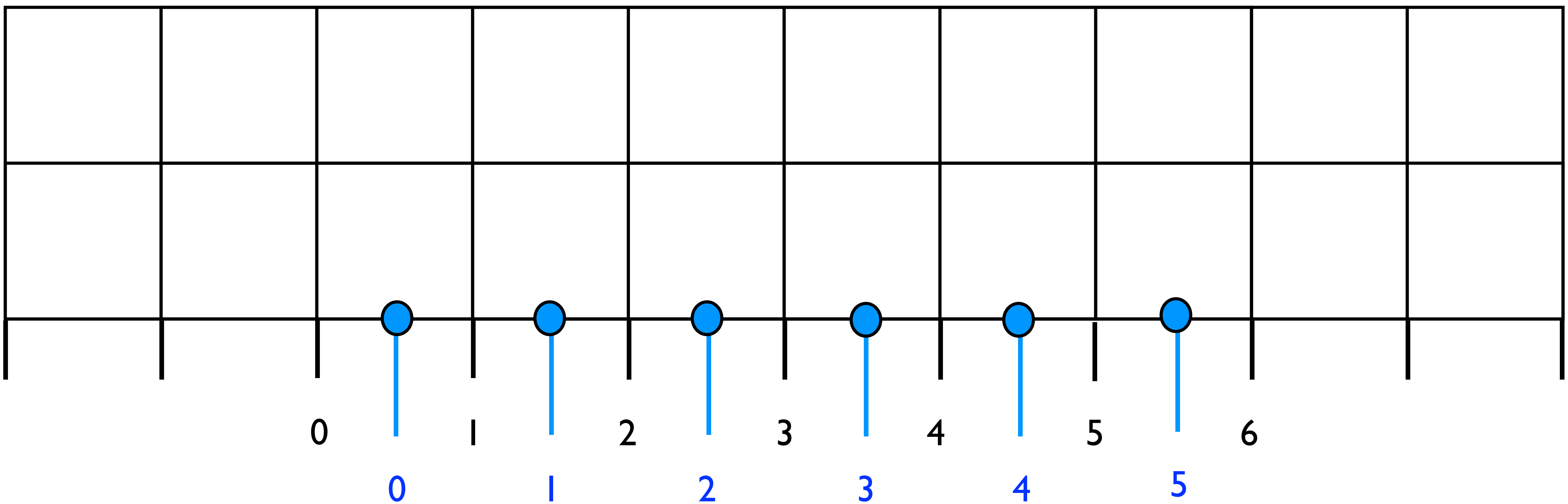
- detailed breakdown of 1D AMR algorithm

- talk about our implementation efforts (ehem)

- some results

AMR steps : initialize values in each cell

(a) initialize coarsest grid (called level $i = 0$)



initial values of q

ncells=6

dx=1/mx

xc=dx/2,3dx/2,...

AMR step : flag, cluster, initialize fine grids

(b) *recurse*: if level ($i > levels_{max}$) *break*

- flag cells for current level i
  - estimate error in each level i cell (compare $D^2(i)$,$D_{2h(i)}$)
    - ∗ if error condition violated, bump $flagged$
- if ($flagged > 0$) construct level $i+1$ grid
  - cluster (based on buffer region and ghost cells) level i grid cells
  - initialize $i+1$ grid for current time
    - ∗ linear interpolation using level i values at current time
- $i \leftarrow i + 1$

AMR step : flag cells at current level for refinement

(b) *recurse*: if level ($i > levels_{max}$) *break*

- flag cells for current level i
  - estimate error in each level i cell (compare $D^2(i)$,$D_{2h(i)}$)
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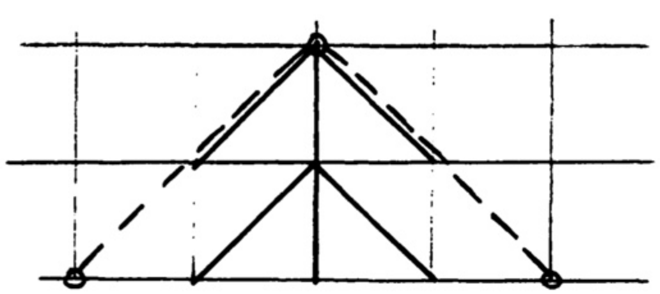
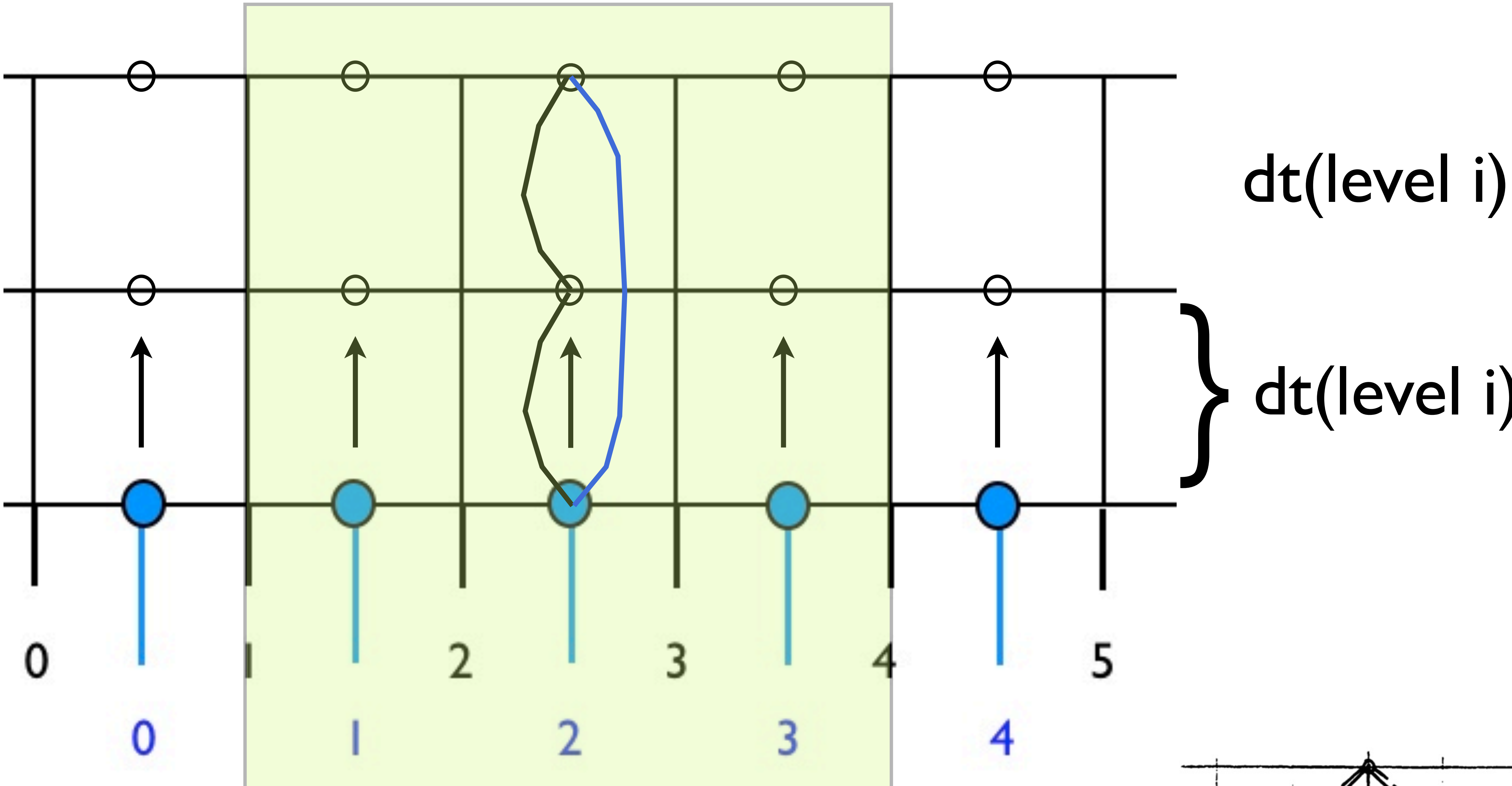
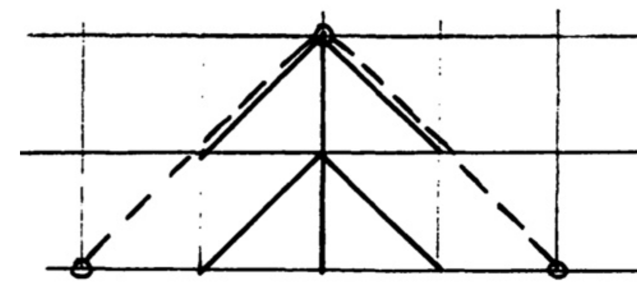
$$\frac{D^2 q(x,t) - D_{2h} q(x,t)}{2^{s+1} - 2} = \tau(x,t) + O(h^{s+2})$$



D := $\quad Q_i^{n+1} = Q_i - \frac{\Delta t}{\Delta x}(A^+ \Delta Q_{i-1/2} + A^- \Delta Q_{i+1/2}) - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2})$

estimate error in each cell





- stored
- temporary

$$\frac{\boxed{D^2 q}(x,t) - \boxed{D_{2h} q}(x,t)}{2^{s+1}-2} = \tau(x,t) + O(h^{s+2})$$

AMR step : cluster flagged cells

(b) *recurse*: if level ($i > levels_{max}$) *break*

- flag cells for current level i
  - estimate error in each level i cell (compare $D^2(i)$,$D_{2h(i)}$)
    - if error condition violated, bump $flagged$
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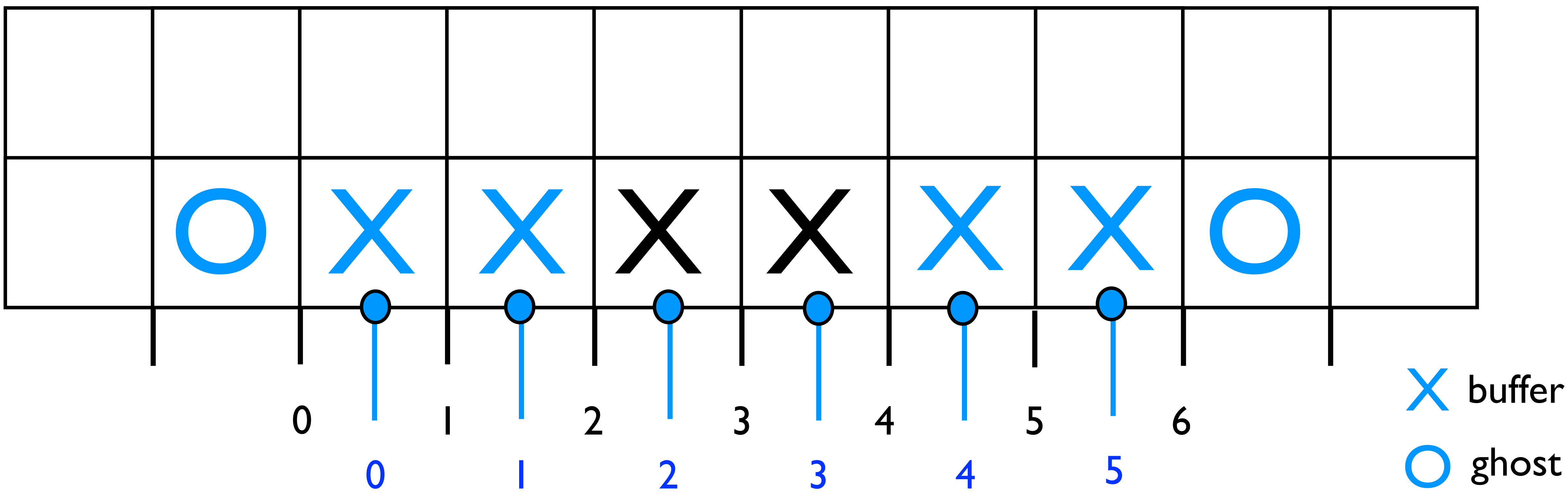
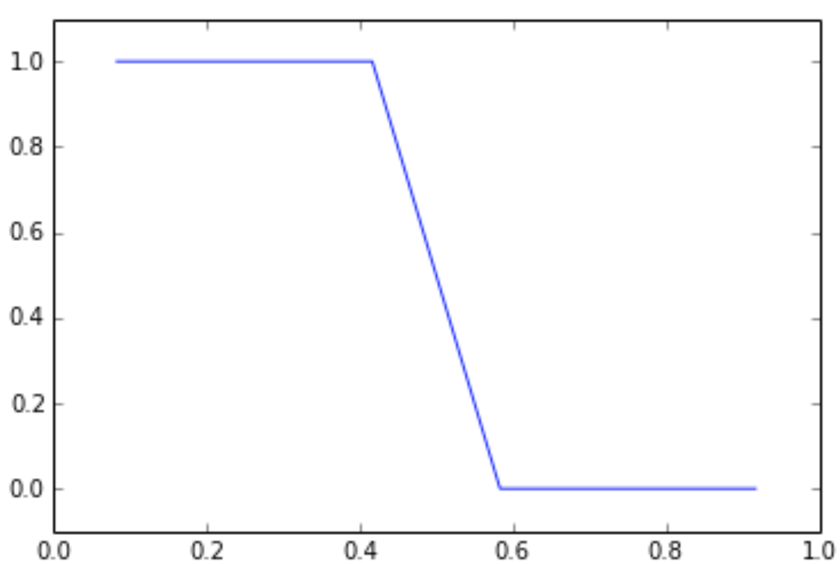
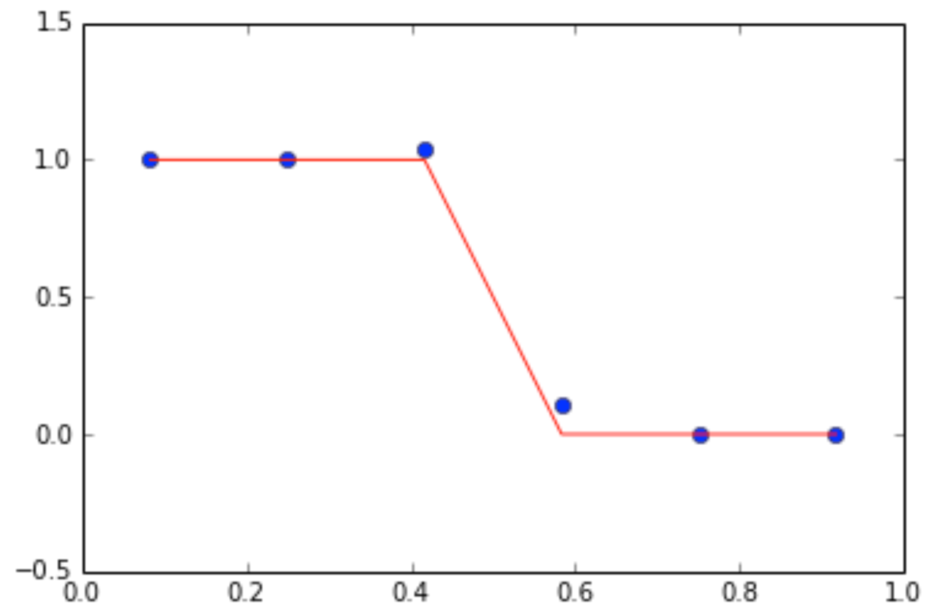
AMR step : cluster flagged cells, determine L(i+1) grid

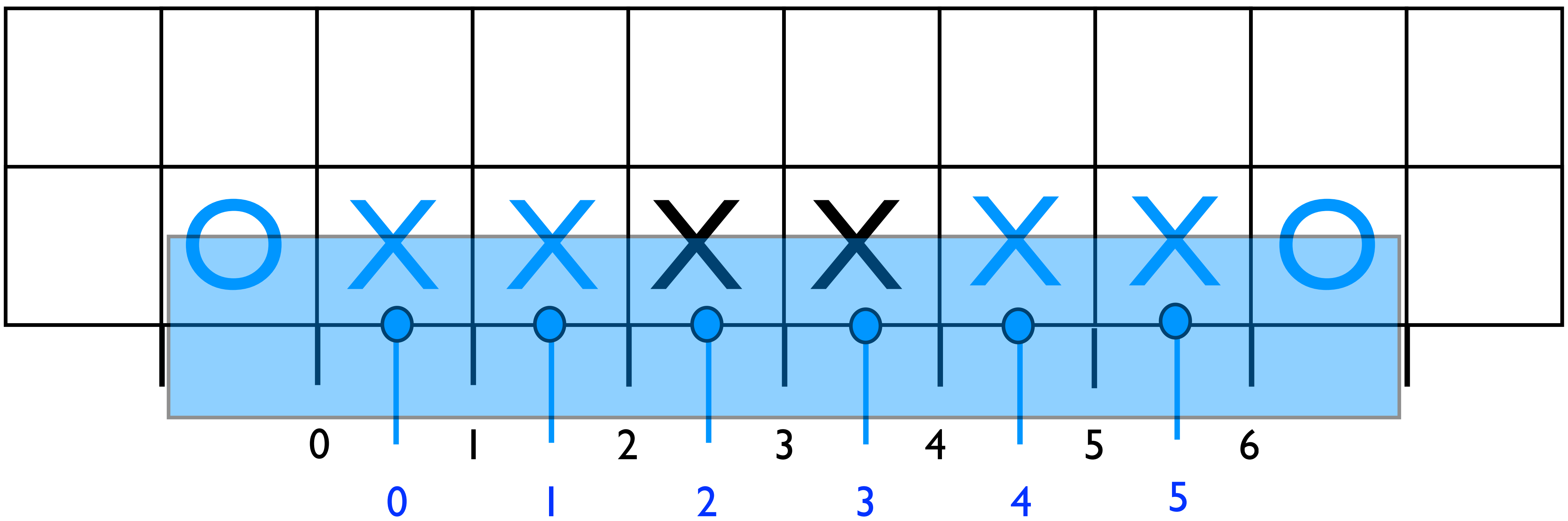


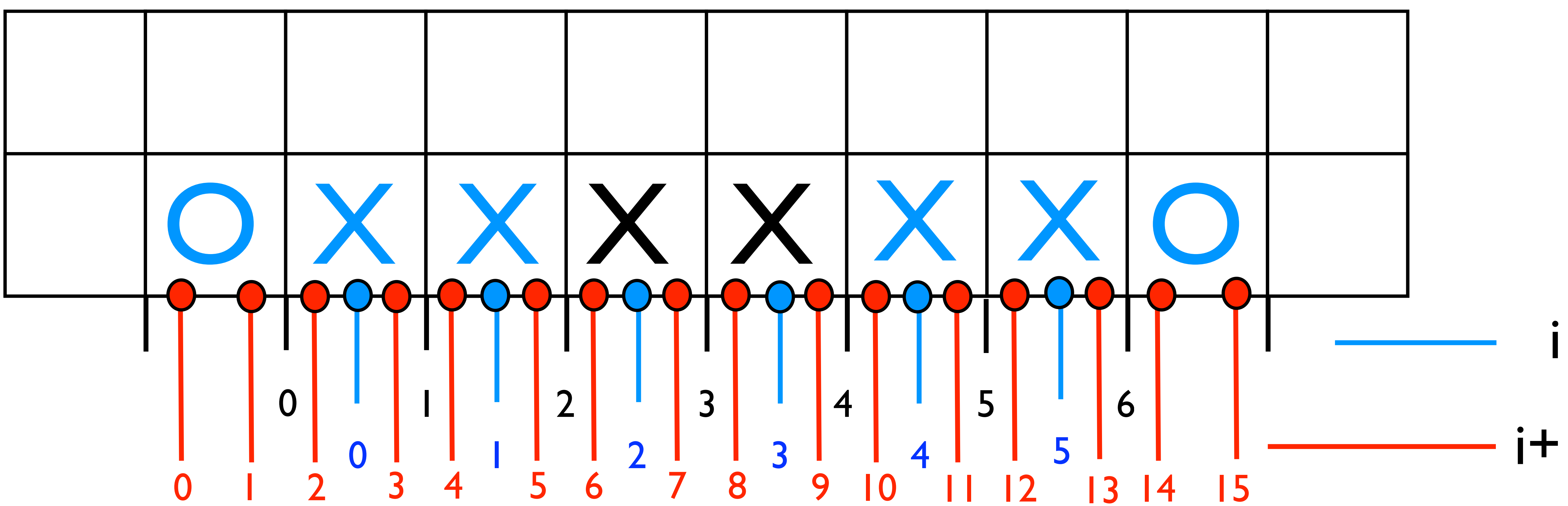




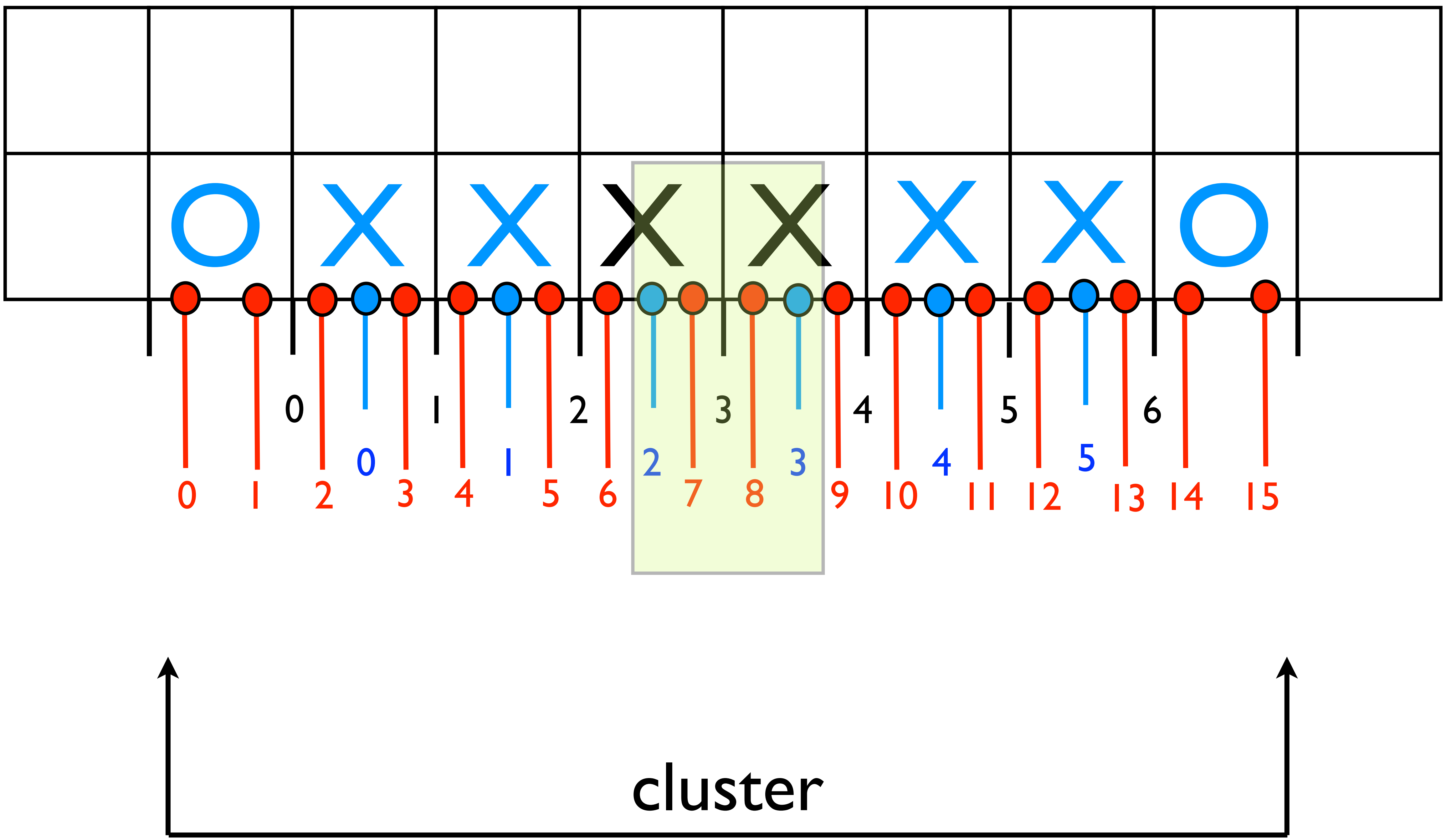
[1. 1. 1. 1. 1. 0. 0. 0. 0. 0.]
2 number of flagged cells
[2. 3.]
start of cluster 0
end of cluster 5
-1 6
cluster, fine grid size 0 16

AMR step : cluster flagged cells, determine L(i+1) grid





AMR step : initialize L(i+1) grid - interpolate L(i)



AMR step : do it all again until happy

(b) *recurse*: if level ($i > levels_{max}$) *break*

- flag cells for current level i
  - estimate error in each level i cell (compare $D^2(i)$,$D_{2h(i)}$)
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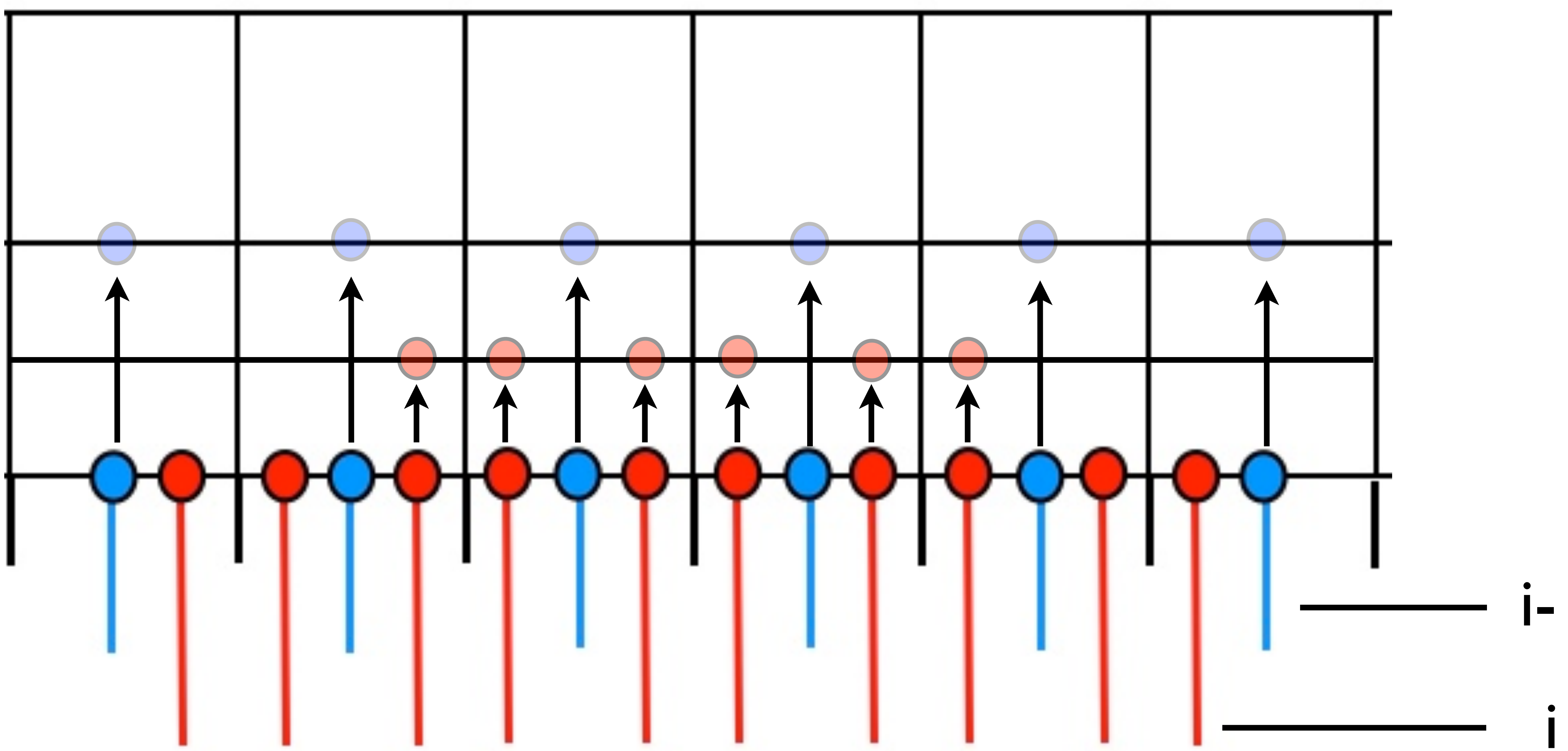
    - linear interpolation using level i values at current time

- $i \leftarrow i + 1$

AMR step : intermediate time stepping

(c) partial time evolution

- $\forall levels(i)$ take single $dt(level(i))$ step by evaluating method
  - at fine-coarse interfaces evaluate points at $dt(level(i))$ time
    - requires interpolating function formed with bilinear interpolation with $level(i-1)$ values



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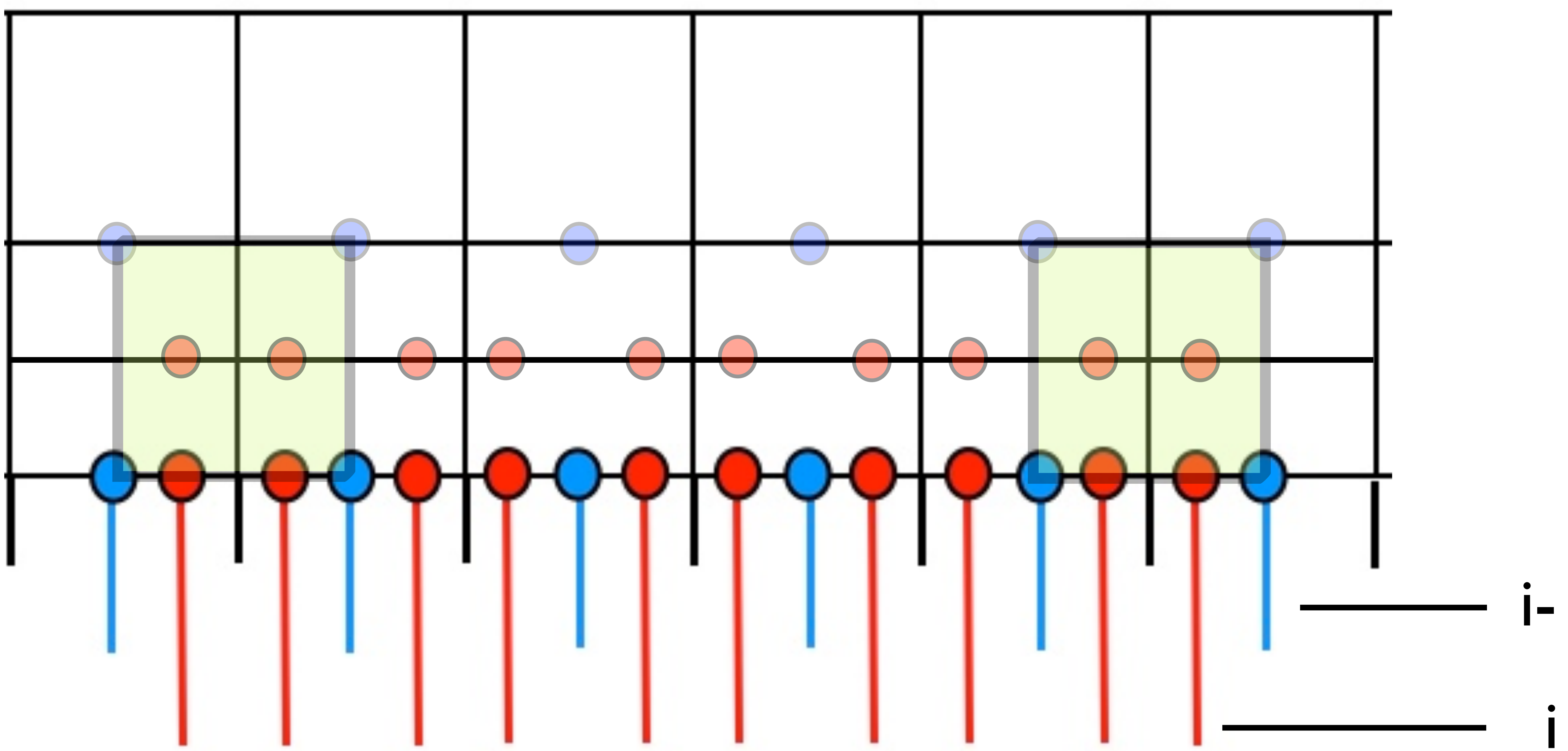
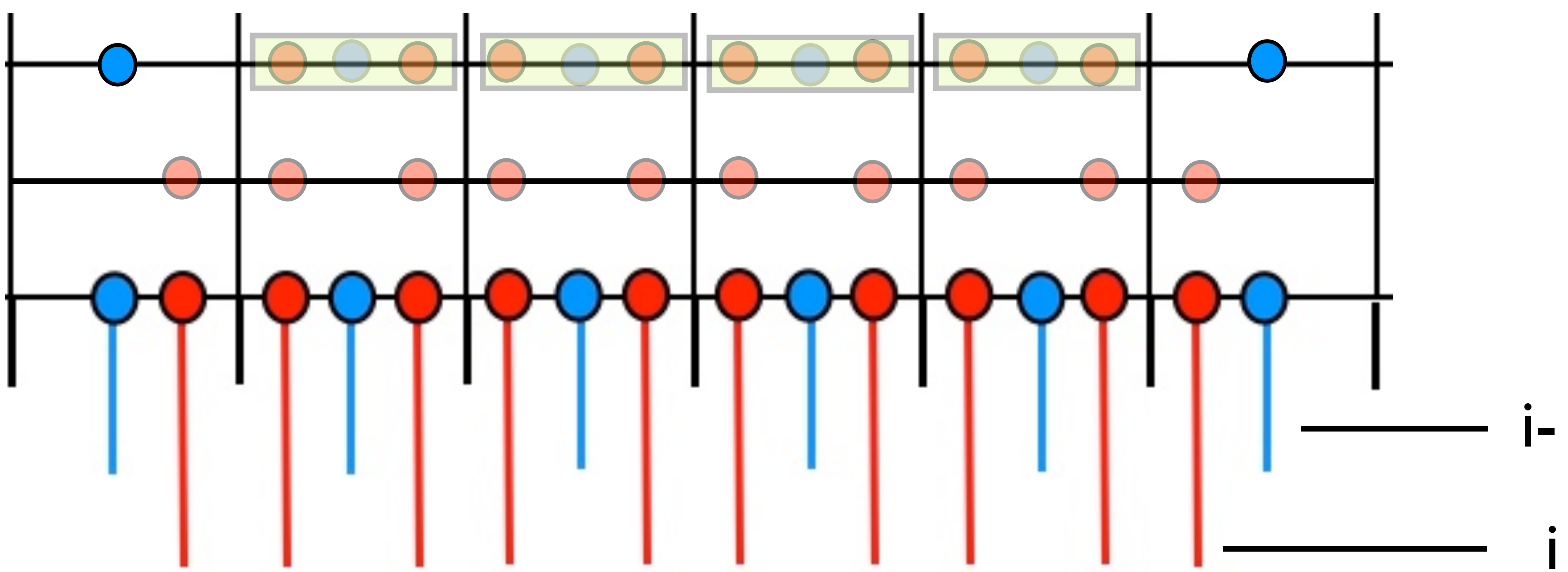


AMR step : promote solutions to full time step

(d) promote grid solutions per grid level to full time step

- $\forall i = L, 1$ $level(i)$ updates interior cells at $level(i-1)$
  - evaluate method using $level(i)$ points and average to form value
- correction step to ensure conservation at fine-coarse interfaces
  - sequentially solve Riemann problems between $level(i)$ values closest to interface and value at $level(i-1)$
  - after each solve, contribution is added to value at $level(i-1)$



AMR step : interface corrections to full time step

(d) promote grid solutions per grid level to full time step

- $\forall i = L, 1$ $level(i)$ updates interior cells at $level(i-1)$
  - evaluate method using $level(i)$ points and average to form value
- correction step to ensure conservation at fine-coarse interfaces
  - sequentially solve Riemann problems between $level(i)$ values closest to interface and value at $level(i-1)$
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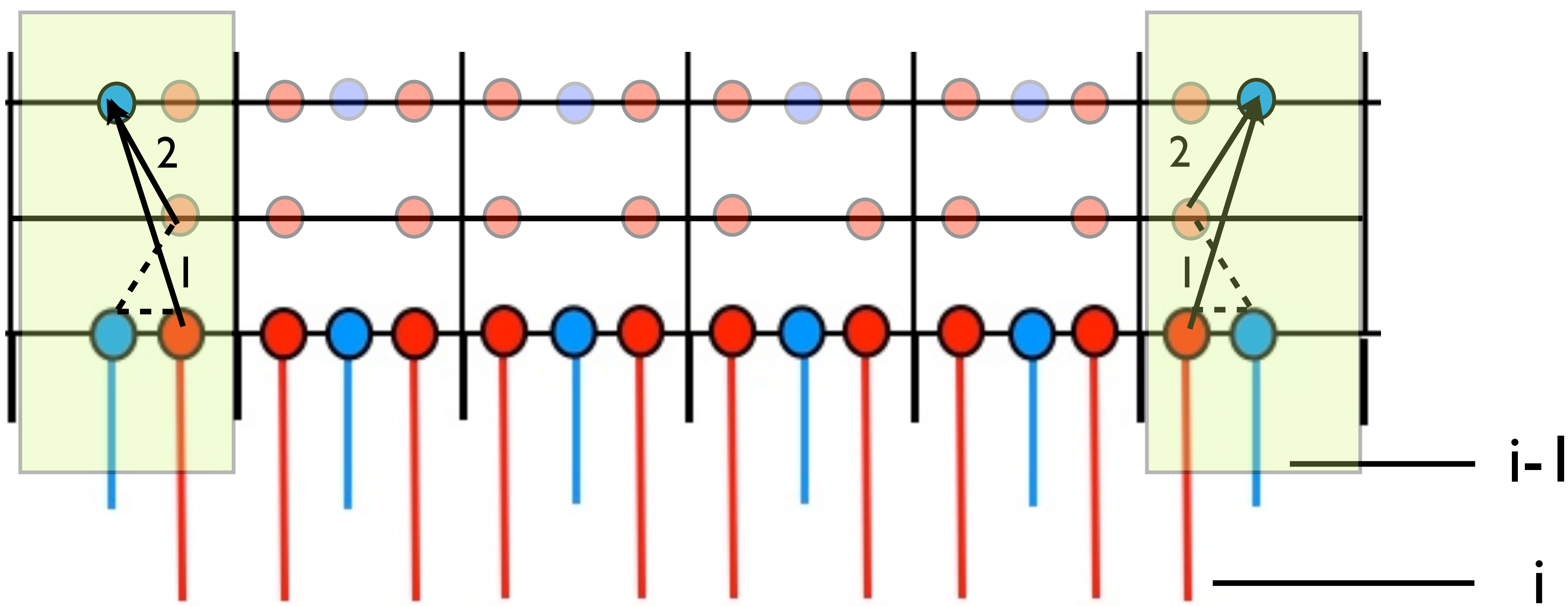
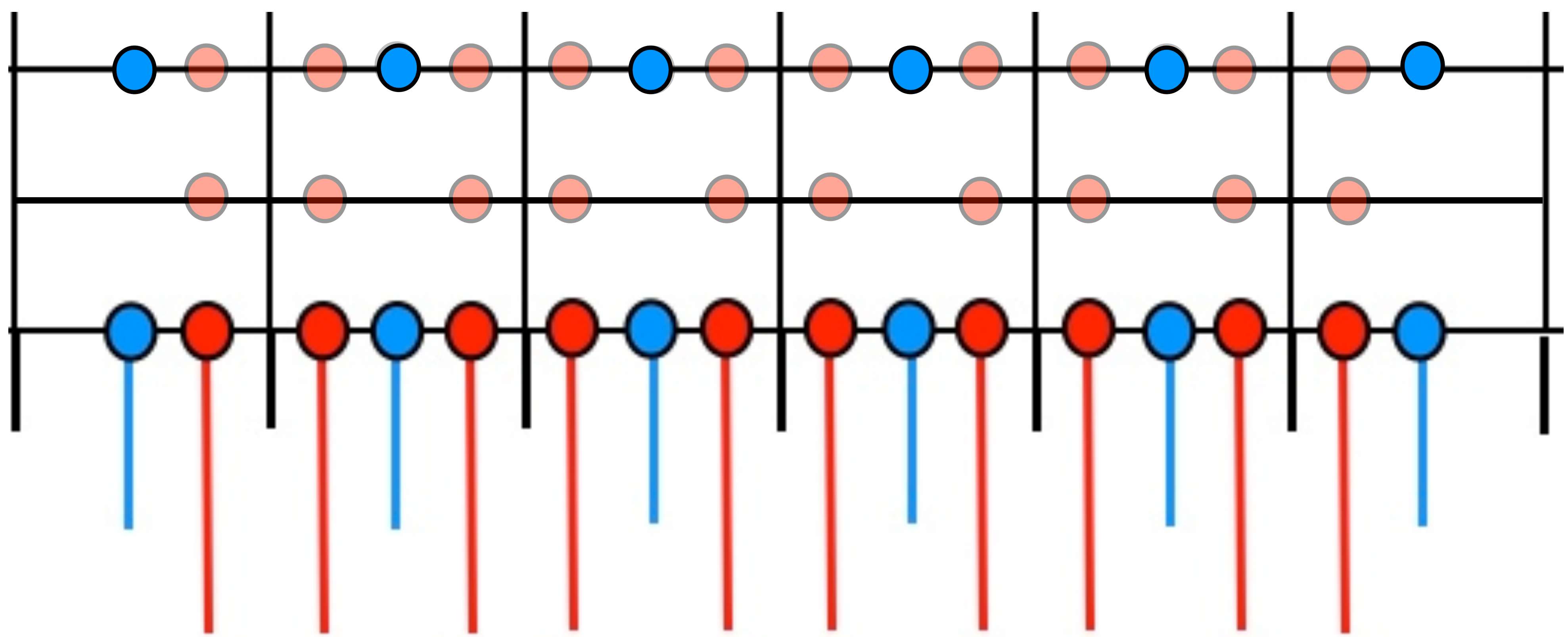


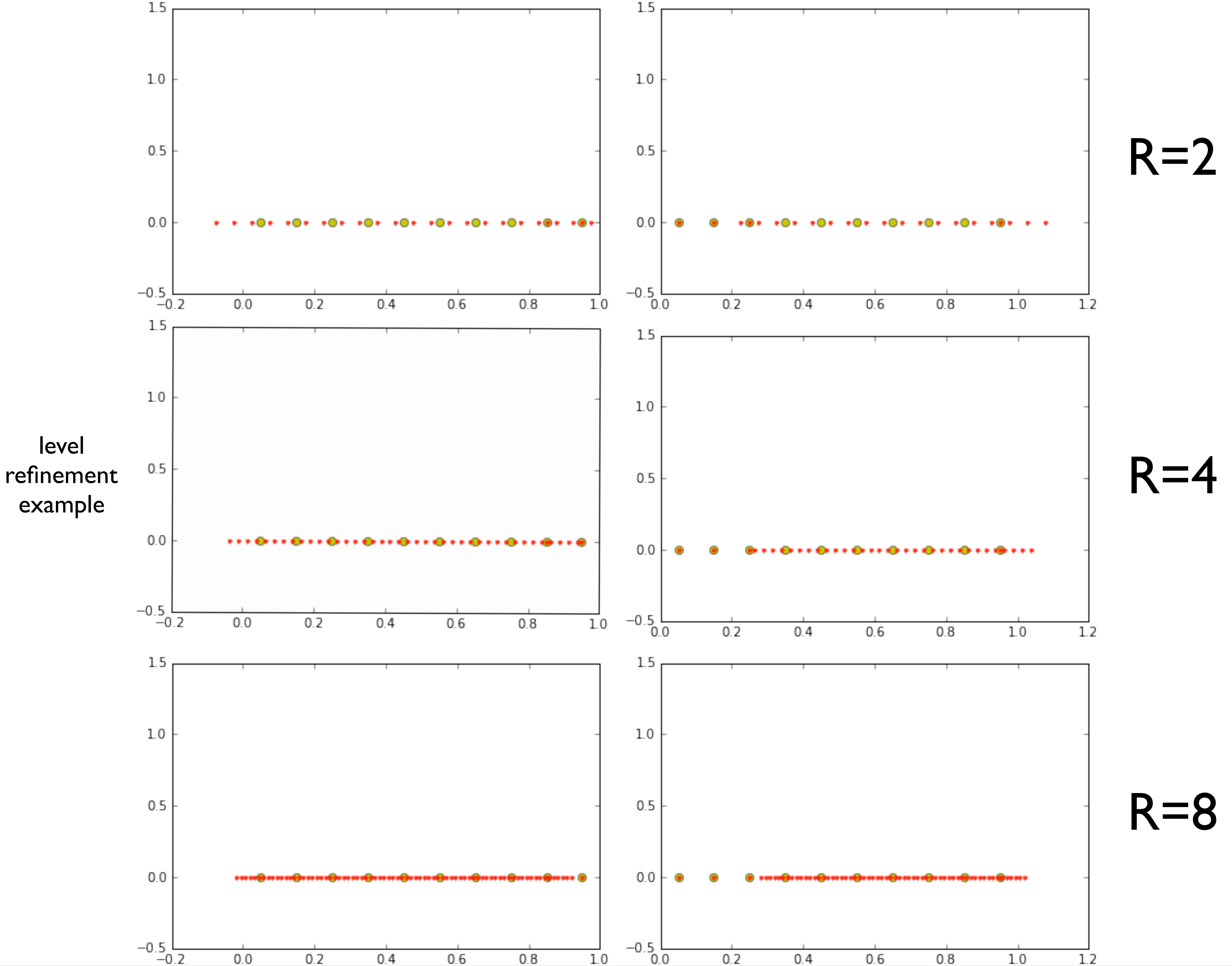
AMR step : take K steps if you like (K ~ buffer size)

(e) completes single time step on coarse grid $level(0)$, go to step b



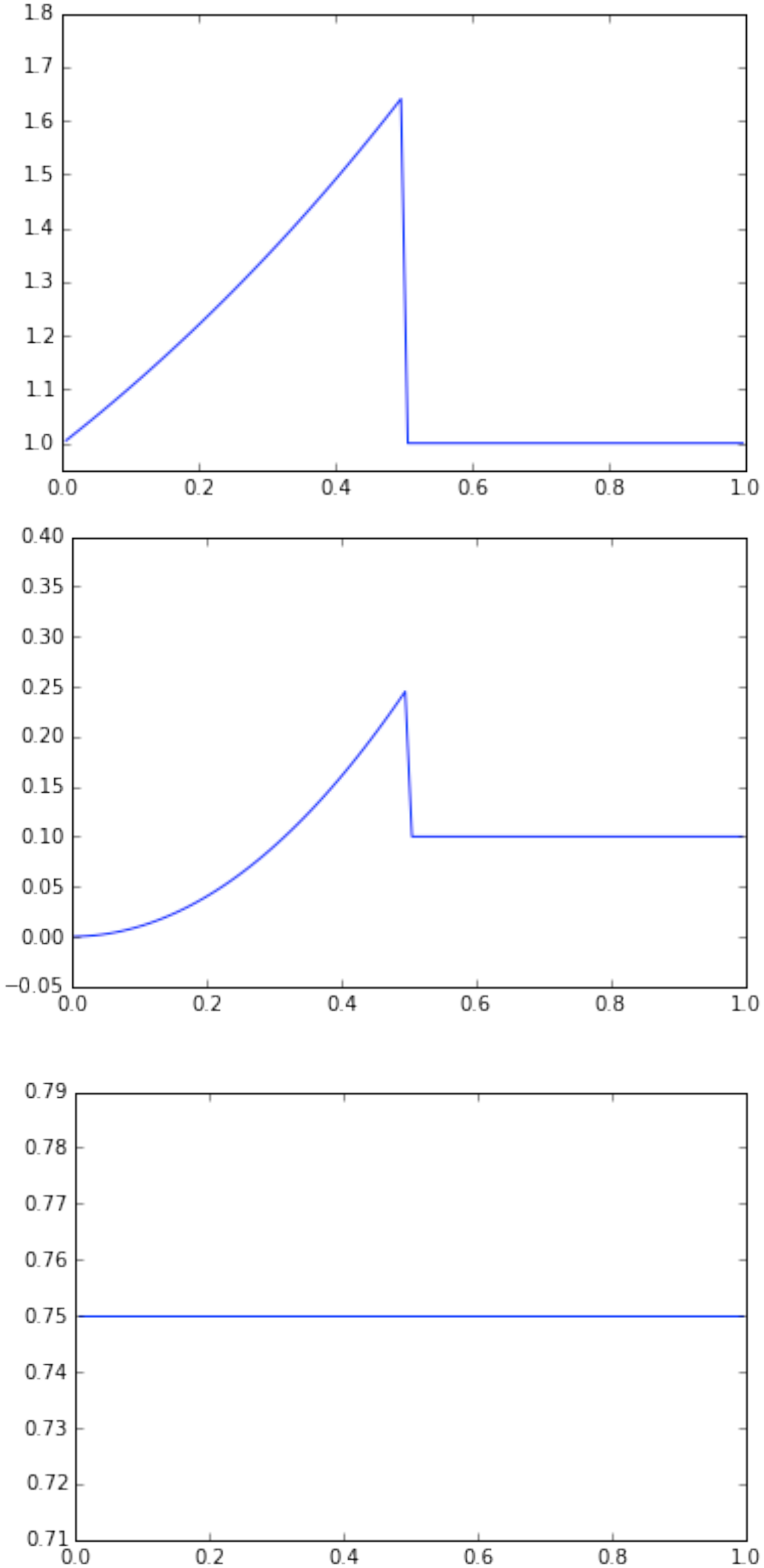
Implementation efforts

- implemented wave propagation method as base solver
    - upwind
    - Lax - Wendroff
    - limiters (we used mostly minmod)
- 1 level amr - 2^p refinement choice
- recursive amr - sadly broken this moment

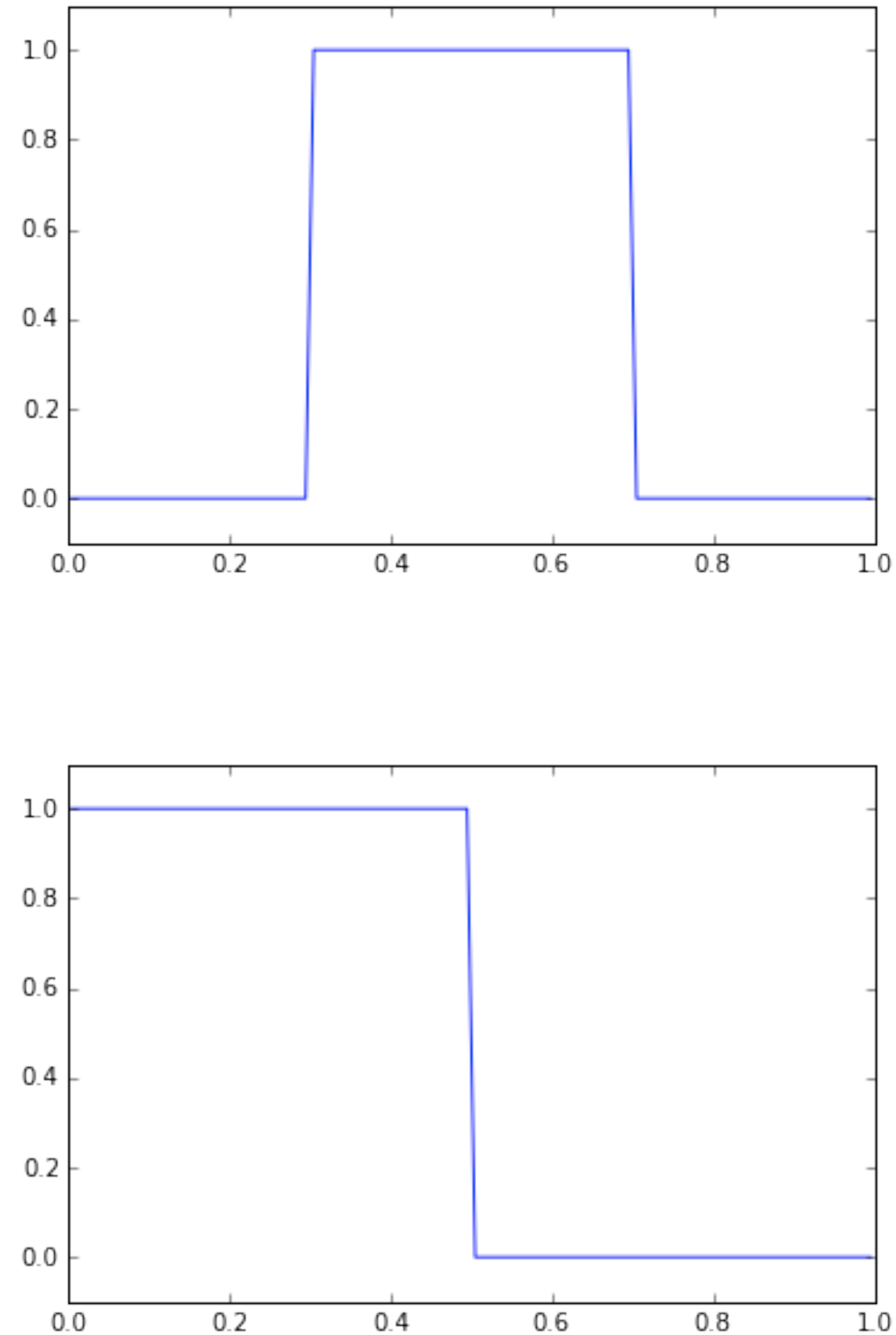


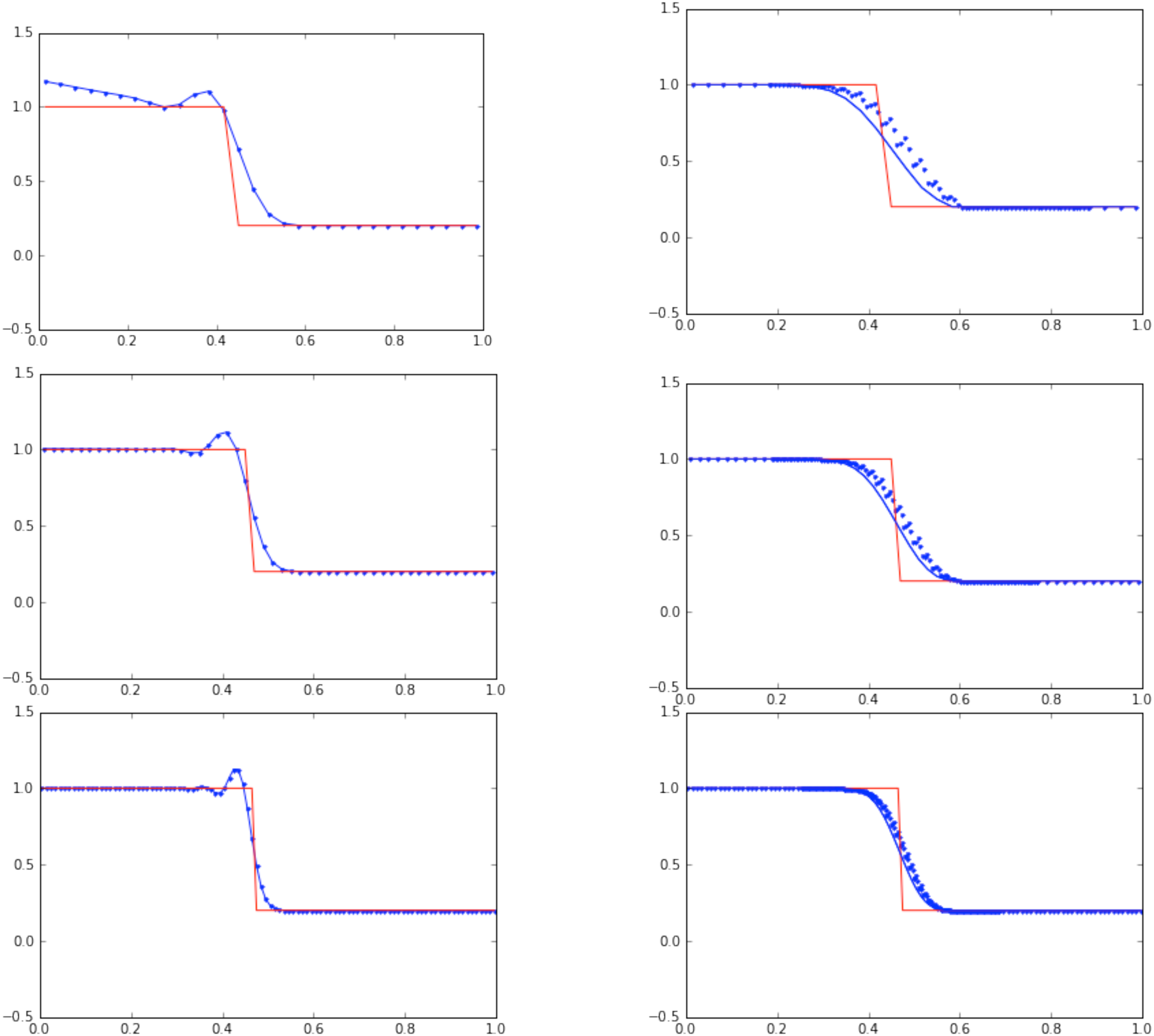
R=2
R=4
R=8
level refinement example

u(x)


1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.0 0.2 0.4 0.6 0.8 1.0
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
0.0 0.2 0.4 0.6 0.8 1.0
0.79
0.78
0.77
0.76
0.75
0.74
0.73
0.72
0.71
0.0 0.2 0.4 0.6 0.8 1.0

q(x)


1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.2 0.4 0.6 0.8 1.0
1.0
0.8
0.6
0.4
0.2
0.0
0.0 0.2 0.4 0.6 0.8 1.0



1.5
1.0
0.5
0.0
−0.5
0.0
0.2
0.4
0.6
0.8
1.0

AMR

studies that we did not present but that matter

- static global refinement versus locally adaptive
    - performance and accuracy tradeoffs

- error improvement as function of level refinement

- role of error tolerance in flagging in grid construction and numerical quality

AMR

difficulties we encountered

- something hosed in recursive implementation
  - close but no cigar

- spreading in refined regions
  - culprit perhaps over-refinement

- overly refined regions - what are the controlling knobs here and how to balance them
  - seems flagging algorithm and clustering but ...

(and questions)

References

[1] http://www.clawpack.org

[2] R. Leveque Finite Volume Methods for Hyperbolic Problems Cambridge University Press (2002)

[3] M. Berger and P. Collela, Local Adaptive Mesh Refinement for Shock Hydrodynamics Journal of Computational Physics 82, 64-84 (1989)

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[5] M. Berger, Adaptive Mesh Refinement for Hyperbolic Partial Differential Equations, *PhD Thesis*, Department of Computer Science, Stanford University, Stanford, CA 94305 (1982)