PHYS 2214 – Prelim 2

community

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Contents

1 Dispersion

Recall that in our discussion of ideal waves, we assumed a constant wave velocity for a given medium that was invariant of the wave's shape or speed. This simplifying assumption does not hold in real life. Instead, wave velocity is a function of a wave's frequency, ω, and wave vector, k.

$$v = \frac{\omega}{k} \tag{1.1}$$

More precisely, we define the *phase velocity*, v_p, and *group velocity*, v_g.

$$v_p = \frac{\omega}{k} \tag{1.2}$$

$$v_g = \left.\frac{d\omega}{dk}\right|_{\omega,k} \tag{1.3}$$

If we consider a graph of ω and k parametrized on v, we can form a graphical description of the phase and group velocity. The phase velocity of a pair (ω, k) is the slope of the line segment connecting the origin to the point (ω, k). The group velocity of a pair (ω, k) is the slope of the line tangent to the curve at (ω, k).

2 Beats

Before we begin our analysis of beats, recall a trigonometric identity.

$$\cos(a) + \cos(b) = 2\left[\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)\right] \tag{2.1}$$

Now, consider two waves propagating in the same medium in the same direction.

$$y_1 = A\cos(k_1 x - \omega_1 t)$$
$$y_2 = A\cos(k_2 x - \omega_2 t)$$

The superposition of the two waves is itself a wave.

$$y = y_1 + y_2 \tag{2.2}$$

$$= 2A\left[\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos\left(\bar{k}x - \bar{\omega}t\right)\right] \tag{2.3}$$

$$\Delta k = k_2 - k_1 \quad \Delta\omega = \omega_2 - \omega_1 \quad \bar{k} = \frac{k_1 + k_2}{2} \quad \bar{\omega} = \frac{\omega_1 + \omega_2}{2} \tag{2.4}$$

If $\Delta k \ll k_1, k_2$, then the first sinusoid oscillates much slower than the second and we get *beats*, as shown in Figure 1. Notice that the enveloping sinusoid has frequency $\frac{\Delta\omega}{2}$. However, the beat frequency is half of the envelope's frequency. This is true because the beat is maximally oscillating at both the peaks and valleys of the envelope.

$$\omega_{\text{beat}} = \Delta\omega \tag{2.5}$$

The beats are produced by the constructive and destructive interference, as shown in Figure 2.

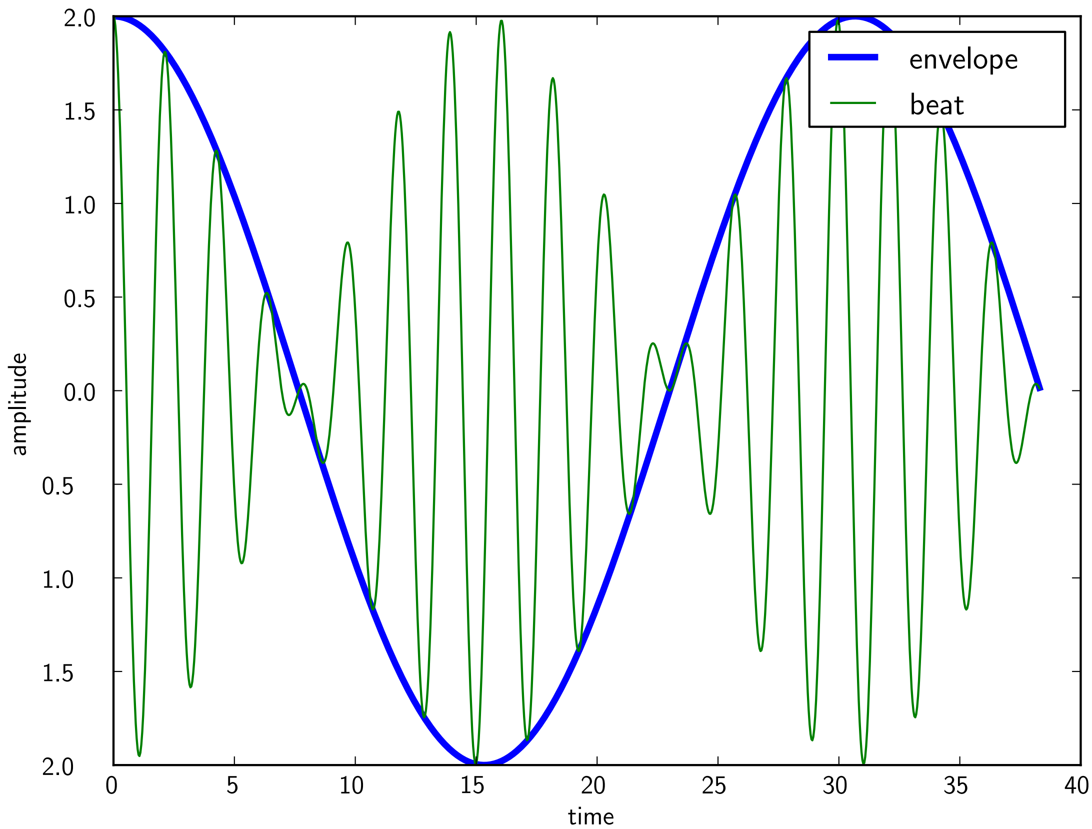

Figure 1: Example beats produced by Listing 1.

3 Doppler Effect

3.1 Mechanical Waves

Consider a source s and observer o. s travels with velocity v_s. o travels with velocity v_o. s emits a wave with frequency f_s and velocity v. From this, we can determine the wave frequency perceived by o, denoted f_o.

$$f_0 = \frac{v + v_o}{v + v_s} f_s \tag{3.1}$$

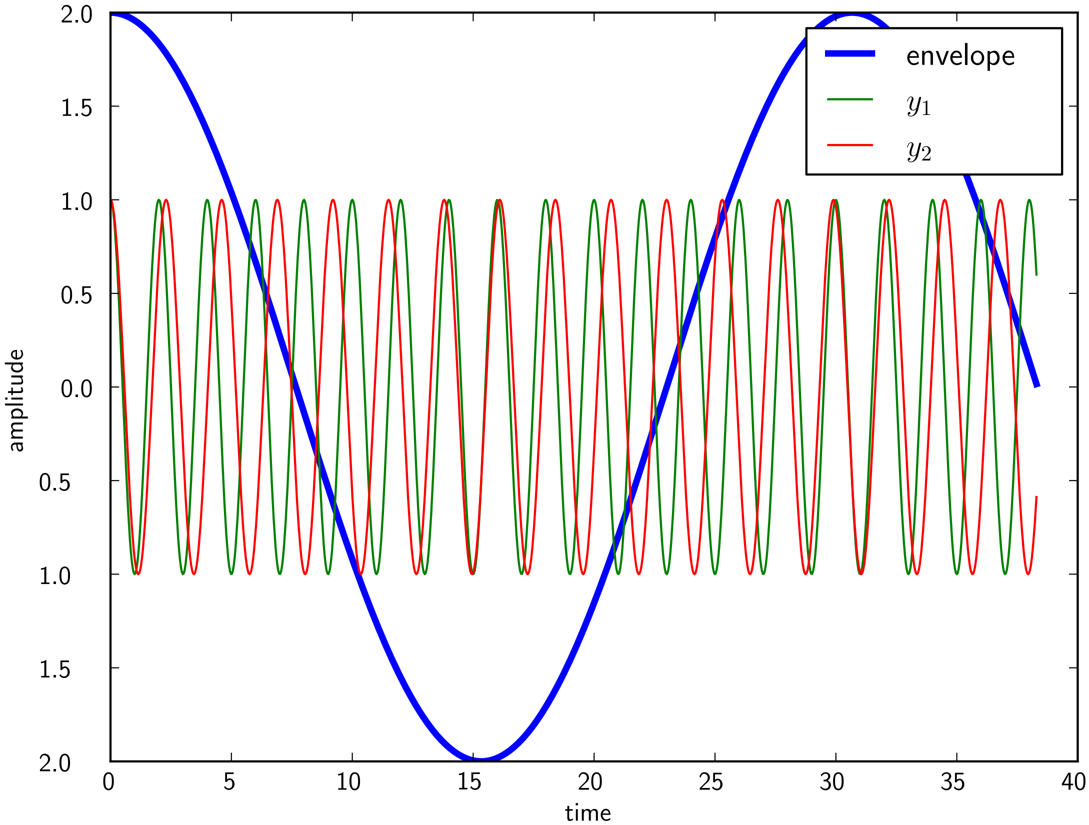

Figure 2: Example beat interference produced by Listing 1.

In equation (3.1), v_o and v_s are positive when in the direction of observer to source. However, there is an intuitive alternative to memorizing this sign convention. Remember that v_o appears in the numerator, and remember that v_s appears in the denominator. When thinking about the velocity of a source or observer, ask yourself what effect their motion would have on the observed frequency. Should the motion increase or decrease the observed frequency? Based on this intuition, the sign of the velocity follows naturally.

For example, consider a source moving away from the observer. If a source is moving away, then the wave he emits will appear to have a lower frequency than f_s. Thus, the observed frequency will decrease. Because v_s appears in the denominator of Equation (3.1), v_s must be positive to decrease the value of f_o.

A few notes.

- v_o and v_s are measured with respect to the medium in which the wave propagates.
- v is always positive.

3.2 Electromagnetic Waves

TODO: There's some controversy on sign convention and a possible typo in the lecture notes.

4 Energy in Mechanical Waves

4.1 Kinetic Energy

Denote kinetic energy U_k and kinetic energy density u_k.

$$U_k = \frac{1}{2}mv^2 = \frac{1}{2}\mu\Delta x\left(\frac{\partial y}{\partial t}\right)^2 \tag{4.1}$$

$$u_k = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \tag{4.2}$$

4.2 Potential Energy

Denote potential energy U_p and potential energy density u_p.

$$U_p = \frac{1}{2}\tau\Delta x\left(\frac{\partial y}{\partial x}\right)^2 \tag{4.3}$$

$$u_p = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 \tag{4.4}$$

4.3 Total Energy

For an arbitrary wave,

$$u = u_k + u_p = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \tag{4.5}$$

For transverse travelling waves, we can use the pulse equation to simplify Equation 4.5.

$$u = \tau\left(\frac{\partial y}{\partial x}\right)^2 = \mu\left(\frac{\partial y}{\partial t}\right)^2 \tag{4.6}$$

5 Power in Mechanical Waves

5.1 Impulse Waves

In transverse mechanical waves, all particle motion is in the y direction. We can use this fact and the relationship $P = Fv$ to find a formula for the power of a transverse mechanical wave.

$$
\begin{aligned}
P &= Fv && (5.1)\\
&= \left(-\tau\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right) && (5.2)\\
&= -\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} && (5.3)
\end{aligned}
$$

For a mechanical transverse pulse, we can apply the pulse equation to derive an alternate formula for power.

$$P = vu \tag{5.4}$$

Here u is the total energy density as in Equation (4.5).

5.2 Standing Waves

To analyze the power of a mechanical standing wave, we can apply the general power equation, $P = -\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$.

Nodes First, analyze the power at a node. At a node, the medium does not vertically oscillate. Thus,

$$\forall t.\, \frac{\partial y}{\partial t} = 0 \tag{5.5}$$

However, the slope of the transverse wave reaches its maximum at a node. Thus

$$\exists t.\, \frac{\partial y}{\partial x} = \max\left(\frac{\partial y}{\partial x}\right) \tag{5.6}$$

From this, we can deduce that kinetic energy density is always zero and potential energy density reaches its maximum at a node. Moreover, we can deduce that power is always zero.

$$
\begin{aligned}
\forall t.\, P &= -\tau\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} && (5.7)\\
&= -\tau\left(\frac{\partial y}{\partial x}\right)(0) && (5.8)\\
&= 0 && (5.9)
\end{aligned}
$$

Antinodes At an antinode, the slope of a standing wave is zero for all time, and the vertical velocity of the wave realizes its maximum at the antinode. Using the same logic as before, we can deduce three equations similar to that for a node.

$$
\begin{aligned}
\forall t.\, \frac{\partial y}{\partial x} &= 0 && (5.10)\\
\exists t.\, \frac{\partial y}{\partial t} &= \max\left(\frac{\partial y}{\partial t}\right) && (5.11)\\
\forall t.\, P &= 0 && (5.12)
\end{aligned}
$$

Thus, the power is always zero at nodes and antinodes. We can now claim that there must not exist any net power transfer along a standing wave. For if there were power transferred along a standing wave, then the power at nodes and antinodes would have to be nonzero at some instant. This is a contradiction.

6 Energy in Sound Waves

The equations for sound wave energy density are analogous to those for mechanical waves. Note that these energy densities are energy per volume, not energy per length.

$$u_k = \frac{1}{2}\rho\left(\frac{\partial s}{\partial t}\right)^2 \tag{6.1}$$

$$u_k = \frac{1}{2}B\left(\frac{\partial s}{\partial x}\right)^2 = \frac{1}{2B}\Delta p^2 \tag{6.2}$$

Here, ρ plays the role of μ. Both are mass densities. B plays the role of τ. τ has units of force, and B has units of pressure.

7 Power in Sound Waves

Before we enter the realm of power derivations, lets recall some background information. I'll color the equations that may come up later. Recall the definitions of B.

$$\Delta p = -B\frac{\Delta V}{V_0} \tag{7.1}$$

$$\Delta p = -B\frac{\partial s}{\partial x} \tag{7.2}$$

Also recall some alternate formulae for v.

$$v = \frac{\omega}{k} = \sqrt{\frac{B}{\rho}} \tag{7.3}$$

Let's assume we have a harmonic sound wave.

$$s = s_0\cos(kx - \omega t) \tag{7.4}$$

$$\Delta p = p_0\sin(kx - \omega t) \tag{7.5}$$

Now for some arithmetic. First, we'll expand the equation $\Delta p = -B\frac{\partial s}{\partial x}$.

$$\begin{aligned}
\Delta p &= -B\frac{\partial s}{\partial x} && (7.6)\\
&= -B\frac{\partial\left(s_0\cos(kx-\omega t)\right)}{\partial x} && (7.7)\\
&= -B(-s_0 k\sin(kx-\omega t)) && (7.8)\\
&= Bs_0 k\sin(kx-\omega t) && (7.9)\\
&= p_0\sin(kx-\omega t) && (7.10)
\end{aligned}$$

From this, we deduce $p_0 = Bs_0k$. With this fact, we can deduce that $u_k = u_p$. The equation for total energy density follows naturally. We can also find the average energy density over a period.

$$\boxed{u = \frac{p_0^2}{B}\sin^2(kx-\omega t)} \tag{7.11}$$

$$\boxed{u_{\text{av}} = \frac{p_0^2}{2B}} \tag{7.12}$$

If we look at $I = \frac{P}{A}$, we can find the average intensity.

$$\boxed{I_{\text{av}} = vu_{\text{av}}} \tag{7.13}$$

8 Intensity and Distance

$$\boxed{I = \frac{P}{4\pi r^2} \propto \frac{1}{r^2}} \tag{8.1}$$

Since $I \propto s_0^2 \propto p_0^2$, both s_0 and p_0 vary as $\frac{1}{r^2}$.

We hear sound on a log scale.

$$\boxed{\beta = 10\log\left(\frac{I}{I_0}\right)} \tag{8.2}$$

9 Fourier

TODO

10 Maxwell's Equations

Integral Form.

$$\oiint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0} \qquad \text{(Gauss' Electric Law)}$$

$$\oiint \vec{B} \cdot d\vec{S} = 0 \qquad \text{(Gauss' Magnetic Law)}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \qquad \text{(Ampere's Law)}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's Law)}$$

Differential Form.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss' Electric Law)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{(Gauss' Magnetic Law)}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(Ampere's Law)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

11 Electromagnetic Wave Equation

11.1 Vacuum

First, in a vacuum. Notice that the Laplacian operator here is the vector Laplacian operator.

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{11.1}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{11.2}$$

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{11.3}$$

11.2 Dielectrics

First, define some constants K and K_m.

$$\epsilon = K\epsilon_0 \quad \mu = K_m \mu_0 \tag{11.4}$$

$$\boxed{\frac{v}{c} = \frac{1}{\sqrt{KK_m}} = \frac{1}{n}} \tag{11.5}$$

11.3 Plane Waves

$$\boxed{B_0 = \frac{E_0}{c}} \tag{11.6}$$

12 Polarization

12.1 Linear Polarization

$$\vec{E} = E_{x0}\cos(\omega t)\hat{\imath} + E_{y0}\cos\omega t\hat{\jmath} \tag{12.1}$$

12.2 Elliptical and Circular Polarization

$$\vec{E} = E_{x0}\cos(\omega t)\hat{\imath} + E_{y0}\sin\omega t\hat{\jmath} \tag{12.2}$$

If $E_{x0} = E_{y0}$, then the polarization is circular. Otherwise, it is elliptical.

12.3 Polarizers

$$\boxed{I_{\text{out}} = I_{\text{in}}\cos^2(\theta)} \tag{Malus' Law}$$

13 Reflection and Transmission

13.1 Characteristic Impedance

$$\boxed{Z = \frac{\mu}{\epsilon}} \tag{13.1}$$

13.2 Normal Incidence

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{13.2}$$

$$T = 1 + R \tag{13.3}$$

13.3 Oblique Incidence, E Parallel

See formula sheet?

13.4 Oblique Incidence, B Parallel

See formula sheet?

13.5 Brewster's Angle

$$\boxed{\tan(\theta_B) = \frac{n_2}{n_1}} \tag{13.4}$$

14 Listings

Listing 1: Beats

```python
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

def main():
    A = 1

    T1 = 2.0
    T2 = 2.3

    w1 = 2 * np.pi / T1
    w2 = 2 * np.pi / T2

    wbeat = w1 - w2
    Tbeat = 2 * np.pi / wbeat

    start = 0
    stop  = 2.5 * Tbeat
    step  = (stop - start) / 1000
    t = np.arange(start, stop, step)

    y1 = A * np.cos(-w1 * t)
    y2 = A * np.cos(-w2 * t)
    y  = y1 + y2

    envelope = 2 * A * np.cos(wbeat / 2 * t)

    plt.figure()
    plt.plot(t, envelope, linewidth=3, label="envelope")
    plt.plot(t, y, label="beat")
    plt.xlabel("time")
    plt.ylabel("amplitude")
    plt.legend()
    plt.savefig("beats.pgf")

    plt.figure()
    plt.plot(t, envelope, linewidth=3, label="envelope")
    plt.plot(t, y1, label="$y_1$")
    plt.plot(t, y2, label="$y_2$")
    plt.xlabel("time")
    plt.ylabel("amplitude")
    plt.legend()
    plt.savefig("beats-interference.pgf")

```

```
main()
```