PHYS 2214 – Prelim 1

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Contents

1 Disclaimer

The math and reasoning presented below is scribed from the lecture notes. I guarantee neither its correctness nor its sanity. If you notice a scribing error or an inconsistency with the notes, please bring it to my attention.

Some math will be hand waved. Bear with me.

2 Oscillation

A **mechanical oscillation** is any repetitive motion. A **free oscillation** is oscillation with two properties:

1. $\exists$ an x_{eq} such that $F(x_{eq}) = 0$.
2. When the object is displaced a small amount, the force restores the object to x_{eq}. This is a **restoring force**. The equilibrium is **stable**.

Mathematically, we can summarize these properties for the two-dimensional case as

$$\boxed{F(x_{eq}) = 0, \quad \left.\frac{dF}{dx}\right|_{x_{eq}} < 0}$$

3 Simple Harmonic Motion (SHM)

SHM is a special type of oscillation where position is a sinusoidal function of time.

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$
$$\boxed{v(t) = x'(t) = -A\omega \sin(\omega t + \phi)}$$
$$\boxed{a(t) = x''(t) = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x(t)}$$

If an object is acted upon by a **linear restoring force**, $F \propto -x$, then it will undergo SHM.

Proof. Assume the constant of proportionality for the linear restoring force is k. From Newton's laws the properties of SHM, we have:

$$\begin{aligned} F &= ma \\ &= -m\omega^2 x \\ &= -kx \\ \omega &= \sqrt{\frac{k}{m}} \end{aligned}$$

■

ω, or ω_0, is the **natural frequency** of oscillation. It is the frequency at which an object oscillates if displaced.

4 Damped Oscillations

A **damping force** is a force that acts opposite of motion. Consider a damping force that varies linearly with velocity.

$$\begin{aligned} F = ma &= -kx - bv \\ mx'' + bx' + kx &= 0 \end{aligned}$$

Ah, the good old fashioned oscillator equation. Let's use complex numbers to solve this familiar beast. Let $z(t) = Ae^{\alpha t}$ where α is some complex number. We'll also use plenty of hand waving.

$$\begin{aligned} mz'' + bz' + kz &= 0 \\ (m\alpha^2 + b\alpha + k)z &= 0 \\ \alpha &= \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \end{aligned}$$

We can now do a case analysis on b, m, and k. Also let $x = \Re(z)$. Final equations for $x(t)$ will be highlighted a distinct color.

4.1 Very Underdamped

Assume $b^2 \ll 4mk$. If b^2 is much much smaller than $4mk$, then we can ignore b^2.

$$\begin{aligned} \alpha &= \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \\ &\approx \frac{-b \pm \sqrt{-4mk}}{2m} \\ &= \frac{-b}{2m} \pm \frac{\sqrt{-4mk}}{2m} \\ &= \frac{-b}{2m} \pm \frac{\sqrt{-4mk}}{\sqrt{4m^2}} \\ &= \frac{-b}{2m} \pm \sqrt{\frac{-4mk}{4m^2}} \\ &= \frac{-b}{2m} \pm \sqrt{\frac{-k}{m}} \\ &= \frac{-b}{2m} \pm i\sqrt{\frac{k}{m}} \end{aligned}$$

Substituting α into z.

$$\begin{aligned}
z &= Ae^{\alpha t} \\
&= Ae^{(\frac{-b}{2m} \pm i\sqrt{\frac{k}{m}})t} \\
&= Ae^{\frac{-b}{2m}t} \pm e^{i\sqrt{\frac{k}{m}}t} \\
x &= Ae^{\frac{-b}{2m}t} \cos\left(\sqrt{\frac{k}{m}}t\right)
\end{aligned}$$

After syntactic reduction,

$$\tau_A = \frac{2m}{b}, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad x(t) = Ae^{-t/\tau_A}\cos(\omega_0 t)$$

τ_A is the **amplitude decay time**.

The very underdamped case is a decaying sinusoid.

4.2 Very Overdamped

Assume $b^2 \gg 4mk$. Now, we ignore the $4mk$ in the numerator.

$$\begin{aligned}
\alpha &= \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \\
&\approx \frac{-b \pm \sqrt{b^2}}{2m} \\
&= \frac{-2b}{2m} \\
&= \frac{b}{m}
\end{aligned}$$

Again substitute α into z.

$$\begin{aligned}
z &= Ae^{\alpha t} \\
&= Ae^{\frac{b}{m}t}
\end{aligned}$$

This time, the syntactic reduction is a bit contrived but yields some nice consistency.

$$\tau_A = \frac{m}{b}, \quad x(t) = Ae^{-t/\tau_A}$$

Unlike in the very underdamped case, the object here does not undergo any oscillation. It decays immediately and monotonically to equilibrium.

4.3 Critically Damped

Assume $\boxed{b^2 = 4mk}$. As I pull the wool over your eyes and wave my hands around, it becomes clear from the divine inspiration of Healey himself that α has a double root. It follows with the utmost triviality:

$$\boxed{\tau_A = \frac{2m}{b}, \quad x(t) = (A + Bt)e^{-t/\tau_A}}$$

As if arbitrated by a deity, the powers that be declare that A and B are determined by initial conditions. What initial conditions? Only God knows now, only God knows. Now, quick onto the next section!

4.4 Energy in Damped Oscillators

Total energy E, kinetic energy K, and potential energy U can be related as $E = K + U$. In an underdamped setting, energy oscillates between K and U. Recall that in an underdamped setting $x(t) = Ae^{-t/\tau_A}\cos(\omega_0 t)$ and consider the following.

$$\begin{aligned} E &\approx U_{\max} \\ &\propto x_{\max}^2 \\ &\propto (e^{-t/\tau_A})^2 \\ &= e^{2t/\tau_A} \\ &=: e^{t/\tau_E} \end{aligned}$$

Tautologically,

$$\boxed{\tau_E = \frac{\tau_A}{2}}$$

τ_E is the **energy decay time**.

5 Driven Oscillations

Drive a damped oscillator with a sinusoidal force at frequency ω_D. The **steady state response** of a linear system will be **sinusoidal at the drive frequency**, not at the natural frequency. We can characterize the response with an amplitude $A(\omega_D)$ and phase $\phi(\omega_D)$, both functions of ω_D.

If we plot $A(\omega_D)$ against ω_D, we'll find a **response curve** that resembles a Gaussian. The response curve has a maximum near ω_0. This maximum is called a **resonance**.

Consider the set of frequencies that produce large amplitudes $S = \left\{\omega_D \mid A(\omega_D) \geq \frac{A_{max}(\omega_D)}{\sqrt{2}}\right\}$. Define the **full width at half energy** $\boxed{\Delta\omega = \max(S) - \min(S)}$. Similarly define the peakiness **quality factor** $\boxed{Q = \frac{\omega_r}{\Delta\omega}}$. An example frequency response curve is given in Figure 1.

Now, we'll dive into the math behind driven oscillations. Consider a driving force $F_D(t) = F_0\cos(\omega_D t)$. From Newton's laws, we get

$$mx'' + bx' + kx = F_D$$

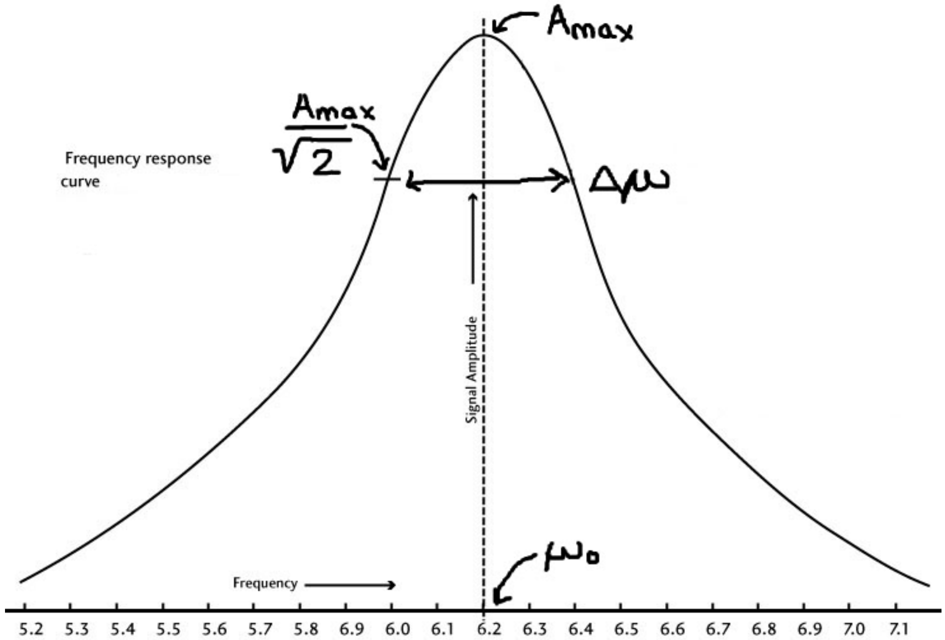

Figure 1: A frequency response curve. $\Delta\omega$ is a measure of the curve's width. Q is a measure of the curve's perkiness.

We'll use our favorite mathematical tool, hand waving, and make things a bit more complex. Convert F_D into $F_0 e^{i\omega_D t}$. Convert x into $Ae^{i\phi}e^{i\omega_D t}$. Now, we substitute these complex valued variables into the differential equation.

$$\begin{aligned}
m(Ae^{i\phi}e^{i\omega_D t})'' + b(Ae^{i\phi}e^{i\omega_D t})' + k(Ae^{i\phi}e^{i\omega_D t}) &= F_0 e^{i\omega_D t} \\
-\omega_D^2 m A e^{i\phi}e^{i\omega_D t} + i\omega_D b A e^{i\phi}e^{i\omega_D t} + kAe^{i\phi}e^{i\omega_D t} &= F_0 e^{i\omega_D t} \\
(-\omega_D^2 m A e^{i\phi} + i\omega_D b A e^{i\phi} + kAe^{i\phi})e^{i\omega_D t} &= F_0 e^{i\omega_D t} \\
-\omega_D^2 m A e^{i\phi} + i\omega_D b A e^{i\phi} + kAe^{i\phi} &= F_0 \\
(-\omega_D^2 m + i\omega_D b + k)Ae^{i\phi} &= F_0
\end{aligned}$$

$$\begin{aligned}
Ae^{i\phi} &= \frac{F_0}{-\omega_D^2 m + i\omega_D b + k} \\
&= \frac{F_0}{-\omega_D^2 m + i\omega_D b + k} \cdot \frac{1/m}{1/m} \\
&= \frac{F_0/m}{-\omega_D^2 + i\omega_D b/m + k/m} \\
&= \frac{F_0/m}{k/m - \omega_D^2 + i\omega_D b/m} \\
&= \boxed{\frac{F_0/m}{\omega_0^2 - \omega_D^2 + i\frac{2\omega_D}{\tau_A}}}
\end{aligned}$$

$\omega_0 = \sqrt{k/m}$ and $\tau_A = 2m/b$.

Again, we can now do a case analysis on ω_0 and ω_D. Final solutions for $x(t)$ will be accented in a similar, distinct color. All solutions will have the form $A\cos(\omega_D t + \phi)$. Why, you ask? Once upon a time, we and God knew. Now, only God knows.

5.1 Low Frequencies

Assume $\omega_D \ll \omega_0$. That is, we are driving at very low frequencies compared to the resonance frequency. Because ω_D is so small, we can neglect those terms and recompute $Ae^{i\phi}$.

$$\begin{aligned}
Ae^{i\phi} &= \frac{F_0/m}{\omega_0^2 - \omega_D^2 + i\frac{2\omega_D}{\tau_A}} \\
&\approx \frac{F_0/m}{\omega_0^2} \\
&\approx \frac{F_0}{\omega_0^2 m} \\
&\approx \frac{F_0}{(\sqrt{k/m})^2 m} \\
&\approx \frac{F_0}{\frac{k}{m}m} \\
&\approx \frac{F_0}{k}
\end{aligned}$$

Substituting into our God equation above,

$$A = \frac{F_0}{k}, \quad \phi = 0, \quad x(t) = \frac{F_0}{k}\cos(\omega_D t)$$

Notice that at low frequencies, the response is determined completely by the spring constant. Inertia and damping are both negligible.

5.2 High Frequencies

Assume $\omega_D \gg \omega_0$. That is, we are driving at very high frequencies compared to the resonance frequency. Since ω_D is much larger that ω_0, we can ignore the ω_0 in the equation for $Ae^{i\phi}$. Also, ω_D^2 dominates ω_D, so we can ignore those terms as well.

$$\begin{aligned}
Ae^{i\phi} &= \frac{F_0/m}{\omega_0^2 - \omega_D^2 + i\frac{2\omega_D}{\tau_A}} \\
&= \frac{F_0/m}{-\omega_D^2} \\
&= -\frac{F_0}{m\omega_D^2} \\
&= \frac{F_0}{m\omega_D^2}e^{-i\pi} && -1 = e^{-i\pi}
\end{aligned}$$

$$\boxed{A = \frac{F_0}{m\omega_D^2}, \quad \phi = -\pi, \quad x(t) = \frac{F_0}{m\omega_D^2}\cos(\omega_D t - \pi)}$$

Notice that at high frequencies, the response is determined completely by the inertia of the system. Damping and spring constants are negligible.

5.3 Resonance

Assume $\boxed{\omega_D = \omega_0}$. Now, $\omega_0^2 - \omega_D^2$ term in the denominator of our $Ae^{i\pi}$ equation is 0.

$$\begin{aligned}
Ae^{i\phi} &= \frac{F_0/m}{\omega_0^2 - \omega_D^2 + i\frac{2\omega_D}{\tau_A}} \\
&= \frac{F_0/m}{i\frac{2\omega_D}{\tau_A}} \\
&= \frac{F_0}{i\frac{2m\omega_D}{\tau_A}} \\
&= -i\frac{F_0}{\frac{2m\omega_D}{\tau_A}} \\
&= -i\frac{F_0\tau_A}{2m\omega_D} \\
&= -i\frac{F_0\frac{2m}{b}}{2m\omega_D} \\
&= -i\frac{F_0}{b\omega_D}
\end{aligned}$$

$$\boxed{A = \frac{F_0}{b\omega_D}, \quad \phi = -\frac{\pi}{2}, \quad \frac{F_0}{b\omega_0}\cos\left(\omega_0 t - \frac{\pi}{2}\right)}$$

Notice that at resonance, the velocity is determined completely by the damping. The inertia and sprint constant are negligible.

5.4 Full Width and Quality Factor

ω_r is not always at ω_0. The lighter the damping, the more true that this becomes. In a heavier damped system, the response may not even have a resonance. After looking to the heavens, we also find

$$\boxed{\Delta\omega = \frac{2}{\tau_A} = \frac{1}{\tau_E}}$$

and

$$\boxed{Q = \frac{\omega_0}{\Delta\omega} = \omega_0\tau_E = \frac{1}{2}\omega_0\tau_A = \frac{\omega_0 m}{b}}$$

Q varies inversely with b.

5.5 Power in Driven Oscillations

Recall instantaneous power $P(t) = F(t)v(t)$. We can average instantaneous power over a single cycle.

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} P(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} F(t)v(t)dt$$

Power is minimized at very low and very high frequencies. It is maximized at resonance. This corresponds to velocity being out of phase and in phase with the driving force respectively. We can also relate the gain of the response curve to Q. In fact, they are equal.

$$\text{gain} = \left|\frac{A(\omega_0)}{A(\omega_D \ll \omega_0)}\right| = Q$$

6 Traveling Waves

A **traveling wave** is defined by its properties:

- A **disturbance** propagates through a medium. Note that the medium itself is not propagating.
- The velocity is determined by the medium

Waves in which the medium disturbance is perpendicular to the wave velocity are **transverse waves**. Waves in which medium displacements are parallel to wave velocity are **longitudinal waves**.

7 Ideal Wave Motion

An ideal wave travels at **constant velocity** and with **unchanging shape**. In order for a medium to realize ideal motion:

- The medium must be **nondispersive**. All waves on a nondispersive medium have the same velocity, regardless of wavelength or amplitude.
- The medium is **linear**
- The wave **must not be damped or amplified** as it travels.

8 Mathematical Description of an Ideal Transverse Traveling Wave

The height of an ideal traveling wave is a function of both position along the medium x and time t. We can express the height as $y(x,t)$. By the properties of a ideal wave motion, it follows that y must be a function of $x - vt$.

$$y(x,t) = f(x \mp vt), \quad v > 0$$

From the definition of f and with the help of partial derivatives, we can derive two equations. The **pulse equation**

$$\frac{\partial y}{\partial t} = \mp v \frac{\partial y}{\partial x}$$

and the **wave equation**

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

9 Harmonic Traveling Waves

A **harmonic traveling wave** is a traveling wave with a sinusoidal displacement. A harmonic traveling wave can be described mathematically as $f(u) = A\cos(ku + \phi)$ where $u = x \mp vt$.

$$y(x,t) = A\cos(k(x \mp vt) + \phi) = A\cos(kx \mp \omega t + \phi)$$

$$k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi f, \quad v = \frac{\omega}{k} = \frac{\lambda}{T} = f\lambda$$

k is the **wavenumber** and the is the parallel of frequency in the x domain. λ is the **wavelength** of the wave.

We can commute our equation to the form

$$y(x,t) = A\cos(\mp\omega t + (kx + \phi))$$

This makes it more clear that at a given fixed x, the medium undergoes vertical simple harmonic motion. Also, $\forall x. \forall x'. A(x) = A(x'), T(x) = T(x')$. In words, the amplitude and period of these vertical simple harmonic motions are all equal. They differ only in their phase.

10 Waves on a String

To simplify our analysis we make the following assumptions.

- Displacements are purely transverse.
- The only force present is tension.
- Tension is constant throughout the string.
- Transverse displacements are small.

The horizontal forces on a small segment of wire sum to 0.

Proof.

$$\begin{aligned}\sum F_x &= \tau_{\text{right}} - \tau_{\text{left}} \\ &= \tau\left[\cos(\theta(x+\Delta x, t)) - \cos(\theta(x,t))\right] \\ &\approx \tau(1-1) \\ &= 0\end{aligned}$$

■

$$\boxed{\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\mu}\frac{\partial^2 y}{\partial x^2}}$$

Proof. Recall a corollary the small angle approximation $\theta \approx 0 \implies \frac{dy}{dx} \approx \tan(\theta)$. Also note that we will substitute $\sum F$ for ma as per Newton's laws.

$$\begin{aligned}
\sum F_y &= \tau\left[\sin(\theta(x+\Delta x, t)) - \sin(\theta(x,t))\right] \\
&\approx \tau\left[y'(x+\Delta x, t) - y'(x,t)\right] \\
ma &= \tau\left[y'(x+\Delta x, t) - y'(x,t)\right] \\
a &= \frac{\tau}{\mu}\frac{\left[y'(x+\Delta x, t) - y'(x,t)\right]}{\Delta x} \\
a &= \frac{\tau}{\mu}y'' \\
\frac{\partial^2 y}{\partial t^2} &= \frac{\tau}{\mu}\frac{\partial^2 y}{\partial x^2}
\end{aligned}$$

■

A proof by pattern matching will show the following corollary holds.

$$\boxed{v = \sqrt{\frac{\tau}{\mu}}}$$

11 Sound Waves

In fluids, waves are purely longitudinal. We can described these longitudinal waves with two properties:

- $s(x,t)$ is the displacement along the axis of displacement.
- $\Delta p(x,t) = p(x,t) - p_0$ is the variation in pressure.

First, we'd like to know the relationship between Δp and s. To find the relationship, we first define the **bulk modulus** B which is the analog of a spring constant.

$$\boxed{\Delta p = \frac{\Delta F}{A} = -B\frac{\Delta V}{V}}$$

$$\boxed{\Delta p = -B\frac{\partial s}{\partial x}}$$

Proof. Consider a column of fluid along the axis of a propagating sound wave. The column of fluid has area A and undisturbed length Δx. After it is perturbed, the left end of the column is displaced to $x+s(x,t)$ and the left end of the column is displaced to $x+\Delta x+s(x+\Delta x,t)$.

The change in volume ΔV is the change in displacement at x and at $x+\Delta x$.

$$\Delta V = A[s(x+\Delta x,t) - s(x,t)]$$

If we substitute this value into our equation for the bulk modulus, we find

$$\begin{aligned}\Delta p &= -B\frac{\Delta V}{V} \\ &= -B\frac{A[s(x+\Delta x,t) - s(x,t)]}{V} \\ &= -B\frac{A[s(x+\Delta x,t) - s(x,t)]}{A\Delta x} \\ &= -B\frac{[s(x+\Delta x,t) - s(x,t)]}{\Delta x} \\ &\approx -B\frac{\partial s}{\partial x}\end{aligned}$$

■

If we use Newton's laws, we can derive the wave equation for sound waves. I elide the proof here, as my arms are tired from waving.

$$\boxed{\frac{\partial^2 s}{\partial t^2} = \frac{B}{\rho}\frac{\partial^2 s}{\partial x^2}}$$

The corollary of which is that

$$\boxed{v = \sqrt{\frac{B}{\rho}}}$$

Comparing the wave equation of sound waves to that of waves on a string, we notice that v can be described as

$$\sqrt{\frac{\text{springiness}}{\text{massiness}}}$$

12 Standing Waves

Compare harmonic traveling waves with harmonic standing waves. First, harmonic traveling waves.

- Travel at a constant speed
- Reflected at boundaries
- $\forall x.\, y(x,t)$ is harmonic

- $\forall x. \forall x'. A(x) = A(x')$
- The phase of vertical harmonic motion varies linearly with x

Now, standing waves.

- $\forall x. y(x,t)$ is harmonic
- The amplitude of the harmonic motion varies sinusoidally with x
- $\forall x. \forall x'. \phi(x) = \phi(x')$

A **standing wave** forms as the superposition of multiple travelling waves. Thus, we can describe a standing wave $y(x,t)$ as the sum of two travelling waves $A_t \cos(kx - \omega t + \phi)$ and $A_t \cos(kx + \omega t + \phi)$. Using a trigonometric identity, it follows

$$y(x,t) = 2A_t \cos(kx + \phi)\cos(\omega t)$$

A few observations drawn from this equation.

- y is a product of functions of x and t. The wave no longer travels.
- $\forall x. y(x,t)$ is harmonic at ω.
- The amplitude varies sinusoidally with x.

13 Standing Waves on a String

Standing waves are not produced at all frequencies. At what frequencies do standing waves form and what determines ω, k, and ϕ? It is the boundary conditions.

For a string with two fixed ends, we have

$$\forall t. y(0,t) = 0 = 2A_t \cos(\phi)$$
$$\forall t. y(L,t) = 0 = 2A_t \cos(kL + \phi)$$

Solving for ϕ and then k, we find

$$\phi = \pi/2, \quad k_n = \frac{n\pi}{L}, \quad \lambda_n = \frac{2L}{n}, \quad f_n = \frac{nv}{2L}, \quad n \in \{1, 2, \ldots\}$$

The set of x where $A(x,t) = 0$ for all time are called **nodes**. The set of x where amplitude is maximized are called **antinodes**. The set of x between nodes are called **loops**. The lowest frequency mode is called the **fundamental frequency**. The other frequencies are **harmonics**.

14 Standing Sound Waves in Pipes

As for harmonic waves on a string, we can write

$$\Delta p(x,t) = A_s \cos(kx + \phi)\cos(\omega t)$$

14.1 Pipe with Two Open Ends

At an open end, $\Delta p = 0$. Just as with waves on a string,

$$k_n = \frac{n\pi}{L}, \quad \lambda_n = \frac{2L}{n}, \quad f_n = \frac{nv}{2L}, \quad n \in \{1, 2, \ldots\}$$

14.2 Pipe with One Closed End

At the closed end, s is a node which implies that Δp is an antinode.

$$\lambda_n = \frac{4L}{n}, \quad f_n = \frac{nv}{4L}, \quad n \in \{1, 3, 5, \ldots\}$$

15 Standing Waves and Resonance

We saw that when we drove an underdamped system near its natural frequency, it experienced its maximum amplitude. Standing waves have a similar pattern. In fact, **standing waves are resonances of an extended system**. Generally, if we create travelling waves at frequencies in which the waves construct, there will be a standing wave. If we create travelling waves at other frequencies, there could be a mess of waves. At certain frequencies, they will destruct. Unlike with the driven underdamped system, the extended system here could have multiple resonances. Though, not all resonances are pretty.

16 Wave Interaction with Boundaries

We'll now examine how travelling waves interact with various boundaries.

16.1 Fixed End

$$g(u) = -f(-u)$$

16.2 Free End

$$g(u) = f(-u)$$

16.3 Matched Termination

Say we attached a massless slip ring to a dashpot. We could null any reflected waves. To do so, the dashpot must have a damping coefficient:

$$\gamma = \frac{\tau}{v}$$

16.4 Conjoined Mediums

$$\boxed{Z = \frac{\tau}{v} = \mu v}$$

$$\boxed{g(u) = Rf(-u)}$$

$$\boxed{R = \frac{Z_1 - Z_2}{Z_1 + Z_2}}$$

$$\boxed{h(u) = Tf(u)}$$

$$\boxed{T = \frac{2Z_1}{Z_1 + Z_2} = 1 + R}$$

$$\boxed{W_2 = \frac{v_2}{v_1} W_1}$$