On the nonlinear dynamics of an inverted double pendulum over a vehicle suspension subject to random excitations

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COBEM 2015
December 6-11, 2015
Rio de Janeiro, RJ, Brazil

Outline

Section 1

Introduction

Horticulture in Brazil

Economical and social aspects:

- Brazil is the world's third largest fruit producer
- annual production of about 40.8 million tonnes
- responsible for 27% of agribusiness workforce

Balanço 2014 e Perspectivas 2015 para o Agronegócio Brasileiro.

Orchard spraying process

- disperse a pulverizing fluid in the orchard
- essential for pest control
  (ensure fruit quality + avoid financial losses)

Orchard tower sprayer

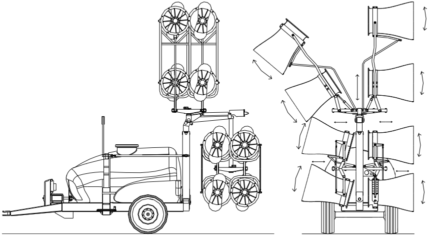
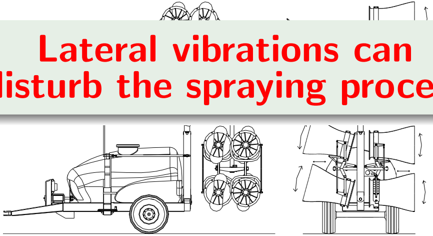


Figure: Schematic representation of an orchard tower sprayer.

Orchard tower sprayer



Lateral vibrations can disturb the spraying process

Figure: Schematic representation of an orchard tower sprayer.

Research objectives

The objectives of this research are:

- construct a stochastic model for orchard sprayer dynamics;
- compute the propagation of uncertainties through this model;
- investigate excitation uncertainties effect in model response.

Section 2

Deterministic Modeling

Idealized mechanical system

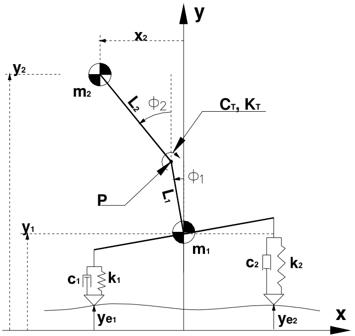


Figure: Inverted double pendulum over a moving base (3 DoF).

Degrees of Freedom:

- y_1 – suspension displacement
- ϕ_1 – suspension rotation
- ϕ_2 – tower rotation

External Excitation:

- y_{e1} – left tire displacement
- y_{e2} – right tire displacement

S. Sartori Junior, J. M. Balthazar, B. R. Pontes Junior, *Non-linear dynamics of a tower orchard sprayer based on an inverted pendulum model.* **Biosystems Engineering**, 103:417–426, 2009..

Modeling of tires displacements

The tires displacements are assumed to be

$$y_{e1}(t) = A \sin(\omega\, t)$$

$$y_{e2}(t) = A \sin(\omega\, t + \rho)$$

- periodic functions in time
- same amplitude
- single frequencial component
- out of phase

S. Sartori Junior, J. M. Balthazar, B. R. Pontes Junior, *Non-linear dynamics of a tower orchard sprayer based on an inverted pendulum model.* **Biosystems Engineering**, 103:417–426, 2009..

Derivation of the equation of motion

To obtain the equations of motion, Lagrangian formalism is used

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{D}}{\partial \dot{q}} = 0, \quad q = \{y_1, \phi_1, \phi_2\},$$

where $\mathcal{L} = \mathcal{T} - \mathcal{V}$.

These energy functionals are given by

$$\mathcal{T} = \frac{1}{2} m_1 \left(\dot{x}_1 + \dot{y}_1\right) + \frac{1}{2} m_2 \left(\dot{x}_2 + \dot{y}_2\right) + \frac{1}{2} I_1 \dot{\phi}_1 + \frac{1}{2} I_2 \dot{\phi}_2$$

$$\mathcal{V} = m_1\, g\, y_1 + m_2\, g\, y_2 + \frac{1}{2} k_1 \left(y_1 - B_1 \,\sin\phi_1 - y_{e1}\right)^2 + \frac{1}{2} k_2 \left(y_1 + B_2 \,\sin\phi_1 - y_{e2}\right)^2 + \frac{1}{2} k_T \left(\phi_2 - \phi_1\right)^2$$

$$\mathcal{D} = \frac{1}{2} c_1 \left(\dot{y}_1 - B_1\, \dot{\phi}_1\, \cos\phi_1 - y_{e1}\right)^2 + \frac{1}{2} c_2 \left(\dot{y}_1 + B_2\, \dot{\phi}_1\, \cos\phi_1 - y_{e1}\right)^2 + \frac{1}{2} c_T \left(\dot{\phi}_2 - \dot{\phi}_1\right)^2$$

S. Sartori Junior, J. M. Balthazar, B. R. Pontes Junior, *Non-linear dynamics of a tower orchard sprayer based on an inverted pendulum model.* **Biosystems Engineering**, 103:417–426, 2009..

Nonlinear dynamical system

The nonlinear dynamics of interest evolves according to

$$[M]\begin{pmatrix} \ddot{y}_1(t) \\ \ddot{\phi}_1(t) \\ \ddot{\phi}_2(t) \end{pmatrix} + [N]\begin{pmatrix} \dot{y}_1^2(t) \\ \dot{\phi}_1^2(t) \\ \dot{\phi}_2^2(t) \end{pmatrix} + [C]\begin{pmatrix} \dot{y}_1(t) \\ \dot{\phi}_1(t) \\ \dot{\phi}_2(t) \end{pmatrix} + [K]\begin{pmatrix} y_1(t) \\ \phi_1(t) \\ \phi_2(t) \end{pmatrix} = \mathbf{g} - \mathbf{h},$$

supplemented by appropriate initial conditions.

Note that $[M]$, $[N]$, $[C]$, $[K]$, $\mathbf{g}$, and $\mathbf{h}$ are configuration dependent.

Integration is done with RKF45 method of MATLAB routine `ode45`.

S. Sartori Junior, J. M. Balthazar, B. R. Pontes Junior, *Non-linear dynamics of a tower orchard sprayer based on an inverted pendulum model.* **Biosystems Engineering**, 103:417–426, 2009..

Section 3

Stochastic Modeling

Uncertainties in the mechanical-mathematical model

The mathematical model is subjected to uncertainties:

- data uncertainty (model parameters variabilites)
- model uncertainty (lack of knowledge of physics)

model uncertainty $\Longrightarrow$ ignored in a first analysis

data uncertainty $\Longrightarrow$ tires excitation (most pronounced)

C. Soize, *Stochastic modeling of uncertainties in computational structural dynamics — recent theoretical advances.* **Journal of Sound and Vibration**, 332: 2379–2395, 2013.

Probabilistic model for tires displacement

A parametric probabilistic approach is adopted: $(\Theta, \Sigma, \mathbb{P})$

The tires displacement model has three parameters:

$$y_{e1}(t) = A \sin(\omega\, t)\,,$$

$$y_{e2}(t) = A \sin(\omega\, t + \rho)\,,$$

It is assumed that A and ω present aleatory behavior, being modeled as (independent) random variables:

$$\mathbb{A} : \Sigma \to \mathbb{R} \quad \text{and} \quad \omega : \Sigma \to \mathbb{R}.$$

C. Soize, *Stochastic modeling of uncertainties in computational structural dynamics — recent theoretical advances*. **Journal of Sound and Vibration**, 332: 2379–2395, 2013.

Construction of the stochastic model

The Shannon entropy of $p_{\mathbb{X}}$ is defined as

$$\mathbb{S}\left(p_{\mathbb{X}}\right) = -\int_{\mathbb{R}} \ln p_{\mathbb{X}}(x)\, p_{\mathbb{X}}(x)\, dx.$$

Maximum Entropy Principle

Among all the probability distributions, consistent with the current known information of a given random parameter, the one which best represents your knowledge about this random parameter is the one which maximizes its entropy.

Jaynes, E. T., *Information theory and statistical mechanics.* **Physical Review Series II**, 106:620–630, 1957.

Shannon, C. E., *A mathematical theory of communication.* **Bell System Technical Journal**, 27:379–423, 1948.

Distribution of the amplitude

Maximizes

$$\mathbb{S}\left(p_{\mathbb{A}}\right) = -\int_{\mathbb{R}} \ln p_{\mathbb{A}}(a)\, p_{\mathbb{A}}(a)\, da$$

such that:

$$\text{Supp}\, p_{\mathbb{A}} = (0, +\infty) \implies \int_{a=0}^{+\infty} p_{\mathbb{A}}(a)\, da = 1$$

$$\mathbb{E}\left[\mathbb{A}\right] = \mu_{\mathbb{A}} \in (0, +\infty) \implies \int_{a=0}^{+\infty} a\, p_{\mathbb{A}}(a)\, da = \mu_{\mathbb{A}} > 0$$

Distribution of the amplitude

Maximizes

$$\mathbb{S}\left(p_{\mathbb{A}}\right) = -\int_{\mathbb{R}} \ln p_{\mathbb{A}}(a)\, p_{\mathbb{A}}(a)\, da$$

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$$\mathbb{E}\left[\mathbb{A}\right] = \mu_{\mathbb{A}} \in (0, +\infty) \implies \int_{a=0}^{+\infty} a\, p_{\mathbb{A}}(a)\, da = \mu_{\mathbb{A}} > 0$$

Exponential distribution

$$p_{\mathbb{A}}(a) = \mathbb{1}_{(0,+\infty)}(a)\, \frac{1}{\mu_{\mathbb{A}}} \exp\left(-\frac{a}{\mu_{\mathbb{A}}}\right)$$

Distribution of the loading frequency

Maximizes

$$\mathbb{S}\left(p_{\omega}\right) = -\int_{\mathbb{R}} \ln p_{\omega}(\omega)\, p_{\omega}(\omega)\, d\omega$$

such that:

$$\text{Supp}\, p_{\omega} = [\omega_1, \omega_2] \subset (0, +\infty) \implies \int_{\omega=0}^{+\infty} p_{\omega}(\omega)\, d\omega = 1$$

Distribution of the loading frequency

Maximizes

$$\mathbb{S}\left(p_{\omega}\right) = - \int_{\mathbb{R}} \ln p_{\omega}(\omega)\, p_{\omega}(\omega)\, d\omega$$

such that:

$$\text{Supp}\, p_{\omega} = [\omega_1, \omega_2] \subset (0, +\infty) \quad \Longrightarrow \quad \int_{\omega=0}^{+\infty} p_{\omega}(\omega)\, d\omega = 1$$

Uniform distribution

$$p_{\omega}(\omega) = \mathbb{1}_{[\omega_1, \omega_2]}(\omega)\, \frac{1}{\omega_2 - \omega_1}$$

Stochastic nonlinear dynamical system

The external excitations become random processes

$$\mathbb{y}_{e1}(t,\theta) = \mathbb{A}\,\sin\left(\mathbb{\omega}\,t\right)$$

$$\mathbb{y}_{e2}(t,\theta) = \mathbb{A}\,\sin\left(\mathbb{\omega}\,t + \rho\right)$$

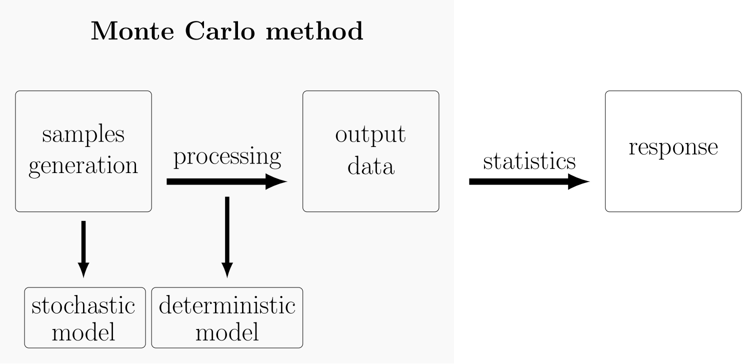
Therefore, stochastic nonlinear dynamics evolves according to

$$[M]\begin{pmatrix}\ddot{\mathbb{y}}_1(t,\theta)\\ \ddot{\mathbb{\phi}}_1(t,\theta)\\ \ddot{\mathbb{\phi}}_2(t,\theta)\end{pmatrix} + [N]\begin{pmatrix}\dot{\mathbb{y}}_1^2(t,\theta)\\ \dot{\mathbb{\phi}}_1^2(t,\theta)\\ \dot{\mathbb{\phi}}_2^2(t,\theta)\end{pmatrix} + [C]\begin{pmatrix}\dot{\mathbb{y}}_1(t,\theta)\\ \dot{\mathbb{\phi}}_1(t,\theta)\\ \dot{\mathbb{\phi}}_2(t,\theta)\end{pmatrix} + [K]\begin{pmatrix}\mathbb{y}_1(t,\theta)\\ \mathbb{\phi}_1(t,\theta)\\ \mathbb{\phi}_2(t,\theta)\end{pmatrix} = \mathbf{g} - \mathbf{h} \quad a.s.$$

supplemented by appropriate initial conditions.

Propagation of uncertainties through the model

Propagation of uncertainties is computed via Monte Carlo method.



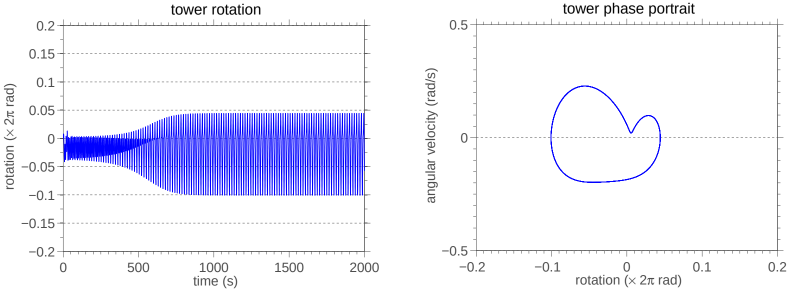
A. Cunha Jr, R. Nasser, R. Sampaio, H. Lopes, and K. Breitman, *Uncertainty quantification through Monte Carlo method in a cloud computing setting*. **Computer Physics Communications**, 185: 1355–1363, 2014.

D. P. Kroese, T. Taimre, Z. I. Botev, **Handbook of Monte Carlo Methods**, Wiley, 2011

Section 4

Numerical Experiments

Periodicity and chaos in the deterministic dynamics

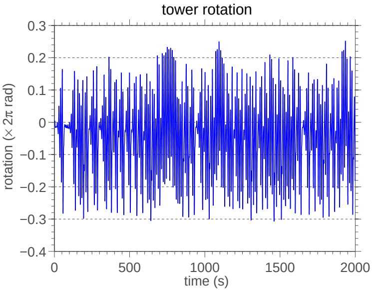


(a) tower rotation (b) phase portrait

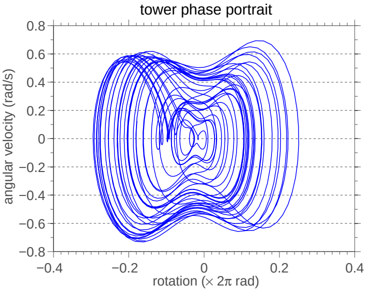
Figure: Tower dynamics for an excitation frequency $\omega = 9\ rad/s$.

A. Cunha Jr, E. S. César, J. L. P. Felix, J. M. Balthazar and P. B. Gonçalves, **On the appearance of regular and chaotic motions in the orchard tower sprayer modeled by an inverted pendulum with vehicular suspension** (in preparation).

Periodicity and chaos in the deterministic dynamics



(a) tower rotation



(b) phase portrait

Figure: Tower dynamics for an excitation frequency $\omega = 13\ rad/s$.

A. Cunha Jr, E. S. César, J. L. P. Felix, J. M. Balthazar and P. B. Gonçalves, **On the appearance of regular and chaotic motions in the orchard tower sprayer modeled by an inverted pendulum with vehicular suspension** (in preparation).

Cases of study for stochastic dynamics

Three study cases are considered:

- Case 1: random amplitude A
- Case 2: random frequency ω
- Case 3: random amplitude A and frequency ω

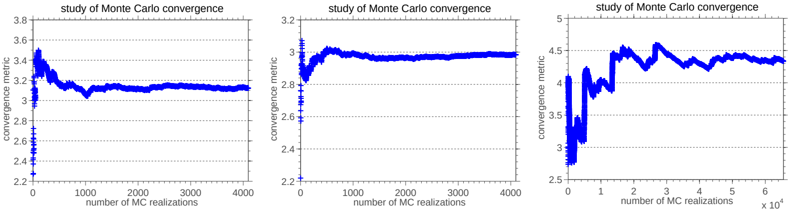
Study of convergence for MC simulation

The convergence of Monte Carlo simulation is measured with the following metric:

$$\texttt{conv}(n_s) = \left(\frac{1}{n_s} \sum_{n=1}^{n_s} \int_{t=t_0}^{t_f} \left(\mathbb{y}_1(t,\theta_n)^2 + \mathbb{\phi}_1(t,\theta_n)^2 + \mathbb{\phi}_2(t,\theta_n)^2 \right) \, dt \right)^{1/2}$$

C. Soize, *A comprehensive overview of a non-parametric probabilistic approach of model uncertainties for predictive models in structural dynamics.* **Journal of Sound and Vibration**, 288: 623–652, 2005.

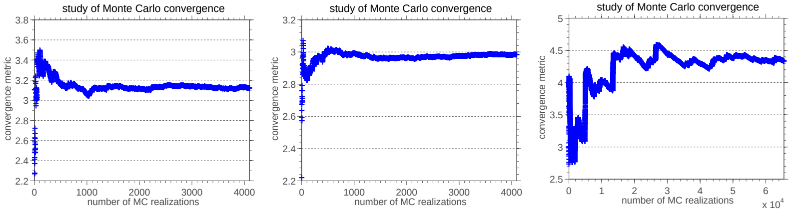
Study of convergence for MC simulation



(a) case 1 (b) case 2 (c) case 3

Figure: MC convergence metric as function of the number of realizations.

Study of convergence for MC simulation

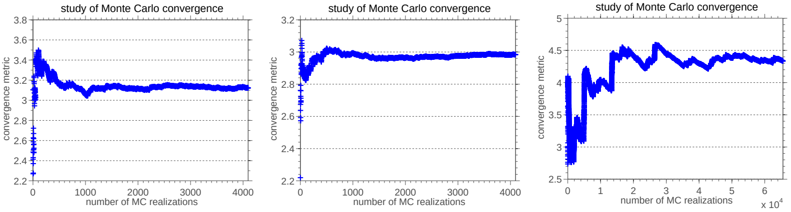


(a) case 1 (b) case 2 (c) case 3

Figure: MC convergence metric as function of the number of realizations.

For cases 1 and 2 sampling is done with 4096 realizations.

Study of convergence for MC simulation



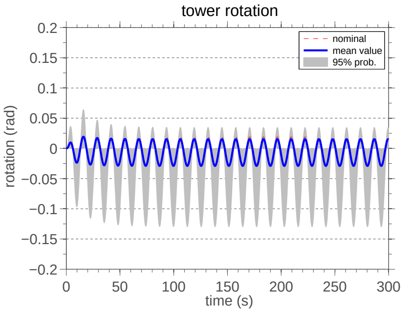
(a) case 1 (b) case 2 (c) case 3

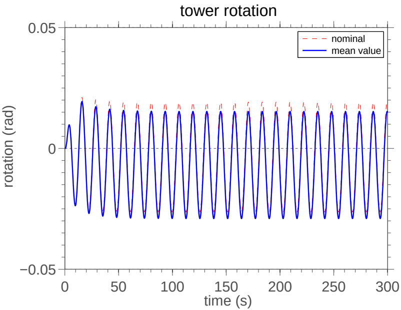
Figure: MC convergence metric as function of the number of realizations.

For cases 1 and 2 sampling is done with 4096 realizations.

For case 3 sampling is done with 65536 realizations.

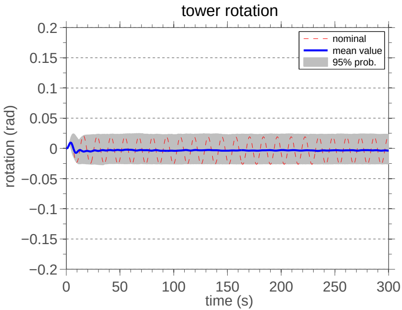
Evolution of tower rotational dynamics

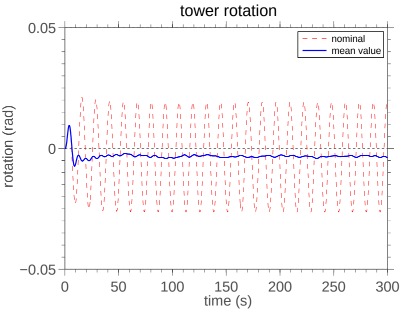




Case 1 (random $\mathbb{A}$)

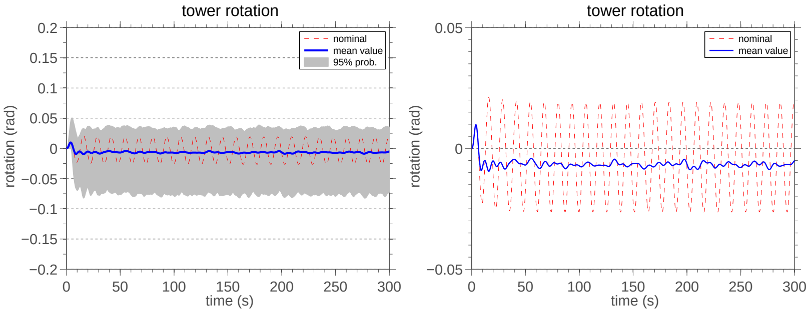
Evolution of tower rotational dynamics





Case 2 (random ω)

Evolution of tower rotational dynamics



Case 3 (random $\mathbb{A}$ and ω)

Temporal average of degrees of freedom

The tower rotation is a random field

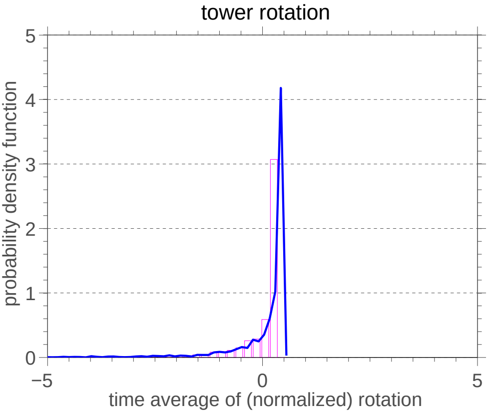
$$(t, \theta) \in [t_0, t_f] \times \Theta \longmapsto \phi_2(t, \theta) \in \mathbb{R}.$$

Its time average is defined as

$$\langle \phi_2 \rangle := \frac{1}{t_f - \tau} \int_{t=\tau}^{t_f} \phi_2(t, \theta) dt$$

which is a random variable.

Distribution of tower rotational dynamics



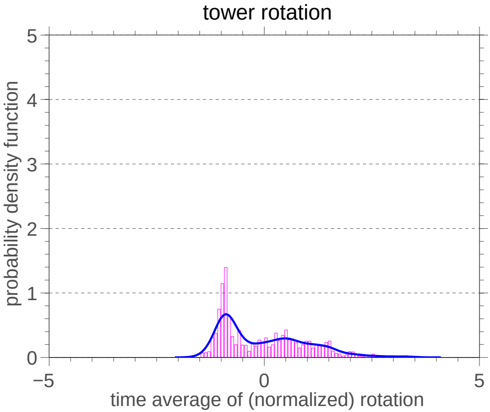
Case 1 (random $\mathbb{A}$)

Statistics of $\langle \phi_2 \rangle$

$$
\begin{aligned}
\text{mean} &= -\,0.01 \\
\text{std. dev.} &= 0.01 \\
\text{skewness} &= -\,4.60 \\
\text{kurtosis} &= 28.82
\end{aligned}
$$

$$\mathbb{P}(\langle \phi_2 \rangle > \textit{mean}) = 0.68$$

Distribution of tower rotational dynamics



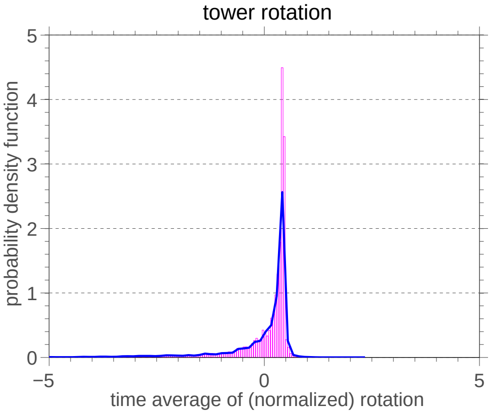
Case 2 (random ω)

Statistics of $\langle \phi_2 \rangle$

$$\begin{aligned} \text{mean} &= 0.00 \\ \text{std. dev.} &= 0.00 \\ \text{skewness} &= 0.78 \\ \text{kurtosis} &= -0.13 \end{aligned}$$

$$\mathbb{P}(\langle \phi_2 \rangle > \mathit{mean}) = 0.43$$

Distribution of tower rotational dynamics



Case 3 (random $\mathbb{A}$ and ω)

Statistics of $\langle \phi_2 \rangle$

$$
\begin{aligned}
\text{mean} &= -0.01 \\
\text{std. dev.} &= 0.01 \\
\text{skewness} &= -4.16 \\
\text{kurtosis} &= 23.57
\end{aligned}
$$

$$\mathbb{P}(\langle \phi_2 \rangle > \textit{mean}) = 0.73$$

Section 5

Final Remarks

Concluding remarks

Contributions and conclusions:

- Construction of a parametric probabilistic model for orchard sprayer dynamics uncertainties;
- Numerical simulation show large discrepancies in the stochastic system response compared nominal (deterministic) model;
- Stochastic numerical experimentation have shown it is probable the system presents great lateral vibrations.

Next steps in this work:

- Investigate the sensitivity of stochastic dynamic system response to the random parameters distribution;
- Modeling and quantification of the model uncertainties, that are due to physics lack of knowledge.

Acknowledgments

Important data supplied:

- Máquinas Agrícolas Jacto S/A

Financial support given to this research:

- CNPq
- CAPES
- FAPERJ
- FAPERGS
- FAPESP

Thank you for your attention!

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Simulation Parameters

Nominal (deterministic) parameters

parameter	value	unit
m_1	6500	kg
m_2	800	kg
L_1	200×10^{-3}	m
L_2	2400×10^{-3}	m
I_1	6850	$kg\ m^2$
I_2	6250	$kg\ m^2$
k_1	465×10^3	N/m
k_2	465×10^3	N/m
c_1	5.6×10^3	$N/m/s$
c_2	5.6×10^3	$N/m/s$
B_1	850×10^{-3}	m
B_2	850×10^{-3}	m
k_T	45×10^3	N/rad
c_T	50×10^3	$Nm/rad/s$
ρ	$\pi/9$	rad

Nonlinear Dynamics Operators

Mass Matrix (configuration dependent)

$$[M] = \begin{bmatrix} m_1 + m_2 & -m_2\, L_1\, \sin\phi_1 & -m_2\, L_2\, \sin\phi_1 \\ -m_2\, L_1\, \sin\phi_1 & I_1 + m_2\, L_1^2 & m_2\, L_1\, L_2\, \cos(\phi_2 - \phi_1) \\ -m_2\, L_2\, \sin\phi_1 & m_2\, L_1\, L_2\, \cos(\phi_2 - \phi_1) & I_2 + m_2\, L_2^2 \end{bmatrix},$$

Dampind Matrix 1 (configuration dependent)

$$[N] = \begin{bmatrix} 0 & -m_2\, L_1\, \cos\phi_1 & -m_2\, L_2\, \cos\phi_2 \\ 0 & 0 & -m_2\, L_1\, L_2\, \sin\left(\phi_2 - \phi_1\right) \\ 0 & -m_2\, L_1\, L_2\, \sin\left(\phi_2 - \phi_1\right) & 0 \end{bmatrix},$$

Damping Matrix 2 (configuration dependent)

$$[C] = \begin{bmatrix} c_1 + c_2 & (c_2\,B_2 - c_1\,B_1)\cos\phi_1 & 0 \\ (c_2\,B_2 - c_1\,B1)\cos\phi_1 & c_T + (c_1\,B_1^2 + c_2\,B_2^2)\cos^2\phi_1 & -c_T \\ 0 & -c_T & c_T \end{bmatrix},$$

Stiffness Matrix (configuration dependent)

$$[K] = \left[\begin{array}{ccc} k_1 + k_2 & 0 & 0 \\ (k_2\, B_2 - k_1\, B1) \cos\phi_1 & k_T & -k_T \\ 0 & -k_T & k_T \end{array} \right],$$

Weight Loading Vector (configuration dependent)

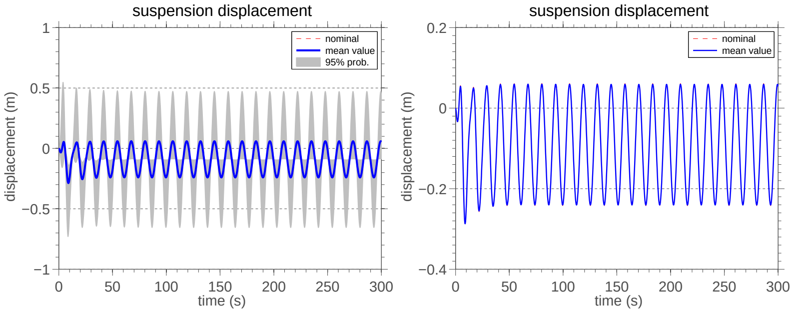
$$\mathbf{g} = \begin{pmatrix} (k_2\, B_2 - k_1\, B_1) \sin\phi_1 + (m_1 + m_2) g \\ (k_1\, B_1^2 + k_2\, B_2^2) \sin\phi_1\, \cos\phi_1 - m_2\, g\, L_1\, \sin\phi_1 \\ -m_2\, g\, L_2\, \sin\phi_2 \end{pmatrix},$$

External Loading Vector (configuration dependent)

$$\mathbf{h} = \begin{pmatrix} k_1\, y_{e1} + k_2\, y_{e2} + c_1\, \dot{y}_{e1} + c_2\, \dot{y}_{e2} \\ -k_1\, B_1\, \cos\phi_1\, y_{e1} + k_2\, B_2\, \cos\phi_1\, y_{e2} - c_1\, B_2\, \cos\phi_1\, \dot{y}_{e1} + c_2\, B_2\, \cos\phi_1\, \dot{y}_{e2} \\ 0 \end{pmatrix}.$$

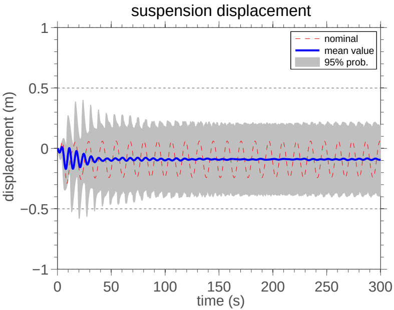
Other Numerical Results

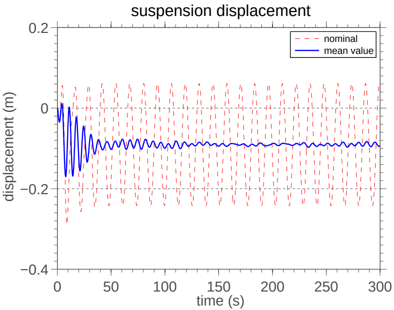
Evolution of suspension translational dynamics



Case 1 (random $\mathbb{A}$)

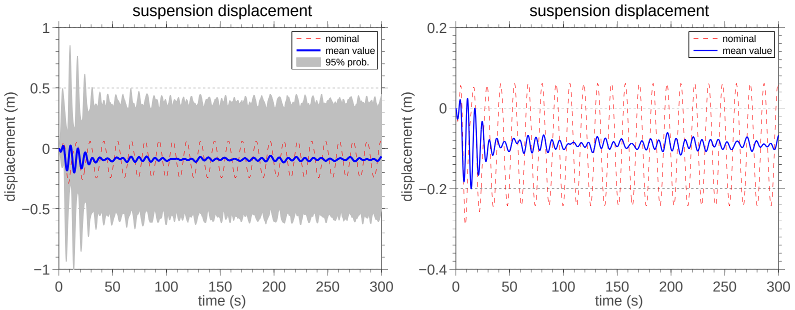
Evolution of suspension translational dynamics





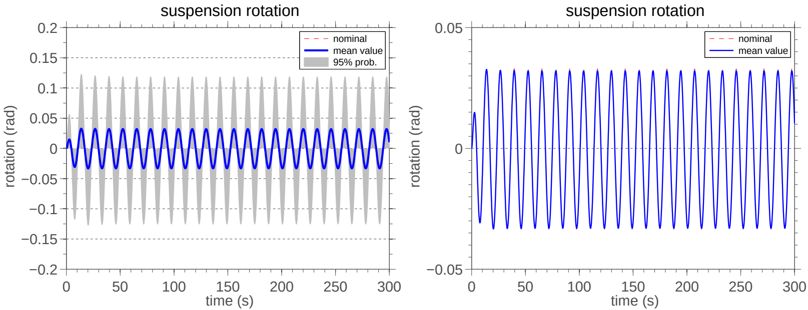
Case 2 (random ω)

Evolution of suspension translational dynamics



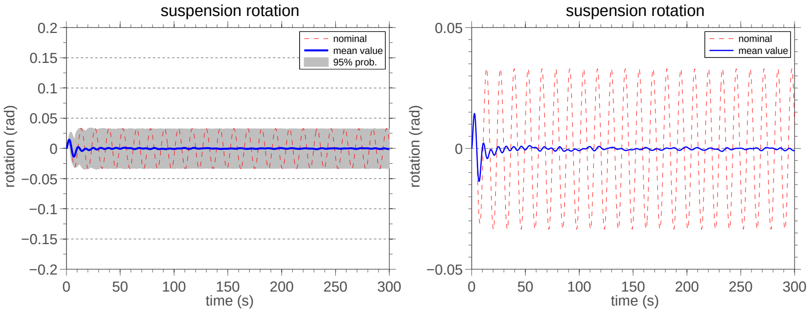
Case 3 (random $\mathbb{A}$ and ω)

Evolution of suspension rotational dynamics



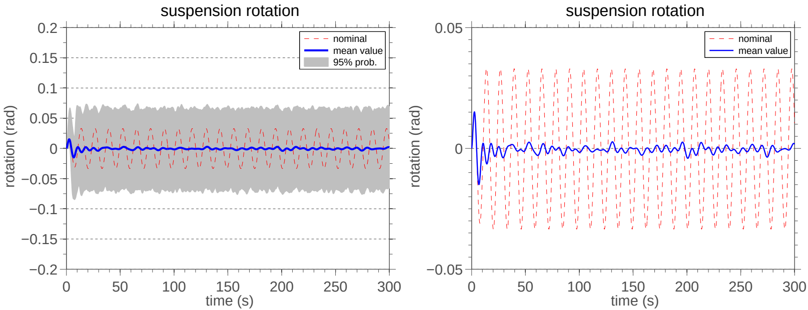
Case 1 (random $\mathbb{A}$)

Evolution of suspension rotational dynamics



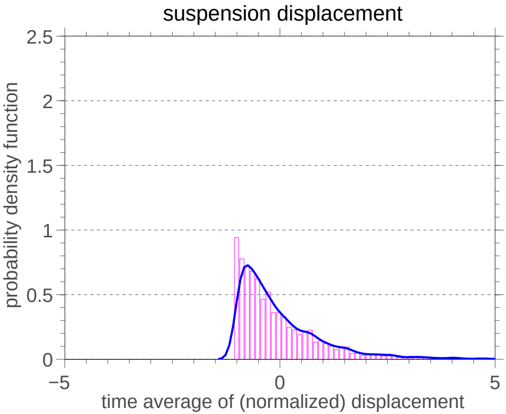
Case 2 (random ω)

Evolution of suspension rotational dynamics



Case 3 (random $\mathbb{A}$ and ω)

Distribution of suspension translational dynamics



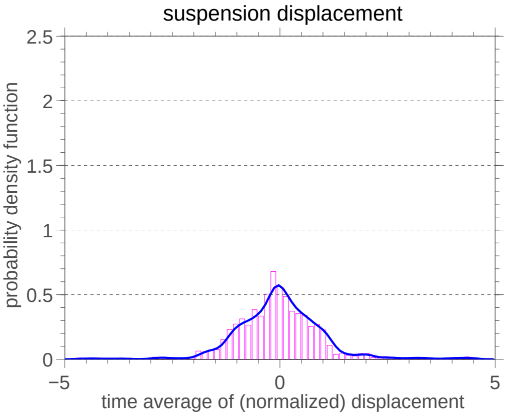
Case 1 (random $\mathbb{A}$)

Statistics of $\langle \mathbb{y}_1 \rangle$

$$
\begin{aligned}
\text{mean} &= -0.09 \\
\text{std. dev.} &= 0.00 \\
\text{skewness} &= 1.89 \\
\text{kurtosis} &= 4.71
\end{aligned}
$$

$$\mathbb{P}(\langle \mathbb{y}_1 \rangle > \mathit{mean}) = 0.37$$

Distribution of suspension translational dynamics



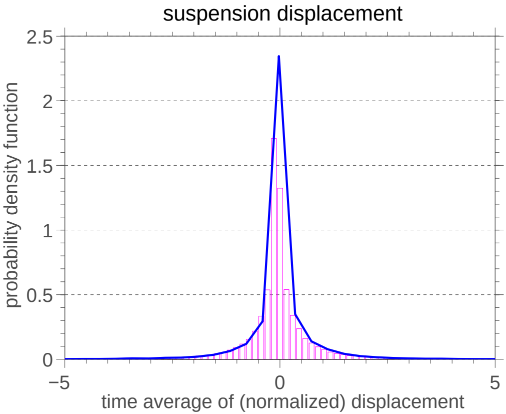
Case 2 (random ω)

Statistics of $\langle \mathbb{y}_1 \rangle$

$$
\begin{aligned}
\text{mean} &= -0.09 \\
\text{std. dev.} &= 0.01 \\
\text{skewness} &= 0.30 \\
\text{kurtosis} &= 4.07
\end{aligned}
$$

$$\mathbb{P}(\langle \mathbb{y}_1 \rangle > \textit{mean}) = 0.37$$

Distribution of suspension translational dynamics



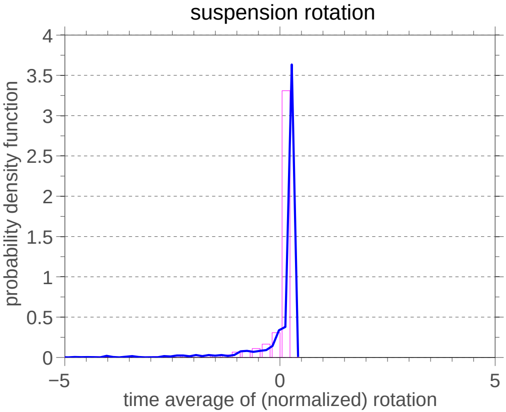
Case 3 (random $\mathbb{A}$ and ω)

<u>Statistics of $\langle \mathbb{y}_1 \rangle$</u>

$$
\begin{aligned}
\text{mean} &= -0.09 \\
\text{std. dev.} &= 0.01 \\
\text{skewness} &= -0.75 \\
\text{kurtosis} &= 42.29
\end{aligned}
$$

$$\mathbb{P}(\langle \mathbb{y}_1 \rangle > \textit{mean}) = 0.38$$

Distribution of suspension rotational dynamics



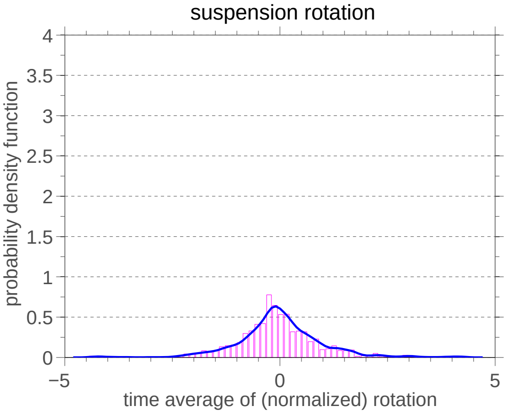
Case 1 (random $\mathbb{A}$)

Statistics of $\langle \phi_1 \rangle$

$$
\begin{aligned}
\text{mean} &= 0.00 \\
\text{std. dev.} &= 0.00 \\
\text{skewness} &= \text{-}5.74 \\
\text{kurtosis} &= 44.12
\end{aligned}
$$

$$\mathbb{P}(\langle \phi_1 \rangle > mean) = 0.38$$

Distribution of suspension rotational dynamics



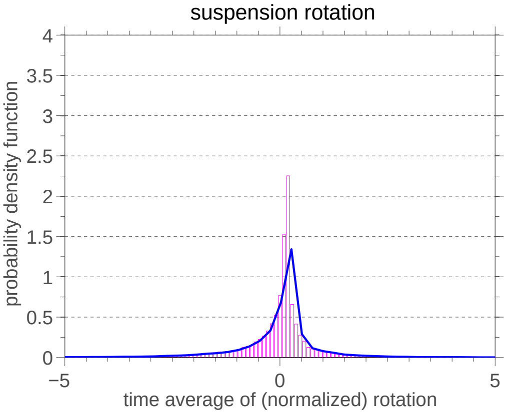
Case 2 (random ω)

Statistics of $\langle \phi_1 \rangle$

$$
\begin{aligned}
\text{mean} &= 0.00 \\
\text{std. dev.} &= 0.00 \\
\text{skewness} &= 0.17 \\
\text{kurtosis} &= 3.38
\end{aligned}
$$

$$\mathbb{P}(\langle \phi_1 \rangle > \mathit{mean}) = 0.42$$

Distribution of suspension rotational dynamics



Case 3 (random $\mathbb{A}$ and ω)

Statistics of $\langle \phi_1 \rangle$

$$\begin{aligned} \text{mean} &= 0.00 \\ \text{std. dev.} &= 0.00 \\ \text{skewness} &= -1.05 \\ \text{kurtosis} &= 14.90 \end{aligned}$$

$$\mathbb{P}(\langle \phi_1 \rangle > mean) = 0.63$$