ON THE NONLINEAR DYNAMICS OF AN INVERTED DOUBLE PENDULUM OVER A VEHICLE SUSPENSION SUBJECTED TO RANDOM EXCITATIONS

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Abstract. *This paper deals with the nonlinear dynamics of a mechanical system which consists of an agricultural tower pulverizer, coupled with a vehicle suspension that is subject to random excitations due to soil irregularities. A deterministic mathematical model, where tower is considered as an inverted double pendulum over an vehicle suspension, with three degrees of freedom (one translation and two rotations) is constructed. To take into account the random loadings due to soil variabilities, a parametric probabilistic approach is employed, where the external force is assumed to be a harmonic random process. This stochastic process has random amplitude and frequency, which are modeled as random variables, and a sinusoidal shape in time. Once experimental data are not available for adjusting consistent distributions for these random variables, their distributions are constructed based only on theoretical information, using the maximum entropy principle. This procedure result in a system of stochastic ordinary differential equations, which defines a stochastic model for the problem. The propagation of uncertainties through the stochastic model is computed using the Monte Carlo method. Numerical simulations show large discrepancies in the system response obtained with the mean of the stochastic model and with the nominal (deterministic) model, and a high level of uncertainty associated. Also, an analysis of the system response probability distributions shows that they present asymmetries with respect to mean and unimodal behavior.*

Keywords: *nonlinear dynamics; inverted double pendulum; agricultural tower pulverizer; uncertainty quantification; parametric probabilistic approach*

1. INTRODUCTION

During the orchards spraying process it is used an equipment called *tower orchard sprayer*. This equipment is composed by two main devices, a vehicle suspension and a support tower equipped with several fans. During typical operating conditions, it is subject to vibration mechanisms that interact nonlinearly. A schematically representation of this device is presented in Figure 1.

Such device has already been studied by Sartori Junior *et al.* (2007), Sartori Junior (2008) and Sartori Junior *et al.* (2009), using a model that considers an inverted double pendulum mounted on a vehicular suspension to emulate the equipment. These works perform parametric analyzes to investigate the influence of certain quantities in the model, for instance, stiffness, torsional damping, etc., and conclude that the developed model respond consistently in all cases it was tested.
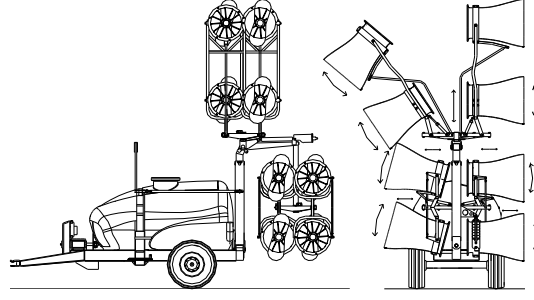
Figure 1. Schematic representation of the tower orchard sprayer. Adapted from Sartori Junior (2008), and courtesy of Máquinas Agrícolas Jacto S/A.

But in the best of authors' knowledge there is no work in open literature which investigate other issues of relevance associated with tower orchard sprayer dynamics, such as its dynamic stability, analysis of the device behavior when it is subjected to a random excitations, a condition which is more representative of the observed in a real system, etc.

Hence, in order to fill the gaps in the study of orchard sprayer nonlinear dynamics, when the mechanical system is subject to random excitations, this work aims to construct a consistent stochastic model to take into account the random loadings, and then compute the propagation of uncertainties through the model. This investigation aims to determine how the system response is affected by the model parameters uncertainties.

The rest of this paper is organized as follows. In section 2 the mechanical system of interest is defined and a deterministic model to describe its nonlinear dynamics of is presented. A stochastic model to to take into account the uncertainties associated to the model parameters is shown in section 3. Numerical experimentation results are presented and discussed in section 4. Finally, in section 5, the main conclusions are highlighted.

2. DETERMINISTIC MODEL

For modeling purposes, the mechanical system is considered an inverted double pendulum, mounted on a vehicular suspension, such as illustrated in Figure 2. This model was developed by Sartori Junior (2008); Sartori Junior *et al.* (2009). The masses of the chassis and the tank are assumed to be concentrated at the bottom of the double pendulum, as a point mass denoted by m_1. On the other hand, the point mass m_2, at the top of the double pendulum, take into account the masses of the fans. The point of articulation between the moving suspension and the tower is denoted by P and its distance to the suspension center of gravity is L_1. The junction P has torsional stiffness k_T, and damping torsional coefficient c_T. The tower has length L_2, and is considered to be massless. The left wheel of the vehicle suspension is represented by a pair spring/damper with constant respectively given by k_1 and c_1, it is located at a distance B_1 from suspension center line, and it is subject to a vertical displacement y_{e1}. Similarly, the right wheel is represented by a pair spring/damper characterized by k_2 and c_2, it is B_2 away from suspension center line, and and displaces vertically y_{e2}. The moments of inertia of the suspension and of the tower, with respect to their centers of gravity, are respectively denoted by I_1 and I_2. Finally, introducing the inertial frame of reference XY, the vertical displacement of the suspension is measured by y_1, while its rotation is computed by ϕ_1, and the rotation of the tower is denoted by ϕ_2.

The left and right tires displacements are respectively assumed to be periodic functions in time, out of phase, with the same amplitude, and a single frequencial component,

$$y_{e1}(t) = A \sin(\omega t), \quad \text{and} \quad y_{e2}(t) = A \sin(\omega t + \rho), \tag{1}$$

where A and ω respectively denote the amplitude and frequency of the tires displacements, and ρ is the phase shift between the two tires.

The nonlinear dynamics of the mechanical system system evolves according to the following system of ordinary differential equations (Sartori Junior *et al.*, 2009)

$$[M] \begin{pmatrix} \ddot{y}_1(t) \\ \ddot{\phi}_1(t) \\ \ddot{\phi}_2(t) \end{pmatrix} + [N] \begin{pmatrix} \dot{y}_1^2(t) \\ \dot{\phi}_1^2(t) \\ \dot{\phi}_2^2(t) \end{pmatrix} + [C] \begin{pmatrix} \dot{y}_1(t) \\ \dot{\phi}_1(t) \\ \dot{\phi}_2(t) \end{pmatrix} + [K] \begin{pmatrix} y_1(t) \\ \phi_1(t) \\ \phi_2(t) \end{pmatrix} = \boldsymbol{g} - \boldsymbol{h}, \tag{2}$$
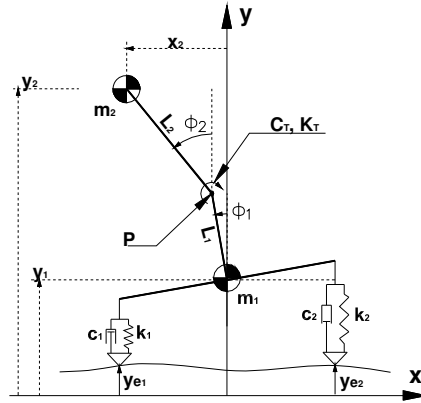
Figure 2. Schematic representation of the mechanical-mathematical model: an inverted double pendulum, mounted on a moving suspension. Adapted from Sartori Junior (2008).

where $[M]$, $[N]$, $[C]$ and $[N]$ are 3×3 (configuration dependent) real matrices, respectively, defined by

$$[M] = \begin{bmatrix} m_1 + m_2 & -m_2 \, L_1 \, \sin \phi_1 & -m_2 \, L_2 \, \sin \phi_1 \\ -m_2 \, L_1 \, \sin \phi_1 & I_1 + m_2 \, L_1^2 & m_2 \, L_1 \, L_2 \, \cos (\phi_2 - \phi_1) \\ -m_2 \, L_2 \, \sin \phi_1 & m_2 \, L_1 \, L_2 \, \cos (\phi_2 - \phi_1) & I_2 + m_2 \, L_2^2 \end{bmatrix}, \tag{3}$$

$$[N] = \begin{bmatrix} 0 & -m_2 \, L_1 \, \cos \phi_1 & -m_2 \, L_2 \, \cos \phi_2 \\ 0 & 0 & -m_2 \, L_1 \, L_2 \, \sin (\phi_2 - \phi_1) \\ 0 & -m_2 \, L_1 \, L_2 \, \sin (\phi_2 - \phi_1) & 0 \end{bmatrix}, \tag{4}$$

$$[C] = \begin{bmatrix} c_1 + c_2 & (c_2 \, B_2 - c_1 \, B_1) \cos \phi_1 & 0 \\ (c_2 \, B_2 - c_1 \, B1) \cos \phi_1 & c_T + (c_1 \, B_1^2 + c_2 \, B_2^2) \cos^2 \phi_1 & -c_T \\ 0 & -c_T & c_T \end{bmatrix}, \tag{5}$$

and

$$[K] = \begin{bmatrix} k_1 + k_2 & 0 & 0 \\ (k_2 \, B_2 - k_1 \, B1) \cos \phi_1 & k_T & -k_T \\ 0 & -k_T & k_T \end{bmatrix}, \tag{6}$$

and let $\boldsymbol{g}$, and $\boldsymbol{h}$ be (configuration dependent) vectors in $\mathbb{R}^3$, respectively, defined by

$$\boldsymbol{g} = \begin{pmatrix} (k_2 \, B_2 - k_1 \, B_1) \sin \phi_1 + (m_1 + m_2) g \\ (k_1 \, B_1^2 + k_2 \, B_2^2) \sin \phi_1 \, \cos \phi_1 - m_2 \, g \, L_1 \, \sin \phi_1 \\ -m_2 \, g \, L_2 \, \sin \phi_2 \end{pmatrix}, \tag{7}$$

and

$$\boldsymbol{h} = \begin{pmatrix} k_1 \, y_{e1} + k_2 \, y_{e2} + c_1 \, \dot{y}_{e1} + c_2 \, \dot{y}_{e2} \\ -k_1 \, B_1 \, \cos \phi_1 \, y_{e1} + k_2 \, B_2 \, \cos \phi_1 \, y_{e2} - c_1 \, B_2 \, \cos \phi_1 \, \dot{y}_{e1} + c_2 \, B_2 \, \cos \phi_1 \, \dot{y}_{e2} \\ 0 \end{pmatrix}. \tag{8}$$

Considering also the appropriate initial conditions, the resulting nonlinear initial value problem is integrated using Runge-Kutta-Fehlberg method (RKF45) (Fehlberg, 1969) that is implemented in MATLAB routine `ode45` (Shampine and Reichelt, 1997).

3. STOCHASTIC MODEL

Consider a probability space $(\Theta, \Sigma, \mathbb{P})$, where Θ is a sample space, Σ is a σ-field over Θ, and $\mathbb{P} : \Sigma \to [0, 1]$ is a probability measure. In this probabilistic space, the amplitude A and the frequency ω are respectively modeled by the random variables $\mathbb{A} : \Sigma \to \mathbb{R}$ and $\omega : \Sigma \to \mathbb{R}$.

To specify the distribution of these random parameters, based only on theoretical information known about them, the maximum entropy principle is employed (Soize, 2005a,b, 2013). For $\mathbb{A}$, that is a positive parameter, it is assumed that: (i) the support of the probability density function (PDF) is the positive real line, i.e., $\mathtt{Supp}\, p_{\mathbb{A}} = (0, +\infty)$; (ii) the mean value is known, i.e. $\mathbb{E}[\mathbb{A}] = \mu_{\mathbb{A}} \in (0, +\infty)$; and (iii) $\mathbb{E}[\ln \mathbb{A}] = q, \ |q| < +\infty$. Besides that, for ω, that is also a positive parameter, the only known information are assumed to be: (i) the support $\mathtt{Supp}\, p_{\omega} = (0, +\infty)$, and (ii) the mean value $\mathbb{E}[\omega] = \mu_{\omega} \in (0, +\infty)$.

Consequently, the distributions which maximize the entropy have the following PDFs

$$p_{\mathbb{A}}(a) = \mathbb{1}_{(0,+\infty)}(a)\, \frac{1}{\mu_{\mathbb{A}}} \left(\frac{1}{\delta_{\mathbb{A}}^2}\right)^{\left(\frac{1}{\delta_{\mathbb{A}}^2}\right)} \frac{1}{\Gamma(1/\delta_{\mathbb{A}}^2)} \left(\frac{a}{\mu_{\mathbb{A}}}\right)^{\left(\frac{1}{\delta_{\mathbb{A}}^2} - 1\right)} \exp\left(-\frac{a}{\delta_{\mathbb{A}}^2 \mu_{\mathbb{A}}}\right), \tag{9}$$

and

$$p_{\omega}(\omega) = \mathbb{1}_{(0,+\infty)}(\omega)\, \frac{1}{\mu_{\omega}} \exp\left(-\frac{\omega}{\mu_{\omega}}\right), \tag{10}$$

which correspond, respectively, to the gamma and exponential distributions. In the above equations $\mathbb{1}_X(x)$ denotes the indicator function of the set X, and $0 \leq \delta_{\mathbb{A}} = \sigma_{\mathbb{A}}/\mu_{\mathbb{A}} < 1/\sqrt{2}$ is a dispersion parameter, being $\sigma_{\mathbb{A}}$ the standard deviation of $\mathbb{A}$.

Due to the randomness of $\mathbb{A}$ and ω, the tire displacements are now described by the following random processes

$$y_{e1}(t, \theta) = \mathbb{A} \sin(\omega t), \quad \text{and} \quad y_{e2}(t, \theta) = \mathbb{A} \sin(\omega t + \rho). \tag{11}$$

Therefore, the dynamics of the mechanical system evolves (almost sure) according to the following system of stochastic differential equations

$$[M] \begin{pmatrix} \ddot{y}_1(t,\theta) \\ \ddot{\phi}_1(t,\theta) \\ \ddot{\phi}_2(t,\theta) \end{pmatrix} + [N] \begin{pmatrix} \dot{y}_1^2(t,\theta) \\ \dot{\phi}_1^2(t,\theta) \\ \dot{\phi}_2^2(t,\theta) \end{pmatrix} + [C] \begin{pmatrix} \dot{y}_1(t,\theta) \\ \dot{\phi}_1(t,\theta) \\ \dot{\phi}_2(t,\theta) \end{pmatrix} + [K] \begin{pmatrix} y_1(t,\theta) \\ \phi_1(t,\theta) \\ \phi_2(t,\theta) \end{pmatrix} = \boldsymbol{g} - \boldsymbol{h}. \quad a.s. \tag{12}$$

The Monte Carlo (MC) method (Robert and Casella, 2010; Cunha Jr et al., 2014) is employed to compute the propagation of uncertainties of the random parameters through the nonlinear dynamics of the mechanical system. This stochastic solver uses stochastic model for the random parameters, constructed above, to generate realizations of them. Each realization defines a new deterministic nonlinear dynamical system, which is integrated using the procedure described in section 2. A large amount of data is generated and statistics of them are calculated to access the stochastic nonlinear dynamics.

4. NUMERICAL EXPERIMENTS

The nominal (deterministic) parameters presented in Table 1 are adopted to simulate the nonlinear dynamics of the mechanical system. For the random variables $\mathbb{A}$ and ω, the parameters which define these random objects are $\mu_{\mathbb{A}} = 300 \times 10^{-3}\ m$, $\delta_{\mathbb{A}} = 0.25$, and $\mu_{\omega} = 9\ rad/s$. For simplicity, all the initial conditions are assumed to be zero. The evolution of this nonlinear dynamic system is investigated for a "temporal window" defined by the interval $[t_0, t_f] = [0, 1] \times 10^3\ s$, using an adaptive time step, which is refined whenever necessary to capture the nonlinear effects.

4.1 Investigating the presence of periodicity and chaos

Depending on the excitation frequency which the mechanical system is subject, it can present periodic or even chaotic behavior. This statement can be verified by observing Figures 3 and 4, which illustrate the tower

Table 1. Nominal (deterministic) parameters for the mechanical system that are used in the simulations.

parameter	value	unit
m_1	6500	kg
m_2	800	kg
L_1	200×10^{-3}	m
L_2	2400×10^{-3}	m
I_1	6850	$kg\,m^2$
I_2	6250	$kg\,m^2$
k_1	465×10^3	N/m
k_2	465×10^3	N/m
c_1	5.6×10^3	$N/m/s$
c_2	5.6×10^3	$N/m/s$
B_1	850×10^{-3}	m
B_2	850×10^{-3}	m
k_T	45×10^3	N/rad
c_T	50×10^3	$Nm/rad/s$
ρ	$\pi/9$	rad

deterministic dynamics, with excitation amplitude $A = 1\ m$, for excitation frequencies equal to $w = 9\ rad/s$ and $w = 13\ rad/s$, respectively. On the left part of the these figures one can see the tower rotation as function of time, and on the right part its phase portrait. Note that a scaling factor of 2π was introduced to allow one read the rotation in "revolution". Clearly, in the first case one has periodic movement, while in the second situation it is noted the the existence of a chaotic behavior (Guckenheimer and Holmes, 1983; Strogatz, 2001).
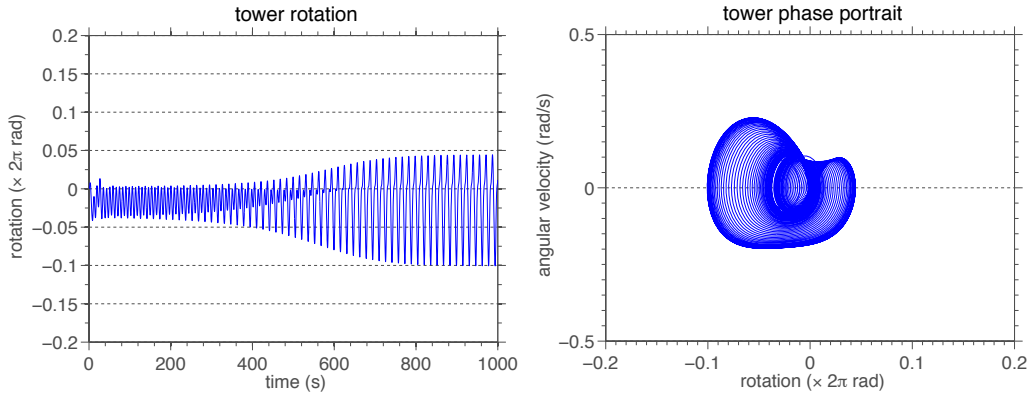

Figure 3. Illustration of tower dynamics for an excitation frequency $w = 9\ rad/s$. The tower rotation as function of time is shown on the left, and the phase portrait on the right.

4.2 Study of convergence for MC simulation

In order to ensure the "quality" of the statistics obtained from MC data, it is necessary to study the convergence of these stochastic simulations. For this purpose, it is taken into consideration the map $n_s \in \mathbb{N} \mapsto \mathtt{conv}(n_s) \in \mathbb{R}$, being

$$\mathtt{conv}(n_s) = \left(\frac{1}{n_s} \sum_{n=1}^{n_s} \int_{t=t_0}^{t_f} \left(y_1(t,\theta_n)^2 + \phi_1(t,\theta_n)^2 + \phi_2(t,\theta_n)^2 \right)\ dt \right)^{1/2}, \tag{13}$$

where n_s is the number of MC realizations. This metric allows one to evaluate the convergence of the approximation $\left(y_1(t,\theta_n), \phi_1(t,\theta_n), \phi_2(t,\theta_n) \right)^T$ in the mean-square sense. See Soize (2005a) for further details.

The evolution of $\mathtt{conv}(n_s)$ as a function of n_s can be seen in Figure 5. Note that for $n_s = 4096$ the metric value has reached a steady value in all cases studied. So, all the stochastic simulations reported in this work use $n_s = 4096$.
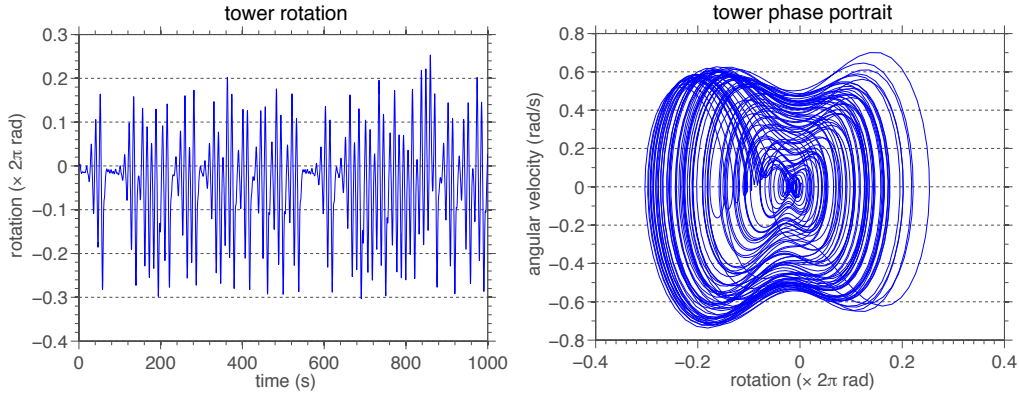
Figure 4. Illustration of tower dynamics for an excitation frequency $w = 13 \ rad/s$. The tower rotation as function of time is shown on the left, and the phase portrait on the right.
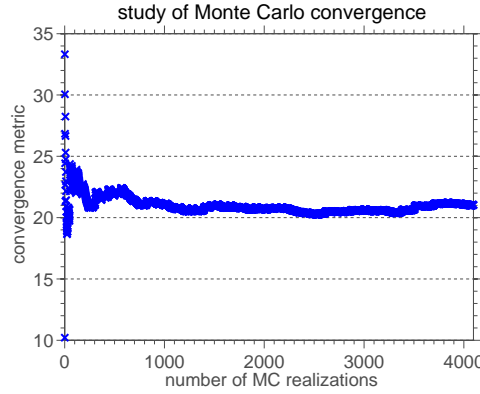

Figure 5. Illustration of MC convergence metric as function of the number of realizations.

In the next section it presented an analysis of how uncertainties (due to randomness of A and w) are propagated through the model.

4.3 Propagation of uncertainties through the nonlinear dynamics

One can analyses the evolution of the suspension stochastic translational dynamics in Figure 6. At this Figure are represented, the mean value (blue line), the nominal value (red line), and an envelope of reliability (grey shadow), wherein a realization of the stochastic system has 90% of probability of being contained, for suspension displacement (left) and velocity (right). Note that in this case the mean value and the nominal value present large discrepancy. While the nominal model predicts that the mechanical system will go into steady state, this is not what is seen when one observe the realizations mean. One can also observe a large amplitude of reliability envelope around the mean, greater than the nominal value amplitude. All these factors indicate that suspension translational dynamics is subject to a high level of uncertainty.

The stochastic rotational dynamics of the suspension can be seen in Figure 7, which presents the mean value (blue line), the nominal value (red line), and a 90% of probability envelope of reliability (grey shadow) for suspension rotation (left) and angular velocity (right). In this figure, scaling factors were introduced to allow one to read rotation in "revolution", and angular velocity in "revolution per minute". Large discrepancies between the nominal value and MC samples mean are observed again for both, rotation and angular velocity. However, now, the order of magnitude of the reliability envelopes are comparable to the nominal value curves amplitudes.

Furthermore, in Figure 8 it is possible to see stochastic rotational dynamics of the tower. As well as Figures 6 and 7, this figure also show a mean value, a nominal value and an envelope of reliability, but now for tower rotation (left) and angular velocity (right). Scaling factor to read rotation in "revolution" and angular velocity in "revolution per minute" were introduced again. It is noted again major differences between nominal and mean values, but with a peculiarity in the amplitudes of reliability envelopes. In case of the angular velocity, nominal value and and envelope of reliability have amplitudes of the same order of magnitude. In the case of rotation, the amplitude of the envelope reliability is much higher, indicating a high level of uncertainty.
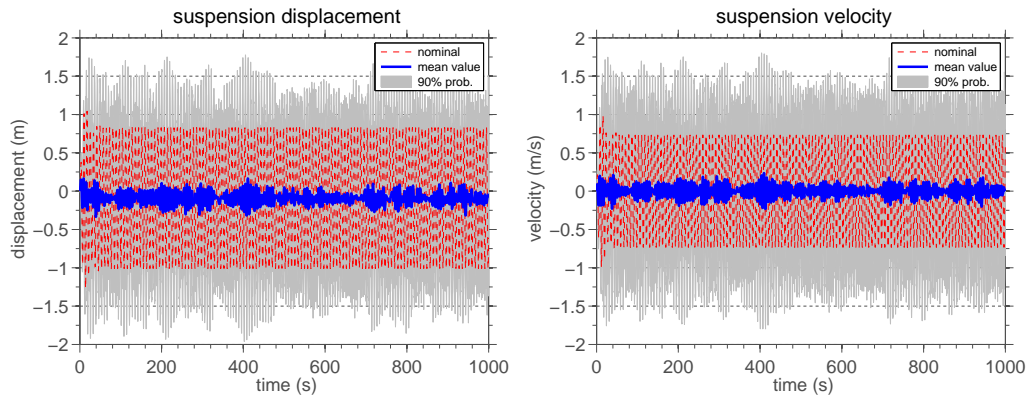
Figure 6. This figure illustrates evolution of suspension translational dynamics. It can be seen the mean value (blue line), the nominal value (red line), and a 90% of probability confidence band (grey shadow), as function of time, for suspension displacement (left) and velocity (right).
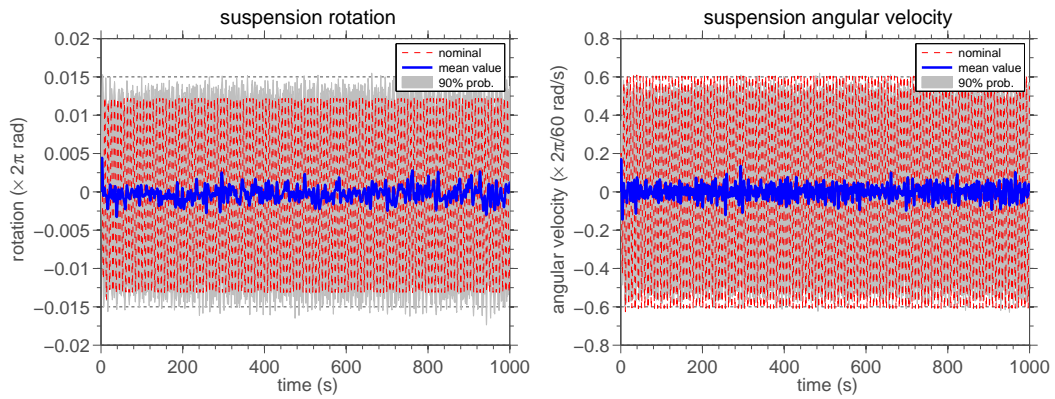

Figure 7. This figure illustrates evolution of suspension rotational dynamics. It can be seen the mean value (blue line), the nominal value (red line), and a 90% of probability confidence band (grey shadow), as function of time, for suspension rotational (left) and angular velocity (right).
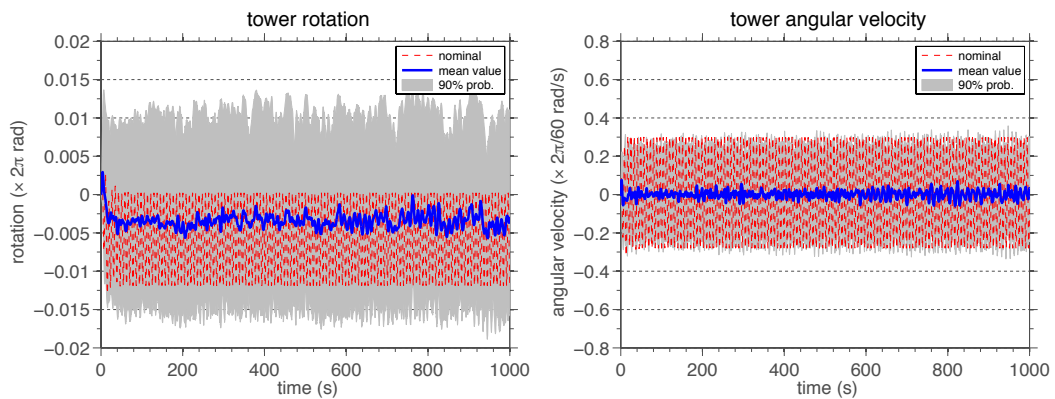

Figure 8. This figure illustrates evolution of tower rotational dynamics. It can be seen the mean value (blue line), the nominal value (red line), and a 90% of probability confidence band (grey shadow), as function of time, for tower rotational (left) and angular velocity (right).
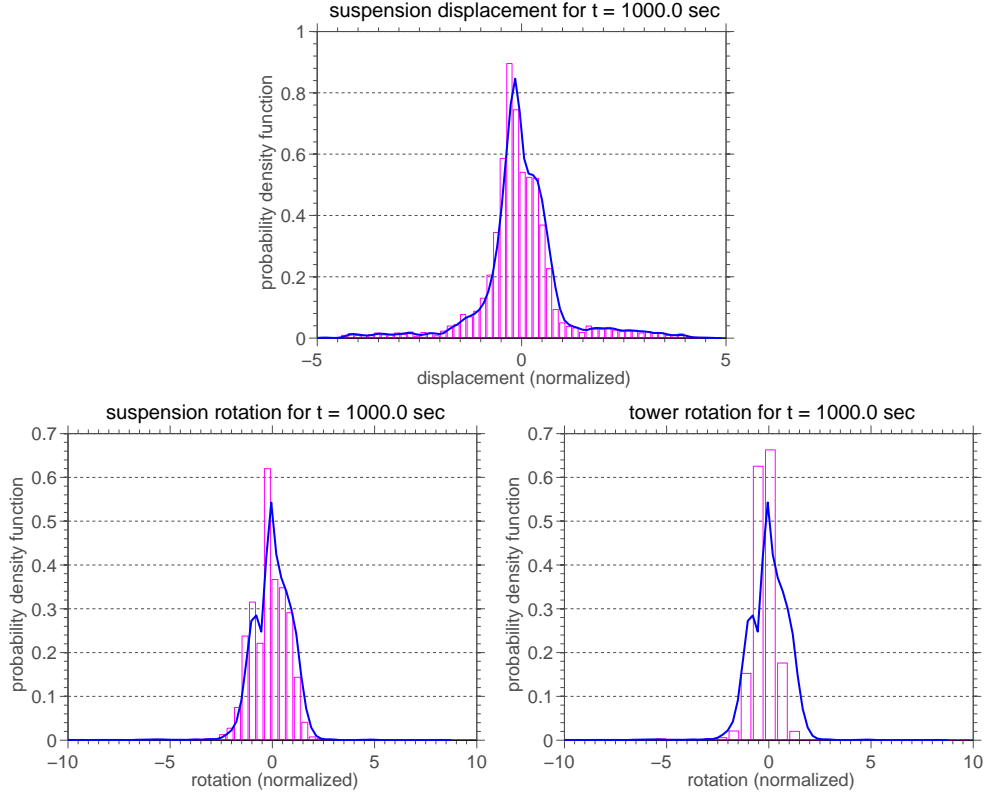
Figure 9. This figure illustrates estimations to the (normalized) PDFs of the: (i) suspension displacement, (ii) suspension rotation, and (iii) tower rotation, at the instant $t = 1000\ s$.

4.4 Probability distribution of the system response

The three degrees of freedom of the system response are the random processes $y_1(t, \theta)$, $\phi_1(t, \theta)$, and $\phi_2(t, \theta)$. Fixing the time t in each one of these random processes, they becomes a random variables. Therefore, for each instant, there is an associated probability distribution to the underlying random process. Accordingly, in what follows it is presented an analysis of the (normalized) probability density functions (PDFs) associated with the mechanical system response. In this context normalized means a random variable with zero mean and unit standard deviation.

In Figure 9, one can observe estimations for the (normalized) PDFs of the: (i) suspension displacement, (ii) suspension rotation, and (iii) tower rotation, at the instant $t = 1000\ s$. In all cases it is possible to observe asymmetries with respect to mean. It is also noted a unimodal behavior in all analyzed distributions, with maximum always occurring in the mean neighborhood.

5. FINAL REMARKS

This work presented the study of the nonlinear dynamics of an agricultural tower pulverizer, coupled with a vehicle suspension, that is subject to random excitations due to soil irregularities, modeled as an inverted double pendulum over a moving suspension, with three degrees of freedom (one translation and two rotations). To take into account the random loadings, a parametric probabilistic approach was employed, where the external force was assumed to be a harmonic random process with random amplitude and frequency. The probability distribution of these random parameters was constructed based on the known information through the maximum entropy principle. The results of numerical simulation show that large discrepancies in the system response can be seen when one compares the mean of the stochastic model with the nominal (deterministic) model. It is also noted that these responses are subject to a high level of uncertainty. Furthermore, an analysis of the system response probability distributions shows that they present asymmetries with respect to mean and unimodal behavior.

6. ACKNOWLEDGEMENTS

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